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ORIGINAL RESEARCH

Algorithmic aspects of Roman domination in graphs

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Abstract

For a simple, undirected graph $G = (V, E)$, a Roman dominating function (RDF) $f: V \rightarrow \{0, 1, 2\}$ has the property that, every vertex u with $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a RDF is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a RDF is called the Roman domination number and is denoted by $\gamma_R(G)$. Given a graph G and a positive integer k, the Roman domination problem (RDP) is to check whether G has a RDF of weight at most k. The RDP is known to be NP-complete for bipartite graphs. We strengthen this result by showing that this problem remains NP-complete for two subclasses of bipartite graphs namely, star convex bipartite graphs and comb convex bipartite graphs. We show that $\gamma_R(G)$ is linear time solvable for bounded tree-width graphs, chain graphs and threshold graphs, a subclass of split graphs. The minimum Roman domination problem (MRDP) is to find a RDF of minimum weight in the input graph. We show that the MRDP for star convex bipartite graphs and comb convex bipartite graphs cannot be approximated within $(1 - \epsilon) \ln |V|$ for any $\epsilon > 0$ unless $NP \subseteq DTIME(|V|^{O(\log \log |V|)})$ and also propose a $2(1 + \ln(\Delta + 1))$-approximation algorithm for the MRDP, where Δ is the maximum degree of G. Finally, we show that the MRDP is APX-complete for graphs with maximum degree 5.

Keywords Roman domination · Tree convex bipartite graph · NP-complete · APX-complete

Mathematics Subject Classification 05C69 · 68Q25

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1 Introduction

Consider $G = (V, E)$ be a simple, undirected and connected graph. For a vertex $v \in V$, the *open neighborhood* of v in G is $N_G(v) = \{u \in V \mid (u, v) \in E\}$ and the *closed neighborhood* of v is defined as $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v is $|N_G(v)|$ and is denoted by $deg(v)$. The *maximum degree* of a graph G, denoted by Δ and the *minimum degree* of a graph, denoted by δ are the maximum and the minimum degree of its vertices. An *induced subgraph* is a graph formed from a subset D of vertices of G and all of the edges in G connecting pairs of vertices in that subset, denoted by $\langle D \rangle$. A *clique* is a subset of vertices of G such that every two distinct vertices in the subset are adjacent. An *independent set* is a set of vertices in which no two vertices are adjacent. A vertex v of G is said to be a *pendant vertex* if $deg(v) = 1$ and is called *isolated vertex* if $deg(v) = 0$. An edge of G is said to be a *pendant edge* if one of its vertices is a pendant vertex. A *star* is a tree on n vertices with one vertex having degree $n-1$, called *central vertex*, and the other $n-1$ vertices having degree 1. A *comb* is a tree obtained by joining a single pendant edge to each vertex of a path. In comb, the path is called *backbone* and the pendant vertices are called *teeth*. A bipartite graph $G = (X, Y, E)$ is called *tree convex* if there exists a tree $T = (X, F)$ such that, for each y in Y, the neighbors of y induce a subtree in T. When T is a star (comb), G is called *star* (*comb*) *convex bipartite* graph [12]. For undefined terminology and notations refer to [21].

A vertex v in G dominates the vertices of its closed neighborhood. A set of vertices $S \subseteq V$ is a *dominating set* (DS) in G if for every vertex $u \in V \setminus S$, there exists at least one vertex $v \in S$ such that $(u, v) \in E$, i.e., $N_G[S] = V$. A vertex $u \in V \setminus S$ is said to be *undominated* if $N_G(u) \cap S = \emptyset$. The *domination number* is the minimum cardinality of a dominating set in G and is denoted by $\gamma(G)$ [8]. The *minimum dominating set* is a dominating set of minimum cardinality. The MINIMUM DOMINATION problem is to find a dominating set of minimum cardinality.

Roman domination was introduced in 2004 by Cockayne et al. in [3]. A function $f : V \to \{0, 1, 2\}$ is a *Roman Dominating Function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a RDF f is the value $f(V) = \sum_{u \in V} f(u)$. The *Roman domination number* is the minimum weight of a RDF on G and is denoted by $\gamma_R(G)$. The minimum Roman domination problem (MRDP) is to find a RDF of minimum weight in the input graph.

