# 5 Coordinate geometry

*A place for everything, and everything in its place.*

Samuel Smiles (1812–1904)

Figure 5.1 shows some scaffolding in which some of the horizontal pieces are 2 m long and others are 1 m. All the vertical pieces are 2 m.

- An ant crawls along the scaffolding from point P to point Q, travelling either horizontally or vertically. How far does the ant crawl?
- A mouse also goes from point P to point Q, travelling either horizontally or along one of the sloping pieces. How far does the mouse travel?
- A bee flies directly from point P to point Q. How far does the bee fly?

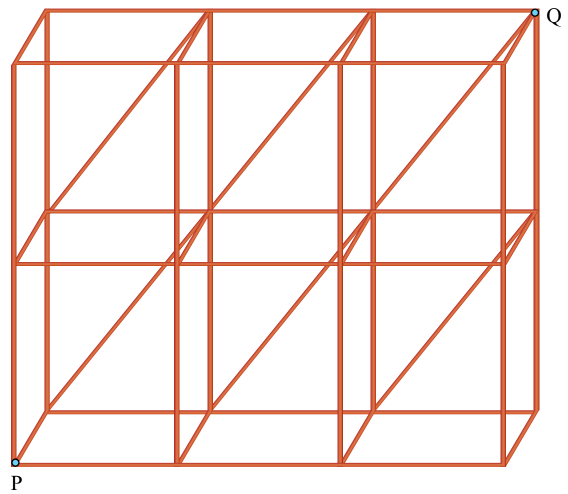



Figure 5.1

**TECHNOLOGY**

When working through this chapter, you may wish to use a graphical calculator or graphing software to check your answers where appropriate.

# 1 Working with coordinates

Coordinates are a means of describing a position relative to a fixed point, or origin.

In two dimensions you need two pieces of information; in three dimensions you need three pieces of information.

In the Cartesian system (named after René Descartes), position is given in perpendicular directions: $x$, $y$ in two dimensions; $x$, $y$, $z$ in three dimensions.

This chapter concentrates exclusively on two dimensions.

Integral resource 1: Coordinate geometry 1: Points and straight lines

Integral resource 2: Explore: Points

## The midpoint and length of a line segment

When you know the coordinates of two points you can work out the midpoint and length of the line segment which connects them.

**ACTIVITY 5.1**

Find

(i) the coordinates of the midpoint, M

(ii) the length AB.

Draw a right-angled triangle with AB as the hypotenuse and use Pythagoras' theorem.

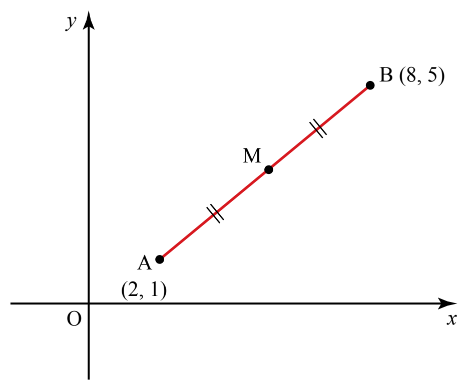



Figure 5.2

You can generalise these methods to find the midpoint and length of any line segment AB.

Let A be the point $(x_1,\ y_1)$ and B the point $(x_2,\ y_2)$.

(i) Find the midpoint of AB.

The midpoint of two values is the mean of those values.

The mean of the $x$ coordinates is $\dfrac{x_1 + x_2}{2}$.

The mean of the $y$ coordinates is $\dfrac{y_1 + y_2}{2}$.

So the coordinates of the midpoint are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

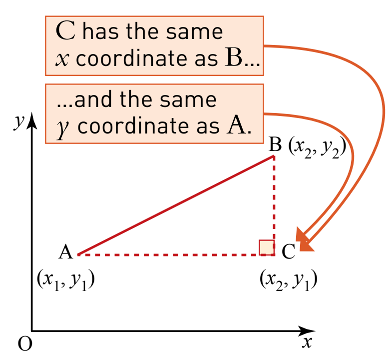



Figure 5.3

(ii) Find the length of AB.

First find the lengths of AC and AB: $AC = x_2 - x_1$

$BC = y_2 - y_1$

By Pythagoras' theorem: $AB^2 = AC^2 + BC^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

So the length AB is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

**Discussion point**

➜ Does it matter which point you call $(x_1, y_1)$ and which $(x_2, y_2)$?

## The gradient of a line

When you know the coordinates of any two points on a straight line, then you can draw that line. The slope of a line is given by its **gradient**. The gradient is often denoted by the letter $m$.

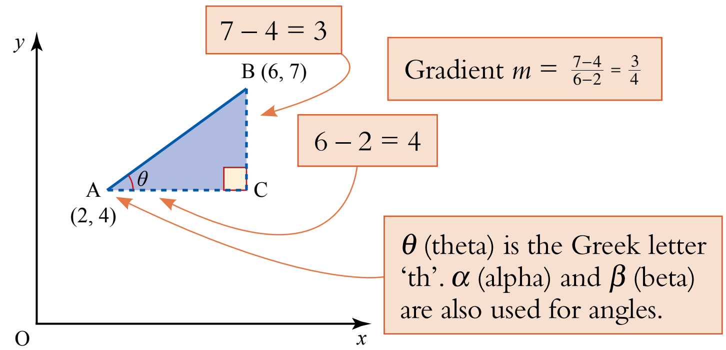



Figure 5.4

In Figure 5.4, A and B are two points on the line. The gradient of the line AB is given by the increase in the $y$ coordinate from A to B divided by the increase in the $x$ coordinate from A to B.

In general, when A is the point $(x_1, y_1)$ and B is the point $(x_2, y_2)$, the gradient is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Gradient = $\frac{\text{change in } y}{\text{change in } x}$

When the same scale is used on both axes, $m = \tan\theta$ (see Figure 5.4).

Integral resource 3: Explore: Parallel and perpendicular lines

## Parallel and perpendicular lines

### ACTIVITY 5.2

It is best to use squared paper for this activity.

Draw the line $L_1$ joining (0, 2) to (4, 4).

Draw another line $L_2$ perpendicular to $L_1$ from (4, 4) to (6, 0).

Find the gradients $m_1$ and $m_2$ of these two lines.

What is the relationship between the gradients?

Is this true for other pairs of perpendicular lines?

When you know the gradients $m_1$ and $m_2$, of two lines, you can tell at once if they are either parallel or perpendicular – see Figure 5.5.

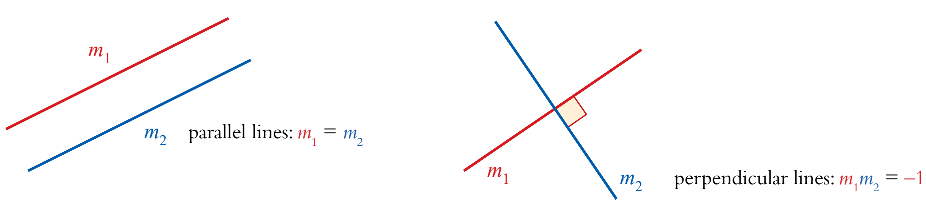



Figure 5.5

! Lines for which $m_1 m_2 = -1$ will only look perpendicular if the same scale has been used for both axes.

So for perpendicular lines:

$$m_1 = -\frac{1}{m_2} \text{ and likewise, } m_2 = -\frac{1}{m_1}.$$

So $m_1$ and $m_2$ are each the negative reciprocal of each other.

**Example 5.1**

A and B are the points (2, 5) and (6, 3) respectively (see Figure 5.6).

Find:

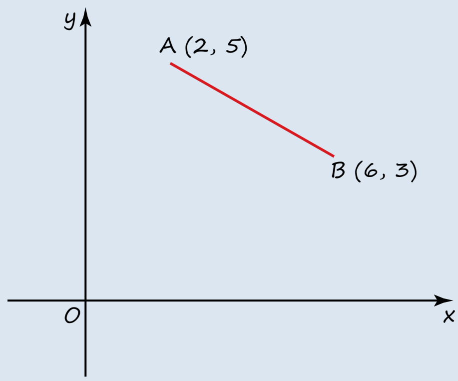
(i) the gradient of AB

(ii) the length of AB

(iii) the midpoint of AB

(iv) the gradient of the line perpendicular to AB.

## Solution




Draw a diagram to help you.

**Figure 5.6**

(i) Gradient $m_{AB} = \dfrac{y_A - y_B}{x_A - x_B}$

$$= \frac{5-3}{2-6}$$

$$= -\frac{1}{2}$$

Gradient is difference in $y$ coordinates divided by difference in $x$ coordinates. It doesn't matter which point you use first, as long as you are consistent!

(ii) Length AB $= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

$$= \sqrt{(6-2)^2 + (3-5)^2}$$

$$= \sqrt{16+4}$$

$$= \sqrt{20}$$

(iii) Midpoint $= \left(\dfrac{x_A + x_B}{2}, \dfrac{y_A + y_B}{2}\right)$

$$= \left(\frac{2+6}{2}, \frac{5+3}{2}\right)$$

$$= (4, 4)$$

(iv) Gradient of AB: $m_{AB} = -\frac{1}{2}$

So gradient of perpendicular to AB is 2.

The gradient of the line perpendicular to AB is the negative reciprocal of $m_{AB}$.

Check: $-\frac{1}{2} \times 2 = -1$ ✓

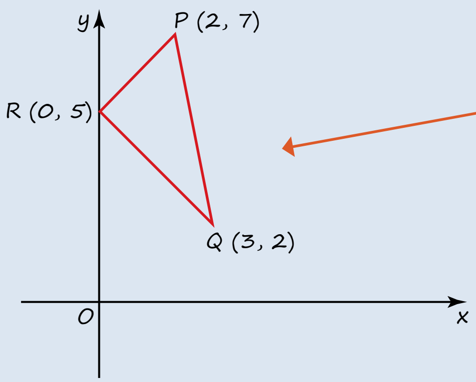
**Example 5.2**

The points P(2, 7), Q(3, 2) and R(0, 5) form a triangle.

(i) Use gradients to show that RP and RQ are perpendicular.

(ii) Use Pythagoras' theorem to show that PQR is right-angled.

## Solution




Always start by drawing a diagram.

Figure 5.7

(i) Show that the gradients satisfy $m_1 m_2 = -1$

$$\text{Gradient of RP} = \frac{2-0}{7-5} = 1$$

$$\text{Gradient of RQ} = \frac{2-5}{3-0} = -1$$

$\Rightarrow$ product of gradients = 1 × (–1) = –1

$\Rightarrow$ sides RP and RQ are at right angles.

