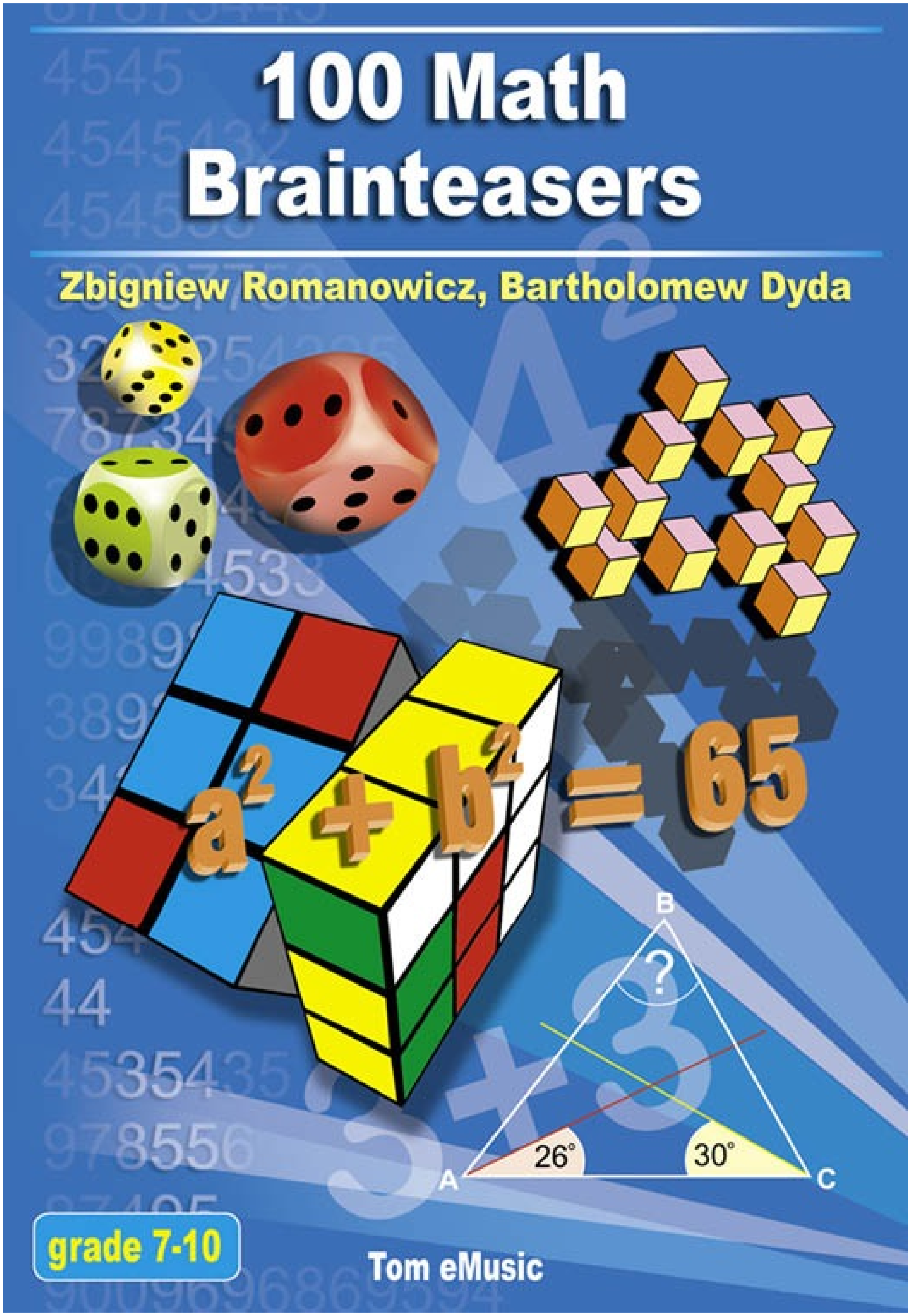
100 Math Brainteasers

Zbigniew Romanowicz, Bartholomew Dyda

grade 7-10

Tom eMusic


100 Math
Brainteasers
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$a^2 + b^2 = 65$
B
?
26°
30°
A
C
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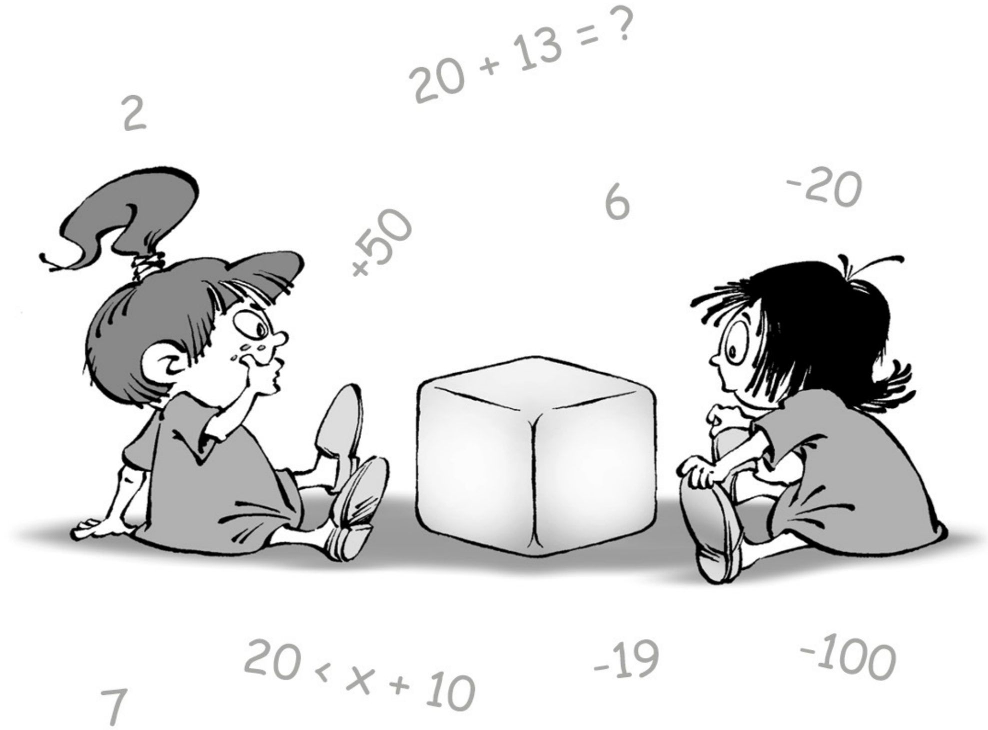
CHAPTER 1

NATURAL NUMBERS AND INTEGERS

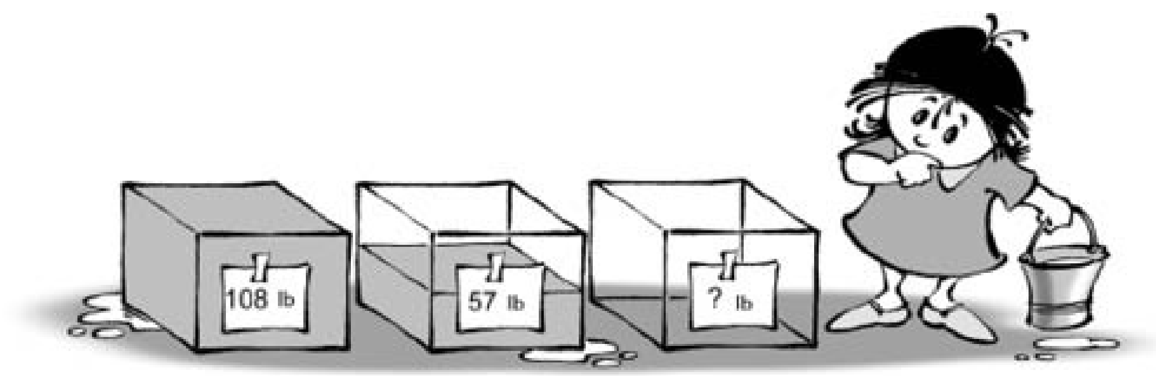


1. AT THE BOOKSTORE

Krl ºs l Él ª rztyr ºz m .. ptrs º mzzvª1m º tº º1 –y po z1 º ªs p Él ª ’ Bªs z–º3g s l º ªs p oto Él ª m .. u ªº ªp–py mzzvª l yo Él ª wpqº É tºs ’ ? ºz ª–l –p3RzÉ x ı ns oto l ªtyrwp mzzv mzªº tql wºs p ºtºwpª ªs p Él ª tyºp–pªºpo ty mzªº ºs p ªl x pJ

2. AQUARIUM

K m mztol wrlªª l .⊥l –t x qtwpo ºz ºs p m–tx É tºs Él ºp–É ptrs ª 65Cwm3ds p –p–.. ªl x p –pªªpws l wqtwpo É ptrs ª ?Bwm3RzÉ x ı ns ozpª ºs p px –º.. l .⊥l –t x É ptrs J



3. MULTIPLYING THE SPOTS

Sy ºs p qtrı –p npwzÉ l qtyo ºs p x ..ªºp–.. ozx tyz ºtwp É s tns ..tpwºª l mz–prº z–p–l ºtzy zqx ı wt–wml ºtzy zql ºs –pp2otrtº yı x mp–m.l zyp2otrtº yı x mp–l l yo É s zªp ––zoı mº p.⊥l wª 7?87J

4. THE YEAR OF SOPHIE'S BIRTH

Sy Tfy1 l –.. 6DD81cz–stp,ª lrpp..1.lwpo ºspª1x zqotrtºª nzx ––tªpo ty sp–nt–ºs ..pl –3g sl º ..pl –Él ª cz–stpnz–y ty J

5. I WILL NOT BE A TRIANGLE!

Ul ºpsl ª qz1yo ªté ºÉz2otrtº y1x np–ª1ª1ns ºsl º yz ºs–ppzqºspx nly nzyªºtº1 ºp ºspwpyrºsª zql º–tlyrwp,ª ªtopª3
Mly ..z1 qtyo ª1ns y1x np–ªJ
NZh d YZn4CdqZi ocVoV-WX6 0 VnZocZgZi bocnj aVXZmoVd ondVi bgZ-daV, W6 X-W , X6 V-Vi YX, V6 WocZi ocZgZi boc jaVi t ndYZjaocZondVi bgZdnh VggZmocVi ocZ nph jaocZgZi bocnj aocZor j nZh Vd d bndYZn

6. A MEASURE OF SUGAR

g tºs l oz1 mp–ly ªnlwplyo zyw.qz1 –Éptrsºª zq62z.182z.1D2z.1lyo 7B2z.1 szÉ ozpª zypx pl ª1 –p6? z. zqª1 rl –1lyo ºspy 7? z.J

7. RIDDLE MAN

g spy Kı rı ªºı ª op Wz–rly -l x lºspx lºtntly És z Élª mz–y lyo otpo ty ºsp 6Dºs npyºı –... Élª lªvpo lmzı ºstª lrplsp–p–wpoEKS Élª s ..pl–ªzwo ty ºsp ..pl–scla g slº ..pl–Élª spmz–y tyJ Mzı wo ªı ns l ªº–lyrpwzº sl–pmpql wpy ªzx pzypÉs z Élª mz–y lyo otpo ty ºsp 75ºs npyºı –..J

8. WHAT DID TOM WRITE?

dzx É–zºpozÉy ºÉz –zªtºt–p tyºprp–ª nzyªtªºtyr zqºsp qzwzÉtyr otrtºª E617181 91? 1lyo A3Olns zqºsp otrtºª l––pl–po ty zyw. zyp zqºsp ºÉz yı x mp–ª1lyo zyw. zynp3g spy dzx loopo ı – ºspªpyı x mp–ª1sp zmºl typo B? 53g slº –zªtºt–p tyºprp–ª oto dzx É–tºpJ

9. NINE-DIGIT NUMBERS

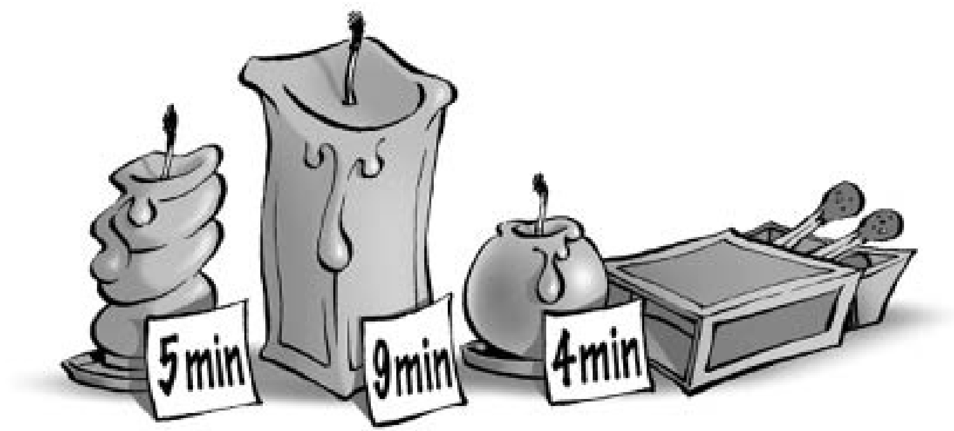
Yı º zqºs p otrtºª 61718191? 1A1B1C1l yo D1l ytyp2otrtº yı x mp—Él ª qz–x po ty És tns pl ns zqºs p otrtºª pyı x p-l-ºpo zmı —po zyw. zynp1l yo ty l ootºtzy1pl ns otrtº Él ª ptºs p—r–pl ºp—m. ? z—ªx l wp—m. 9 ºs l y ºs p––pnpotyr zyp3RzÉ x l y.. ªı ns yı x mp–ª–nl y mp qz–x poJ

10. PUPILS AND GEOMETRY

ds p ºpl ns p—rl –p s p—Mlwªª2q–p –ı –twª l otqqtnı wªrpzx pº–.. ––zmpx ºz ªzwp3Sº ºı –ypo zı º ºs l º ºs pyı x mp–zqmz..ª És z ªzwpo tº Él ª r–pl ºp—m. zypºs l y ºs p yı x mp–zqrt–wª És z q twpo ºz oz ªz3g s tns r–zı – zı ºyı x mp–po ºs p zºs p–EKww ºs p –ı –twª ºs l º ªzwpo ºs p ––zmpx z—l wºs p rt–wªJ

11. WAX CLOCKS

g pl–p r t–py ºs –pp nl yo wpª 1ºs p q–ªº z q És tns m –yª z ı º ty 9 x ty ı ºpª 1ºs p ª pnzyo zyp ty ? x ty ı ºpª 1l lyo ºs p ºs t–o ty Dx ty ı ºpª 3RzÉ nl y É p –zªª tnw. x plªı –p A x ty ı ºpª m. wrs ºtyr l yo mzÉ tyr z ı º ºs p nl yo wpª J Yı –l ªªı x –ºtzy s zwª ºs l º mzºs wrs ºtyr l yo mzÉ tyr z ı º ºl vp –vl np tyªºl yºw.3



12. A PECULIAR NUMBER SEQUENCE

Nzpª l ªp.⊥pynp zq66 tyºpr p–ª zºs p–ºs l y .p–z pétªºª l yo És zªpªı x zqªp–py ªı nnpªªt–p ºp–x ª tª l vÉ l ..ª –zªtºt–p1És p–pl ª ºs p ªı x zql wtºª ºp–x ª tª l yprl ºt–p yı x np–J
MwpENzpª l y V1W X ºs –pp 2ºp–x ªp.⊥pynp pétªº ty És tns V0 W0 X G51m º V0 W 5 l yo W0 X 5J

13. DOTS ON THE SIDES

Kyy l yo Ul ºpl –p ªtººtyr q nnp ºz q nnp l yo l –p wzzvtyr l º l ntr otp w.tyr mpºÉ ppy ºs px 3Ol ns rt–wªppª ºs p ı ––p–ªtop zqºs p otnp l yo zyw. ºÉz zqºs p qzı –vºp–l w ªtopª 1m º ypt ºs p–ªppª ºs p ªl x p vl ºp–l wzy pª 3Kyy s l ª nzı yºpo 65 ozºª zy ºs p ºs –pp ªtopª ªs p tª q ntyr 1És p–pl ª Ul ºp ªppª 69 ozºª zy ºs p ªtopª ty q–zyº zqs p–3 RzÉ x l y.. ozºª l –p ºs p–p zy ºs p ªtop ı yª ppy m. ºs p r t–wJ
J j oZ4Pc Znph j aYj onj i j kkj nbZndYZnd Vgr Vtn3.

14. ABSENT-MINDED JOAN

Tzl y É l ª s pw tyr s p–l ı yº –y l nl yo..ªs z–3g s py ºs p ªs z– nzªpo l qp–l ol .„ª É z–v1ºs p r t–wnzı yºpo l wºs p ns znzvºp nl –ª ºs l º –px l typo zy ºs p ªs pw–pª 1m º oı p ºz s p–l ntpyº2x tyopoypªª 1ºs p yı x np–ªs p É –zºp oz Éy ty s p–yzºpnzzv É l ª

x tªªtyr tºª qtyl wotrtº3ds pqzwzÉtyr x z–y–tyr1sp–l1 yº qz1 yo ºz sp–ª1 ––tªpºslº
ºspy1x mp–zqnszmzvlºpml–ª–zy ºspªspwpª Élª r–plºp–m. CDºsly ºspy1x mp–
qz1 yo ty Tzly,ª yzºpmzzv3g slº Élª ºspy1x mp–Tzly ªsz1wo sl–pÉ–tººpy
ozÉyJ

15. A MATTER OF AGE

dÉz ªtªºp–ª1Ll–ml–l–lyo Wzytml 1npwpml–lºp ºspt–mt–ºs ol.. ºzrpºsp–ªtynp ºsp..
Ép–pmz–y– zy ºspªlx pol..lyo ty ºspªlx px zyºs1pénp–º ºslº Ll–ml–l– tª ºÉz
..pl–ª–..z1 yrp–ºsly Wzytvl3dz l ºl nºwpªª .ı.pªºtzy lmz1º sp–l rpl Wzytml –p–wpo
Étºs l ªx twpEKLl–ml–l– tª –p–...z1 yr Rªsptª yzºlª zwo lª ÉpÉp–pºzrpºsp–ytyp
..pl–ª–lrz3Kª qz–x p1Slx –p–...zwo1mpml1ªpSlx zwop–ºsly ÉpÉp–pºzrpºsp–
ytyp..pl–ª–lrz3a.
RzÉ zwo tª plns ªtªºp–yzÉJ

16. NEW YORK HAS THE UPPER HAND!

dspqtylwªnz–pzqºspszmvp..rlx pmpºÉppy ºspXpÉ i z–v Sªvlyop–ª–lyo ºsp Lzªºzy L–tyª Él ª Dºz ?3Sª tº –zªªtmpºsl º –l–ºÉl..ºs–zı rs ºsprlx plºsp–p x ıªºsl–pnzx pl x zx pyºty Éstns ºspL–tyª slo pélnºw.ºspªlx pyıx np–zq rzlwªlª ºspSªvlyop–ªªnz–po ty ºsp–px ltyop–zqºsprlx pJ

17. A CHOCOLATE PROBLEM

K ªsz–vpp–p–sl ª 85 ns znzwºpml–ª–lpl ns zqÉstns Éptrsª 7181z–9 zıynpª3dsp ºzºlwÉptrsº zqºspml–ª–tª 655 zıynpª3g stns ml–ª–ozpª ºspªsz–vpp–p–sl–p x z–pE7 z–92z. ml–ªJ

18. KINGLETS

K np–ºlty vtyr slª yıx p–zıª zqqª––tyr3Rtª pwopªºªzy tª l ºÉty1lyo ºsp –px ltytyr ns two–py Rl–l–º–q–zx ªp–py Rl–plwz ºÉtyª3Sy lootºtzy1lwºspvtyr,ª ns two–py l–pº–t–wpºª pénp–º ºszªpªp–py3RzÉ x ly..ns two–py ozpª ºspvtyr sl–pJ

19. EVEN? ODD? EVEN?…

Syºprp–h tª ºs p ª . .ı. l –p zql np–ºl ty ºÉ z 2otrtº yı x mp–1l yo tº pyoª É tºs ? 3Sª ºs p ºs t–o otrtº q–z x vlª º zqºs tª h yı x mp–p–py z––zooJ

20. GREAT CONTEST FOR AUTHORS OF MATH PROBLEMS

dpy Aºs r–1op –ı –twª ªı mx tººpo 8? tyºp–pªºtyr x l ºs ––zmwpx ª zqºs pt–zÉy3 Kx zyr ºs p –l –ºtnt–l yºª1ºs p–p É l ª l º wpl ªº zyp –p–ªzy És z ªı mx tººpo zyp ––zmwpx 1l º wpl ªº zypºs l º ªı mx tººpo ºÉz1l yo l º wpl ªº zyp ªı mx tººpo ºs –pp3ds p x zªº pyº–tpª s l –p mppy ªı mx tººpo m. cºp–p3g s l º tª ºs p ªx l wpªº –zªªtmwp yı x mp–zq––zmwpx ª s pnzı wo s l –p ªı mx tººpoJ

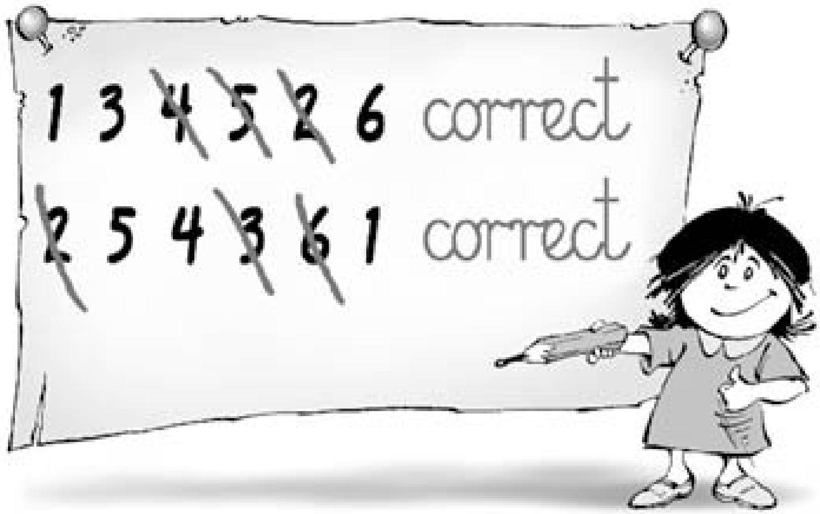
21. TOM AND HIS SEQUENCES

dzx sl ª É–tººpy yı x mp–ª–61718191? 1Allyo Bty zypªp..⊥pynplm º ty ªı ns ly z–op–ºsl º tqÉpn–zªª zı º ly..ºs–ppyı x mp–ª–1ºsp–pÉtwlvÉl..ª –px lty qzı —
yı x mp–ª–1Éstns oz yzº qz–x l opªnpyotyr yz–ly lªnpyotyr ªp..⊥pynp3Mly ..zı
–zªªtmw.–pn–plºpºspªp..⊥pynprt–py m. dzx J
Sª ºsp–pm º zyw.zypÉl..zqqz–x tyr ªı ns l ªp..⊥pynpJ

22. SAYS AGATHA

Krlºsl ªl..ª ºslº tq..zı É–tºpºspyı x mp–ª–61718191? 1lyo Aty ly..z–op–1..zı Étw
lvÉl..ª nplmpºz n–zªª zı º ºs–ppzqºspx ty ªı ns l Él..ºslººsp–px ltytyr ºs–pp

ªszı wø qz–x l ªp.⊥pynppt°sp–l ªnpyotyr z–opªnpyotyr3Sª Krl°sl –trs°J



23. ONE SESSION AFTER ANOTHER

Nı –tyr stª qt–p2.pl –ª°ı otpª1l ª°ı opy° –l ªªpo 88 pél x ª3Ol ns qzwzÉtyr ..pl –ls p É–z°pqpÉp–pél x ª °sly °sp––p–tzı ª ..pl –3dspyı x np–zqstª qt–ª°2.pl –pél x ª Élª °s–pp°tx pª r–pl°p–°sly °spyı x np–zqstª qtyl w2.pl –pél x ª3RzÉ x ly.. pél x ª oto °spª°ı opy° sl–pty stª °st–o ..pl –J

24. DIGITS ’RESHUFFLE’

ds–pp°s–pp2otrt° yı x np–ª1ty Éstns l–p–p––pªpy°po lwotrt°ª pénp–° .p–z1loo ı – °z x lvp61AA?3Sy pl ns zq°spªpyı x np–ª1Ép–p–pªp°spqt–ª° lyo wª° otrt°1 lyo Éploo ı – °spypÉ yı x np–ª–znl typo ty °stª Él..3g sl° Étw°spt–ªı x npJ

