The Wavelet Transform

Part 1

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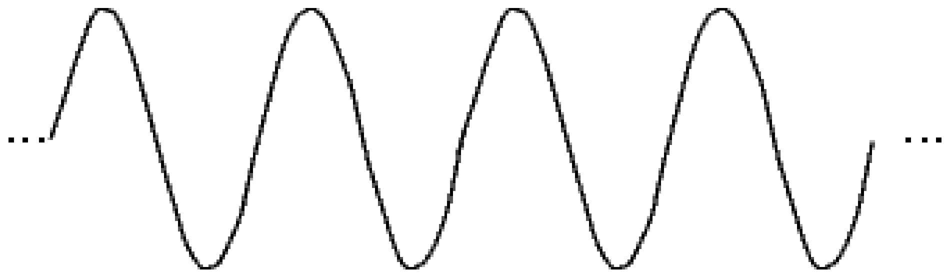
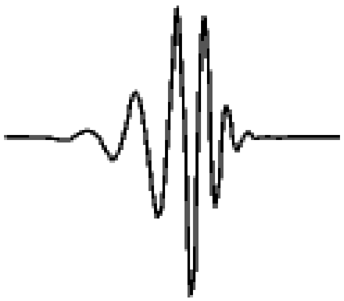
Introduction

- **The wavelet analysis represents the next logical step beyond the short time Fourier transform: a windowing technique with a *window length that is dependent on frequency***

- **A wavelet function is a waveform of a limited duration that has an average value of zero (from French: a small wave)**



Sine Wave



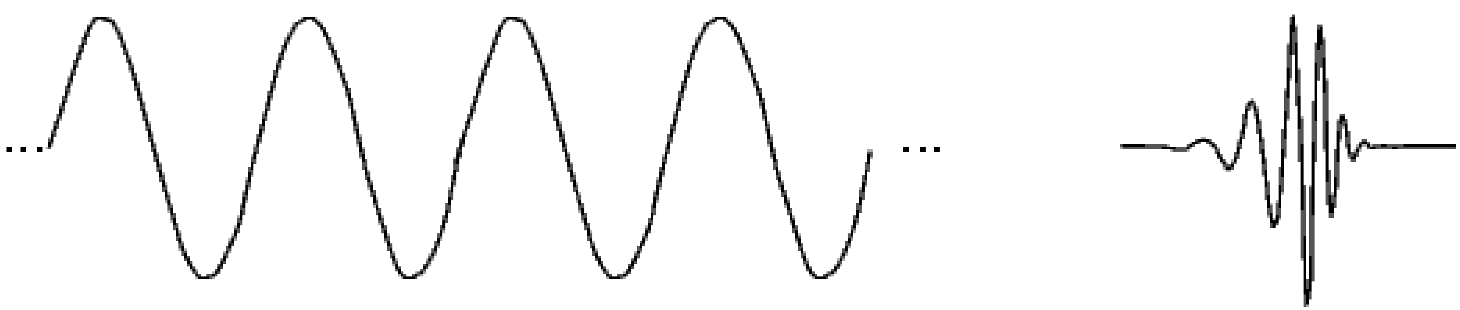
Wavelet (db10)

Wavelet and Sine Wave

- Let us compare wavelets with sine waves

- Sinusoids do not have a limited duration -- they extend from minus to plus infinity

- Where sinusoids are regular, wavelets tend to be irregular



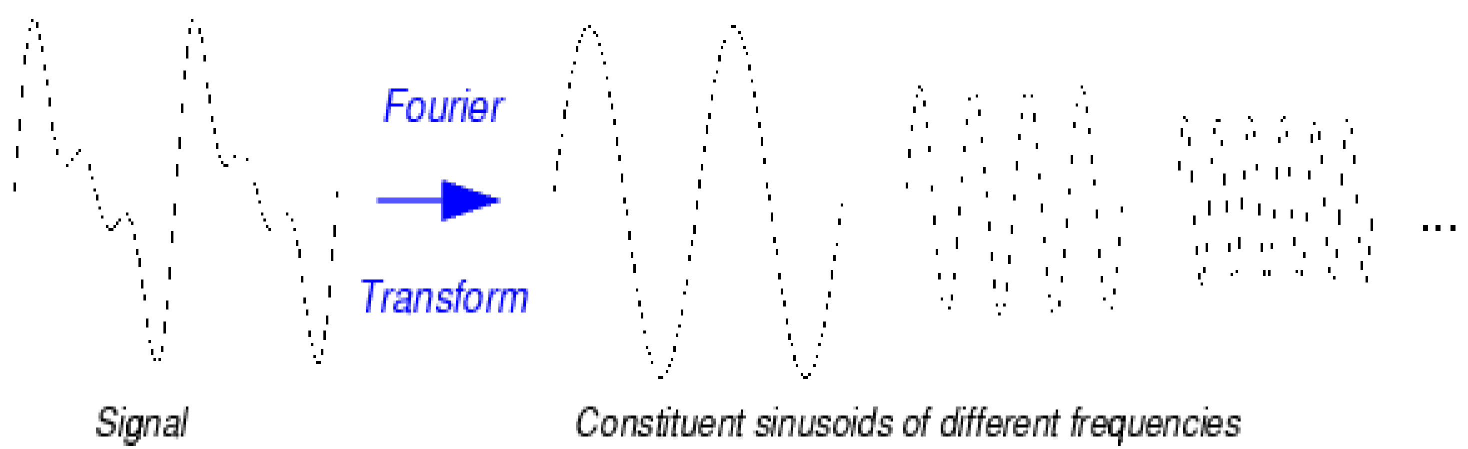
Sine Wave

Wavelet (db10)

From Continuous Fourier Transform

- The results of the continuous Fourier transform are the Fourier coefficients, which when multiplied by a sinusoid of frequency ω yield the constituent sinusoidal components of the original signal

- Graphically, the process looks like



To the Continuous Wavelet Transform

- **Similarly, the *continuous wavelet transform* (CWT) is defined as the sum over all time of the signal multiplied by *scaled* and *shifted* versions of the *mother wavelet* ψ**

$$C\left(scale, shift\right) = \frac{1}{\sqrt{\left|scale\right|}} \int_{-\infty}^{\infty} s(t)\psi^{*}(scale, shift, t)\,dt$$

where the asterisk indicates the complex conjugate

The Continuous Wavelet Transform

The continuous wavelet transform is defined as follows:

$$C(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} s(t)\psi^*\left(\frac{t-b}{a}\right)dt = s \otimes \psi^*_c$$

where

$$\psi_c(u) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{-u}{a}\right)$$

the parameter a represents the *scale factor*, determining the *centre frequency* of the wavelet function

the parameter b indicates the *shift factor*, determining the *centre time* of the wavelet function

$\otimes$ denotes the convolution

The Continuous Wavelet Transform: a Convolution Between Signal and Wavelet

•**For convenience of students, familiar with filter theory notation, let us re-write the above equation:**

$$C(a,t) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} s(\tau)\psi^*\left(\frac{\tau - t}{a}\right) d\tau = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} s(\tau)\psi^*\left(\frac{-(t-\tau)}{a}\right) d\tau$$

•**As is well known from the filter theory, the last equation can be viewed as a convolution:**

