The Wavelet Transform

Part 3

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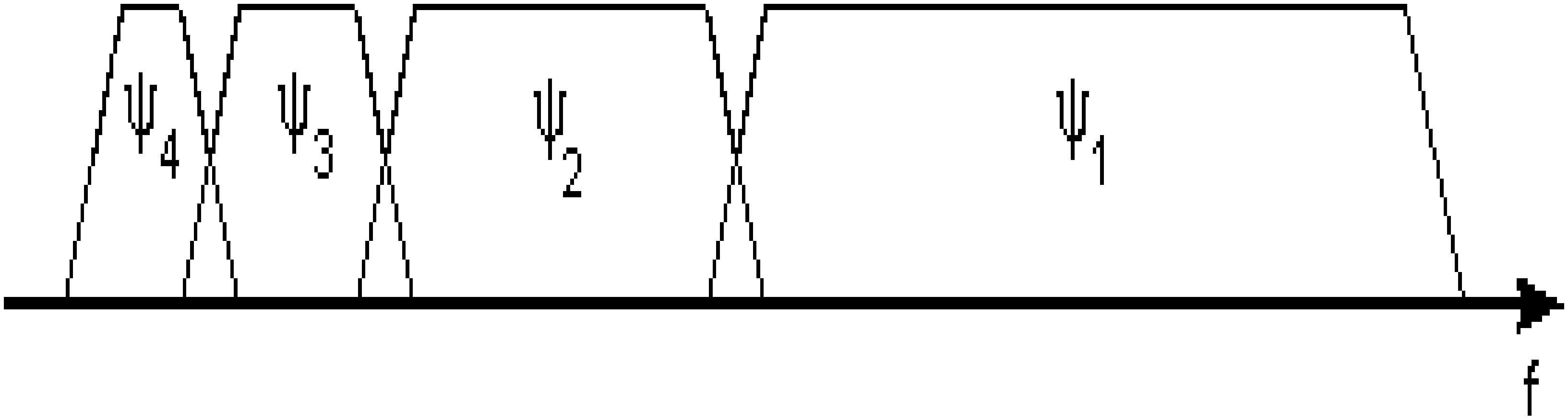
The Discrete Wavelet Transform via a Filter Bank

The Wavelet Transform is a Filter Bank

- **A series of scaled wavelets can be seen as *a* *bandpass filter bank***
- **One can see that the frequency bandwidth (i.e. frequency resolution) of the wavelet transform increases as frequency increases**



The Discrete Wavelet Transform Through a Filter Bank

- **We can consider the wavelet transform as a filter bank**

- **The discrete wavelet transform (DWT) through a filter bank provides sufficient information both for analysis and synthesis of the original signal, with a significant reduction in the computation time**

The Discrete Wavelet Transform Through a Filter Bank

- **The outputs of the different filters are the wavelet transform coefficients**

- **Analyzing a signal by passing it through a filter bank is not a new idea and has been known many years under the name *subband coding***

- **A filter bank in subband coding can be built in two main ways**

The Discrete Wavelet Transform Through a Filter Bank

- ***One way*** **is to build many band-pass filters to split the spectrum into frequency bands**

- ***The advantage*** **is that the width of every band can be chosen freely, in such a way that the spectrum of the signal is covered in the places where it might be interesting**

- **The disadvantage is that we will have to design every filter separately and this can be a time consuming process**

The Discrete Wavelet Transform Through a Filter Bank

- **Another way is to split a signal in two equal parts, a low pass part and a high pass part**

- **The high pass part contains the signal details**

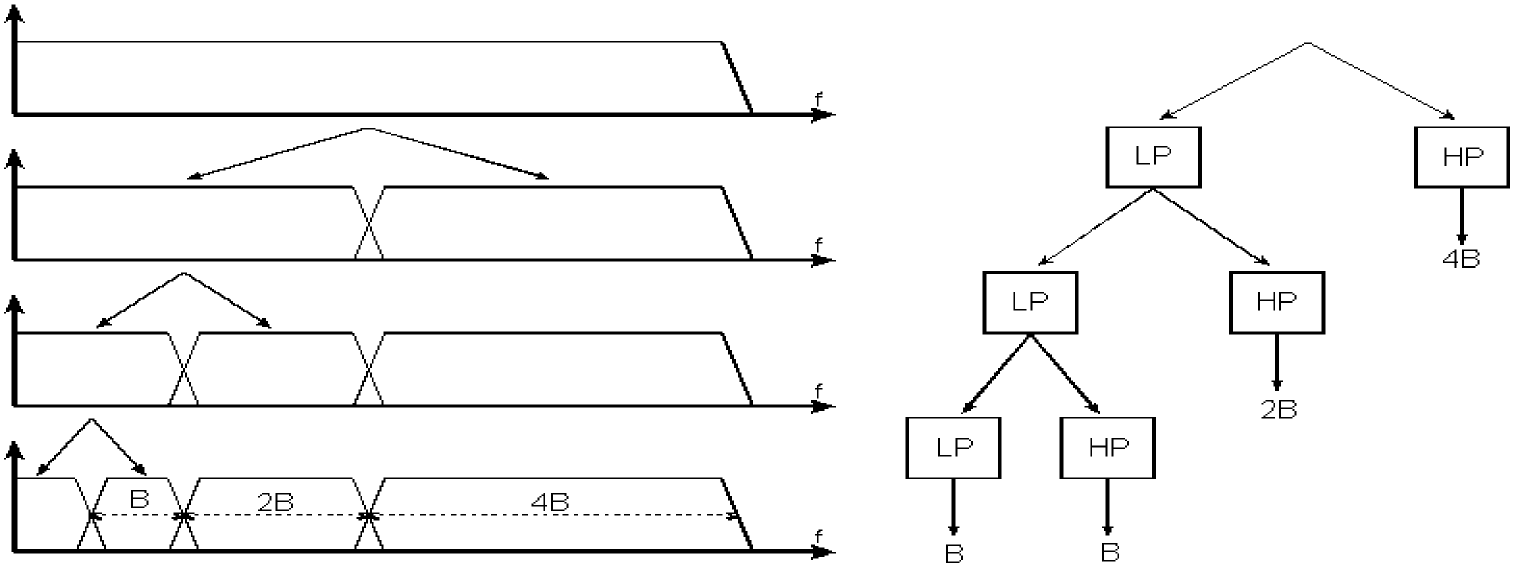
- **However, the low pass part still contains some signal details and, therefore, we can split it again and again, until we are satisfied with the number of bands that we have created**

- **In this way, we have created an *iterated filter bank*.**

- **Usually, the number of bands is limited by, for instance, the amount of data or computation power available.**

