Contents lists available at ScienceDirect

Discrete Applied Mathematics

Algorithmic results for weak Roman domination problem in graphs

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ARTICLE INFO

ABSTRACT

Consider a graph $G = (V, E)$ and a function $f : V \rightarrow \{0, 1, 2\}$. A vertex u with $f(u) = 0$ is defined as *undefended* by f if it lacks adjacency to any vertex with a positive f-value. The function f is said to be a *weak Roman dominating function* (WRD function) if, for every vertex u with $f(u) = 0$, there exists a neighbor v of u with $f(v) > 0$ and a new function $f' : V \rightarrow \{0, 1, 2\}$ defined in the following way: $f'(u) = 1, f'(v) = f(v) - 1$, and $f'(w) = f(w)$, for all vertices w in $V \setminus \{u, v\}$; so that no vertices are undefended by f'. The total weight of f is equal to $\sum_{v \in V} f(v)$, and is denoted as $w(f)$. The *Weak Roman domination number* denoted by $\gamma_r(G)$, represents $min\{w(f)|f$ is a WRD function of $G\}$. For a given graph G, the problem of finding a WRD function of weight $\gamma_r(G)$ is defined as the *Minimum Weak Roman domination problem*. The problem is already known to be NP-hard for bipartite and chordal graphs. In this paper, we further study the algorithmic complexity of the problem. We prove the NP-hardness of the problem for star convex bipartite graphs and comb convex bipartite graphs, which are subclasses of bipartite graphs. In addition, we show that for the bounded degree star convex bipartite graphs, the problem is efficiently solvable. We also prove the NP-hardness of the problem for split graphs, a subclass of chordal graphs. On the positive side, we present a polynomial-time algorithm to solve the problem for P_4-sparse graphs. Further, we have presented some approximation results.

1. Introduction

Cockayne et al. [5], introduced the concept of *Roman Dominating function* (RDF) for a graph G. Given a graph $G = (V, E)$, RDF is a function $f : V \rightarrow \{0, 1, 2\}$ that satisfies the following condition: for every vertex u with $f(u) = 0$, there exists a vertex v adjacent to u with $f(v) = 2$. The weight of f is defined as the summation of the function values over all elements in V, and is denoted as $w(f)$. Furthermore, for a subset S of V, we define $f(S)$ as the summation of the function values over all elements in S. The *Roman domination number*, represented as $\gamma_R(G)$, is defined as $min\{w(f) \mid f$ is a RDF of $G\}$. A function that represents an RDF of weight $\gamma_R(G)$ is referred as a $\gamma_R(G)$-function. Extensive research has been conducted on the topic of Roman domination on graphs, as evidenced by the work referenced in [5,8,9].

The impetus for formulating this description of a Roman dominating function stemmed from an article authored by Ian Stewart in Scientific American, named "Defend the Roman Empire!" [12]. Every vertex inside the graph corresponds to a distinct geographical region within the historical context of the Roman Empire. A place, denoted as node v, is classified

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[1] Research of Arti Pandey is supported by CRG project, Grant Number - CRG/2022/008333, Science and Engineering Research Board (SERB), India
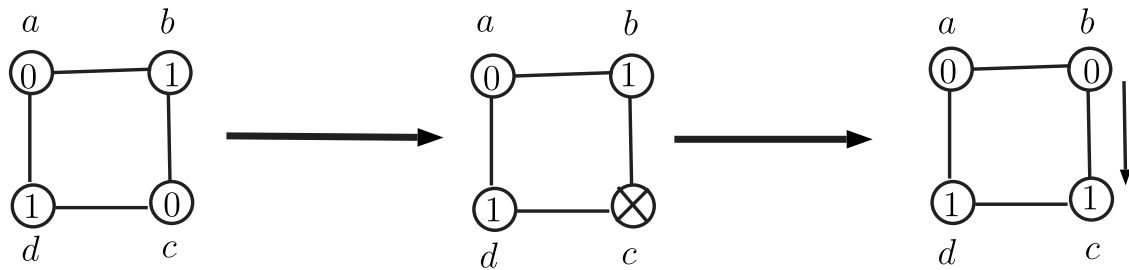
Fig. 1. Example of a WRD function on C_4.

as *unsecured* if there are no legions stationed there, that is $f(v) = 0$. Conversely, a place is deemed *secured* if there are one or two legions stationed there, represented by $f(v) \in \{1, 2\}$. One possible method of securing an insecure area, (or a vertex v), is by deploying a legion from a neighboring location. In the fourth century A.D., Emperor Constantine the Great issued an edict stipulating that the transfer of a legion from a fortified position to an unfortified one is prohibited if such an action would result in leaving the latter unsecured. Therefore, it is necessary to have two legions positioned at a specific place ($f(v) = 2$) in order to afterward deploy one of the legions to a neighboring place. Through this approach, Emperor Constantine the Great was able to provide a robust defense for the Roman Empire. Due to the considerable costs associated with maintaining a legion in a specific area, the Emperor aims to strategically deploy the minimum number of legions necessary to safeguard the Roman Empire.

The notion of *Weak Roman domination* was initially introduced by Henning et al. [9] to reduce the expense associated with the placement of legions. This problem involves easing some limitations that are imposed in the Roman Domination problem. Consider a graph $G = (V, E)$ and a function f defined as $f : V \to \{0, 1, 2\}$. Let V_0, V_1, and V_2 denote the sets of vertices that are allocated the values 0, 1, and 2, respectively, under the function f. A vertex $u \in V_0$ is said to be *undefended* by f (or simply *undefended* if the function f is evident from the context), if it is not adjacent to any vertex in $V_1 \cup V_2$.

The function f is referred to as a *weak Roman dominating function* (WRD function) if, for every vertex u in the set V_0, there exists a neighbor v of u in the set $V_1 \cup V_2$ and a function $f' : V \to \{0, 1, 2\}$ defined in the following way: $f'(u) = 1$, $f'(v) = f(v) - 1$, and $f'(w) = f(w)$ for all vertices w in the set $V \setminus \{u, v\}$, such that no vertices are undefended by f'. We define the weight $w(f)$ of f to be $\sum_{v \in V} f(v)$. The *Weak Roman domination number*, denoted as $\gamma_r(G)$, is the minimum weight of a WRD function in G; that is, $\gamma_r(G) = min\{w(f)|f$ is a WRD function in $G\}$. A WRD function of weight $\gamma_r(G)$ is denoted as a $\gamma_r(G)$-function.

For example refer to Fig. 1, a graph $G = C_4$ is considered, where $V(G) = \{a, b, c, d\}$ and $E(G) = \{ab, bc, cd, da\}$. Consider the function $f : V(G) \to \{0, 1, 2\}$, defined as follows : $f(a) = f(c) = 0$ and $f(b) = f(d) = 1$. There is no undefended vertex by f. Now let an unsecured vertex be attacked (without loss of generality, say c; the "crossed" vertex in Fig. 1). The legion placed at b can move to c and the new function f' ($f'(d) = f'(c) = 1$ and $f'(a) = f'(b) = 0$) is formed; but still G has no vertex that is undefended by f'. Hence f is a WRD function of G.

The optimization and decision version of the problem are the following:

MINIMUM WEAK ROMAN DOMINATION problem (MIN-WRD)

> **Instance**: A graph $G = (V, E)$.
> **Solution**: A WRD function f of G with $w(f) = \gamma_r(G)$.