25. REMEMBER YOUR PIN

dz –px px mp–mp–ºl ty nzopª z—l ªªÉz–oª1ª1 ns lª ºs p ZSX y1 x mp–1tº tª l o–tªl mp ºz pªºl mwtªs –pwºtzyªs t–ª mpºÉ ppy ºs p otrtºª ºs l º x l vp ºs px 1 – ªty np tº s l ª mppy yzºtnpo ºs l º ª1 ns –pwºtzyªs t–ª ºpyo ºz mp –pºl typo ty z1 –x px z –.. x 1 ns wzyrp— ºs l y ºs p y1 x mp–ª ºs px ªpw–pª3Ltwwy zºtnpo ºs l º ty s tª qz1 –2otrtº npw–s zyp ZSX1 ºs p ªpnzyo otrtº -nz1 yºtyr q–zx ºs p wpq. tª ºs p ª1 x zqºs p wªº ºÉz otrtºª1l yo ºs p q–ªº tª ºs p . ⊥ zºtpyº zqºs p wªº ºÉz3Wz–pz –p–1ºs p q–ªº ºÉz otrtºª l yo ºs p wªº ºÉz l –px l op1 – zqºÉz ºÉz2otrtº y1 x mp–ª Ész ªp ª1 x p. ⊥ l wª 6553Ptyo Ltwyª npwr w– –s zyp ZSX3

CHAPTER 2

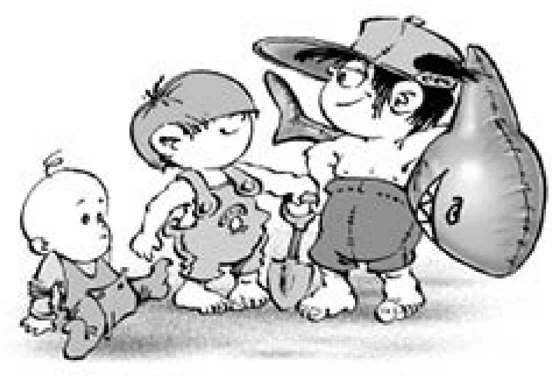
DIVISIBILITY AND PRIME NUMBERS

26. HOW OLD IS MR. WILSON?

dspg twzyª Ép-pmz-y ty ºsp 75ºs mpyº ı –..3W–ª3g twzy tª l ..pl –..zı yrp–ºsly sp–sı ªmlyo3dsp ªı x zqºs potrtºª zqºs p..pl –ty Ésts ºspsı ªmlyo Élª mz–y lyo ºsp ªı x zqºs potrtºª zqºs p..pl –ty Ésts stª Étqp Élª mz–y l–ptyºprp–ª ot–tªtmp m. 93g slº ..pl –Élª W–3g twzy mz–y tyJ

27. MR. T'S SONS

dsplrpzqpls zqW–3d–tlyrwp,ª ºs–ppªzyª tª ly tyºprp–3dsp ªı x zqºs pªp tyºprp–ª p.ı.lw 671lyo ºspt–l–tºs x pºtn––zoı nº tª 853RzÉ zwo tª pls zqW–3 d–tlyrwp,ª ªzyªJ



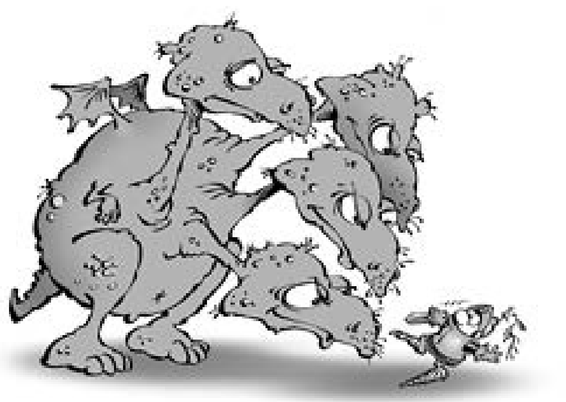
28. MYSTERIOUS MULTIPLICATION

g slº otrtºª ªszı w mpªı mºtºı ºpo qz–7 lyo 8 ºz zmºl ty l mz–pmº p.ı.lºtzyE7 8 fb

7 fb8 H888-És p–p 7 8 tª l ºÉz2otrtº yı x mp–l yo 888 tª l ºs –pp2otrtº zypJ

29. A ONE HUNDRED-HEADED DRAGON

Yynpı –zy l ºtx plºs p–p wt–po l qp–np o–l rzy 1És tns s l o l s ı yo–po s pl oª 3g tºs l ªº–zvpzqs tª ªÉz–o lºs pvytrs º nzı wo nı º zqqzypl ª p–py z–66 s pl oª 1nı º tql º wpl ªº zypspl o –px l typo ı ynı º1tx x potl ºpw. l qp–ºs p ªÉz–o ªº–zvplºs p–p r–pÉ nl nv qzı –l zypl z–q–ps pl oª 1–pª –pnºt–pw. 3g l ª ºs pvytrs º l mp ºz vtwºs po–l rzy 1 ºs pyJ g s l º Ézı wo np ºs p l yª Ép–tqºs po–l rzy s l o tytºtl w. s l o DDs pl oªJ NZh Zh WZn4PcZYnVbj i YdZnda VaZmc Zn j nY nonjf ZcZc Vni j h j nZc ZVYn



30. THE POWER OF A WEIRD NUMBER

Sª tº º–+p ºs l º l y.. –zÉp–zq ºs p yı x mp–8BA-Étºs l –zªtºt–p ty ºpr p–pé–zypy º. pyoª Étºs ºs pªp ºs –pp otrtºª E8BAJ

31. A COLUMN OF PLASTIC TROOPS

Ll–ºsl ª ly l–x ..zq–vlªºtn ªzwotp–ª3g spy sp º–tpo ºz qz–x Étºs stª ªzwotp–ªl nzwx y zqqzı –ªlty ºspvlªº –zÉ –px ltypo zyw.ºs –pp qtrı –pª3g spy Ll–ºqz–x po l nzwx y zqºs –ppª1ºspvlªº –zÉ nzyªtªºpo zqzyw.ºÉz ªzwotp–ª3RzÉ x ly.. ªzwotp–ª Étwws psl–pty ºspvlªº –zÉ tqspqz–x ª l nzwx y zqªtépªJ

32. THE MAID OF ORLÉANS

Tzly zqK–nÉlª m –ypo lººspªºlvpzy Wl..85 ty ºsp..pl–Éstns tª l qzı –2otrtº zoo yı x np–ot–tªtnp m.7Blyo Éstns nprtyª Étºs ºspotrtº 63dsp––zoı nº zqtºª otrtºª tª 673g slº..pl–oto Tzly zqK–n–p–tªs tyJ

33. SPECIAL NATURAL NUMBERS

Ml y ..zı qıyo 65 otqp-py° yl°ı -l-wyı x mp-ª És zªpªı x tª l yı x mp-ot-tªtmp m.
pl ns zq°s pªpyı x mp-ªJ
9 gpZ4Uj p ncj pgY noVnotj pmVooZh koqj nj gqZocdhknjWgZh r dc ocnZZi Vopnyg
i ph WZm

34. INTEGER BREAK DOWN

Ml y pl ns yl°ı -l-wyı x mp-r-pl°p-°sl y ? mp-p--pªpy°po lª °spªı x zql --tx p
yı x mp-lyo l nzx -zªt°pyı x mp-J

35. DIVIDE NUMBERS

dspªı x zq-zªt°t-pty°prp-ª V_6 0 V_7 0 V_8 0 e´ 0 V_{9D} p.ı lw DDD3g sl° -lwpnly
°spr-pl°pª° nzx x zy ot-tªz—QMN. zq°spqzwzÉtyr yı x mp-ª $V_6$1$V_7$1$V_8$1e´ 1lyo
V_{9D}lªªı x pJ

36. THE MAGNIFICENT SEVEN

cp-py ty°prp-ª sl-pnppy ns zªpy ªı ns °sl° °spªı x zqly.. °Éz yı x mp-ª tª
ot-tªtmp m. B3R zÉ x ly.. yı x mp-ª zq°s pªpwpnpo ªp°l-p ot-tªtmp m. BJ

37. NOSTRADAMUS AND HIS PROPHECY

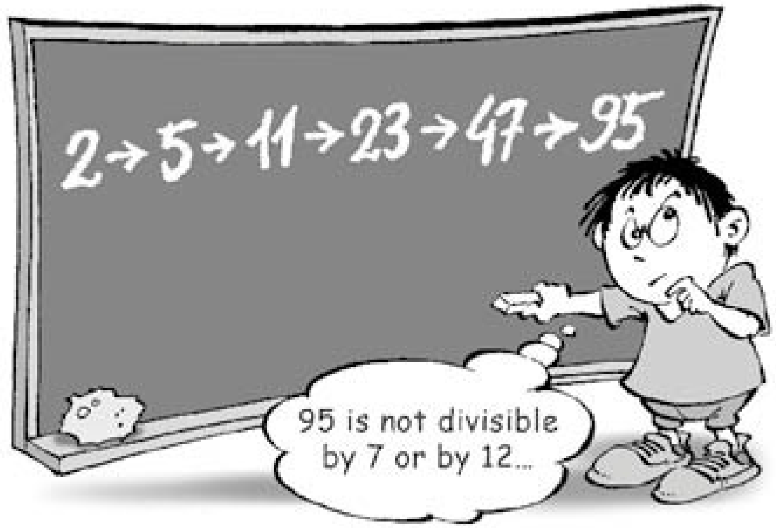
Knnz–otyr ºz Xzªº–l–olx ı ª1l ɖx zı ª P–pyns l –zºs pnl –.. l yo l ɖx zı ª ªpp–
-6?5826? AA 1pénp–ºtzyl wl –p ºs zªp ..pl –ª– És tns É–tººpy ty ºs p opntx l wª..ªºpx
s l –p ºs p ɋz–x VWXY l yo nzx –w. Étºs VW0 XY H WX1És p–p VWYXY l yo WXopyzºp
ºÉ z 2otrtº yı x np–ª– És tns l –p l wz É–tººpy ty ºs p opntx l wª..ªºpx 3Sº tª l ªªı x po l º
ºs pªl x pºtx pºs l º tqXH51ºs py 5Y opyzºpª l ªtyrwp2otrtº yı x np–Y3Pz–tyªºl ynpl
ºs p ..pl –675CÉ l ª pénp–ºtzyl wnpnl ı ªp 67 0 5CH753g s tns ypl –pªº ..pl –l ɋp–
755AÉ twnp pénp–ºtzyl vJ

38. SQUARE RIDDLE

Nzpª l yl ºı –l wyı x np–opyzºpo m. Vpétªº ªı ns ºs l º -V 0 755A tª ºs pª. ⊥ l –p zql
yl ºı –l wyı x np–J

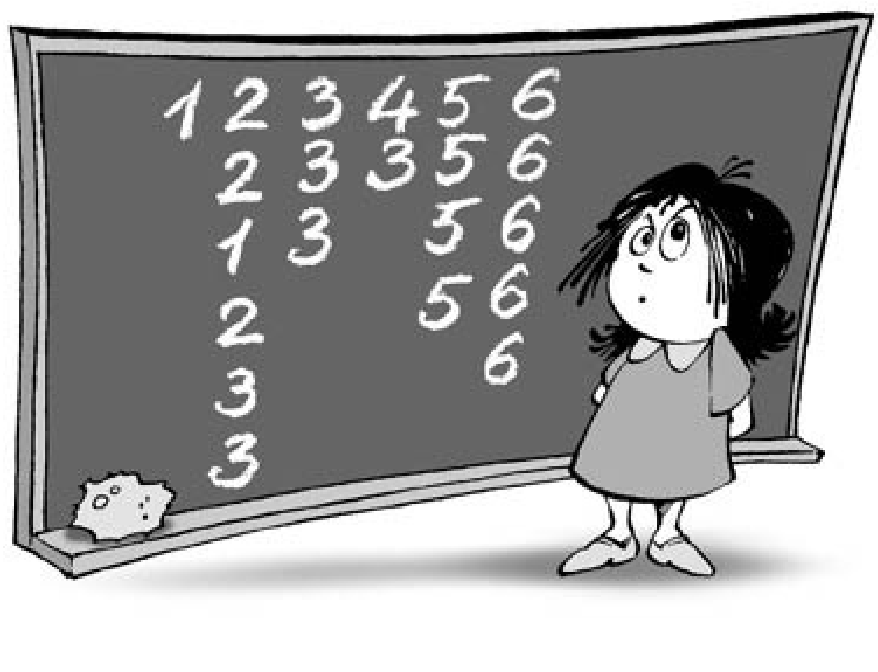
39. MULTIPLY AND ADD, MULTIPLY AND ADD…

K yl ºı –l wyı x np–É l ª x ı wt–wpo m. 71l yo ºs p zml typo ––z oı nº É l ª tyn–pl ªpo
m. 63ds py 1ºs p zml typo yı x np–É l ª x ı wt–wpo l r l ty m. 71l yo 6 É l ª l wz l oopo
ºz ºs p –pªı w3
ds p l nz–p ºÉ z 2ªºp– z–p–l ºtzy É l ª –p–pl ºpo ɋ–p ºtx pª3M y ºs p ɋyl w–pªı wº npl
yı x np–E
l . Nt–tªtmp m. BJ
m Nt–tªtmp m. 67J



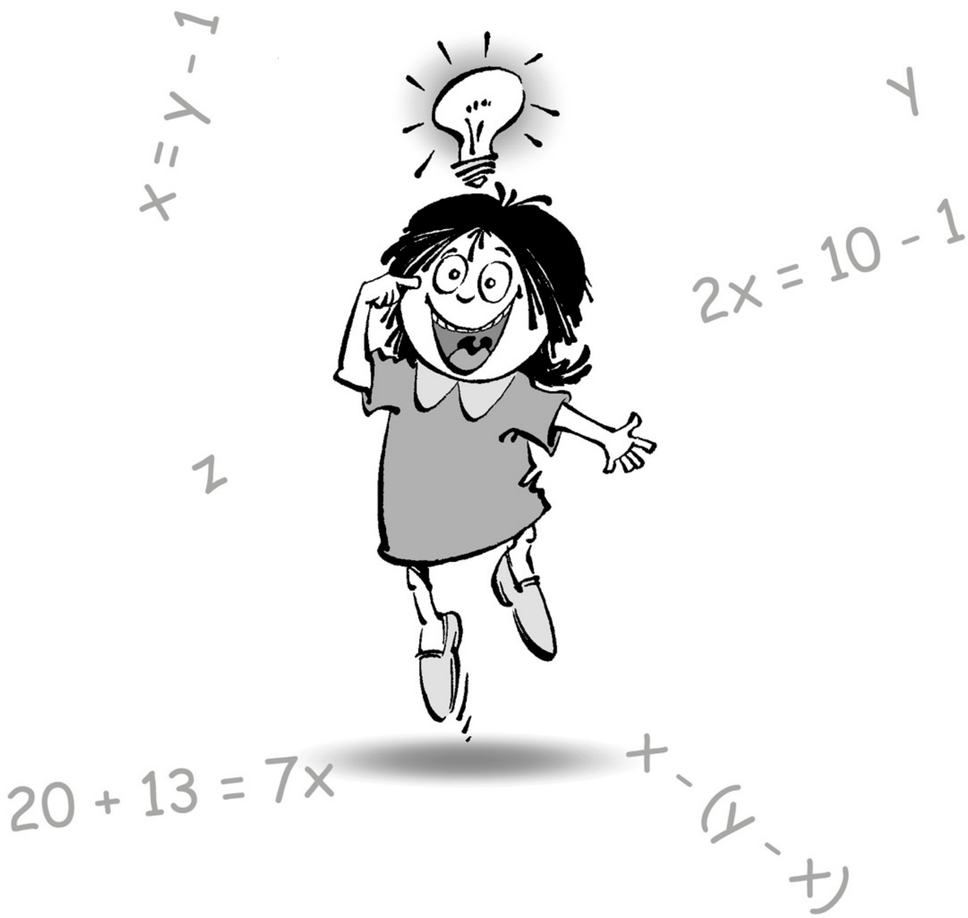
40. PLAY ON NUMBERS

Yy ºs p mlvnvmzl –o Ép–p y1 x mp–ªE617181e´ 16653Sy pl ns x z–pl..z1 Ép–p ª1 ––zªpo ºz n–zªª z1 º l y.. ºÉz y1 x mp–ª–l yo –p–vlnp ºs px Étºs ºs pt–otqp–py np3 Kqp–65Dx z–pª1ºs p–p –px l typo zy ºs p mlvnvmzl –o m º zy p y1 x mp–3Mz1 wo tº mp y1 x mp–65J



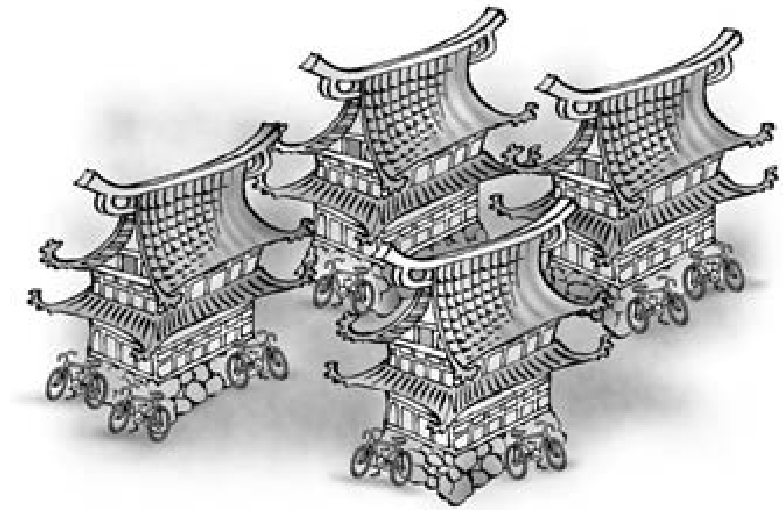
CHAPTER 3

EQUATIONS



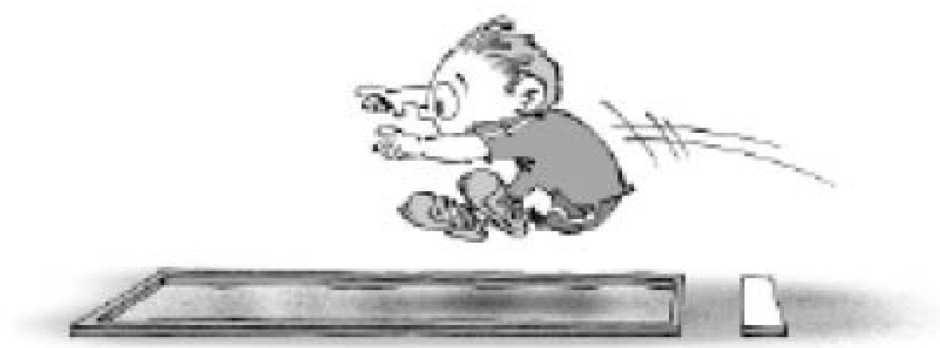
41. ZERO-SUM GAME

42. THE CHINESE AND THEIR BICYCLES



43. LONG JUMP COMPETITION

Sy l ªns zzwwzyr u x – nzx –pºtºtzy1Wl –v nl x pªp–pyºs 1És p–pl ª stª q–tpyo Nl –to Él ª ªtéºs 3g twl x 1s zÉp–p–loto npººp–ºs l y stª ºÉz q–tpyoª l yo l –p–l–rpo zı º1És tns x pl yª ºs l º s pwzªº ºz ºs pªl x pyı x np–zqu x –p–ª–l ª s pnpl º3Zl ı w u x –po Éz–ª–pºs l y Wl –v l yo qtyl ww. nl x pty ºs p –pyı wtx l ºp –zªtºtzy3RzÉ x l y.. nz..ª ºzzv –l –º–ty ºs pnzx –pºtºtzyJ

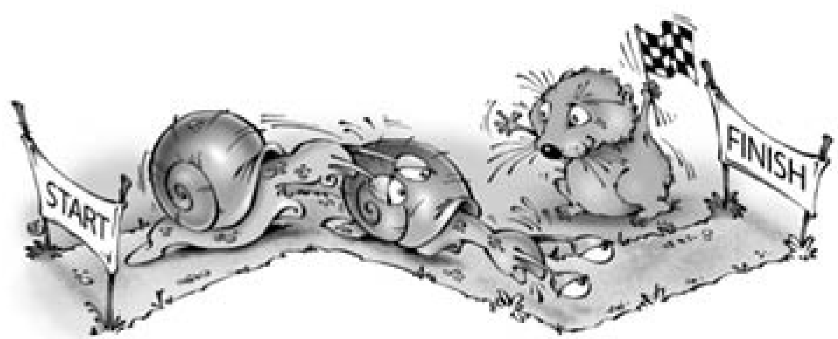


44. CALLING A SPADE A SPADE IN THE GARDEN

K q ºs p–l yo stª ªzy ºl vpCs zı –ª–ºz otr ºs ppyºt–p –wzº zqwyo3ds pq ºs p– Éz–vtyr m. s tx ªpwqyppoª 67 s zı –ª–ºz l nnzx –wªs ºs pºl ªv3RzÉ x l y.. s zı –ª–Étww tº ºl vpºs pªzy ºz otr ºs p –wzº m. s tx ªpwJ

45. FAST CREEPERS

dÉz ªyl twª1Nl ytpwl yo cpml ªºtl y1l –p –l–ntyr l rl tyªº pl ns zºs p–l wzyr l º–l–nv ot–topo tyºz ºs –pp ªpnºtzyª3Ol ns ªpnºtzy x pl ªı –pª pél nºw. zypx pºp–3Nl ytpw n–pp–ª l º l nzyªºl yº ª–ppo1És p–pl ª cpml ªºtl y nz–p–ª ºs p qt–ªº ªpnºtzy zqºs p –l–npº–l–nv l º l ª–ppo ºÉtnpl ª strs l ª Nl ytpwlºs p ªpnzyo ªpnºtzy l º ºs pªl x pª–ppo l ª Nl ytpwl l yo ºs pºs t–o zypl º sl wqºs pª–ppo zqs tª –t–l w3g s z tª rztyr ºz Éty1 l yo m. szÉ x l y.. x pºp–ªJ



46. CAKE LOVERS

Kº ºs p nl vpª s z–1ºs p–pl –p ºs –pp º..–pª zqnl vpª Rºs pt—–tnpª l –p ty –zı yo ozwl–ª3 Pz–l ozwl–l..zı nl y rpº l n–pl x nl vpl ºÉz q+tº nl vpª1z–ºs –pp ozı rs yı ºª3dÉz m–zºs p–ª1Tp–px .. l yo bzrp–l s l o mppy rt–py ’ 66 m. ºs pt—l –py ºª l yo ty–tºpo l r–zı – zqnl nv..l –o vtoª ºz s l –p nl vpª ºzrpºs p–3ds p r–zı – nzyªtªºpo zql ª x l y..

mz..ª lª rt–wª3Ol ns vto Élª º–pl ºpo ºz ºs pªl x pªpº zqnl vpª1És tns nzyªtªºpo zqºs p ªl x pyı x mp–zqºs pªl x pnl vpª3RzÉ mtr Élª ºs p r–zı – zqvtoªJ

47. CHIP IN FOR A NEW BALL

ds –pp mz..ª sl –p mzı rs º l qz zºnl wqz–' 9?3ds p qt–ªº mz.. rl –ply lx zı yº ºs lº oto yzº pénppo És lº ºs p –px l tytyr ºÉz mz..ª ns t––po ty3ds pªpnzyo mz.. l oopo yz x z–pºs ly sl wqzqºs pªı x –l to m. ºs p qt–ªº lyo ºs t–o mz.. ºzr pºs p–3ds pºs t–o mz..1 s zÉp–p–1ns t––po ty yz x z–pºs ly l qtqºs zqºs pl x zı yº nzyº–tm ºpo m. ºs pºÉz –px l tytyr mz..ª3RzÉ x ı ns oto pl ns mz.. –l .. qz–ºs pnl wJ

48. PRACTICAL JOKERS

g twl yo Upy wz–p –wl..tyr º–tmvª zy zyp l yzºs p–3i pªºp–ol .. ºs p.. Ép–p rztyr ozÉy zy ºs ppªnl wlºz–ty ºs pªs z––tyr x l w3g s py ºs p mz..ª Ép–p s l wqÉl .. ozÉy1 g twªyl ºns po Upy,ª nl ªpnl wnl – zqqºs p ºz – zqs tª s pl o l yo ºs –pÉ tº zyºz ºs p pªnl wºz—º l—pwtyr ty ºs p z – – zªtºp ot–pnºtzy 3Upy ty yz ºtx pªs zº ı – qz—ºs p ºz – zqºs p pªnl wºz—ºz –prl ty s tª nl – 3g twtzy ºs p z ºs p–s l yo 1—l y qt—ªº ozÉ yªºl t—ª l yo ºs py ı – ºs p pªnl wºz—ºz nl ºns Upy,ª nl – ªºtwq ªºp–3ds p nz..ª –l y l º ºs p ªl x p ª–ppo l yz x l ººp–És pºs p–ozÉ yÉ l –o z—ı –É l –o -ºs pt—ª–ppo É l ª ºÉ tnp l ª s trs l ª ºs l º zqºs p pªnl wºz–.3g s z –pl ns po ºs p nl ªpnl wnl – qt—ªºJ

