

Substitution & Formulae

In mathematics, engineering and science, formulae are used to relate physical quantities to each other. They provide rules so that if we know the values of certain quantities, we can calculate the values of others. In this unit we discuss several formulae and illustrate how they are used.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- substitute numbers into formulae in order to calculate various physical quantities

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1. Introduction

A **formula** is a recipe, or a rule, for doing something. In mathematics, a formula gives a relationship between different quantities. When there is more than one, we use the word **formulae**. There are many standard formulae which have been derived by mathematicians and scientists. In this unit we will see lots of formulae and practice using them.

2. The formula for the area of a circle

The formula which tells us the area of a circle is $A = \pi r^2$. Here, r is the radius of the circle, and A is its area. To calculate the area of a circle we square its radius and multiply the result by π . (In the examples which follow we shall approximate π by 3.142.)



Key Point

the area, A , of a circle with radius r :

$$A = \pi r^2$$

Example

Suppose we wish to find the area of a circular lawn, of radius 3m, in order that we can turf it. We can calculate the area by substituting the value $r = 3$ in the formula:

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \times 3^2 \\ &= \pi \times 9 \end{aligned}$$

Taking π as 3.142 we can use a calculator to obtain $A = 28.3$ correct to 1 decimal place. So we would need to purchase approximately 29m² of turf in order to cover the lawn.

3. The formula for the volume of a sphere

The formula for the volume, V , of a sphere, or ball, having radius r is $V = \frac{4}{3}\pi r^3$.



Key Point

the volume, V , of a sphere of radius r :

$$V = \frac{4}{3}\pi r^3$$

Example

Suppose we wish to find the volume of a football which has radius 10cm. Then

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi \times 10^3 \\ &= 4189 \quad (\text{to 4 significant figures}) \end{aligned}$$

So the volume of the football is 4189 cm³.

Exercises 1

Given below are a selection of formulae relating to areas, perimeters and volumes of geometric shapes.

Circles: $A = \pi r^2 = \pi \left(\frac{d^2}{4}\right)$ and $C = 2\pi r = \pi d$

where A is the area, C is the circumference, r is the radius and d is the diameter.

Rectangles: $A = wh$ and $P = 2(w + h)$

where A is the area, P is the perimeter, w is the width and h is the height.

Spheres: $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$

where V is the volume, A is the surface area and r is the radius.

Cylinders: $V = \pi r^2 h$, $C = 2\pi r h$ and $A = 2\pi r(r + h)$

where V is the volume, C is the curved surface area, A is the surface area of a solid cylinder (including the ends), r is the radius and h is the height.

In the following exercises give your answer correct to 1 decimal place.

1. What is the area of a circle with radius 3cm?
2. What is the circumference of a circle with diameter 5cm?
3. What is the perimeter of a rectangle with width 5cm and height 8 cm?
4. What is the volume of a sphere with radius 2cm?
5. What is the curved surface area of a cylinder with radius 10cm and height 40cm?
6. What is the area of a rectangle with width 3.5 cm and height 5.2 cm?
7. What is the surface area of a sphere with radius 4.7 cm?
8. What is the total surface area of a cylindrical can with radius 5.3 cm and height 8.1 cm?
9. What is the total surface area of a solid hemisphere with radius 7.3 cm?
10. A strong man's barbell is made up of two spheres, each with radius 15cm joined by a cylinder of length 85cm and radius 2cm. What is the volume of the barbell?

4. Newton's second law of motion

This law relates the quantities force F , mass m , and acceleration a , and states $F = ma$.



Key Point

Newton's second law

$$F = ma$$

where F = force, m =mass, a =acceleration.

Example

We wish to calculate the force on a circus performer who is fired from a cannon.

His mass is 60 kg and his acceleration on leaving the cannon is 2.5 ms^{-2} .

Using Newton's second law:

$$\begin{aligned} F &= ma \\ &= 60 \times 2.5 \\ &= 150 \end{aligned}$$

So the force on the performer is 150 newtons. (The newton is the unit of force, when the mass is measured in kilograms and the acceleration is measured in metres per second per second).

5. The equations of motion

One of the so-called 'equations of motion' is $v = u + at$ where v = final speed, u = initial speed, a = acceleration, and t = time.

Example

Suppose an object has an initial speed 5 ms^{-1} and moves with an acceleration of 2 ms^{-2} for 3 seconds. We want to calculate its final speed.

$$\begin{aligned} v &= u + at \\ &= 5 + 2 \times 3 \\ &= 11 \end{aligned}$$

So its final speed is 11 ms^{-1} .

