VARMA

Vector ARMA

$$\Phi_i = \begin{pmatrix} \varphi_{i,1,1} & \cdots & \varphi_{i,1,n} \\ \vdots & \ddots & \vdots \\ \varphi_{in1} & \cdots & \varphi_{inn} \end{pmatrix}$$

$$\Psi_i = \begin{pmatrix} \psi_{i11} & & \psi_{i1n} \\ & & \\ \psi_{in1} & & \psi_{inn} \end{pmatrix}$$

$y_t \in \mathbb{R}^n \quad \mu \in \mathbb{R}^n \quad \vartheta_t \sim iid\ N(0, \Omega)$

$$y_t = \mu + \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{i=1}^{q} \Psi_i \vartheta_{t-i}$$

$$\Phi_p(L) y_t = \mu + \Psi_q(L) \vartheta_t$$

$$\Phi_p = I - \Phi_1 L - \Phi_2 L^2 - \ldots - \Phi_p L^p$$

VAR(P)

$$y_t = \mu + \sum_{i=1}^{p} \phi_i y_{t-i} + \vartheta_t$$

Reduced-form VAR

$$VAR(1)$$

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \varphi_{111} & \varphi_{112} \\ \varphi_{121} & \varphi_{122} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \nu_{1,t} \\ \nu_{2,t} \end{pmatrix}$$

$VAR(p)$, n ур-й $\qquad \nu_t \sim N(0, \Omega)$

$$n^2 \cdot p + n + n + \frac{n(n-1)}{2}$$

$$2 + \frac{2}{1} = 3$$

Как выбрать p?

$$S = \max(p, q)$$

$$AIC(p) = \log|\hat{V}(p)| + \frac{2}{T-S} n^2$$

$$BIC(p) = SC(p) = \log|\hat{V}(p)| + \frac{\log(T-S)}{T-S} n^2$$

$$HQ(p) = \log|\hat{V}(p)| + \frac{2\log\log(T-S)}{T-S} n^2$$

$$\hat{p}(SC) \le \hat{p}(HQ) < \hat{p}(AIC)$$

$$\hat{V}(p) = \frac{1}{T} \sum_{t=1}^{T} \underset{n\times 1}{\hat{\nu}_t} \, \underset{1\times n}{\hat{\nu}_t^T}$$

Box-Pierce, Ljung-Box

H_0: $E_t(\nu_t, \nu_{t-i}^T) = 0$, $i = 1, \dots, h > p$

H_A: $\exists i$ s.t. $E_t(\nu_t, \nu_{t-i}^T) \neq 0$

$$Q_h = T \sum_{j=1}^{h} tr(\hat{C}_j^T \hat{C}_0^{-1} \hat{C}_j \hat{C}_0^{-1}) \sim \chi^2(n^2(h-p))$$

$$C_i = \frac{1}{T} \sum_{t=i+1}^{T} \hat{\nu}_t \hat{\nu}_{t-i}^T$$

Forecasting.

$$y_t = \mu + \Phi_1 y_{t-1} + \nu_t$$

$$y_{T+1|T} = E(y_{T+1}|T) = \mu + \Phi_1 y_T$$

$$\hat{y}_{T+1|T} = \hat{\mu} + \hat{\Phi}_1 y_T$$

Granger causality

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \sum_{i=1}^{p} \begin{pmatrix} \varphi_{i11} & \overset{\overset{!}{=}0}{\varphi_{i12}} \\ \varphi_{i21} & \varphi_{i22} \end{pmatrix} \begin{pmatrix} y_{1,t-i} \\ y_{2,t-i} \end{pmatrix} + \nu_t$$

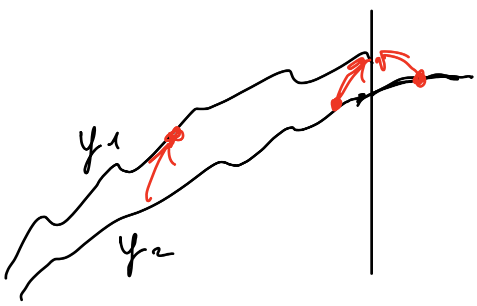
y_{2t} is not Granger-causal if its lags do not appear in y_{1t} equation

$H_0: \varphi_{1,1,2} = \varphi_{2,1,2} = \dots = \varphi_{p12} = 0$

H_1: хотя бы один коэф $\neq 0$

Аналогичный тест.