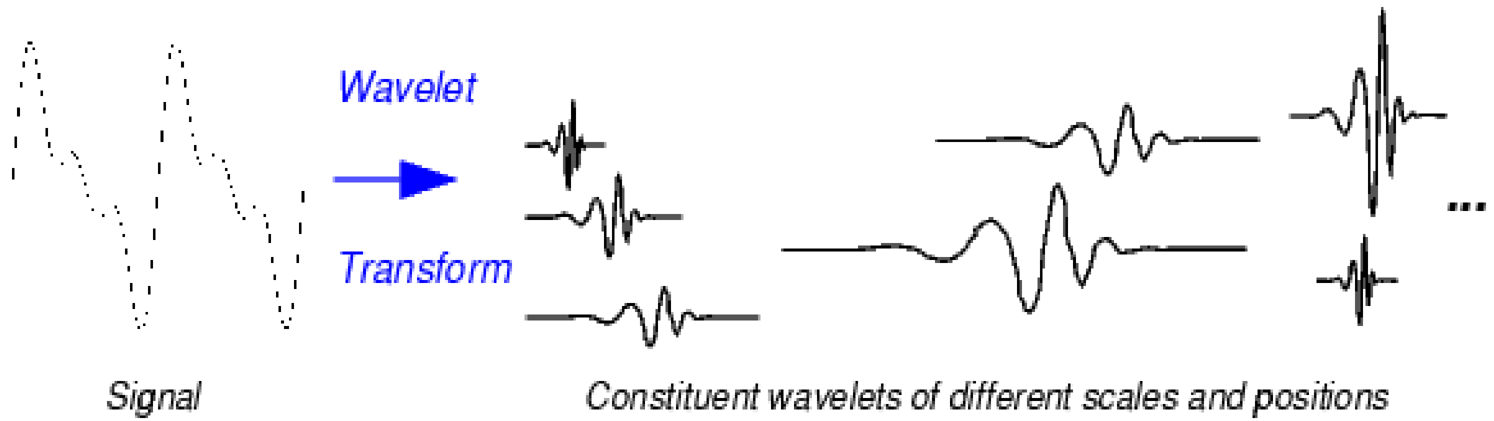
$$C(a,b) = s \otimes \psi^*{}_c \qquad \psi_c(u) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{-u}{a}\right)$$

The Continuous Wavelet Transform

• **The shifted and scaled wavelet is called a *baby* wavelet**

• **The result of the CWT is many wavelet coefficients , which are a function of scale and shift**

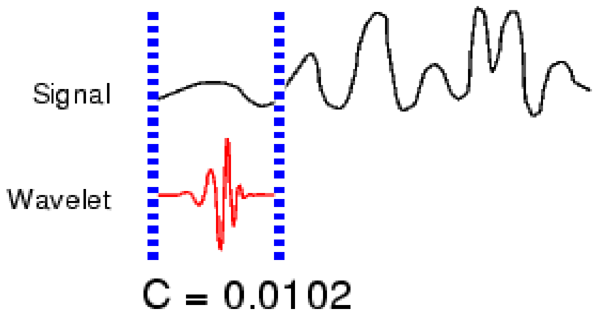
• **Multiplying each coefficient by the appropriately scaled and shifted wavelet yields the constituent wavelets of the original signal. Graphically, the process looks like**

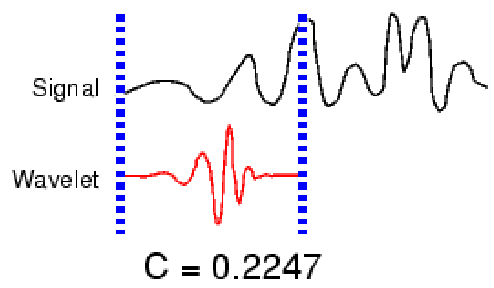


The Continuous Wavelet Transform: Correlation Between Signal and Wavelet

- The continuous wavelet transform is the ***cross correlation*** of the signal with the scaled and shifted wavelet

- This cross correlation is a measure of the similarity between signal and the scaled and shifted wavelet

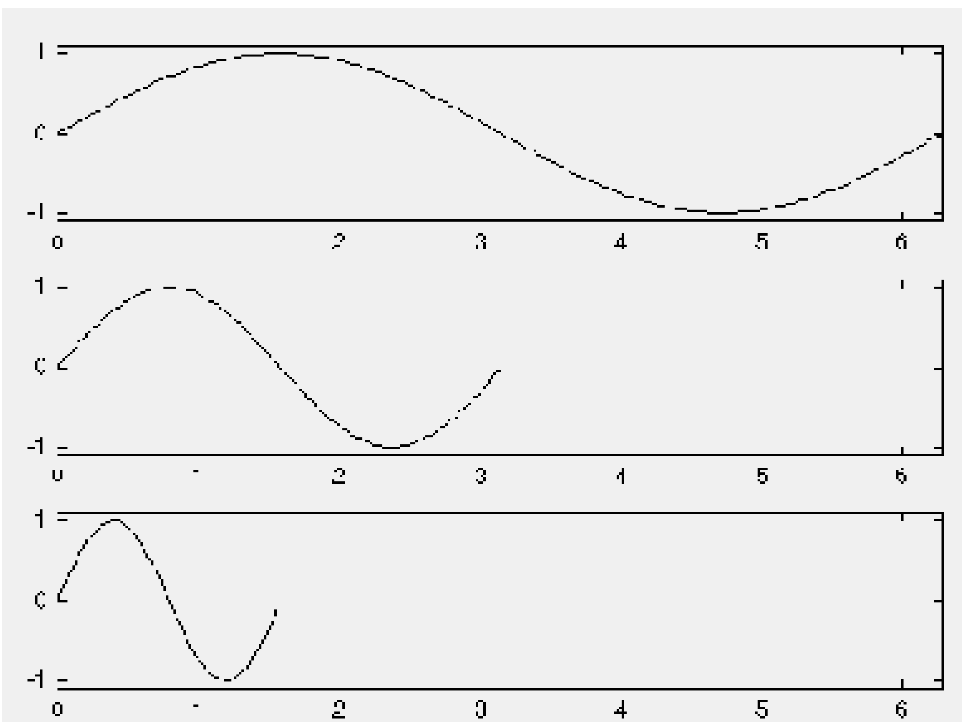




Scaling a Wavelet Function

- **What do we mean?**
- **Scaling a wavelet function means *stretching or compressing* it**

•If we're talking about sinusoids, the effect of the scaling is also very easy to see



