mathtutor differentiation

Tangents and normals

This unit explains how di?erentiation can be used to calcula te the equations of the tangent and normal to a curve. The tangent is a straight line which just to uches the curve at a given point. The normal is a straight line which is perpendicular to the tangent.

To calculate the equations of these lines we shall make use ofthe fact that the equation of a straight line passing through the point with coordinates (x_1, y_1) and having gradient m is given by

$$\frac{y - y_1}{x - x_1} = m$$

We also make use of the fact that if two lines with gradients m_1 and m_2 respectively are perpendicular, then $m_1 m_2 = -1$.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial o n this topic, you should be able to:

- calculate the equation of the tangent to a curve at a given point
- calculate the equation of the normal to a curve at a given point

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1. Introduction

Consider a function $f(x)$ such as that shown in Figure 1. When we calculate the derivative, f', of the function at a point $x = a$ say, we are finding the gradient of the tangent to the graph of that function at that point. Figure 1 shows the tangent drawn at $x = a$. The gradient of this tangent is $f'(a)$.

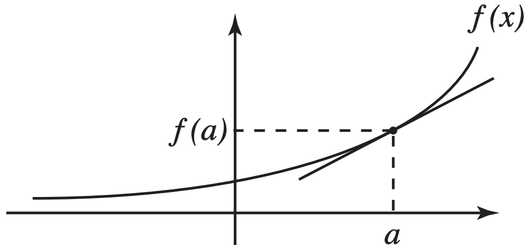

Figure 1. The tangent drawn at $x = a$ has gradient $f'(a)$.

We will use this information to calculate the equation of the tangent to a curve at a particular point, and then the equation of the normal to a curve at a point.

Key Point

$f'(a)$ is the gradient of the tangent drawn at $x = a$.

2. Calculating the equation of a tangent

Example

Suppose we wish to find the equation of the tangent to

$$f(x) = x^3 - 3x^2 + x - 1$$

at the point where $x = 3$.

When $x = 3$ we note that

$$f(3) = 3^3 - 3.3^2 + 3 - 1 = 27 - 27 + 3 - 1 = 2$$

So the point of interest has coordinates $(3, 2)$.

The next thing that we need is the gradient of the curve at this point. To find this, we need to differentiate $f(x)$:

$$f'(x) = 3x^2 - 6x + 1$$

We can now calculate the gradient of the curve at the point where $x = 3$.

$$f'(3) = 3.3^2 - 6.3 + 1 = 27 - 18 + 1 = 10$$

So we have the coordinates of the required point, $(3, 2)$, and the gradient of the tangent at that point, 10.

What we want to calculate is the equation of the tangent at this point on the curve. The tangent must pass through the point and have gradient 10. The tangent is a straight line and so we use the fact that the equation of a straight line that passes through a point (x_1, y_1) and has gradient m is given by the formula

$$\frac{y - y_1}{x - x_1} = m$$

Substituting the given values

$$\frac{y - 2}{x - 3} = 10$$

and rearranging

$$\begin{aligned} y - 2 &= 10(x - 3) \\ y - 2 &= 10x - 30 \\ y &= 10x - 28 \end{aligned}$$

This is the equation of the tangent to the curve at the point $(3, 2)$.

Key Point

The equation of a straight line that passes through a point (x_1, y_1) and has gradient m is given by

$$\frac{y - y_1}{x - x_1} = m$$

Example

Suppose we wish to find points on the curve $y(x)$ given by

$$y = x^3 - 6x^2 + x + 3$$

where the tangents are parallel to the line $y = x + 5$.

If the tangents have to be parallel to the line then they must have the same gradient. The standard equation for a straight line is $y = mx + c$, where m is the gradient. So what we gain from looking at this standard equation and comparing it with the straight line $y = x + 5$ is that the gradient, m, is equal to 1. Thus the gradients of the tangents we are trying to find must also have gradient 1.