WEAK ROMAN DOMINATION DECISION problem (DECIDE-WRD)

> **Instance**: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
> **Question**: Does there exist a WRD function f of G with weight $w(f) \leq k$?

1.1. Notations and definitions

This paper only considers simple, undirected, finite and nontrivial graphs. Let $G = (V, E)$ be a graph. n and m will be used to denote the cardinalities of V and E, respectively. $N(v)$ stands for the set of neighbors of a vertex v in V and $N[v] = N(v) \cup \{v\}$. The number of neighbors of a vertex $v \in V$ defines its *degree*, which is represented by the symbol $deg(v)$. The maximum degree of the graph will be denoted by Δ. For a set $U \subseteq V$, the notation $deg_U(v)$ is used to represent the number of neighbors that a vertex v has within the subset U. Additionally, we use $N_U(v)$ to refer to the set of neighbors of vertex v within U. A vertex u is called a *private neighbor* of v with respect to $S \subseteq V$, if $N[u] \cap S = \{v\}$. The set $pn(v; S) = N[v] - N[S \setminus \{v\}]$ of all private neighbors of v with respect to S is called the *private neighbor set* of v with respect to S.

A vertex of degree one is known as a *pendant vertex*. A set $I \subseteq V$ is called an *independent set* if no two vertices of I are adjacent. A set $S \subseteq V$ is said to be a *dominating set* if every vertex of $V \setminus S$ is adjacent to some vertex of S. The cardinality

of a minimum sized dominating set of G is said to be the *domination number* of G (denoted by $\gamma(G)$). A graph G is said to be a *complete graph* if any two vertices of G are adjacent. A set $S \subseteq V$ is said to be a clique if the subgraph of G induced on S is a complete graph. The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$, for every non negative integer n.

The *join* of two graphs G_1 and G_2 refers to a graph formed by taking separate copies of G_1 and G_2 and connecting every vertex in $V(G_1)$ to each vertex in $V(G_2)$ using edges. The symbol $\oplus$ will denote the join operation. Similarly, *disjoint union* of two graphs H_1 and H_2 is the graph $H = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$. The disjoint union will be denoted with the symbol $\cup$.

A vertex v of a graph $G = (V, E)$ is said to be a *universal vertex* if $N[v] = V$. A path with n vertices is denoted as P_n. A *comb* graph $G = (V, E)$ is defined as follows: $V = V_1 \cup V_2$, where $V_1 = \{u_1, \ldots, u_k\}$ and $V_2 = \{v_1, \ldots, v_k\}$ and $E = \{u_i u_{i+1} \mid i \in [k-1]\} \cup \{u_i v_i \mid i \in [k]\}$. Note that $G[V_1]$ is a path, and it is said to be the *backbone* of the comb.

A graph $G = (V, E)$ is said to be P_4-*sparse* if a subgraph induced on any 5 vertices of G contains at most one P_4. A *spider* is a graph $G = (V, E)$, where V admits a partition in three subsets S, C and R such that

- $C = \{c_1, \ldots, c_l\}$ $(l \geq 2)$ is a clique.
- $S = \{s_1, \ldots, s_l\}$ is a stable set.
- Every vertex in R is adjacent to every vertex in C and nonadjacent to all vertex of S.

A spider $G(S, C, R)$ is said to be a *thin spider* if for every $i \in [l]$, $N(s_i) = \{c_i\}$ and it is called a *thick spider* if for every $i \in [l]$, $N(s_i) = C \setminus \{c_i\}$.

A graph $G = (V, E)$ is said to be *bipartite* if $V(G)$ can be partitioned into two independent sets X and Y. A bipartite graph $G = (X \cup Y, E)$ is called a *complete bipartite graph* or a *bi-clique* if every vertex of X is adjacent to every vertex of Y. A complete bipartite graph $G = (X \cup Y, E)$ is called *star* if any of X or Y has cardinality one.

A bipartite graph $G = (X \cup Y, E)$ is *comb convex* (or *star convex*) if a tree $T = (X, F)$ can be defined, such that T is a comb (or a star) and for every vertex $y \in Y$, the neighborhood of y induces a subtree in T.

A graph is *split* if its vertex set can be partitioned in an independent set and a clique.

1.2. Existing literature

Henning et al. introduced the idea of weak roman domination in [9]. In the same paper, they have provided the following bound for $\gamma_r(G)$ in terms of $\gamma(G)$: $\gamma(G) \leq \gamma_r(G) \leq 2\gamma(G)$ and characterization of graphs is given in which lower bound is attained. Additionally, it presents characterizations of forests in which the upper bound $\gamma_r(G) = 2\gamma(G)$ is met. It has also been proved that the problem the DECIDE-WRD is NP-complete, even when restricted to bipartite and chordal graphs. On the positive side, they have shown that the MIN-WRD can be solved in linear time for trees.

Liu et al. further studied the MIN-WRD problem, and proved that the problem can be solved in polynomial-time for block graphs [10]. Chapelle et al. proved that the MIN-WRD problem can be solved for general graphs with a time complexity of $O^*(2^n)$, accompanied by an exponential space complexity. The authors have also presented an algorithm with a time complexity of $O^*(2.2279^n)$, which takes polynomial space. In addition they proposed a polynomial algorithm to solve the MIN-WRD problem in interval graphs [4]. To the best of our current understanding, no further algorithmic advancements are made regarding the weak roman domination problem.

1.3. Our results

The subsequent sections of this work are organized in the following manner:

- In Section 2, a polynomial-time algorithm has been proposed for solving the MIN-WRD problem for P_4-sparse graphs.
- In Section 3, the DECIDE-WRD problem is shown to be NP-complete for comb convex bipartite graphs.
- In Section 4, the DECIDE-WRD problem is shown to be NP-complete for star convex bipartite graphs. In [9], it has been shown that the DECIDE-WRD problem is NP-complete for bipartite graphs, but using the same reduction, it can also be observed that the same problem is also NP-complete for bounded degree bipartite graphs. We have shown that the MIN-WRD problem is polynomial-time solvable for bounded degree star convex bipartite graphs.
- In Section 5, the DECIDE-WRD problem is shown to be NP-complete for split graphs.
- In Section 6, a $2(1 + ln(\Delta + 1))$ approximation algorithm is given for the MIN-WRD problem. In the same section we have shown the APX-completeness of the problem for graphs with maximum degree 4. Section 7 concludes our work.

2. Algorithm for P_4-sparse graphs

In this section, we give a polynomial-time algorithm to solve MIN-WRD for P_4-sparse graphs. The class of P_4-sparse graphs can be considered as a generalization of the class of cographs. Below we present a characterization theorem for P_4-sparse graphs.

Theorem 2.1 ([3])**.** *A graph G is said to be P_4-sparse if and only if one of the following conditions hold*

- G is a single vertex graph.
- $G = G_1 \cup G_2$, where G_1 and G_2 are P_4-sparse graphs.
- $G = G_1 \oplus G_2$, where G_1 and G_2 are P_4-sparse graphs.
- G is a spider which admits a spider partition (S, C, R) where either $G[R]$ is a P_4-sparse graph or $R = \phi$.

Hence by Theorem 2.1, a connected graph that is P_4-sparse and contains at least two vertices can be classified as either a join of two P_4-sparse graphs or a particular type of spider (thick or thin). In this section, we discuss the Weak Roman domination number for both of these cases.