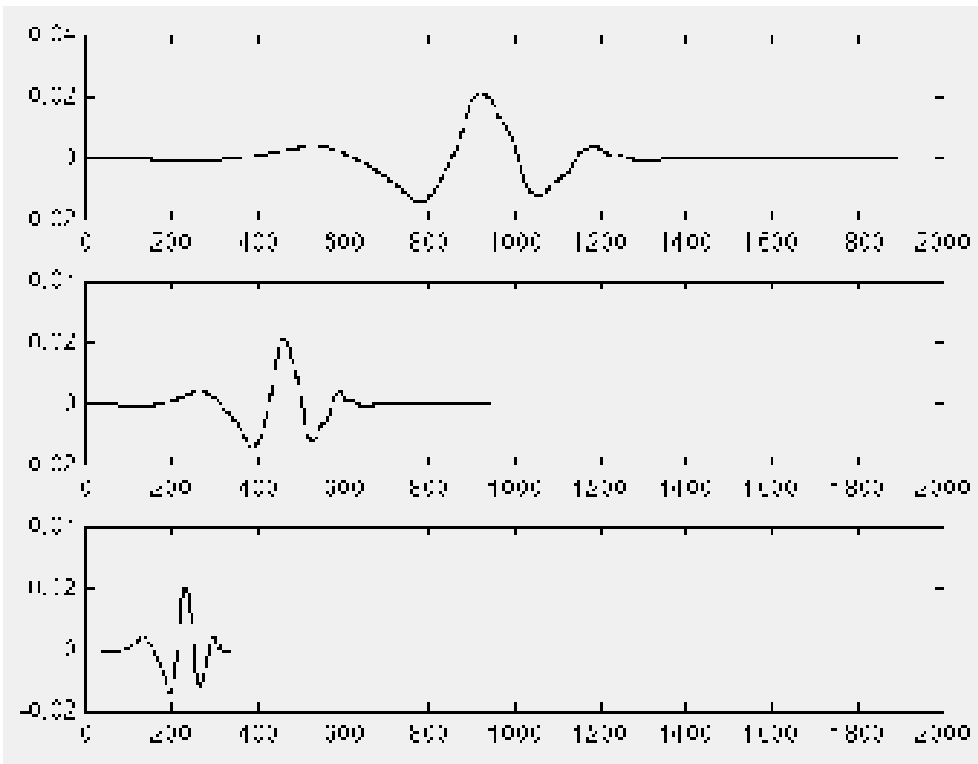
$$f(t) = \sin(t); \quad a = 1$$

$$f(t) = \sin(2t); \quad a = \frac{1}{2}$$

$$f(t) = \sin(4t); \quad a = \frac{1}{4}$$

Scaling a Wavelet Function

- **The scale factor works exactly the same with wavelets**
- **The smaller the scale factor, the more "compressed" the wavelet function**



$$f(t) = \psi(t) \quad ; \quad a = 1$$

$$f(t) = \psi(2t) \; ; \quad a = \frac{1}{2}$$

$$f(t) = \psi(4t) \; ; \quad a = \frac{1}{4}$$

Scaling a Wavelet Function

- **It is clear from figures that, for a sinusoid the scale is related *inversely* to the Fourier frequency**

- **Similarly, with wavelets, the scale is related inversely to the frequency of the wavelet function**

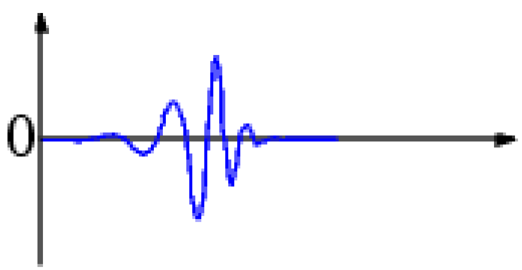
- **If $a \leq 1$, then wavelets look like a compressed versions of the mother wavelet**

- **If $a \geq 1$, then wavelets look like a stretched versions of the mother wavelet**

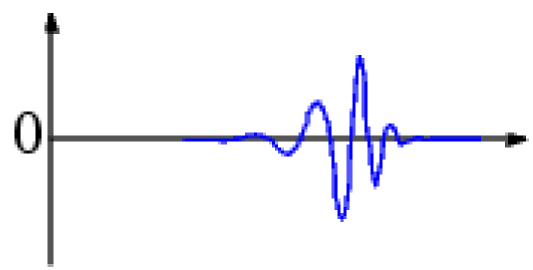
Shifting a Wavelet Function

- **Shifting a wavelet function simply means delaying its onset**

•**Mathematically, delaying a function by** k **is represented by** $\psi(t-k)$



Wavelet function
$\psi(t)$



Shifted wavelet function
$\psi(t-k)$

Scaling and Shifting a Wavelet Function

- **Suppose that mother wavelet is centred at** $t = t_0$ **and its Fourier transform has frequency centre at** $f = f_c$

- **Therefore, the time and frequency centres of the scaled and shifted wavelet are**

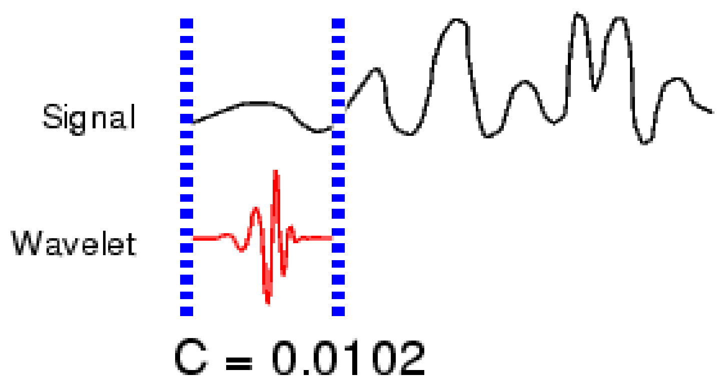
$$at_0 + b, \qquad \textbf{and} \qquad \frac{f_c}{a}$$

Five Easy Steps to the Continuous Wavelet Transform

1. Compare wavelet to a section at the start of the original signal

2. Calculate a wavelet coefficient which represents how closely correlated the wavelet is with this section of the signal

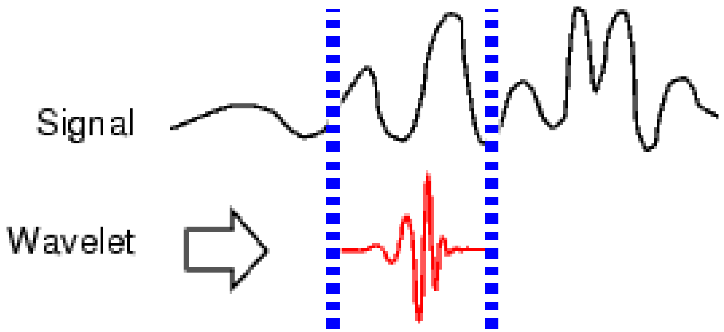
The higher coefficient is, the more the similarity



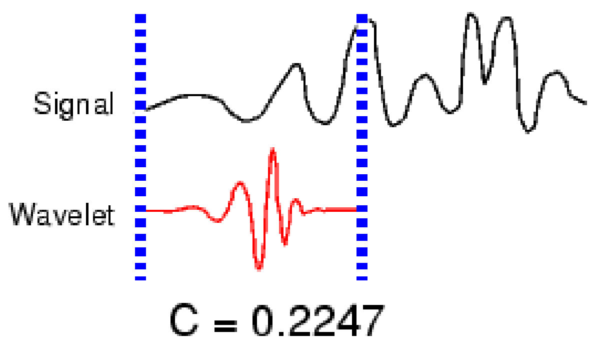
- **The results will depend on the shape of the wavelet you choose**

Five Steps to the Continuous Wavelet Transform

3. **Shift** the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal



4. **Scale** (stretch) the wavelet and repeat steps 1 through 3



Five Steps to the Continuous Wavelet Transform

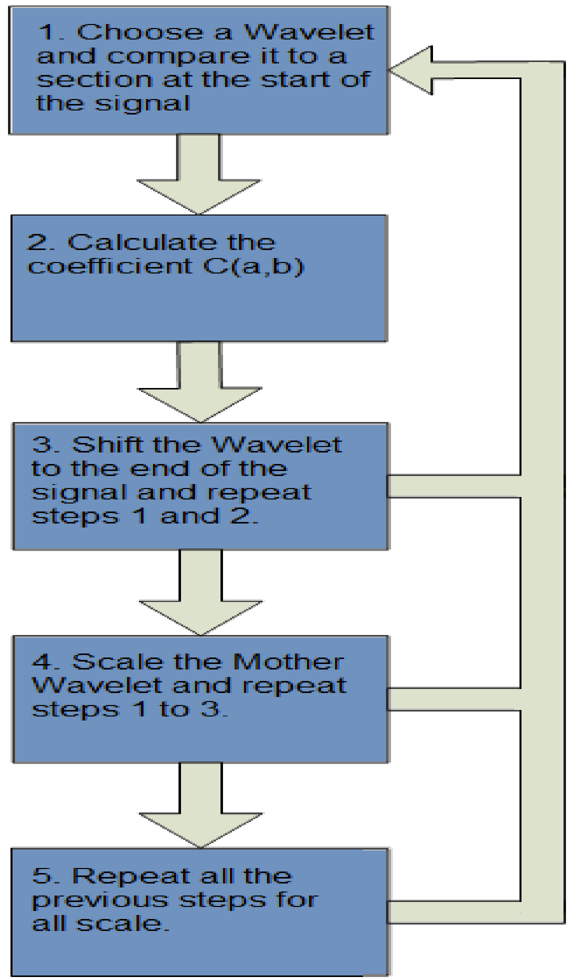
5. Repeat steps 1 through 4 for all scales