Given a graph G and a positive integer k, the Roman domination problem (RDP) is to check whether G has a RDF of weight at most k. We refer to [3–5,9–11,14,17–19] for the liturature on Roman domination in graphs.

It is known that RDP is linear time solvable for interval graphs and is NP-complete for planar graphs, bipartite graphs and split graphs [14]. The results obtained in the paper are structured as follows. In Sect. 2, we strengthen the result for bipartite graphs by showing that this problem remains NP-complete for two subclasses of bipartite graphs, i.e., star convex and comb convex bipartite graphs. In Sects. 3, 4 and 5, respectively, we show that RDP is linear time solvable for threshold graphs, chain graphs and bounded tree-width graphs. In Sect. 6, we show that the MRDP for star convex bipartite graphs and comb convex bipartite graphs cannot be approximated within $(1 - \epsilon) \ln |V|$ for any $\epsilon > 0$ unless $NP \subseteq DTIME(|V|^{O(\log \log |V|)})$. In Sect. 7, we show that the MRDP is APX-complete for graphs with $\Delta = 5$. Finally, in Sect. 8, we give a conclusion.

2 Complexity results

In this section, we show that the decision version of the Roman domination problem is NP-complete for star convex bipartite graphs and comb convex bipartite graphs by giving a polynomial time reduction from a well-known NP-complete problem, Exact-3-Cover ($X3C$) [7], which is defined as follows.
EXACT-3-COVER (X3C)
INSTANCE A finite set X with $\mid X \mid = 3q$ and a collection C of 3-element subsets of X.
QUESTION Is there a subcollection C' of C such that every element of X appears in exactly one member of C'?
The decision version of Roman domination problem is defined as follows.
ROMAN DOMINATION PROBLEM (RDP)
INSTANCE A simple, undirected graph $G = (V, E)$ and a positive integer $k \leq |V|$.
QUESTION Does G have a RDF of weight at most k?

Theorem 1 *RDP is NP-complete for star convex bipartite graphs.*

Proof Given a graph G and a function f, whether f is a RDF of size at most k can be checked in polynomial time. Hence RDP is a member of NP. Now we show that RDP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3C, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{c_1, c_2, \ldots, c_t\}$, to an instance $\langle G, k\rangle$ of RDP as follows.

Create vertices x_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a_1, a_2 and a_3. Add edges (a_i, a) for each a_i and (c_i, a) for each c_i. Also add edges (c_j, x_i) if $x_i \in c_j$.

The graph constructed is shown in the Fig. 1. Let $A = \{a\} \cup \{x_i : 1 \leq i \leq 3q\}$ and $B = \{c_i : 1 \leq i \leq t\} \cup \{a_1, a_2, a_3\}$. Assume the set A induces a star with vertex a as central vertex, as shown in the Fig. 2, and the neighbors of each element in B induce a subtree of star.

Therefore G is a star convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3C in polynomial time. Next we need to prove the following claim. □

Claim $X3C$ has a solution if and only if G has a RDF with weight at most $2q + 2$.

Proof Suppose C' is a solution for $X3C$ with $|C'| = q$. We define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in C' \text{ or } v = a \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Clearly, f is a RDF and $f(V) = 2q + 2$. Let $k = 2q + 2$.

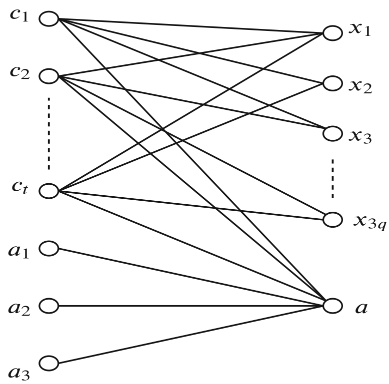

Fig. 1 Construction of a star convex bipartite graph from an instance of X3C

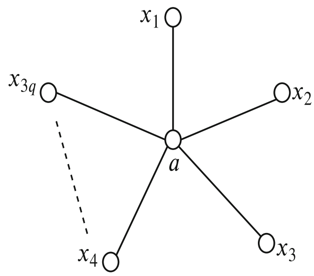

Fig. 2 Star graph

Conversely, suppose that G has a Roman dominating function g with weight at most k. Clearly, $g(a) + g(a_1) + g(a_2) + g(a_3) \geq 2$. Without loss of generality, let $g(a) = 2$ and $g(a_1) = g(a_2) = g(a_3) = 0$. Since $(a, c_j) \in E$, it follows that each vertex c_j may be assigned the value 0. We have the following claim. □

Claim If $g(V) \leq k$ then for each $x_i \in X$, $g(x_i) = 0$.

