Задача 1

№1. ξ - Uniform $[0,1]$.

1) $\eta_1 = 2\xi - \mathbb{I}\{\xi > 1/2\}$; $\Rightarrow$

$$\varphi_{\eta_1}(u) = \mathbb{E}[e^{iu\eta_1}] = \mathbb{E}[e^{iu2\xi} \cdot e^{iu(-\mathbb{I}\{\xi > \frac{1}{2}\})}] =$$

$$= \int_0^{\frac{1}{2}} e^{2iux} dx + \int_{\frac{1}{2}}^{1} e^{2iux - iu} dx = \frac{1}{2iu}(e^{iu} - 1) + \frac{e^{-iu}}{2iu}(e^{2iu} - e^{iu}) =$$

$$= \frac{1}{iu}(e^{iu} - 1) \quad \in \mathbb{C}.$$

2) $\eta_2 = \ln(\xi)$; $\Rightarrow$

$$\varphi_{\eta_2}(u) = \mathbb{E}[e^{iu\eta_2}] = \mathbb{E}[e^{iu\ln\xi}] = \mathbb{E}(\xi^{iu}) =$$

$$= \int_0^1 x^{iu} dx = \frac{x^{iu+1}}{iu+1}\Big|_0^1 = \frac{1}{iu+1} \quad \text{– не действит-но значна} \quad \in \mathbb{C}$$

3) $$\eta_3 = \begin{cases} -1, & 0 \le \xi \le \frac{1}{3} \\ 0, & \frac{1}{3} \le \xi \le \frac{2}{3} \\ 1, & \frac{2}{3} \le \xi \le 1 \end{cases} ; \Rightarrow$$

$$\varphi_{\eta_3}(u) = \mathbb{E}[e^{iu\eta_3}] = \int_0^{1/3} e^{-iu} dx + \int_{1/3}^{2/3} 1\, dx + \int_{\frac{2}{3}}^{1} e^{iu} dx =$$

$$= e^{-iu} x\Big|_0^{\frac{1}{3}} + x\Big|_{\frac{1}{3}}^{\frac{2}{3}} + e^{iu} x\Big|_{\frac{2}{3}}^{1} = \frac{1}{3}(1 + e^{-iu} + e^{iu}) =$$

$$= \frac{1}{3}(1 + 2\cos u) \leftarrow \text{действительнозначная}$$

Задача 2

№2 1) $X_t = \sum_{i=1}^{N_t} Y_i$ — составной процесс Пуассона, где

$Y_i \sim i.i.d \sim exp\left(\frac{1}{5000}\right)$ $\quad Y_i$ — i-ая выплата

$Y_i \perp\!\!\!\perp N_t.$ $\quad N_t$ — кол-во пришедших заявок до момента t

процесс Пуассона с $\lambda = 100$

2) • $\mathbb{E}(X_t) = \lambda t\, \mathbb{E}(Y_1) = 100t \cdot 5000 = 5 \cdot 10^5 t$

• $\mathbb{D}(X_t) = \lambda t\, \mathbb{E}(Y_1^2) = 100t \cdot 2 \cdot (5000)^2 = 10^9 \cdot 5t$

$\mathbb{E}(Y_1^2) = \frac{2}{\lambda^2}$

• $\mathbb{P}(X_t = 0) = \mathbb{P}(N_t = 0) + \mathbb{P}(N_t > 0)\, \mathbb{P}(Y_1 = 0 \dots Y_{N_t} = 0) =$

$= \mathbb{P}(N_t = 0) = e^{-100t} \frac{(100t)^0}{0!} = e^{-100t}$

• Нам известна формула:

$$\mathcal{L}_{X_t - X_s}(u) = e^{\lambda(t-s)(\mathcal{L}_Y(u) - 1)}$$

Найдем $\mathcal{L}_Y(u) = \int_0^\infty e^{-xu} \frac{1}{5000} e^{-\frac{1}{5000}x}\, dx =$

$$= \frac{1}{5000}\left(\frac{1}{\frac{1}{5000} + u}\right) = \frac{1}{5000u + 1}$$

$$\mathcal{L}_{X_t}(u) = e^{100t\left(\frac{1}{5000u+1} - 1\right)} = e^{-100tu\left(\frac{5000}{5000u+1}\right)} =$$

$$= e^{-100tu\left(\frac{1}{u + 1/5000}\right)}.$$

Задача 3

(3.) (a) N_t – однородный процесс Пуассона. $\lambda = 10$.