- **When you're done, you'll have the wavelet coefficients produced at different scales by different sections of the signal**

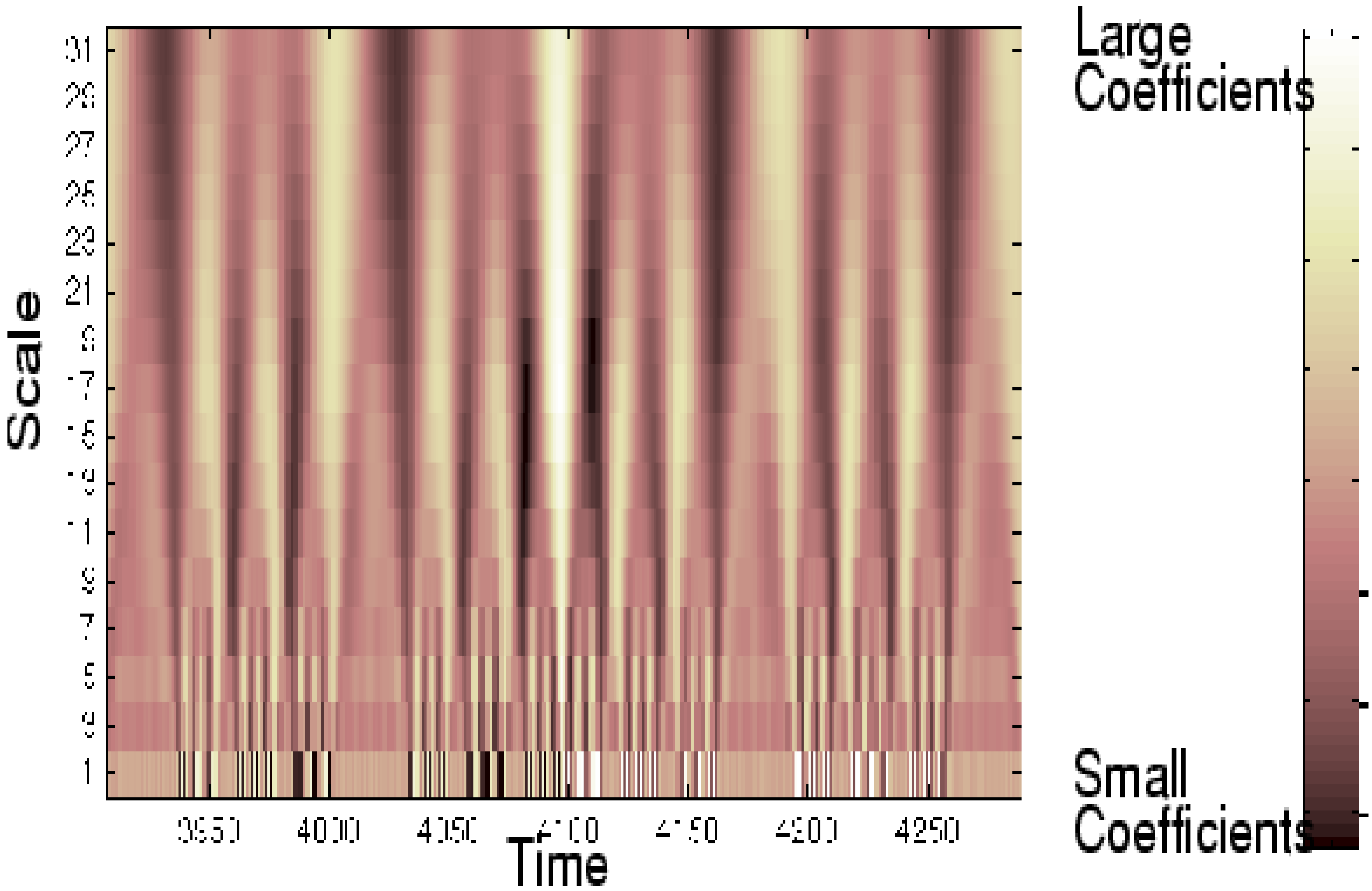
- **How to make sense of all these coefficients?**

- **You could make a plot on which the x-axis represents position along the signal (time), the y-axis represents scale, and the colour at each point represents the magnitude of the coefficient**

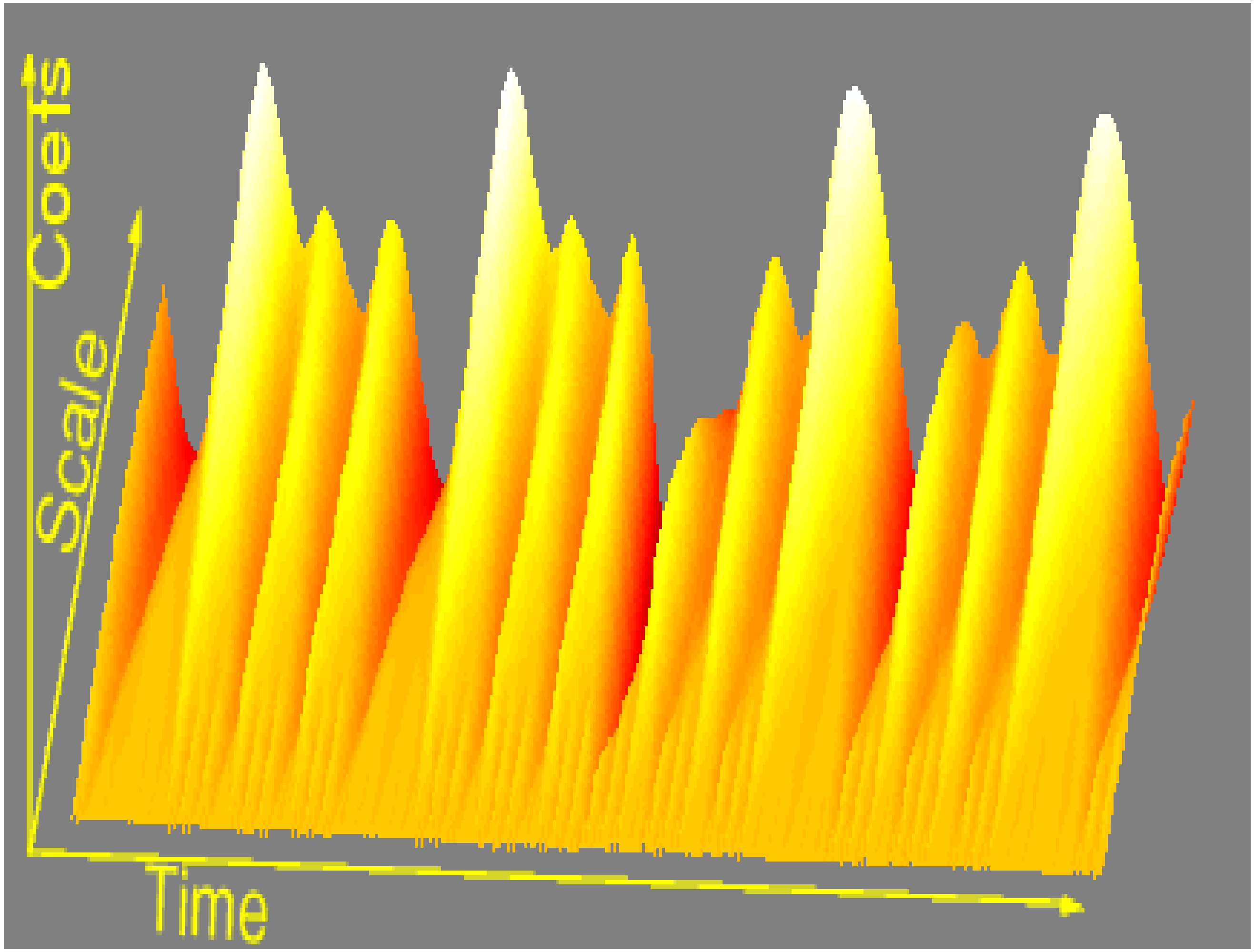
Five Steps to a Continuous Wavelet Transform