(ii) Pythagoras' theorem states that for a right-angled triangle with hypotenuse of length $a$ and other sides of lengths $b$ and $c$, $a^2 = b^2 + c^2$.

Conversely, when $a^2 = b^2 + c^2$ for a triangle with sides of lengths $a$, $b$ and $c$, then the triangle is right-angled and the side of length $a$ is the hypotenuse.

$$\text{length}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$PQ^2 = (3-2)^2 + (2-7)^2 = 1 + 25 = 26$$

$$RP^2 = (2-0)^2 + (7-5)^2 = 4 + 4 = 8$$

$$RQ^2 = (3-0)^2 + (2-5)^2 = 9 + 9 = 18$$

Since 26 = 8 + 18, $PQ^2 = RP^2 + RQ^2$

PQ is the hypotenuse since RP and RQ are perpendicular.

$\Rightarrow$ sides RP and RQ are at right angles.

## Exercise 5.1

① For the following pairs of points A and B, calculate:

(a) the midpoint of the line joining A to B

(b) the distance AB

(c) the gradient of the line AB

(d) the gradient of the line perpendicular to AB.

(i) A(2, 5) and B(6, 8)

(ii) A(−2, −5) and B(−6, −8)

(iii) A(−2, −5) and B(6, 8)

(iv) A(−2, 5) and B(6, −8)

② The gradient of the line joining the point P(3, −4) to Q($q$, 0) is 2. Find the value of $q$.

③ The three points X(2, −1), Y(8, $y$) and Z(11, 2) are collinear. Find the value of $y$.

They lie on the same straight line.

④ For the points P($x$, $y$), and Q($3x$, $5y$), find in terms of $x$ and $y$:

(i) the gradient of the line PQ

(ii) the midpoint of the line PQ

(iii) the length of the line PQ.

⑤ The points A, B, C and D have coordinates (1, 2), (7, 5), (9, 8) and (3, 5) respectively.

(i) Find the gradients of the lines AB, BC, CD and DA.

(ii) What do these gradients tell you about the quadrilateral ABCD?

(iii) Draw an accurate diagram to check your answer to part (ii).

PS ⑥ The points A, B, and C have coordinates (−4, 2), (7, 4) and (−3, −1).

(i) Draw the triangle ABC.

(ii) Show by calculation that the triangle ABC is isosceles and name the two equal sides.

(iii) Find the midpoint of the third side.

(iv) Work out the area of the triangle ABC.

⑦ The points A, B and C have coordinates (2, 1), ($b$, 3) and (5, 5), where $b > 3$, and $\angle ABC = 90°$.

Find:

(i) the value of $b$

(ii) the lengths of AB and BC

(iii) the area of triangle ABC.

⑧ Three points A, B and C have coordinates (1, 3), (3, 5), and (−1, $y$). Find the value of $y$ in each of the following cases:

(i) AB = AC

(ii) AB = AC

(iii) AB is perpendicular to BC

(iv) A, B and C are collinear.

⑨ The triangle PQR has vertices P(8, 6), Q(0, 2) and R(2, $r$).

Find the values of $r$ when the triangle PQR:

(i) has a right angle at P

(ii) has a right angle at Q

(iii) has a right angle at R

(iv) is isosceles with RQ = RP.

PS ⑩ A quadrilateral has vertices A(0, 0), B(0, 3), C(6, 6), and D(12, 6).

(i) Draw the quadrilateral.

(ii) Show by calculation that it is a trapezium.

(iii) EBCD is a parallelogram. Find the coordinates of E.

PS ⑪ Show that the points with coordinates (1, 2), (8, −2), (7, 6) and (0, 10) are the vertices of a rhombus, and find its area.

PS ⑫ The lines AB and BC in Figure 5.8 are equal in length and perpendicular.

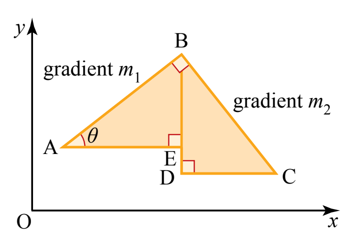



Figure 5.8

(i) Show that triangles ABE and BCD are congruent.

(ii) Hence prove that the gradients $m_1$ and $m_2$ satisfy $m_1 m_2 = -1$.

Integral resource 4: Explore: Straight lines

# 2 The equation of a straight line

## Drawing a line, given its equation

There are several standard forms for the equation of a straight line, as shown in Figure 5.9.

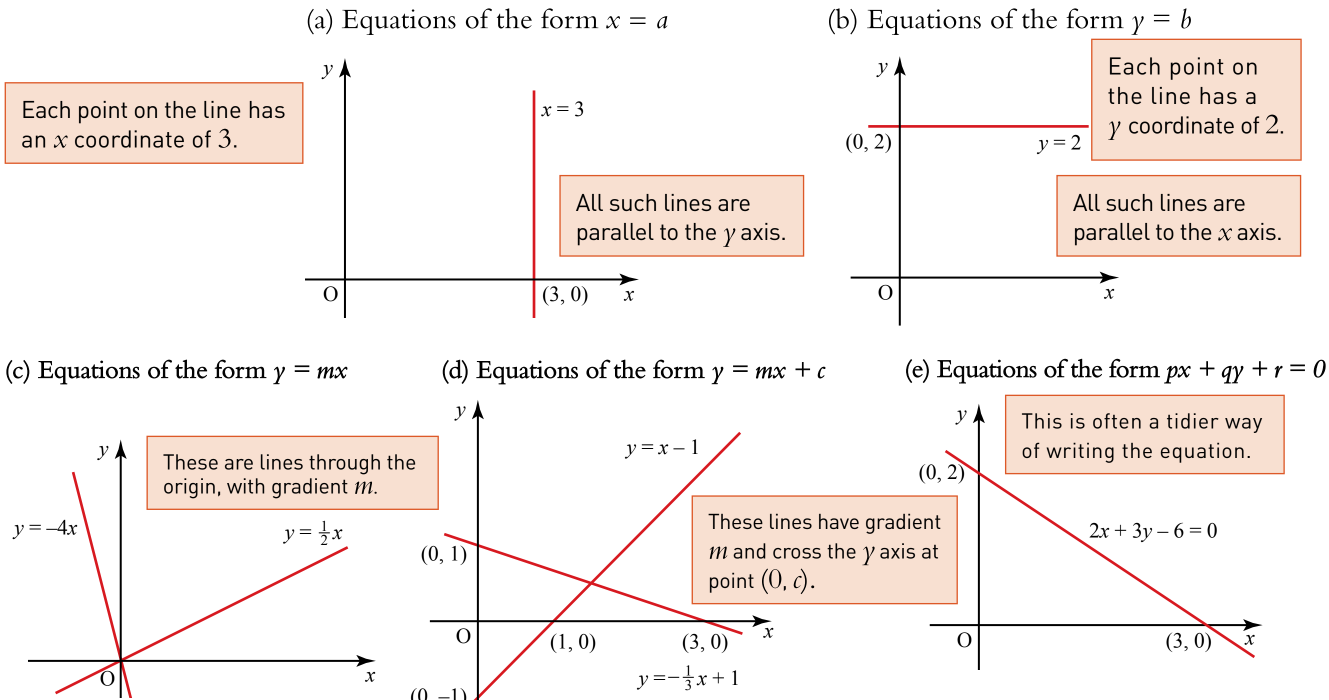



Figure 5.9

**Example 5.3**

(i) Sketch the lines (a) $y = x - 1$ and (b) $3x + 4y = 24$ on the same axes.

(ii) Are these lines perpendicular?

### Solution

(i) To draw a line you need to find the coordinates of two points on it.

Usually it is easiest to find where the line cuts the $x$ and $y$ axes.

(a) The line $y = x - 1$ passes through the point $(0, -1)$.

The line is already in the form $y = mx + c$.

Substituting $y = 0$ gives $x = 1$, so the line also passes through $(1, 0)$.

(b) Find two points on the line $3x + 4y = 24$.

Set $x = 0$ and find $y$ to give the $y$-intercept.

Then set $y = 0$ and find $x$ to give the $x$-intercept.

Substituting $x = 0$ gives $4y = 24 \Rightarrow y = 6$

substituting $y = 0$ gives $3x = 24 \Rightarrow x = 8$.

So the line passes through $(0, 6)$ and $(8, 0)$. (Figure 5.10 overleaf)

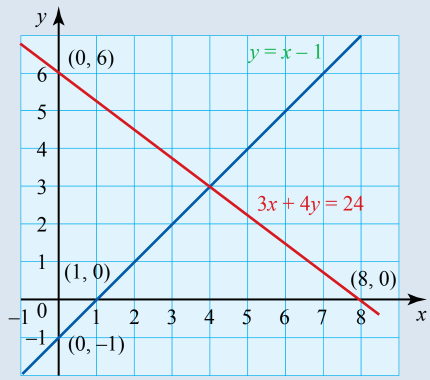



**Figure 5.10**

(ii) The lines look almost perpendicular but you need to use the gradient of each line to check.

Gradient of $y = x - 1$ is 1.

Gradient of $3x + 4y = 24$ is $-\frac{3}{4}$.

Rearrange the equation to make $y$ the subject so you can find the gradient.
$4y = -3x + 24$
$y = -\frac{3}{4}x + 6$

Therefore the lines are not perpendicular as $1 \times \left(-\frac{3}{4}\right) \neq -1$.

**Warning**

When you draw two perpendicular lines on a diagram, they will be at right angles if, and only if, both axes are to the same scale.

## Finding the equation of a line

To find the equation of a line, you need to think about what information you are given.

(i) **Given the gradient, $m$, and the coordinates $(x_1, y_1)$ of one point on the line**

$y - y_1 = m(x - x_1)$

Take a general point $(x, y)$ on the line, as shown in Figure 5.11.

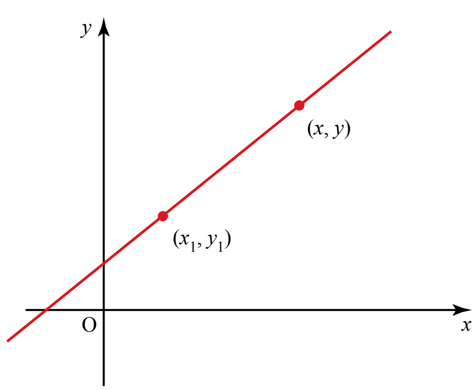



**Figure 5.11**

The gradient, $m$, of the line joining $(x_1, y_1)$ to $(x, y)$ is given by

$$m = \frac{y - y_1}{x - x_1}$$

$$\Rightarrow y - y_1 = m(x - x_1)$$

This is a very useful form of the equation of a straight line.