Lemma 2.1. *Let $G = (V, E)$ be a headless thin spider ($R = \phi$ in the spider partition (S, C, R)), then, $\gamma_r(G)$ is equal to $|C|$.*

Proof. Consider a graph $G = (V, E)$ that is a thin spider. Let the spider partition of G be represented as (S, C, R), where R is an empty set. In [9], it is proved that $\gamma(G) \leq \gamma_r(G)$. Note that for the graph G, $\gamma(G) = |C|$. Now it is enough to show the existence of a WRD function with weight equal to $|C|$ in order to establish the lemma. We define a function $f : V \rightarrow \{0, 1, 2\}$, as follows: for any element u in the set S, $f(u)$ equals 1, and for any other element u, $f(u)$ equals 0. Clearly, f is a WRD function and $w(f) = |C|$. □

Lemma 2.2. *Let $G = (V, E)$ be a thin spider with a nonempty head, ($R \neq \phi$ in the spider partition (S, C, R)). Then, $\gamma_r(G)$ is equal to $|C| + 1$.*

Proof. Consider a graph $G = (V, E)$ that is a thin spider. Let the spider partition of G be represented as (S, C, R), where R is a nonempty set, and let $|C| = |S| = k$. Note that for the graph G, $\gamma(G) = |C|$. Now to complete the proof of this lemma we show the following two statements:

   i. There exists a WRD function f on G with $w(f) = |C| + 1$.
   ii. $\gamma(G) < \gamma_r(G)$.

Initially, we establish a WRD function with a weight of $|C| + 1$. Consider the function $f : V \rightarrow \{0, 1, 2\}$, defined as follows: $f(c_i) = 2$ for a fixed $i \in [k]$ and $f(c_j) = 1$ for all c_j in the set $C \setminus \{c_i\}$. For any other element u, $f(u)$ is equal to 0. It can be observed that the function f exhibits the property of being a WRD function with a weight of $|C| + 1$.

Now we show that $\gamma(G) < \gamma_r(G)$. For sake of contradiction, assume that $\gamma(G) = \gamma_r(G) = |C|$. Then there exists a WRD function of G with $w(f) = \gamma(G) = |C|$. In this case we can say that $f(c_j) + f(s_j) = 1$ holds for all $j \in [k]$. This statement suggests that $f(r)$ equals zero for every r in the set R. Therefore, an attack on a certain vertex r belonging to the set R can be defended by sending a legion from a certain element c_j from the set C, where $f(c_j)$ equals 1 (that means one legion is placed at c_j). However, following the defense, the function is transformed into f', where $f'(r) = 1$, $f'(c_j) = 0$, and $f(u) = f'(u)$ for all $u \in V \setminus \{c_j, r\}$. This results in the vertex $s_j \in S$ being left undefended, which contradicts the initial assumption. Therefore, the value of $\gamma_r(G)$ is equal to $|C| + 1$. □

Lemma 2.3. *Let $G = (V, E)$ be a thick spider with an empty head, ($R = \phi$ in the spider partition (S, C, R)). Then, $\gamma_r(G) = 2$ if $|C| = 2$ and $\gamma_r(G) = 3$, for $|C| \geq 3$.*

Proof. Consider a graph $G = (V, E)$ that is a thick spider, such that $R = \phi$. Let $|C| = |S| = k$. If $k = 2$, then G is a P_4, hence $\gamma_r(G) = 2$. For each integer k greater than or equal to 3, we show the existence of a WRD function f such that the weight of f is equal to 3. The function $f : V \rightarrow \{0, 1, 2\}$ is defined as follows : $f(c_i) = 2$ (for a fixed $c_i \in C$), $f(s_i) = 1$ (for $s_i \in S$) and $f(u) = 0$ for every $u \in V \setminus \{c_i, s_i\}$. It can be observed that the function f exhibits the properties of being a WRD function and $w(f) = 3$.

Next to show that $\gamma_r(G) = 3$ for $|C| \geq 3$, we show that $\gamma_r(G)$ cannot be 2. Assuming $\gamma_r(G) = 2$ and $|C| \geq 3$, the following cases can be considered.
Case 1: In this case we assume that $f(c_i) = f(c_j) = 1$ for some fixed $c_i, c_j \in C$, and $f(u) = 0$ for every $u \in V \setminus \{c_i, c_j\}$. If the attack is on the vertex s_b (where s_b is not a member of the set $\{s_i, s_j\}$), it can be assumed without loss of generality that the legion stationed at c_i will be required to move to s_b, resulting in the vertex s_j being undefended. Therefore, a contradiction emerges.
Case 2: In the second case, the function f is defined so that $f(c_i) = f(s_i) = 1$ for some c_i in the set C and corresponding s_i in the set S. Additionally, $f(u) = 0$ for all elements u in the set $V \setminus \{c_i, s_i\}$. If the attack is on s_j (where s_j is not equal to s_i), it necessitates the movement of the legion stationed at c_i to s_j, resulting in the vertex s_b being undefended (s_b being an element of $S \setminus \{s_i, s_j\}$). This gives rise to a contradiction.
Case 3: The last case being, there exists a WRD function f defined as: $f(s_i) = f(s_j) = 1$ for some fixed $s_i, s_j \in S$, and $f(u) = 0$ for every $u \in V \setminus \{s_i, s_j\}$. Now there exists a vertex $s_b \in S$ (where $b \notin \{i, j\}$) as $|S| \geq 3$; hence s_b is undefended by f, which contradicts the fact that f is a WRD function.
Consequently, it follows that $\gamma_r(G)$ is greater than or equal to 3, which in turn implies that $\gamma_r(G) = 3$. □

Lemma 2.4. *Let $G = (V, E)$ be a thick spider with a nonempty head, ($R \neq \phi$ in the spider partition (S, C, R)). Then, $\gamma_r(G)$ is equal to 3.*

Proof. Consider a graph $G = (V, E)$ that is a thick spider, such that $R \neq \phi$. Let $|C| = |S| = k$. At first, we will show the existence of a WRD function f such that the weight of f is equal to 3. Define the function $f : V \rightarrow \{0, 1, 2\}$ as follows: for some fixed $i \in [k]$, $f(c_i) = 2$ and $f(s_i) = 1$, where $c_i \in C$ and $s_i \in S$ and $f(u) = 0$ for every $u \in V \setminus \{c_i, s_i\}$. It can be observed that the function f exhibits the properties of being a WRD function and $w(f) = 3$. If we assume that $\gamma_r(G) = 2$, we can analyze three different possibilities.

Case 1: In the first scenario, it is seen that $f(c_i) = f(c_j) = 1$ for two fixed c_i and c_j belonging to the set C, whereas $f(u) = 0$ for all u in the set V excluding c_i and c_j. In the event of an attack on a vertex $r \in R$, it is reasonable to suppose, without loss of generality, that the legion positioned at c_i will need to relocate to r. Consequently, this relocation will leave the vertex s_j undefended. Hence, a contradiction arises.

Case 2: In the second scenario, the function f is defined so that $f(c_i) = f(s_i) = 1$ for a fixed c_i in the set C and corresponding s_i in the set S. Additionally, $f(u) = 0$ for all elements u in the set $V \setminus \{c_i, s_i\}$. In the event that an attack is on $r \in R$, it becomes imperative to relocate the legion now stationed at c_i to r. Consequently, this relocation leaves all the vertices of $S \setminus \{s_i\}$ undefended. This results in a contradiction.