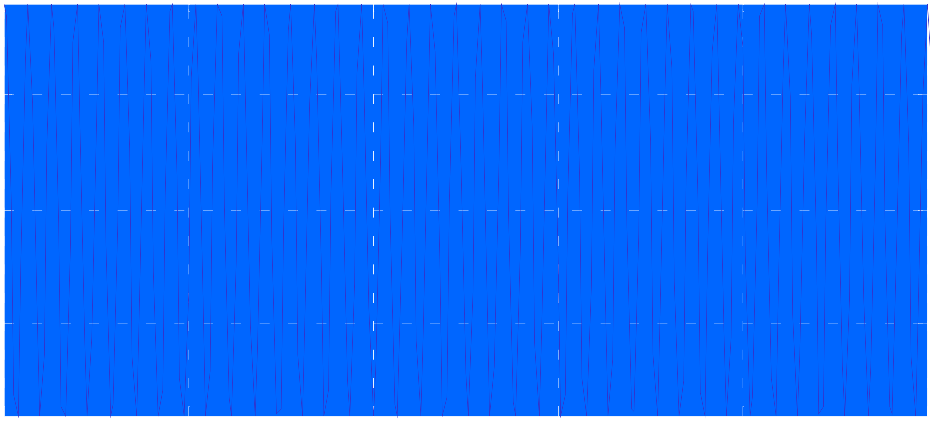
2D Wavelet Transform Plot

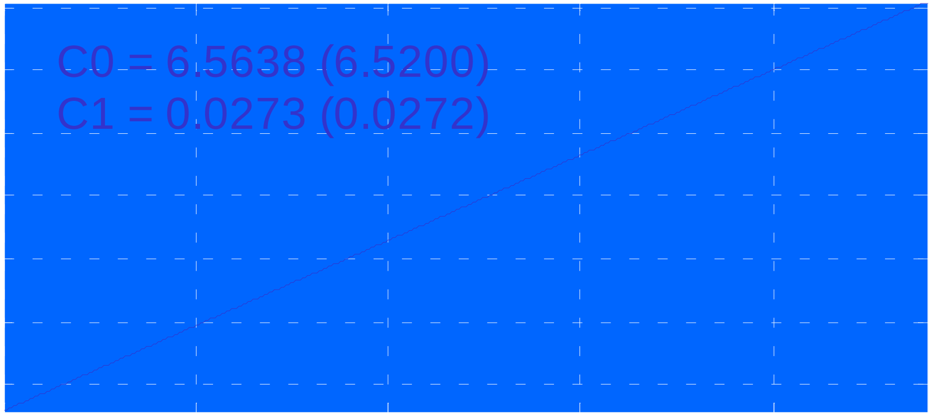


3D Wavelet Transform Plot

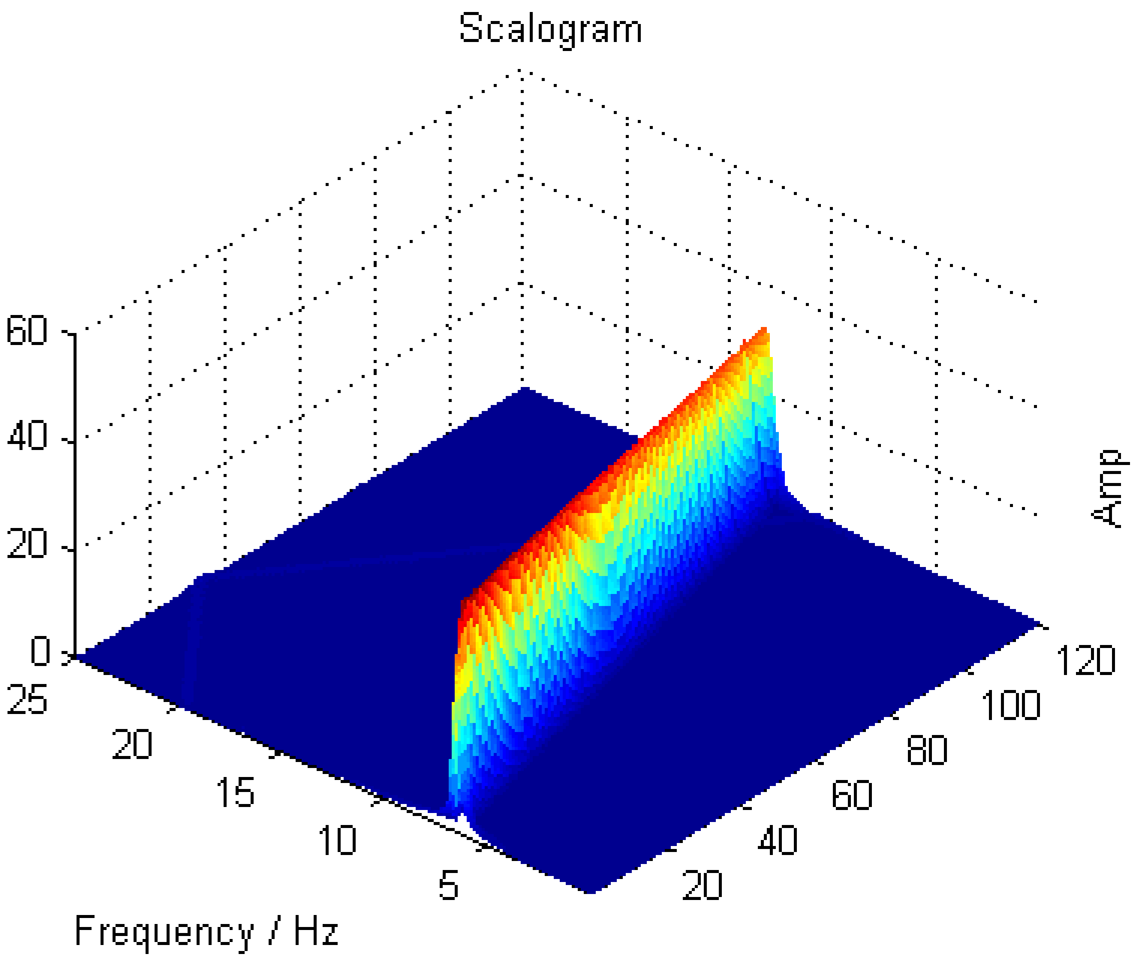


Tacho Signal and Its Instantaneous Frequency Estimation





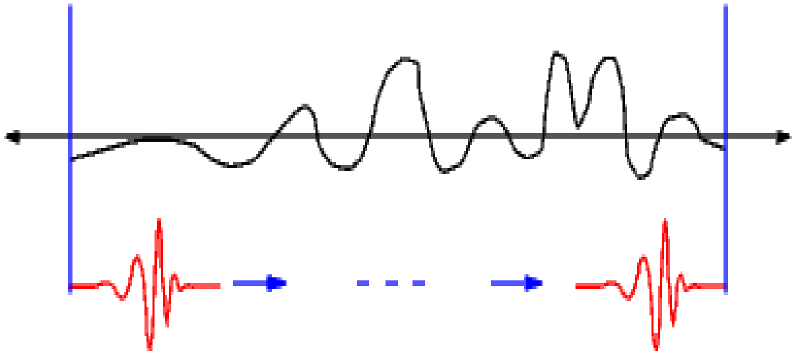
Tacho Signal and Its Instantaneous Frequency Estimation by the Scalogram



What's *Continuous* in the Continuous Wavelet Transform?