(i) $\mathbb{P}\{\xi_{101} \le t \mid S_{100} = \frac{28}{3}\} = \mathbb{P}\{S_{101} - S_{100} \le t \mid S_{100} = \frac{28}{3}\} = \mathbb{P}\{S_{101} \le t + \frac{28}{3} \mid S_{100} = \frac{28}{3}\} =$

$= \mathbb{P}\{N_{t+\frac{28}{3}} \geq 101 \mid N_{\frac{28}{3}} = 100\} \overset{\text{в силу независимости приращений}}{=} \mathbb{P}\{N_{t+\frac{28}{3}} - N_{\frac{28}{3}} \geq 1\} = 1 - \mathbb{P}\{N_{t+\frac{28}{3}} - N_{\frac{28}{3}} = 0\} =$

$= \boxed{1 - e^{-10t}, \text{ если } t \geq 0}$

(ii) $\mathbb{P}\{S_{150} - S_{100} \le t \mid S_{100} = \frac{28}{3}\} = \mathbb{P}\{S_{150} \le t + \frac{28}{3} \mid S_{100} = \frac{28}{3}\} =$

$= \mathbb{P}\{N_{t+\frac{28}{3}} \geq 150 \mid N_{\frac{28}{3}} = 100\} = \mathbb{P}\{N_{t+\frac{28}{3}} - N_{\frac{28}{3}} \geq 50\} = 1 - \mathbb{P}\{\underbrace{N_{t+\frac{28}{3}} - N_{\frac{28}{3}}}_{\sim \mathrm{Pois}(\lambda t)} < 50\} =$

$= 1 - \mathbb{P}\{S_{50} > t\} = \mathbb{P}\{S_{50} \le t\} = F_{S_{50}}(t) = \begin{cases} 1 - e^{-10t} \sum\limits_{k=0}^{49} \frac{(10t)^k}{k!}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

a)

(b) неоднородный с $\Lambda(t) = 10t^{5/4} \Longrightarrow F_{\xi_{101} | S_{100} = 9\frac{1}{3}}(t) = \boxed{1 - e^{-10(t + 9\frac{1}{3})^{5/4} + 10 \cdot (9\frac{1}{3})^{5/4}}}$

(ii) $P\{S_{150} - S_{100} \leqslant t \mid S_{100} = 9\frac{1}{3}\} = P\{S_{150} \leqslant t + 9\frac{1}{3} \mid S_{100} = 9\frac{1}{3}\} = P\{N_{t+9\frac{1}{3}} \geqslant 150 \mid N_{9\frac{1}{3}} = 100\} =$

$= P\{N_{t+9\frac{1}{3}} - N_{9\frac{1}{3}} \geqslant 50 \mid N_{9\frac{1}{3}} = 100\} = P\{N_{t+9\frac{1}{3}} - N_{9\frac{1}{3}} \geqslant 50\} =$

независ. сл.

$= 1 - P\{N_{t+9\frac{1}{3}} - N_{9\frac{1}{3}} < 50\} = 1 - \sum\limits_{k=0}^{49} P\{N_{t+9\frac{1}{3}} - N_{9\frac{1}{3}} = k\} =$

$= 1 - \sum\limits_{k=0}^{49} e^{-\Lambda(t+9\frac{1}{3}) + \Lambda(9\frac{1}{3})} \cdot \frac{(\Lambda(t+9\frac{1}{3}) - \Lambda(9\frac{1}{3}))^k}{k!} = 1 - e^{-\Lambda(t+9\frac{1}{3}) + \Lambda(9\frac{1}{3})} \sum\limits_{k=0}^{49} \frac{(\Lambda(t+9\frac{1}{3}) - \Lambda(9\frac{1}{3}))^k}{k!}$

b)

Задача 4

(i) Докажем утверждение методом математической индукции.