Proof (Proof by contradiction) Assume $g(V) \leq k$ and there exist m (≥ 1) x_i's such that $g(x_i) \neq 0$. The number of x_i's with $g(x_i) = 0$ is $3q - m$. Since g is a RDF, each x_i with $g(x_i) = 0$ should have a neighbor c_j with $g(c_j) = 2$. So the number of c_j's required with $g(c_j) = 2$ is $\lceil \frac{3q-m}{3} \rceil$. Hence $g(V) = 2 + m + 2\lceil \frac{3q-m}{3} \rceil$, which is greater than k, a contradiction. Therefore for each $x_i \in X$, $g(x_i) = 0$.

Since each c_i has exactly three neighbors in X, clearly, there exist q number of c_i's with weight 2 such that $\left(\bigcup_{g(c_i)=2} N_G(c_i)\right) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 2\}$ is an exact cover for C. □

Theorem 2 *RDP is NP-complete for comb convex bipartite graphs.*

Proof Clearly, RDP is a member of *NP*. We transform an instance $\langle X, C \rangle$ of $X3C$, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{c_1, c_2, \ldots, c_t\}$, to an instance $\langle G, k \rangle$ of RDP as follows.

Create vertices x_i, x_i' for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a', a_1, a_2 and a_3. Add edges (a_i, a) for each a_i and (c_j, x_i) if $x_i \in c_j$. Next add edges (c_j, a) and (c_j, a') for each c_j. Also add edges by joining each c_j to every x_i'.

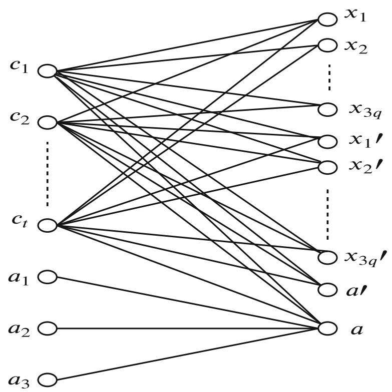

Fig. 3 Construction of a comb convex bipartite graph from an instance of X3C

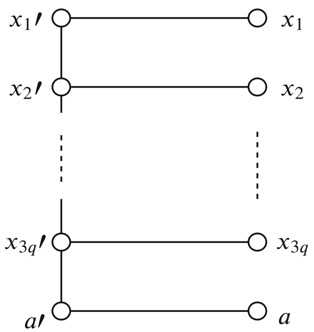

Fig. 4 Comb graph

The graph constructed is shown in the Fig. 3. Let $A = \{a, a'\} \cup \{x_i, x_i' : 1 \leq i \leq 3q\}$ and $B = V \setminus A$. Assume, the set A induces a comb with elements $\{x_i' : 1 \leq i \leq 3q\} \cup \{a'\}$ as backbone and $\{x_i : 1 \leq i \leq 3q\} \cup \{a\}$ as teeth, as shown in the Fig. 4, and the neighbors of each element in B induce a subtree of the comb.

Therefore G is a comb convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3C in polynomial time. Next we need to prove the following claim. □

Claim $X3C$ has a solution if and only if G has a RDF with weight at most $2q + 2$.

Proof Suppose C' is a solution for $X3C$ with $|C'| = q$.

The forward proof is same as in first claim of Theorem 1.

Conversely, suppose that G has a RDF g with weight k. This proof is obtained with similar arguments as in the converse proof of Theorem 1 and by using the assignment $g(v) = 0$ if $v \in \{x_i' : 1 \leq i \leq 3q\} \cup \{a'\}$. □

Now, the following result is immediate from Theorems 1 and 2 .

Theorem 3 *RDP is NP-complete for tree convex bipartite graphs.*

3 Threshold graphs

In this section, we determine the Roman domination number of threshold graph.

Definition 1 A graph $G = (V, E)$ is called a *threshold graph* if there is a real number T and a real number $w(v)$ for every $v \in V$ such that a set $S \subseteq V$ is independent if and only if $\sum_{v \in S} w(S) \leq T$.