$y_{1,t}$ is not granger causal



y_2 – causal for y_1

Instantenous Causality

y_{2t} instantenously causal для y_{1t}, если знание y_2 в прогнозном периоде помогает спрогнозировать y_1

y_{1t}

y_{2t}

y_{3t}

SVAR

B-mat $\qquad\qquad$ A-mat

$$B_0 y_t = \lambda + B_1 y_{t-1} + \dots + B_p y_{t-p} + A \varepsilon_t$$

$\varepsilon_t \sim iid N(0, \Sigma)$

quar.

$$\text{SVAR}: n + n^2(p+1) + n$$

$$\text{VAR}: n + n^2 p + n + \frac{n(n-1)}{2}$$

$$= \frac{n(n+1)}{2}$$

$$y_t = B_0^{-1}\lambda + B_0^{-1} B_1 y_{t-1} + \dots + B_0^{-1} B_p y_{t-p} + B_0^{-1}\varepsilon_t$$

$$y_t = \mu + \Phi_1 y_{t-1} + \dots + \Phi_p y_{t-p} + v_t$$

$$\mu = B_0^{-1}\lambda \quad \Phi_i = B_0^{-1} B_i \quad v_t = B_0^{-1}\varepsilon_t$$

$v_t \sim (0, \Omega)$

$\lambda = B_0 \mu$

1) recursive
2) short-run
3) long-run
4) sign restrictions
5) Bayesian prior
6) through heteroscedasticity

Recursive identification

$$B_0 = \begin{pmatrix} x & 0 & 0 & 0 \\ & x & & \\ & & x & \\ & & & \ddots \\ & & & & x \end{pmatrix}$$

1)

$$\frac{n(n-1)}{2}$$

$$\Sigma = I \quad 2$$

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

2×2

$$B_0 = \begin{pmatrix} b_{011} & 0 \\ b_{021} & b_{022} \end{pmatrix} \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} =$$

$$b_{011}\, y_{1t} = f_1(\dots)$$

$$b_{021}\, y_{1t} + b_{022}\, y_{2t} = f_2(\dots)$$

$$y_{2t} = f_3(\dots)$$

$$v_t = B_0^{-1} \varepsilon_t$$

$$\underbrace{E(v_t v_t^T)}_{\Omega} = B_0^{-1} \overset{I}{\overset{\shortparallel}{E(\varepsilon_t \varepsilon_t^T)}} (B_0^{-1})^T$$

$$\Omega = B_0^{-1} (B_0^{-1})^T$$

Choleski decomp: симметричная положит. опр. матрица A:

$$A = LL^T$$

$\hat{\Omega}$ $\quad$ $\hat{B}_0^{-1} = chol(\hat{\Omega})$

B_0

$\hat{\lambda} = \hat{B}_0 \hat{\mu}$ $\quad$ $\hat{B}_i = \hat{B}_0 \hat{\Phi}_i$

IRF, FEVD

Impulse Response Function

1) $VAR(p) \to VMA(\infty)$

2) Подставить шоки вместо ε

VAR(1)

$$y_t = \mu + \Phi_1 y_{t-1} + \nu_t =$$

$$= \ldots =$$

$$= (I + \Phi_1 + \Phi_1^2 + \ldots)\mu + \nu_t + \Phi_1 \nu_{t-1} +$$
$$+ \Phi^2 \nu_{t-2} + \ldots =$$

$$(I - \Phi_1)(I + \Phi_1 + \Phi_1^2 + \ldots) = I$$

$$I - \Phi_1 + \Phi_1 - \quad + \quad - \quad + \quad = I$$

$$= (1 - \Phi_1)^{-1}\mu + \nu_t + \Phi_1 \nu_{t-1} + \ldots$$

VAR(p)

$$\begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix} = \begin{pmatrix} \mu_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \Phi_1 & \cdots & & & \Phi_p \\ I & 0 & \cdots & & 0 \\ 0 & \ddots & & & \vdots \\ 0 & & \ddots & & \vdots \\ 0 & 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ \vdots \\ \vdots \\ y_{t-p} \end{pmatrix} + \begin{pmatrix} \nu_t \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$Y_t = M + \Phi Y_{t-1} + U_t$$

VAR(p)