1. Для $n = 1$ утверждение верно:

$$F_{S_1}(t) = \mathbb{P}\{S_1 \le t\} = \mathbb{P}\{N_t \ge 1\} = 1 - \mathbb{P}\{N_t = 0\} = 1 - e^{-\Lambda(t)}$$
$$f_{S_1}(t) = \lambda(t) \cdot e^{-\Lambda(t)}$$

2. Пусть для $n - 1$ верно

$$f_{S_{n-1}}(t) = e^{-\Lambda(t)} \cdot \frac{(\Lambda(t))^{n-2}}{(n-2)!} \cdot \lambda(t)$$

3. Покажем, что для n верно

$$f_{S_n}(t) = e^{-\Lambda(t)} \cdot \frac{(\Lambda(t))^{n-1}}{(n-1)!} \cdot \lambda(t)$$

Пользуясь подсказкой в условии, запишем

$$\begin{aligned}
\mathbb{P}\{S_n \le t\} &= \mathbb{P}\{S_{n-1} \le t\} - \mathbb{P}\{N_t = n-1\} \\
f_{S_n}(t) &= f_{S_{n-1}}(t) - \left(e^{-\Lambda(t)} \cdot \frac{(\Lambda(t))^{n-1}}{(n-1)!}\right)'_t \\
&= f_{S_{n-1}}(t) - \left(-e^{-\Lambda(t)}\lambda(t) \cdot \frac{(\Lambda(t))^{n-1}}{(n-1)!} + e^{-\Lambda(t)}\frac{(n-1)(\Lambda(t))^{n-2} \cdot \lambda(t)}{(n-1)!}\right) \\
&= e^{-\Lambda(t)}\lambda(t) \cdot \frac{(\Lambda(t))^{n-1}}{(n-1)!}
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathbb{P}\{\xi_{k+1} \le t | S_k = s\} &= \mathbb{P}\{S_{k+1} - S_k \le t | S_k = s\} \\
&= \mathbb{P}\{S_{k+1} \le t + s | S_k = s\} \\
&= \mathbb{P}\{N_{t+s} \ge k+1 | N_s = k\} \\
&= \mathbb{P}\{N_{t+s} - N_s \ge 1 | N_s = k\} \\
&\overset{(1)}{=} \mathbb{P}\{N_{t+s} - N_s \ge 1\} \\
&= 1 - \mathbb{P}\{N_{t+s} - N_s = 0\} \\
&\overset{(2)}{=} 1 - e^{-\Lambda(t+s)+\Lambda(s)}
\end{aligned}$$

При переходе (1) было использовано свойство независимости приращений, при переходе (2) — факт из определения о том, что $N_a - N_b \sim Pois(\Lambda(a) - \Lambda(b))$.

(iii)

$$\begin{aligned}
F_{\xi_k}(t) &= \mathbb{P}\{\xi_k \le t\} \\
&= \mathbb{P}\{S_k - S_{k-1} \le t\} \\
&= \int_{\mathbb{R}} \mathbb{P}\{S_k - S_{k-1} \le t | S_{k-1} = s\} f_{S_{k-1}}(s) ds \\
&= \int_0^{+\infty} \mathbb{P}\{S_k \le t + s | S_{k-1} = s\} f_{S_{k-1}}(s) ds \\
&= \int_0^{+\infty} \mathbb{P}\{N_{t+s} - N_s \ge 1\} f_{S_{k-1}}(s) ds \\
&= \int_0^{+\infty} \left(1 - \mathbb{P}\{N_{t+s} - N_s = 0\}\right) f_{S_{k-1}}(s) ds \\
&= \int_0^{+\infty} f_{S_{k-1}}(s) ds - \int_0^{+\infty} e^{-\Lambda(t+s)+\Lambda(s)} e^{-\Lambda(s)} \lambda(s) \cdot \frac{(\Lambda(s))^{k-2}}{(k-2)!} ds \\
&= 1 - \int_0^{+\infty} e^{-\Lambda(t+s)} \lambda(s) \cdot \frac{(\Lambda(s))^{k-2}}{(k-2)!} ds
\end{aligned}$$