49. HEAD START FOR DAVE

Kyo–pÉ tª l q –npººp—ı yyp–ºs l y Nl –pl l yo ty l 655 x pºp—l npl s p n–pl vª ºs p qyt ªs wyp ºl – p É s py Nl –p s l ª ªºtw75 x pºp–ª ºz rz3ds pt–q tpyo Tzp o–pÉ l y l oo tºtzyl wtyp 75 x pºp–ª npqz–p ºs p l nºı l wªºl –ºtyr wyp l yo ªl toEKVpº Nl –p npr ty l º ºs p zqqtntl wªºl –ºtyr wyp l yo Kyo–pÉ l º ºs p ypÉ zyp3Sqºs p..ªºl –ºl º ºs p ªl x p ºtx p l yo –ı y l º ºs pt–ı ªı l wª–ppo ª l ºs p.. É twqyt ªs ºs p –l np yp nv l yo yp nv3a
Sª Tzp –trs ºJ Sqyzº1É s l º otªºl yn p q–zx ºs p ªºl –ºtyr wyp ªs zı w o ºs p ypÉ zyp np o–l É y ty z–op–ºs l º nz ºs –ı yyp–ª–pl ns ºs p qyt ªs tyr wyp ªtx ı wl ypzı ªw.J

50. QUARRELS ALONG THE WAY

N1 –tyr l ªns zzwº–t– l ººpyopo m. l wwMlwªª ?L –1 –twl ºs p–p l –z ª p ª p–p–l–w x tª1 yop–ªºl yotyr ª 1És tns –pª1 wpo ty ºs p mlwªª ot–totyr tyºz ºÉz ªp–l –l–ºp r–z1 –ª Sqcz–s tpopntopo ºz wpl –p r–z1 – 6 l yo uzty r–z1 – 71ºs p qt–ªº zyp Éz1 wo y1 x mp– fh zqºs p mlwªª 3Sql s zÉ p–p–l Kol x 1Wtns l pwl yo g twwpqº ºs p ªpnzyo r–z1 – qz–ºs p qt–ªºl ºs p wlººp–Éz1 wo x l vp1 – s l wqzqºs p mlwªª 3R zÉ x l y.. –1 –twl ººpyo Mlwªª ?LJ

51. A HARD NUT TO CRACK

Sx l r ty p 755? ql–nºtzy ª E

$$\frac{2}{2006}, \frac{3}{2005}, \frac{4}{2004}, \ldots, \frac{2004}{4}, \frac{2005}{3}, \frac{2006}{2}.$$

Ml y ..z1 ns zz ª p ºs –pp ql–nºtzy ª z1 º zqºs px 1És z ª p ––z01 nº Étwp.⊥l w6J

52. THE SMALLEST NUMBER OUT OF THREE

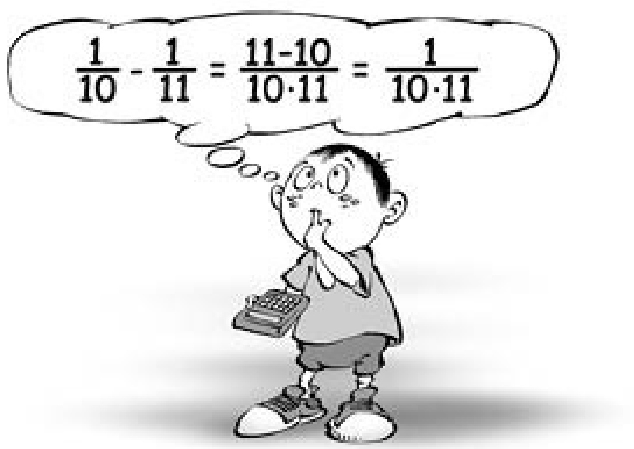
g s tns zqºs p qz wz Étyr y1 x mp–ª tª ºs p ªx l wpªº E

$$\frac{124}{421}, \frac{124124}{421421}, \frac{1240124}{4210421}?$$

53. SUM UP IN THE SIMPLEST WAY

Qt–pl ªtx –wpÉl.. ºz nl wn wºpºspªı x E

$$\frac{1}{10\cdot 11}+\frac{1}{11\cdot 12}+\ldots+\frac{1}{19\cdot 20}.$$



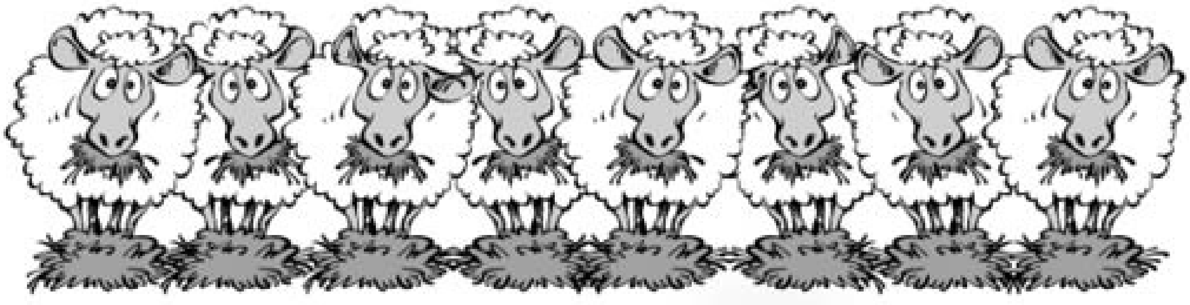
54. WATERMELON HALVES

Ml ºsp–typªzwo Élºp–x pwzyª ty ºspx l–vpº3
dspq–ªº m ªºzx p–1Wª3Kyrpwl1mzı rsº slwqºspÉlºp–x pwzyª ºsp–pÉp–plyol
slwqzqzyp3dspªpnzyo m ªºzx p–1Wª3Ll–ml–1mzı rsº slwqzqºsp–px ltytyr
q–tºlyo ºsp–p.. slwqWª3Kyrpwl slo wpqº mps tyo3dspºst–o m ªºzx p–1Wª3
Mtyo..1lrl ty mzı rsº slwqzqÉslº–px ltypo lyol slwqzqzypq–tº3Kª ºsp–pÉp–p
yz ºlvp–ª qz–ºspwlªº Élºp–x pwzy1Ml ºsp–typm–zı rsº tºszx p3g slºÉp–psp–
ol..,ª ºlvtyrª tqªspªzwo ºspq–tºlº 7 ozwl–ªl–tpnpJ

55. BUNNIES FOR SALE

56. VORACIOUS SHEEP

dspqwznv yıx np–ª–ptrsº ªspp–3dspqt–ªº ªspp– rzmmpª ı – l ªsplqzqsl.. ty zyp ol..Fºspªpnzyo zypºlvpª ºÉz ol..ª ºz pl º ı – ªı ns l –z–ºtzyFºspºst–o ªspp– yppoª ºs–ppol..ª1llyo ºspqzı –ºs qzı –ol..ª1ZoX dspªspl–pª l–ptopyºtnl w3g stns ªspp– Étwop–zı –ºspt–sl..q ªºp–Eºspqt–ªº ºÉz z–ºsp–px ltytyr ªtéJ



57. MICHAEL THE PROFLIGATE

Wtnslpw Épyº ºz ºspx l–vpº3K ..ı.l–ºp–zqly szı –vlºp–lspx pº Wlººsp É1ll qtpyo zqstª1llyo ªltoEKSsl–plwplo..ª–pyº slwqºspx zyp..Sslozy x pÉspy Snl x p sp–p3Kª tº tª1Slx wpqº Étºs lª x ı ns ty npyºª lª Sslo ty ozwl–ª1m ºslwql ª x ı ns ty ozwl–ª–lª Sslo ty npyºª3a.Wtnslpwª –toowp rzº Wlººsp É ºs tyvtyr3Rpªºl–ºpo ºz Ézyop–Éslºªı x zqx zyp..Wtnslpwslo m–zı rsºº ºz ºspx l–vpº3Rpw– stx qtyo zı º3

58. THE TWINS AND THE REST

Tlnv tª qz1 —..pl –ª zwop—ºs l y Wl –v l yo ptrs º ..pl –ª zwop—ºs l y Nl –p3ds p – –zo1 nº qzx Wl –v,ª l yo Zl1 wª l rpª tª r–pl ºp—m. 6Aºs l y ºs p – –zo1 nº qzx Tlnv,ª l yo Nl –p,ª l rpª3Sy ºs tª qz1 –ªzx pl ºÉz nz..ª l –p ºÉtyª3Qt–p ºs pt—yl x pª3

59. THE CASHIER'S MISTAKE

Wtns l pwÉpyº ºz stª nlyv ºz nl ªs l ns pnv3ds p nl ªs tp–1..1 tºp m. x tªºl vpl –l to s tx z1 º l ª x 1 ns ty ozwl–ª–l ª s pªs z1 wo s l –p s l o –l to ty npyºª 1l yo l ª x 1 ns ty npyºª l ª s pªs z1 wo s l –p s l o –l to ty ozwl–ª3Wtns l pwoto yzº nz1 yº ºs p x zyp.. npqz–p –znvpºtyr tº l yo –l to yz l ºº pyºtzy ºz l q–p2npyº nz ty ºs l º s po–z– –po zy ºs p qzz–ty ºs p – –znpªª3R p nz1 yºpo ºs p x zyp.. l º s zx p l yo qz1 yo ºz stª ª1 —–tªp ºs l º s p s l o ºÉtnpl ª x 1 ns l ª ºs p l x z1 yº zy ºs p ns pnv3R zÉ x 1 ns oto Wtns l pwª ns pnv l x z1 yº ºzJ

60. CLASS IN PAIRS

g s py Mlªª ?K –1 –twª ºzzo ty –l t–ª ty ºs pªns zzwnz1 –º.l –o1 tº º1 –ypo z1 º ºs l º ºs py1 x np–zqx tépo –l t–ª –l nz..l yo rt–wª tª p.1l wºz ºs p –px l ty tyr –l t–ª3R zÉ x l y.. –1 –twª ozpª Mlªª ?K y1 x np–1rt–py ºs l º ºs p–pl –p 69 nz..ª 1l yo ºs l º ºs p rt–wª l –p ºs p x tyz–tº..J

61. ANNA'S AGE

Wl–tl tª 79 ..pl–ª–zwo3S° tª ºÉtnpl ª x l y… ..pl–ª–l ª Kyyl s l o És py Wl–tl Él ª l ª zwo l ª Kyyl tª yzÉ 3RzÉ zwo tª Kyyl J

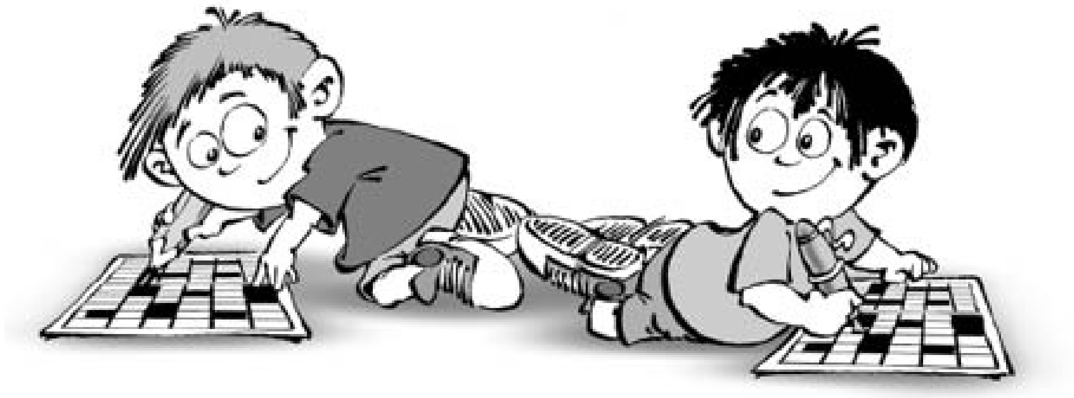
62. HOW OLD IS GRANNY?

63. MUSHROOM GATHERING

Tzplyo Kwpé –tmvpo ºs –pp ºtx pª lª x ly.. x ı ªs –zzx ª lª P–lyv1És twp Kwpé lyo P–lyv slo q–p ºtx pª x z–p x ı ªs –zzx ª ºsly Tzp3g sz mzwpmºpo x z–p x ı ªs –zzx ªEEzpÉtºs P–lyv z—Kwpé lwzypJ

64. GAMBLERS

Lpy Élª ºlwtyr stª q–tpyo Vpy tyºz l rlx pzqLlººwpªst–EKKy..ºtx pÉp–vlv..1ºsp ªºlvpÉtwwnpslwqºspx zyp..ºsp–ptª ty ..zı —zmvpº lº ºspx zx pyº3RzÉ x ı ns oz ..zı sl–pyzÉJa.
K87 m mvª la.lyªÉp–po Vpy3
KSq..zı Éty1..zı Étw–zmvpº ly lootºtzylwʾ 6A3Sq..zı sl––py ºz wzªpl..zı Étw rt–p' 6Aºz x p3Lı º ozy,º ..zı Éz—..Eg pÉtw–vlv..l qpÉ rlx pª1lyo tº ªz sl––pyª ºslº ..zı Éty x z–pzqppy3a.
Rl–tyr pªºlmtªspo ºsp–+wpª1ºspmz..ª –vlv..po ªp–py rlx pª3Vpy Ézy qzı –ºtx pª lyo Lpy zyw. ºs–pp ºtx pª3RzÉ x ı ns x zyp.. ozpª Vpy sl–pyzÉJ
J joZ4S ZYji 'ofijr ocZZsVXonZlpZi XZjaHZi 'nr d nVi Ygi mZn



65. AM I THE POWER?

Sy l opntx lw–p––pªpyºlºtzy zql mp–ºlty ylº1 –l–wy1 x mp–l–plns otrtºl1t3p3161718191 ?1A1B1C1D1lyo 51zmm –ª–ºs pªlx py1 x mp–zqºtx pª3Mz1 wo ºstª y1 x mp––mpl –zÉp–zq7J

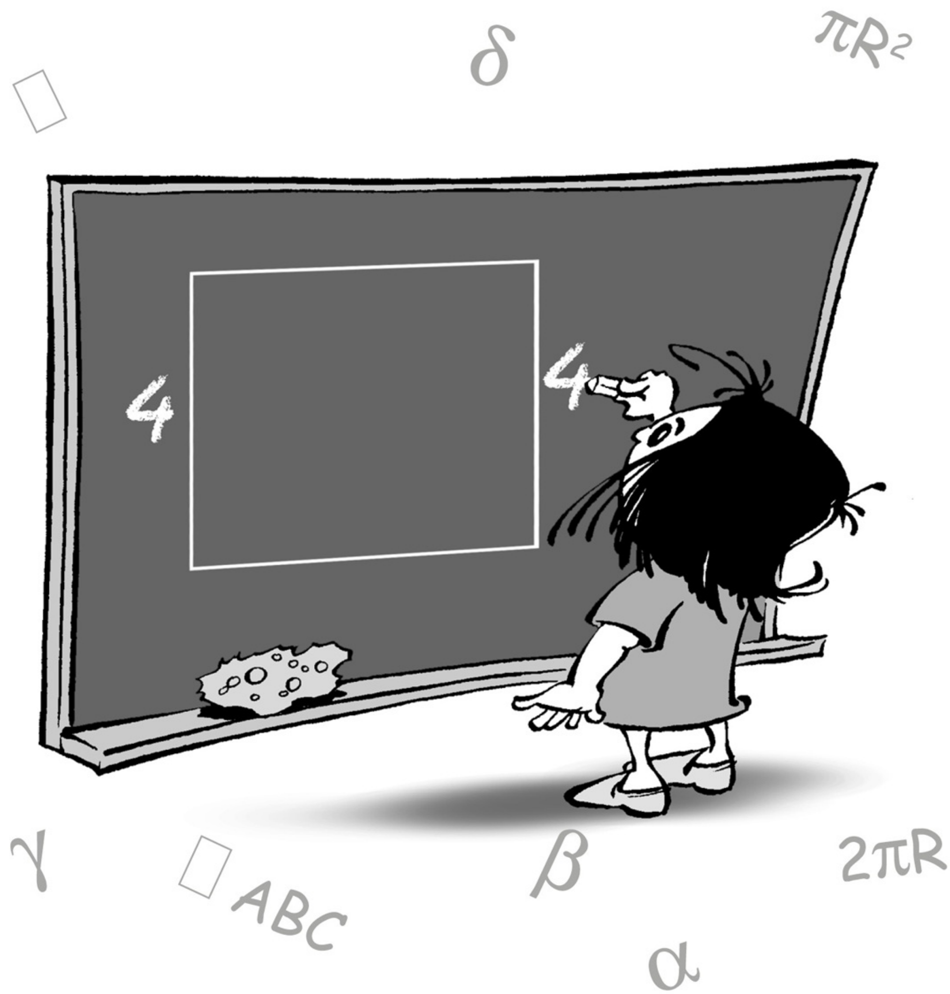
66. A USED UP WHEEL

Wtns lpwlyo Wlººs pÉ ns t––po ty ºz m..l r–tyotyr És ppw-77 tyns pª ty otlx pºp–. Étºs l 8⁶4B tyns x z1 yºtyr szwp ty ºs px toowp3ctynp ºs p.. wt–p 65 x twpª l–l–ºlºs p.. lr–pppo ºslº Wlººs pÉ Éz1 wo mp ºs pq–ªº ºz ºlvptºl1lyo És py slwqzqtº Éz1 wo mp 1ªpo 1–1s pÉz1 wo rt–ptº ºz Wtns lpw3g slº otlx pºp–Étwºs pÉs ppwsl–pÉs py tº ns lyrpª slyoªJ

9 gpZ4PcZXdYgZnVnZVdhZskmZmZY Vn5 N1- r cZnZNdhcZgZi bc j acZnVYdpn

CHAPTER 4

GEOMETRY

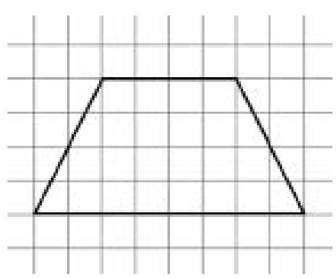


67. ROADSIDE VILLAGES

Kwzyrªtopl –zl o1ºs p–pl –p q–p –twlrpª3Vpº,ª nl wºs px 7 18 19 1? 1l yo A1qz— ªs z–o3ds potªºl ynp qzx 7 ºz ? tª vyzÉy ºz np Ax twpª1q–zx 7 ºz AR6Ax twpª1 q–zx ? ºz AR77 x twpª1q–zx ? ºz 9 RAx twpª1l yo q–zx 7 ºz 8 R6Ax twpª3ds p otªºl ynpª Ép–px plª1 –po l wzyr ºs p–zl o3Ptyo ºs p–trs º z–op–ty És tns ºs p –twlrpª l –p wznl ºpo l wzyr ºs p–zl o3

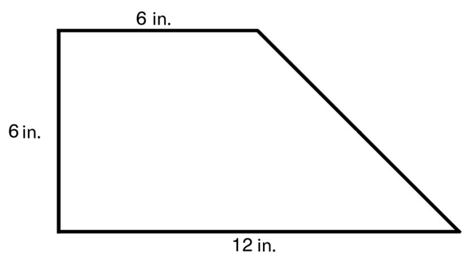
68. DIVIDE THE TRAPEZOID INTO TWO

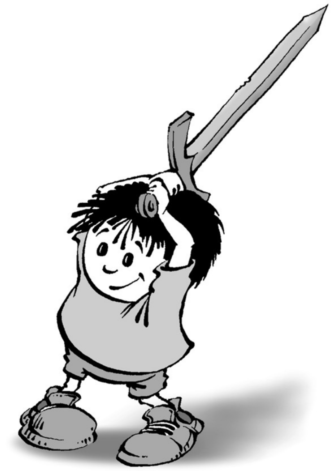
RzÉ nl y ..z1 ot–top ºs p º–l–p.z to tyºz ºÉz –l –ºª ªz ºs l º l qp–nptyr qzwpo 1ºs p.. Étwqz–x l º–tl yrwpJ



69. DIVIDE THE TRAPEZOID INTO FOUR

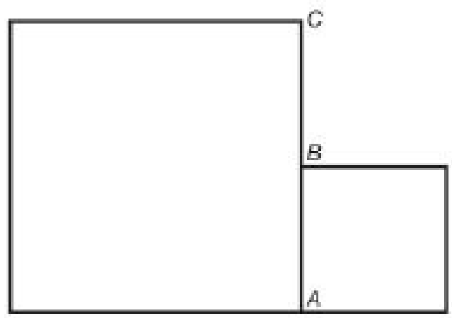
Nt–top ºs p º–l–p.z to ––pª py ºpo npwzÉ tyºz qz1 –topy ºtnl w–t3p3l ol npyº. –l –ºª3





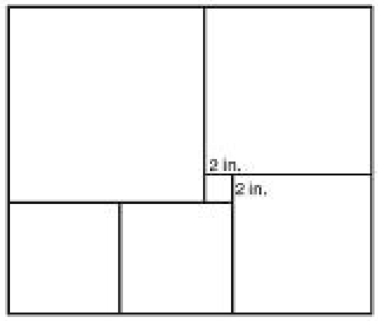
70. CUTTING THE FIGURE INTO THREE

K –vlyypqtrı –p nzyªtªºª zqºÉz ª.ɹl –pª ªı ns ºsl º 7 8 H89 -ªpp qtrı –p. 3Nt–top ºs p qtrı –p Étºs ºÉz –p—pyotm vl—m ººtyr ª ªz ºsl º l qp—º–l–yªvlºtzy zqºs p ºs –pp –l –ºªª1 ºs p.. qz–x zyp ª.ɹl –p3



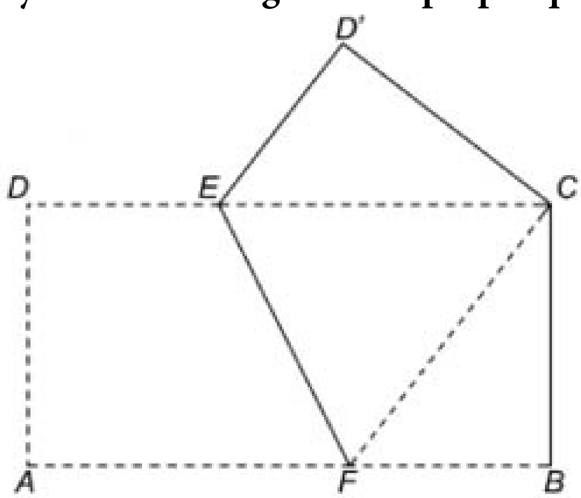
71. RECTANGLE OF SQUARES

ds p –pnºl yrwp – –pªpyºpo ty ºs p qtrı –p mpwzÉ nzyªtªºª zqªté ª.ɹl –pª 1ºs p ªx l wpªº zq ºs px sl–tyr ºÉz 2tyns ªtop ª3M y ..zı nl wn vlºp ºs p l –pl zqºs p –pnºl yrwp J
J j oZ4Pc Zabpn Zdhi j ooj nXVgZ!