Here it is worth pointing out the correct order of carrying out the operations in the calculation of $5 + 2 \times 3$. The multiplication is carried out before the addition. You may recall the BODMAS rule which states the order in which operations should be carried out.

Brackets
powers
Division
Multiplication
Addition
Subtraction

and so multiplication is carried out before addition.

Another equation of motion states $v^2 = u^2 + 2as$. As before, v = final speed, u = initial speed, a = acceleration, s = distance travelled.

Example

Suppose a stone is dropped from a cliff which is 100m high. Because the stone is simply dropped, rather than thrown, its initial speed is zero, and so $u = 0$. Its acceleration will be the acceleration due to gravity which is $a = 9.8\text{ms}^{-2}$. Suppose we wish to calculate the final speed just before the stone hits the water at the foot of the cliff. The stone travels a distance of 100m before hitting the water and so $s = 100$. We must substitute all these known values into the formula.

$$\begin{aligned}
 v^2 &= u^2 + 2as \\
 &= 0^2 + 2 \times 9.8 \times 100 \\
 &= 1960
 \end{aligned}$$

So to calculate v , we need to find the square root of this result: $v = \sqrt{1960} = 44 \text{ ms}^{-1}$ (to two significant figures).

A third equation of motion is $s = ut + \frac{1}{2}at^2$ where s = distance travelled, t = time, u = initial speed, a = acceleration.

Example

We wish to determine the depth of a well by dropping a stone down it.

We find that it takes 3 seconds for the stone to reach the bottom of the well. As in the previous example, the initial speed is zero and the acceleration due to gravity is 9.8ms^{-2} .

We substitute the given values into the formula.

$$\begin{aligned}
 s &= ut + \frac{1}{2}at^2 \\
 &= 0 \times 3 + \frac{1}{2} \times 9.8 \times 3^2 \\
 &= 44
 \end{aligned}$$

So the depth of the well is 44m (to the nearest whole number).



Key Point

The equations of motion with constant acceleration:

$$v = u + at \quad v^2 = u^2 + 2as \quad s = ut + \frac{1}{2}at^2$$

where u = initial speed, v = final speed, a = acceleration, t = time, s = distance travelled.

6. The formula for kinetic energy

If an object has mass m and moves with speed v , its kinetic energy (K.E.) is given by the formula $\text{K.E.} = \frac{1}{2}mv^2$.



Key Point

$$\text{Kinetic Energy} = \frac{1}{2}mv^2$$

where m = mass, and v = speed.

Example

A sprinter of mass 70 kg runs at a speed of 10ms^{-1} .

A truck has mass 2000 kg and moves with a speed of 20ms^{-1} .

Let us compare their kinetic energies:

For the sprinter: $\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 70 \times 10^2 = 3500$

For the truck: $\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 2000 \times 20^2 = 400\,000$

So the kinetic energy of the truck is more than 100 times greater than that of the sprinter.

When m is measured in kilograms, and v is measured in metres per second, the unit of kinetic energy is the joule.

7. The period of a pendulum

The formula which tells us the time for a complete swing of a pendulum, i.e. its **period**, T , is $T = 2\pi\sqrt{\frac{\ell}{g}}$. Here ℓ is the length of the pendulum, and g is the acceleration due to gravity which is 9.8 ms^{-2} .



Key Point

$$\text{The period } T \text{ of a pendulum} = 2\pi\sqrt{\frac{\ell}{g}}$$

where ℓ = length of pendulum, and g = acceleration due to gravity.

Example

Suppose we wish to find the period of a grandfather clock which has a pendulum of length 1m.

$$\begin{aligned} T &= 2\pi\sqrt{\frac{\ell}{g}} \\ &= 2\pi\sqrt{\frac{1}{9.8}} \\ &= 2\pi \times 0.319 \end{aligned}$$

Approximating π by 3.142 we find $T = 2.007$. So the period of the pendulum is 2 seconds, to the nearest second.

Exercises 2

Some formulae from mechanics are given below.

Constant Acceleration

$$v = u + at, v^2 = u^2 + 2as \text{ and } s = ut + \frac{1}{2}at^2$$

where u is the initial speed, v is the final speed, t is time, s is distance travelled and a is acceleration. In the SI system of units u and v are measured in metres/second (m/s or ms^{-1}), t in seconds (s), s in metres (m) and a in metres per second squared (m/s^2 or ms^{-2}).