Case 3: The third scenario involves $f(r) = 1$ and $f(u) = 1$, for some fixed $r \in R$ and $u \in C \cup S$ and $f(v) = 0$ for every $v \in V \setminus \{r, u\}$. Assume that there exists an element c_i in the set C such that $f(c_i) = 1$. However, it follows that the element s_i remains undefended. Likewise, in the case where $f(s_i) = 1$ for a fixed $s_i \in S$, it follows that all vertices in S except for s_i remain undefended. Therefore, this statement presents a contradiction. Hence $\gamma_r(G) = 3$. □

Now the only case that remains to be analyzed is the case when G is a join of two P_4-sparse graphs G_1 and G_2. In this case, $\gamma(G) \leq 2$. Since $\gamma_r(G) \leq 2\gamma(G)$ [9]; $\gamma_r(G) \leq 4$. Now it is easy to observe that for a complete graph G, $\gamma_r(G) = 1$ and vice versa. So from now on, we will consider that G is not a complete graph. Hence, it is enough to characterize G for $\gamma_r(G) = 2, 3$. Here, we are stating a theorem from [9], which will be useful in upcoming proofs.

Theorem 2.2 ([9])**.** *For any graph G, $\gamma(G) = \gamma_r(G)$ if and only if there exists a minimum dominating set S of G, such that*

(a) *$pn(v; S)$ induces a clique for every $v \in S$,*
(b) *for every vertex $u \in V(G) \setminus S$ that is not a private neighbor of any vertex of S, there exists a vertex $v \in S$ such that $pn(v; S) \cup \{u\}$ induces a clique.*

Hence by Theorem 2.2, the following observation follows,

Observation 2.1. *Let $G = G_1 \oplus G_2$, where G_1 and G_2 are P_4-sparse graphs. Then $\gamma_r(G) = 2$ if and only if one of the following conditions satisfies*

(i) *$\gamma(G) = 1$ and G is not a complete graph.*
(ii) *$\gamma(G) = 2$ and there exists a minimum dominating set S of G that satisfies condition (a) and (b) of Theorem 2.2.*

Note that it takes $O(|S|n^2)$ (resp. $O((n - |S|)|S|(n^2))$) time to check whether a dominating set S satisfies the condition (a) (resp. condition (b)) of Theorem 2.2 or not. Hence, to check $\gamma_r(G) = \gamma(G)$, it takes $O(\binom{n}{|S|}(n - |S|)|S|(n^2))$ time, where S is any minimum dominating set. Hence, to verify the condition (ii) of Observation 2.1, it takes $O(\binom{n}{2} \cdot (n-2) \cdot 2 \cdot (n^2)) = O(n^5)$ time.

Hence, in the following lemma, we will only consider those graphs G, which are not a complete graph and do not satisfy the conditions (i) and (ii) of Observation 2.1.

Lemma 2.5. *Let $G = G_1 \oplus G_2$, where G_1 and G_2 are P_4-sparse graphs. Then $\gamma_r(G) = 3$ if and only if one of the following conditions satisfies*

(i) *There exists one vertex in $u \in V(G)$ such that $G[V(G) \setminus N[u]]$ is a clique.*
(ii) *There exists a set of two vertices $S = \{u_1, u_2\} \subseteq V(G^*)$ (where $G^* \in \{G_1, G_2\}$) such that $G^*[V(G^*) \setminus (N[u_1] \cup N[u_2])]$ is a clique.*

Proof. If (i) holds, define a function $f : V(G) \rightarrow \{0, 1, 2\}$ in following way: $f(u) = 2$ and $f(v) = 1$ where $v \in V(G) \setminus N[u]$ and $f(x) = 0$ for every $x \in V(G) \setminus \{u, v\}$. Clearly, f is a WRD function, hence $\gamma_r(G) = w(f) = 3$. If (ii) holds, then let without loss of generality there exists a set of two vertices $S = \{u_1, u_2\} \subseteq V(G_1)$ such that $G_1[V(G_1) \setminus (N[u_1] \cup N[u_2])]$ is a clique. Now define a function $g : V(G) \rightarrow \{0, 1, 2\}$ in following way: $g(u_1) = g(u_2) = 1$, $g(u_3) = 1$ where $u_3 \in V(G_2)$ and $g(x) = 0$ for every $x \in V(G) \setminus \{u_1, u_2, u_3\}$. Clearly, g is a WRD function, hence $\gamma_r(G) = w(g) = 3$.

Conversely, let $\gamma_r(G) = 3$ and f be a WRD function of weight 3. Now two cases may appear.

Case 1: There exist two vertices u_1, u_2 in $V(G)$ such that $f(u_1) = 2$ and $f(u_2) = 1$. Let $S = \{u_1, u_2\}$. Now if $G[pn(u_2, S)]$ is not a clique then there exist two nonadjacent vertices $v, v' \in pn(u_2, S)$. In that case, any attack on v must result in the legion placed at u_2 being moved to v, which leaves v' undefended. Hence $G[pn(u_2, S)]$ is a clique, implying $G[V(G) \setminus N[u_1]]$ is a clique.

Case 2: There exist three vertices u_1, u_2 and u_3 in $V(G)$ such that $f(u_1) = 1, f(u_2) = 1$ and $f(u_3) = 1$. Before analyzing this case we will prove the following claim.

Claim 2.1. *Let f be a minimum WRD function of $G = G_1 \oplus G_2$ with $f(u_1) = f(u_2) = f(u_3) = 1$, where $u_1, u_2, u_3 \in V(G_1)$ and $f(v) = 0$, otherwise; then there exists a minimum WRD function f' on G such that $f'(u_1) = f'(u_2) = 1$ and $f'(u_3') = 1$ for $u_3' \in V(G_2)$ and $f(v) = 0$, otherwise.*

Proof. We need to show that the function f' effectively defends against all potential attacks.

Consider an arbitrary $v \in V(G_2)$ with $f'(v) = 0$. Suppose that this vertex is attacked. In response, the legion stationed at vertex u_1 moves to defend vertex v. It is easy to observe that after this move, all vertices in the graph remain defended.

Let us consider an unsecured vertex v (in the context of f') belonging to the set $V(G_1)$, which is subjected to an attack. In the configuration f, it is necessary for one of the vertices u_1, u_2, u_3 to be moved at vertex v in order to protect against the attack. Following this move, no unsecured vertices will be undefended (as f is a WRD function). If a vertex from the set $\{u_1, u_2\}$ relocates to vertex v in order to defend against the attack in the context of the function f, then the same vertex relocates to vertex v in the case of the function f'. Similarly, if vertex u_3 relocates to vertex v in the context of function f, then vertex u_3' relocates to vertex v in the case of function f'. □

Hence by Claim 2.1, it is enough to check the case when $u_1, u_2 \in V(G_1)$ (or $V(G_2)$) and $u_3 \in V(G_2)$ (or $V(G_1)$). Without loss of generality, let $u_1, u_2 \in V(G_1)$ and $u_3 \in V(G_2)$. Let $G_1[V(G_1) \setminus (N[u_1] \cup N[u_2])]$ be not a clique, then there exist two vertices $v, v' \in V(G_1) \setminus (N[u_1] \cup N[u_2])$ which are non adjacent. Then any attack on v must be defended by the legion stationed at u_3 by moving to v; which leaves v' undefended, which is a contradiction. Hence $G_1[V(G_1) \setminus (N[u_1] \cup N[u_2])]$ is a clique. □

Since the checks mentioned in Lemma 2.5 and Observation 2.1 can be done in time $O(n^5)$, the following theorem can be stated.