- What's "continuous" in the CWT is the set of scales and shifts at which it operates

- Unlike the discrete WT, the CWT operates at every scale and shift; analysing wavelet is shifted smoothly over the full domain of the analysed signal



Similarities between the Fourier, Short Time Fourier and the Wavelet Transforms

- The Fourier and the wavelet transforms are ***linear*** operations

- The basis functions are **localized *in frequency*** for these transforms, making mathematical tools such as power spectral densities (i.e. square magnitudes of the Fourier transform) and scalograms (i.e. square magnitudes of the wavelet transform) useful at picking out frequencies and calculating power distributions

- For these transforms we are using **scaled** basis functions

- For the short time Fourier and wavelet transforms we are using **time windows**

Dissimilarities between the Fourier, the Short Time Fourier and the Wavelet Transforms

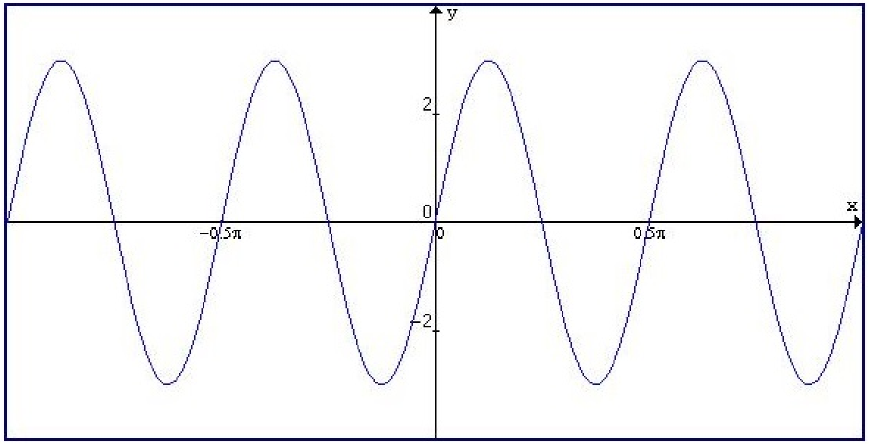
- **The most important dissimilarity between the Fourier and wavelet transforms is that wavelet functions are *localized in time (space);* Fourier functions *are not***

For time localization of the short time Fourier transform we are using constant time window

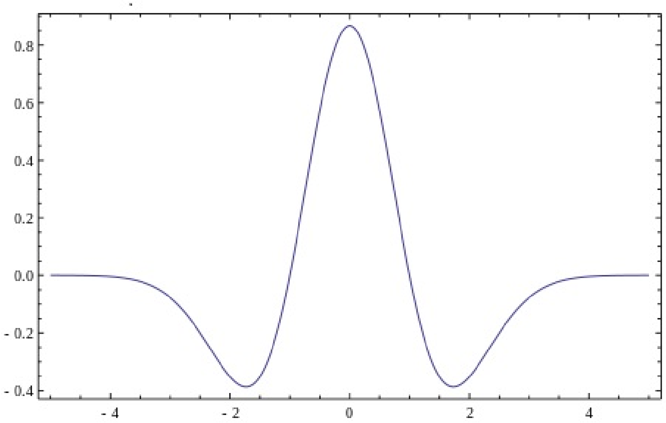
For time localization of the wavelet transform we are using variable time window.

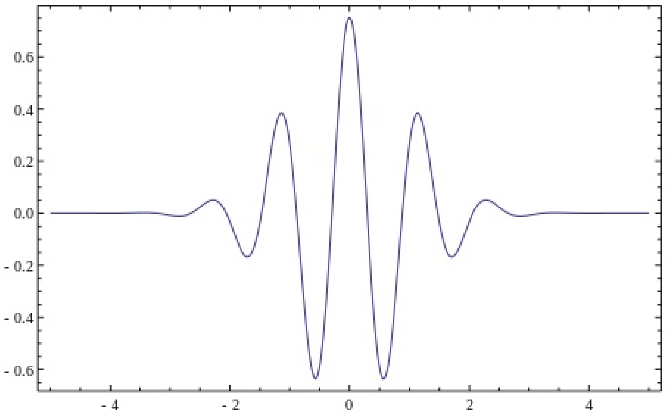
Dissimilarities between the Fourier Transform and the Wavelet Transform

Sine wave (goes on forever)



Wavelets

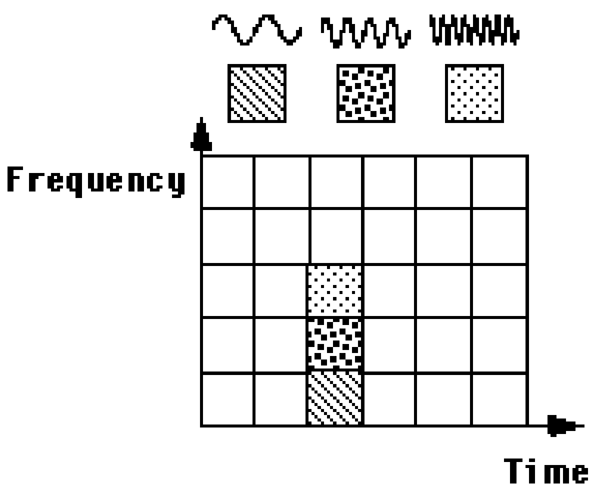


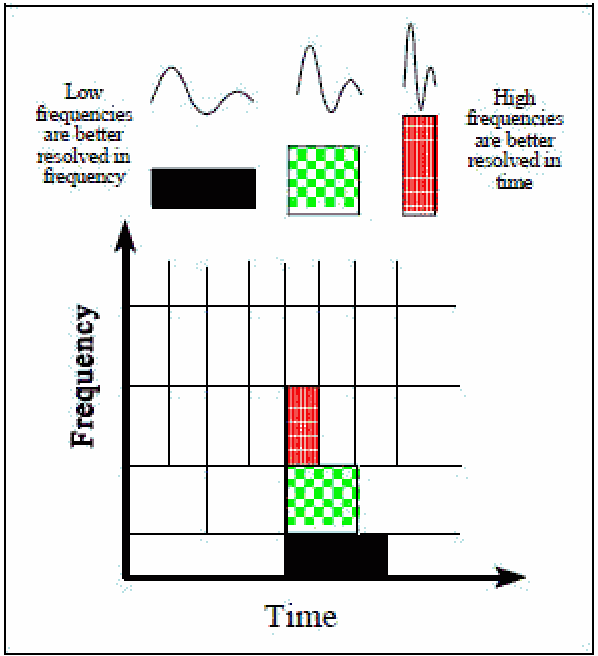


Dissimilarities between Short Time Fourier and Wavelet Transforms

• **Dissimilarities in time-frequency resolution**

One way to see these dissimilarities is to look at the basis function coverage of the time-frequency plane