Although several characterizations are defined for threshold graphs, we use the following characterization of threshold graphs given in [15] to prove that Roman domination number can be computed in linear time for threshold graphs.

A graph G is a threshold graph if and only if it is a split graph and, for split partition (C, I) of V where C is a clique and I is an independent set, there is an ordering $\{x_1, x_2, \ldots, x_n\}$ of vertices of C such that $N_G[x_1] \subseteq N_G[x_2] \subseteq N_G[x_3] \subseteq \cdots \subseteq N_G[x_n]$, and there is an ordering $\{y_1, y_2, \ldots, y_m\}$ of the vertices of I such that $N_G(y_1) \supseteq N_G(y_2) \supseteq N_G(y_3) \supseteq \cdots \supseteq N_G(y_m)$.

Theorem 4 *Let G be a threshold graph. Then*

$$\gamma_R(G) = \begin{cases} 1, & \text{if } |V(G)| = 1 \\ k+1, & otherwise \end{cases} \tag{2}$$

where k is the number of connected components in G.

Proof If $|V(G)| = 1$ then, clearly, $\gamma_R(G) = 1$. Otherwise, let G be a threshold graph with n clique vertices such that $N_G[x_1] \subseteq N_G[x_2] \subseteq N_G[x_3] \subseteq \cdots \subseteq N_G[x_n]$. Now, define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 1, & \text{if } deg(v) = 0 \\ 2, & \text{if } v = x_n \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Clearly, f is a RDF and $\gamma_R(G) \leq k+1$. From the definition of RDF, it follows that $\gamma_R(G) \geq k+1$. Therefore $\gamma_R(G) = k+1$. □

Now, the following result is immediate from Theorem 4.

Theorem 5 *RDP can be solvable in linear time for threshold graphs.*

Proof Since the ordering of the vertices of the clique and the number of connected components in a threshold graph can be determined in linear time [13,15], the result follows. □

4 Chain graphs

In this section, we propose a method to compute the Roman domination number of a chain graph. A bipartite graph $G = (X, Y, E)$ is called a *chain graph* if the neighborhoods of the vertices of X form a *chain*, that is, the vertices of X can be linearly ordered, say $x_1, x_2, ..., x_p$, such that $N_G(x_1) \subseteq N_G(x_2) \subseteq \cdots \subseteq N_G(x_p)$. If $G = (X, Y, E)$ is a chain graph, then the neighborhoods of the vertices of Y also form a chain. An ordering $\alpha = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$ of $X \cup Y$ is called a *chain ordering* if $N_G(x_1) \subseteq N_G(x_2) \subseteq \cdots \subseteq N_G(x_p)$ and $N_G(y_1) \supseteq N_G(y_2) \supseteq \cdots \supseteq N_G(y_q)$. Every chain graph admits a chain ordering [22]. The following proposition is stated in [3].

Proposition 1 *Let $G = K_{m_1,\ldots,m_n}$ be the complete n-partite graph with $m_1 \leq m_2 \leq \cdots \leq m_n$.*

(a) If $m_1 \geq 3$ then $\gamma_R(G) = 4$.
(b) If $m_1 = 2$ then $\gamma_R(G) = 3$.
(c) If $m_1 = 1$ then $\gamma_R(G) = 2$.

If $G(X, Y, E)$ is a complete bipartite graph then $\gamma_R(G)$ is obtained directly from Proposition 1. Otherwise, the following theorem holds.

Theorem 6 *Let $G(X, Y, E)(\ncong K_{r,s})$ be a chain graph. Then,*

$$\gamma_R(G) = \begin{cases} 3, & \textit{if } |X| = 2 \textit{ or } |Y| = 2 \\ 4, & otherwise \end{cases} \tag{4}$$

Proof If $G \cong K_1$ then $\gamma_R(G) = 1$. Otherwise, let $G(X, Y, E)$ be a connected chain graph with $|X| = p$ and $|Y| = q$. Now, define a function $f : V \to \{0, 1, 2\}$ as follows.

$$\textit{Case } (1) : |X| \geq 2 \text{ and } |Y| = 2 \text{ then } f(v) = \begin{cases} 2, & \text{if } v = y_1 \\ 1, & \text{if } v = y_2 \\ 0, & \text{otherwise} \end{cases}$$

$$\textit{Case } (2) : |X| = 2 \text{ and } |Y| \neq 2 \text{ then } f(v) = \begin{cases} 2, & \text{if } v = x_2 \\ 1, & \text{if } v = x_1 \\ 0, & \text{otherwise} \end{cases}$$

