Computation of Cournot-Nash Equilibria by Entropic Regularization

Computational Optimal Transport

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Presentation of the problem

Cournot-Nash Equilibrium Definition: A joint distribution $\gamma \in \mathcal{P}(\Theta \times \mathcal{Y})$ satisfying:

- $P_1^{\#}\gamma = \mu$ (Feasibility Condition)
- $P_2^{\#}\gamma = \nu$ (Self-consistency Condition)
- $\gamma\left(\{(x,y) \in \Theta \times \mathcal{Y} : \Phi(x,y,\nu) = \min_{z\in\mathcal{Y}} \Phi(x,z,\nu)\}\right) = 1$

Cost Function: The function $\Phi(x,y,\nu)$ is decomposed as:

$$\Phi(x,y,\nu) = c(x,y) + V(\nu)(y),$$

where:

- $c(x,y)$: Individual cost associated with strategy y for type x,
- $V(\nu)(y)$: Aggregated cost of strategy y, including:
  - **Repulsive effects:** Higher cost when many players choose similar strategies,
  - **Attractive (mimetic) effects:** Higher cost for outlier strategies.

Variational Reformulation

Let define E an energy function:

$$E(\nu) = \int_{\mathcal{Y}} F(y) d\nu(y) + \frac{1}{2} \int_{\mathcal{Y}^2} \phi(y_1, y_2) d\nu(y_1) d\nu(y_2),$$

and the optimal transport cost:

$$\mathcal{W}_c(\mu, \nu) = \inf_{\gamma \in \Pi(\mu, \nu)} \int_{\Theta \times \mathcal{Y}} c(x, y) d\gamma(x, y)$$

The equilibrium corresponds to solving:

$$\inf_{\nu \in \mathcal{P}(\mathcal{Y})} \mathcal{W}_c(\mu, \nu) + E(\nu).$$

Entropic Regularization

Discrete Problem: Assuming strategies μ and ν are discrete, problem becomes:

$$\inf_{\nu \in \mathcal{P}(\mathcal{Y})} MK(\nu) + E(\nu)$$

where:

- $MK(\nu) = \inf_{\gamma \in \Pi(\mu,\nu)} \sum_{i,j} c_{i,j} \gamma_{i,j}, \quad$ (transport cost)
- $E(\nu) := \sum_j F_j(\nu_j) + \frac{1}{2} \sum_{k,j} \varphi_{kj} \nu_k \nu_j, \quad$ (energy)

Entropic Regularization:

$$MK_\epsilon(\nu) := \inf_{\gamma \in \Pi(\mu,\nu)} \left\{ \sum_{i,j} c_{i,j} \gamma_{i,j} + \epsilon \sum_{i,j} \gamma_{i,j} (\ln \gamma_{i,j} - 1) \right\}$$

Bregman Proximal Problem

Smooth Approximation: The problem becomes:

$$\inf_{\nu \in \mathcal{P}(\mathcal{Y})} MK_\epsilon(\nu) + E(\nu)$$

Simplified Representation: With $\bar{\gamma}_{i,j} = e^{-c_{i,j}/\epsilon}$, the problem reduces to:

$$\inf_{\nu \in \mathcal{P}(\mathcal{Y}), \gamma \in \pi(\mu,\nu)} \epsilon \left(KL(\gamma \| \bar{\gamma}) + E(\nu)\right) \iff \inf_{\gamma \in \mathbb{R}_+^{I \times J}} \{KL(\gamma \| \bar{\gamma}) + G(\nu)\}$$

where $G(\nu) = G_1(\nu) + G_2(\nu) + G_3(\nu)$

Dykstra's Algorithm for Proximal Problems

Problem Formulation: Given a divergence $D_\Gamma(\pi\|\xi)$ and two convex functions $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$, solve:

$$\min_{\pi\in\mathcal{D}}\{D_\Gamma(\pi\|\xi)+\varphi_1(\pi)+\varphi_2(\pi)\}$$

Key Assumptions:

- φ_1 and φ_2 are proper, convex, lower semi-continuous, and coercive.
- Qualification condition on domains

Algorithm 1 Dykstra's Algorithm for KL Divergence

Initialize: $\pi^{(0)}=\bar{\gamma}$, $z^{(0)}=\mathbf{1}$, $z^{(-1)}=\mathbf{1}$ $l>0$ Compute:

$$\pi^{(l)}=\mathrm{Prox}_{G_l}\left(\pi^{(l-1)}\odot z^{(l-2)}\right)$$

Update:

$$z^{(l)}=z^{(l-2)}\odot\pi^{(l)}$$

Numerical Experiment 1: One-Dimensional Problem

Experiment Setup:

- Player types (μ) modeled as a mixture of two Gaussians
- Cost matrix: $c_{i,j} = |x_i - y_j|^2$
- Energy function:

$$E(\nu) = \sum_j \frac{1}{2}\nu_j^2 + \frac{1}{2}\sum_{k,j} \phi_{k,j}\nu_k\nu_j + \sum_j |y_j - 2|^2,$$

with $\phi_{k,j} = 10^{-2}|x_k - y_j|^2$

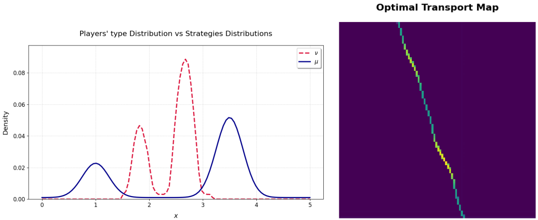

Figure: Solution of Non-Regularized Problem (Left) and Optimal Transport Map (Right)

Convergence Comparison depending on ϵ

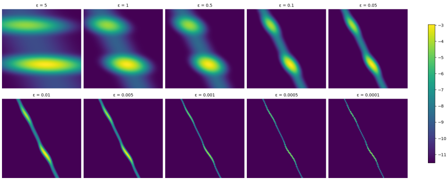

Figure: Comparison of Transport Plans (ϵ)

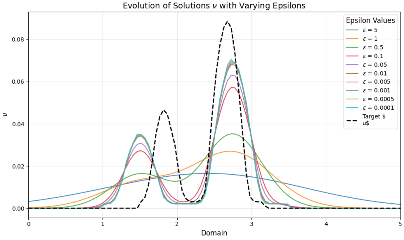

Figure: Strategy Distributions (ϵ)

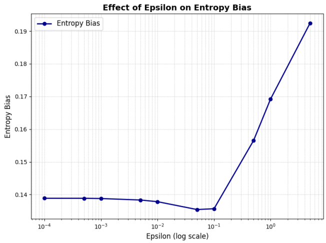

Figure: Entropic Bias ($\log(\epsilon)$)

Case of Two Populations

Setup and Parameters

- **Distributions:** Two Gaussian mixtures represent the types of players in each population.
- **Energy Functions:**

$$E_l(\nu_l) = \sum_{j \in J} (\nu_j^l)^3 + \sum_{k,j \in J \times J} \varphi_{kj}^l \nu_j^l \nu_k^l + \sum_{j \in J} |y_j - 1|^4,$$

  where $\varphi_{kj}^l = 10^{-2} |y_k - y_j|^2$.
- **Congestion Term:**

$$F_j(\nu_j^1 + \nu_j^2) = (\nu_j^1 + \nu_j^2)^3.$$

- **Cost Function:**

$$c_{ij} = |x_i - y_j|^2.$$

Numerical Experiment 2

Equilibrium Solutions for Different ϵ

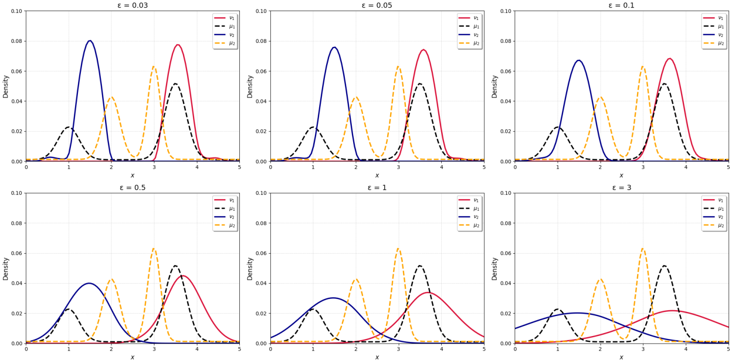

Figure: Comparison of equilibrium solutions for two populations under different values of ϵ

Conclusion and Connection with the Course

Conclusion

- Reformulated the Cournot-Nash equilibrium as an optimal transport problem
- Used entropic regularization to approximate solutions efficiently via proximal splitting
- Observed that as $\epsilon \to 0$, solutions became sharper, but convergence slowed
- Double approximations in the semi-implicit scheme