For example, the equation of the line with gradient 2 that passes through the point $(3, -1)$ can be written as $y - (-1) = 2(x - 3)$

which can be simplified to $y = 2x - 7$.

(ii) **Given the gradient, $m$, and the $y$-intercept $(0, c)$**

$y = mx + c$

A special case of $y - y_1 = m(x - x_1)$ is when $(x_1, y_1)$ is the $y$-intercept $(0, c)$.

The equation then becomes

$y = mx + c$ ← Substituting $x_1 = 0$ and $y_1 = c$ into the equation

as shown in Figure 5.12.

When the line passes through the origin, the equation is

$y = mx$ ← The $y$-intercept is $(0, 0)$, so $c = 0$

as shown in Figure 5.13.

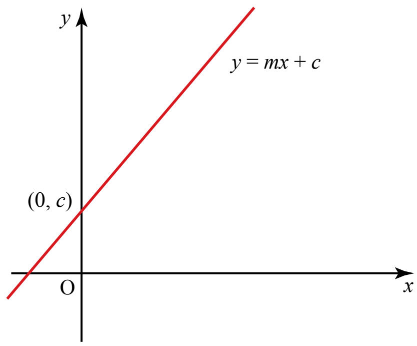



Figure 5.12

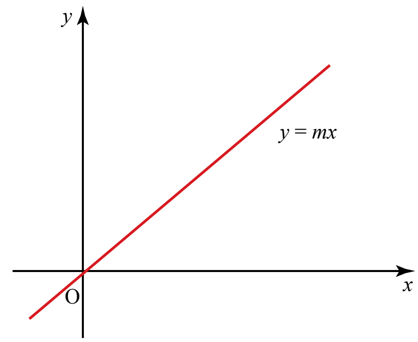



Figure 5.13

## ACTIVITY 5.3

A Show algebraically that an equivalent form of
$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$
is
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

B Use both forms to find the equation of the line joining $(2, 4)$ to $(5, 3)$ and show they give the same equation.

(iii) **Given two points, $(x_1, y_1)$ and $(x_2, y_2)$**

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

The two points are used to find the gradient:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This value of $m$ is then substituted in the equation

$$y - y_1 = m(x - x_1)$$

This gives

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or $$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

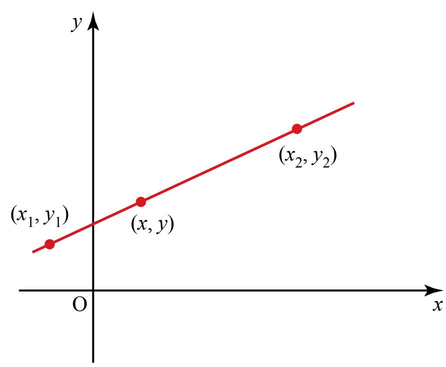



Figure 5.14

## Discussion points

- ➜ How else can you write the equation of the line?
- ➜ Which form do you think is best for this line?

**Example 5.4**

Find the equation of the line perpendicular to $4y + x = 12$ which passes through the point P(2, −5).

## Solution

First rearrange $4y + x = 12$ into the form $y = mx + c$ to find the gradient.

$$4y = -x + 12$$
$$y = -\frac{1}{4}x + 3$$

For perpendicular gradients $m_1 m_2 = -1$
So $m_2 = -\frac{1}{m_1}$

So the gradient is $-\frac{1}{4}$

The negative reciprocal of $-\frac{1}{4}$ is 4.

Check: $-\frac{1}{4} \times 4 = -1$ ✓

So the gradient of a line perpendicular to $y = -\frac{1}{4}x + 3$ is 4.

Using $y - y_1 = m(x - x_1)$ when $m = 4$ and $(x_1, y_1)$ is (2, −5)

$$\Rightarrow y - (-5) = 4(x - 2)$$
$$\Rightarrow y + 5 = 4x - 8$$
$$\Rightarrow y = 4x - 13$$

Straight lines can be used to model real-life situations. Often simplifying assumptions need to be made so that a linear model is appropriate.

**Example 5.5**

The diameter of a snooker cue varies uniformly from 9 mm to 23 mm over its length of 140 cm.

Varying uniformly means that the graph of diameter against distance from the tip is a straight line.

(i) Sketch the graph of diameter ($y$ mm) against distance ($x$ cm) from the tip.

(ii) Find the equation of the line.

(iii) Use the equation to find the distance from the tip at which the diameter is 15 mm.

## Solution

(i) The graph passes through the points (0, 9) and (140, 23).

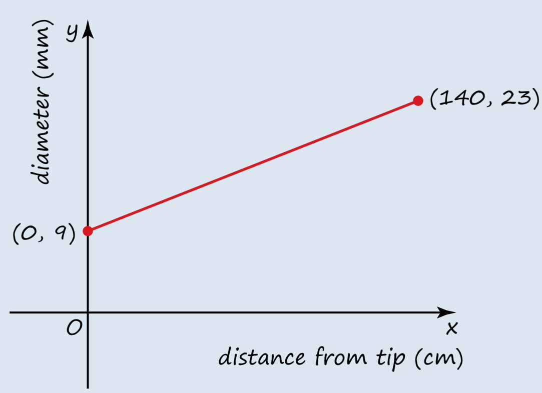



Figure 5.15

(ii) Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{23 - 9}{140 - 0} = 0.1$

Using the form $y = mx + c$, the equation of the line is $y = 0.1x + 9$.

(iii) Substituting $y = 15$ into the equation gives

$$15 = 0.1x + 9$$

$$0.1x = 6$$

$$x = 60$$

$\Rightarrow$ The diameter is 15 mm at a point 60 cm from the tip.

## Discussion points

➜ Which of these situations in Figure 5.16 could be modelled by a straight line?

➜ For each straight line model, what information is given by the gradient of the line?

➜ What assumptions do you need to make so that a linear model is appropriate?

➜ How reasonable are your assumptions?

| | | |
|---|---|---|
| Interest earned on savings in a bank account against time | Height of ball dropped from a cliff against time | Profit of ice cream seller against number of sales |
| Tax paid against earnings | Cost of apples against mass of apples | Value of car against age of car |
| Mass of candle versus length of time it is burning | Distance travelled by a car against time | Mass of gold bars against volume of gold bars |
| Population of birds on an island against time | Mobile phone bill against number of texts sent | Length of spring against mass of weights attached |

Figure 5.16

## Exercise 5.2

① Sketch the following lines:

(i) $y = -2$ (ii) $x = 2$

(iii) $y = -2x$ (iv) $y = x + 2$

(v) $y = 2x + 5$ (vi) $y = 5 - 2x$

(vii) $2x - y = 5$ (viii) $y + 2x + 5 = 0$

② Find the equations of the lines (i)–(v) in Figure 5.17.

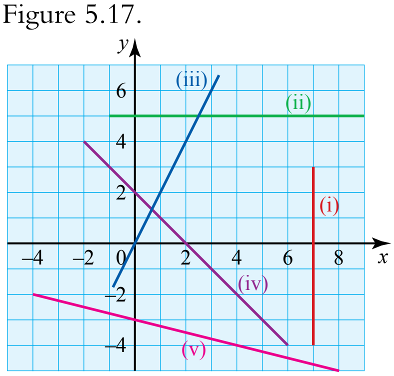



Figure 5.17

③ Find the equations of the lines

(i) parallel to $y = 3x - 2$ and passing through $(0, 0)$

(ii) parallel to $y = 3x$ and passing through $(2, 5)$

(iii) parallel to $2x + y - 3 = 0$ and passing through $(-2, 5)$

(iv) parallel to $3x - y - 2 = 0$ and passing through $(5, -2)$

(v) parallel to $x + 2y = 3$ and passing through $(-2, -5)$.

④ Find the equations of the lines

(i) perpendicular to $y = 3x$ and passing through $(0, 0)$

(ii) perpendicular to $y = 2x + 3$ and passing through $(4, 3)$

(iii) perpendicular to $2x + y = 4$ and passing through $(4, -3)$

(iv) perpendicular to $2y = x + 5$ and passing through $(-4, 3)$

(v) perpendicular to $2x + 3y = 4$ and passing through $(-4, -3)$.

5. Find the equations of the line AB in each of the following cases.

(i) A(3, 1), B(5, 7)

(ii) A(−3, −1), B(−5, −7)

(iii) A(−3, 1), B(−5, 7)

(iv) A(3, −1), B(5, −7)

(v) A(1, 3), B(7, 5)

PS 6. Show that the region enclosed by the lines

$$y = \frac{2}{3}x + 1,\ y = 1 - \frac{3x}{2},$$

$$3y - 2x + 1 = 0 \text{ and } 2y + 3x + 5 = 0$$

forms a rectangle.

The **perpendicular bisector** is the line at right angles to AB (**perpendicular**) that passes though the midpoint of AB (**bisects**).

7. Find the equation of the perpendicular bisector of each of the following pairs of points.

(i) A(2, 4) and B(3, 5)

(ii) (A(4, 2) and B (5, 3)

(iii) A(−2, −4) and B(−3, −5)

(iv) A(−2, 4) and B(−3, 5)

(v) A(2, −4) and B(3, −5)

8. A median of a triangle is a line joining one of the vertices to the midpoint of the opposite side.

In a triangle OAB, O is at the origin, A is the point (0, 6), and B is the point (6, 0).

(i) Sketch the triangle.

(ii) Find the equations of the three medians of the triangle.

(iii) Show that the point (2, 2) lies on all three medians. (This shows that the medians of this triangle are concurrent.)

PS 9. A quadrilateral ABCD has its vertices at the points (0, 0), (12, 5), (0, 10) and (−6, 8) respectively.

(i) Sketch the quadrilateral.

(ii) Find the gradient of each side.

(iii) Find the length of each side.

(iv) Find the equation of each side.

(v) Find the area of the quadrilateral.

PS 10. A firm manufacturing jackets finds that it is capable of producing 100 jackets per day, but it can only sell all of these if the charge to the wholesalers is no more than £20 per jacket. On the other hand, at the current price of £25 per jacket, only 50 can be sold per day. Assuming that the graph of price $P$ against number sold per day $N$ is a straight line:

(i) sketch the graph, putting the number sold per day on the horizontal axis (as is normal practice for economists)

(ii) find its equation.

Use the equation to find:

(iii) the price at which 88 jackets per day could be sold

(iv) the number of jackets that should be manufactured if they were to be sold at £23.70 each.

PS 11. To clean the upstairs window on the side of a house, it is necessary to position the ladder so that it just touches the edge of the lean-to shed as shown in Figure 5.18. The coordinates represent distances from O in metres, in the $x$ and $y$ directions shown.