Clearly, f is a RDF and $\gamma_R(G) \leq 3$. From the definition of RDF, it follows that $\gamma_R(G) \geq 3$. Therefore $\gamma_R(G) = 3$.

$$\textit{Case } (3) : |X| \neq 2 \text{ and } |Y| \neq 2 \text{ then } f(v) = \begin{cases} 2, & \text{if } v \in \{x_p, y_1\} \\ 0, & \text{otherwise} \end{cases}$$

Clearly, f is a RDF and $\gamma_R(G) \leq 4$.

Since $p \geq 2$ and $q \geq 2$, in any RDF of G, $f(X) \geq 2$ and $f(Y) \geq 2$. Therefore $\gamma_R(G) \geq 4$. Hence $\gamma_R(G) = 4$. □

If the chain graph G is disconnected with k connected components $G_1, G_2, \ldots, G_k$ then it is easy to verify that $\gamma_R(G) = \sum_{i=1}^{k} \gamma_R(G_i)$.
Now, the following result is immediate from Theorem 6.

Theorem 7 *RDP can be solvable in linear time for chain graphs.*

Proof Since the chain ordering and the connected components can be computed in linear time [13,20], the result follows. □

5 Bounded tree-width graphs

Let G be a graph, T be a tree and v be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T. The pair (T, v) is called a *tree-decomposition* of G if it satisfies the following three conditions: (i) $V(G) = \bigcup_{t \in V(T)} V_t$, (ii) for every edge $e \in E(G)$ there exists a $t \in V(T)$ such that both ends of e lie in V_t, (iii) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_1, t_2, t_3 \in V(T)$ and t_2 is on the path in T from t_1 to t_3. The *width* of (T, v) is the number $max\{|V_t| - 1 : t \in T\}$, and the *tree-width* $tw(G)$ of G is the minimum width of any tree-decomposition of G. By Courcelle's Thoerem, it is well known that every graph problem that can be described by counting monadic second-order logic (CMSOL) can be solved in linear-time in graphs of bounded tree-width, given a tree decomposition as input [2]. We show that RDP can be expressed in CMSOL.

Theorem 8 (Courcelle's Theorem) [2] *Let P be a graph property expressible in CMSOL and k be a constant. Then, for any graph G of tree-width at most k, it can be checked in linear-time whether G has property P.*

Theorem 9 *Given a graph G and a positive integer k, RDP can be expressed in CMSOL.*

Proof Let $f : V \to \{0, 1, 2\}$ be a function on a graph G, where $V_i = \{v | f(v) = i\}$ for $i \in \{0, 1, 2\}$. The CMSOL formula for the RDP is expressed as follows.

$Rom\_Dom(V) = (f(V) \leq k) \wedge \exists V_0, V_1, V_2, \forall p(p \in V_1 \vee p \in V_2 \vee (p \in V_0 \wedge \exists q \in V_2 \wedge adj(p, q))),$

where $adj(p, q)$ is the binary adjacency relation which holds if and only if, p, q are two adjacent vertices of G. □

Now, the following result is immediate from Theorems 8 and 9 .

Theorem 10 *RDP can be solvable in linear time for bounded tree-width graphs.*

6 Lower bound on the approximation ratio of MRDP in star convex and comb convex bipartite graphs

In Sect. 2, it has been shown that the RDP is NP-complete for the star convex and the comb convex bipartite graphs. In this section, we prove an approximation hardness result for the MRDP in star convex and comb convex bipartite graphs. To show the hardness result for the MRDP, we provide an approximation preserving reduction from the MIN SET COVER problem which is stated below.

Min set cover problem Let X be any non-empty set and C be a family of subsets of X. For the set system (X, C), a set $C' \subseteq C$ is called a cover of X, if every element of X belongs to at least one element of C'. The MIN SET COVER problem is to find a minimum cardinality cover of X for a given set system (X, C). The following result is proved in [6].

Theorem 11 [6] *The MIN SET COVER problem for the input instance (X, C) does not admit a $(1 - \epsilon) \ln |X|$-approximation algorithm for any $\epsilon > 0$ unless $NP \subseteq$*

$DTIME(|X|^{O(\log\log|X|)})$. Furthermore, this inapproximability result holds for the case when the size of the input collection C is no more than the size of the set X.