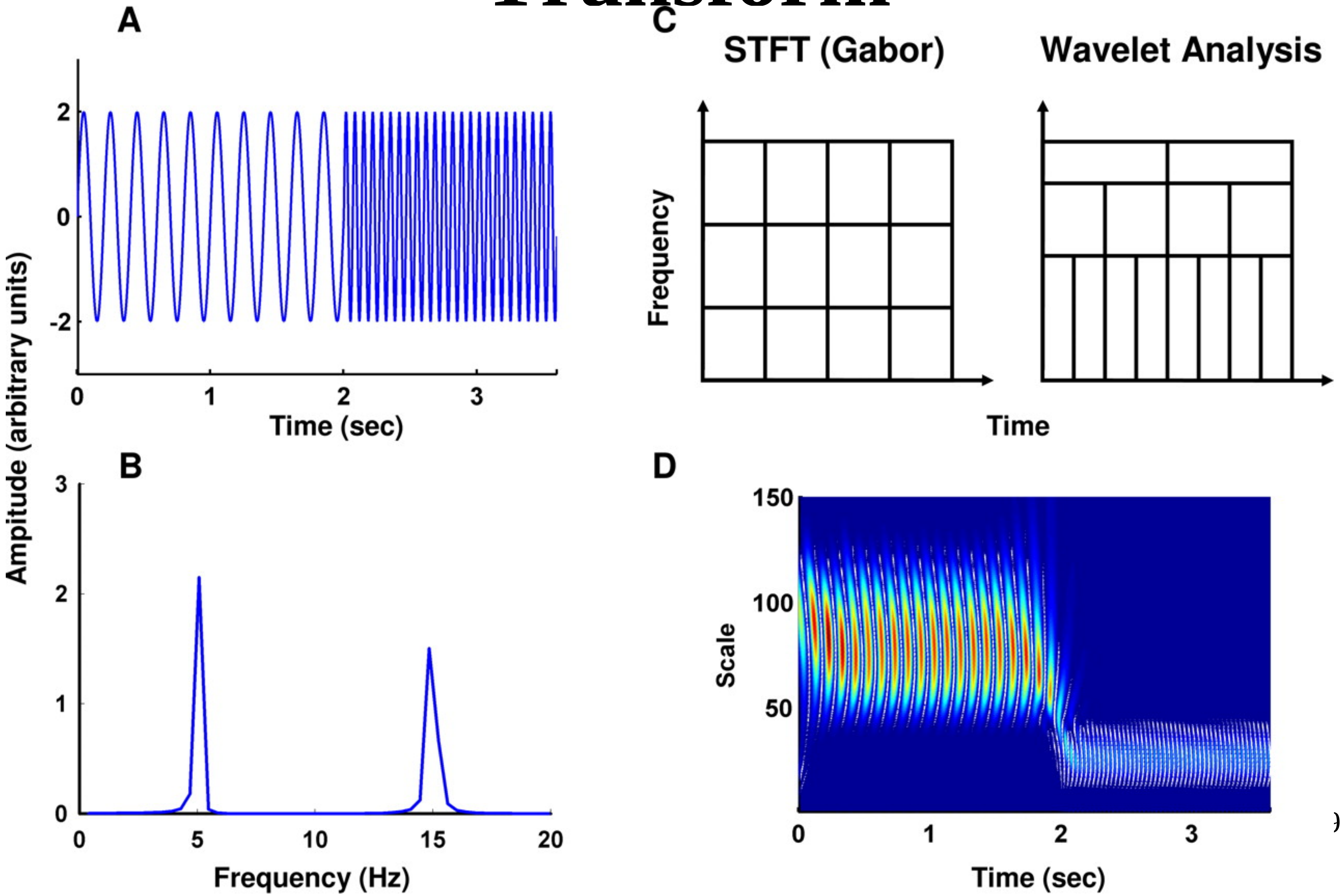
Dissimilarities between the Short Time Fourier Transform and the Wavelet Transform

- **For the short time Fourier transform window truncates the sine or cosine function to obtain a particular time width**

- **Because a single window is used for all frequencies in the short time Fourier transform, the time and frequency resolutions of the analysis are the same at all locations in the time-frequency plane**

- **A feature of wavelet transforms is that the window time width *varies with frequency***

Dissimilarities between the Short Time Fourier Transform and the Wavelet Transform



Dissimilarities between the Short Time Fourier Transform and Wavelet Transforms

- **In order to isolate signal discontinuities, impulses, one would like to have short duration windows**

- **At the same time, in order to obtain detailed frequency analysis, one would like to have long duration windows**

- **A way to achieve this is to have short time high-frequency windows and long time low-frequency windows**

- **This can be achieved by window length variation with frequency**

Dissimilarities between the Fourier Transform and the Wavelet Transform

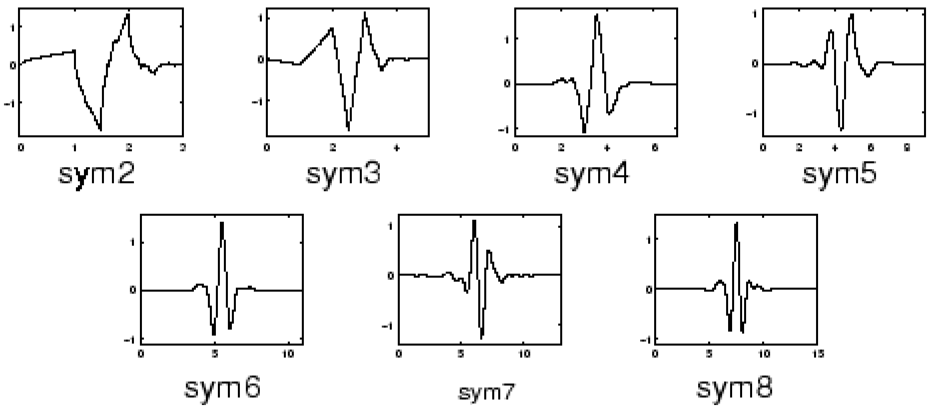
- **Wavelet transforms do not have a single set of basis functions like the Fourier transform, which utilizes just the sine and cosine functions**

Instead, wavelet transforms have an infinite set of possible mother wavelet functions

- **Thus, wavelet analysis provides immediate access to information that can be missed by other time-frequency methods**

- **In other words, the wavelet transform actually represents entire families of transforms, i.e. we can obtain different transform results using different mother wavelets.**

Wavelet Families



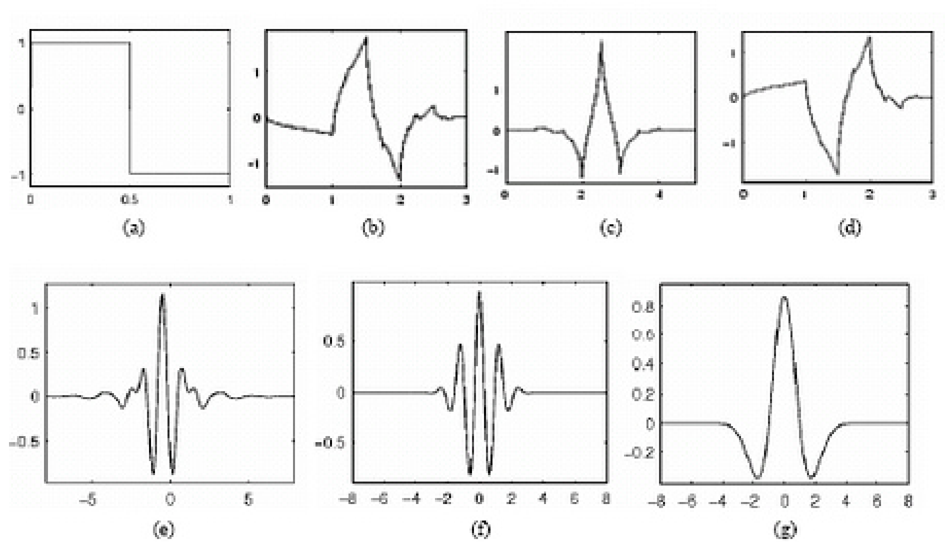


Figure 2.4 Wavelet families (a) Haar (b) Daubechies4 (c) Coiflet1 (d) Symlet2 (e) Meyer (f) Morlet (g) Mexican Hat.