We know that if we differentiate $y(x)$ we will obtain an expression for the gradients of the tangents to $y(x)$ and we can set this equal to 1. Differentiating, and setting this equal to 1 we find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x + 1 = 1$$

from which

$$3x^2 - 12x = 0$$

This is a quadratic equation which we can solve by factorisation.

$$\begin{aligned} 3x^2 - 12x &= 0 \\ 3x(x-4) &= 0 \end{aligned}$$

$$3x = 0 \qquad \text{or} \qquad x - 4 = 0$$

$$x = 0 \qquad \text{or} \qquad x = 4$$

Now having found these two values of x we can calculate the corresponding y coordinates. We do this from the equation of the curve: $y = x^3 - 6x^2 + x + 3$.

when $x = 0$: $\quad y = 0^3 - 6.0^2 + 0 + 3 = 3$.

when $x = 4$: $\quad y = 4^3 - 6.4^2 + 4 + 3 = 64 - 96 + 4 + 3 = -25$.

So the two points are $(0, 3)$ and $(4, -25)$

These are the two points where the gradients of the tangent are equal to 1, and so where the tangents are parallel to the line that we started out with, i.e. $y = x + 5$.

Exercise 1

1. For each of the functions given below determine the equation of the tangent at the points indicated.

a) $f(x) = 3x^2 - 2x + 4$ at $x = 0$ and 3.

b) $f(x) = 5x^3 + 12x^2 - 7x$ at $x = -1$ and 1.

c) $f(x) = xe^x$ at $x = 0$.

d) $f(x) = (x^2 + 1)^3$ at $x = -2$ and 1.

e) $f(x) = \sin 2x$ at $x = 0$ and $\dfrac{\pi}{6}$.

f) $f(x) = 1 - 2x$ at $x = -3$, 0 and 2.

2. Find the equation of each tangent of the function $f(x) = x^3 - 5x^3 + 5x - 4$ which is parallel to the line $y = 2x + 1$.

3. Find the equation of each tangent of the function $f(x) = x^3 + x^2 + x + 1$ which is perpendicular to the line $2y + x + 5 = 0$.

3. The equation of a normal to a curve

In mathematics the word 'normal' has a very specific meaning. It means 'perpendicular' or 'at right angles'.

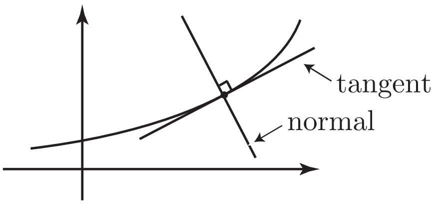

Figure 2. The normal is a line at right angles to the tangent.

If we have a curve such as that shown in Figure 2, we can choose a point and draw in the tangent to the curve at that point. The normal is then at right angles to the curve so it is also at right angles (perpendicular) to the tangent.

We now find the equation of the normal to a curve. There is one further piece of information that we need in order to do this. If two lines, having gradients m_1 and m_2 respectively, are at right angles to each other then the product of their gradients, $m_1 m_2$, must equal -1.

Key Point

If two lines, with gradients m_1 and m_2 are at right angles then $m_1 m_2 = -1$

Example

Suppose we wish to find the equation of the tangent and the equation of the normal to the curve

$$y = x + \frac{1}{x}$$

at the point where $x = 2$.

First of all we shall calculate the y coordinate at the point on the curve where $x = 2$:

$$y = 2 + \frac{1}{2} = \frac{5}{2}$$

Next we want the gradient of the curve at the point $x = 2$. We need to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Noting that we can write y as $y = x + x^{-1}$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

Furthermore, when $x = 2$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{1}{4} = \frac{3}{4}$$

This is the gradient of the tangent to the curve at the point $(2, \frac{5}{2})$. We know that the standard equation for a straight line is

$$\frac{y - y_1}{x - x_1} = m$$

With the given values we have

$$\frac{y - \frac{5}{2}}{x - 2} = \frac{3}{4}$$

Rearranging

$$
\begin{aligned}
y - \frac{5}{2} &= \frac{3}{4}(x-2) \\
4\left(y - \frac{5}{2}\right) &= 3(x-2) \\
4y - 10 &= 3x - 6 \\
4y &= 3x + 4
\end{aligned}
$$

So the equation of the tangent to the curve at the point where $x = 2$ is $4y = 3x + 4$.