Now we are ready to prove the following result:

Theorem 12 *The MRDP for a star convex bipartite graph G with n vertices does not admit a $(1-\epsilon)\ln n$-approximation algorithm for any $\epsilon > 0$ unless $NP \subseteq DTIME(n^{O(\log\log n)})$.*

Proof In order to prove the theorem, we propose the following approximation preserving reduction. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $C = \{c_1, c_2, \ldots, c_q\}$ be an instance of the MIN SET COVER problem. From this, with similar arguments as in Theorem 1, we construct an instance $G = (V, E)$ of MRDP for star convex bipartite graphs. Next, we state the following claim. □

Claim MIN SET COVER instance (X, C) has a cover of cardinality m if and only if G has a RDF of size $2m + 2$.

Proof The proof is obtained with similar arguments as in first claim of Theorem 1. □

If f is a minimum RDF of G and C^* is a minimum set cover of X for the set system (X, C), then $f(V) = 2|C^*| + 2$. Suppose that the MRDP can be approximated within a ratio of α, where $\alpha = (1-\epsilon)\ln n$ for some fixed $\epsilon > 0$, by using some approximation algorithm, say Algorithm P, that runs in polynomial time. Let k be a fixed positive integer. Then the algorithm SET-COVER-APPROX constructs solution for MIN SET COVER problem. Our algorithm is given in Algorithm 1.

Algorithm 1 SET-COVER-APPROX(X, C)

Require: A set X and a collection C of subsets of X.
Ensure: A cover of X.
1: **if** there exists a cover C' of X of cardinality $\leq k$ **then**
2: $C_x = C'$;
3: **else**
4: Construct the graph G;
5: Compute a RDF g on G by using algorithm P;
6: Construct a cover C' of X from RDF g (as illustrated in the proof of the Claim in Theorem 12);
7: $C_x = C'$;
8: **end if**
9: return C_x;

Clearly, SET-COVER-APPROX runs in polynomial time. If the cardinality of a minimum cover of X is at most k, then it can be computed in polynomial time. Next, we analyze the case, where the cardinality of a minimum cover of X is greater than k. Let C^* denotes a minimum cover of X and f be a minimum RDF of G. So, $|C^*| > k$. If C_x is a cover of X computed by the algorithm SET-COVER-APPROX, then, $|C_x| < g(V) \leq \alpha(f(V)) \leq \alpha(2 + 2|C^*|) \leq \alpha(2 + \frac{2}{|C^*|})|C^*|$. Therefore, SET-COVER-APPROX approximates a cover of X within a ratio of $\alpha(2 + \frac{2}{|C^*|})$.

If $\frac{1}{|C^*|} < \epsilon/2$, then the approximation ratio becomes $\alpha(2 + \frac{2}{|C^*|}) < (1-\epsilon)(2 + 2(\epsilon/2)) \ln n = (1-\epsilon)(2+\epsilon) \ln n = (1-\epsilon') \ln n \approx (1-\epsilon') \ln p$ (since $\ln n \approx \ln p$ for sufficiently large values of p), where $\epsilon' = \epsilon^2 + \epsilon - 1$.

This proves that the algorithm APPROX-SET-COVER approximates set cover of X within ratio $(1-\epsilon') \ln p$ for some fixed $\epsilon' > 0$. By Theorem 11, if the MIN SET COVER problem can be approximated within a ratio of $(1-\epsilon') \ln p$, then $NP \subseteq DTIME(p^{O(\log \log p)})$. It follows that, if MRDP can be approximated within a ratio of $(1-\epsilon) \ln n$ for any $\epsilon > 0$, then $NP \subseteq DTIME(n^{O(\log \log n)})$.

Hence, for a star convex bipartite graph $G = (V, E)$, the MRDP cannot be approximated within a ratio of $(1-\epsilon) \ln n$ for any $\epsilon > 0$ unless $NP \subseteq DTIME(n^{O(\log \log n)})$. □

Theorem 13 *The MRDP for a comb convex bipartite graph G with n vertices does not admit a $(1-\epsilon) \ln n$-approximation algorithm for any $\epsilon > 0$ unless $NP \subseteq DTIME(n^{O(\log \log n)})$.*

Proof The proof is obtained with similar arguments as in Theorem 12, in which replacing the Theorem 1 by Theorem 2 and the first Claim in Theorem 1 by Claim in Theorem 2. □

7 Approximation results

In this section, we obtain an upper on the approximation ratio of the MRDP. We also show that the MRDP is in APX-complete for graphs with maximum degree 5.

