The Discrete Fourier Transform

Part 1

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Module “SIGNAL ANALYSIS AND PROCESSING”
NME3523

The Discrete-Time Fourier Transform

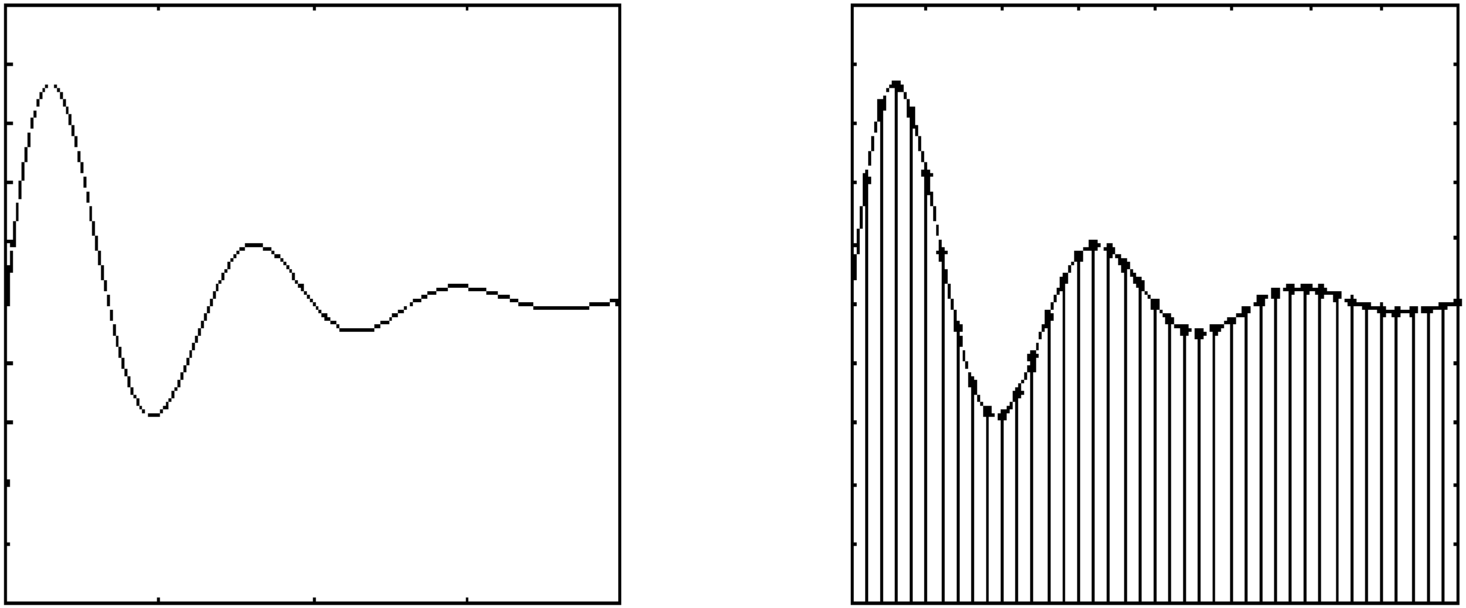
- **The continuous Fourier transform of an analogue signal x(t) is:**

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-i\omega t)\,dt$$

- **Assume that we have sampled the analogue signal with the sample rate** Δt

$$t_n = n\Delta t$$

The Discrete-Time Signals



- **Consider an analogue signal *x(t)* that can be viewed as a continuous function of time, as shown above in the left graph**

- **We can represent this signal as a discrete-time signal by using values of *x(t)* at intervals of Δt to form *x[n]* as shown above in the right graph**

The Discrete-Time Fourier Transform

Using the discrete time signal, we obtain the *discrete-time* Fourier transform (DTFT)

$$X_c(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-i\omega n\Delta t)$$

The Discrete-Time Fourier Transform

$$X_c(\omega) = \sum_{n=-\infty}^{\infty} x(n) \exp(-i\omega n)$$

- **As can be seen from the last equation, the DTFT is a continuous function of ω and a periodic function with frequency period 2π**