Now we need to find the equation of the normal to the curve.

Let the gradient of the normal be m_2. Suppose the gradient of the tangent is m_1. Recall that the normal and the tangent are perpendicular and hence $m_1 m_2 = -1$. We know $m_1 = \frac{3}{4}$. So

$$\frac{3}{4} \times m_2 = -1$$

and so

$$m_2 = -\frac{4}{3}$$

So we know the gradient of the normal and we also know the point on the curve through which it passes, $\left(2, \frac{5}{2}\right)$.

As before,

$$
\begin{aligned}
\frac{y - y_1}{x - x_1} &= m \\
\frac{y - \frac{5}{2}}{x - 2} &= -\frac{4}{3}
\end{aligned}
$$

Rearranging

$$
\begin{aligned}
3\left(y - \frac{5}{2}\right) &= -4(x-2) \\
3y - \frac{15}{2} &= -4x + 8 \\
3y + 4x &= 8 + \frac{15}{2} \\
3y + 4x &= \frac{31}{2} \\
6y + 8x &= 31
\end{aligned}
$$

This is the equation of the normal to the curve at the given point.

Example

Consider the curve $xy = 4$. Suppose we wish to find the equation of the normal at the point $x = 2$. Further, suppose we wish to know where the normal meet the curve again, if it does.

Notice that the equation of the given curve can be written in the alternative form $y = \dfrac{4}{x}$. A graph of the function $y = \dfrac{4}{x}$ is shown in Figure 3.

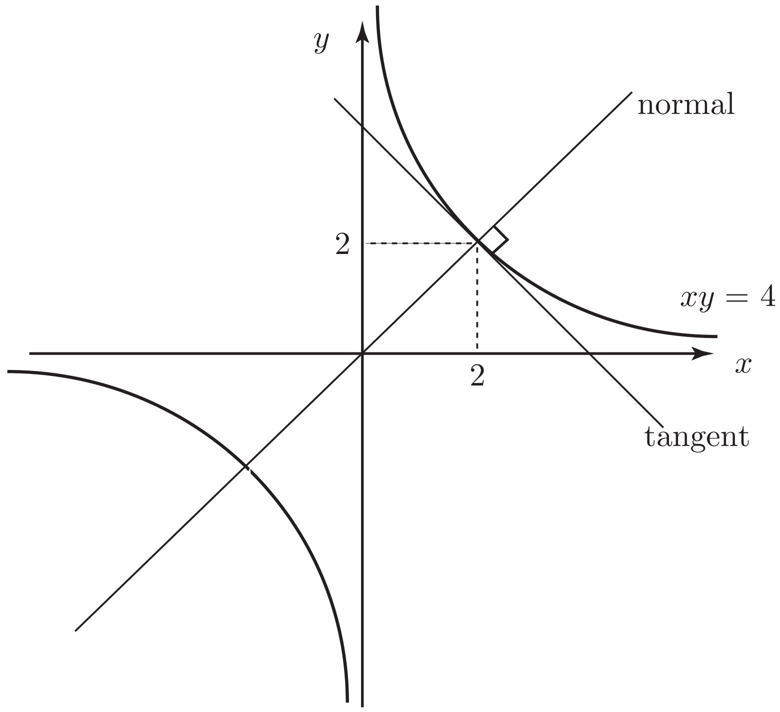

Figure 3. A graph of the curve $xy = 4$ showing the tangent and normal at $x = 2$.

From the graph we can see that the normal to the curve when $x = 2$ does indeed meet the curve again (in the third quadrant). We shall determine the point of intersection. Note that when $x = 2$, $y = \dfrac{4}{2} = 2$.

