Trigonometric waveforms

14.1 Graphs of trigonometric functions

By drawing up tables of values from 0° to 360°, graphs of $y = \sin A$, $y = \cos A$ and $y = \tan A$ may be plotted. Values obtained with a calculator (correct to 3 decimal places—which is more than sufficient for plotting graphs), using 30° intervals, are shown below, with the respective graphs shown in Fig. 14.1.

(a) $y = \sin A$

A	0	30°	60°	90°	120°	150°	180°
$\sin A$	0	0.500	0.866	1.000	0.866	0.500	0

A	210°	240°	270°	300°	330°	360°
$\sin A$	−0.500	−0.866	−1.000	−0.866	−0.500	0

(b) $y = \cos A$

A	0	30°	60°	90°	120°	150°	180°
$\cos A$	1.000	0.866	0.500	0	−0.500	−0.866	−1.000

A	210°	240°	270°	300°	330°	360°
$\cos A$	−0.866	−0.500	0	0.500	0.866	1.000

(c) $y = \tan A$

A	0	30°	60°	90°	120°	150°	180°
$\tan A$	0	0.577	1.732	∞	−1.732	−0.577	0

A	210°	240°	270°	300°	330°	360°
$\tan A$	0.577	1.732	∞	−1.732	−0.577	0

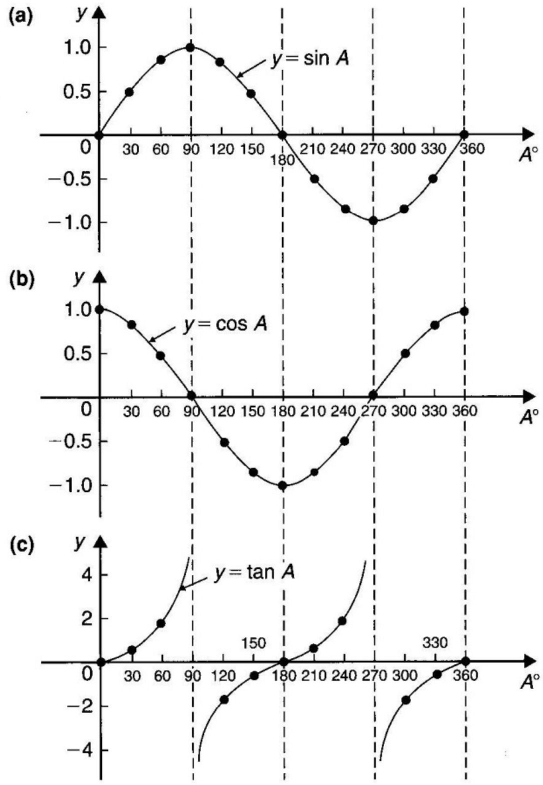

Figure 14.1

From Fig. 14.1 it is seen that:

(i) Sine and cosine graphs oscillate between peak values of ±1.

(ii) The cosine curve is the same shape as the sine curve but displaced by 90°.

(iii) The sine and cosine curves are continuous and they repeat at intervals of 360°; the tangent curve appears to be discontinuous and repeats at intervals of 180°.

14.2 Angles of any magnitude

(i) Figure 14.2 shows rectangular axes XX' and YY' intersecting at origin 0. As with graphical work, measurements made to the right and above 0 are positive while those to the left and downwards are negative. Let OA be free to rotate about 0. By convention, when OA moves anticlockwise angular measurement is considered positive, and vice-versa.

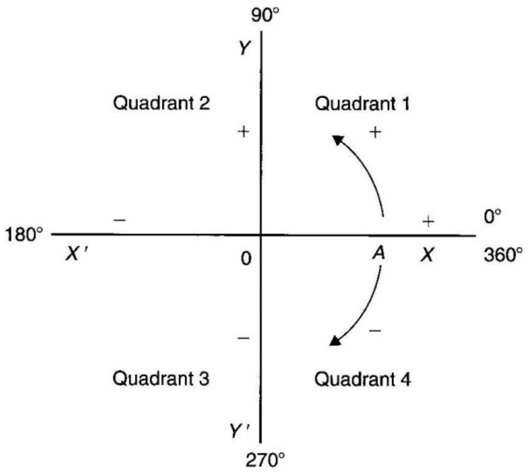

Figure 14.2

(ii) Let OA be rotated anticlockwise so that θ_1 is any angle in the first quadrant and let perpendicular AB be constructed to form the right-angled triangle OAB (see Fig. 14.3). Since all three sides of the triangle are positive, all six trigonometric ratios are positive in the first quadrant. (Note: OA is always positive since it is the radius of a circle.)