7.1 Approximation algorithm

In this subsection, we design an approximation algorithm for MRDP based on the well known optimization problem called MINIMUM DOMINATION problem. The following theorem has been proved in [13].

Theorem 14 [13] *The MINIMUM DOMINATION problem in a graph with maximum degree Δ can be approximated with an approximation ratio of* $1 + \ln(\Delta + 1)$.

Let APPROX-DOM-SET be an approximation algorithm that gives a dominating set D of a graph G such that $|D| \leq (1 + \ln(\Delta + 1))\gamma(G)$, where Δ is the maximum degree of a graph G.

Next, we propose an algorithm APPROX-RDF to compute an approximate solution of MRDP. In our algorithm, first we compute a dominating set D of the input graph G using the approximation algorithm APPROX-DOM-SET. Next, we construct a triple T_r in which every vertex in D will be assigned with weight 2 and the remaining vertices will be assigned with weight 0.

Now, let $T_r = (D', \emptyset, D)$ be the triple obtained by using the APPROX-RDF algorithm. It can be easily seen that every vertex $v \in V$ is assigned with weight either 0 or 2. Since D is a dominating set of G, every vertex $v \in D'$ having weight 0 is adjacent

to a vertex $u \in D$ having weight 2. Thus, T_r gives a Roman dominating function of G.
We note that the algorithm APPROX-RDF computes a Roman dominating triple T_r of a given graph G in polynomial time. Hence, we have the following result.

Algorithm 2 APPROX-RDF(G)

Input: A simple, undirected graph G.
Output: A Roman dominating triple (V_0, V_1, V_2) of vertices of G.
1: $D \leftarrow$ APPROX-DOM-SET(G)
2: $T_r \leftarrow (V \setminus D, \emptyset, D)$
3: return T_r.

Theorem 15 *The MRDP in a graph with maximum degree Δ can be approximated with an approximation ratio of* $2(1 + \ln(\Delta + 1))$.

Proof Let D be the dominating set produced by the algorithm APPROX-DOM-SET, T_r be the Roman dominating triple produced by the algorithm APPROX-RDF and W_r be the weight of T_r.
It can be observed that $W_r = 2|D|$. It is known that $|D| \leq (1 + \ln(\Delta + 1))\gamma(G)$. Therefore, $W_r \leq 2(1 + \ln(\Delta + 1))\gamma(G)$. Since $\gamma(G) \leq \gamma_R(G)$ [3], it follows that $W_r \leq 2(1 + \ln(\Delta + 1))\gamma_R(G)$. Hence the result. □

Since the MRDP in a graph with maximum degree Δ admits an approximation algorithm that achieves the approximation ratio of $2(1 + \ln(\Delta + 1))$, we immediately have the following corollary of Theorem 15.

Corollary 1 *The MRDP is in the class of APX when the maximum degree Δ is fixed.*

7.2 APX-completeness

In this subsection, we prove that the MRDP is APX-complete for graphs with maximum degree 5. This can be proved using an L-reduction, which is defined as follows.

Definition 1 (L-reduction) ([16]) Given two NP optimization problems F and G and a polynomial time transformation f from instances of F to instances of G, one can say that f is an L-*reduction* if there exists positive constants α and β such that for every instance x of F

1. $opt_G(f(x)) \leq \alpha.opt_F(x)$.
2. For every feasible solution y of $f(x)$ with objective value $m_G(f(x), y) = c_2$ in polynomial time one can find a solution y' of x with $m_F(x, y') = c_1$ such that $|opt_F(x) - c_1| \leq \beta|opt_G(f(x)) - c_2|$.

