The Short-Time Fourier Transform

Part 2

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The Short-Frequency Time Transform

- **If we relate the window function in time with the window function in frequency by the Fourier transform:**

$$H(f) = \int h(t) e^{-i2\pi ft} dt$$

then $$X(t, f) = x(t, f) e^{-i2\pi ft}$$

- **The *short-time Fourier transform* is the same as the *short-frequency time transform* except for the phase factor**

- **Since the distribution is the square modulus, the phase factor does not enter into it**

The Short-Frequency Time Transform

- **Thus, either the *short-time Fourier transform or short-frequency time transform* can be used to define the time-frequency distribution:**

$$\left| X(t, f) \right|^2 = \left| x(t, f) \right|^2$$

- **This shows that the spectrogram can be used to study the behavior of time properties at a particular frequency, which is useful for condition monitoring and diagnostics**

- **This is done by choosing an** $H(f)$ **narrow, or equivalently by taking an** $h(t)$ **broad**

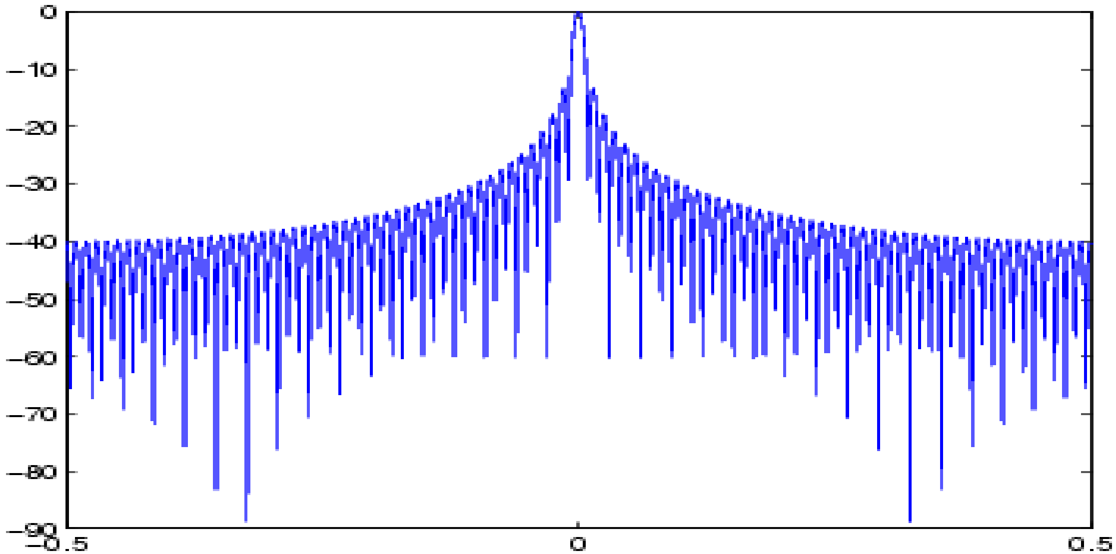
Window Selection

- **The choice of the *analysis window* is important, since it directly affects the trade-off between the frequency resolution and side-lobe attenuation, as seen below**

- **To understand the effect of the window, let us consider its effect on a complex sinusoidal signal**

- **It is well-known that the Fourier transform of a windowed sinusoid is the Fourier transform of the window function shifted to be centred at the frequency of the sinusoid**