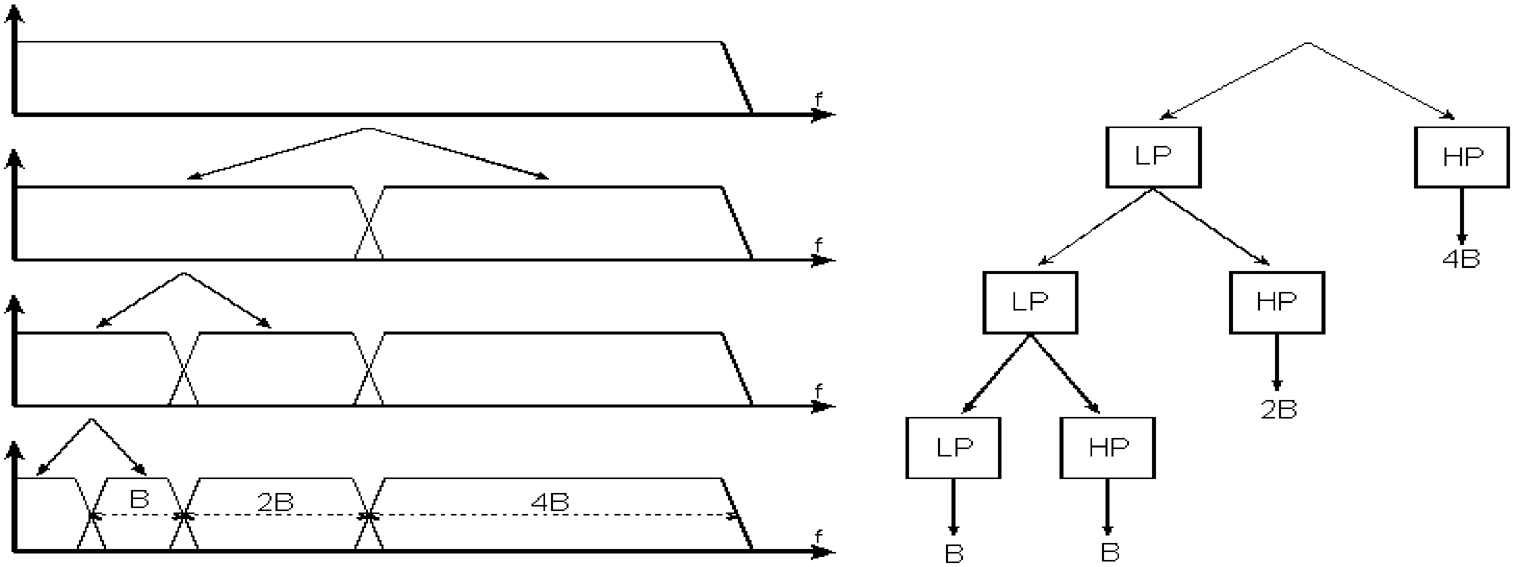
The Discrete Wavelet Transform Through a Filter Bank

The process of splitting a signal is displayed below:



The Discrete Wavelet Transform Through a Filter Bank

- **The advantage of this scheme is that we have to design *only two filters***
- ***The disadvantage* is that the width of every band *can not* be chosen freely**



Approximations and Details

- **The *low-frequency* content of a broadband signal is the most important signal part. It is what gives the signal its identity**

- **The *high-frequency* content, on the other hand, imparts nuances**

- **Consider the human voice. If you remove the high-frequency components, the voice sounds different, but you can still tell what's being said.**

- **However, if you remove enough of the low-frequency components, you hear gibberish**

- **In the wavelet analysis, we speak of *approximations* (low frequency components) and *details* (high frequency components)**

The DWT: Filtering and Subsampling

- **The procedure starts with passing a signal with time shift** b_0 **through two filters: *half band* lowpass (band pass) filter and half band highpass filter**

- **The lowpass filter removes all frequencies that are above half of the highest frequency in a signal**

- **After passing a signal through this filter, half of the samples can be eliminated according to the Nyquist's rule *without loss of information,* since half of the samples is redundant**

- **Simply discarding every other sample will *downsample a* signal by 2, and, therefore, a signal will have half of samples**

The DWT: Filtering and Subsampling

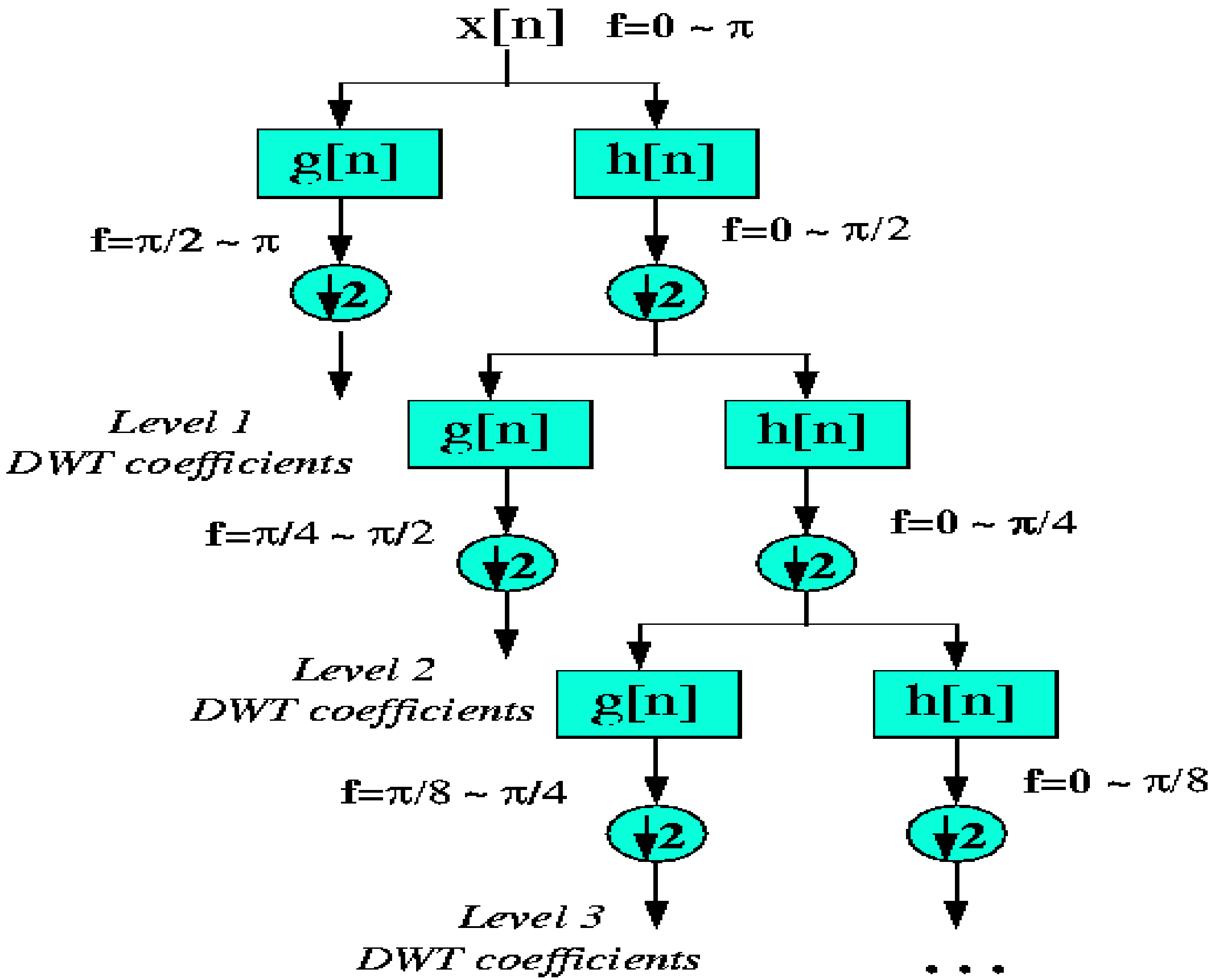
- **The DWT analyzes a signal at different frequency bands with different resolutions by decomposing a signal into approximations and details**