- **To verify this latter property, observe that**

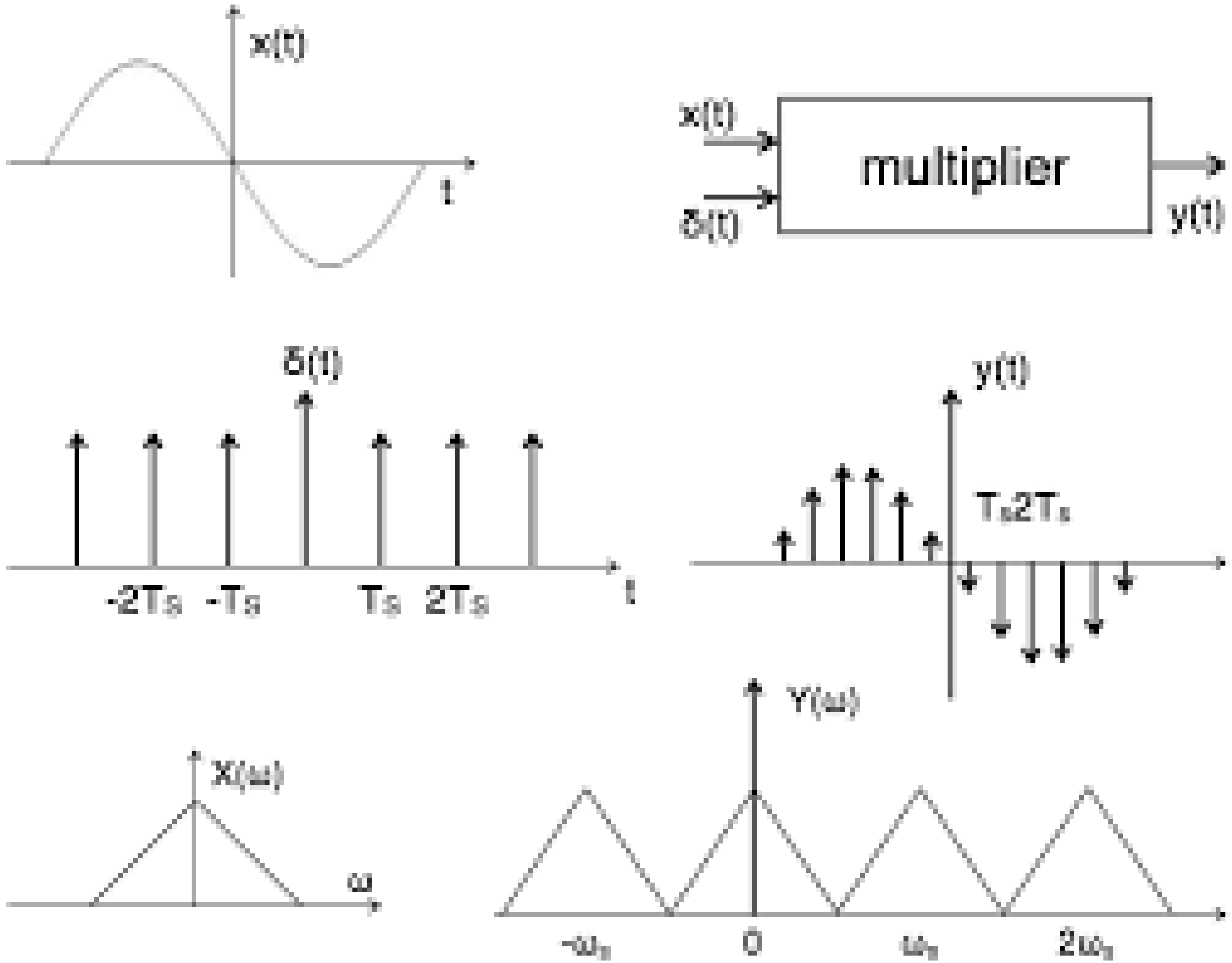
$$X_c(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n) e^{-i(\omega+2\pi k)n} = \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n} e^{-i2\pi kn} = \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n} = X_c(\omega)$$

- ***This property is just a consequence of the fact that the frequency range for any discrete-time sine signal is limited to $(-\pi, \pi)$ or $(0, 2\pi)$ and any frequency outside this interval is equivalent to a frequency within the interval***

The Discrete-Time Fourier Transform

The DTFT becoming a periodic function in frequency domain and effect of uniform time sampling on the Fourier transform is that Fourier transform is copied and displaced an infinite number of times in the positive and negative frequency directions

The Discrete-Time Fourier Transform



The Discrete-Time Fourier Transform

- **In the case of a finite-length signal** $x(n)$ **,** $0 \le n \le N-1$

we approximate the infinite summation with a finite summation:

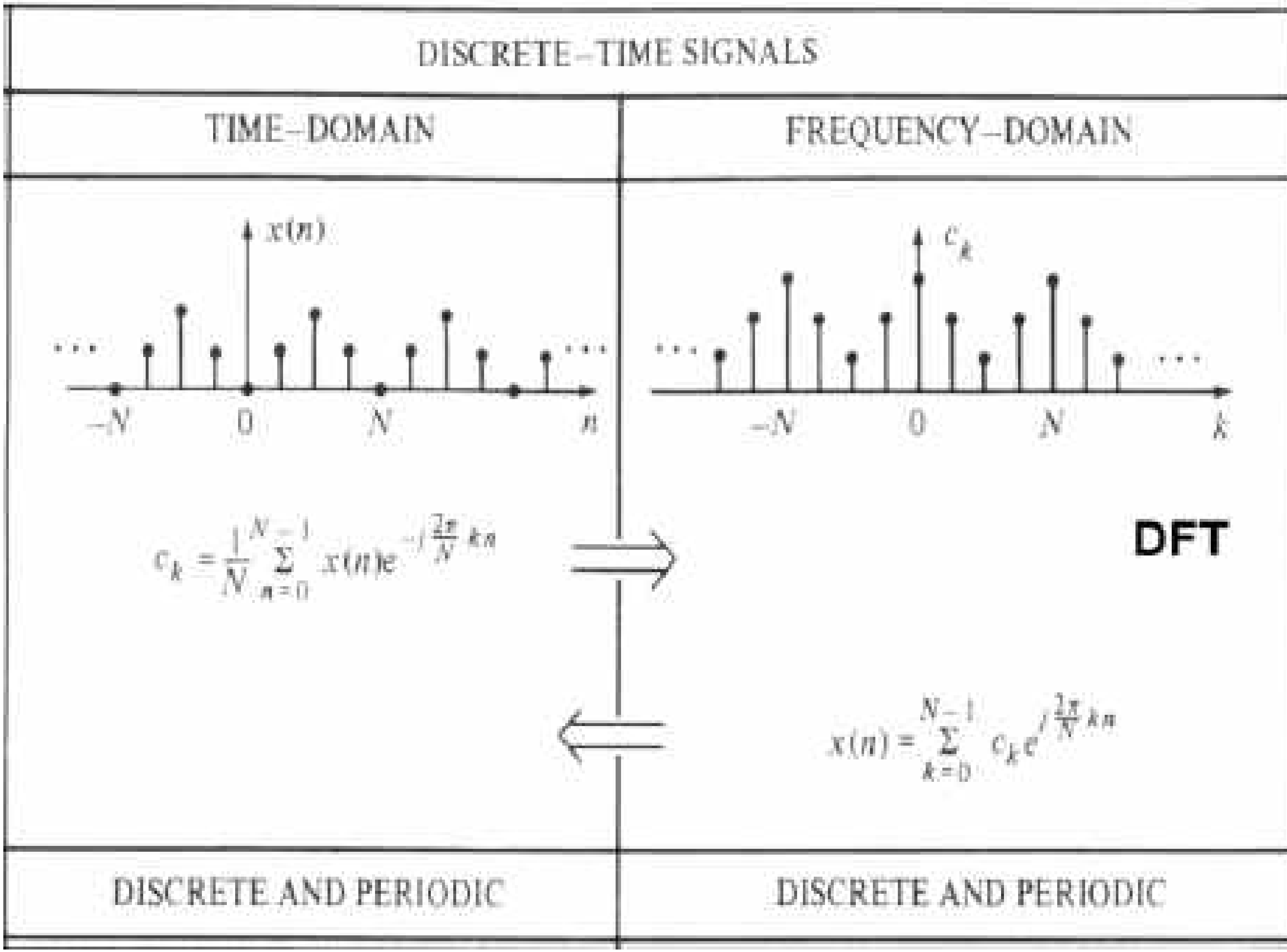
$$X_c(\omega) = \sum_{n=0}^{N-1} x(n)\exp(-i\omega n)$$

The Discrete Fourier Transform

- **In fact, for a finite-length** N **signal** $x(n)$ **, only** N **values of DTFT, called the *frequency samples*, at** N **distinct frequency points,** $\omega = \omega_k$ **,** $k = 0, 1, 2, \ldots, N-1$ **, are sufficient to determine** $x(n)$ **, and hence the Fourier transform, *uniquely***

- **This leads to the concept of the discrete Fourier transform that is *applicable only to a finite-length signal***

The Discrete Fourier Transform



The Discrete Fourier Transform

- **Since DTFT is periodic with period 2π , it is convenient to sample it taking N uniformly spaced samples between 0 and 2π i.e. at**

$$\omega_k = \frac{2\pi k}{N}$$

- **We can now modify the DTFT to highlight this sampling in the frequency domain by replacing the continuous ω by ω_k :**

$$X(k) = \sum_{n=0}^{N-1} x(n) \exp(-\frac{i2\pi kn}{N}), \; k = 0, 1, ..., N-1$$

- **The finite-length sequence $X(k)$ is called the *discrete Fourier transform* (DFT) of the finite length signal $x(n)$**

The DFT and the Inverse DFT

- **Using the commonly employed notation**

$$W_N = e^{-i\frac{2\pi}{N}}$$

we can re-write the DFT as

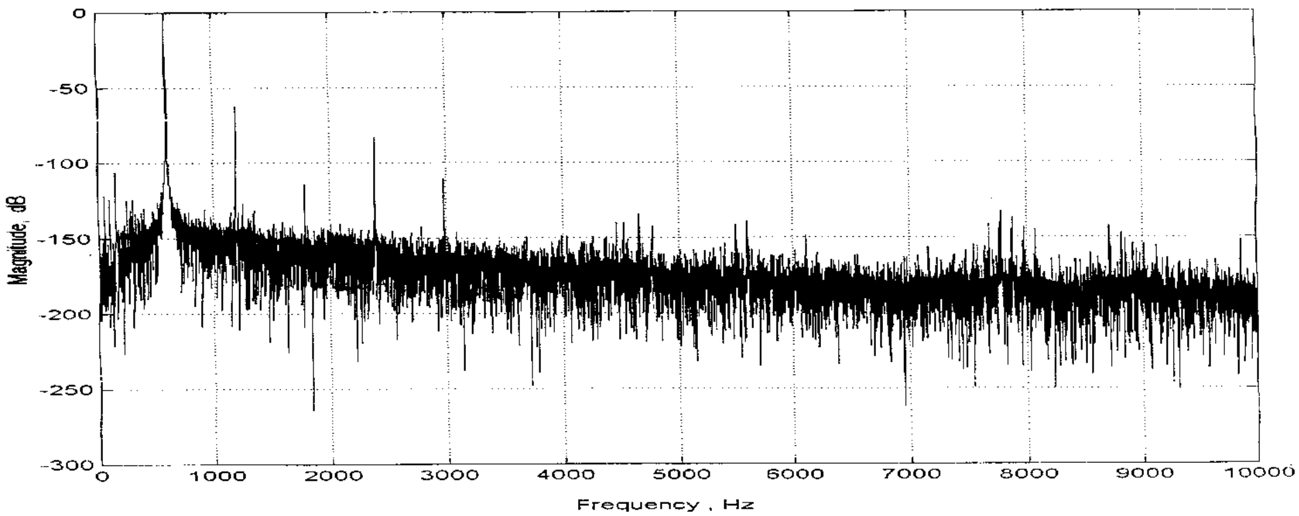
$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{\,kn}$$