Theorem 2.3. *Given a connected P_4-sparse graph G; $\gamma_r(G)$ can be computed in $O(n^5)$ time.*

3. NP-hardness for comb convex bipartite graphs

In this section, we establish that the DECIDE-WRD problem is NP-hard for comb convex bipartite graphs. To prove this result we present a polynomial-time reduction from the Restricted Exact Cover by 3-Sets problem.

The Restricted Exact Cover by 3-Sets (RXC3) problem is defined as follows: Given a finite set $X = \{x_1, \ldots, x_{3q}\}$, along with a collection $C = \{c_1, \ldots, c_{3q}\}$ (where $c_i \subseteq X$) comprising subsets of X, each containing exactly three elements from X, and ensuring that every element $x_i \in X$ is included in exactly three sets belonging to C. The objective of this problem is to check whether a subcollection, denoted as $C'(\subseteq C)$, such that each element $x_i \in X$ is contained in exactly one $c_i \in C'$, exists or not. This problem, denoted as RXC3, was defined and proved to be NP-hard in [7]. The formal definition of the problem is following:

Restricted Exact Cover by 3-Sets problem (RXC3)

Instance: A universe X and a collection C of 3-elements subsets of X.
Question: Does there exist a subcollection $C' \subseteq C$ such that every element of X belongs to exactly one set in C'?

We have shown a reduction from an instance of RXC3 problem to an instance of the DECIDE-WRD problem for comb convex bipartite graph. The reduction is explained in Construction $\mathcal{A}_1$.

Construction $\mathcal{A}_1$: Consider an instance (X, C) of the RXC3 problem such that $X = \{x_1, \ldots, x_{3q}\}$ and $C = \{c_1, \ldots, c_{3q}\}$. We proceed to construct a graph $H = (V', E')$ using the following procedure. First, we create three copies of the set X, denoted as X_1, X_2, and X_3, as well as two copies of the set C, labeled as C_1 and C_2. Specifically, we define $X_i = \{x_1^i, \ldots, x_{3q}^i\}$ for $i \in [3]$, $C_i = \{c_1^i, \ldots, c_{3q}^i\}$, for $i \in [2]$. Now $V' = X_1 \cup X_2 \cup X_3 \cup C_1 \cup C_2$ and $E' = \{c_i^1 x_j^1, c_i^1 x_j^2 \mid i, j \in [3q]\} \cup \{c_i^1 x_j^3 \mid x_j \in c_i\} \cup \{c_i^1 c_i^2 \mid i \in [3q]\}$.

Example 1. (X, C) be an instance of exact cover where $X = \{x_1, x_2 \ldots, x_6\}$ and $C = \{c_1 = \{x_1, x_4, x_5\}, c_2 = \{x_2, x_4, x_3\}, c_3 = \{x_2, x_3, x_5\}, c_4 = \{x_1, x_4, x_6\}, c_5 = \{x_1, x_2, x_6\}, c_6 = \{x_3, x_5, x_6\}\}$. The corresponding H from this specific instance has been given in Fig. 2.

Clearly, H is a bipartite graph, as $X_1 \cup X_2 \cup X_3 \cup C_2$ and C_1 are two partite sets. To establish that the graph H is indeed a comb convex bipartite graph, we create a comb graph denoted by F as follows:

Let $V(F) = X_1 \cup X_2 \cup X_3 \cup C_2$, with the backbone of the comb being $X_1 \cup X_2$, and the vertices in $X_3 \cup C_2$ acting as pendant vertices. It can be readily verified that for each $c_i^1 \in C_1$, the neighborhood $N(c_i^1)$ in graph F remains connected. Consequently, we conclude that H is a comb convex bipartite graph.

Theorem 3.1. *(X, C) is a "yes" instance if and only if the graph H possesses a WRD function with weight $4q$.*

Proof. Given an instance (X, C) of the RXC3 problem, we can construct a comb convex bipartite graph H using construction $\mathcal{A}_1$. It is important to note that this construction can be done in polynomial-time.
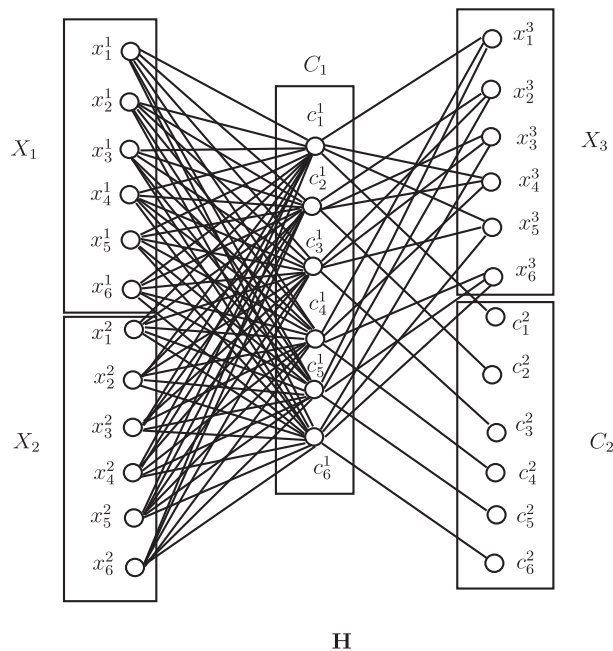
Fig. 2. The Construction $\mathcal{A}_1$ applied on Example 1.

Consider a "yes" instance of the RXC3 problem (X, C) such that $X = \{x_1, \ldots, x_{3q}\}$ and $C = \{c_1, \ldots, c_{3q}\}$. Let $C' \subseteq C$ be an exact cover of the instance (X, C). It is important to note that for any such instance, the size of an exact cover (if one exists) is $\frac{|C|}{3}$. Hence $|C'| = q$. Now, we define a function $f : V(H) \to \{0, 1, 2\}$ as follows:

1. Define $C_1' = \{c_i^1 \mid c_i \in C'\}$. For every vertex u in C_1', we set $f(u) = 2$.
2. For every vertex u in $C_1 \setminus C_1'$ (the corresponding sets in C, which are not included in the exact cover), we set $f(u) = 1$.
3. For all other vertices u in H, $f(u) = 0$.

It is evident that this function f constitutes a WRD function with a weight precisely equal to $4q$.

Conversely, consider a WRD function f with a weight of at most $4q$. It is important to note that for any vertex $c_i^1 \in C_1$ and its corresponding counterpart $c_i^2 \in C_2$ in the constructed graph H, we have the property $f(c_i^1) + f(c_i^2) \geq 1$. Therefore, the sum of the labels for all c_i^1 and c_i^2 is at least $3q$. As a consequence, we can deduce that $f(X_3)$ must be less than or equal to q.

Now, let us assume that there exist l vertices in X_3 with nonzero labels (where $0 < l \leq q$). This implies that $3q - l$ vertices in the set X_3 must have a label of 0. Now, the following claim will help us to complete the proof.