- **The decomposition of the signal into different frequency bands is simply obtained by successive high bandpass ("highpass") and low bandpass ("lowpass") filtering of a signal**

- **This procedure should be repeated for each discrete time shift**

- ***The difference of DWT from the usual filtering is that the time localization of these frequencies will not be lost***

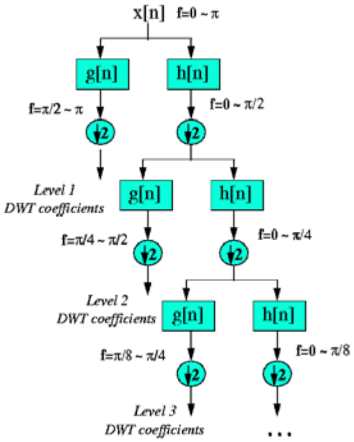
Filtering and Subsampling



Filtering and Subsampling

- **Suppose that the original signal has 512 time points, spanning a frequency band of zero to π rad/s. At the first decomposition level, the signal is passed through the highpass and lowpass filters, followed by subsampling by 2**

- **The output of the highpass filter has 256 points, but it only spans the frequencies $\pi/2$ to π rad/s**

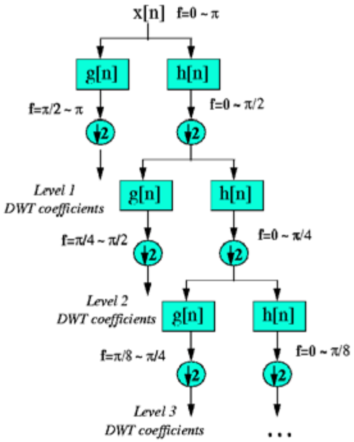


These 256 samples constitute the *first level* of DWT coefficients

Filtering and Subsampling

- **The output of the lowpass filter also has 256 samples, but it spans the other half of the frequency band from 0 to $\pi/2$ rad/s**

- **This signal is passed through the lowpass and highpass filters for further decomposition**

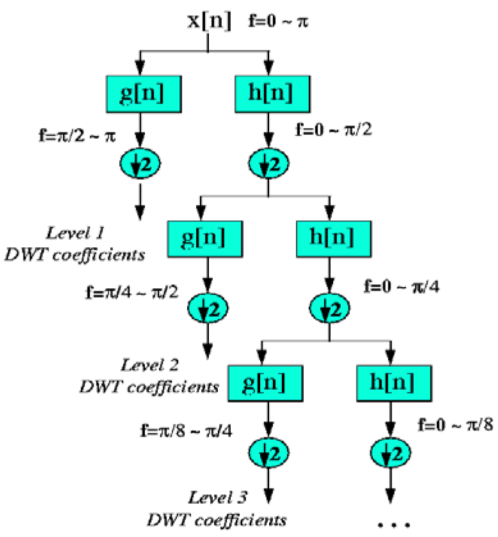


The output of the second lowpass filter followed by subsampling has 128 samples spanning a frequency band of 0 to $\pi/4$ rad/s, and the output of the second highpass filter followed by subsampling has 128 samples spanning a frequency band of $\pi/4$ to $\pi/2$ rad/s

Filtering and Subsampling

- **The second highpass filtered signal constitutes the *second level* of DWT coefficients.**

- **The lowpass filter output is then filtered once again for further decomposition**



Filtering and Subsampling

- **This process continues until some specific number of samples is left**

- **For this specific example, there would some levels of decomposition, each having half the number of samples of the previous level**

- **The DWT of the original signal is then obtained by combining all coefficients starting from the last level of decomposition (i.e. remaining two samples) in this case**

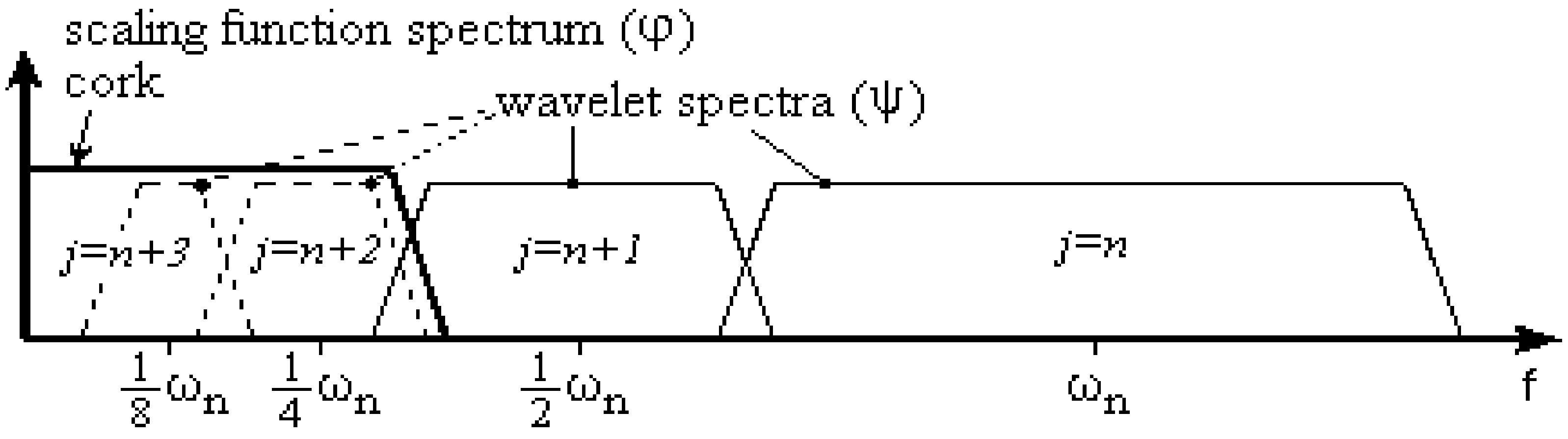
- **The DWT will then have the same number of coefficients as the original signal**