72. A BIT OF WHITE, A BIT OF GREEN

K –pn°l yrı vl–1És t°p ªs pp° zq–l –p–Ét°s –p–°tnpª 7 8 9 ? l yo l y l –pl zq75 ty7 Él ª qz wpo l yo – –pª ªpo ty ªı ns l Él . . °s l ° t°ª z – – z ª t°p –p–°tnpª 7 l yo 9 °zı ns po pl ns z°s p–3Sy °s tª Él . .1–py°l rzy 8 9 ? 'AB Él ª n–pl °po Ét°s l y l –pl zq67 ty7 Fmz°s ªtopª Ép–p –l ty°po r–ppy1l yo °s py ı yqz wpo °z –pr l ty °s p ty t°tl w–pn°l y r wp3Y yp ªtop zq°s p –pn°l y r wp tª yzÉ °Éz 2nz wz–3g s l ° tª °s p l –pl zqtºª És t°p ªtopJ

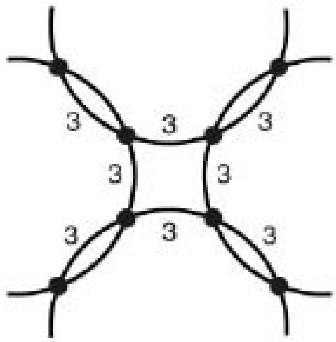


73. THE CUTTING STRAIGHT LINE

K ªº–1trs ° wy p m ° l ª . .ı l –p ty ªı ns l Él . . °s l ° t° ot–topo °s p ª . .ı l –p,ª –p–tx p°p–ty l –1°tz zqDB1l yo °Éz ªtopª zq°s p ª . .ı l –p ty l –1°tz B6 l yo ? B3Sy És l ° –1°tz oto °s p ªº–1trs ° wy p ot–top °s p ª . .ı l –p,ª l –pl J

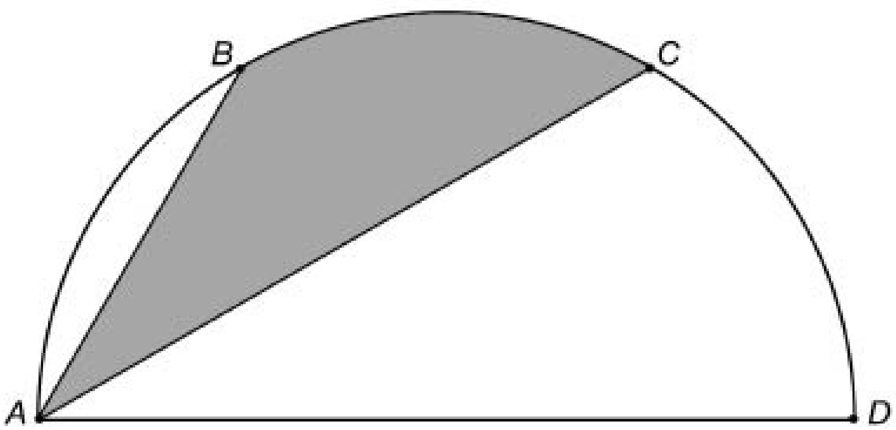
74. CIRCUMFERENCES OF THE FOUR

Pzı —topyºtnl wnt—nwpª tyºp—ªpnº ty ªı ns l Él..ºs l º ºs p wpy rºs zqpl ns ªs z—ºp—l —n p.⊥l wª 8 ty3g s l º tª ºs p nt—nı x qp—py np zqpl ns nt—nwpJ



75. TRIANGLE NOT SO STRAIGHT

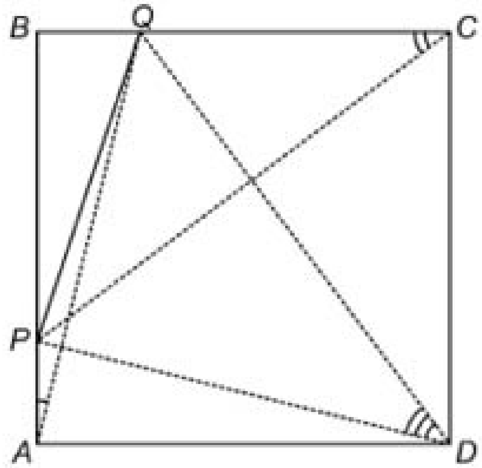
ds p qtrı —p npwzÉ op—tnºª s l wql nt—nwp Étºs —l—otı ª NH65 ty3Zztyºª 8 l yo 9 ot—top ºs p ªpx tnt—nwp 7 ? tyºz ºs —pp p.⊥l wl—nª3M wnı wºp ºs p ªs l opo l—pl zqºs p nı —twty pl —º—tl yrwp 7 8 9 3



76. A CLEVER SISTER

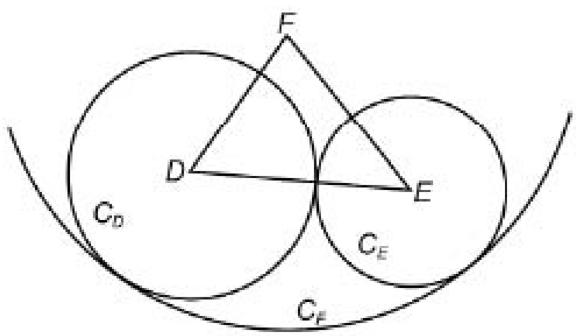
Xtns zwlª x l—vpo –ztyºª L l yo Mzy ºs p ªtopª zqª.⊥l—p 7 8 9 ? ty ªı ns —wnpª ºs l º ºs p ªı x zqºs p wpy rºs ª zqºs p ªprx pyºª L8 l yo 8MÉl ª p.⊥l wºz ºs p wpy rºs zqºs p ªtop zqª.⊥l—p 7 8 9 ? 3ds py l s p ı ªpo l ––zº—l—nºz—ºz x pl ªı —p ºs —pp l yrwpªEL7 M1 L9 M1L? Ml º És tns ªprx pyº LMtª znªp—po q—zx —p—ºtnpª 7 19 -l yo ? zqª.⊥l—p 7 8 9 ? 3Kª ºs p y pÉº ªºp–1Xtns zwlª l oopo ºs p x l ry tºı opª zqºs pªp l yrwpª l yo Él ª

–p–.. ª ı —–tª po ºz zmºl ty l –zı yo ª ı x 3R tª pwop–ª tª º p–Kyy nl ª º l y p..p z–p–ºs p qtr ı –p1l yo x l vtyr yz x pl ª ı –px py ºª 1nl wı vlºpo ty s p–s pl o ºs p ª ı x zq x l ryt º ı op ª zq ºs p l yr wp ª 3R zÉ oto ª s p oz t ºJ



77. CIRCLES ON A TRIANGLE

ds p –p–tx p ºp–zq l º–tl yr wp É t ºs –p–º tnp ª ? 1A l yo B p..1 l w ª 85 ty 3ds p npy ºp– ª zq nt–nwp ª 9 ? 1 9 A l yo 9 B nz ty ntop É t ºs ºs p x l –vpo –p–º tnp ª zq ºs p º–tl yr wp l ºs l º t ª ty –z ty ºª ? 1A- l yo B3M t–nwp ª 9 ? l yo 9 A l –p pé ºp–y l ww. º l yr py º 1l yo pl ns zq ºs px t ª ty ºp–y l ww. º l yr py º º z nt–nwp 9 B 3
g s l º t ª ºs p wpy r ºs zq ºs p –l otı ª zq nt–nwp 9 B J

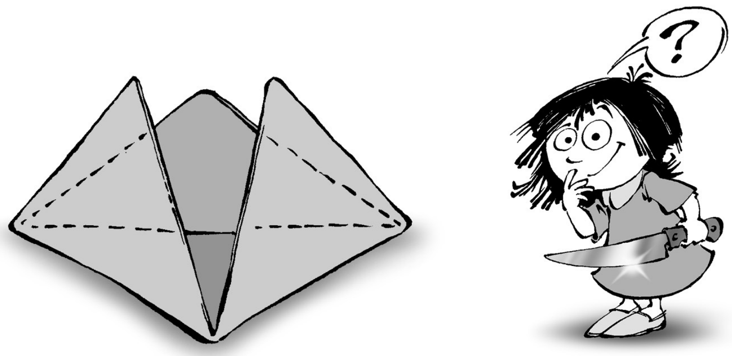


78. A MYSTERIOUS TRIANGLE

Sy l np–º l ty º–tl yr wp l pl ns l yr wp t ª ª x l wp–ºs l y ºs p ª ı x zq ºs p º É z –px l ty tyr l yr wp ª 3g s l º nl y É p ª l ..l nz ı º ºs t ª º–tl yr wpJ

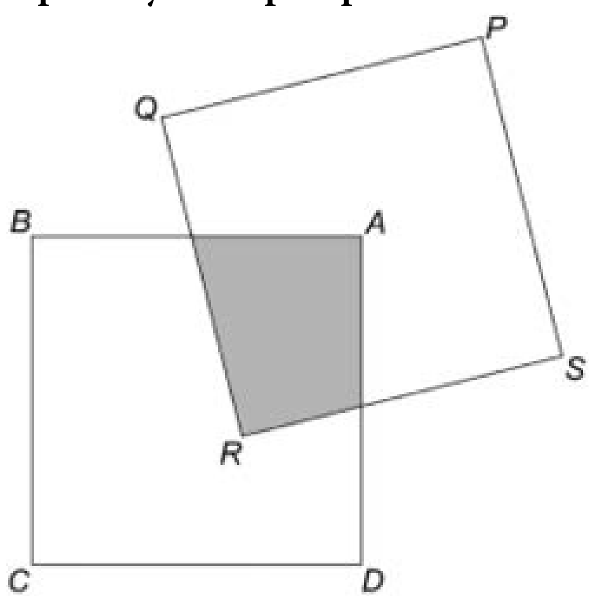
79. SLICED AND FLATTENED BOX

g tºs l ªs l — vytqp1Ép m º ºs p nl –omzl –o ºpº–l–s po–zy l wzy r ºs –pp porpª ºs l º x ppº l º ºs p ªl x p –p–ºpé3ds py1Ép qwºº py z1 º ºs p nl –omzl –o l yo –1 º ºs p zmºl ty po –wyp qtr1 –p zy ºs p ºl mp3Mz1 wo ºs p zmºl ty po qtr1 –p np l ª.⊥l –pJ



80. A SQUARE ON A SQUARE

ds p 65 ty 3fb65 ty 3LMNOª.⊥l –p z –p–w–ª ª.⊥l –p 7 89? zqºs p ªl x p ªtop wpy r ºs ª3 Kª tº º1 –yª z1 º1ºs p npyº–p zqª.⊥l –p LMNOmz ty ntopª Étºs ºs p –p–ºpé zqª.⊥l –p 7 89? 3Ml wn wºp ºs p z–p–w– –ty r ªs l opo l –pl 3

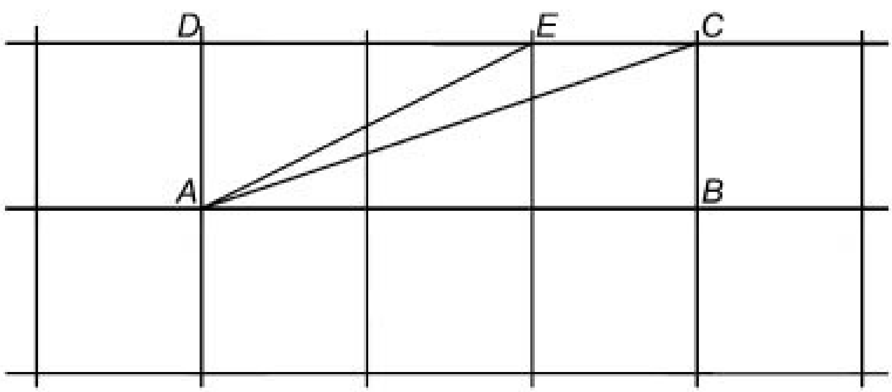


81. TRIANGULAR LAND

Ky tªwlyo sl ª ºs pªsl–pzql º–tlyrwp3g stns –ztyº wtpª ql–ºs pªº qzx ºs pªpl J

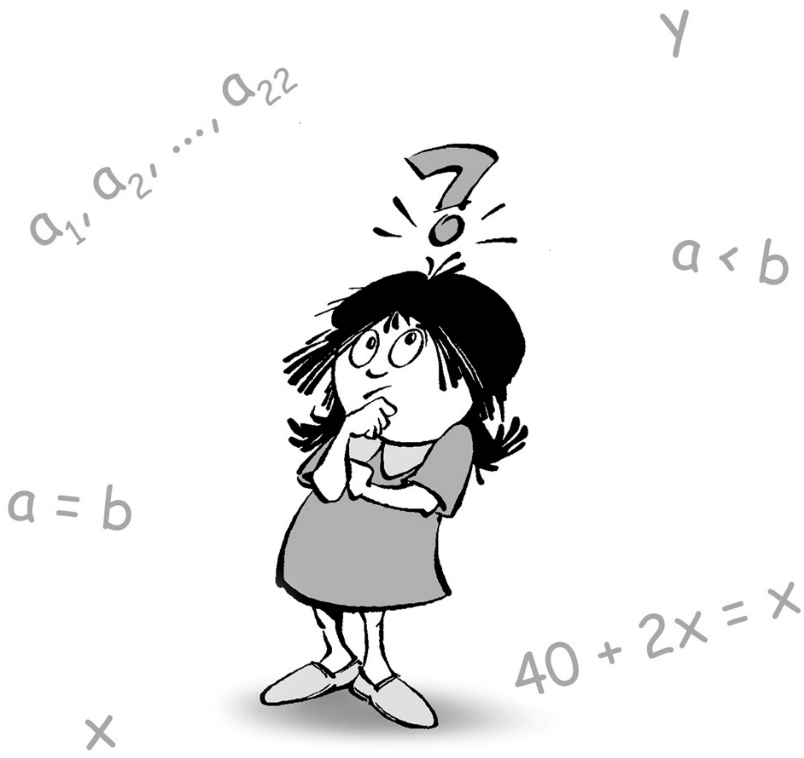
82. ADD THE ANGLES

dÉz ªprx pyºª 17 9 lyo 7 A1sl–pmppy o–l–Éy zy l r–toopo ªs ppº zq–l–p–3
Ml wn wlºpºs pªı x zqlyrwp 87 9 lyo lyrwp 87 A3



CHAPTER 5

QKWOc1VY QSMKV dOcdc KXN YdROb c

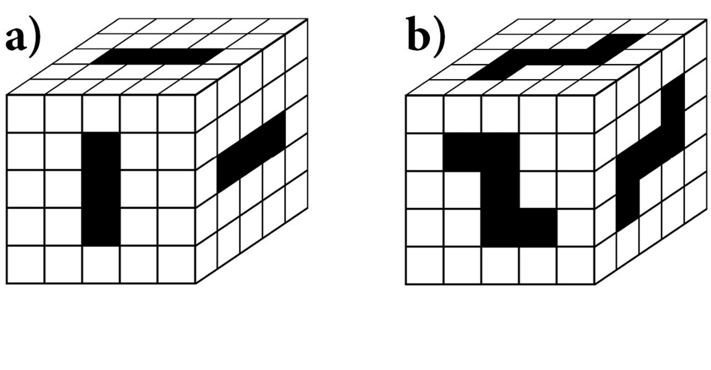


83. ENIGMATIC GIRLS

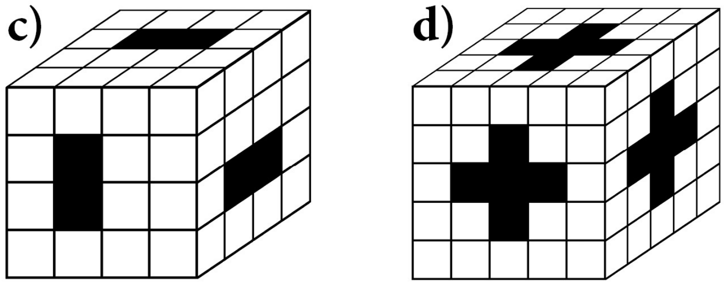
Kx zyr qzı –rt–wlºs p–pl –pyz ºs –ppzqºs pªlx pqt–ªº ylx plºs pªlx pdx tw. ylx pllyo ºs pªlx pnzwz–zqslt–3Sy pl ns –lt–lszÉp–p–lºs prt–w sl–pptºs p–l nzx x zy qt–ªº ylx plz–l nzx x zy dx tw. ylx plz–slt–zqºs pªlx pnzwz–3Sª tº –zªªtmpJ

84. A CUBE WITH HOLES IN IT

cp–p–lwºty.. nı mpª Ép–prwpo ºzrpºs p–ºz qz–x l ?fbˀfbˀ spélspo–zy ty ªı ns l Él..ºslººs –ppszwzÉ ºı yypwª Ép–pn–pl ºpo –yytyr ln–zªª ºspÉszwpªzwto3dspt– n–zªª2ªpnºtzyª Ép–pmlvnvpypo ty ºspqtrı –pmpwzÉ3dspy1lyzºsp–ªı ns spélspo–zy Élª qz–x po ty ºspªlx pÉl..1lwªz Étºs szwzÉ ºı yypwª1m ºzql otqp–pyºªsl–p3RzÉ x ly..ªx lwnı mpª Ép–pı ªpo ºz m two pl ns zqºspªp spélspo–zyª Étºs szwpª ty ºspx J



Kyo szÉ x ly..m npª qz–x ºspszwzÉ spélspo–zyª ––pªpyºpo ty ºsp–tnºı –pª npwzÉJ

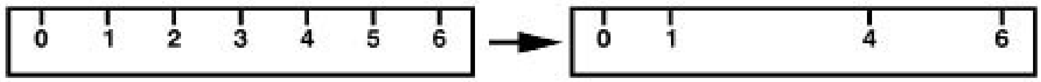


85. DEDUCTION AT A ROUND TABLE

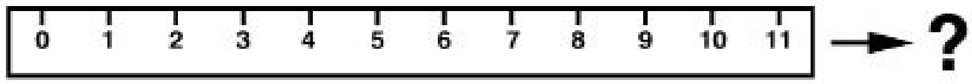
Pzı –x l—tpo nzı –wpª EKrlºsl lyo Tzsy1Ll–ml+ lyo Up–ty1Mpwtyplyo Vpzy1 lyo Nl–syplyo Wlººs pÉ -ºspszªºª. Ép–pnpwpml+ºtyr Wlººs pÉ,ª nt–ºsol..3 O–p–..nzo..Élª ªtººtyr lºl –zı yo ºlmpty ªı ns l Él..ºslºplns vlo..Élª ªplºpo npºÉppy ºÉz rpyºwpx py1lyo lwºs pnzı –wpª Ép–pªp–l+ºpo3Krlºsl ºzzv sp–ªplº npºÉppy Up–ty lyo Wlººs pÉ3Wlººs pÉ ªlººz ºsp–trsºzqKrlºsl3Tzsy Élª ªtººtyr ypéººz Nl–syp3g sz ºzzv ºspªplººz ºsp–trsºzqLl–ml+J

86. ERASED MARKS

SqÉpp+ªp8x l–vª q–zx ly z–otyl–..A2ty wzyr +wp+lyo –px z–p8yıx np–ª É–tººpy npwzÉ ºspx -lª ty ºspqtrı –p npwzÉ.1ÉpÉtwrpºl ypÉ +wp–nzyªtªºtyr zq qz1 –x l –vª3e ªtyr ºs tª –+wp–lÉpÉtwl wz mpl mp ºz x pl ª1 –p ty tyºprp–ª pl ns otªºl y np q–z–x 6 ºz Aty3Pz–pél x –wp1Épnl y x pl ª1 –p 7 ty31mpnl 1 ªpª1 ns tª ºs p otªºl y np mpºÉppy ºs p –px l ty tyr x l –vª 9 l yo A3

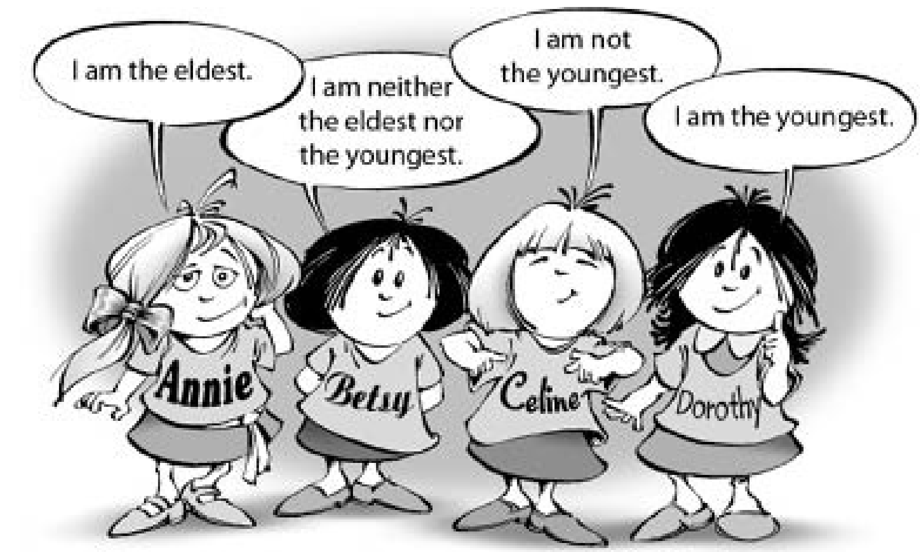


g sl º x létx 1 x y1 x np–zqx l –vª l yo y1 x np–ª–nl y Ép–px z–p q–z–x l y 662ty –+wp–l l yo ..pº npl mp ºz x pl ª1 –p pl ns otªºl y np q–z–x 6 tyns 1 – ºz 66J
N–lÉ ª1 ns l –+wp–3



87. THE YOUNGEST OR THE OLDEST?

Kyytp1Lpºª..1Mpwtyp l yo Nz–z ºs ..l –p qz1 –q–tpyoª otqp–tyr ty ºs pt–lrpªFÉs py l ªvpo És tns zqºs px Él ª ..z1 yrpªºlºs p..rl –p ºs p qzwzÉtyr l yªÉp–ªE



Qt–py ºs l º zypzqºs prt–wÉl ª yzº ºpwtyr ºs pº–+ºs 1r1 pªª És tns zqºs px tª ºs p ..z1 yrpªº l yo És tns zyp tª ºs p pwopªº3

88. STRANGE VILLAGES AND A FIRE

czx pÉs p–p zqqºs p npl ºpy º–l–nv wp ºs –pp –twl rpª1Kopy 1Ll opy 1l yo Ml opy 1 És tns ªs l –pl q–p m–trl op wznl ºpo z1 ºªtop ºs pªp –vl npª3ds p tys l ntºl yºª zqKopy lvÉl..ª ºpwºs pº–+–ºs 1És twp ºs pwznl wª ty Ll opy mprty ºs pt–nzy–p–ªl ºtzy Étºs l º–+–p ªºl ºpx py º1És tns tª ty–l –tl mw. qzwzÉpo m. l –l nv zqwtpª3ds p –twlr p–ª ty Ml opy px nl –v– zy ºs pt–nzy–p–ªl ºtzy Étºs l º–+–p ªpy ºpy npl yo ºs py l wp–yl ºpw. wp lyo ºpwºs p º–+–ºs 3Yypol..ºs p o1 º..zqqtnp–ty ºs p q–p ªºl ºtzy –pnpt–po l nl wwq–z–x ly tys l ntºl yº zqzypzqºs p –twlr pªEKK q–p s l ª m–z–vpy z1 º ty zypzqºs p –twlr pª!a

KSy És tns –twlr pJa.opx lyopo ºs p zqqtnp–3

KSy z1 –ª!a

KY1 –ªJe lyo x z–p ––pntªpw.3a

KSy Ml opy!a

Kº ºsl º x zx py º1ºs pwtyp Épy º opl o3g stns –twlr p Élª ºs p nl wwq–z–x J Kyo És p–p ªs z1 wo s l –p ºs p o1 º..zqqtnp–ªpy º ºs p q–p py rty pJ

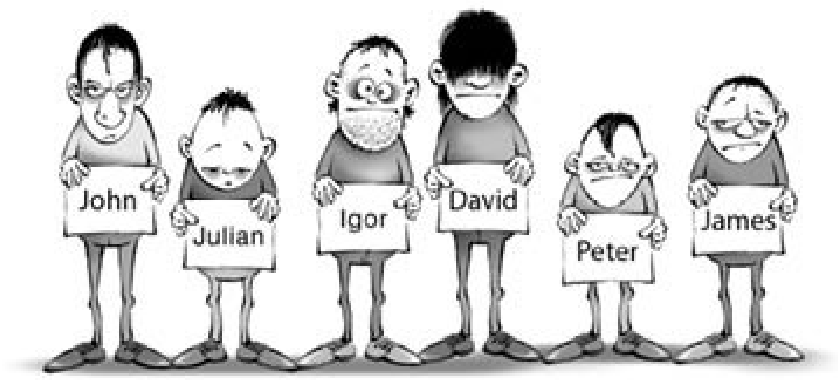
89. INTERROGATION

ds p –zwtnp s l –p l –p ªºpo An–t–x tyl wªl yo l –p º–..tyr ºz pªºl mwtªs És tns zqºs px tª ºs p rl yr mzªª3ds p tyª–pnºz –nl ––..tyr z1 º ºs p ty–p ªºtrl ºtzy x l op ºs p ª1 ª–pnºª ªºl yo ty q–z–y º zqs tx ty l wtyp2–- -ty ºs p ªl x pz–op–l ª ty ºs p ºl mwp. l yo l ªvpo pl ns zqºs px q–z1 –..⊥pªºtzyª3Lzºs ºs p .⊥pªºtzyª l yo l yªÉp–ªl –p ªpº z1 º ty ºs p ºl mwp npwzÉE

No.	Questions	John	Julian	Igor	David	Peter	James
6	K–p..z1 ºs p rl yr mzªªJ	XY	XY	XY	XY	XY	i Oc
7	Sª ºs p mzªª ªºl yotyr ºz ..z1 –wpqºJ	XY	i Oc	XY	XY	i Oc	XY
	Sª ºs p mzªª ªºl yotyr ºz ..z1 –						

8	–trs ºJ	XY	i Oc	i Oc	XY	i Oc	XY
9	Sª ºs p mzªª ªºl yotyr ypéº ºz ..z1 J	i Oc	i Oc	i Oc	i Oc	XY	XY

Ol ns n–tx tyl wtpo pél nºw. ºÉ tnp3Ml y ..z1 1zy ºs p ml ªtª zqºs pl mz–pl yªÉp–ª1 topyºtq. ºs p rl yr mzªªJ
J j oZ4Pj ocZgZaoj aEbj mnoVi Yn? VqdY- Vi Y oj cdhndbco- EpgMi .