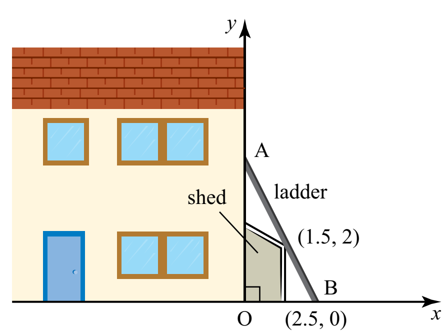



**Figure 5.18**

(i) Find the equation of the line of the ladder.

(ii) Find the height of the point A reached by the top of the ladder.

(iii) Find the length of the ladder to the nearest centimetre.

⑫ A spring has an unstretched length of 10 cm. When it is hung with a load of 80 g attached, the stretched length is 28 cm. Assuming that the extension of the spring is proportional to the load:

PS

(i) draw a graph of extension $E$ against load $L$ and find its equation

(ii) find the extension caused by a load of 48 g

(iii) find the load required to extend the spring to a length of 20 cm.

This particular spring passes its elastic limit when it is stretched to four times its original length. (This means that if it is stretched more than that it will not return to its original length.)

(iv) Find the load which would cause this to happen.

PS ⑬

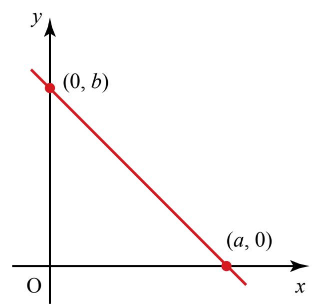



**Figure 5.19**

Show that the equation of the line in Figure 5.19 can be written

$\frac{x}{a} + \frac{y}{b} = 1$.

Integral resource 5: Test C1

## 3 The intersection of two lines

The intersection of any two curves (or lines) can be found by solving their equations simultaneously. In the case of two distinct lines, there are two possibilities:

**(i)** they are parallel, or **(ii)** they intersect at a single point.

You often need to find where a pair of lines intersect in order to solve problems.

**Example 5.6**

The lines $y = 5x - 13$ and $2y + 3x = 0$ intersect at the point P.

Find the coordinates of P.

### Solution

You need to solve the equations

$y = 5x - 13$ ①

and $2y + 3x = 0$ ②

simultaneously.

Substitute equation ① into ② : $2(5x - 13) + 3x = 0$

$10x - 26 + 3x = 0$ ← Multiply out the brackets.

$13x - 26 = 0$ ← Simplify

$13x = 26$

$x = 2$

Substitute $x = 2$ into equation ① to find $y$. ← Don't forget to find the $y$ coordinate.

$y = 5 \times 2 - 13$

$y = -3$

So the coordinates of P are $(2, -3)$.

**Discussion point**

→ The line $l$ has equation $2x - y = 4$ and the line $m$ has equation $y = 2x - 3$. What can you say about the intersection of these two lines?

## Exercise 5.3

① Find the coordinates of the point of intersection of the following pairs of lines.

(i) $y = 2x + 3$ and $y = 6x + 1$

(ii) $y = 2 - 3x$ and $2y + x = 14$

(iii) $3x + 2y = 4$ and $5x - 4y = 3$

② (i) Find the coordinates of the points where the following pairs of lines intersect.

(a) $y = 2x - 4$ and $2y = 7 - x$

(b) $y = 2x + 1$ and $2y = 7 - x$

The lines form three sides of a square.

(ii) Find the equation of the fourth side of the square.

(iii) Find the area of the square.

PS ③ (i) Find the vertices of the triangle ABC whose sides are given by the lines

AB: $x - 2y = -1$

BC: $7x + 6y = 53$

and AC: $9x + 2y = 11$.

(ii) Show that the triangle is isosceles.

④ A(0, 1), B(1, 4), C(4, 3) and D(3, 0) are the vertices of a quadrilateral ABCD.

(i) Find the equations of the diagonals AC and BD.

(ii) Show that the diagonals AC and BD bisect each other at right angles.

(iii) Find the lengths of AC and BD.

(iv) What type of quadrilateral is ABCD?

⑤ The line $y = 5x - 2$ crosses the $x$ axis at A.

The line $y = 2x + 4$ crosses the $x$ axis at B.

The two lines intersect at P.

PS

(i) Find the coordinates of A and B.

(ii) Find coordinates of the point of intersection, P.

(iii) Find the exact area of the triangle ABP.

PS ⑥ Triangle ABC has an angle of 90° at B. Point A is on the $y$ axis, AB is part of the line $x - 2y + 8 = 0$, and C is the point (6, 2).

(i) Sketch the triangle.

(ii) Find the equations of the lines AC and BC.

(iii) Find the lengths of AB and BC and hence find the area of the triangle.

(iv) Using your answer to (iii), find the length of the perpendicular from B to AC.

PS ⑦ Two rival taxi firms have the following fare structures:

Firm A: fixed charge of £1 plus 40p per kilometre;

Firm B: 60p per kilometre, no fixed charge.

(i) Sketch the graph of price (vertical axis) against distance travelled (horizontal axis) for each firm (on the same axes).

(ii) Find the equation of each line.

(iii) Find the distance for which both firms charge the same amount.

(iv) Which firm would you use for a distance of 6 km?

PS ⑧ Two sides of a parallelogram are formed by parts of the lines

$2x - y = -9$

and $x - 2y = -9$.

(i) Show these two lines on a graph.

(ii) Find the coordinates of the vertex where they intersect.

Another vertex of the parallelogram is the point (2, 1).

(iii) Find the equations of the other two sides of the parallelogram.

(iv) Find the coordinates of the other two vertices.

PS ⑨ The line with equation $5x + y = 20$ meets the $x$ axis at A and the line with equation $x + 2y = 22$ meets the $y$ axis at B. The two lines intersect at a point C.

(i) Sketch the two lines on the same diagram.

(ii) Calculate the coordinates of A, B and C.

(iii) Calculate the area of triangle OBC where O is the origin.

(iv) Find the coordinates of the point E such that ABEC is a parallelogram.

⑩ Figure 5.20 shows the supply and demand of labour for a particular industry in relation to the wage paid per hour.

Supply is the number of people willing to work for a particular wage, and this increases as the wage paid increases. Demand is the number of workers that employers are prepared to employ at a particular wage: this is greatest for low wages.

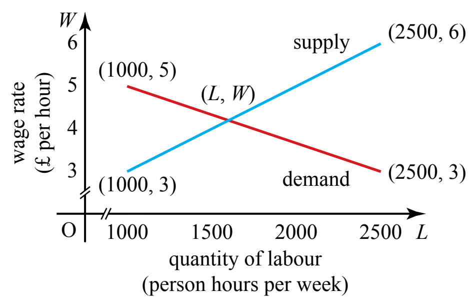



**Figure 5.20**

(i) Find the equation of each of the lines.

(ii) Find the values of $L$ and $W$ at which the market 'clears', i.e. at which supply equals demand.

(iii) Although economists draw the graph this way round, mathematicians would plot wage rate on the horizontal axis. Why?

⑪ When the market price £$p$ of an article sold in a free market varies, so does the number demanded, $D$, and the number supplied, $S$.

In one case $D = 20 + 0.2p$ and $S = -12 + p$.

(i) Sketch both of these lines on the same graph. (Put $p$ on the horizontal axis.)

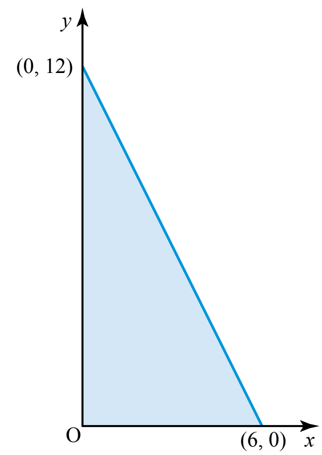
The market reaches a state of equilibrium when the number demanded equals the number supplied.

(ii) Find the equilibrium price and the number bought and sold in equilibrium.

PS ⑫ A median of a triangle is a line joining a vertex to the midpoint of the opposite side. In any triangle, the three medians meet at a point called the centroid of the triangle.

Find the coordinates of the centroid for each triangle shown in Figure 5.21.

(i)




(ii)

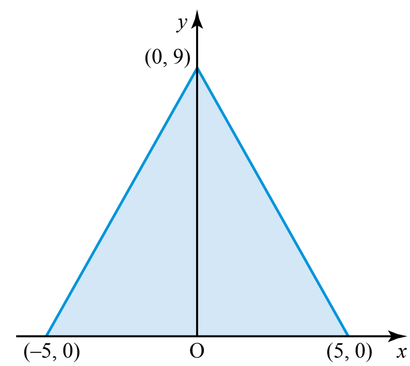



**Figure 5.21**

PS ⑬ Find the exact area of the triangle whose sides have the equations $x + y = 4$, $y = 2x - 8$ and $x + 2y = -1$.

**Prior knowledge**

You should be able to complete the square, which is covered in Chapter 3.

# 4 The circle

You are, of course, familiar with the circle, and have done calculations involving its area and circumference. In this section you are introduced to the equation of a circle.

The circle is defined as the **locus** of all the points in a plane which are at a fixed distance (the radius) from a given point (the centre).

**Locus** means possible positions subject to given conditions. In two dimensions the locus can be a path or a region.

This definition allows you to find the equation of a circle.

Remember, the length of a line joining $(x_1, y_1)$ to $(x_2, y_2)$ is given by

$$\text{length} = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$$

This is just Pythagoras' theorem.

Integral resource 6: Coordinate geometry 2: Circles

Integral resource 7: Explore: Circles

For a circle of radius 3, with its centre at the origin, any point $(x, y)$ on the circumference is distance 3 from the origin.

So the distance of $(x, y)$ from $(0, 0)$ is given by

$$\sqrt{(x-0)^2 + (y-0)^2} = 3$$

$$\Rightarrow \qquad x^2 + y^2 = 3^2$$

Squaring both sides.

$$\Rightarrow \qquad x^2 + y^2 = 9$$

This is the equation of the circle in Figure 5.22.

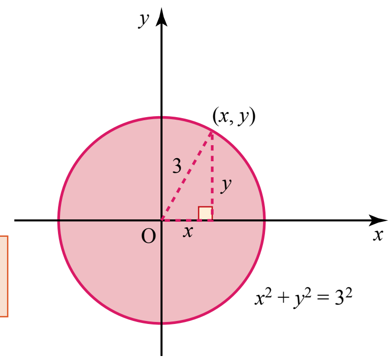



**Figure 5.22**

The circle in Figure 5.23 has a centre (9, 5) and radius 4, so the distance between any point on the circumference and the centre (9, 5) is 4.