Scaling Function (Father Wavelet)

- **When we use a scaling function instead of wavelets *we lose information***
- **From a *signal representation* point of view we do not lose any information, since it will still be possible to reconstruct the signal**
- **However, from a wavelet analysis point of view we discard possible valuable scale information**
- **The *width* of the scaling function spectrum is therefore an important parameter. The shorter *width* the more wavelet coefficients you will have and the more scale information**
- **There are practical limitations on the coefficient number**

Scaling Function (Father Wavelet)

- **Since we selected the scaling function in such a way that its spectrum fitted in the space left open by the wavelets, we use *a finite* number of wavelets up to a certain scale *j***



- **In this way, we have limited the number of wavelets from an *infinite number to a finite number***

Frequency Width of the Scaling Function

- **Since the analysis process is iterative, in theory it can be continued indefinitely**

- **It should be clear where the iteration definitely has to stop and *this determines the width of the power spectral density of the scaling function***

- **In practice, you can select a suitable number of levels based on the nature of the signal**

The Multiresolution Analysis

- **This approach is called the *multiresolution analysis (MRA)***

- **MRA, as implied by its name, analyzes the signal at different frequencies with different resolutions**

  Every spectral component is not resolved equally

- ***MRA is designed to give good time resolution and poor frequency resolution at high frequencies and good frequency resolution and poor time resolution at low frequencies***

- **This approach makes sense especially when a signal has high frequency components of short durations and low frequency components of long durations**

Low Bandpass and High Bandpass Filters

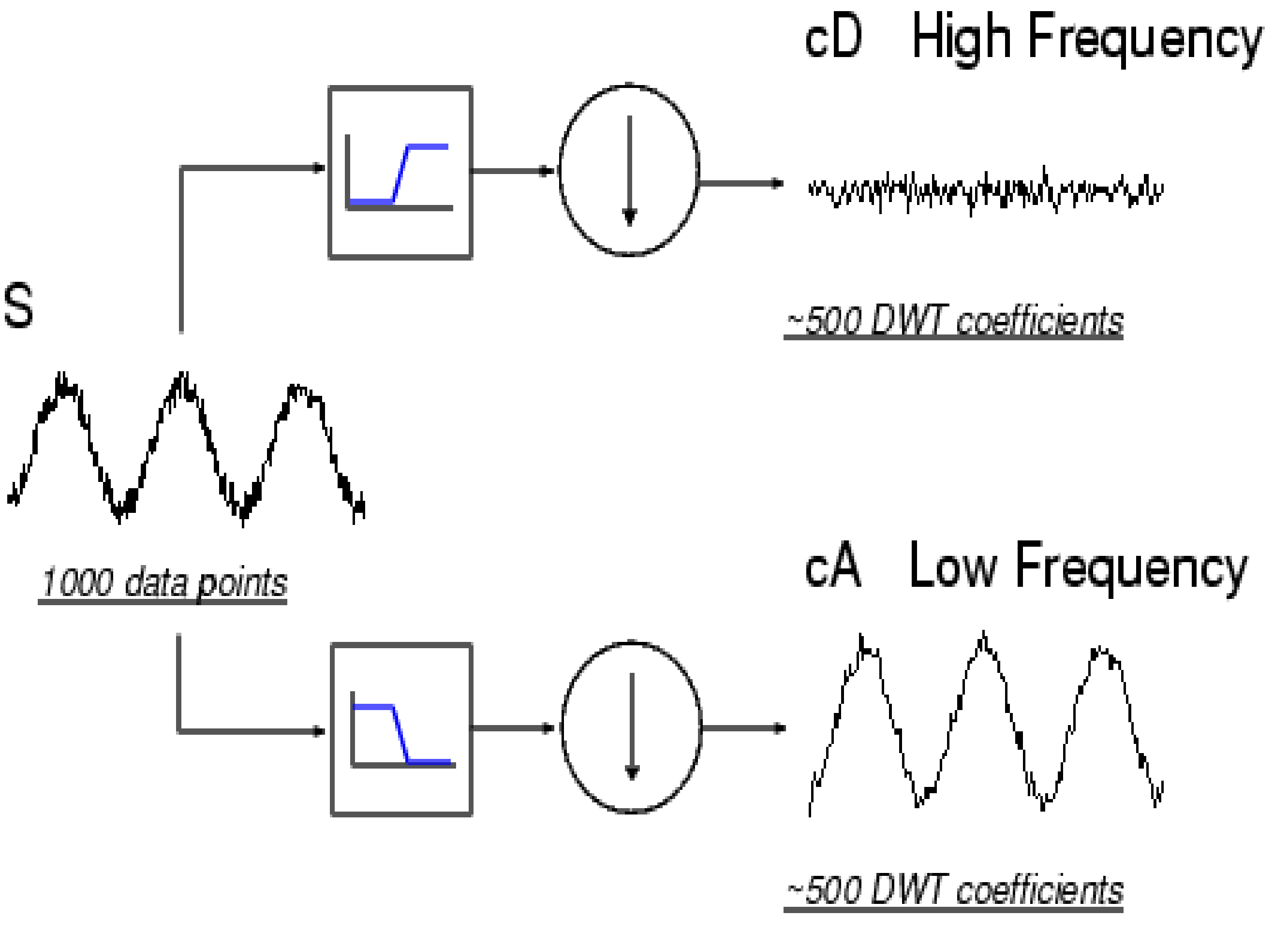
- **One important property of the discrete wavelet transform is the *relationship* between the impulse responses of the high bandpass and low bandpass filters.**