- **The inverse DFT (IDFT) is given by**

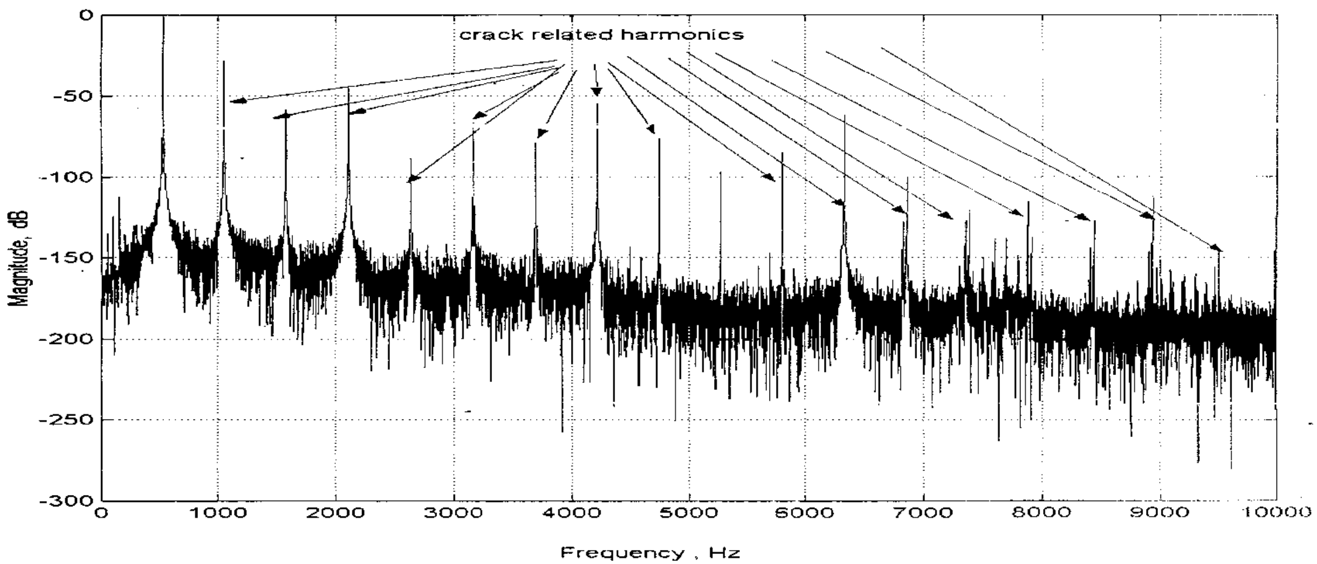
$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{\,-kn}$$

Experiments with Aircraft Compressor Blades

Modulus of the Discrete Fourier Transform of the Resonant Oscillations of Blades



Without Crack



With Crack

DFT Properties

Periodicity

If $x(n)$ **and** $X(k)$ **are an** N**-point DFT pair, then**

$$x(n+N) = x(n) \quad \textbf{for all} \quad n$$

$$X(k+N) = X(k) \quad \textbf{for all} \quad k$$

DFT Properties

Linearity

The DFT of a linear combination of two sequences is a linear combination of the DFT of the individual sequences

i.e. if $$x(n) = a_1 x_1(n) + a_2 x_2(n),$$

then $$X(k) = a_1 X_1(k) + a_2 X_2(k)$$

- **The DFT is a linear transform**

DFT Properties

Time shift of a sequence

- **The DFT of a sequence shifted in time is such that**

$$x(n+l) \leftrightarrow W_N^{-lk} X(k)$$

Frequency shift (modulation)

- **The multiplication of the sequence $x(n)$ with the complex exponential sequence is equivalent to the circular shift of the DFT by l units in frequency, i.e.**

$$x(n)e^{i2\pi \ln/N} \leftrightarrow X(k-l)$$

DFT Properties

Time-reversal

- **The DFT of** $x(-n)$ **is such that**

$$x(-n) \leftrightarrow X(-k)$$

Convolution in time

- **If** $x_1(n) \leftrightarrow X_1(k)$ **and** $x_2(n) \leftrightarrow X_2(k)$ **then**

$$x_1(n) \otimes x_2(n) = \sum_{l=0}^{N-1} x_1(l) x_2(n-l) = \sum_{l=0}^{N-1} x_1(n-l) x_2(l) \leftrightarrow X_1(k) X_2(k)$$

- **This is the basis of one *of the most important applications* of the DFT: to compute a discrete convolution in time through the application of the IDFT to the multiplication of the DFTs of the two sequences**