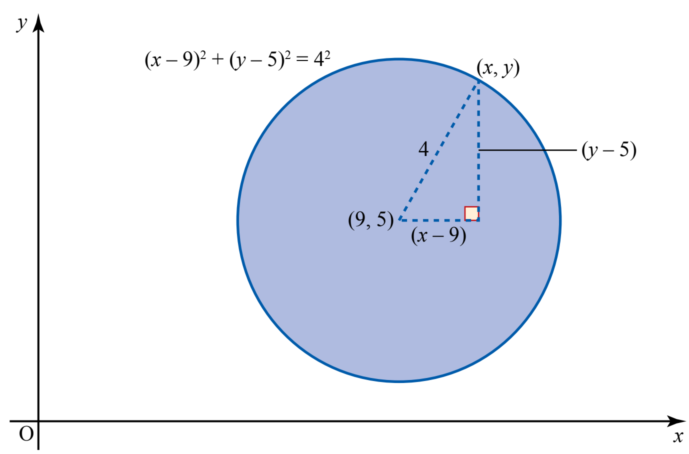



**Figure 5.23**

**TECHNOLOGY**

Graphing software needs to be set to equal aspect to get these graphs looking correct.

The equation of the circle in Figure 5.23 is:

$$\sqrt{(x-9)^2 + (y-5)^2} = 4$$

$$\Rightarrow (x-9)^2 + (y-5)^2 = 16.$$

T

## ACTIVITY 5.4

Sophie tries to draw the circle $x^2 + y^2 = 9$ on her graphical calculator.
Explain what has gone wrong for each of these outputs.

(i)

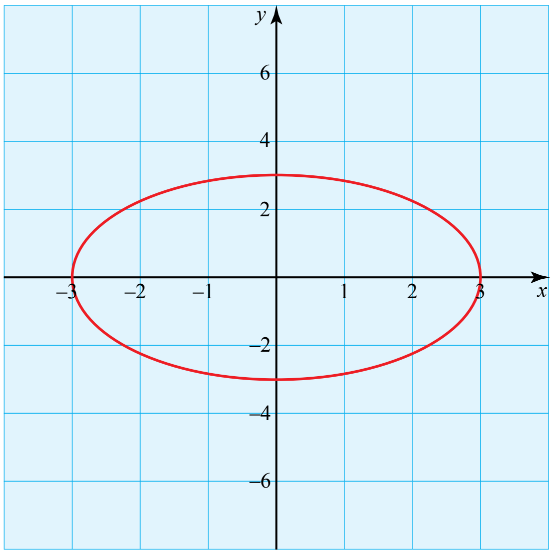



Figure 5.24

(ii)

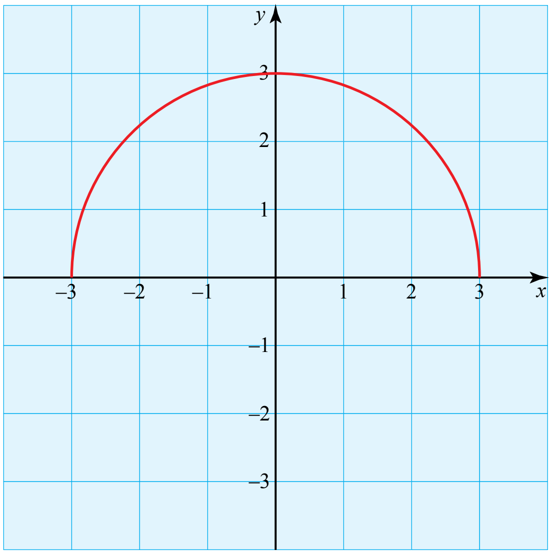



Figure 5.25

**Note**

In the form shown in the activity, the equation highlights some of the important characteristics of the equation of a circle. In particular:

(i) the coefficients of $x^2$ and $y^2$ are equal

(ii) there is no $xy$ term.

## ACTIVITY 5.5

Show that you can rearrange $(x - a)^2 + (y - b)^2 = r^2$ to give
$x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - r^2) = 0$

These results can be generalised to give the equation of a circle as follows:

centre $(0, 0)$, radius $r$: $x^2 + y^2 = r^2$

centre $(a, b)$, radius $r$: $(x - a)^2 + (y - b)^2 = r^2$.

**Example 5.7**

Find the centre and radius of the circle $x^2 + y^2 - 6x + 10y - 15 = 0$.

### *Solution*

You need to rewrite the equation so it is in the form $(x - a)^2 + (y - b)^2 = r^2$.

$$x^2 - 6x \quad + \quad y^2 + 10y \quad - 15 = 0$$

Complete the square on the terms involving $x$ …

$$(x - 3)^2 - 9 + (y + 5)^2 - 25 - 15 = 0$$

… then complete the square on the terms involving $y$.

$$(x - 3)^2 \quad + \quad (y + 5)^2 \quad = 49$$

So the centre is $(3, -5)$ and the radius is 7.

$7^2 = 49$

Integral resource 8: Explore: Circle properties

## Circle geometry

There are some properties of a circle that are useful when solving coordinate geometry problems.

**1** The angle in a semicircle is a right angle (see Figure 5.26).

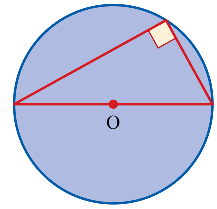



Figure 5.26

**2** The perpendicular from the centre of a circle to a chord bisects the chord (see Figure 5.27).

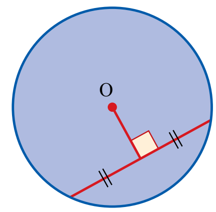



Figure 5.27

**3** The tangent to a circle at a point is perpendicular to the radius through that point (see Figure 5.28).

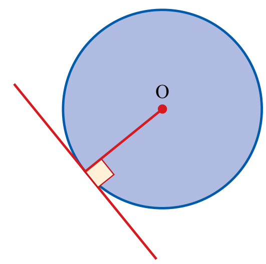



Figure 5.28

### Discussion points

- How can you prove these results?
- State the converse of each of these results.

The converse of '$p$ implies $q$' is '$q$ implies $p$'.

The converse of each of the three circle properties above is also true.

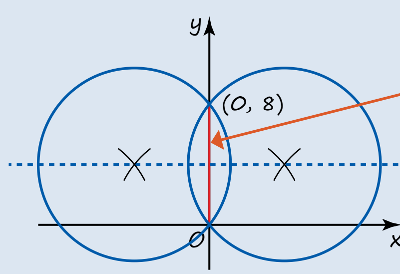
The next three examples use these results in coordinate geometry.

**Example 5.8**

A circle has a radius of 5 units, and passes through the points $(0, 0)$ and $(0, 8)$.

Sketch the two possible positions of the circle and find their equations.

### Solution




The line joining $(0, 0)$ to $(0, 8)$ is a chord of each circle.

The perpendicular bisector of a chord passes through the centre of each circle.

Figure 5.29

The midpoint of the chord is $(0, 4)$.

The equation of the bisector is $y = 4$.

The chord is along the $y$ axis, so the perpendicular bisector passes through $(0, 4)$ and is parallel to the $x$ axis.

So the centre of the circle lies on the line $y = 4$.

Let the centre be the point $(a, 4)$.

Using Pythagoras' theorem $a^2 + 16 = 25$

$\Rightarrow \quad a^2 = 9$

$\Rightarrow \quad a = 3$ or $a = -3$.

The two possible equations are therefore

$(x - 3)^2 + (y - 4)^2 = 25$ and

$(x + 3)^2 + (y - 4)^2 = 25$.

The radius of the circle is 5 and the circle passes through the origin ...

... so the distance between the centre $(a, 4)$ and the origin is 5.

$(x - (-3))^2 + (y - 4)^2 = 25$

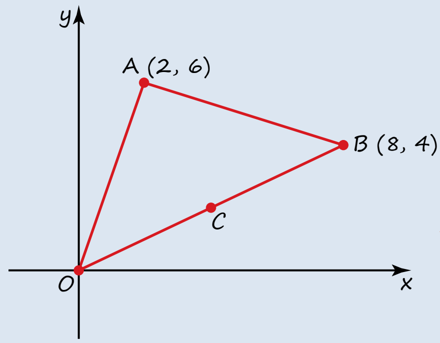
**Example 5.9**

(i) Show that OB is a diameter of the circle which passes through the points O(0, 0), A(2, 6) and B(8, 4).

(ii) Find the equation of the circle.

## Solution

(i)




Always draw a sketch.

**Figure 5.30**

If OB is the diameter of the circle, and A lies on the circle then $\angle$OAB is 90°.

The angle in a semicircle is a right angle.

So to show OB is the diameter you need to show that OA and AB are perpendicular.

$$\text{Gradient of OA} = \frac{6}{2} = 3$$

$$\text{Gradient of AB} = \frac{6-4}{2-8} = \frac{2}{-6} = -\frac{1}{3}$$

$$\text{Product of gradients} = 3 \times -\frac{1}{3} = -1$$

by the converse of the theorem that the angle in a semicircle is a right angle

$\Rightarrow$ Lines OA and AB are perpendicular so angle OAB = 90°.

$\Rightarrow$ OAB is the angle in a semicircle where OB is the diameter, as required.

(ii) The centre C of the circle is the midpoint of OB.

$$C = \left(\frac{0+8}{2}, \frac{0+4}{2}\right) = (4, 2)$$

To find the equation of a circle you need the centre and radius.

The radius of the circle, CO $= \sqrt{4^2 + 2^2} = \sqrt{20}$.

So the radius, $r = \sqrt{20} \Rightarrow r^2 = 20$

Hence the equation of the circle is $(x - 4)^2 + (y - 2)^2 = 20$.

**Example 5.10**

Figure 5.31 shows the circle $x^2 + y^2 = 25$.

The point P(4, 3) lies on the circle and the tangent to the circle at P cuts the coordinate axes at the points Q and R.

Find

(i) the equation of the tangent to the circle at P

(ii) the exact area of triangle OQR.

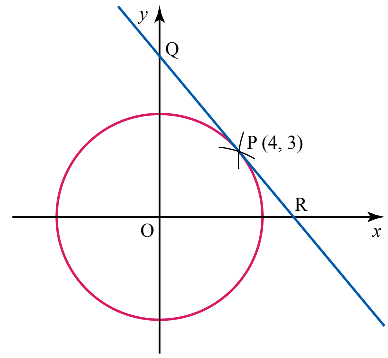



**Figure 5.31**

## Solution

(i) The gradient of OP is $\frac{3}{4}$.

OP is the radius of the circle ...

To find the equation of the tangent you need the gradient.

So the gradient of the tangent is $-\frac{4}{3}$.