- **These filters are not independent; they are related by**

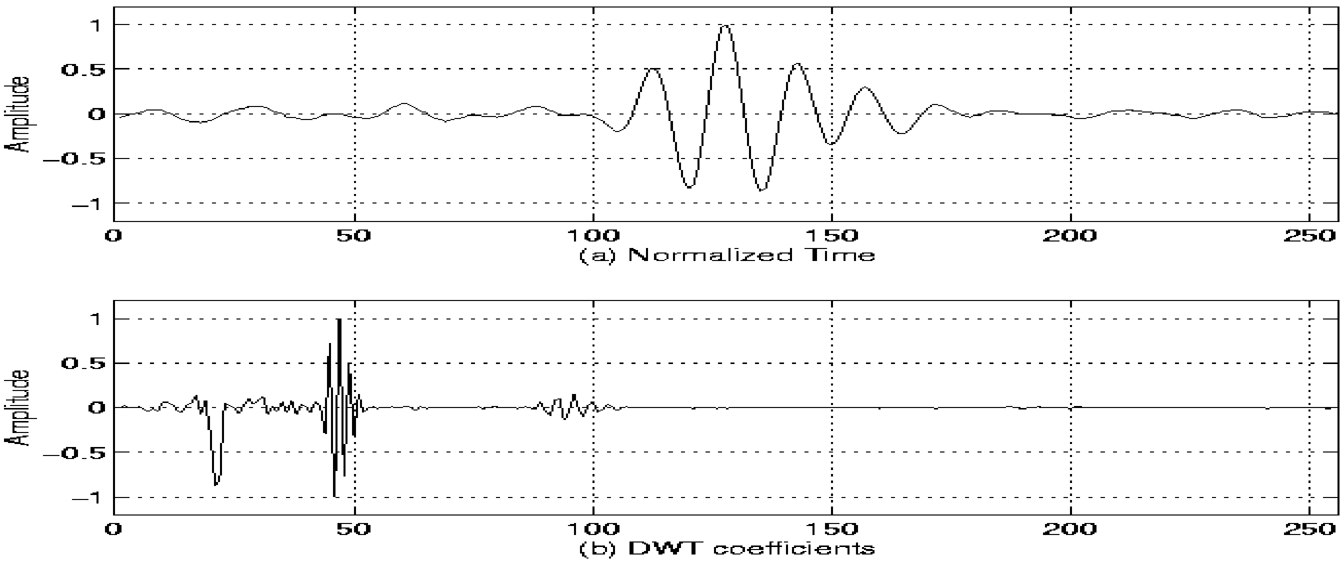
$$g(n) = (-1)^n h(L-n)$$

- **where $h(n)$ is the low bandpass filter, $g(n)$ is the highpass filter and L is the filter length**

The DWT: Case Study 1



The DWT: Case Study 2



Case Study 2

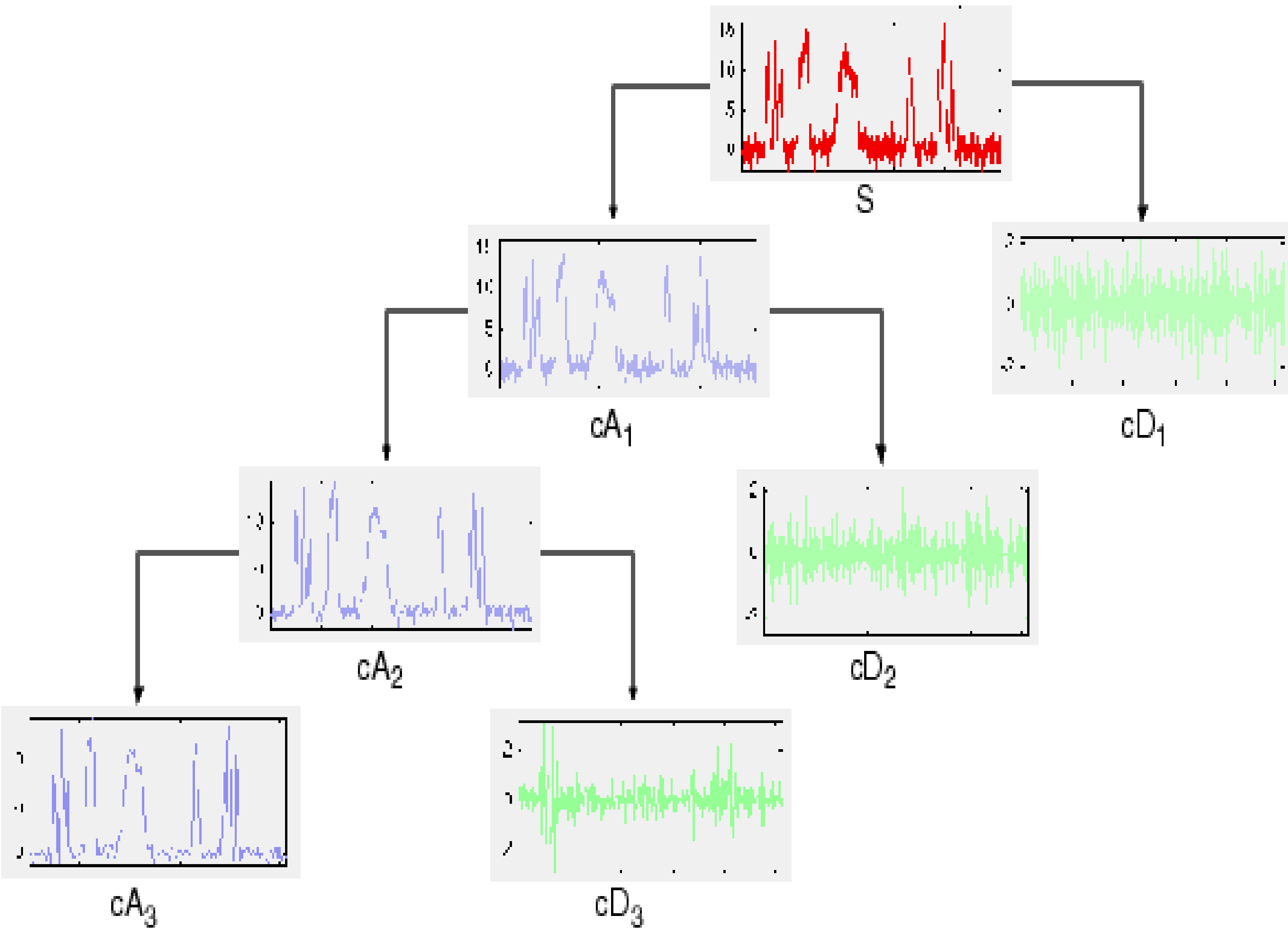
- **Figure** a **shows a 256 sample signal**
- **Figure** b **shows the 8 level DWT of the signal.**
- **The last 128 samples of the DWT correspond to the highest frequency band in the signal**
- **The previous 64 samples correspond to the second highest frequency band and so on.**
- **It should be noted that only the first 128 samples, which correspond to lower frequencies of the analysis, carry relevant information and the rest of DWT has no information**

Case Study 2: Data Reduction

- **Therefore, the last 128 samples can be discarded without any loss of information**

- **This is how the discrete wavelet transform provides a very effective *data reduction technique***

The DWT: Case Study 3



Signal Reconstruction

- **We've learned how the discrete wavelet transform can be used to analyze, or *decompose*, signals and images**

- **This process is called a *decomposition* or *analysis***

- **The other half of the "wavelet story" is how decomposed components can be assembled back into the original signal *without loss of information***

- **This process is called a *reconstruction* or a *synthesis***

- **The mathematical manipulation, that performs a synthesis is called the *inverse wavelet transform* (IWT)**

Signal Reconstruction from CWT

- **From the continuous wavelet transform, the original function can be recovered by utilizing**