DFT Properties

Multiplication of two sequences

- **If** $x_1(n) \leftrightarrow X_1(k)$ **and** $x_2(n) \leftrightarrow X_2(k)$ **then**

$$x_1(n)x_2(n) \leftrightarrow \frac{1}{N} X_1(k) \otimes X_2(k) = \frac{1}{N}\sum_{n=0}^{N-1} X_1(n) X_2(k-n) = \frac{1}{N}\sum_{n=0}^{N-1} X_1(k-n) X_2(n)$$

- **This property is the circular convolution (convolution in frequency)**

Symmetry Properties of the DFT

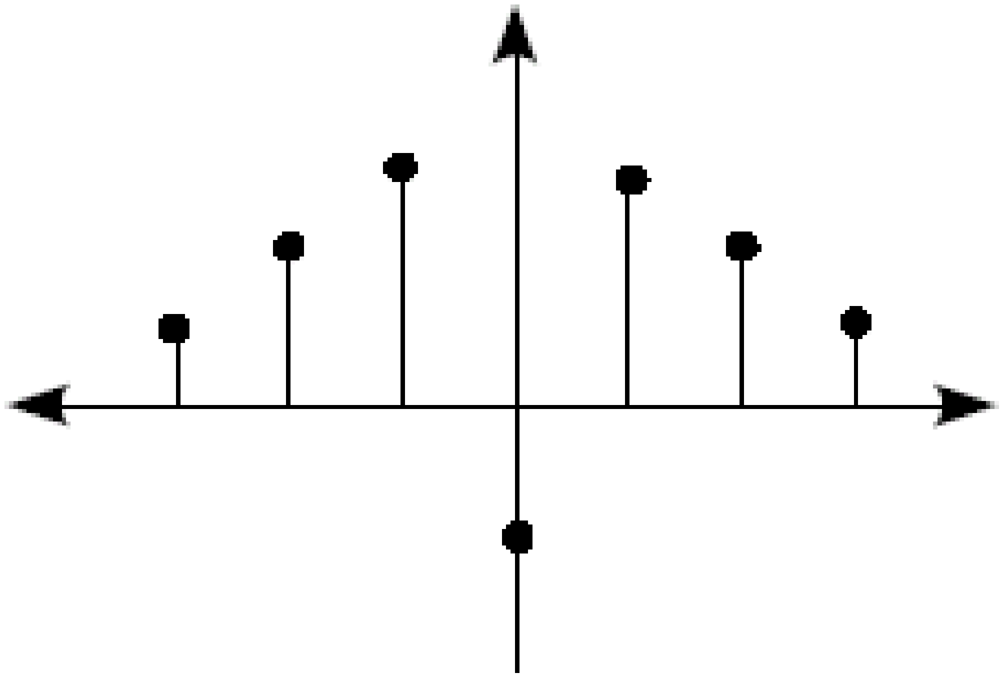
- **The DFT and IDFT are transforms of complex-valued signals to complex-valued spectra.**

- **However, in practice, digital signals are normally real-valued**

- **The DFT of a real-valued digital signal has a special symmetry, in which:**

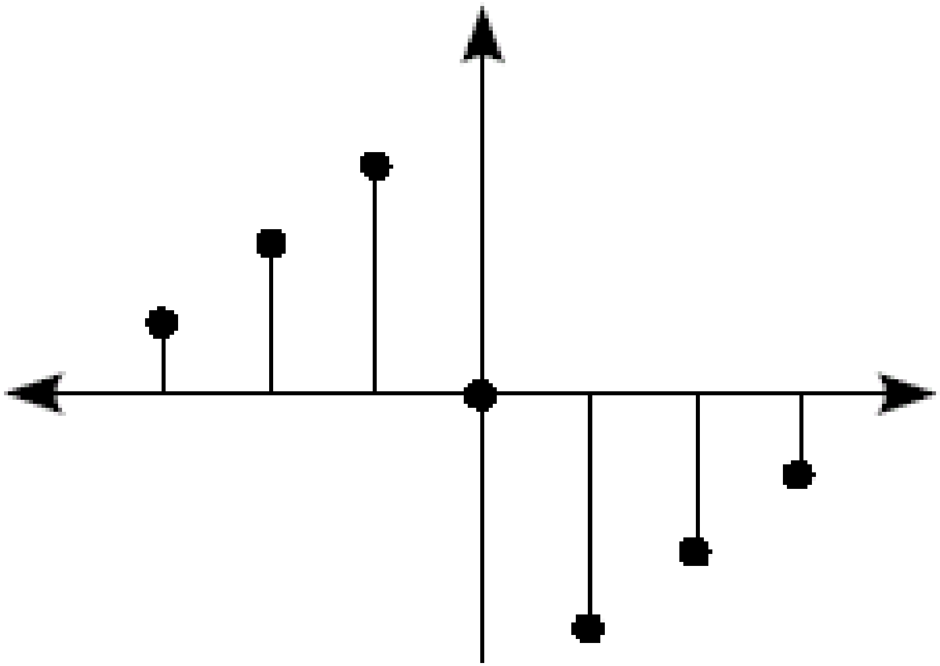
- **the real part of the transform is *even symmetric***
- **the imaginary part is *odd symmetric***