Here, $opt_F(x)$ represents the size of an optimal solution for an instance x of an NP optimization problem F.
An optimization problem π is APX-complete if:

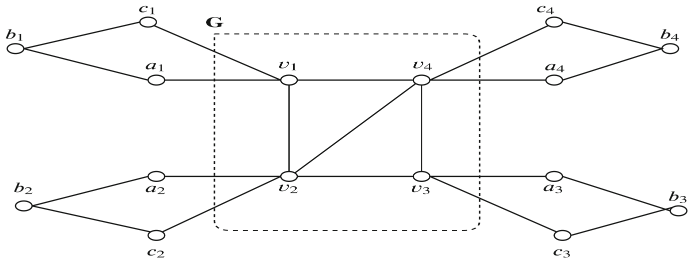

Fig. 5 An illustration to the construction of G' from G

1. $\pi \in$ APX, and
2. $\pi \in$ APX-hard, i.e., there exists an L-reduction from some APX-complete problem to π.

By using Corollary 1, we can say that MRDP is in APX for graphs with maximum degree 5. To show APX-hardness of MRDP, we give an L-reduction from MINIMUM DOMINATING SET problem in graphs with maximum degree 3 (DOM-3) which has been proved as APX-complete [1].

Theorem 16 *The MRDP is APX-complete for graphs with maximum degree 5.*

Proof It is known that MRDP is in APX. Given an instance $G = (V, E)$ of DOM-3, where $V = \{v_1, v_2, \ldots, v_n\}$, we construct an instance $G' = (V', E')$ of MRDP as follows.

Create n copies of P_3 with b_i as the central vertex and a_i, c_i as terminal vertices. Add the edges $\{(v_i, a_i), (v_i, c_i) : 1 \leq i \leq n\}$. Example construction of G' from G is shown in Fig. 5.

Note that G' is a graph with maximum degree 5. First we need to prove the following claim. $\square$

Claim If D^* is a minimum dominating set of G then $\gamma_R(G') = 2n + |D^*|$, where $n = |V(G)|$.

Proof Let $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ be a graph and $G' = (V', E')$ is a graph constructed from G.

Let D^* be a minimum dominating set of G and $f : V \to \{0, 1, 2\}$ be a function on graph G', which is defined as below.

$$f(v) = \begin{cases} 2, & \text{if } v \in \{v_i : v_i \in D^*\} \text{ or } v \in \{b_i : v_i \notin D^*\} \\ 1, & \text{if } v \in \{b_i : v_i \in D^*\} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Clearly, f is a RDF and $\gamma_R(G') \leq 2n + |D^*|$.

Next, we show that $\gamma_R(G') \geq 2n + |D^*|$. Let g be a RDF on graph G'. Clearly if $g(v_i) = 0$ then $g(a_i) + g(b_i) + g(c_i) \geq 2$ and if $g(v_i) \geq 1$ then $g(v_i) + g(a_i) + g(b_i) + g(c_i) \geq 3$. Therefore $\gamma_R(G') \geq 2n + |D^*|$, where $D^* = \{v_i : g(v_i) \geq 1\}$ is a minimum dominating set of G. Hence $\gamma_R(G') = 2n + |D^*|$. $\square$

Let D^* be a minimum dominating set of G and $f : V' \rightarrow \{0, 1, 2\}$ be a minimum RDF of G'. It is known that for any graph $G = (V, E)$ with maximum degree Δ, $\gamma(G) \geq \frac{n}{\Delta+1}$, where $n = |V|$. Thus, $|D^*| \geq \frac{n}{4}$. From the above Claim it is evident that $f(V') = |D^*| + 2n \leq |D^*| + 8|D^*| = 9|D^*|$.

Now consider a RDF $g{:}V' \rightarrow \{0, 1, 2\}$ of G'. Clearly, the set $D = \{v_i : g(v_i) \geq 1$ or $g(a_i) \geq 1$ or $g(c_i) \geq 1\}$ is a dominating set of G. Therefore, $|D| \leq g(V') - 2n$. Hence, $|D| - |D^*| \leq g(V') - 2n - |D^*| \leq g(V') - f(V')$. This implies that there exists an L-reduction with $\alpha = 9$ and $\beta = 1$. □

8 Conclusion

In this paper, we have shown that the RDP is NP-complete for star convex bipartite graphs and comb convex bipartite graphs. Investigating the algorithmic complexity of these problems for other subclasses of bipartite graphs remains open. Next, we have shown that determining $\gamma_R(G)$ is linear time solvable for threshold graphs, chain graphs and bounded tree-width graphs. From approximation point of view, we have given polynomial time approximation algorithm for MRDP and shown that MRDP is APX-hard for graphs with maximum degree 5. The complexity status of this problem is still open for graphs with maximum degree 3 or 4.

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