Dissimilarities between the Fourier Transform and the Wavelet Transform

•**The problem associated with continuous Fourier basis is that when we analyse discontinuous functions, we run into Gibb's phenomenon (i.e. a leakage in the power spectral density)**

•**Since wavelets can use discontinuous wavelet functions (Haar, etc.), edge effects are reproduced much better for analysis of discontinuous functions**

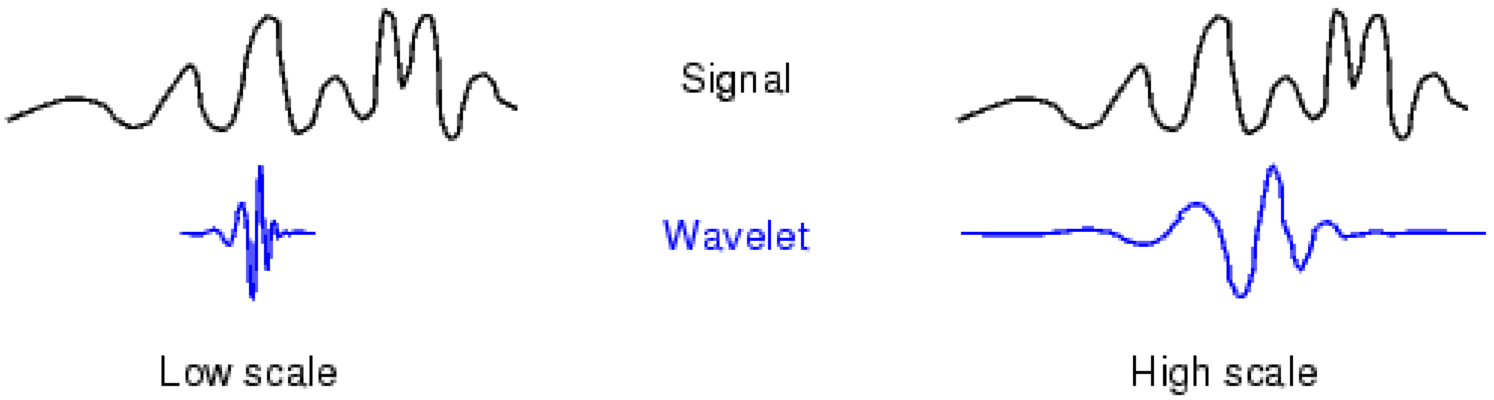
Dissimilarities between the Fourier Transform and the Wavelet Transform

- **The Fourier transform is always complex valued, the wavelet transform could be real valued or complex valued depending on the mother wavelet**

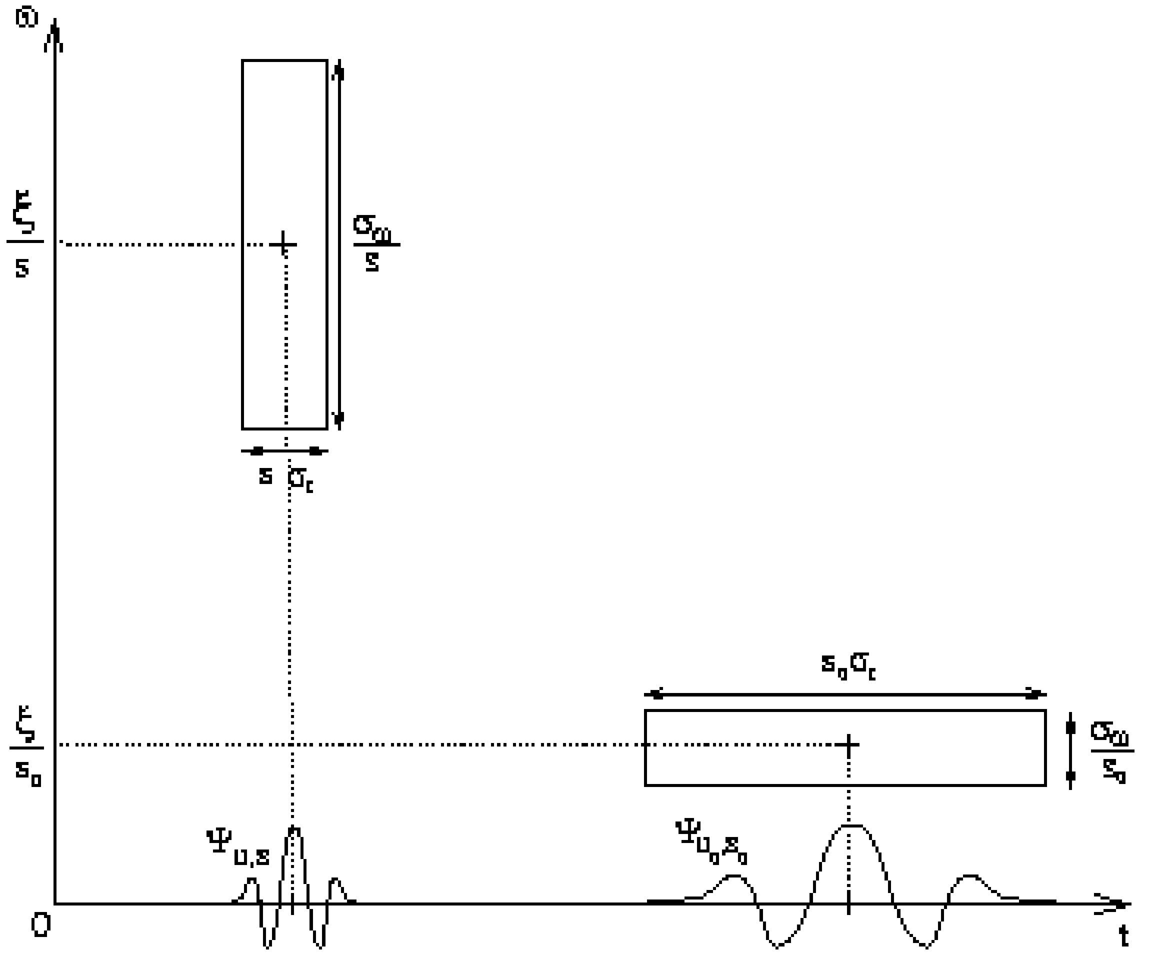
Wavelet Scale and Frequency

- The higher scales correspond to the "stretched" wavelets

- The lower scales correspond to the “compressed" wavelets



Wavelet Scale and Frequency



Wavelet Scale and Frequency

- **Thus, there is a correspondence between wavelet scales and frequency as revealed by wavelet analysis:**

Low scale **⇒Compressed wavelet ⇒Rapidly changing details ⇒*High frequency***

High scale **⇒Stretched wavelet ⇒Slowly changing ⇒*Low frequency***