Symmetry Properties of the DFT for Real Signals



the DFT real part is **even** ***symmetric***

Symmetry Properties of the DFT for Real Signals



the DFT imaginary part is ***odd symmetric***

Summary: the DFT Symmetry Properties

Real and imaginary components of the DFT

If $x(n)$ **is a real signal, then**

$$X_R(-k) = X_R(k)$$ **(even symmetry)**

$$X_I(-k) = -X_I(k)$$ **(odd symmetry)**

DFT Properties

Complex-conjugate properties

- **If** $x(n) \leftrightarrow X(k)$ **, then** $x^*(n) \leftrightarrow X^*(-k) = X^*(N-k)$

where the asterisk indicates the complex conjugate of the function

Correlation

- **For complex sequences, if** $x(n) \leftrightarrow X(k)$ **and** $y(n) \leftrightarrow Y(k)$ **then**

$$r_{xy}(l) \leftrightarrow X(k)Y^*(k)$$

where $r_{xy}(l)$ **is the un-normalized cross correlation function**

DFT Properties

❑ *The Parseval's theorem*

- **For complex-valued sequences, if**

$$x(n) \leftrightarrow X(k) \quad \textbf{and} \quad y(n) \leftrightarrow Y(k)$$

then

$$\sum_{n=0}^{N-1} x(n) y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) Y^*(k)$$

- **In the case where** $y(n) = x(n)$ **we obtain**

$$\sum_{n=0}^{N-1} \left| x(n) \right|^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left| X(k) \right|^2$$

which expresses the energy of the finite-duration sequence in terms of the DFT components

Aliasing in Frequency Domain

- **The sampling theorem assumes that a signal is band limited**

- **In practice, however, signals are *time limited*, not band limited. As a result, determining an adequate sampling frequency which does not lose desired information can be difficult**

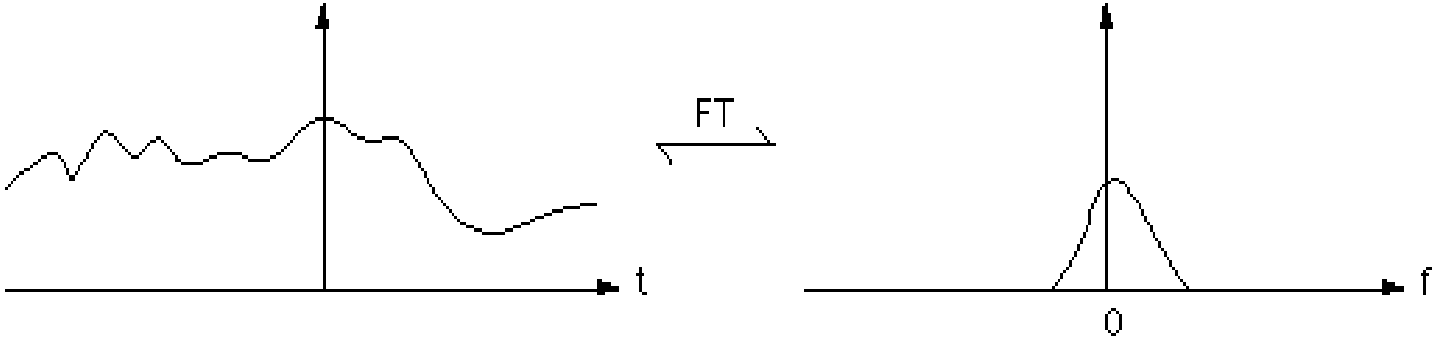
- **When a signal is under-sampled, its power spectral density has overlapping tails**

- **So, the estimated power spectral density no longer has complete information about the true power spectral density, and it is no longer possible to recover signal from the sampled signal**

Aliasing in Frequency Domain

Aliasing in Frequency Domain

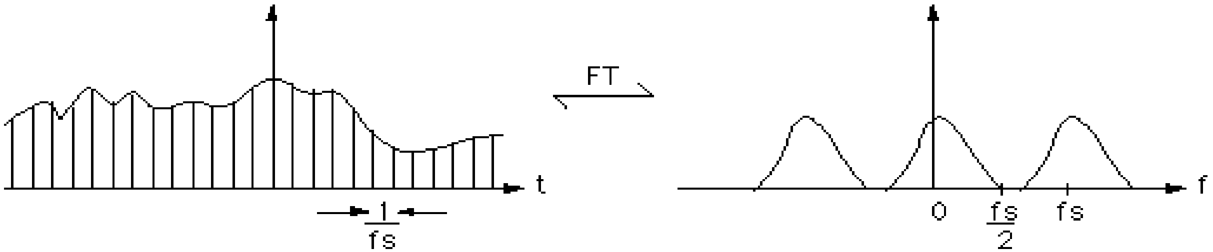
Consider a continuous signal in the time and frequency domain