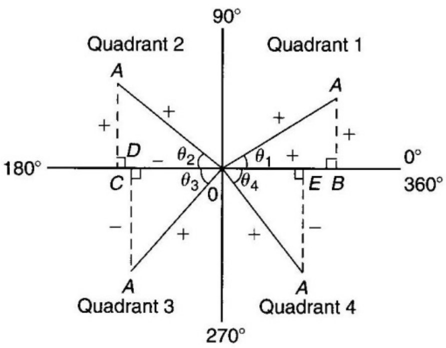

Figure 14.3

(iii) Let OA be further rotated so that θ_2 is any angle in the second quadrant and let AC be constructed to form the right-angled triangle OAC. Then:

$$\sin\theta_2 = \frac{+}{+} = + \qquad \cos\theta_2 = \frac{-}{+} = -$$

$$\tan\theta_2 = \frac{+}{-} = - \qquad \operatorname{cosec}\theta_2 = \frac{+}{+} = +$$

$$\sec\theta_2 = \frac{+}{-} = - \qquad \cot\theta_2 = \frac{-}{+} = -$$

(iv) Let OA be further rotated so that θ_3 is any angle in the third quadrant and let AD be constructed to form the right-angled triangle OAD. Then:

$$\sin\theta_3 = \frac{-}{+} = - \text{ (and hence } \operatorname{cosec}\theta_3 \text{ is } -)$$

$$\cos\theta_3 = \frac{-}{+} = - \text{ (and hence } \sec\theta_3 \text{ is } +)$$

$$\tan\theta_3 = \frac{-}{-} = + \text{ (and hence } \cot\theta_3 \text{ is } -)$$

(v) Let OA be further rotated so that θ_4 is any angle in the fourth quadrant and let AE be constructed to form the right-angled triangle OAE. Then:

$$\sin\theta_4 = \frac{-}{+} = - \text{ (and hence } \operatorname{cosec}\theta_4 \text{ is } -)$$

$$\cos\theta_4 = \frac{+}{+} = + \text{ (and hence } \sec\theta_4 \text{ is } +)$$

$$\tan\theta_4 = \frac{-}{+} = - \text{ (and hence } \cot\theta_4 \text{ is } -)$$

(vi) The results obtained in (ii) to (v) are summarized in Fig. 14.4. The letters underlined spell the word CAST when starting in the fourth quadrant and moving in an anticlockwise direction.

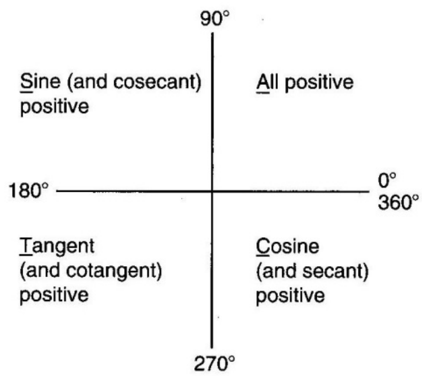

Figure 14.4

(vii) In the first quadrant of Fig. 14.1 all the curves have positive values; in the second only sine is positive; in the third only tangent is positive; in the fourth only cosine is positive (exactly as summarized in Fig. 14.4).

A knowledge of angles of any magnitude is needed when finding, for example, all the angles between 0° and 360° whose sine is, say, 0.3261. If 0.3261 is entered into a calculator and then the inverse sine key pressed (or $\sin^{-1}$ key) the answer 19.03° appears. However there is a second angle between 0° and 360° which the calculator does not give. Sine is also positive in the second quadrant (either from CAST or from Fig. 14.1(a)). The other angle is shown in Fig. 14.5 as angle θ where $\theta = 180° - 19.03° = 160.97°$. Thus 19.03° **and** 160.97° are the angles between 0° and 360° whose sine is 0.3261 (check that $\sin 160.97° = 0.3261$ on your calculator).

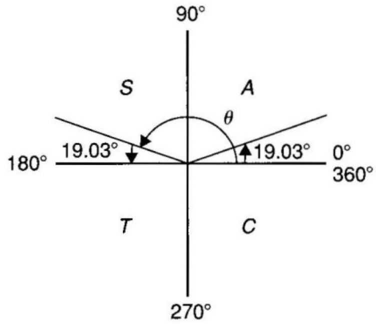

Figure 14.5

Be careful! Your calculator only gives you one of these answers. The second answer needs to be deduced from a knowledge of angles of any magnitude, as shown in the following problems.

Problem 1. Determine all the angles between 0° and 360° whose sine is −0.4638

The angles whose sine is −0.4638 occurs in the third and fourth quadrants since sine is negative in these quadrants (see Fig. 14.6(a)). From Fig. 14.6(b), $\theta = \sin^{-1} 0.4638 = 27°38'$.

Measured from 0°, the two angles between 0° and 360° whose sine is −0.4638 are $180° + 27°38'$, i.e. **207°38′** and $360° - 27°38'$, i.e. **332°22′**. (Note that a calculator generally only gives one answer, i.e. −27.632588°).