$$\Phi(L) y_t = \mu + \nu_t$$

$$\Phi(L) = I - \Phi_1 L - \ldots \Phi_p L^p$$

$$C(L)\Phi(L) = I$$

$$C(L) = C_0 + C_1 L + C_2 L^2 + \ldots$$

$$\cdot\, C(L)\, \Phi(L)\, y_t = \mu + \nu_t$$

$$y_t = C(L)\mu + C(L)\nu_t$$

$$\tilde{\mu} = \left(\sum_{i=0}^{\infty} C_i\right)\mu$$

$$C(L)\,\Phi(L) = I$$

$$(C_0 + C_1 L + \dots)(I - \Phi_1 L - \dots - \Phi_p L^p) = I$$

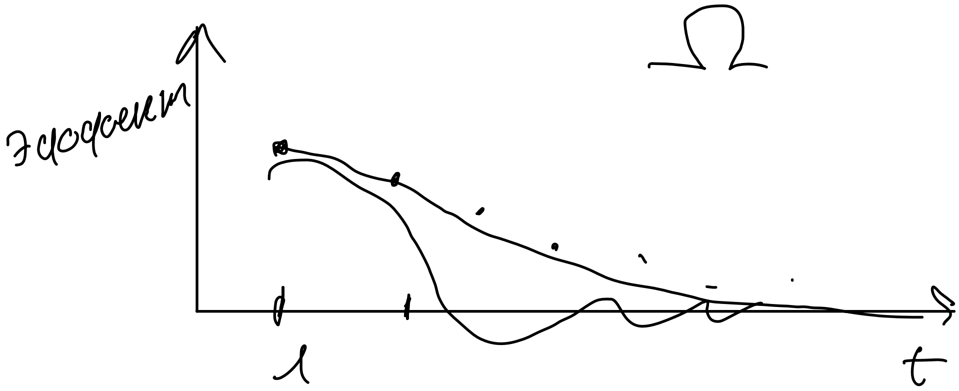
$$I = C_0$$
$$0 = C_1 - C_0 \Phi_1$$
$$0 = C_2 - C_1 \Phi_1 - C_0 \Phi_2$$

$$C_0 = I$$

$$C_i = C_{i-1}\Phi_1 + C_{i-2}\Phi_2 + \dots + C_0 \Phi_i$$

$$y_t = \tilde{\mu} + C_0 \nu_t + C_1 \nu_{t-1} + \dots C_2 \nu_{t-2} \dots$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$



Ω

$$\nu_t = B_0^{-1}\varepsilon_t$$

$$y_t = \tilde{\mu} + \underbrace{C_0 B_0^{-1}}_{\tilde{\Psi}_0}\varepsilon_t + \underbrace{C_1 B_0^{-1}}_{\tilde{\Psi}_1}\varepsilon_{t-1} + \dots$$

$\dots$

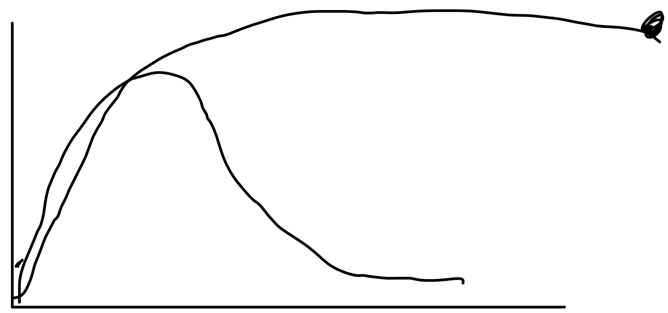
(i, j) матрицы $\tilde{\Psi}_s$ это отклик $y_{i,t+s}$ на единичное изменение ε_{jt}

$$\tilde{\Psi}_s \to 0, \quad s \to \infty$$

$$\tilde{\Psi} = \tilde{\Psi}_0 + \tilde{\Psi}_1 + \dots =$$

$$= (C_0 + C_1 + \dots)\, B_0^{-1}$$

$$\Psi = (I - \Phi_1 - \dots - \Phi_p)^{-1} B_0^{-1}$$



FEVD

$$y_{T+h} = \tilde{\mu} + C_0 v_{T+h} + C_1 v_{T+h-1} + \dots + C_h v_T$$

$$y_{T+h|T} = \tilde{\mu} + C_h v_T$$

$$\parallel$$

$$C_0 v_{T+h} + C_1 v_{T+h-1} + \dots + C_{h-1} v_{T+1}$$

$$y_{T+h} - y_{T+h|T} =$$

$$= C_0 B_0 \varepsilon_t + \dots$$

$$\Psi_0 \varepsilon_{t+h} + \Psi_1 \varepsilon_{T+h-1} + \dots + \Psi_{h-1} \varepsilon_{T+1}$$

VAR(1) $\quad y_1, y_2 \quad h=2$

$$\begin{pmatrix} y_{1,T+2} - y_{1,T+2|T} \\ y_{2,T+2} - y_{2,T+2|T} \end{pmatrix} =$$

$$= \begin{pmatrix} \psi_{011} & \psi_{012} \\ \psi_{021} & \psi_{022} \end{pmatrix} \begin{pmatrix} \varepsilon_{1,T+2} \\ \varepsilon_{2,T+2} \end{pmatrix} + \begin{pmatrix} \psi_{111} & \psi_{112} \\ \psi_{121} & \psi_{122} \end{pmatrix} \begin{pmatrix} \varepsilon_{1,T+1} \\ \varepsilon_{2,T+1} \end{pmatrix}$$

$$FEV_1 = (\psi_{011}^2 + \psi_{111}^2) + (\psi_{012}^2 + \psi_{112}^2)$$

$$FEV_2 \quad (\psi_{021}^2 + \psi_{121}^2) + (\psi_{022}^2 + \psi_{122}^2)$$

	ε_1	ε_2	Σ
y_1	$\dfrac{\psi_{011}^2 + \psi_{111}^2}{FEV_1}$	$\dfrac{\psi_{012}^2 + \psi_{112}^2}{FEV_1}$	1
y_2	$\dfrac{\psi_{021}^2 + \psi_{121}^2}{FEV_2}$	$\dfrac{\psi_{022}^2 + \psi_{122}^2}{FEV_2}$	