... the tangent and radius meet at right angles ...

... so the gradient of the tangent is the negative reciprocal of the gradient of the radius.

The equation of the tangent at P(4, 3) is

$$y - 3 = -\frac{4}{3}(x - 4)$$

$$\Rightarrow \quad 3y - 9 = 16 - 4x$$

$$\Rightarrow \quad 4x + 3y - 25 = 0$$

Area is $\frac{1}{2}$ × base × height.

(ii) OQR forms a right-angled triangle.

The base is the $x$ coordinate of R and the height is the $y$ coordinate of Q.

Find Q:

$$3y - 25 = 0$$

$$\Rightarrow \quad y = \frac{25}{3}$$

Substitute $x = 0$ into the tangent equation to find Q.

Find R:

$$4x - 25 = 0$$

$$\Rightarrow \quad x = \frac{25}{4}$$

Substitute $y = 0$ into the tangent equation to find R.

Area of triangle OQR is

$$\frac{1}{2} \times \frac{25}{4} \times \frac{25}{3} = \frac{625}{24} \text{ square units.}$$

Exact means leave your answer as a fraction (or a surd).

## Exercise 5.4

① Find the equations of the following circles.

(i) centre (2, 3), radius 1

(ii) centre (2, −3), radius 2

(iii) centre (−2, 3), radius 3

(iv) centre (−2, −3), radius 4

② For each of the following circles state

(a) the coordinates of the centre

(b) the radius.

(i) $x^2 + y^2 = 1$

(ii) $x^2 + (y - 2)^2 = 2$

(iii) $(x - 2)^2 + y^2 = 3$

(iv) $(x + 2)^2 + (y + 2)^2 = 4$

(v) $(x - 2)^2 + (y + 2)^2 = 5$

③ The equation of a circle is $(x - 3)^2 + (y + 2)^2 = 26$.
Complete the table to show whether each point lies inside the circle, outside the circle or on the circle.

| Point | Inside | Outside | On |
|---|---|---|---|
| (3, −2) | ✓ | | |
| (−2, −5) | | | |
| (6, −6) | | | |
| (4, 3) | | | |
| (0, 2) | | | |
| (−2, −3) | | | |

④ Draw the circles $(x - 4)^2 + (y - 5)^2 = 16$ and $(x - 3)^2 + (y - 3)^2 = 4$.
In how many points do they intersect?

PS ⑤ Sketch the circle $(x + 2)^2 + (y - 3)^2 = 16$, and find the equations of the four tangents to the circle which are parallel to the coordinate axes.

⑥ Find the coordinates of the points where each of these circles crosses the axes.

(i) $x^2 + y^2 = 25$

(ii) $(x - 4)^2 + (y + 5)^2 = 25$

(iii) $(x + 6)^2 + (y - 8)^2 = 100$

PS ⑦ Find the equation of the circle with centre (1, 7) passing through the point (−4, −5).

⑧ Show that the equation $x^2 + y^2 + 2x - 4y + 1 = 0$ can be written in the form $(x + 1)^2 + (y - 2)^2 = r^2$, where the value of $r$ is to be found.
Hence give the coordinates of the centre of the circle, and its radius.

PS ⑨ Draw the circle of radius 4 units which touches the positive $x$ and $y$ axes, and find its equation.

⑩ A(3, 5) and B(9, −3) lie on a circle.
Show that the centre of the circle lies on the line with equation $4y - 3x + 14 = 0$.

⑪ For each of the following circles find

(a) the coordinates of the centre

(b) the radius.

(i) $x^2 + y^2 - 6x - 2y - 6 = 0$

(ii) $x^2 + y^2 + 2x + 6y - 6 = 0$

(iii) $x^2 + y^2 - 2x + 8y + 8 = 0$

PS ⑫ A circle passes through the points A(3, 2), B(5, 6) and C(11, 3).

(i) Calculate the lengths of the sides of the triangle ABC.

(ii) Hence show that AC is a diameter of this circle. State which theorems you have used, and in each case whether you have used the theorem or its converse.

(iii) Calculate the area of triangle ABC.

⑬ (i) Find the midpoint, C, of AB where A and B are (1, 8) and (3, 14) respectively. Find also the distance AC.

(ii) Hence find the equation of the circle which has AB as a diameter.

PS ⑭ A(1, −2) is a point on the circle $(x - 3)^2 + (y + 1)^2 = 5$.

(i) State the coordinates of the centre of the circle and hence find the coordinates of the point B, where AB is a diameter of the circle.

(ii) C(2, 1) also lies on the circle. Use coordinate geometry to verify that angle ACB = 90°.

PS ⑮ The tangent to the circle $x^2 + (y + 4)^2 = 25$ at the point (−4, −1) intersects the $x$ axis at A and the $y$ axis at B. Find the exact area of the triangle AOB.

PS ⑯ A circle passes through the points (2, 0) and (8, 0) and has the $y$ axis as a tangent. Find the two possible equations for the circle.

PS ⑰ A(6, 3) and B(10,1) are two points on a circle with centre (11, 8).

(i) Calculate the distance of the chord AB from the centre of the circle.

(ii) Find the equation of the circle.

PS ⑱ A(6, 6), B(6, −2) and C(−1, −1) are three points on a circle.
Find the equation of the circle.

**Prior knowledge**

You need to be able to

- solve a quadratic equation
- use the discriminant to determine the number of roots of a quadratic equation.

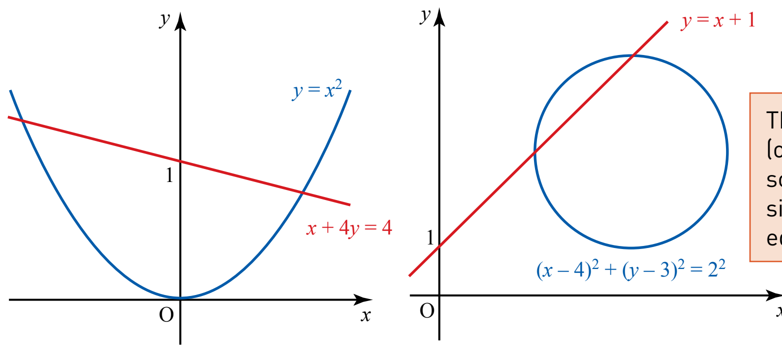
These are covered in Chapter 3.

# 5 The intersection of a line and a curve

When a line and a curve are in the same plane, the coordinates of the point(s) of intersection can be found by solving the two equations simultaneously.

There are three possible situations.

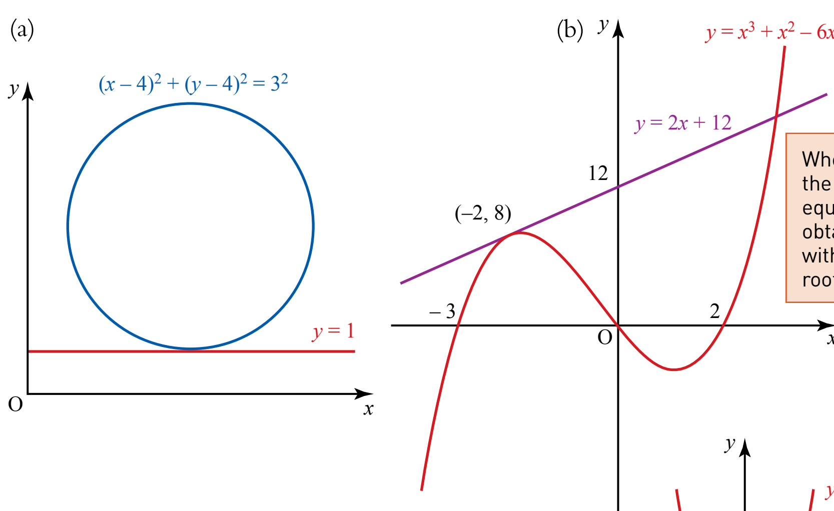
**(i)** All points of intersection are distinct (see Figure 5.32).




There are 2 (or more) distinct solutions to the simultaneous equations.

**Figure 5.32**

**(ii)** The line is a tangent to the curve at one (or more) point(s) (see Figure 5.33). In this case, each point of contact corresponds to two (or more) coincident points of intersection. It is possible that the tangent will also intersect the curve somewhere else (as in Figure 5.33b).




When you solve the simultaneous equations you will obtain an equation with a repeated root.

**Figure 5.33**

**(iii)** The line and the curve do not meet (see Figure 5.34).

When you try to solve the simultaneous equations you will obtain an equation with no roots. So there is no point of intersection.

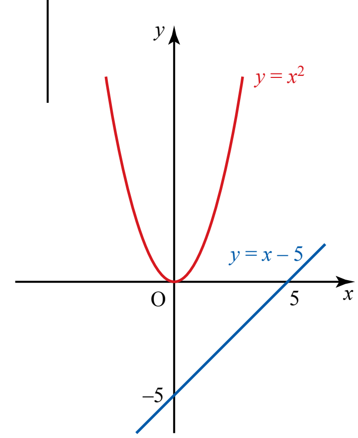



**Figure 5.34**

**Example 5.11**

A circle has equation $x^2 + y^2 = 8$.

For each of the following lines, find the coordinates of any points where the line intersects the circle.

(i) $y = x$ (ii) $y = x + 4$ (iii) $y = x + 6$

## Solution

(i) Substituting $y = x$ into $x^2 + y^2 = 8$ gives

Simplify.

$$x^2 + x^2 = 8$$
$$2x^2 = 8$$
$$x^2 = 4$$
$$x = \pm 2$$

Don't forget the negative square root!

Since $y = x$ then the coordinates are $(-2, -2)$ and $(2, 2)$.

The line intersects the circle twice.

(ii) Substituting $y = x + 4$ into $x^2 + y^2 = 8$ gives

$$x^2 + (x + 4)^2 = 8$$

Multiply out the brackets.

$$\Rightarrow x^2 + x^2 + 8x + 16 = 8$$
$$\Rightarrow 2x^2 + 8x + 8 = 0$$
$$\Rightarrow x^2 + 4x + 4 = 0$$

Divide by 2.

$$\Rightarrow (x + 2)^2 = 0$$
$$x = -2$$

This is a repeated root, so $y = x + 4$ is a tangent to the circle.

When $x = -2$ then $y = -2 + 4 = 2$

So the coordinates are $(-2, 2)$.

(iii) Substituting $y = x + 6$ into $x^2 + y^2 = 8$ gives

$$x^2 + (x + 6)^2 = 8$$
$$\Rightarrow x^2 + x^2 + 12x + 36 = 8$$
$$\Rightarrow 2x^2 + 12x + 28 = 0$$

Check the discriminant: $b^2 - 4ac$

$$12^2 - 4 \times 2 \times 28 = -80$$

Since the discriminant is less than 0, the equation has no real roots.