Claim 3.1. *For each $x \in X_3$ with $f(x) = 0$, there exists a neighbor c_i^1 of x with $f(c_i^1) + f(c_i^2) \geq 2$.*

Proof. Let there exist $x \in X_3$ such that x has no neighbors $c_i^1 \in C_1$, with the property: $f(c_i^1) + f(c_i^2) \geq 2$ (which means $f(c_i^1) + f(c_i^2) = 1$ for every $c_i^1 \in C_1$). Now consider an attack on x. A legion must come from one of its neighbors in C_1. Let from c_j^1 one legion comes to x, but this move leaves the vertex c_j^2 undefended (as $f(c_j^1) + f(c_j^2) = 1$), contradicting the fact that f is a WRD function. Hence, the claim holds. $\square$

Since there are $3q - l$ vertices in the set X_3 with a label of 0, and for each of these vertices, there exists a neighbor in C_1 (say c_i^1) that satisfies $f(c_i^1) + f(c_i^2) \geq 2$, hence by this fact and Claim 3.1, we can infer that there must be at least $\frac{3q-l}{3}$ vertices among the c_i^1s that satisfy $f(c_i^1) + f(c_i^2) \geq 2$. This implies that the weight of the WRD function f is greater than or equal to $\frac{3q-l}{3} + 3q + l$ which simplifies to $4q + \frac{2l}{3}$. However, this conclusion contradicts the initial assumption that f has a weight of at most $4q$. Hence it follows that l must be equal to 0. Therefore, $f(X_3) = 0$, and there exist exactly $\frac{3q}{3} = q$ vertices in the set C_1 that satisfy the condition $f(c_i^1) + f(c_i^2) = 2$. As a result, $C' = \{c \in C \mid f(c^1) + f(c^2) = 2\}$ forms an exact cover of the set X. $\square$

Since checking the membership of the DECIDE-WRD problem in NP is easily verifiable, we can conclude by Theorem 3.1, that the DECIDE-WRD is NP-complete.
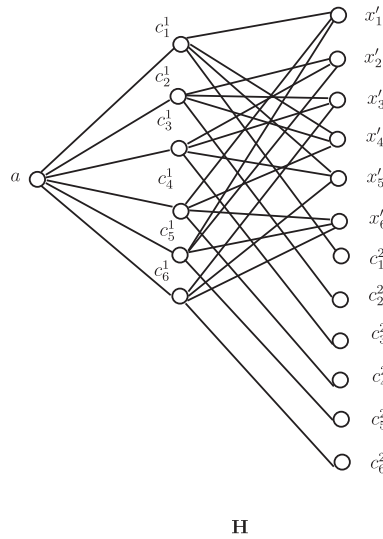
H

Fig. 3. The Construction $\mathcal{A}_2$ applied on Example 1.

4. NP-hardness for star convex bipartite graphs

In this section, our objective is to establish the NP-completeness of the DECIDE-WRD problem for star convex bipartite graphs. To accomplish this, we will provide a polynomial-time reduction from the RXC3 problem to the DECIDE-WRD problem for star convex bipartite graphs. The construction is delineated as follows:

Construction $\mathcal{A}_2$: Given an instance (X, C) of the RXC3 problem, where $X = \{x_1, \ldots, x_{3q}\}$ and $C = \{c_1, \ldots, c_{3q}\}$; define $V' = \{a\} \cup C_1 \cup C_2 \cup X'$. Here, $X' = \{x'_1, \ldots, x'_{3q}\}$, $C_1 = \{c_1^1, \ldots, c_{3q}^1\}$, and $C_2 = \{c_1^2, \ldots, c_{3q}^2\}$. The edge set E' is defined as follows, $E' = \{x'_j c_i^1 \mid x_j \in c_i\} \cup \{ac_i^1 \mid c_i^1 \in C_1\} \cup \{c_i^1 c_i^2 \mid i \in [3q]\}$.

Now a star F can be made with vertices of $\{a\} \cup X' \cup C_2$ where a is the central vertex of the star; such that $F[N(c_i^1)]$ is connected for every $i \in [3q]$. So, H is a star convex bipartite graph. In summary, given an instance of the RXC3 problem, we can effectively construct a star convex bipartite graph H using the defined methodology. Importantly, this construction can be executed in polynomial-time. Please refer to Fig. 3 for a representation of the construction.

Theorem 4.1. (X, C) *has an exact cover if and only if* H *has a WRD function of weight* $4q$.

Proof. Given an instance (X, C) of the RXC3 problem, we can construct a star convex bipartite graph H using construction $\mathcal{A}_2$. It is important to note that this construction can be done in polynomial-time.

Consider a "yes" instance of the RXC3 problem (X, C) such that $X = \{x_1, \ldots, x_{3q}\}$ and $C = \{c_1, \ldots, c_{3q}\}$. Let $C' \subseteq C$ be an exact cover of the instance (X, C). It is important to note that for any such instance, the size of an exact cover (if one exists) is $\frac{|C|}{3}$. Hence $|C'| = q$. Now, we define a function $f : V(H) \to \{0, 1, 2\}$ as follows:

1. Define $C'_1 = \{c_i^1 \mid c_i \in C'\}$. For every vertex u in C'_1, we set $f(u) = 2$.
2. For every vertex u in $C_1 \setminus C'_1$ (the corresponding sets in C, which are not included in the exact cover), we set $f(u) = 1$.
3. For all other vertices u in H, $f(u) = 0$.

It is evident that this function f constitutes a WRD function with a weight precisely equal to $4q$.

Conversely, consider a WRD function f with a weight of at most $4q$. It is important to note that for any vertex $c_i^1 \in C_1$ and its corresponding counterpart $c_i^2 \in C_2$ in the constructed graph H, we have the property $f(c_i^1) + f(c_i^2) \geq 1$. Therefore, the sum of the labels for all c_i^1 and c_i^2 is at least $3q$. As a consequence, we can deduce that $f(X_3)$ must be less than or equal to q.

Now, let us assume that there exist l vertices in X' with nonzero labels (where $0 < l \leq q$). This implies that $3q - l$ vertices in the set X' must have a label of 0. Now, the following claim will help us to complete the proof.

Claim 4.1. *Given a WRD function* f *on* H, *for each* $x \in X'$ *with* $f(x) = 0$, *there exists a neighbor* c_i^1 *of* x *with* $f(c_i^1) + f(c_i^2) \geq 2$.

Proof. Let there exist $x \in X'$ such that x has no neighbors $c_i^1 \in C_1$, with the property: $f(c_i^1) + f(c_i^2) \geq 2$ (which means $f(c_i^1) + f(c_i^2) = 1$ for every $c_i^1 \in C_1$). Now consider an attack on x. A legion must come from one of its neighbors in C_1.

Let from c_j^1 one legion comes to x, but these move leaves the vertex c_j^2 undefended (as $f(c_i^1) + f(c_i^2) = 1$), contradicting the fact that f is a WRD function. Hence the claim holds. $\square$

Since there are $3q - l$ vertices in the set X' with a label of 0, and for each of these vertices, there exists a neighbor in C_1 (say c_i^1) that satisfies $f(c_i^1) + f(c_i^2) \geq 2$, hence by this fact and Claim 4.1, we can infer that there must be at least $\frac{3q-l}{3}$ vertices among the c_i^1s that satisfy $f(c_i^1) + f(c_i^2) \geq 2$. This implies that the weight of the WRD function f is greater than or equal to $\frac{3q-l}{3} + 3q + l$ which simplifies to $4q + \frac{2l}{3}$. However, this conclusion contradicts the initial assumption that f has a weight of at most $4q$. Hence it follows that l must be equal to 0. Therefore, $f(X') = 0$, and there exist exactly $\frac{3q}{3} = q$ vertices in the set C_1 that satisfy the condition $f(c_i^1) + f(c_i^2) = 2$. As a result, $C' = \{c_i \in C \mid f(c_i^1) + f(c_i^2) = 2\}$ forms an exact cover of the set X. $\square$

Hence, by Theorem 4.1 and the fact that membership of the problem in NP is easily verifiable, it can be concluded that the DECIDE-WRD is NP-complete.