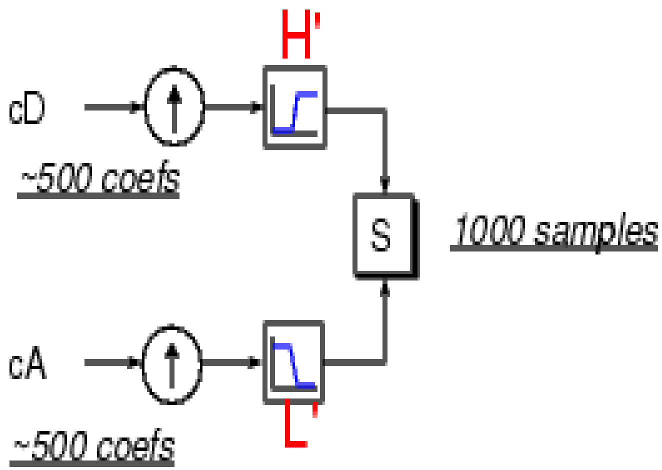
$$s(t) = \frac{1}{C_g} \int_a \int_b C(a,b)\, \psi_n(a,b) \frac{da\,db}{a^2}$$

- **This allows the original signal to be recovered from its wavelet transform by integrating over all scales and shifts**

- **If we *limit* the integration over a range of scales rather than all scales, we can perform a basic filtering of the original signal**

Signal Reconstruction from Filter Bank

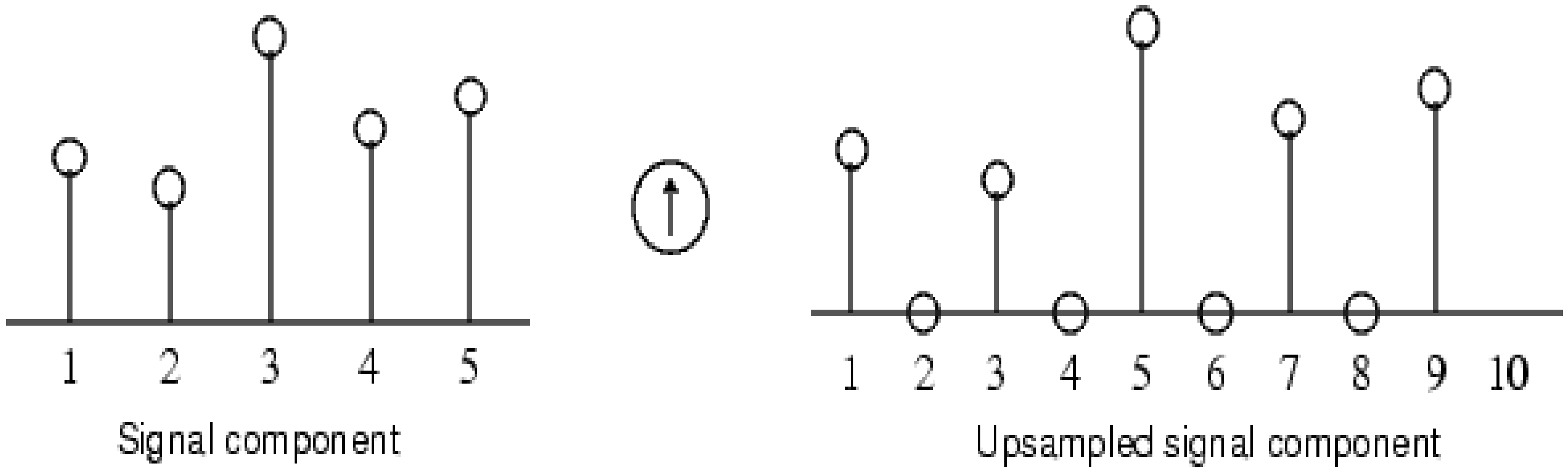
- **To synthesize a signal, we reconstruct it from the wavelet coefficients of the approximations and details**



- **Where wavelet analysis involves filtering and downsampling, the wavelet reconstruction consists of *upsampling* and *filtering***

Signal Reconstruction from Filter Bank

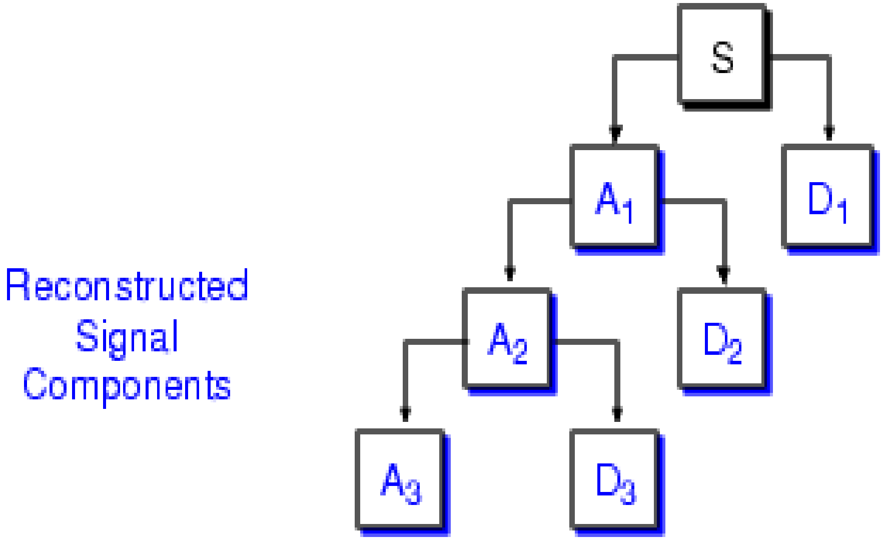
- ***Upsampling*** **is the process of lengthening a signal component by** **inserting zeros** **between samples**



Separate Reconstruction

- **Extending this technique to the components of a multilevel analysis, we find that similar relationships hold for all the reconstructed signal components**

- **That is, there are several ways to reassemble the original signal:**



$$S = A_1 + D_1$$

$$= A_2 + D_2 + D_1$$

$$= A_3 + D_3 + D_2 + D_1$$

Reconstruction Filters

- **The choice of filters is crucial in achieving perfect reconstruction of the original signal**

- **Ideal brick-wall filters are not realizable**

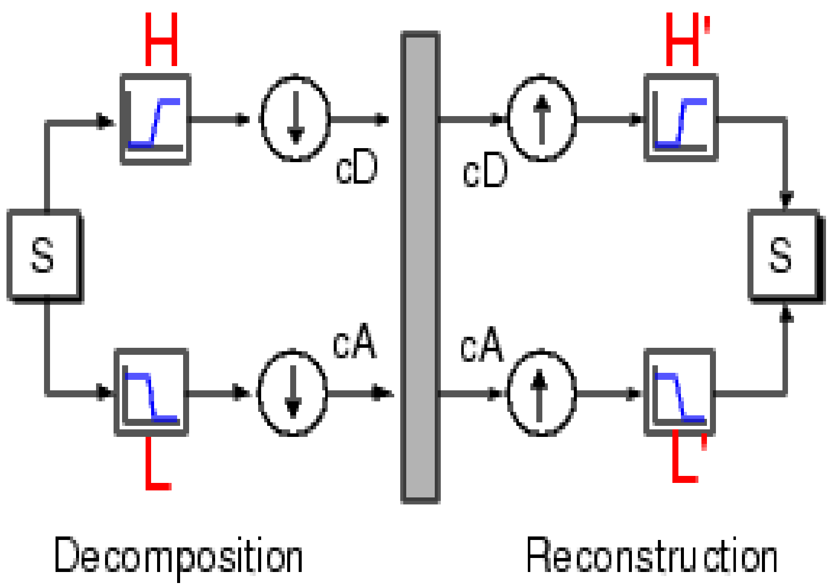
- **Therefore, the downsampling of the signal components performed during the decomposition may introduce aliasing**

