The Correlation of the Digital Signals

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The Correlation of the Digital Signals

- **Correlation has the main application in the area of comparison of signals**

- **Objective of correlation is to measure a degree to which two signals are similar**

- **The higher correlation, the more similar two signals are**

- **Example of its use is in radar systems for a target detection**

The Correlation and the Covariance of Energy Signals

- The **cross-correlation** between these signals is defined as

$$r_{xy}(n,m) = \sum_{n,m=-\infty}^{n,m=\infty} \left[x(n) y^*(m) \right]$$

where $*$ **denotes the complex conjugate**

- The **cross-covariance** between these signals is defined as

$$c_{xy}(n,m) = \sum_{n,m=-\infty}^{n,m=\infty} \left[(x(n) - \bar{x}(n))(y^*(m) - \bar{y}^*(m) \right]$$

where $\bar{x}(n)$ **and** $\bar{y}^*(m)$ **are the means of the signals** $x(n)$ **and** $y^*(m)$ **respectively**

The Correlation and the Covariance of Energy Signals

- **The covariance is the mean-removed cross-correlation**

- **If the means of signals are zero, the cross-correlation and the cross-covariance are equal**

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- **The autocorrelation and autocovariance are the special cases when** $y(n) = x(n)$

- **In general, the mean, the variance, the correlation and the covariance of the random signals are *time-varying functions***

Wide-Sense Stationary (WSS) Signals

- **The class of random signals often encountered in signal processing is the wide-sense stationary (WSS) random signals for which some of the key statistical properties (functions) are *independent of time***

- **More specifically, for WSS signals the mean and the variance have constant values for all values of the time indices and the autocorrelation and the autocovariance depend only on the difference of the time indices and not on the actual values of indices**

- **This means that regardless of which time point of the signal the statistical properties of the signal are studied, they all are the same**

The Cross-Correlation of the WSS

- **Let us suppose that we have two *real* WSS ergodic energy signals. The cross-correlation is defined as**

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) = \sum_{n=-\infty}^{\infty} x(n+l)y(n),\ l = 0,\ \pm 1, \ldots$$

where the index l is the time shift (or lag) parameter; the order of the subscripts, with x preceding y, indicates the direction in which one signal is shifted, relative to the other

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- **If we reverse the roles of signals, we obtain another cross-correlation sequence:**

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)x(n-l) = \sum_{n=-\infty}^{\infty} y(n+l)x(n)$$

The Cross-Correlation of WSS: Properties

- **By comparing the cross-correlation sequences, we conclude that**

$$r_{xy}(l) = r_{yx}(-l)$$

i.e., ***it is the symmetry property***

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- **Hence, both expressions provide the same information with respect to the similarity of two signals**

The Cross-Correlation of WSS: Properties

- **Let us assume that we have two real WSS energy (i.e. with a finite energy) signals $x(n)$ and $y(n)$**

- **The energies in these signals are respectively**

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$$E_x = \sum_{n=-\infty}^{\infty} x^2(n) = r_{xx}(0) \qquad E_y = \sum_{n=-\infty}^{\infty} y^2(n) = r_{yy}(0)$$

- **The cross-correlation satisfies the condition that**

$$|r_{xy}(l)| \le \sqrt{r_{xx}(0) r_{yy}(0)}$$

- **In the case of the autocorrelation:**

$$|r_{xx}(l)| \le r_{xx}(0)$$

The Cross-Correlation of WSS: Properties

- **This means that the autocorrelation sequence of a signal attains its maximum value at zero lag**

- **This result is consistent with the notion that a signal matches perfectly with itself at zero shift**

- **It is often desirable, in practice, to normalize the autocorrelation and cross-correlation to the range from -1 to 1**

- **The normalized autocorrelation and cross-correlation are defined as**

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)} \qquad \rho_{xy}(l) = \frac{r_{xy}(l)}{\sqrt{r_{xx}(0)r_{yy}(0)}}$$

The Cross-Correlation of the WSS: Properties

- **The normalized cross and auto correlations are less or equal unity and, hence, these sequences are independent of amplitude scaling**
- **The autocorrelation function is an even function, i.e.**

$$r_{xx}(l) = r_{xx}(-l)$$

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- **Two signals are linearly independent or uncorrelated if the cross-covariance is zero**

- **Two signals are statistically independent if**

$$W\left[x(n)y(k)\right] = W\left[x(n)\right]W\left[y(k)\right]$$

where W **is the operator of the probability density function**

The Cross-Correlation of WSS: Properties

- **Statistically independent signals are always uncorrelated but uncorrelated signals may be statistically dependent**

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- **If two signals are jointly Gaussian, than the terms uncorrelated and independent are equivalent**

The Correlation of Power Signals

- **We considered the correlation of the *energy* signals**

- **Let us consider two *power* signals.**

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- **The cross-correlation is defined as**

$$r_{xy}(l) = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n)y(n-l) = \lim_{M\to\infty} \sum_{n=-M}^{M} x(n+l)y(n), \; l = 0, \; \pm 1, \ldots (*)$$

- **If** $x(n) = y(n)$ **, we can easily obtain from equation (*) the autocorrelation function**

Correlation Estimates

There are two ways to compute correlation estimates

- **The first is the direct method, involving the computation of average products among the sampled signals**

* - **The second way is the indirect approach based on the Weiner-Khinchine theorem: first, computing a power spectral density estimate using the FFT, and then computing the inverse Fourier transform of the power spectral density**

- **The second approach takes advantage of the dramatic computational efficiency of the FFT and, hence, is much less expensive and less time consuming to execute**

Correlation Estimates: the Direct Method

- **In order to compute the cross-correlation for signals:**

- **For positive lags, we simply shift *y(n)* to the right relative to *x(n)* by *l* units, compute the multiplications and sum over all values of discrete time .**

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- **For negative lags, we simply shift *y(n)* to the left relative to *x(n)* by *l* units, compute the multiplications and sum over all values of discrete time**