So the line $y = x + 6$ does not meet the circle.

**Note**

This example shows you an important result. When you are finding the intersection points of a line and a quadratic curve, or two quadratic curves, you obtain a quadratic equation.

If the discriminant is:

- positive – there are **two** points of intersection
- zero – there is **one repeated** point of intersection
- negative – there are **no** points of intersection.

## The intersection of two curves

The same principles apply to finding the intersection of two curves, but it is only in simple cases that it is possible to solve the equations simultaneously using algebra (rather than a numerical or graphical method).

**Example 5.12**

Sketch the circle $x^2 + y^2 = 16$ and the curve $y = x^2 - 4$ on the same axes. Find the coordinates of any points of intersection.

**Discussion points**

- How else could you solve the simultaneous equations in the example?
- Which method is more efficient in this case?

### Solution

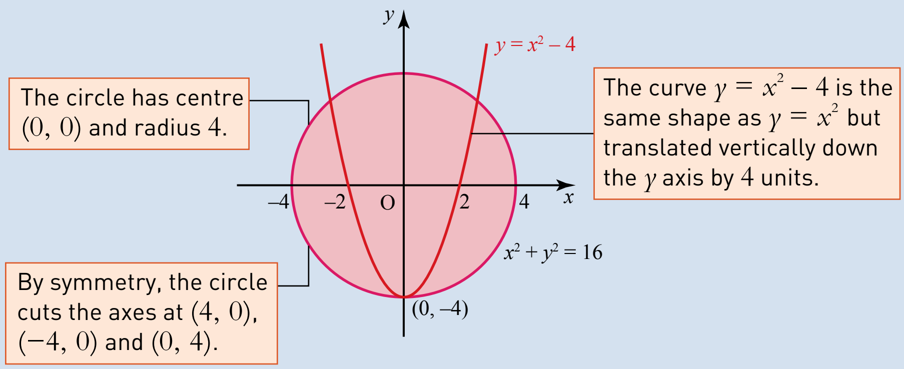



**Figure 5.35**

Substituting $y = x^2 - 4$ into $x^2 + y^2 = 16$ gives

$$x^2 + (x^2 - 4)^2 = 16$$

$$\Rightarrow x^2 + x^4 - 8x^2 + 16 = 16 \quad \text{Simplify.}$$

$$\Rightarrow \quad x^4 - 7x^2 = 0$$

$$\Rightarrow \quad x^2(x^2 - 7) = 0 \quad \text{Factorise.}$$

$$\Rightarrow \quad x^2 = 0 \Rightarrow x = 0 \text{ (twice).}$$

$$\text{or} \Rightarrow \quad x^2 = 7 \Rightarrow x = \pm\sqrt{7}$$

Don't forget the negative square root.

Substitute back into $y = x^2 - 4$ to find the $y$ coordinates.

$x = 0 \quad \Rightarrow y = -4$

$x = \pm\sqrt{7} \Rightarrow y = 7 - 4 = 3$

So the points of intersection are $(-\sqrt{7}, 3)$, $(\sqrt{7}, 3)$ and $(0, -4)$ (twice).

Integral resource 9: Test C2

**Exercise 5.5**

① Show that the line $y = 3x + 1$ crosses the curve $y = x^2 + 3$ at $(1, 4)$ and find the coordinates of the other point of intersection.

② Find the coordinates of the points where the line $y = 2x - 1$ cuts the circle $(x - 2)^2 + (y + 1)^2 = 5$.

③ Find the coordinates of the points of intersection of the line $2y = x - 5$ and the circle $(x + 1)^2 + (y - 2)^2 = 20$.
What can you say about this line and the circle?

④ (i) Show that the line $x + y = 6$ is a tangent to the circle $x^2 + y^2 = 18$.
(ii) Show the line and circle on a diagram.
Find the point of contact of the tangent parallel to the line $x + y = 6$, and the equation of the tangent.

PS

⑤ (i) Find the coordinates of the points of intersection of the line $y = 2x$ and the curve $y = x^2 + 6x - 5$.
(ii) Show also that the line $y = 2x$ does not cross the curve $y = x^2 + 6x + 5$.

PS

6 Find the coordinates of the points A and B where the line $x - 3y + 15 = 0$ cuts the circle $x^2 + y^2 + 2x - 6y + 5 = 0$.

Also find the coordinates of the points where the line $y = x + 1$ meets the curve $y = x^3 - 3x^2 + 3x + 1$.

PS 7 Figure 5.36 shows the cross-section of a goldfish bowl. The bowl can be thought of as a sphere with its top removed and its base flattened.

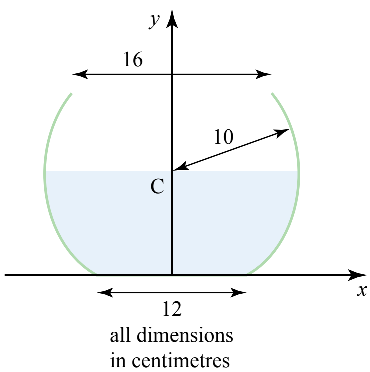



**Figure 5.36**

Assume the base is on the $x$ axis and the $y$ axis is a line of symmetry.

(i) Find the height of the bowl.

(ii) Find the equation of the circular part of the cross-section.

(iii) The bowl is filled with water to a depth of 12 cm.

Find the area of the surface of the water.

PS 8 The line $y = 1 - x$ intersects the circle $x^2 + y^2 = 25$ at two points A and B.

(i) Find the coordinates of the points and the distance AB.

(ii) Is AB a diameter of the circle?

Give a reason for your answer.

PS 9 (i) Find the value of $k$ for which the line $2y = x + k$ forms a tangent to the curve $y^2 = 2x$.

(ii) Hence find the coordinates of the point where the line $2y = x + k$ meets the curve for the value of $k$ found in part (i).

10 The equation of a circle is $(x + 2)^2 + y^2 = 8$ and the equation of a line is $x + y = k$, where $k$ is a constant.

Find the values of $k$ for which the line forms a tangent to the curve.

PS

PS 11 The equations of two circles are given below.

$$(x + 1)^2 + (y - 2)^2 = 10$$

$$\text{and } (x - 1)^2 + (y - 3)^2 = 5$$

The two circles intersect at the points A and B.

Find the area of the triangle AOB where O is the origin.

Problem solving

# Integer point circles

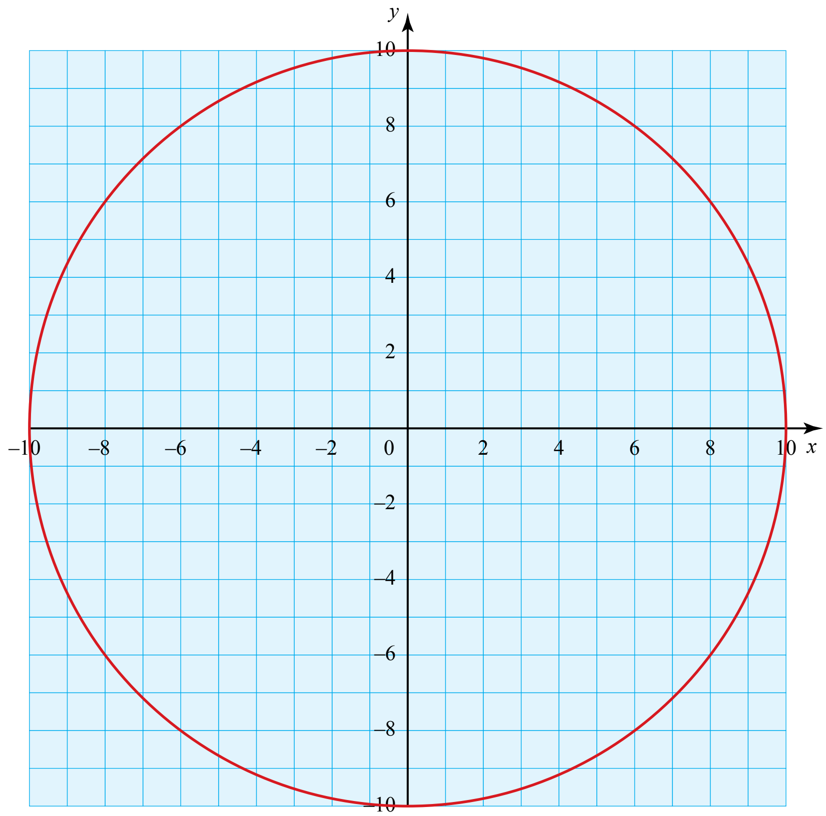



Figure 5.37

Look at the circle in Figure 5.37. Its equation is $x^2 + y^2 = 100$.

It goes through the point (6, 8). Since both 6 and 8 are integers, this is referred to as an **integer point** in this question. This is not the only integer point this circle goes through; another is (−10, 0) and there are others as well.

(i) How many integer points are there on the circle?

(ii) How many circles are there with equations of the form $x^2 + y^2 = N$, where $0 < N < 100$, that pass through at least one integer point?

How many of these circles pass through at least 12 integer points?

(iii) Devise and explain at least one method to find the equation of a circle with radius greater than 10 units that passes through at least 12 integer points.

1 **Problem specification and analysis**

Parts (i) and (ii) of the problem are well defined and so deal with them first. Start by thinking about possible strategies. There are several quite different approaches, based on geometry or algebra. You may decide to try more than one and see how you get on.

Part (iii) is more open ended. You have to devise and explain at least one method. Leave this until you get to the last stage of the problem solving cycle. By then your earlier work may well have given you some insight into how to go about it.

2 **Information collection**

In this problem there will probably be a large amount of trial and error in your data collection. As well as collecting information, you will be trying out different possible approaches.

There are a number of cases that you could try out and so you need to be on the lookout for patterns that will cut down on your work. You have to think carefully about how you are going to record your findings systematically.

3 **Processing and representation**

The work you need to do at this stage will depend on what you have already done at the Information collection stage.

You may have already collected all the information you need to answer parts (i) and (ii) by just counting up the numbers. Alternatively, however, you may have found some patterns that will help you to work out the answers.

You then need to find a good way to present your answers. Think of someone who is unfamiliar with the problem. How are you going to show such a person what you have found in a convincing way?

4 **Interpretation**

So far you have been looking at parts (i) and (ii) of the problem. They are well defined and all the answers are numbers.

In part (iii), you are now expected to interpret what you have been doing by finding not just numbers but also a method, so that you can continue the work with larger circles.

To give a good answer you will almost certainly need to use algebra but you will also need to explain what you are doing in words.