Work and Energy

$$W = Fd, K.E. = \frac{1}{2}mv^2, P.E. = mgh$$

where W is work done, $K.E.$ is kinetic energy, $P.E.$ is potential energy, F is force, d is distance moved by the force, m is mass, v is speed, g is acceleration due to gravity and h is height. In

the SI system of units W , $K.E.$ and $P.E.$ are measured in Joules (J), F in Newtons (N), d in metres (m), m in kilograms (kg), v in metres per second (m/s or ms^{-1}), g in metres per second squared (m/s^2 or ms^{-2}) and h in metres (m).

Throughout these exercises you should take the acceleration due to gravity, g , as 9.8 m/s^2 .

Give your answers correct to 2 decimal places.

1. A car, starting at rest, accelerates at 5 m/s^2 for 3 seconds. What is its final speed (in m/s)?
2. How far does the car in the previous question travel?
3. A stone is dropped from the top of a building 32 m tall. Its acceleration is that due to gravity. What is its speed (in m/s) when it hits the ground?
4. If the mass of the stone in the previous question is 0.1 kg, what is its potential energy when it is released?
5. What is the kinetic energy of the stone from the previous two questions when it hits the ground?
6. What is the work done by a horizontal force of 10N in moving an object horizontally through 5m.
7. Determine the kinetic energy of a mass of 6.2 kg moving at a speed of 3.1 m/s.

8. Formulae used to generate terms of a sequence

There are two simple ways of generating sequences of numbers.

- (a) by a rule which tells you how to calculate the next term from the previous term (or possibly from the previous two or more terms)
- (b) by a formula which tells you how to calculate each term independently.

Consider the sequence

$$2, 4, 6, 8, \dots$$

This is generated by the rule ‘add 2 to the previous term to get the next term, given that the first term is 2’. Alternatively it can be generated from the formula $u_n = 2n$ (where u_n means the n th term of the sequence). So, for example, to find the 5th term we put $n = 5$ to get $u_5 = 2 \times 5 = 10$.

When using a rule to generate a sequence, if you want to work out a specific term in the sequence you must work out all the previous terms. But when you have a formula you can work out any term you like without having to calculate others.

A famous sequence is the Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, ...

Here the rule is ‘add the previous two terms together to get the next term, given that the first two terms are 0 and 1’. There is a formula which can be used to generate each term independently but it requires some quite complex mathematics to determine it.

The accompanying video shows how a formula can be used in an impressive magic trick.

Exercises 3

In the exercises below, give answers which are not whole numbers correct to 2 decimal places. Note that questions 7 to 10 involve the exponential, natural logarithm and sine functions. If you are not familiar with these functions then simply omit these questions.

1. Calculate the 5th term of the sequence generated by the formula
 $u_n = 3n + 1$.
2. Calculate the 10th term of the sequence generated by the formula
 $u_n = 54 - 2n$.
3. Calculate the 12th term of the sequence generated by the formula
 $u_n = n^2 - 10n + 1$.
4. Calculate the 5th term of the sequence generated by the formula
 $u_n = \frac{1}{2}n(n + 1)$.
5. Calculate the 8th term of the sequence generated by the formula
 $u_n = \frac{1}{6}n(n + 1)(2n + 1)$.
6. Calculate the 6th term of the sequence generated by the formula
 $u_n = 2\sqrt{n + 3}$ (take the positive square root).
7. Calculate the 4th term of the sequence generated by the formula
 $u_n = e^{-n/4}$.
8. Calculate the 9th term of the sequence generated by the formula
 $u_n = \ln\left(\frac{n^2}{10}\right)$.
9. Calculate the 3rd term of the sequence generated by the formula
 $u_n = \sqrt{\frac{\ln n^3}{n^2 - 1}}$.
10. Calculate the 11th term of the sequence generated by the formula
 $u_n = \sin\left(\frac{n\pi}{3}\right)$, where the angle is measured in radians.

Answers

Exercise 1

- | | | | |
|---------------------------|-----------------------------|--------------------------|--------------------------|
| 1) 28.3 cm ² | 2) 19.6 cm | 3) 26 cm | 4) 33.5 cm ³ |
| 5) 2513.3 cm ² | 6) 18.2 cm ² | 7) 277.6 cm ² | 8) 446.2 cm ² |
| 9) 502.2 cm ² | 10) 29342.5 cm ³ | | |

Exercise 2

- 1) 15 m/s 2) 22.5 m 3) 25.04 m/s 4) 31.36 J
5) 31.36 J 6) 50 J 7) 29.79 J

Exercise 3

- 1) 16 2) 34 3) 25 4) 15
5) 204 6) 6 7) 0.37 8) 2.09
9) 0.64 10) -0.87