Now, in contrast to the aforementioned result, we show that the MIN-WRD problem is polynomial-time solvable for star convex bipartite graphs with bounded degrees. Now, we state a theorem that is used in proving the above statement.

Theorem 4.2 ([11]). *A bipartite graph $G = (X, Y, E)$ is a star convex bipartite graph if and only if there exists a vertex x in X such that every vertex y in Y is either adjacent to x or is a pendant vertex.*

Theorem 4.3. *The MIN-WRD problem is polynomial-time solvable for star convex bipartite graph of bounded degree.*

Proof. Consider a connected star convex bipartite graph $G = (X \cup Y, E)$, where $\Delta(G) \leq k$. By Theorem 4.2, there exists a vertex x in X, such that every $y \in Y$ is either adjacent to y or is a pendant vertex. Now let $Y_0 = N(x)$ and $X_0 = N_G(Y_0)$. It is easy to observe that $X_0 = X$, because G is connected. Note that $|Y_0| = |N(x)| \leq k$ and $|X_0| \leq k(k-1) + 1$. Hence $|V(G)| = |Y_0| + |X| + |Y \setminus Y_0| \leq k + k(k-1) + 1 + k^2(k-1) = k^3 + 1$. Since the size of $V(G)$ is bounded by $k^3 + 1$, the MIN-WRD problem can be solved in constant time. $\square$

5. NP-hardness for split graphs

In this section, we establish the NP-hardness of the problem for split graphs. This result is proved by providing a polynomial-time reduction from the Red Blue Dominating Set problem. Given a bipartite graph $G = (B \cup R, E)$, the Red Blue Dominating Set problem asks to identify a minimum cardinality set $S \subseteq R$ that dominates the set B. The decision version of this problem is formulated as follows:

RED BLUE DOMINATING SET DECISION problem (DECIDE-RBD)

> **Instance**: A bipartite graph $G = (R \cup B, E)$ and a positive integer $k \leq |R|$.
> **Question**: Does there exist $S \subseteq R$ such that $N(S) = B$ and $|S| \leq k$?

The DECIDE-RBD problem is known to be NP-hard [1]. Now we describe a reduction which maps an instance of the DECIDE-RBD problem to an instance of the DECIDE-WRD problem.

Construction $\mathcal{A}_3$: Given a bipartite graph $G = (R \cup B, E)$, where the partition consists of $R = \{r_1, \ldots, r_l\}$ and $B = \{b_1, \ldots, b_h\}$. We create the graph $H(V(H), E(H))$ in the following way: $V(H) = R \cup B \cup R'$ where $R' = \{r_1', \ldots, r_l'\}$. $E(H)$ is defined as follows:

1. Include all the edges from the original graph G, preserving its structure (E).
2. Introduce edges connecting each pair of vertices r_i and r_j, where $i \neq j$ and $i, j \in [l]$. This step ensures that all vertices in R are pairwise connected.
3. Add edges between each vertex r_i and its corresponding counterpart r_i', where $i \in [l]$.

Note that in the resulting graph H, R is a clique and $R' \cup B$ is an independent set. Hence H is a split graph. Also, the entire construction process can be accomplished in polynomial-time. The reduction can be understood more precisely by the example shown in Fig. 4.

Theorem 5.1. *(G, k) is a "yes" instance of the RBD problem if and only if $(H, k + |R|)$ is a "yes" instance of the DECIDE-WRD problem.*

Proof. Given that $(G = (R \cup B, E), k)$ is a "yes" instance of the DECIDE-RBD problem, implying the existence of a set $S \subseteq R$ with $N(S) = B$ and $|S| \leq k$, we can construct a WRD function $f : V(H) \rightarrow \{0, 1, 2\}$ as follows:

1. For each vertex u in the set $R \setminus S$, set $f(u) = 1$.
2. For each vertex u in the set S, set $f(u) = 2$.
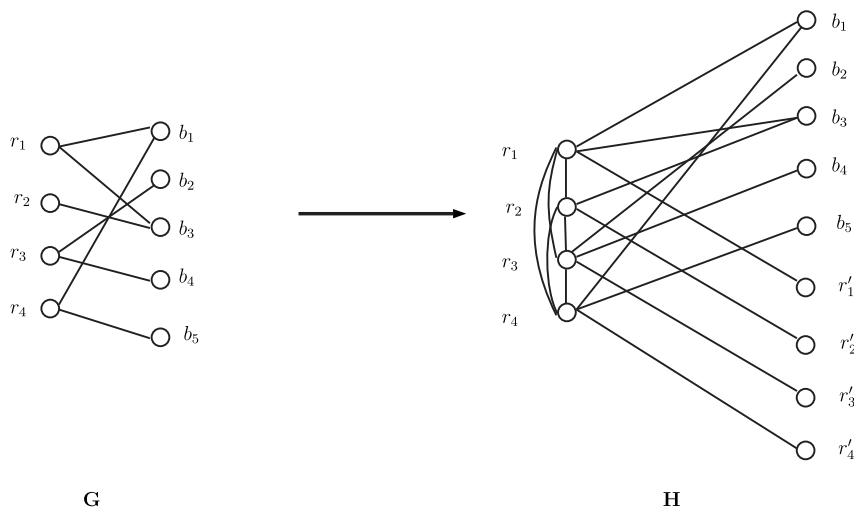3. For all other vertices u in H, set $f(u) = 0$.

Fig. 4. The Construction $\mathcal{A}_3$ applied on Example 1.

This construction guarantees that f satisfies the properties of a WRD function and $w(f) \leq k + |R|$. Consequently, we have established that $(H, k + |R|)$ is a "yes" instance of the DECIDE-WRD problem.

Conversely, let $(H, k + |R|)$ be a "yes" instance of the DECIDE-WRD problem, which implies the existence of a WRD function g with $w(g) \leq k + |R|$. Without loss of generality we assume that $w(g) = \gamma_r(G)$.

Now, it can be observed that for each vertex $r_i \in R$, $g(r_i) + g(r_i') \geq 1$. Additionally, if we have $g(b_i) \geq 1$ for a vertex $b_i \in B$, then there must exist at least one vertex $r_j \in N_R(b_i)$ such that $g(r_j) + g(r_j') = 1$; as if $g(r_j) + g(r_j') \geq 2$, we may assume $g(r_j) = 2$ and $g(r_j') = 0$ for every $r_j \in N_R(b_i)$. In that case we can define a WRD function f, with $f(b_i) = 0$ and $f(v) = g(v)$ for every $v \in V(H) \setminus \{b_i\}$. This implies $w(f) < w(g)$, leading to a contradiction.

We can leverage this observation to construct another WRD function g' as follows:

1. Pick a vertex $b_i \in B$ with non zero g' label. Set $g'(r_j) = 2$ and $g'(r_j') = 0$ for a vertex $r_j \in N_R(b_i)$ with $g(r_j) + g(r_j') = 1$.
2. Set $g'(b_i) = 0$ for the vertex b_i associated with the above condition.
3. For all other vertices u in H, set $g'(u) = g(u)$.