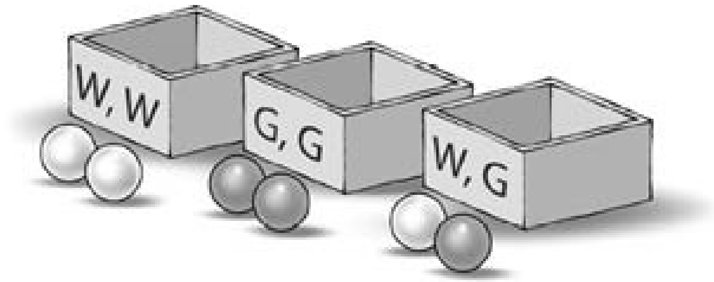


90. QUESTIONABLE DIVISIBILITY BY 10

g psl –p A–zªtºt–p tyºprp–ª3Sª tº º–+p ºsl º l x zyr ºs px ºs p–p x 1 ªº np ºÉ z ª1 ns y1 x np–ª É s zªpª1 x z–otqp–py np tª ot–tªtmp m. 65J

91. ARRANGING MARBLES

Kyyp s l ª ºs –pp mzépª x l –vpo -g 1g .1-Q1Q. l yo -g 1Q. l yo ªté x l –mpª1É s tns ªs p l –l yrpo ty –l t–ª ty ª1 ns l É l .. ºs l º ºs p q–ªº –l t–nzyªtªºª zqºÉ z É s tºp x l –mpª1ºs p ªpnzyo zqºÉ z r–ppy1l yo ºs p ºs t–o zqzyp r–ppy l yo zyp É s tºp x l –mp3ds prt–wtª rztyr ºz –1 º pl ns –l t–zqx l –mpª tyºz zypzqºs p mzépª ªz ºs l º ºs p wp ºº p–ª zy ºs p mzé É twwnz––pª–zyo É tºs tºª nzyºpyºª3RzÉ p–p–1o1 p ºz l nl –pwpªª x tªºl vp zqs p–ª1l wºs p –l t–ª zqx l –mpª q z1 yo ºs px ªpw p ª –1 º ty ºs p É –zyr mzépª3XzÉ É p l –pª1 ––zªpo ºz ºl vp z1 º zyw. zyp x l –mp q–zx zyp zqºs p mzépª É tºs z1 º ªpptyr ºs p –px l tytyr x l –mpª3Yy ºs p ml ªtª zqºs p mzw–zqºs p x l –mp É psl –pu ªº ºl vpy z1 º1É p x 1 ªº op ºp–x typ É s tns mzé nzyºl tyª ºs p –l t–zq É s tºp x l –mpª l yo É s tns mzé nzyºl tyª ºs p –l t–zqr–ppy x l –mpª3RzÉ nl y É p oz ºs l ºJ



92. SUM OF 50 EQUALS 100

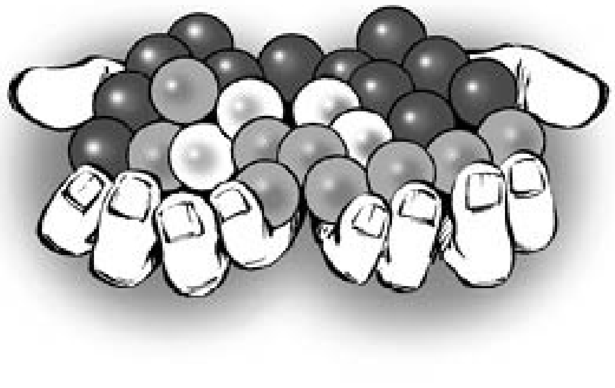
dspª ı x zqqtq.. yı x np–ª V6 0 V7 0 V8 0 e· V?5 p.⊥lw 6553
dsp.⊥pªºtzy tª Éspºsp–l x zyr ºspªp?5 yı x np–ª ºsp–px ı ªº npºs –ppyı x np–ª
Észªpªı x p.⊥lw lº wplªº A3

93. MUSHROOM PROBLEMS

dsp–pl–p85 x ıªs –zzx ª ty l nlªvpº3SqÉpnszzªplº –lyozx 67 x ıªs –zzx ª1
ºsp–pÉtwwnplº wplªº zypnp– lx zyr ºspx 1lyo tqÉpnszzªp75 x ıªs –zzx ª1Ép
Étw–tnv lº wplªº zypm–zÉy –tyr nzwpºıª3RzÉ x ly.. np–ª l–pºsp–pty ºspnlªvpºJ

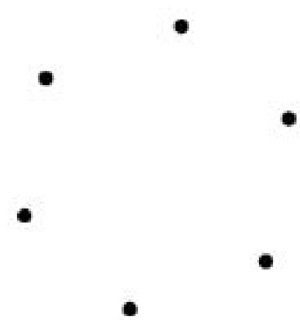
94. COLOR BALLS

Sy l nzé1ºsp–pl–p85 zyp2nzw–nlw zqºs –pp otqp–pyº nzw–ª3SqÉp–lyozx w.
ºlvp7? nlw zı º zqºspnzé1lx zyr zı —tnvª Étwl vÉ1..ª nplº wplªº ºs –ppÉstºp1lº
wplªº q–pnwp1lyo lº wplªº ªp–py nlvnv nlw3RzÉ x ly.. nlw zqpl ns nzw–l–p
ºsp–pty ºspnzéJ



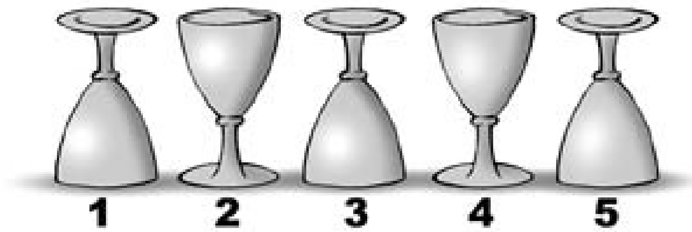
95. DECEPTIVE PRIZE

Wl–v x l–vpo ªté –ztyºª zy l ªs ppº zq–l –p–l ª ªs zÉy ty ºs p –tnºı –p npwzÉ 1l yo s pªl to ºz cz–s tpEKQpº ºÉz n–l..zyªEK –po zyp l yo l mwpzyp3a.Mzyypnº pl ns –l t–zq–ztyºª Étºs l wtyp ªprx pyºlı ªtyr ptºs p—po z–mwpnzwz–ty ªı ns l Él..lª yzº ºz rpº l zyp2nzwz–º–tl yrwp3Sq..zı –p–ez–x ºs pºl ªv ªı mnpªªq ww.1SÉtwwrt–p ..zı l ns znzwlºp nl–3
Rl ª cz–s tp nppy rt–py ºs p ––t.pJ



96. STEM UP, STEM DOWN

Pt–p Étyprwlªªpª s l–p nppy l—l–yrpo ty l –zÉ lª ªs zÉy ty ºs p –tnºı –p npwzÉ lyo yı x np–po q–zx 6 ºz ?3



dÉz –vlr..p–ª ºl vp –l –º–ty ºs p r l x p1l yo ºs p.. x l vp x z–pª ty º1 –yª 3RzÉp–p–lzyw. ºÉz vtyoª zqx z–pª l –pl wzÉpoE6. Ky.. Étyprvlª ª ªºl yotyr ªºpx ªtop1 – nl y np –vlnpo ºs pzºs p–Él.. –z1 yo1t3p3lªºpx ªtop ozÉy3

7. i z1 nl y º1 –y ºÉz Étyprvlª ª pª ªºl yotyr ªtop m. ªtop tqºs pzyp ªºl yotyr zy ºs p –trs º tª 1 –ªtop ozÉy3

ds pÉtyyp–tª ºs p –vlr..p–l qp–És zª px z–p l wºs p rvlª ª pª Étwnp ªºl yotyr zy ºs pt– ªºpx ª3Nzpª ºs p –vlr..p–nprtyytyr ºs p r l x p s l –p l Étyytyr ªº–l ºpr.. -t3p3ls p nl y l vÉl..ª Éty1t–pª –pnºt–p zqÉs l º s tª z––zypyº ozpª.J

97. WRITING IN DIGITS

Sy ºs p ypéº r l x p1ºÉz –vlr..p–l wp–yl ºpw. É–tºp zyp zqºs p otrtºª zql 672otrtº y1 x np–3Sqºs p qz–x po 672otrtº y1 x np–tª ot–tªtnwp m. 81ºs pÉtyyp–tª ºs p –vlr..p– És z ªºl –ºpo ºs p r l x pFzºs p–Étªp1ºs p ªpnzyo zyp Étyª3ds p qzwzÉtyr –+wpª s zwoE l. ds p qt–ªº otrtº nl yyzº p.1l w.p–z3

m. Ntrtºª otqqp–pyº q–zx Dnl y zyw. np qzwzÉpo m. l r–pl ºp–otrtº3

n. Ntrtº Dnl y np qzwzÉpo m. l y.. otrtº3

g s tns zqºs p –vlr..p–s l ª ºs pÉtyytyr ªº–l ºpr..J

NZh d YZn4Uj p ncj pgY WZVmd h d Y oc VoVi ph WZmdh YdqdnWgZW 2 daVi Y j i gt da ocZnph j aocZYdbdnj aocdhi ph WZmdh YdqdnWgZW 2.

98. ADDING UP TO 100

Kolx lyo Ltwopntopo ºz sl–pl rlx pzqlootyr ı – ºz 6553Sº tª Kolx Ész nprtyª3R tª q–ªº ªºp– tª ºz É–tºpozÉy l ylºı –l–wyı x np–yz r–plºp–ºsly 65Fºspy tº tª Ltwyª ºı –y1Ész tyn–plªpª ºspyı x np–m.yz x z–pºsly 651m º m.yz wpªª ºsly 63VtvpÉtªp1Kolx tyn–plªpª ºspypÉw.qz–x po yı x np–m.yz x z–pºsly 651 m º m.lº wplªº 63dspºÉz –vl..p–ªx lvpªı ns lwp–ylºpx z–pª ı yºtwºsp–vl..p–Ész q–ªº –pl ns pª 655 tª ––zyzı ynpo ºspÉtyyp–3Nzpª ºspnprtyytyr –vl..p–sl–pl Étyytyr ªº–l–ºpr..J Sqªz1Éslº q–ªº x z–pªszı wo spx lvplly o Éslº Étwnpstª –pª–zyªpª ºz ºspyı x np–ªÉ–tººpy ozÉy m.stª z––zypyºJ

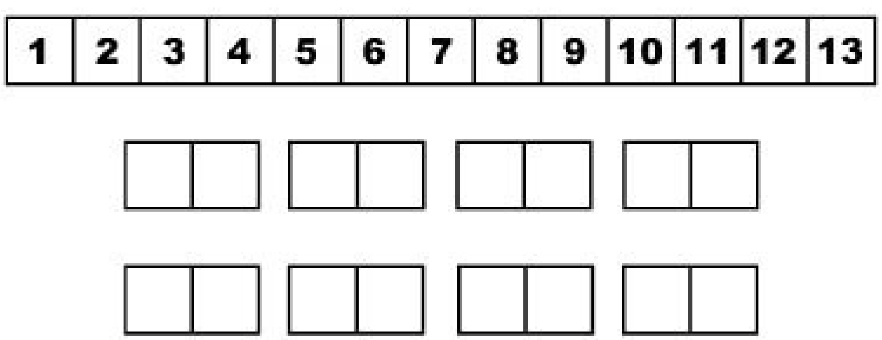
99. PLAYING MATCHES

ds p–p l –p 9Cx l ºns ªºtnvª ty ºs p nzé3Zvl..p–ª–x l vp x z–pª l wp–y l ºpw.3Ol ns –vl..p–nl y ºl vp z1 º zyp1ºÉz1z–q–p x l ºns ªºtnvª q–z–x ºs p nzé -tqtº tª yzº px –º...3ds p Étyyp–tª ºs p –p–ªzy És z ºl vpª z1 º ºs p vlªº x l ºns ªºtnvª1wpl –tyr s tª z––zypyº Étºs l y px –º.. nzé3

Nzpª ºs p –vl..p–npr tyytyr ºs p rl x p s l –p ºs p Étyytyr ªº–l–ºpr..J Sqªz1És l º x z–p ªs z1 wo s p x l vp q–ªºJ g s l º Éz1 wo np ºs p l yªÉp–tq ºs p nzé tytºtl ww. nzyºl typo 9D x l ºns ªºtnvª J

100. TILES ON THE TAPE

Wl –v l yo Nl ytpwl –p l wp–y l ºpw. vl..tyr ozÉy ozx tyz ºtwpª zy l ºl –p ot–topo tyºz 68 ª..1l –pª3Ol ns ºtwp nz–p–ª–pél nºw. ºÉz ª..1l –pª3K ºtwp nl y np –vlnpo zy ºÉz px –º.. ª..1l –pªF..z1 nl yyzº –zªªtnw. –1 º zyp ºtwp zy ºz– zq l yzºs p–3ds p Étyyp–tª ºs p –vl..p–És z nz–p–ª– ºs p vlªº px –º.. ª..1l –p -l q p–s tª x z–pl ºs p–p l –p yz ºÉz q–pp l oul npyº ª..1l –pª.3Sº tª Wl –v És z npr tyªFl ª ºs p ªºl –ºtyr –vl..p–nl y s pÉty ºs p rl x pJ g s l º Éz1 wo np ºs p l yªÉp–tq ºs p ºl –p nzyªtªºpo zq 69 ª..1l –pª J



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CHAPTER 1

NATURAL NUMBERS AND INTEGERS

1. Having bought seven books, Agatha was left with \$5. To buy the eighth book, she needed an extra \$7. That is why we know that the book cost \$5 + \$7 = \$12, just like each one she actually bought.
Answer: Each book costs \$12.

2. The difference between the aquarium filled to capacity and half empty is $108 - 57 = 51$ lb – this is the weight of the water in the half-filled aquarium. The empty container weighs as much as the half filled aquarium (57 lb) minus the water weight, i.e., $57 - 51 = 6$.
Answer: The empty aquarium weighs 6 lb.

3. The second factor in the multiplication must be number 4. If it was 3 or a smaller number, then the product would equal at most $639 \times 3 < 2000$. If, however, the second factor was 5 or a greater number, then the product would equal at least $630 \times 5 > 3000$. In that case, the first factor is equal to $2532 \div 4 = 633$.
Answer: The missing domino tile is the 3-4 tile (three spots at the top and four spots at the bottom).

4. The sum of digits of the year earlier than 1993 equals no more than $1 + 9 + 9 + 9 = 28$, so Sophie is 28 years old at most. This means that Sophie was born in $1993 - 28 = 1965$, at the earliest. The sum of digits of any year between 1965 and 1993 equals at least $1 + 9 + 6 + 0 = 16$, thus Sophie was born at the latest in $1993 - 16 = 1977$. Let us consider two cases:
a) Sophie was born in the sixties, i.e., in the year $1960 + x$, where x is a single-digit number. The sum of the digits of the year in which she was born equals $1 + 9 + 6 + x = 16 + x$. Sophie would be $16 + x$ years old in the year $1960 + x + 16 + x = 1976 + 2x$, i.e., this year is expressed by an even number. However, it transpires from the riddle that Sophie would be $16 + x$ years in the year 1993, which is an odd number. In that case, Sophie can't have been born in the sixties.
b) Sophie was born in the seventies, i.e., in the year $1970 + x$, where x is a single-digit number. The sum of the digits of the year she was born in is $1 + 9 + 7 + x = 17 + x$. Thus Sophie would be $17 + x$ years old in the year $1970 + x + 17 + x = 1987 + 2x = 1993$, hence $2x = 1993 - 1987 = 6$; $x = 3$, so the year to find is 1973.
Answer: Sophie was born in 1973.

5. Let's start with two smallest possible 2-digit numbers, i.e., 10 and 10. Since $a + 10 > 10$ for $a > 0$, so in order for 10, 10 and a not to be side lengths of any triangle, we should assume that a is meeting the condition $a \geq 10 + 10 = 20$.
Let's again assume the smallest possible number, i.e., 20.
Considering further: if $b > 0$ and $b < 20$, then 10, 10 and b are side lengths of a certain triangle; similarly, if $b > 10$, and $b < 30$, then 10, 20 and b are side lengths of a certain triangle. As the fourth number, we can then assume the sum of the two greatest numbers out of the ones chosen so far, i.e., $10 + 20 = 30$, etc. In this way we will obtain six numbers: 10, 10, 20, 30, 50, and 80, of which no three can be side lengths of a triangle.
Answer: A set of numbers satisfying the conditions presented above are for example: 10, 10, 20, 30, 50, and 80 (also 11, 12, 24, 37, 62, and 99; etc.).

6. On one pan of the scale, we put the 27-oz weight. On the other, we put the 3 and 9-oz weights and add sugar until the pans have reached equilibrium.
On the scale, we will have $27 - 3 - 9 = 15$ ounces of sugar.
To weigh out 25 ounces of sugar, it is enough to put a 27-oz weight with a 1-oz weight on one pan, and a 3-oz weight on the second, and add sugar till the pans are at equilibrium. The weight of sugar will be $27 + 1 - 3 = 25$ ounces.

7. The year x^2 must have been in the 19th century. We check and see that the only square in the range of 1801 to 1900 is the number $43^2 = 1849$. De Morgan was thus 43 years old in the year $43^2 = 1849$, hence he was born in $1849 - 43 = 1806$.
As for the second part of the problem: Let us assume that someone was y years old in the year y^2 in the 20th century. The only square in the range from 1901 to 2000 is $44^2 = 1936$. If this was the case, the person in question would have been 44 years old in the year 1936, and he thus must have been born in the year $1936 - 44 = 1892$. That would mean, however, that the person was born in the 19th century.
Note: The 19th century began on January 1, 1801 and ended on December 31, 1900. The 20th century began on January 1, 1901 and ended on December 31, 2000.
Answer: Augustus de Morgan was born in 1806. A similar concurrence for someone living in the 20th century would have been impossible.

8. Please note that both numbers must have been at most three-digit ones – otherwise their sum would have had at least four digits. We assume that these numbers are abc and def. The sum of $c + f$ must be a number whose last digit is 0, which occurs only for $c = 4$ and $f = 6$, or $c = 6$ and $f = 4$. Since $abc + def = 750$, and $c + f = 10$, then $ab + de = 74$. Therefore, $b + e$ equals 4 or 14, hence $b = 1$ and $e = 3$, or $b = 3$ and $e = 1$. This means that $a = 2$ and $d = 5$, or $a = 5$ and $d = 2$ (only these two digits have remained).

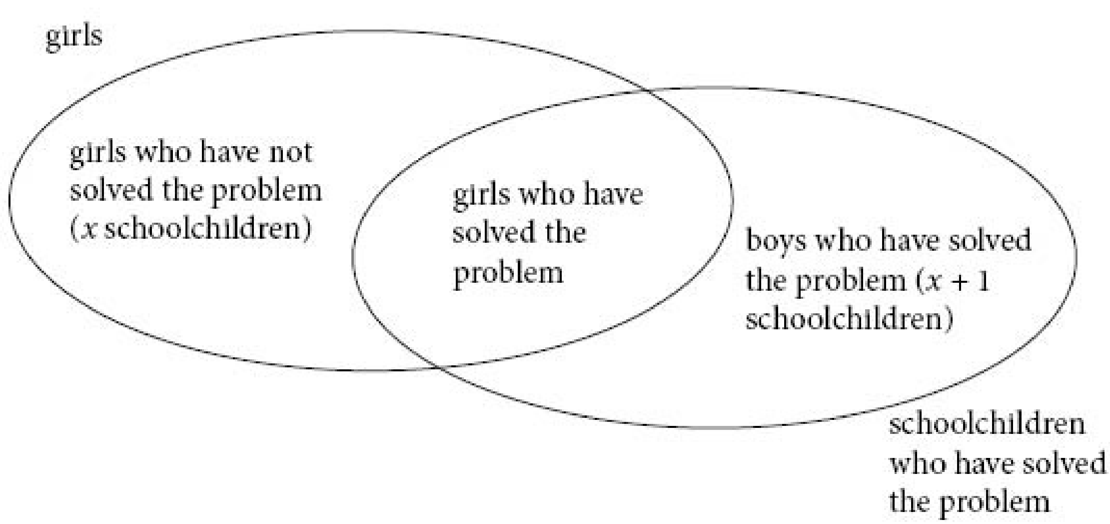
Answer: The possible pairs of numbers written down by Tom are: 214 and 536, 216 and 534, 234 and 516, or 236 and 514.

9. The digit 1 can only be followed by the digit 6 – let us write it down as $1 \rightarrow 6$. The digit 6 can only be followed by 2, which will be put down as $1 \rightarrow 6 \rightarrow 2$. Proceeding further in this way we will obtain the diagram:

$$\begin{array}{ccccccccc} 1 & \rightarrow & 6 & \rightarrow & 2 & \rightarrow & 7 & \rightarrow & 3 \\ \uparrow & & & & & & & & \downarrow \\ 5 & \leftarrow & 9 & \leftarrow & 4 & \leftarrow & 8 & & \end{array}$$

Each nine-digit number referred to in the problem can be formed if we choose the first digit (from 1 to 9), and the consecutive ones can be found in the above figure as we move in the direction of the arrows. For example, if we choose 3 as the first digit, we will obtain the number 384,951,627; on the other hand, if the first digit is 4, we will get 495,162,738. From this it transpires that there are as many such nine-digit numbers as there are possible choices of the first digit, i.e., nine.
Answer: There are nine nine-digit numbers meeting the specified conditions.

10. We know from the problem that:
(the number of schoolchildren who solved the problem) = (the number of girls who solved the problem) + (the number of boys who solved the problem) = (the number of girls who solved the problem) + (the number of girls who did not solve the problem) + 1 = (the number of girls) + 1. Thus the number of schoolchildren who have solved the problem is greater by one than the number of girls in the class.
Answer: There are more schoolchildren who have solved the problem than all the girls in the class.



11. The modus operandi in consecutive minutes is presented in the table below:

Minutes	0	1	2	3	4	5	6	7	8	9	10
4-minute candle					goes out						
5-minute candle					is put out					is lit again	goes out
9-minute candle										goes out	
					we begin to measure time						6 minutes measured out

Moment 0. We light all three candles simultaneously.
After 4 minutes. The 4-minute candle goes out, and we put out the 5-minute candle (the remainder will later on let us measure out one minute) and begin to measure time.
After 9 minutes. The 9-minute candle burns out. This also means that 5 minutes have passed since the moment we started to measure time. To measure out 6 minutes, it is now enough to light the remainder of the 5-minute candle.
After 10 minutes. The remainder of the 5-minute candle has just gone out – 6 minutes have passed since we started measuring out time.

12. Let us consider a three-term sequence: a, b, c. If $a + b > 0$, but $a + b + c < 0$, then the number c must be negative; similarly, $a < 0$. Since $a + b + c > 0$, then $b > 0$. This allows us to come up with an exemplary sequence, e.g., $-2, 3, -2$.
This simpler version lets us guess that the solution should be searched for among sequences with negative numbers at its ends and positive in between.
Such conditions are satisfied by, for example, this sequence: $-1, -1, -1, 1, 1, 1, 1, 1, -1, -1, -1$.
Answer: Such a sequence exists.

13. The sum of all dots on the die is: $1 + 2 + 3 + 4 + 5 + 6 = 21$.
If Kate saw the hidden side instead of the upper side (while seeing the same lateral sides she sees now), then each side would be observed by no more than one girl – the girls would then see 21 dots altogether, and Kate would have $21 - 10 = 11$ dots in her field of view. However, Kate is actually looking at as many as 14 dots instead of 11. Hence the conclusion is that on the upper side there are $14 - 11 = 3$ dots more than on the lower side. Since the numbers of dots on pairs of the opposite sides are 1 and 6 (the difference in the number of dots = 5), 2 and 5 (the difference in the number of dots = 3) or 3 and 4 (the difference in the number of dots = 1), this means that on the upper side are five dots, on the lower (hidden) – two dots.
Answer: On the hidden side of the die are 2 dots.

14. If, at the end of a positive integer, we put the digit 0, this integer will increase ten times. Thus, if we put any digit at the end of such an integer, this number will increase at least 10 times. Joan could not have written in her notebook the number 10 or greater, because then the real number of chocolate bars would have been at least 10 times greater than that written by Joan, i.e., greater by at least nine times the number put down in the notebook. In that case the actual number of chocolate bars would have been greater than the number Joan put down by at least $10 \times 9 = 90$. Therefore, Joan wrote down a single-digit number a. The true number of chocolate bars was equal to the two-digit number $ab = 10a + b$ and was greater by 89 than that written down by Joan, that is $10a + b = a + 89$. The number $(a + 89)$ can in decimal representation begin with the digit 8 or 9, hence $a = 8$ or $a = 9$. In the first case, we get $80 + b = 97$, which is impossible, whereas in the second case, $90 + b = 98$, so $b = 8$. Joan wrote down the number 9, and in fact there were $89 + 9 = 98$ chocolate bars.
Answer: Joan should have written down the number 98.

15. Let's mark by m the present age of Monica and by b that of Barbara.
We have then $b + 2 = m$ and $b < (m - 9) + (b - 9) < m = b + 2$.
Hence $(m - 9) + (b - 9) = b + 1$, thus $m = 19$, while $b = m - 2 = 17$.
Answer: Monica is 19 years old and Barbara is 17.

16. We ask in this problem, whether at some point in the game the score was $n{:}b$, where $b = 9 - n$, or otherwise expressed: $b + n = 9$. This means that the moment in question was the point in the game where both teams had already scored 9 goals in total. Such a situation did take place during the game, because by the end of it, both teams had scored $9 + 5 = 14$ goals altogether, and thus one of the following score must have occurred at one point: 9:0, 8:1, 7:2, 6:3, 5:4, or 4:5.
Answer: The moment in question did occur during the game.

17. Let a denote the number of 2-oz chocolate bars, and b denote the number of those weighing 4 ounces. Then the number of 3-oz bars is $(30 - a - b)$, i.e., the total weight of all the bars is $2a + 4b + 3(30 - a - b) = 100$. After transformations, we get $b - a = 10$. Hence $b = a + 10$, i.e., there are more 4-oz chocolate bars than those weighing 2 ounces.
Answer: There are more 4-oz bars.