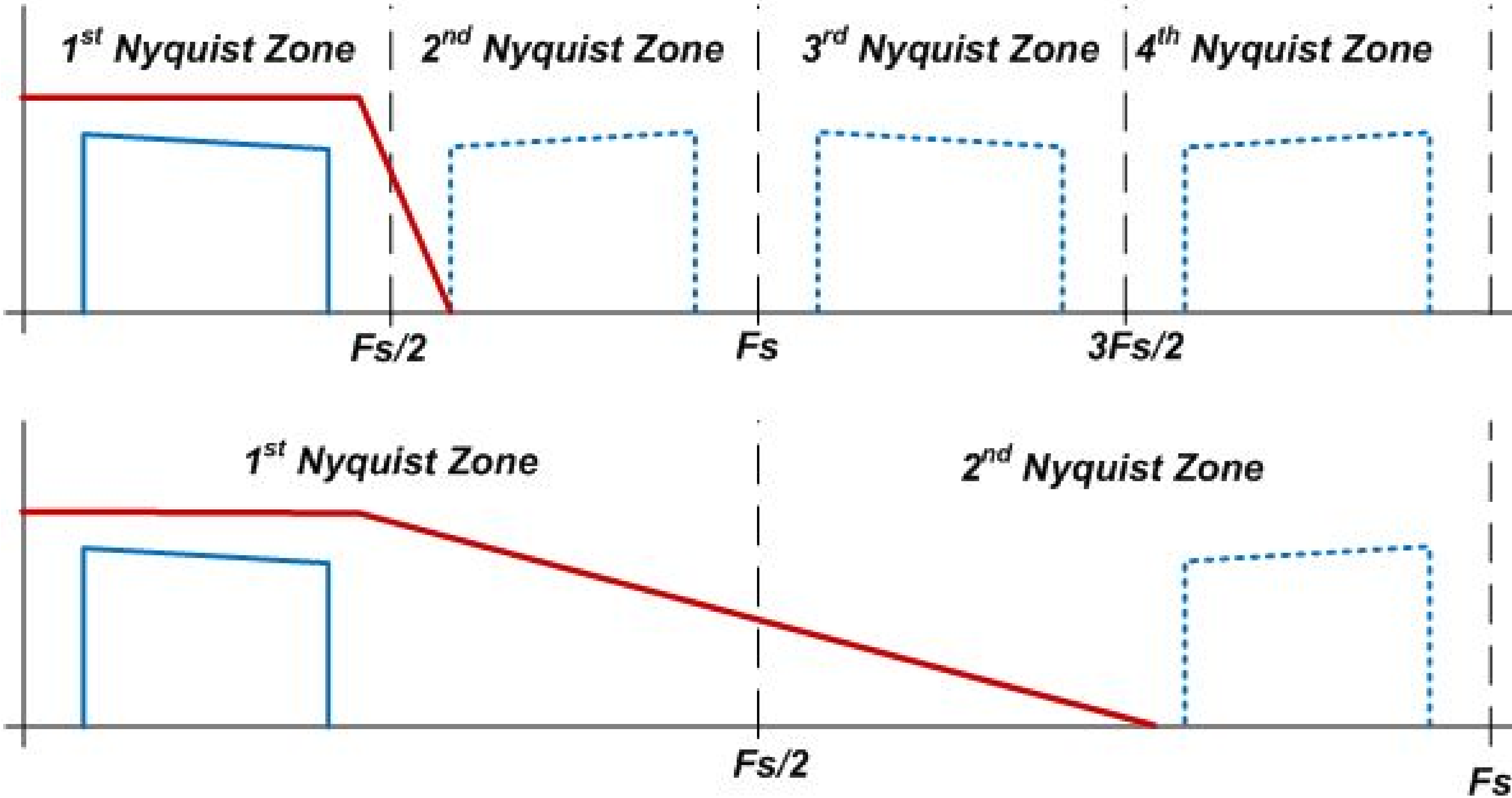
Sampling of Signals

- **We sample this signal with sampling frequency fs, time between samples is 1/fs**

- **As it was shown earlier, the DFT is a periodic function in frequency domain with period 2π and effect of uniform time sampling on the Fourier transform is that Fourier transform is copied and displaced by f_s an infinite number of times in the positive and negative frequency directions**