This construction ensures that g' is a WRD function of same weight as g. By repeating this process a finite number of times, we can conclude that there exists a WRD function f with the same weight as g, such that $f(B) = 0$, and for each vertex $b_i \in B$, there exists a vertex $r_j \in N_R(b_i)$ such that $f(r_j) + f(r_j') = 2$.

Hence, the set $S = \{r_j \in R \mid f(r_j) + f(r_j') = 2\}$ dominates the set B, that is $N(S) = B$ and $|S| \leq k$, implying that (G, k) is indeed a "yes" instance of the DECIDE-RBD problem. □

Hence, by Theorem 5.1, it can be concluded that the DECIDE-WRD problem is NP-complete for split graphs.

6. Approximation results

To the best of our knowledge, there does not exist any approximation result in the literature for the MIN-WRD problem. In this section, we give some approximation results. First, we define the minimum dominating set problem.

Minimum Dominating Set problem

> **Instance**: A graph $G = (V, E)$.
> **Solution**: A subset S of $V(G)$ with $|S| = \gamma(G)$.

The following theorems are from previous literature.

Theorem 6.1 ([6]). *The MINIMUM DOMINATION SET problem in a graph with maximum degree Δ can be approximated with an approximation ratio of $1 + ln(\Delta + 1)$.*

In [6], a $1 + ln(\Delta + 1)$ factor approximation algorithm is given for the MINIMUM DOMINTING SET problem. Let us name this approximation algorithm as *APPROX-DOM*. Next, we show that the MIN-WRD problem can also be approximated within a logarithmic factor.

Theorem 6.2. *The MIN-WRD problem in a graph with maximum degree Δ can be approximated with an approximation ratio of $2(1 + ln(\Delta + 1))$.*

Proof. Given a graph $G = (V, E)$ with maximum degree Δ, we apply APPROX-DOM on G and get a dominating set S. Clearly $|S| \leq (1 + ln(\Delta + 1))\gamma(G)$. Now we define $f : V \to \{0, 1, 2\}$ as: $f(u) = 2$ for each $u \in S$ and $f(u) = 0$, otherwise. It is easy to observe that f is a weak Roman dominating function and $w(f) = 2|S| \leq 2(1 + ln(\Delta + 1))\gamma(G) \leq 2(1 + ln(\Delta + 1))\gamma_r(G)$. Hence we get a polynomial time $2(1 + ln(\Delta + 1))$-factor approximation algorithm for the MIN-WRD problem. □

Hence by Theorem 6.2 it can be concluded that the MIN-WRD problem is in APX for bounded degree graphs. Now we show the APX-hardness of the problem. Before showing the APX-hardness we state the definition of an L-reduction.

Given two NP optimization problems F and G and a polynomial time transformation $\mathcal{A}$ from instances of F to instances of G, we say that $\mathcal{A}$ is an *L-reduction* if there are positive constants α and β such that for every instance x of F

1. $OPT_G(h(x)) \leq \alpha OPT_F(x)$.
2. For every feasible solution y of $h(x)$ with objective value $m_G(h(x), y) = c_2$ we can find in polynomial time a solution y' of x with $m_F(x, y') = c_1$ such that $|OPT_F(x) - c_1| \leq \beta|OPT_G(h(x)) - c_2|$.

Theorem 6.3. *The MIN-WRD problem is APX-hard for graphs with maximum degree* 4.

Proof. The MINIMUM DOMINATING SET problem is APX-hard for graphs with maximum degree 3 [2]. It is enough to construct an L-reduction $\mathcal{A}$ from the MINIMUM DOMINATING SET problem to the MIN-WRD problem. Let G be a graph with maximum degree 3, then we construct G' by adding to each vertex $v \in V(G)$ a path $P_4(wxyz)$. Clearly, G' is a graph with maximum degree 4. This reduction in given in [9] to show the NP-completeness of the MIN-WRD problem. We show that this is also an L-reduction.

Let D be a dominating set of G of cardinality $\gamma(G)$ and g be a WRD function of G' with $w(g) = \gamma_r(G')$. By Lemma 23 of [9], $w(g) = |D| + 2|V(G)|$. Note that G is a graph with maximum degree 3, hence $|V(G)| \leq \sum_{v \in D}(deg_G(v) + 1) \leq 4|D|$. So, $w(g) = |D| + 2|V(G)| \leq 9|D|$. Hence $\gamma_r(G') \leq 9\gamma(G)$.

Now, Let f be a WRDF of G'. Then by the proof of Lemma 23 in [9], $f(P_v) \geq 2$, for every $v \in V(G)$.

Note that, if $f(P_v) = 2$, then $f(v) = 0$. If $f(x) = 1$, then $f(w) = 0$, which implies that v is adjacent to some $u \in (V_1 \cup V_2) \cap V(G)$, where V_i is the set of vertices v with $f(v) = i$, for $i \in \{0, 1, 2\}$. If $f(x) = 0$, then clearly $f(w) = 1$ and $f(v) = 0$. But an attack on x can only be defended by moving the legion from w to x. This implies v is defended by some $u \in (V_1 \cup V_2) \cap V(G)$.

Now we define a function f' as follows:

- $f'(v) = f'(w) = f'(y) = 1, f'(x) = f'(z) = 0$; if $f(P_v) = 3$ (where $P_v = \{w, x, y, z\}$).
- $f'(v) = 2, f'(w) = f'(y) = 1, f'(x) = f'(z) = 0$; if $f(P_v) \geq 4$.
- $f'(v) = f(v), f'(w) = f(w), f'(x) = f(x), f'(y) = f(y)$ and $f'(z) = f(z)$; if $f(P_v) = 2$.

If $f(P_v) = 3$, then $f(v) + f(w) + f(x) + f(y) + f(z) = 3$. Hence, we reconfigure as: $f'(v) = f'(w) = f'(y) = 1$, $f'(x) = f'(z) = 0$.

If $f(P_v) \geq 4$, then we reconfigure as: $f'(v) = 2, f'(w) = f'(y) = 1, f'(x) = f'(z) = 0$. It is easy to observe that f' is a WRDF on G', with $w(f') \leq w(f)$.

Now let $S = (V_1 \cup V_2) \cap V(G)$, from the above cases, it can be concluded that S is a dominating set of G. Then $|S| \leq w(f') - 2|V(G)| \leq w(f) - 2|V(G)|$. Hence $|S| - |D| \leq w(f) - 2|V(G)| - (w(g) - 2|V(G)|) \leq w(f) - w(g)$. As a result, $\mathcal{A}$ is an L-reduction with $\alpha = 9$ and $\beta = 1$. Hence the theorem follows. □

7. Conclusion

This work presents an extension of the existing algorithmic research conducted on the weak roman domination problem. The NP-completeness of the problem has been proved for specific classes of graphs. Also, a polynomial-time algorithm has been proposed for solving the problem in the context of P_4-sparse graphs, a well-known extension of the class of cographs. Later we have shown that the problem admits a $2(1 + ln(\Delta + 1))$ approximation algorithm for graphs with maximum degree Δ. We have also shown that the problem is APX-complete for graphs with maximum degree 4. It would be of interest to investigate the computational complexity of the problem for other graph classes, such as distance hereditary graphs, permutation graphs, convex bipartite graphs etc.

Data availability

No data was used for the research described in the article.

Acknowledgment

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