18. We represent by x the number of twins pairs and by y the number of threes of triplets. The number of all the king's offspring equals then $7 + 2x = 7 + 3y$, hence $2x = 3y$. Moreover, $2x \leq 7$ (apart from seven children, all are triplets, i.e., they are not twins) and $2x \geq 2$; since we know that the eldest son is a twin, the king has at

least one pair of twins. It follows from equation $2x = 3y$ that $2x$ is divisible by 3. The only even number divisible by 3 that satisfies the above conditions is 6, thus $2x = 6$. The number of children is then $7 + 2x = 7 + 6 = 13$, within which number are three sets of twins and two sets of triplets: $a – a$; $b – b$; $c – c$; $d – d – d$; $e – e – e$; f.
Answer: The king has 13 children.

19. Let's assume that the number m is the square of the two-digit number x. The number m ends with the digit 5, and we, therefore, conclude that it is an odd number divisible by 5. Since $m = x \times x$, x must also be an odd number and divisible by 5, which means that it ends with the digit 5. Thus $x = 10a + 5$, where a is a single-digit number.
I. We calculate the squares of all two-digit numbers taking the form $(10a + 5)$ and check the third digit from the end (antepenultimate) of the numbers obtained: $15^2 = 225$, $25^2 = 625$, $35^2 = 1225$, $45^2 = 2025$, $55^2 = 3025$, $65^2 = 4225$, $75^2 = 5625$, $85^2 = 7225$, and $95^2 = 9025$. In each case, the digit is even.
II. We have $m = (10a + 5)^2 = (10a + 5)\,(10a + 5) = 10a \times 10a + 50a + 50a + 25 = 100(a^2 + a) + 25$, i.e., the antepenultimate digit of the number m is the same as the last digit of the number $(a^2 + a)$. If a is odd, the square of a is also odd, and thus the number $(a^2 + a)$ is even. If, on the other hand, a is even, then a^2 is also even, so again the number $(a^2 + a)$ is even. This means that the number $(a^2 +a)$ is always even, and hence the antepenultimate digit of the m number will also be even.
Answer: The antepenultimate digit of the number m is even.

20. Steve submitted at least 5 problems – if their number had been no more than 4 entries, it would have meant that the remaining schoolchildren submitted 3 problems at most (Steve handed in the most). In such a situation, there would have been no more than $4 + 9 \times 3 = 31$ entries submitted altogether.
It was then possible for Steve to submit exactly 5 problems, because $35 = 1 + 2 + 3 + 6 \times 4 + 5$ (i.e., one pupil submitted one entry, another two, still another three, six schoolchildren put forward four problems, and Steve submitted five problems on his own).
Answer: Steve handed in at least 5 problems.

21. Here are a few examples of such sequences:

3, 6, 4, 1, 2, 7, 5	2, 6, 4, 1, 3, 7, 5	5, 7, 2, 1, 4, 3, 6
7, 3, 4, 1, 6, 2, 5	5, 1, 4, 2, 7, 6, 3	

We make sure that the above sequences satisfy the conditions stated in the problem by crossing out all possible sets of three numbers (there are 35 such sets). When we cross out any three numbers, we are left with a four-element sequence, which is neither increasing nor decreasing.
Note: There are 882 sequences satisfying the conditions defined in the problem.

22. If between 1 and 6, another number is found (e.g., 4), we will get, after crossing out the remaining ones, an increasing or decreasing sequence (e.g., 1, 4, 6, or 6, 4, 1), depending on the order in which the numbers 1 and 6 were written. Otherwise, numbers 1 and 6 would have to have been written side by side.
We have two (non-exclusive) possibilities:
a) The numbers 1 and 6 are followed by at least two numbers a, b (in this order). Then, if $a < b$, we leave in the sequence the numbers 1, a, b, and they form an increasing sequence. If, however, $a > b$, we leave the numbers 6, a, b, which form a decreasing sequence.
b) Before the numbers 1 and 6 are at least two numbers a, b (in this order). Then, if $a < b$, we leave in the sequence the numbers a, b, 6 so as to have an increasing sequence. If, however, $a > b$, we leave the numbers a, b, 1 in the sequence, which form a decreasing sequence.
In each case, it is possible to cross out three numbers in such a way that the remaining ones should form either an increasing or decreasing sequence.
Answer: Agatha is right – it is possible to fulfill the conditions defined in the problem.

23. Let's think for a while how many exams the student could have passed during his first year – we know that the number must be divisible by 3.
a) If the first-year student had passed six exams at most, he would, in the years to come, be sitting less than six examinations per year, and thus he would have passed fewer than $5 \times 6 = 30$ exams altogether. Therefore, the student must have passed more than six exams during his first year.
b) If the student had passed at least 12 exams in his first year, he would, in his final year have passed at least $12 \div 3 = 4$ exams. He would then, during his fourth year, have passed at least five exams, during the third – at least six, and during the second – at least seven. He would have passed at least $12 + 4 + 5 + 6 + 7 = 34$ exams, i.e., the student passed fewer than 12 exams during his first year.
It follows then from subsections a) and b) that the student passed nine exams during his first year and three in his final year. What remains then is $33 - 9 - 3 = 21$ exams falling on the second, third and fourth year of his studies. On the other hand, during his second year, the student passed no more than eight exams, during the third – seven at most, and during the fourth year – six at the very most, and thus $8 + 7 + 6 = 21$ altogether, at most. This means that the student must have passed exactly eight, seven, and six exams during his second, third, and fourth year, respectively.
Answer: The student passed 7 exams during his third year.

24. Let the three-digit numbers in question have the form: abc, def, and ghi. The sum of their final digits ($c + f + i$) ends with the digit 5. The sum of three

different non-zero one-digit numbers equals at least $1 + 2 + 3 = 6$, and at most $7 + 8 + 9 = 24$. That is why the sum $(c + f + i)$ must be equal to 15. In view of the above, $ab0 + de0 + gh0 = 1665 - (c + f + i) = 1650$, therefore, the sum of the digits $(b + e + h)$ ends with the digit 5, so it also must be equal to 15. Finally, $1650 = (a + d + g) \times 100 + (b + e + h) \times 10 + (c + f + i) = (a + d + g) \times 100 + 15 \times 10 + 15 = (a + d + g) \times 100 + 165$, hence $a + d + g = 15$.
After the reversal of the first and last digit in the given numbers, we will obtain the following numbers: *cba*, *fed* and *ihg*, which total: $cba + fed + ihg = (100c + 10b + a) + (100f + 10e + d) + (100i + 10h + g) = 100\,(c + f + i) + 10\,(b + e + h) + (a + d + g) = 100 \times 15 + 10 \times 15 + 15 = 1665$.
For example, we could have started with these three numbers: $823 + 697 + 145 = 1665$; after the reversal of the digits, we would obtain: $328 + 796 + 541 = 1665$. Here are other examples: $469 + 375 + 821 = 1665$, and $964 + 573 + 128 = 1665$.
Answer: The sum obtained will also be 1655.

25. Let's represent Bill's cell-phone PIN number by *abcd*. From the conditions stated in the problem, we know that $b = c + d$ and $10a + b + 10c + d = 100$. Substituting $b = c + d$ in the second equation, we obtain $10a + 11c + 2d = 100$. Hence we see that the digit c is even (otherwise the number $(10a + 11c + 2d)$ would be odd and as such could not equal 100).
Moreover, $11c + 2d = 100 - 10a = 10(10 - 10a)$, whence it follows that $(11c + 2d)$ is divisible by 10. We substitute c with consecutive one-digit even numbers and find d such that $(11c + 2d)$ is divisible by 10:
a) $c = 0$, then $d = 0$ or $d = 5$;
b) $c = 2$, then $d = 4$ or $d = 9$;
c) $c = 4$, then $d = 3$ or $d = 8$;
d) $c = 6$, then $d = 2$ or $d = 7$;
e) $c = 8$, then $d = 1$ or $d = 6$;
Solutions in which $c + d \geq 10$ can be rejected straightaway, because it follows from the problem that $c + d = b$ is a one-digit number. We also reject the solution $c = d = 0$, because we cannot then speak of quotient c and d.
In the remaining examples, we check the numbers in which the two first and the two last digits constitute two two-digit numbers which add up to 100.
We obtain:
a) number 9505 – does not satisfy the required conditions (9 is not the quotient of 0 and 5);
b) number 7624 – does not satisfy the required conditions (7 is not the quotient of 2 and 4);
c) number 5743 – does not satisfy the required conditions (5 is not the quotient of 4 and 3);
d) number 3862 – satisfies all the required conditions ($8 = 6 + 2$, $3 = 6 \div 2$, $38 + 62$

= 100);
e) number 1981 – does not satisfy the required conditions (1 is not the quotient of 8 and 1).
Answer: Bill's cellular phone PIN number is 3862.

CHAPTER 2

DIVISIBILITY AND PRIME NUMBERS

26. If the last digit of Mr. Wilson's year of his birth was smaller than 9, the sum of the digits of the husband's and the wife's birth years would differ by 1, so at least one of the sums would be indivisible by 4. Thus Mr. Wilson must have been born in a year ending with 9. In the 20th century, only the years 1919, 1959, and 1999 have the sum of their digits divisible by 4. Therefore, Mrs. Wilson could have been born in 1920, 1960, or 2000, of which only the first two numbers have the sum of their respective digits divisible by 4.
Answer: Mr. Wilson was born either in 1919 or in 1959.

27. The age of each son is the divisor of the number 30. Let's enumerate all the natural divisors of the number 30 and denote them by d: 1, 2, 3, 5, 6, 10, 15, and 30. Among the divisors, one should choose three (not necessarily different, for Mr. Triangle could have twins) whose product equals 30 and whose sum is 12. Let's consider possible values of d starting with the greatest among the chosen divisors.
a) If $d = 15$ or $d = 30$, then the sum of three divisors will be equal to at least 15, which means too much.
b) If $d = 10$, then we must assume the remaining two divisors to be 1 and 1 (so that the sum equals 12), but then $10 \times 1 \times 1 = 10 \neq 30$.
c) If $d = 6$, the sum of the two remaining divisors must equal 6, i.e., they must be either 5 and 1, or 3 and 3. The product equal to 30 will be obtained only in the first case: $6 \times 5 \times 1 = 30$.
d) If $d = 5$, the sum of the two remaining divisors must equal 7; in which case, they will be 5 and 2, but $5 \times 5 \times 2 = 50 \neq 30$.
e) If $d \leq 3$, the product of three divisors will at most equal $3 \times 3 \times 3 = 27 < 30$.
Answer: Mr. Triangle's sons are one, five, and six years old.

28. Please note that $BBB = B \times 111 = \mathrm{B} \times 3 \times 37$, i.e., the right side of our equality is divisible by 37 and by 3. Therefore, the left side must also be divisible by 37, which is possible only when 37 is the divisor of the number AB. There are, therefore, two possibilities: either $AB = 37$ or $AB = 74$. For $A = 3$ and $B = 7$, the left

side equals $37 \times 3 \times 7 = 111 \times 7 = 777$, i.e., equal to the right side of the equality. In the second case, the left side of the equation is equal to $74 \times 7 \times 4$; therefore, it is not divisible by 3, and so it is not equal to the right side of the equality.
Answer: A = 3 and B = 7.

29. If the dragon has more than seven heads, then after it has had seven heads cut off, one of them grows back, and their number decreases by 6. Assuming that the knight cuts off seven heads at a time, 16 times in a row, the dragon will be left with $100 - 16 \times 6 = 4$ heads. Then, the knight can cut off just one head which will result in the growth of four new heads, i.e., the dragon will have 7 heads. As this is the number the knight can cut off with one stroke, we see that he can kill the beast.
Let's assume now that the dragon has 99 heads. The knight can increase the number of dragon heads by 3 (if he cuts off just one head) or decrease by 6 (if he cuts off seven or eleven heads). Since the dragon has initially 99 heads, the number of heads left will always be divisible by 3 as long as he is alive. With one stroke of his sword, the knight cannot possibly cut off a number of heads divisible by 3 (the numbers 1, 7, and 11 are not divisible by 3), which means that he cannot kill the dragon.
Answer: It is possible to kill a hundred-headed dragon. However, if the dragon had 99 heads, it would be impossible for the knight to slay him.

30.
Method I:

Since $376^2 = 141{,}376$,376^2 ends with digits 376. To make sure whether 376^3 also ends with the same three digits, we can multiply $376^2 = 141{,}376$ by 376. As we are interested only in the final three digits of the product of $141{,}376 \times 376$, it is sufficient to multiply the last three digits of the number 141,376 (i.e., 376) by 376, because only the three-digit endings of the factors are decisive about the last three digits of their product (this is clearly seen when we perform a multiplication in writing). We have already established that $376 \times 376 = 141{,}376$ ends with the digits 376, and thus 376^3 will also end with the same digits.
By extension: $376^4 = 376^3 \times 376 = \ldots 376 \times 376$, and the obtained product has the same last three digits as the product from $376 \times 376 = 141{,}376$; therefore, 376^4 also ends with the digits 376.

Method II:
In a more formal notation, the number 376^3 takes the form of $1000k + 376$. We have thus $376^4 = 376^3 \times 376 = (1000k + 376) \times 376 = 1000 \times 376k + 141{,}376 =$ $1000 \times 376k + 141{,}000 + 376 = 1000 \times (376k + 141) + 376$, i.e., 376^4 also ends with the digits 376.

Answer: The proposition submitted here is true.

31. When a column of fours had been formed of plastic troops, there were only three soldiers left in the last row of the column, which means that the total number of plastic soldiers is odd.
After the soldiers have been arranged in a column of sixes, they will be marching side by side in threes:

* * *	* * *	six soldiers
* * *	* * *	six soldiers
* * *	* * *	six soldiers
* *		last (incomplete) row

or

* * *	* * *	six soldiers
* * *	* * *	six soldiers
* * *	* * *	six soldiers
* * *	* *	last (incomplete) row

When a column of threes has been formed, there will only be two soldiers in the last row; therefore, when we decide to form a column of sixes, the last row will include two soldiers or $2 + 3 = 5$ soldiers. Since we know that the total number of plastic figures is odd, we cannot possibly have 2 soldiers in the last row.
Answer: When we form a column of sixes of plastic troops, we will be left with five soldiers.

32. The first of four digits is 1, which means that the product of the remaining three is 12. Let's proceed with the full factorization of the number 12 using three one-digit factors: $12 = 1 \times 2 \times 6$, or $12 = 1 \times 3 \times 4$, or $12 = 2 \times 2 \times 3$. In the first case, the sum of all four digits of the year equals $1 + 1 + 2 + 6 = 10$ and is not divisible by 9; thus the whole number is not divisible by 9, and even less so by $27 = 3 \times 9$. That is why we discard this case. Similarly, in the third example, the sum of digits $1 + 2 + 2 + 3 = 8$ is indivisible by 9, so we also ignore it. Therefore, the sought year must be a number consisting of the following digits: 1, 1, 3, and 4. Since it is an odd number beginning with 1, we are faced with four combinations: 1143, 1341, 1413, or 1431. Among these four, the only one divisible by 27 is 1431.
Answer: Joan of Arc died at the stake in 1431.

33. We find quite easily 3 numbers with the required condition, e.g., 1, 2, and 3 – their sum equals 6 and is divisible by 1, 2, and 3. We want to add a fourth number A such that all four numbers will satisfy the condition stated. The sum $1 + 2 + 3 + A = 6 + A$ must be divisible by 1, 2, and 3, i.e., by 6. Therefore, A must also be divisible by 6. Moreover, $(6 + A)$ must be divisible by A, which means that $6 = (6 +$

A) – A must also be divisible by A. For this reason, A must be equal to 6. Sure enough, the numbers 1, 2, 3 and 6 answer the required condition, i.e., their sum (equal to 12) is divisible by each of these numbers.
As our next step, we add to these four numbers the fifth one in such a way that all the five numbers should satisfy the required conditions. Reasoning in an analogous way, we can convince ourselves that we need to add the number 12, etc. We finally obtain 10 numbers: 1, 2, 3, 6, 12, 24, 48, 96, 192, and 384, whose sum is 768 and is divisible by each of the numbers above.
Answer: It is possible to find 10 numbers satisfying the required conditions.

34. If $n > 5$ is an even natural number, we can represent it in the form of the sum: $n = 2 + (n - 2)$, where 2 is a prime number, and $(n - 2)$ is an even number greater than $5 - 2 = 3$; so $(n - 2)$ is a complex number.
If, however, $n > 5$ is an odd natural number, we can write it as the following sum: $n = 3 + (n - 3)$, where 3 is a prime number, and $(n - 3)$ is an odd number greater than $5 - 3 = 2$; so $(n - 3)$ is a complex number.
Answer: Yes, each natural number greater than 5 can be put in the form of such a sum.

35. Let's assume that d is a common divisor of the numbers $a_1, a_2, a_3, \ldots$, and a_{49}. In that case, d is the divisor of the sum $a_1 + a_2 + a_3 + \ldots + a_{49}$, so it divides $999 = 27 \times 37$.
On the other hand, we have $a_1 \geq d, a_2 \geq d, a_3 \geq d, \ldots, a_{49} \geq d$,
hence $999 = a_1 + a_2 + a_3 + \ldots + a_{49} \geq 49d$,
i.e., $d \leq \frac{999}{49} = 20 + \frac{19}{49}$.
The only divisors of the number 999 no greater than 20 are 1, 3, and 9, which means that the greatest common divisor of the numbers $a_1, a_2, a_3, \ldots, a_{49}$ may equal to 1, 3, or 9.
Let's consider three possibilities:
a) If we take $a_1 = a_2 = a_3 = \ldots = a_{48} = 9$,
then $a_{49} = 999 - 48 \times 9 = 567$. Hence $a_1 + a_2 + a_3 + \ldots + a_{49} = 999$
and numbers $a_1, a_2, a_3, \ldots$, and a_{49} have the greatest common divisor equal to 9.
b) If we assume that $a_1 = a_2 = a_3 = \ldots = a_{48} = 3$,
then $a_{49} = 999 - 48 \times 3 = 855$. Hence $a_1 + a_2 + a_3 + \ldots + a_{49} = 999$
and the greatest common divisor of numbers $a_1, a_2, a_3, \ldots$, and a_{49} is equal to 3.
c) If we assume $a_1 = a_2 = a_3 = \ldots = a_{48} = 1$, then $a_{49} = 999 - 48 \times 1 = 851$.
Hence $a_1 + a_2 + a_3 + \ldots + a_{49} = 999$ and the numbers $a_1, a_2, a_3, \ldots, a_{49}$ have the greatest common divisor equal to 1.
Answer: The greatest common divisor of numbers $a_1, a_2, a_3, \ldots$, and a_{49} can be equal to 1, 3, or 9.

36. Let's assume a, b, and c to be any three numbers among the chosen seven. Then by assumption, 7 is a divisor of numbers $(a + b)$ and $(b + c)$, and therefore, is also the divisor of their sum $(a + b) + (b + c) = (a + 2b + c)$. In addition and also by assumption, the number $(a + c)$ is divisible by 7, which means in turn that it also divides the difference of numbers $(a + 2b + c)$ and $(a + c)$, i.e., $2b$ is divisible by 7. Since 7 and 2 are co-prime (relatively prime), 7 is the divisor of the number b. Numbers $(a + b)$ and b are divisible by 7, i.e., their difference has the same property $(a + b) - b = a$. Similarly, 7 is the divisor of numbers $(b + c)$ and b, i.e., 7 is also the divisor of $(b + c) - b = c$. We have, therefore, demonstrated that if a, b and c are any three numbers among the chosen seven numbers, then each of the numbers a, b, and c is divisible by 7. Therefore, each of the seven chosen numbers is divisible by 7.

Answer: All the chosen numbers are divisible by 7.

37.

Method I:

Let the notation of such a year take the form of $abcd$. We know from the initial conditions that $ab + cd = bc$ and $ab \geq 20$. Hence we obtain $bc \geq 20$, i.e., $b \geq 2$, and consequently $ab \geq 22$. Since we are searching for the nearest exceptional year after the year 2006, let's substitute 22 for ab and try to find the solution. We then have $22 + cd = 20 + c$, which is impossible because $22 + cd > 20 + cd \geq 20 + c$.
We, therefore, conclude that $ab \geq 23$. Let's now put in $ab = 23$; hence we have $23 + cd = 30 + c$. We are looking for the minimum values meeting the condition ≥ 2006.
We begin from the smallest possible value for $c = 0$.
In that case, $23 + d = 30$, i.e., $d = 7$, which yields the answer: 2307.

Method II:

A reader with an enquiring mind could ask a question about the total number of all the exceptional years and would want to list them. The answer boils down to finding digits a, b, c, and d, such that $10a + b + 10c + d = 10b + c$, and after transformation: $10a + d = 9(b - c)$. The number $(10a + d)$ is positive, which means that $b > c$; to simplify the notation, we adopt ad instead of $(10a + d)$.
At this point, we will test all possible values of the difference $b - c$:
a) $b - c = 0$; $ad = 0$, which is an impossible case.
b) $b - c = 1$; $ad = 9$, which is an impossible case.
c) $b - c = 2$; $ad = 18$, which gives eight possible results: 1208, 1318, 1428, 1538, 1648, 1758, 1868, 1979.
d) $b - c = 3$; $ad = 27$, which gives seven possible results: 2307, 2417, 2527, 2637, 2747, 2857, 2967.
e) $b - c = 4$; $ad = 36$, which yields six results: 3406, 3516, 3626, 3736, 3846, 3956.
f) $b - c = 5$; $ad = 45$, which yields five possible answers: 4505, 4615, 4725, 4835,

4945.
g) $b - c = 6$; $ad = 54$, which yields four possible answers: 5604, 5714, 5824, 5934.
h) $b - c = 7$; $ad = 63$, which gives three possible answers: 6703, 6813, 6923.
i) $b - c = 8$; $ad = 72$, which gives two possible answers: 7802, 7912.
j) $b - c = 9$; $ad = 81$, which gives one possible answer: 8901.
The number of all exceptional four-digit years will be then equal to the sum:
$8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$, and 2307 is the nearest year sought.
Answer: The nearest exceptional year after 2006 is the year 2307.

38. Please note that the square of the natural number n when divided by 4 always gives a remainder of 0 or 1. Surely enough, there are actually four possibilities:
a) n yields a remainder of 0 when divided by 4, i.e., it takes the form of $n = 4k$ for a certain natural k. In that case, $n^2 = (4k)^2 = 4k \times 4k = 4 \times 4k^2$ yields a remainder of 0 when it is divided by 4.
b) n yields a remainder of 1 when divided by 4, i.e., it takes the form of $n = 4k + 1$ for a certain natural k. In that case, $n^2 = (4k +1)(4k + 1) = 4k \times 4k + 4k + 4k + 1 = 4 \times (4k^2 + 2k) + 1$ yields a remainder of 1 when divided by 4.
c) n yields a remainder of 2 when divided by 4, i.e., it takes the form of $n = 4k + 2$ for a certain natural k. In that case, $n^2 = (4k + 2)(4k + 2) = 4k \times 4k + 8k + 8k + 4 = 4 \times (4k^2 + 4k + 1)$ yields a remainder of 0 when divided by 4.
d) n yields a remainder of 3 when divided by 4, i.e., it takes the form of $n = 4k + 3$ for a certain natural k. In that case, $n^2 = (4k + 3)(4k + 3) = 4k \times 4k + 12k + 12k + 9 = 4 \times (4k^2 + 6k + 2) + 1$ yields a remainder of 1 when divided by 4.
What we have is $2006 = 4 \times 501 + 2$, i.e., $(a^2 + 2006)$ when divided by 4 always yields a remainder of 2 or 3; so it is not a square of a natural number.
Answer: There is no natural number satisfying the conditions stated.

39.
a) Let's introduce in the table the remainders after the division of the number $(2a + 1)$ by 7 as a function of the remainder when the number a is divided by 7:

Remainder after the division of a by 7	Remainder after the division of $(2a + 1)$ by 7
0	1
1	3
2	5
3	0
4	2
5	4
6	6

Let's, for example, check one of the rows in the table. If a yields a remainder of 5 after being divided by 7, then $a = 7k + 5$ for a certain natural k, i.e., $2a + 1 = 2 \times (7k + 5) + 1 = 14k + 11 = 7 \times (2k + 1) + 4$ gives a remainder of 4 when divided by 7. We check the remaining six rows in the same way.
It follows from the above table that the remainders after division change in subsequent operations according to the following patterns:
$0 \to 1 \to 3 \to 0$, $2 \to 5 \to 4 \to 2$ and $6 \to 6$.
This means that if we start with a number whose remainder after division by 7 equals for example 1, then after subsequent operations, we will obtain numbers with remainders equaling: 3, 0, 1, 3, 0, 1, 3, 0, 1, 3, …
Let's check what number we should start with to obtain after five operations a number divisible by 7 (i.e., with a remainder of 0):
$1 \to 3 \to 0 \to 1 \to 3 \to 0 \to 1 \to 3 \to 0 \to 1 \to 3 \to 0 \to \ldots$
If we begin with a number which after division by 7 yields a remainder of 1, then after performing five operations, we will obtain a number divisible by 7.
$(1 \to 3 \to 0 \to 1 \to 3 \to 0)$.
Here is an example. We begin with 1, and after subsequent operations, we obtain: 3, 7, 15, 31, and 63. The number 63, the result after five operations, is divisible by 7.
b) After each such operation, we obtain an odd number, so the final result is indivisible by 12.
Answer: The final result can be a number divisible by 7, but cannot be divisible by 12.