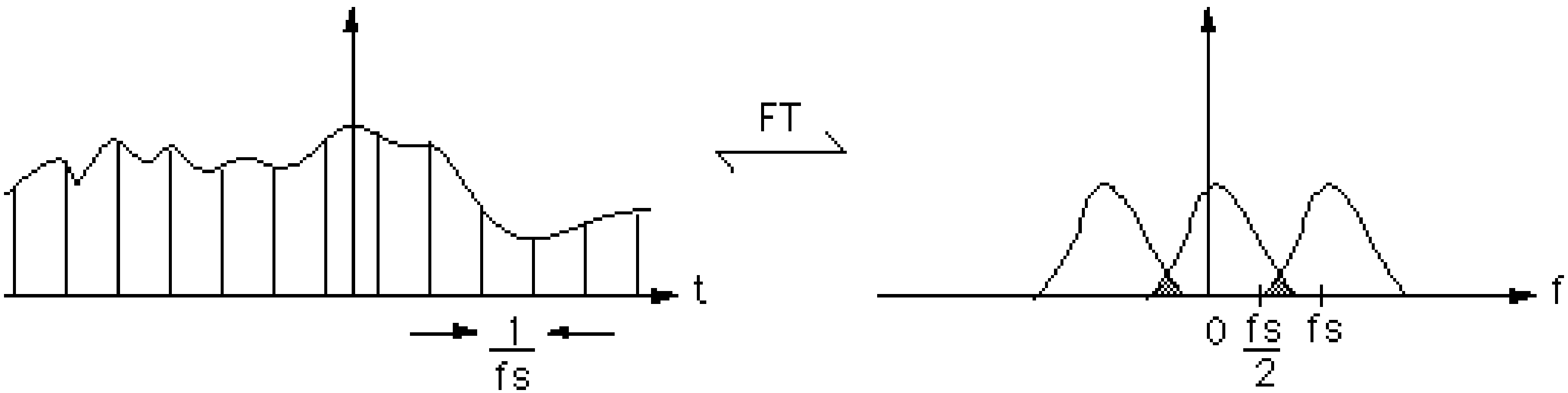


Signal Sampling



Aliasing in Frequency Domain

- **If the sampling frequency is too low the power spectral densities overlap, and become corrupted**

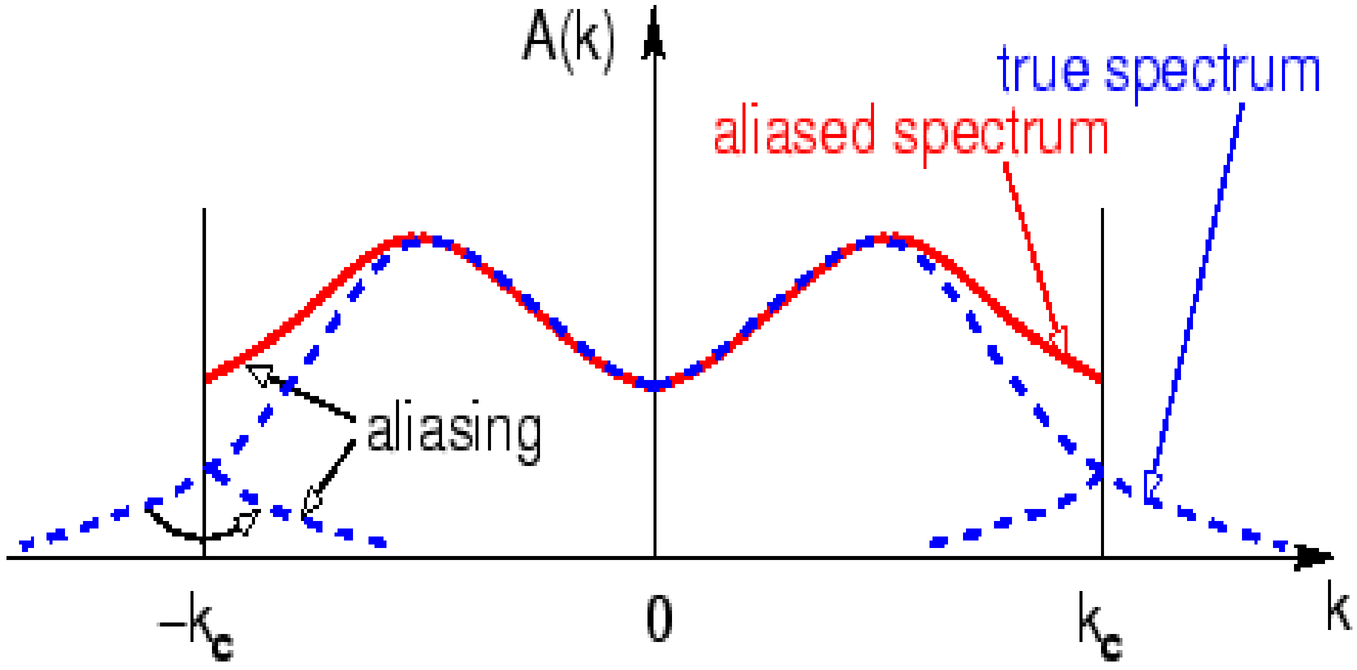


- **The unique part of the DFT graph occupies the frequency range** $|f| \leq \frac{f_s}{2}$

- **Higher frequencies show spurious Fourier coefficients which are repetitions of those which apply at frequencies in the range** $|f| \leq \frac{f_s}{2}$

Aliasing in Frequency Domain

Figure illustrates an aliasing due to undersampling in time and that even with the greatest precautions, such an aliasing can sometimes not be avoided, and needs then to be correctly interpreted



Aliasing in Frequency Domain

- **If there are frequencies above this range present in the original signal, these introduce a distortion of a graph called *aliasing***
- **The high frequency components falsely distort the power spectral density calculated for frequencies in the range** $|f| \leq \frac{f_s}{2}$
- **If** f_0 **is the maximum frequency present in signal, then aliasing can be avoided by ensuring that**

$$\frac{f_s}{2} \geq f_0$$

- **or**

$$f_s \geq 2 f_0$$

Aliasing in Frequency Domain

- **The frequency $f_s = 2f_0$ is called Nyquist rate**
- **If sampling frequency is smaller than Nyquist rate, the shifted versions of power spectral density *overlap***
- **Nyquist's theorem says that if we sample a signal in which the highest frequency we wish to reproduce correctly is *f* then we must sample the signal at a minimum frequency of *2f*. For this sampling rate, the frequency *f* is known as the Nyquist frequency**