- **By carefully choosing filters for the decomposition and reconstruction phases (*that are closely related but not identical*), we can cancel out the effects of aliasing**

- **One way to reduce possible aliasing is to use quadrature mirror filters**

Quadrature Mirror Filters

- **The low- and highpass decomposition filters (L and H), together with their associated reconstruction filters (L' and H'), form a system *quadrature mirror filters***



- **It is desirable for the QMF to have linear phase**

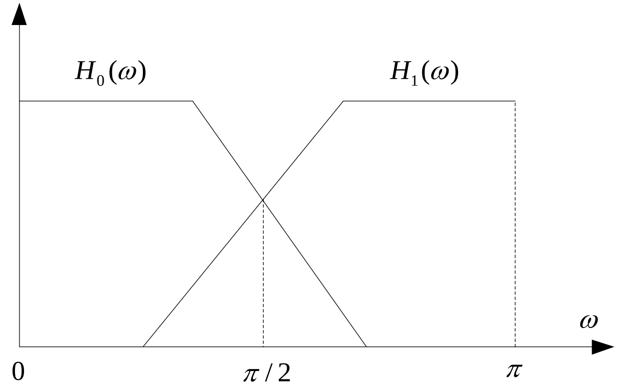
Quadrature Mirror Filters

- **In order to eliminate aliasing, these filters should satisfy the following properties:**

$$L(z) = G(z) \qquad H(z) = G(-z)$$

$$L'(z) = 2G(z) \qquad H'(z) = -2G(-z)$$

- **where** $G(z)$ **is the transfer function of a suitable lowpass filter,** $L(z)$ **and** $H(z)$ **are the transfer functions of lowpass and highpass *analysis* filters respectively;** $L'(z)$ **and** $H'(z)$ **are the transfer functions of lowpass and highpass *synthesis* filters respectively**



Quadrature Mirror Filters

Notice that under the above conditions, we need only to design one filter (a suitable lowpass filter) from which all the other filters can be obtained

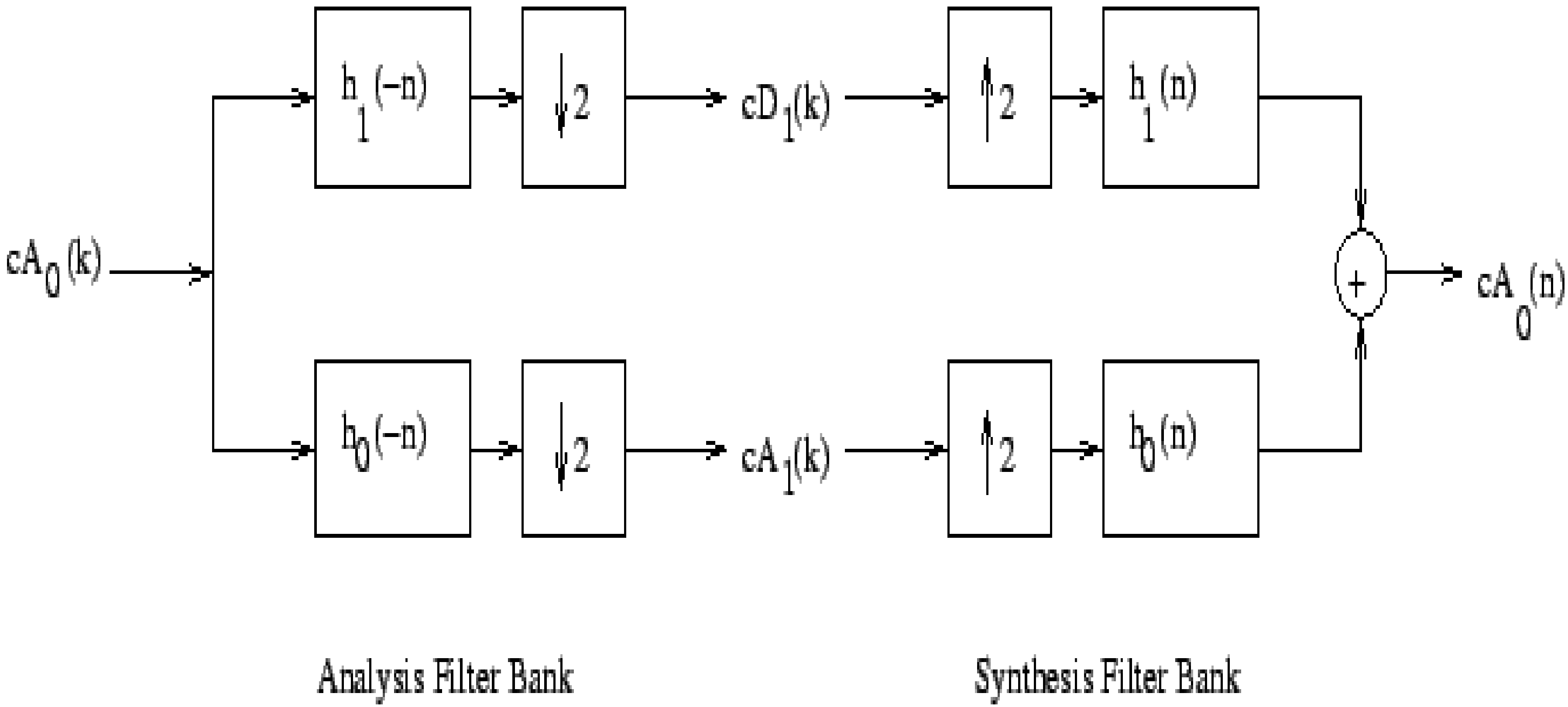
Perfect Reconstruction Filter Bank

- **In the absence of quantization errors the output of the system “analysis filter bank-synthesis filter bank” could be exactly the same as the input of this system. This condition is called *perfect reconstruction***

- **Perfect reconstruction can be performed in the case where the reconstruction filters are time reversed versions of the analysis filters**