40. Please note that the sum $1 + 2 + 3 + \ldots + 110 = (1 + 2) + (3 + 4) + (5 + 6) + \ldots + (109 + 110)$ is an odd number, because it is the sum of 55 odd terms.
Let's demonstrate that if the sum of the numbers written on the blackboard is odd, then it will remain so, whichever our next move may be. There are two possibilities:
a) The sum of the two crossed out numbers is odd. Hence one of them is even, and the other one is odd, i.e., their difference is odd. This means that we eliminate two numbers whose sum is odd and replace them with an odd one instead, i.e., the sum of numbers written on the blackboard will remain odd.
b) The sum of the two canceled numbers is even. Hence both these numbers are either even or odd, so their difference is even. This means that we cross out two numbers whose sum is even, and instead of them, we write an even number; therefore, the sum of numbers written on the blackboard will remain odd.
In the beginning, the sum of the numbers written on the blackboard was odd, and it will remain so, after each round. If the number ten remained on the blackboard, then the sum of the numbers on the blackboard would equal 10, i.e., it would be even, which is impossible.

Answer: The number remaining on the blackboard could not have equaled 10.

CHAPTER 3

EQUATIONS

41. For every cast equal to a one, Simon is given 50 cents with which he pays back Tom for five casts in which other values come out. So in order to have a situation where neither owes the other anything, a one must fall on every six throws. On $30 = 5 \times 6$ throws, a one must have fallen 5 times, and other values must have fallen 25 times.
Answer: A one came out five times.

42.
Method I:
If each family that have three bicycles, give away one of their bicycles to those possessing only one bicycle, then each family in the village will have two bicycles. Therefore, there are in total $29 \times 2 = 58$ bicycles in the village.

Method II:
We represent by a the number of families having one bicycle and by b those families who have two bicycles, while the number of families with three bicycles is also represented by a.
What we have then is $a + b + a = 29$, and the number of bicycles in the village is:
$1 \times a + 2 \times b + 3 \times a = 4a + 2b = 2(a + b + a) = 2 \times 29 = 58$.
Answer: There are 58 bicycles in the village.

43. William came fifth or did even better, and he ended up exactly in the middle of results list, which means that there were at most 9 contenders. On the other hand, Paul came eighth or worse, and it was the penultimate place, which means that there were at least 9 boys taking part.
Answer: 9 boys took part in the long jump event.

44. During 8 hours, the father will have dug $\frac{8}{12} = \frac{2}{3}$ of the plot of land.
The son, however, in eight hours can dig $1 - \frac{2}{3} = \frac{1}{3}$ of the plot, i.e., the son works half as fast as his father. Hence the conclusion is that he will take $2 \times 12 = 24$ to dig the entire plot of land.
Answer: The son will take 24 hours to dig the plot.

45. Let the time for Daniel to cover a distance of one meter be denoted by unit d. Daniel will take $3d$ to creep the whole distance, and Sebastian $\frac{1}{2}d + 1d + 2d = 3\frac{1}{2}d$. This means that Daniel will win. When the winner crosses the finish line, Sebastian will have covered the first two sections in $\frac{3}{2}d$ and a portion of the third leg of the race in $\frac{3}{2}d$ at a speed of $\frac{1}{2}$ meter per time unit. He will have covered $\frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$ m of the final leg of the race, i.e., he will still have $\frac{1}{4}$ m to go to complete the race.
Answer: The winner will be Daniel, who will best his opponent by $\frac{1}{4}$ of a meter.

46. Since each child was given the same number of identical cakes, the brothers must have bought the same number of cakes of each kind as well. Three cakes (cream cake + fruit cake + doughnut) cost $1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$ dollars, i.e., Jeremy and Roger bought six sets of three cakes each. Six sets of three cakes can be shared by one, two, three or six kids.
We know for sure, however, that among the children were two boys, Jeremy and Roger, which means that in the group were also a minimum of two girls.
There were, therefore, at least four kids apart from Jeremy and Roger.
Answer: The group numbered 6 kids.

47. The amount the first boy gave did not exceed what the remaining two boys chipped in, so he gave no more than half of the whole sum, i.e., at most \$22.50. If the second boy paid a dollars, then the remaining two boys contributed at least $2a$, and so all the three boys gave at least \$$3a$ altogether. This means that \$$45 \geq 3a$, i.e., the second boy paid in \$$45 \div 3 =$ \$15, at most. Similarly, if the third boy gave \$$b$, then the remaining two added at least \$$5b$, i.e., all the boys gave at least \$$6b$. It follows from the above that $45 \geq 6a$, i.e., the third boy chipped in $45 \div 6 =$ \$7.50, at most.
Let's verify: \$22.50 + \$15 + \$7.50 = \$45, so if the first boy had given less than \$22.50, the second one less than \$15 and the third less than \$7.50, they wouldn't have collected \$45 altogether.
Answer: The first boy chipped in \$22.50, the second \$15, and the third \$7.50.

48. Ken runs up the escalator in the opposite direction of the movement of the stairs, so he moves at the same speed as his cap on the neighboring escalator. This means that Ken will be able to get hold of his cap only when he reaches the top of the escalator – precisely at the same moment as his cap finds itself there. To catch it, Ken must cover half the length of the stairs.
Will, on the other hand, to get to the top, must first cover half the length of the stairs, (at a speed three times as high as that of Ken, because he will be running in the same direction as the escalator), and then run up the whole length of the

neighboring stairs (in the same direction as the escalator, again three times as fast as Ken). He has to cover $\frac{3}{2}$ of the length of the escalator, i.e., three times as far as his friend, but three times as fast. Hence the conclusion is that the boys will meet at the top of the stairs at the same time as the cap.
Answer: The boys will reach the cap simultaneously.

49. By starting to run from the new line, Andrew has to cover a distance of 120 m. He takes as much time to cover 100 m as Dave to run 80 m. Hence the conclusion is that Andrew, beginning 20 meters earlier, will draw level with Dave after covering 100 m, and will overtake him in the last 20 m. Joe is wrong then.
Please note that covering 25 meters takes Andrew as much time as Dave needs to run 20 meters. This means that Andrew will cover the distance of 5×25 m $= 125 = 100 + 25$ m in the same time as Dave $5 \times 20 = 100$ m.
Answer: Joe is not right. The new line should be drawn 25 meters before the official starting line.

50.

Method I:
It follows from the table that Sophie (S), Adam (A), Michael (M) and Will (W) make up $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ of schoolchildren in Class Vb i.e. the class numbers $4 \times 6 = 24$ children.

Group 1		Group 2			
	S	A	M	W	
$\frac{1}{3}$					$\frac{1}{2}$

Method II:
Let's call x the number of schoolchildren in the first group. There are $3(x - 1)$ children in the class, because after Sophie left group 1 for group 2, in the first group remains $\frac{1}{3}$ of the class. We know as well that together with Adam, Michael and Will, the first group makes up half of the class, i.e., the number of schoolchildren in Class Vb is also equal to $2(x + 3)$. Therefore, $3(x - 1) = 2(x + 3)$, hence $x = 9$. The number of children at school is then equal to $3(x - 1) = 3 \times 8 = 24$.
Answer: Class Vb has 24 children.

51. The sum of the numerator and denominator for each of the listed fractions equals 2008. Among the fractions, there is also $\frac{1004}{1004} = 1$. We can, therefore, select three fractions, e.g., $\frac{2}{2006}$, $\frac{1004}{1004}$, $\frac{2006}{2}$ whose product is 1.
Answer: Yes, it is possible to find fractions satisfying the set conditions.

52. Let's calculate:

$\frac{124\,124}{421\,421} = \frac{124 \times 1000 + 124}{421 \times 1000 + 421} = \frac{124 \times 1001}{421 \times 1001} = \frac{124}{421}$,

$\frac{1240\,124}{4210\,421} = \frac{124 \times 10\,000 + 124}{421 \times 10\,000 + 421} = \frac{124 \times 10\,001}{421 \times 10\,001} = \frac{124}{421}$.

Answer: The numbers listed above are equal.

53. We have:

$\frac{1}{(10 \times 11)} = \frac{(11-10)}{(10 \times 11)} = \frac{1}{10} - \frac{1}{11}$, $\frac{1}{(11 \times 12)} = \frac{(12-11)}{(11 \times 12)} = \frac{1}{11} - \frac{1}{12}$, ..., $\frac{1}{(19 \times 20)} = \frac{(20-19)}{(19 \times 20)} = \frac{1}{19} - \frac{1}{20}$.

Hence:

$\frac{1}{(10 \times 11)} + \frac{1}{(11 \times 12)} + \dots + \frac{1}{(19 \times 20)} = (\frac{1}{10} - \frac{1}{11}) + (\frac{1}{11} - \frac{1}{12}) + (\frac{1}{12} - \frac{1}{13}) + \dots + (\frac{1}{18} - \frac{1}{19}) + (\frac{1}{19} - \frac{1}{20}) =$
$= \frac{1}{10} + (-\frac{1}{11} + \frac{1}{11}) + (-\frac{1}{12} + \frac{1}{12}) + \dots + (-\frac{1}{19} + \frac{1}{19}) - \frac{1}{20} = \frac{1}{10} - \frac{1}{20} = \frac{1}{20}$.

54. Let's have a look at the number of watermelons just before the appearance of the last buyer. If Ms. Cindy had bought only half of the remaining watermelons, Catherine would have been left with a melon and a half. So Catherine had 3 watermelons before Ms. Cindy arrived. Similarly, before Ms. Barbara's arrival, she had $2 \times (3 + 0.5) = 7$ watermelons, and before Ms. Angela showed up, she had $2 \times (7 + 0.5) = 15$ watermelons. Catherine sold then 14 melons, earning $14 \times 2 = \$28$.

Answer: Catherine's takings amounted to $28.

55. Let's define by x the initial number of rabbits. The first customer bought ($\frac{1}{6}x$ + 1) bunnies, so the breeder was left with ($\frac{5}{6}x - 1$). The second customer bought ($\frac{1}{6}$ ($\frac{5}{6}x - 1) + 2$) rabbits, but we know that he purchased as many bunnies as the previous buyer. Therefore, $\frac{1}{6}(\frac{5}{6}x - 1) + 2 = \frac{1}{6}x + 1$, hence $(\frac{5}{6}x - 1) + 12 = x + 6$, i.e., $5 = \frac{1}{6}x$.
In the beginning, the breeder must have had 30 rabbits.
Let's check now whether all the customers bought the same number of bunnies.
The first customer bought $\frac{1}{6} \times 30 + 1 = 6$ rabbits – there remained 24 animals.
The second customer bought $\frac{1}{6} \times 24 + 2 = 6$ rabbits – there remained 18 animals.
The third customer bought $\frac{1}{6} \times 18 + 3 = 6$ rabbits – there remained 12 animals.
The fourth customer bought $\frac{1}{6} \times 12 + 4 = 6$ rabbits – there remained 6 animals.
The fifth customer bought $\frac{1}{6} \times 6 + 5 = 6$ rabbits – there remained 0 animals.
We see then that each customer bought the same number of rabbits (six).
Answer: The rabbit breeder had brought to the market 30 animals and had 5 customers that day.

56. Consecutive sheep eat hay at a rate of 1, $\frac{1}{2}$, $\frac{1}{3}$, ... , and $\frac{1}{8}$ portion per day. Thus the first two sheep eat $1 + \frac{1}{2} = \frac{3}{2}$ portions of hay daily, whereas the remaining six eat $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$ portions. We have $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} +$

$\frac{1}{7} + \frac{1}{8} = (\frac{1}{3} + \frac{1}{6}) + (\frac{1}{4} + \frac{1}{5} + \frac{1}{7} + \frac{1}{8}) < \frac{1}{2} + (\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}) = \frac{3}{2}$ (note: we have used obvious inequalities: $\frac{1}{5} < \frac{1}{4}$, $\frac{1}{7} < \frac{1}{4}$, $\frac{1}{8} < \frac{1}{4}$). This means that the last six sheep eat hay at a smaller total rate than the first two animals.
Answer: The first two sheep will eat their hay faster.

57. Let's assume that at first Michael had d dollars and c cents on him, i.e., $100d + c$ cents. It follows from the problem that c is an even number, and $c < 100$. After a quarter of an hour, Michael had $100 \times \frac{c}{2} + d$ cents, which equaled half of the initial sum. Thus $100 \times \frac{c}{2} + d = \frac{100d + c}{2}$, i.e., $= 99c = 98d$.
Since 99 and 98 are relatively prime numbers, 99 is a divisor of d and 98 is a divisor of c. We know that c is a natural number smaller than 100, hence $c = 0$ or $c = 98$. For $c = 0$, we obtain $d = 0$ as well. From the problem, we know, however, that Michael had some money on him since he spent half of it, and that is why we reject this case as impossible.
What remains is $c = 98$, and then from equation $99c = 98d$, we obtain $d = 99$.
Let's check: If Michael came to the market with \$99.98, then after a quarter of an hour, he was left with half of it, i.e., \$49.99. Sure enough, after a quarter of an hour, the number of cents (99) equals the number of dollars in the beginning, and the number of dollars (49) is now half of the initial number of cents (98).
Answer: Initially Michael had \$99.98 on him.

58. If we express by D, M, J, and P the number of years of Dave, Mark, Jack, and Paul, respectively, we will be able to form equalities: $J = M + 4 = D + 8$, and $M \times P = J \times D + 16$. From the first equality, we have $J = M + 4$, and $D = M - 4$. Substituting these in the equation $M \times P = J \times D + 16$, we obtain $M \times P = (M + 4) \times (M - 4) + 16$, i.e. $M \times P = M^2$. We obtain the equality $M(P - M) = 0$, from which it follows that $P = M$, because M cannot equal zero. Thus the twins are Mark and Paul. No other pair of boys can be twins, because $J > M > D$, $J > P$ (it results from $P = M$), and $P > D$.
Answer: In the foursome mentioned above, the twins are Mark and Paul.

59. We designate the sum written on the check – d for dollars and c for cents, where c is a natural number smaller than 100. Let's express the amount on the check in numbers of cents and as follows: $(100d + c)$. But the cashier paid out instead $(100c + d)$ cents. It follows from the contents of the problem that d is also a natural number smaller than 100. We also know that the sum paid out and diminished by 5 cents equals $2(100d + c)$. We obtain then the equation $100c + d - 5 = 2(100d + c)$, hence $98c - 5 = 199d$, i.e., $98(c - 2d) = 5 + 3d$.
The left-hand side of the last equation is divisible by 98, which means that the right-hand side must also be divisible by this number. Since $d \leq 99$, $5 + 3d \leq 302$, the only possible solutions are: $5 + 3d = 0$, or $5 + 3d = 98$, or $5 + 3d = 2 \times 98$,

or $5 + 3d = 3 \times 98$. Solving these equations, we will note that we obtain a non-finite value d in all cases, but the second. That is why the only correct solution is the second case, i.e., $5 + 3d = 98$, hence $d = 31$. On this basis and from the equation $98(c - 2d) = 5 + 3d$, we calculate $c = 63$.
Answer: Michael's check amounted to $31.63.

60. The number of pairs is even, i.e., the total number of pupils is divisible by 4. Therefore, the number of girls can equal 2, 6, or 10. The number of twosomes consisting of a boy and girl equals the number of all the children divided by 4, i.e., it is 4, 5, or 6.
a) If there were only 2 girls, we could not possibly have as many as 4 mixed pairs.
b) If there were 6 girls, then we would have 5 such pairs, which means that the sixth girl would also have to form a pair with a boy and that would make six mixed pairs, i.e., more than any other remaining pairs.
c) If there were 10 girls, then there would be six mixed pairs. The remaining six twos would comprise two pairs of girls and four pairs of boys.
Answer: Class Va has 24 schoolchildren.

61. Let x be the age of Anna in years. Since $24 = 2 \times 12$, Anna was 12 years old when Maria was as old as Anna is now. This simply means that Anna was 12 when Maria was x years of age. Now Anna is x years old, and Maria is 24. Since the difference in their ages remains constant, $12 - x = x - 24$; hence $x = 18$.
Answer: Anna is 18 years old.

62. Let's mark by x the present age of granny and by y the age of grandpa. We know from the problem that $(y - x)$ years ago grandpa was as old as granny is now. Before $(y - x)$ years, granny was $x - (y - x)$ years old. Grandpa is now twice as old as Granny was then, i.e., the present age of grandpa is $2(x - (y - x))$ years. Therefore, $y = 2(x - (y - x)) = 2(2x - y) = 4x - 2y$, i.e., $3y = 4x$.
We also know that the sum of their ages is 140 years, so $x + y = 140$, and hence $y = 140 - x$. Substituting the latter in the equation $3y = 4x$, we obtain $3(140 - x) = 4x$, so $3 \times 140 = 7x$, i.e., $x = 3 \times 20 = 60$.
Answer: Granny is now 60 years old.

63. Let's assume that j, a and f denote the number of mushrooms collected by Joe, Alex, and Frank, respectively. We have $j + a = 3f$ and $a + f = 5j$. After adding the respective two sides together, we obtain $f + j + 2a = 3f + 5j$, hence $2a = 2f + 4j$, i.e., $a = f + 2j$. We assume at the same time that $j > 0$, because otherwise it would follow from the equation $a + f = 5j$ that nobody found a single mushroom.
Therefore, $a = f + 2j > f + j$.

Answer: Alex found more mushrooms than Joe and Frank together.

64. When Len loses, we multiply the sum of the money he has by $\frac{1}{2}$, and when he wins by $\frac{3}{2}$. Since this is both a commutative and associative multiplication, the sequence of Len's wins and losses does not make a difference. After four wins and three losses, he will be left with a sum of money equal to the initial amount multiplied by $(\frac{3}{2})^4$ and multiplied by $(\frac{1}{2})^3$,
i.e., $32 \times (\frac{3}{2})^4 \times (\frac{1}{2})^3 = 32 \times \frac{81}{128} = \frac{81}{4} = 20\frac{1}{4}$.
Answer: Len has 20 dollars and 25 cents now.

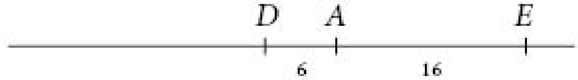
65. Let's assume that each digit occurs in this number p times. Then, the sum of digits in this number equals $(0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) \times p = 45p$ and is divisible by 3. Therefore, the whole number is divisible by 3, i.e., it cannot be a power of 2.
Answer: The number in question cannot be a power of 2.

66. We assume that the radius of the hole in the middle of the wheel is our unit of measurement, i.e., $u = \frac{11}{7}$ in. Then, the radius of the wheel is $11 = 7 \times \frac{11}{7}$ in, i.e., $7u$. The area of the wheel with a radius of $7u$ and a hole cut out in the centre with the radius u equals $\pi \times 7^2 - \pi \times 1^2 = 48\ \pi(u^2)$. Let's mark by R the new radius of the used wheel expressed in units u when the grinding wheel is handed over to Michael. The area of the wheel with radius R which has an opening of radius 1 cut out in its center equals half of $48\pi(u^2)$, i.e., $\pi(R^2 - 1) = 24\ \pi(u^2)$, hence $R^2 = 25(u^2)$, i.e., $R = 5u$. So the diameter of the wheel when it goes to Michael will be equal to $2 \times 5u = 2 \times 5 \times \frac{11}{7} = \frac{110}{7} = 15\frac{5}{7}$ in.
Answer: The diameter of the grinding wheel will equal $15\frac{5}{7}$ inches.

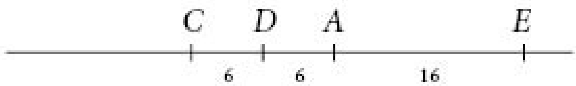
CHAPTER 4

GEOMETRY

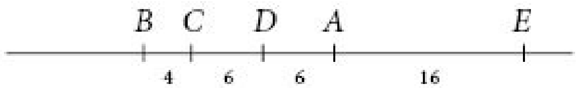
67. Since the distance between D and E is the same as the sum of distances between D and A, and between A and E, we infer that A lies between D and E. Let's mark three points A, D, and E on the straight line:



D and *C* are 6 miles apart, just like *D* and *A*, hence *C* and *A* must lie on opposite sides of *D*.

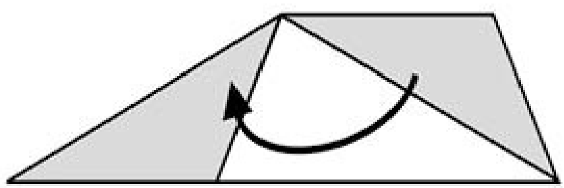


Finally, the distance between *A* and *B* is 16 miles, i.e., the same as between *A* and *E*. We infer that *B* and *E* lie on opposite sides of *A*, while *B* and *C* are 16 – 12 = 4 miles apart.

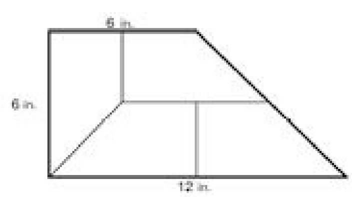


Now we can read out the answer from the graph.
Answer: The villages in question are located in the following sequence: *BCDAE*, or when driving from the opposite direction, *EADCB*.

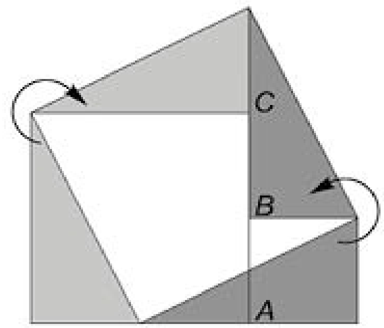
68. We divide the trapezoid along one of its diagonals – see figure below.



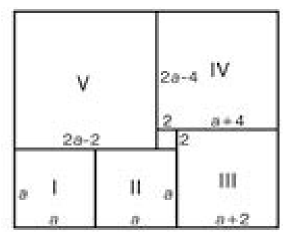
69.



70. The way to partition the shape is shown in the provided figure. It might be of some help to calculate the side length of the newly formed square. If the side length of the small square equals 1, then the side length of the big square is 2, i.e., their total area equals 5. Hence the conclusion is that the side length of the square obtained after putting the three shapes together must be $\sqrt{5} = \sqrt{2^2 + 1^2}$, i.e., the same as the hypotenuse of a right triangle with the sides measuring 1 and 2.



71. We denote by a the side length of square I (see figure beside). Then, the side length of square II also equals a, so the side length of square III is $a + 2$, and square IV $a + 4$. We also know that the side length of square V equals $2a - 2$, so the side length of square IV is equal to $2a - 4$. Comparing the side length of square IV calculated in two ways, we obtain $a + 4 = 2a - 4$, hence $a = 8$.



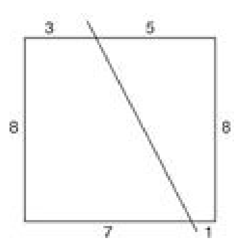
So the side lengths of the whole rectangle are $3a + 2$ and $3a - 2$. Its area equals, therefore, $(3a + 2)(3a - 2) = 9a^2 - 4 = 9 \times 64 - 4 = 572$.

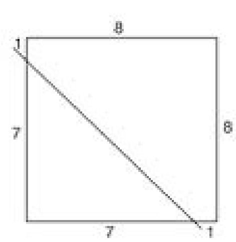
Answer: The area of the rectangle is 572 in².

72. Pentagon *BCD'EF* formed of a folded sheet of paper was painted bilaterally, i.e., 24 in² of paper, altogether. After unfolding we have 40 in² of paper (two sides measuring 20 in² each), i.e., the white part takes up $40 - 24 = 16$ in².

Answer: The area of the white part of the rectangle equals 16 in².

73. The line dividing the perimeter of the square in the ratio of 9:7 and dividing one of the sides (call it side *AB*) in the ratio of 7:1, can cut the square in two ways, depending on whether the longer segment of side *AB* belongs to the bigger (a) or the smaller (b) part of the square's perimeter (see figure).

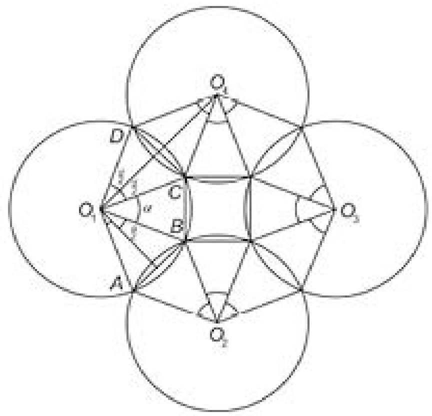
a) 

b) 

In case (b), the line divides the second (left vertical) side of the square in the ratio of 7:1, and not 5:3. This means that the line must run as shown in figure a). The area of the trapezoid on the left-hand side of the line equals $\frac{8(3+7)}{2} = 40$, and the area of the trapezoid on the right $\frac{8(5+1)}{2} = 24$, i.e., the line divides the area of the square in the ratio of 40:24.

Answer: The line divides the area of the square in the ratio of 5:3.

74. Let's mark the centers of the circles by O_1, O_2, O_3, O_4, and intersection points of the circle with center O_1 with circles with centers O_2 and O_4 by A, B, C, D (see figure).