The wording of the questions suggests there is more than one method and that is indeed the case. So a really good answer will explore the different possibilities.

## LEARNING OUTCOMES

Now you have finished this chapter, you should be able to

- solve problems involving finding the midpoint of two points
- solve problems involving finding the distance between two points
- understand gradient as the rate of change
- find the gradient of a line joining two points
- recall and use the relationships between gradients for parallel and perpendicular lines
- solve problems with parallel and perpendicular lines
- draw or sketch a line, given its equation
- find the equation of a line
- solve real-life problems that can be modelled by a linear function
- find the intersection of two lines
- find the centre and radius of a circle from its equation
  - when the equation of the circle is given in its standard form
  - when the equation of the circle needs to be rewritten in completed square form
- find the equation of a circle
  - given the radius and the centre
  - using circle theorems to find the centre and radius
- find the equation of a tangent to a circle using the circle theorems
- find the points of intersection of a line and a curve
  - understanding the significance of a repeated root in the case of a line which is a tangent to the curve
  - understanding the significance of having no roots in the case of a line which does not intersect the curve
- find the intersection points of curves in simple cases.

## KEY POINTS

1 For a line segment A($x_1$, $y_1$) and B($x_2$, $y_2$) (Figure 5.38) then:

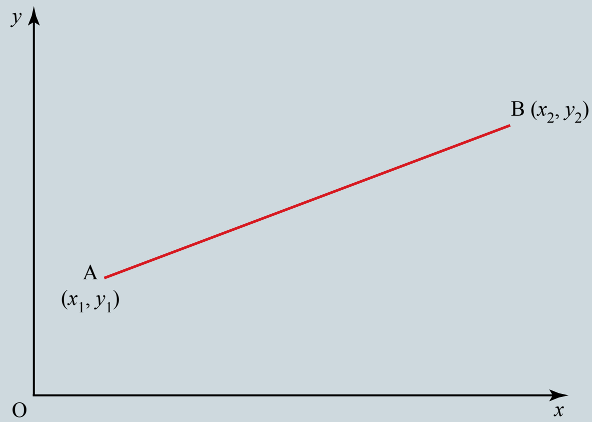



**Figure 5.38**

- the gradient of AB is $\dfrac{y_2 - y_1}{x_2 - x_1}$
- the midpoint is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$
- the distance AB is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

using Pythagoras' theorem

2 Two lines are parallel ⇔ their gradients are equal.

3 Two lines are perpendicular ⇔ the product of their gradients is –1.

4 The equation of a straight line may take any of the following forms:
- line parallel to the $y$ axis: $x = a$
- line parallel to the $x$ axis: $y = b$
- line through the origin with gradient $m$: $y = mx$
- line through (0, $c$) with gradient $m$: $y = mx + c$
- line through ($x_1$, $y_1$) with gradient $m$: $y - y_1 = m(x - x_1)$

5 The equation of a circle is
- centre (0, 0), radius $r$: $x^2 + y^2 = r^2$
- centre ($a$, $b$), radius $r$: $(x - a)^2 + (y - b)^2 = r^2$.

6 The angle in a semicircle is a right angle (Figure 5.39).

7 The perpendicular from the centre of a circle to a chord bisects the chord (Figure 5.40).

8 The tangent to a circle at a point is perpendicular to the radius through that point (Figure 5.41).

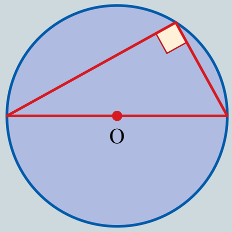



**Figure 5.39**

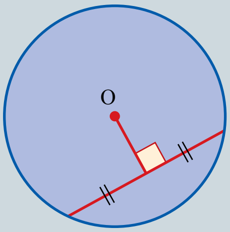



**Figure 5.40**

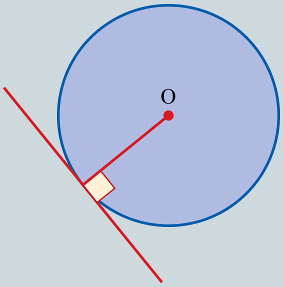



**Figure 5.41**

9 To find the points of intersection of two curves, you solve their equations simultaneously.

## PRACTICE QUESTIONS PURE MATHEMATICS 1

MP ① (i) Prove that $\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}}$. [2 marks]

(ii) Show that $\dfrac{\sqrt{3}+1}{\sqrt{3}-1} = \sqrt{3}+2$. [2 marks]

PS ② (i) Solve the equation $2^{3x} = 4^{x+4}$. [3 marks]

MP (ii) Find a value of $x$ which is a counter-example to $10^x > 2^x$. [1 mark]

MP ③ Do not use a calculator in this question.

Figure 1 shows the curves $y = x^2 - 4x + 1$ and $y = 7 - x^2$.

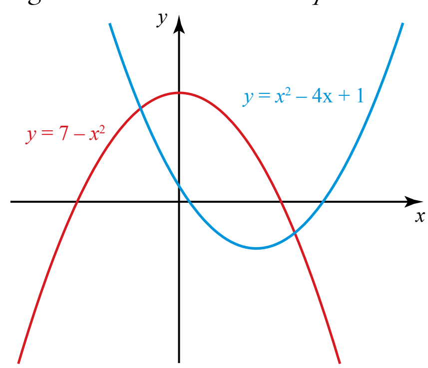



**Figure 1**

(i) Find the coordinates of their points of intersection. [5 marks]

(ii) Prove that $y = -2x$ is a tangent to $y = x^2 - 4x + 1$ and state the coordinates of the point of contact. [4 marks]

④ Do not use a calculator in this question.

(i) Write $x^2 + 6x + 7$ in the form $(x + a)^2 + b$. [3 marks]

(ii) State the coordinates of the turning point of $y = x^2 + 6x + 7$ and whether it is a minimum or maximum. [3 marks]

(iii) Sketch the curve $y = x^2 + 6x + 7$ and solve the inequality $x^2 + 6x + 7 > 0$. [4 marks]

⑤ Figure 2 shows a circle with centre C which passes through the points A(2, 4) and B(−1, 1).

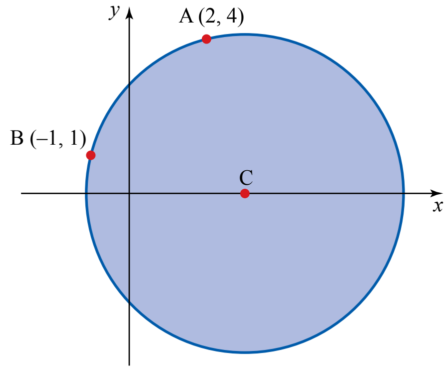



**Figure 2**

(i) AB is a chord of the circle. Show that the centre of the circle must lie on the line $x + y = 3$, explaining your reasoning. [7 marks]

(ii) The centre of the circle also lies on the $x$ axis. Find the equation of the circle. [5 marks]

PS ⑥ Figure 3 shows an equilateral triangle, ABC with A and B on the $x$ axis and C on the $y$ axis.

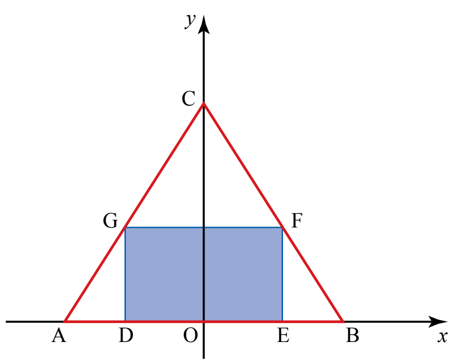



**Figure 3**

Each side of triangle ABC measures 4 units.

(i) Find the coordinates of points A, B and C in exact form. [4 marks]

(ii) Show that the equation of line BC can be written as $y = \sqrt{3}(2 - x)$. [2 marks]

A rectangle DEFG is drawn inside the triangle, as also shown in Figure 3.

D and E lie on the $x$ axis, G on AC and F on BC.

(iii) Find the greatest possible area of rectangle DEFG. [7 marks]

M T

⑦ Figure 4 shows a spreadsheet with the information about typical stopping distances for cars from the Highway Code. Figure 5 has been drawn using the spreadsheet.

Home Insert Page Layout Formulas

*fx*

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Speed (mph)** | **Thinking distance (m)** | **Braking distance (m)** | **Total stopping distance (m)** |
| 2 | 20 | 6 | 6 | 12 |
| 3 | 30 | 9 | 14 | 23 |
| 4 | 40 | 12 | 24 | 36 |
| 5 | 50 | 15 | 38 | 53 |
| 6 | 60 | 18 | 55 | 73 |
| 7 | 70 | 21 | 75 | 96 |

**Figure 4**

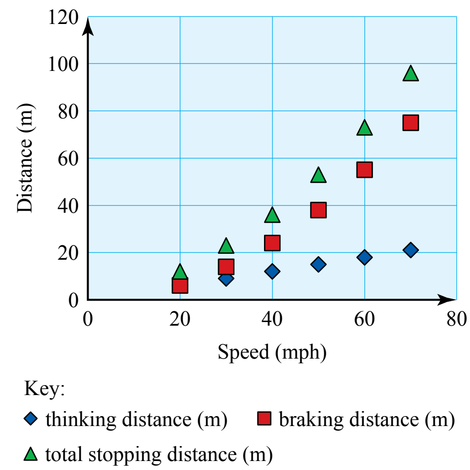



**Figure 5**

(i) (a) What feature of the scatter diagram (Figure 5) suggests that the thinking distance is directly proportional to speed? [1 mark]

(b) What does this tell you about the thinking time for different speeds?
Comment, with a brief explanation, on whether this is a reasonable modelling assumption. [2 marks]

(c) Write down a formula connecting the speed, $x$ mph and the thinking distance $d$ m. [1 mark]

(ii) The spreadsheet (Figure 4) gives the following linear best fit model for the total stopping distance, $y$ m in terms of the speed $x$ mph.

$$y = 1.6771x - 26.38$$

(a) Use the model to find the total stopping distance for a speed of 10 mph. [1 mark]

(b) Explain why this is not a suitable model for total stopping distance. [1 mark]

(iii) The spreadsheet gives the following quadratic best fit model for the total stopping distance.

$$y = 0.0157x^2 + 0.2629x + 0.6$$

The values for total stopping distance using this model are shown in Table 1.

**Table 1**

| **Speed (mph)** | 20 | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|
| **Quadratic model (m)** | | 22.617 | 36.236 | 52.995 | 72.894 | 95.933 |

(a) Calculate the missing value for 20 mph. [1 mark]

(b) Give one possible reason why the model does not give exactly the same total stopping distances as those listed in the Highway Code. [1 mark]