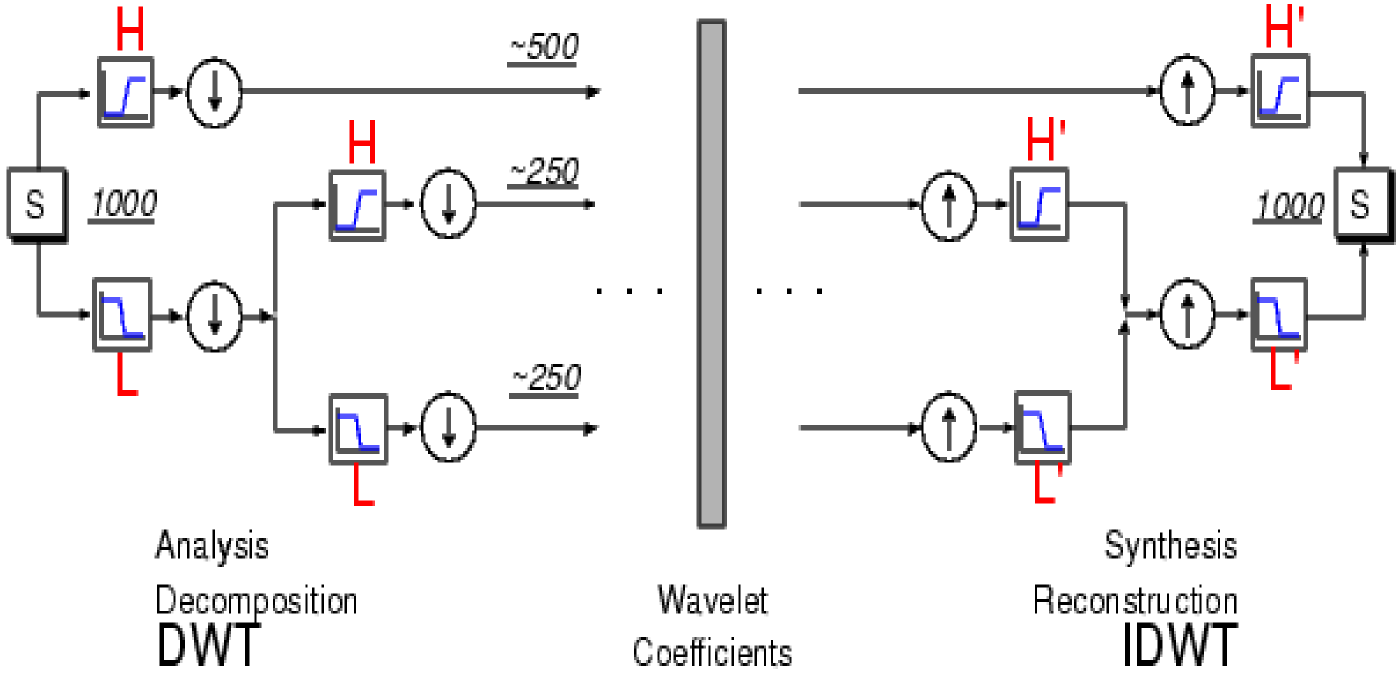
- **Notice that under these conditions we also need only design one analysing filter (lowpass or highpass) from which all the other filters can be obtained**

Perfect Reconstruction Filter Bank



Perfect Reconstruction Filter Bank

Multi-step Analysis and Reconstruction



Process involves two phases: breaking up a signal to obtain the wavelet coefficients, and reconstructing the signal from the coefficients

Analysis and Reconstruction

- **Of course, there is no point breaking up a signal merely to have the satisfaction of immediately reconstructing it**

- **We may *modify the wavelet coefficients* before performing the reconstruction step**

- **Modification of the wavelet coefficients before reconstruction is the main idea of "de-noising"**

Denoising

- **Denoising includes three main steps:**

- **The wavelet analysis**

- **A thresholding (i. e. reduction or complete removal of selected wavelet coefficients) in order to separate out noise and signal**

- **A wavelet reconstruction**

- **A thresholding removes the smallest amplitude coefficients regardless of scale**

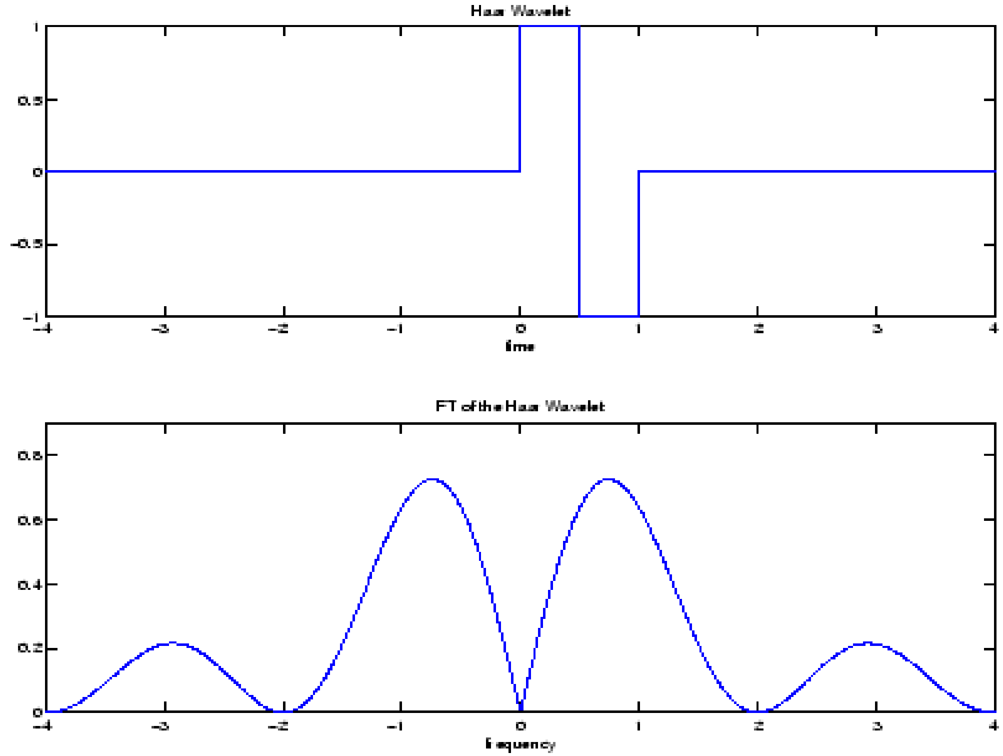
Denoising

Wavelet Functions: Real Wavelets

- **Wavelet family consists of *real* wavelets and *complex* wavelets**

Real wavelets

- **The Haar wavelet is discontinuous step function**



The Haar Wavelet

- **It is a compactly supported wavelet function**

- **The Haar wavelet is suitable for analyzing discontinuous functions and not suitable for analyzing continuous functions**

The Daubechies Wavelets

- **Ingrid Daubechies invented *compactly supported* wavelets**

- **Thus, making discrete wavelet analysis through filter bank practicable**