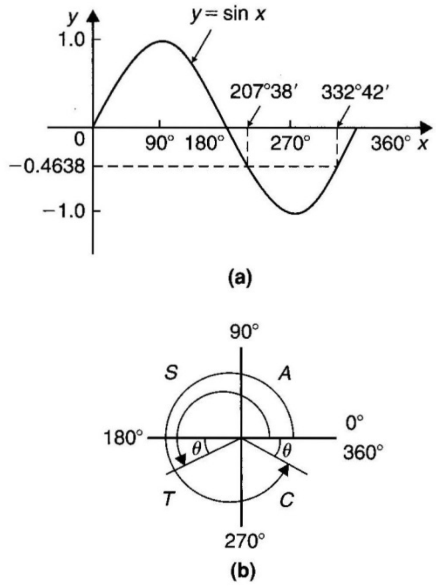

Figure 14.6

Problem 2. Determine all the angles between 0° and 360° whose tangent is 1.7629

A tangent is positive in the first and third quadrants (see Fig. 14.7(a)). From Fig. 14.7(b), $\theta = \tan^{-1} 1.7629 = 60°26'$. Measured from 0°, the two

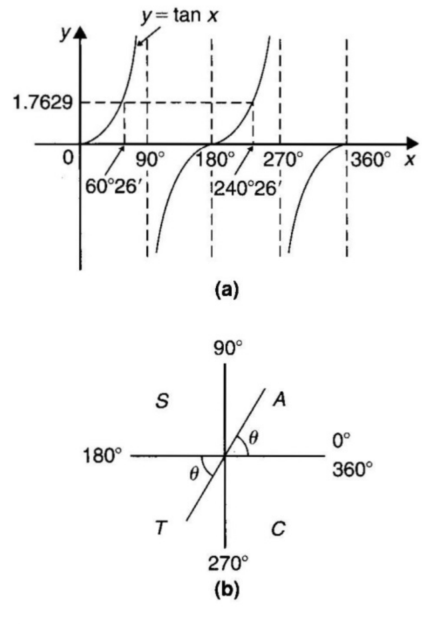

Figure 14.7

angles between 0° and 360° whose tangent is 1.7629 are **60°26′** and 180° + 60°26′, i.e. **240°26′**.

Problem 3. Solve $\sec^{-1}(-2.1499) = \alpha$ for angles of α between 0° and 360°.

Secant is negative in the second and third quadrants (i.e. the same as for cosine). From Fig. 14.8, $\theta = \sec^{-1} 2.1499 = \cos^{-1}\left(\frac{1}{2.1499}\right) = 62°17'$.
Measured from 0°, the two angles between 0° and 360° whose secant is −2.1499 are

$\alpha = 180° - 62°17' = \mathbf{117°43'}$ and

$\alpha = 180° + 62°17' = \mathbf{242°17'}$

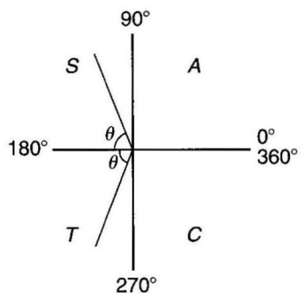

Figure 14.8

Problem 4. Solve $\cot^{-1} 1.3111 = \alpha$ for angles of α between 0° and 360°.

Cotangent is positive in the first and third quadrants (i.e. same as for tangent). From Fig. 14.9, $\theta = \cot^{-1} 1.3111 = \tan^{-1}\left(\frac{1}{1.3111}\right) = 37°20'$.

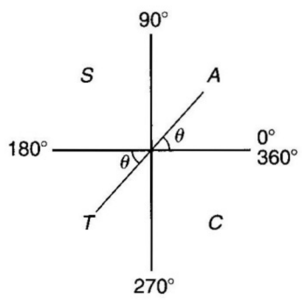

Figure 14.9

Hence $\alpha = \mathbf{37°20'}$

and $\alpha = 180° + 37°20' = \mathbf{217°20'}$

Now try the following exercise

Exercise 61 Further problems on evaluating trigonometric ratios of any magnitude

1. Find all the angles between 0° and 360° whose sine is −0.7321.
   [227°4′ and 312°56′]
2. Determine the angles between 0° and 360° whose cosecant is 2.5317.
   [23°16′ and 156°44′]
3. If cotangent $x = -0.6312$, determine the values of x in the range $0° \le x \le 360°$.
   [122°16′ and 302°16′]

In Problems 4 to 6 solve the given equations.

4. $\cos^{-1}(-0.5316) = t$
   [$t = 122°7'$ and $237°53'$]
5. $\sec^{-1} 2.3162 = x$
   [$x = 64°25'$ and $295°35'$]
6. $\tan^{-1} 0.8314 = \theta$
   [$\theta = 39°44'$ and $219°44'$]