We first determine the gradient of the tangent at the point $x = 2$. Writing

$$\begin{aligned} y &= \frac{4}{x} \\ &= 4x^{-1} \end{aligned}$$

and differentiating, we find

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= -4x^{-2} \\ &= -\frac{4}{x^2} \end{aligned}$$

Now, when $x = 2$ $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{4}{4} = -1.$

So, we have the point $(2, 2)$ and we know the gradient of the tangent there is -1. Remember that the tangent and normal are at right angles and for two lines at right angles the product of their gradients is -1. Therefore we can deduce that the gradient of the normal must be $+1$. So, the normal passes through the point $(2, 2)$ and its gradient is 1.

As before, we use the equation of a straight line in the form:

$$\begin{aligned}
\frac{y-y_1}{x-x_1} &= m \\
\frac{y-2}{x-2} &= 1 \\
y-2 &= x-2 \\
y &= x
\end{aligned}$$

So the equation of the normal is $y = x$.

We can now find where the normal intersects the curve $xy = 4$. At any points of intersection both of the equations

$$xy = 4 \qquad \text{and} \qquad y = x$$

are true at the same time, so we solve these equations simultaneously. We can substitute $y = x$ from the equation of the normal into the equation of the curve:

$$\begin{aligned}
xy &= 4 \\
x \cdot x &= 4 \\
x^2 &= 4 \\
x &= \pm 2
\end{aligned}$$

So we have two values of x where the normal intersects the curve. Since $y = x$ the corresponding y values are also 2 and -2. So our two points are $(2,2)$, $(-2,-2)$. These are the two points where the normal meets the curve. Notice that the first of these is the point we started off with.

Exercise 2

1. For each of the functions given below determine the equations of the tangent and normal at each of the points indicated.

a) $f(x) = x^2 + 3x + 1$ at $x = 0$ and 4.

b) $f(x) = 2x^3 - 5x + 4$ at $x = -1$ and 1.

c) $f(x) = \tan x$ at $x = \dfrac{\pi}{4}$.

d) $f(x) = 3 - x$ at $x = -2$, 0 and 1.

2. Find the equation of each normal of the function $f(x) = \dfrac{1}{3}x^3 + x^2 + x - \dfrac{1}{3}$ which is parallel to the line $y = -\dfrac{1}{4}x + \dfrac{1}{3}$.

3. Find the x co-ordinate of the point where the normal to $f(x) = x^2 - 3x + 1$ at $x = -1$ intersects the curve again.

Answers

Exercise 1

1. a) $y=-2x+4$, $y=16x-23$ b) $y=-16x-2$, $y=32x-22$ c) $y=x$,
d) $y=-300x-0475$, $y=24x-16$, e) $y=2x$, $y=x+\dfrac{\sqrt{3}}{2}-\dfrac{\pi}{6}$,
f) $y=1-2x$, $y=1-2x$, $y=1-2x$

2. $y=2x-\dfrac{95}{27}$, $y=2x-13$

3. $y=2x+2$, $y=2x+\dfrac{22}{27}$

Exercise 2

1. a) At $x=0$: $y=3x+1$, $y=-\dfrac{1}{3}x+1$, At $x=4$: $y=11x-15$, $y=-\dfrac{1}{11}x+\dfrac{323}{11}$
b) At $x=-1$: $y=x+8$, $y=-x+6$, At $x=1$: $y=x$, $y=-x+2$
c) At $x=\dfrac{\pi}{4}$: $y=2x+1-\dfrac{\pi}{2}$, $y=-\dfrac{1}{2}x+1+\dfrac{\pi}{8}$
d) At $x=-2$: $y=3-x$, $y=x+7$, At $x=0$: $y=3-x$, $y=x+3$,
At $x=1$: $y=3-x$, $y=x+1$

2. $y=-\dfrac{1}{4}x+\dfrac{9}{4}$, $y=-\dfrac{1}{4}x-\dfrac{49}{12}$

3. $\dfrac{21}{5}$