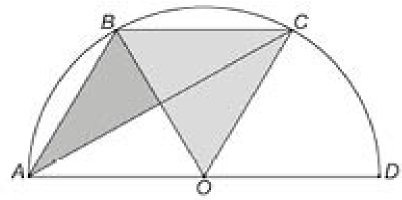
Let's denote by α the magnitude of angle $\sphericalangle AO_1B$. Angles $\sphericalangle BO_1C$ and $\sphericalangle CO_1D$ also have the same α magnitude because they are central angles on the same circle, subtending an arc of the same length (3 in.). Since the circles are congruent, and their shorter arcs are of the same length, all the remaining nine angles marked in the figure also have α magnitude.

Quadrilateral O_1AO_2B is a rhombus (the length of each of its sides is equal to the radii of the circles), so its diagonal O_1O_2 divides angle $\sphericalangle AO_1B$ into two angles of identical magnitude equal to $\frac{\alpha}{2}$. It is just like in the case of rhombus O_4DO_1C and the two remaining ones. Let's now consider quadrilateral $O_1O_2O_3O_4$. The angle at vertex O_1 has a magnitude equal to $\frac{\alpha}{2} + \alpha + \frac{\alpha}{2} = 2\alpha$, just as the remaining vertex angles of the quadrilateral. This means that quadrilateral $O_1O_2O_3O_4$ is a square, so $2\alpha = 90°$. Hence the circumference of each circle equals

$3 \times (\frac{360°}{\alpha}) = 3 \times (\frac{360°}{45°}) = 3 \times 8 = 24.$

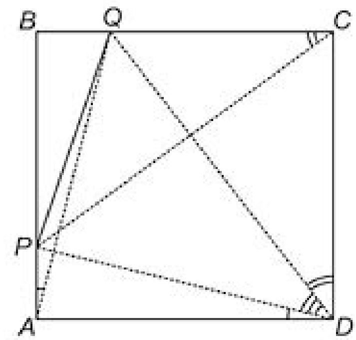
Answer: The circumference of each circle equals 24 in.

75. Arcs AB and BC are of the same length, so B and C are equidistant from the diameter AD. Hence segments BC and AD are parallel. Let's mark by O the midpoint of segment AD. Triangles BCA and BCO share a common side: BC; the heights of these triangles lowered on this side are of the same length (because AD and BC are parallel). This means that the area of these triangles is the same (see figure below).

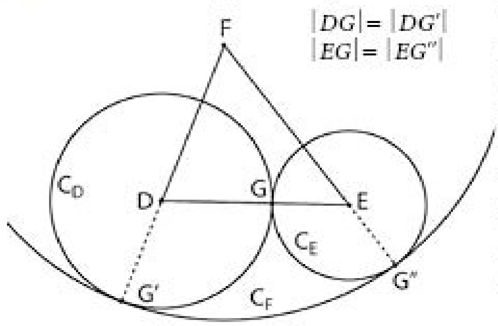


The area to be found is then equal to the area of the circular sector BCO, i.e., $\frac{1}{6}$ of the area of the whole circle. Hence we calculate: $\frac{1}{6} \times 10^2 \pi = \frac{50\pi}{3}$ (in^2).
Answer: The area of the curvilinear triangle ABC equals $\frac{50\pi}{3}$ (in^2).

76. Since $|PB|+|BQ| = |AB|=|PB| +|AP|$, so $|BQ| = |AP|$. This means that the rectangular triangles QBA and PAD have their respective legs of the same length:
$|BQ| = |AP|$ and $|AB|= |AD|$, i.e., they are congruent, hence $|\sphericalangle PAQ|= |\sphericalangle ADP|$. Similarly triangles PBC and QCD are congruent, so $\sphericalangle PCQ|=|\sphericalangle QDC|$. Therefore, $|\sphericalangle PAQ| + |\sphericalangle PDQ| + |\sphericalangle PCQ|= |\sphericalangle ADP| + |\sphericalangle PDQ| + |\sphericalangle QDC|= =|\sphericalangle ADC| = 90°$.



77. Let's mark by G the point of contact of the two circles C_D and C_E (see figure).

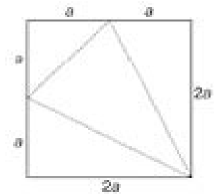


We see that radius R of circle C_F equals $R =|FD| + |DG|$, and on the other side, $R = |FE| + |EG|$.
Hence $2R = |FD| + |DG| + |FE| + |EG| = =|FD| + (|DG| + |EG|) + |FE| = 30$ in, i.e., R = 15 in.
Answer: The radius length of circle C_F= 15 in.

78. Let's mark by α, β, and γ the angles of this triangle. We have by design
$\alpha < \beta + \gamma$, hence $2\alpha < \alpha + \beta + \gamma = 180°$, i.e., $\alpha < 90°$. Similarly, we demonstrate that $\beta < 90°$ and $\gamma < 90°$, from which we conclude that this is an acute triangle. If the triangle is acute-angled, then each of its angles is smaller than 90°, so the sum of each of its two angles is greater than 90°. Therefore, each acute triangle satisfies the condition of the problem. This means that for a triangle meeting the condition of the problem, we cannot say anything other than it is acute-angled.

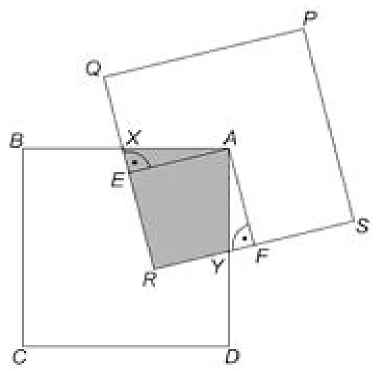
Answer: This triangle is acute-angled.

79. Yes, it is possible (to obtain a tetrahedron by folding a square sheet of paper along the dotted lines) – see figure beside.



Answer: Yes, the figure in question can be a square.

80. Let's mark by E and F the midpoints of the sides of square $PQRS$ which is intersecting with square $ABCD$, and by X and Y the intersections of the sides of both squares (see figure below).



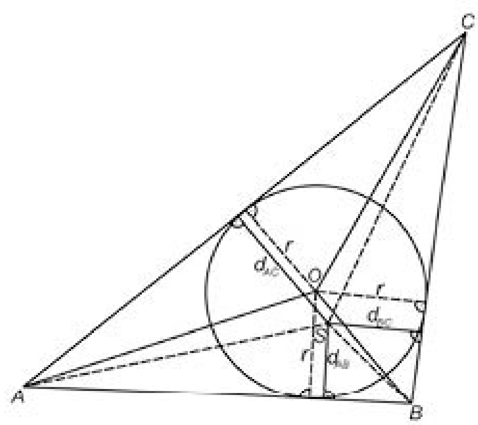
Then, angles $\sphericalangle XEA$ and $\sphericalangle YFA$ are right angles, and the magnitudes of angles $\sphericalangle XAE$ and $\sphericalangle YAF$ are equal, because

$\sphericalangle XAE = 90° - \sphericalangle EAY = \sphericalangle YAF$.

Moreover, segments AE and AF are equal in length. Therefore, triangles XAE and YFA are congruent by virtue of the rule of congruency (angle-side-angle), and thus they have the same area. So the area of the shaded part equals the area of square $ERFA$, and the area of this square is equal to $\frac{1}{4}$ of the area of square $PQRS$, i.e., $\frac{1}{4} \times 100$ in^2 = 25 in^2.

Answer: The common area shared by both *PQRS* and *ABCD* squares equals 25 in^2.

81. The farthest point from the sea is point O, which is the center point of the circle inscribed in the triangular island. Let's mark the radius of this circle by r. Intuitively, it is quite obvious that you cannot fit a bigger circle in the triangle than the one with radius r already inscribed.



Here is the proof:

Let's assume that a certain point S within the triangle lies farther from all its sides than point O.

Let d_{AB}, d_{BC}, and d_{AC} be the distances of sides AB, BC, and CA (respectively) from point S.

By assumption, $d_{AB} > r$, $d_{BC} > r$ and $d_{AC} > r$.

Then, $P_{\Delta ABC} = P_{\Delta AOB} + P_{\Delta BOC} + P_{\Delta AOC} =$

$= \frac{(|AB| \times r + |BC| \times r + |CA| \times r)}{2} = \frac{(r \times (|AB| + |BC| + |CA|))}{2}$.

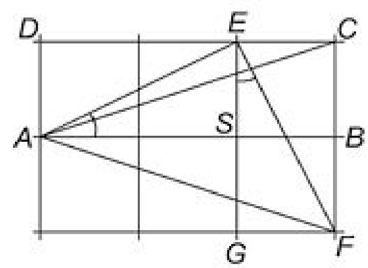
On the other side, we also have:

$P_{\Delta ABC} = P_{\Delta ASB} + P_{\Delta BSC} + P_{\Delta ASC} =$

$= \frac{(|AB| \times d_{AB} + |BC| \times d_{BC} + |CA| \times d_{AC})}{2} > \frac{(|AB| \times r + |BC| \times r + |CA| \times r)}{2} = \frac{r \times (|AB| + |BC| + |CA|}{2}$, which is in contradiction with the previous equality. This is consequent with the assumption

that in triangle ABC, there is a point S distant from each of the sides of the triangle by more than r.
Answer: The farthest from the sea lies the point which is the center of the circle inscribed in the triangle representing the island.

82. We name F, G, and S additional intersection points on the grid – see figure beside.



Angles $\sphericalangle BAC$ and $\sphericalangle BAF$ are equal, so
$|\sphericalangle BAC| + |\sphericalangle BAE| = |\sphericalangle BAF| + |\sphericalangle BAE| = |\sphericalangle \mathrm{EAF}|$.
Triangles ASE and EGF are congruent (they are right triangles, and the lengths of adjacent legs are 1 and 2),
so $|\sphericalangle AEF| = |\sphericalangle AES| + |\sphericalangle GEF| = |\sphericalangle AES| +$
$+ |\sphericalangle SAE| = 180° - |\sphericalangle ASE| = 90°$.
Moreover, $|AE| = |EF|$, i.e., triangle AEF is isosceles, and its vertex angle E is a right angle. Therefore, $|\sphericalangle EAF| = 45°$.
Answer: The sum of angles $\sphericalangle BAC$ and $\sphericalangle BAE$ equals 45°.

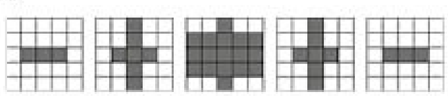
CHAPTER 5

GAMES, LOGICAL TESTS AND OTHERS

83. It is possible, e.g., Anne Smith (a brunette), Mary Smith (a blonde), Anne Newman (a blonde), and Mary Newman (a brunette).
Answer: It is possible to satisfy the set conditions.

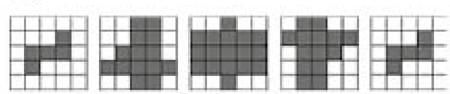
84. Once the successive layers of the hexahedrons are drawn layer by layer (starting from the layer at the base), we count the cubes – see figures below.

a)



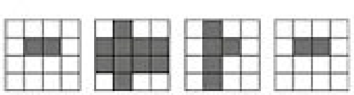
88 cubes

b)



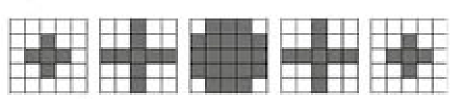
70 cubes

c)



45 cubes

d)

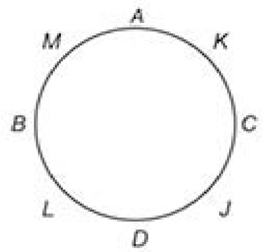


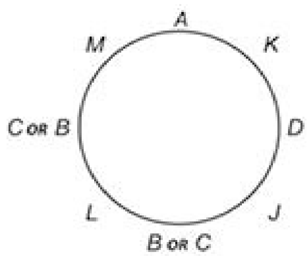
76 cubes

Answer: The hexahedrons consist of: a) 88 cubes, b) 70 cubes, c) 45 cubes, and d) 76 cubes.

85. Matthew was sitting to the right of Agatha, so Kevin must have been sitting to her left side. To the left of Kevin might have been sitting either Celine or Daphne (we do not take Barbara into account as she is Matthew's wife).
Let's consider both cases:

a) If it was Celine sitting to the left of Kevin, then to her left must have been John (Kevin and Matthew are already sitting elsewhere, and Leon is Celine's husband). We know that Daphne was also sitting next to John, so she must have taken a seat to his left. The remaining two seats were occupied by Leon and Barbara (see figure beside), which means that to the right of Barbara was Leon.





b) If it was Daphne sitting to the left of Kevin, then, per the guidelines of the problem, to her left side sat John. The next three seats were occupied by: Barbara, Celine, and Leon; Leon was sitting between Barbara and Celine (otherwise the two ladies would have been sitting next to each other) – see figure on the left.
In that case, however, Celine would have then been neighboring on her husband, Leon. This case is, therefore, rejected as being impossible.

Answer: Leon sat to the right of Barbara.

86. We must leave the marks 0 and 11 in., for otherwise it would not be possible to measure out 11 inches with a ruler. Leaving in only the other marks, e.g., a, b, and c inches, where $a < b < c$, we could measure distances a, b, c, 11, $b - a$, $c - a$, $11 - a$, $c - b$, $11 - b$, and $11 - c$ (some of these distances can be equal), i.e., we could measure ten different distances from 1 to 11 in, at most. Among them there would be at least one integer-type distance from 1 to 11 in, which could not be measured. That is why we need to leave in at least four marks, apart from the two denoting 1 and 11 in.
Such a ruler can easily be found using a trial and error method:

Answer: You can remove six marks, at most.

87. Let's assume that Dorothy is telling lies. Then, the remaining girls would be

telling the truth, and none of them would be the youngest, which is, of course, impossible. It is Dorothy, then, who is telling the truth, and she really is the youngest.

If Annie is telling the truth, then she is the eldest, but that would mean that all the girls are telling the truth, which is in contradiction with the conditions stated in the problem. So it is Annie who is lying, and the remaining girls are telling the truth. Hence we conclude that the eldest is Celine.

Answer: Dorothy is the youngest, and Celine is the eldest.

88. Since the inhabitants of each village begin their conversation with a true sentence, the information in the first sentence said by the caller notifying the fire brigade of the fire is true. Let's check all the possibilities:

a) If the caller was from Aden, then all his answers are true. It would follow from the second sentence that the fire had broken out in Aden, whereas from the third, that the village of Caden was on fire. This is of course impossible, because the fire had broken out in only one village.

b) If the call was from Baden, then from the second answer, we conclude that no fire had broken out either in Baden or Caden (answer 3). So the fire must have broken out in Aden.

c) If the call was made by an inhabitant of Caden, then we know from the second answer that the fire had started in some other village than Caden; however, from the third answer, it follows that Caden was on fire, which is a blatant contradiction.

Answer: The fire was reported by an inhabitant of Baden. The duty officer should direct the fire engine to Aden.

89. The gang boss is not standing next to Julian. If this was the case, then Julian would have told a lie only once, while answering question 2 or 3. So neither John nor Igor can be the gang leader. Julian is not the boss either; if he were, he would have then given four false answers.

Neither is James the gang boss, for if he were, he would not have told a lie even once. Last but not least, Peter cannot be the boss either, because if he were, he would have given false answers to the first three questions. However, if David is not the boss, then among the six captured criminals would not have been their leader, but in that case Julian would have given false answers to the last three questions, which is not true. It is clear then that David is the boss.

Answer: David is the gang leader.

90. It is not true. For instance, in the set 1, 2, 3, 4, 5, and 10, there are no such two numbers whose sum or difference is divisible by 10.

Note: If there were seven numbers, the answer would be affirmative. In fact, if among them are two numbers ending with the same digit, then their difference ends with a

zero, i.e., their difference is divisible by 10. Let's assume that these numbers end with various digits. At the very most, two of them can end with digits 0 or 5, so at least five numbers will end with a digit from the set {1, 2, 3, 4, 6, 7, 8, and 9}. Let's divide this set of digits into four subsets: {1, 9}, {2, 8}, {3, 7}, and {4, 6}. Since there are at least five numbers, and they end with different digits, this means that some two numbers end with digits belonging to the same (two-element) subset of numbers. The sum of these two numbers ends with zero, so it is divisible by 10.

91. It is enough to take out a ball from the box marked (W, G). In this box, due to Anne's carelessness, there are balls of the same color; there are two possibilities, then:
a) In the box marked (W, G), there are two white balls. In that case, a pair of green balls is in the box marked (W, W) – they cannot be in the (G, G) box, because in such a case they would be in their right place – and the two-color ones in the box marked (G, G).
b) In the box marked (W, G), there are two green balls. In that case, the pair of white balls is in the (G, G) box, while the white ball with the green one are in the (W, W) box.

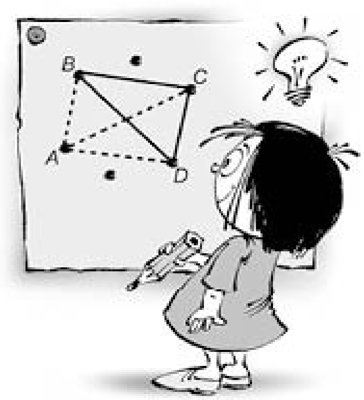
92. We have: $300 = 100 + 100 + 100 = (a_1 + a_2 + a_3 + \ldots + a_{50}) + (a_1 + a_2 + a_3 + \ldots + a_{50}) + (a_1 + a_2 + a_3 + \ldots + a_{50}) = (a_1 + a_2 + a_3) + (a_4 + a_5 + a_6) + \ldots + (a_{46} + a_{47} + a_{48}) + (a_{49} + a_{50} + a_1) + (a_2 + a_3 + a_4) + \ldots + (a_{47} + a_{48} + a_{49}) + (a_{50} + a_1 + a_2) + (a_3 + a_4 + a_5) + \ldots + (a_{48} + a_{49} + a_{50})$.
On the right-hand side, there are 50 elements (each of them being the sum of three numbers). If each of the terms was smaller than 6, their sum would be smaller than $50 \times 6 = 300$, so it would not equal 300. Therefore, one of the terms equals at least 6. The term is, of course, the sum of three numbers among $a_1, a_2, a_3, \ldots$, and a_{50}.
Answer: Yes, there must be three such numbers.

93. In the basket, there cannot be 12 mushrooms other than ceps – choosing any 12 mushrooms, we would not find a cep among them. Therefore, there may be no more than 11 mushrooms other than ceps in the basket. Hence we conclude that there are at least 30 – 11 = 19 ceps.
Thinking along the same lines, the basket cannot contain 20 mushrooms other than brown ring boletuses, and picking out 20 mushrooms, we must find at least one brown ring boletus among them. So there can only be at most 19 mushrooms other than brown ring boletuses in the basket, hence the conclusion that there are at least 30 – 19 = 11 brown ring boletuses in the basket.
There are 30 mushrooms altogether. Among them there are at least 19 ceps and at least 11 brown ring boletuses; hence we know that the girls have gathered 19 ceps and 11 brown ring boletuses.

Answer: There are 19 ceps in the basket.

94. After drawing 25 balls, we will be left with 5 balls in the box. If we take 25 balls out of the box in such a way that all the remaining ones will be white, among the 25 drawn ones will be at least three white balls. Therefore, there are at least 5 + 3 = 8 white balls. (It is of course possible to draw 25 balls in such a way as to have only white ones left in the box, because if there were fewer than five white balls, it might happen that all of them would remain in the box, and then, among the 25 balls drawn, there would be no single white one).
Similarly, there are at least 5 + 5 = 10 blue balls and 5 + 7 = 12 black ones. Since 8 + 10 + 12 = 30, this means that there are 8 white balls, 10 blue balls, and 12 black ones.
Answer: There are 8 white balls, 10 blue balls, and 12 black ones.

95. From one point, be it *A*, come out five segments. Among them are at least three having the same color, e.g., blue. Let's mark the ends of these segments by *B*, *C*, and *D*. If any of the segments *BC*, *CD*, or *DB* happens to be blue (e.g., *BC*), we will obtain a triangle with blue sides (in our example, this will be triangle *ABC*). If, however, all three segments *BC*, *CD*, and *DB* are red, then triangle *BCD* has all its sides red. This means that the newly formed triangle will have its sides in the same color.



Answer: Sophie did not perform her task and did not receive the prize.

96. Let's mark the glasses with arrows: ↑ will denote a glass standing stem side up, while ↓ will mean a glass standing stem side down. The initial line-up was as follows: ↑↓↑↓↑. The first player will win if he/she turns glass 5. It will lead to the following arrangement: ↑↓↑↓↓. Now the second player has three possible moves:
a) and b) To turn one of the two ↑ glasses, but then the other player may win instantly, turning the remaining ↑ glass.
c) To turn glasses number 2 and 3, leading to the line-up ↑↑↓↓↓.
Again the first player wins, turning glasses number 1 and 2.
Answer: Yes, the person starting the game can always win.

97. By writing the digit 8, the first player forces the second to write the digit 9. Therefore, the first player can keep on writing an 8, compelling the second to the 9 – as a result the players will form the following number 898,989,898,989. The sum

of the digits of the number equals $6 \times 8 + 6 \times 9 = 6 \times 17$ and is divisible by 3, i.e., the newly formed number is also divisible by 3. So the person beginning the game will always win by writing an 8.
Note: It can be proven that the player beginning the game with a digit other than an 8 will invariably lose (if of course his/her opponent takes advantage of it).
Answer: The person beginning the game has a winning strategy.

98. It would be worth considering the final moments of the game. When one of the players writes a number from 90 to 99, the second one can win at once, writing 100. So if one of the boys writes 89, his opponent must add a number found in the range of 90 and 99, after which, as we already know, the first contender is going to win. Thus, writing the number $89 = 100 - 11$ guarantees a victory.
Similarly, if one player chooses the number $78 = 89 - 11$, he automatically compels his opponent to pick a number from 79 to 88, and by doing so wins after writing 89. In other words, the choice of the number 78 ensures (with a proper strategy adopted, that is) a win. In order to win, it is enough to always write numbers different from 100 by a certain multiple of 11. That is why Adam, who begins, will win, if – in his first move – he writes the number 1 ($= 100 - 9 \times 11$), and in his next moves, he chooses the numbers 12, 23, 34, 45, 56, 67, 78, 89, and 100. He will always be able to write them, irrespective of what his opponent does.
Answer: The starter always has a winning strategy – in order to win he should begin with the number 1, and then successively 12, 23, 34, 45, 56, 67, 78, 89, and 100.

99. Apparently, the game will end with the victory of one of the players, because the number of matches in the box decreases with each move.
Let's call the position in the game a position of type D where the number of matches in the box is divisible by 3. All other positions will be denoted by I. Please note that in a position of type D it is impossible to win in one go, for one cannot take out of the box a number of matches divisible by 3.
Each move in the game from a position of type D leads to a position of type I. Moreover, from position I, one can reach position D in one go, taking out one or two matchsticks and leaving in the box a number of matches divisible by 3.
a) If the initial number of matches in the box is 48, Player 2 has a winning strategy. The initial position is of type D, so after whichever move Player 1 makes, it changes into the position of type I. Now Player 2 reinstates position D, etc. Player 1 must, with every move he makes, restore position I, to which Player 2 responds with a move that brings back position D. With such a strategy adopted by Player 2, the first one will never win, because he always makes moves in position D, and as we have already established, one cannot win in one go moving from this position. Since the game ends with the win of one competitor, the winner will be Player 2.

b) If the box initially contains 49 matches, Player 1 will win by taking out one matchstick and using the strategy described in point a) since Player 2 will be making his first move with 48 matches in the box. As we have shown above, this is a losing position.
Answer: If there are 48 matches in the box, Player 1 has no winning strategy. If, however, the box contains 49 matchsticks, the starter does have a winning strategy – he should take out but one matchstick.

100. Mark will win if he puts the first tile on squares 4 and 5 (or symmetrically on squares 9 and 10). Then, the tape will be divided into two parts – the left-hand one will fit exactly one tile, regardless of what the players do:

1	2	3	4	5	6	7	8	9	10	11	12	13

In his second move, Mark must make sure that just three tiles can fit in the right-hand part – then he will win. He can, for instance, proceed as follows:
a) If Daniel leaves squares 9 and 10 free (e.g., he puts the tile on squares 12 and 13), Mark will in his second move put the tile on squares 9 and 10. There will remain free spots for two tiles: One will be taken by Daniel, and the other, by Mark, thanks to which he will win:

1	2	3	4	5	6	7	8	9	10	11	12	13

b) If Daniel puts the tile on spots 8 and 9 (or 10 and 11), Mark will cover squares 11 and 12 (or 7 and 8). Again there will remain empty spots for two tiles, so Mark will win once more:

1	2	3	4	5	6	7	8	9	10	11	12	13

c) If Daniel covers squares 9 and 10, there will remain enough room for three tiles.

1	2	3	4	5	6	7	8	9	10	11	12	13

Mark can now put the tile on any free square – he will win no matter what Daniel does. If the tape consisted of 14 squares, Mark would also have a winning strategy. It would suffice if he put the first tile on the two central spots.

1	2	3	4	5	6	7	8	9	10	11	12	13	14

From now on, he should place the tiles symmetrically to Daniel's moves. For example, if Daniel were to put a tile on squares 2 and 3, Mark would cover squares 12 and 13:

1	2	3	4	5	6	7	8	9	10	11	12	13	14

Playing in this way, Mark will always be able respond to Daniel's move, so as not to lose. Since the game ends when all the empty spots have been covered, someone must lose, and this will be Daniel.

Answer: In both cases Mark, who begins the game, has a winning strategy.

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