Case Study: the Windowed Sinusoid

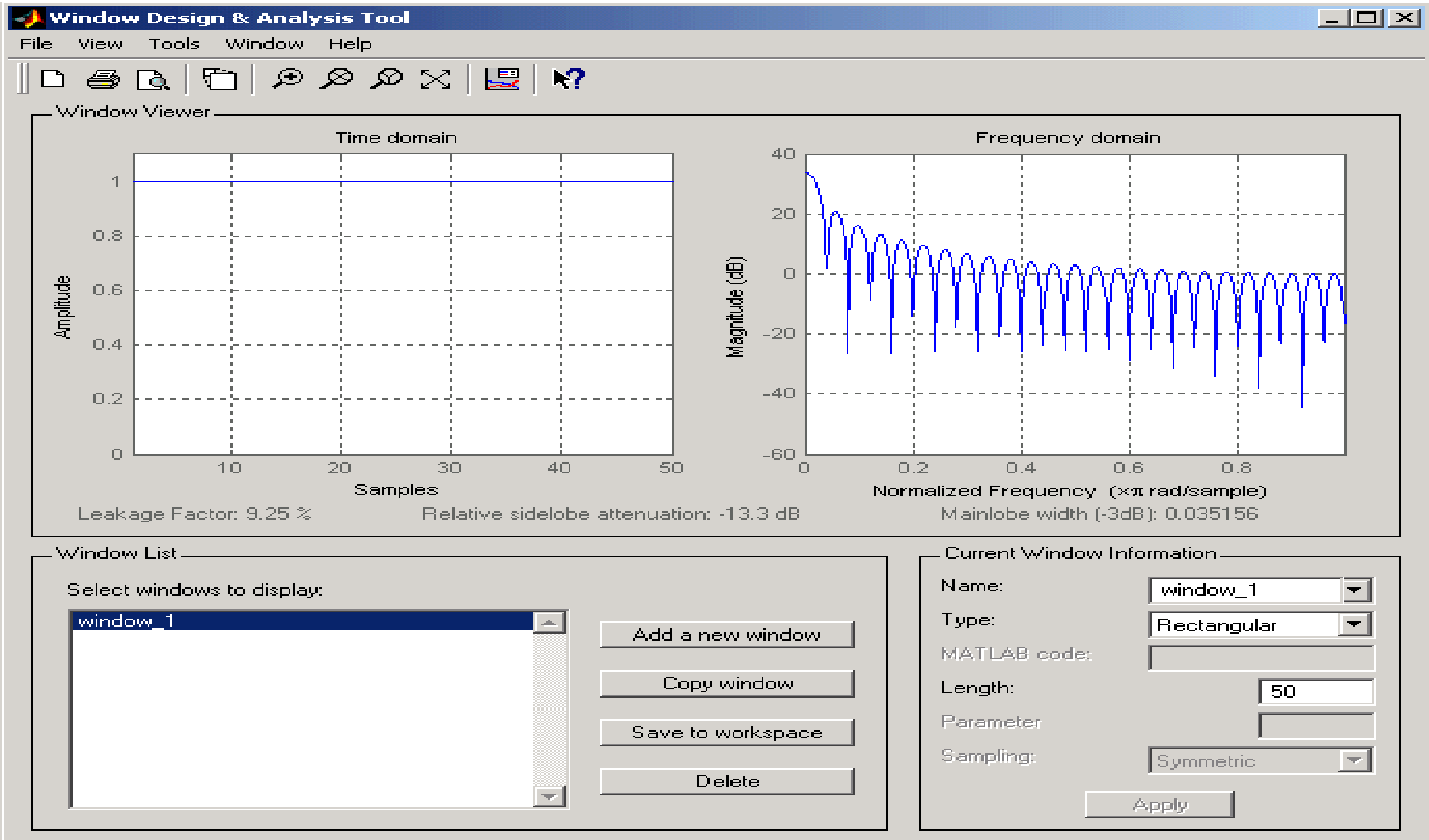


Window Selection

A good window function must present a magnitude response (i.e. magnitude of the Fourier transform of the window function) characterized by the ratio of the main-lobe amplitude to the largest side-lobe amplitude

This ratio must be as large as possible

The Rectangular Window



The main problem associated with the rectangular window is the relatively low level (13.5dB) of the ratio of the main lobe to the largest side lobe

The Rectangular Window

- **Such a problem is due to the inherent discontinuity of the rectangular window in the time domain.**

- **One way to reduce such a discontinuity is to employ windows, which contain a taper and decays toward zero gradually, instead of abruptly, and therefore present only small discontinuities near its edges**

- **Literature lists several window functions that possess desirable magnitude responses: Bartlett (triangular), Blackman, Hamming, Hanning, Flat Top, Kaizer**

Non-Rectangular Windows

- **All of these functions have significantly lower side lobes (e.g. higher ratio of the main-lobe amplitude to the largest side-lobe amplitude) compared with the rectangular window**

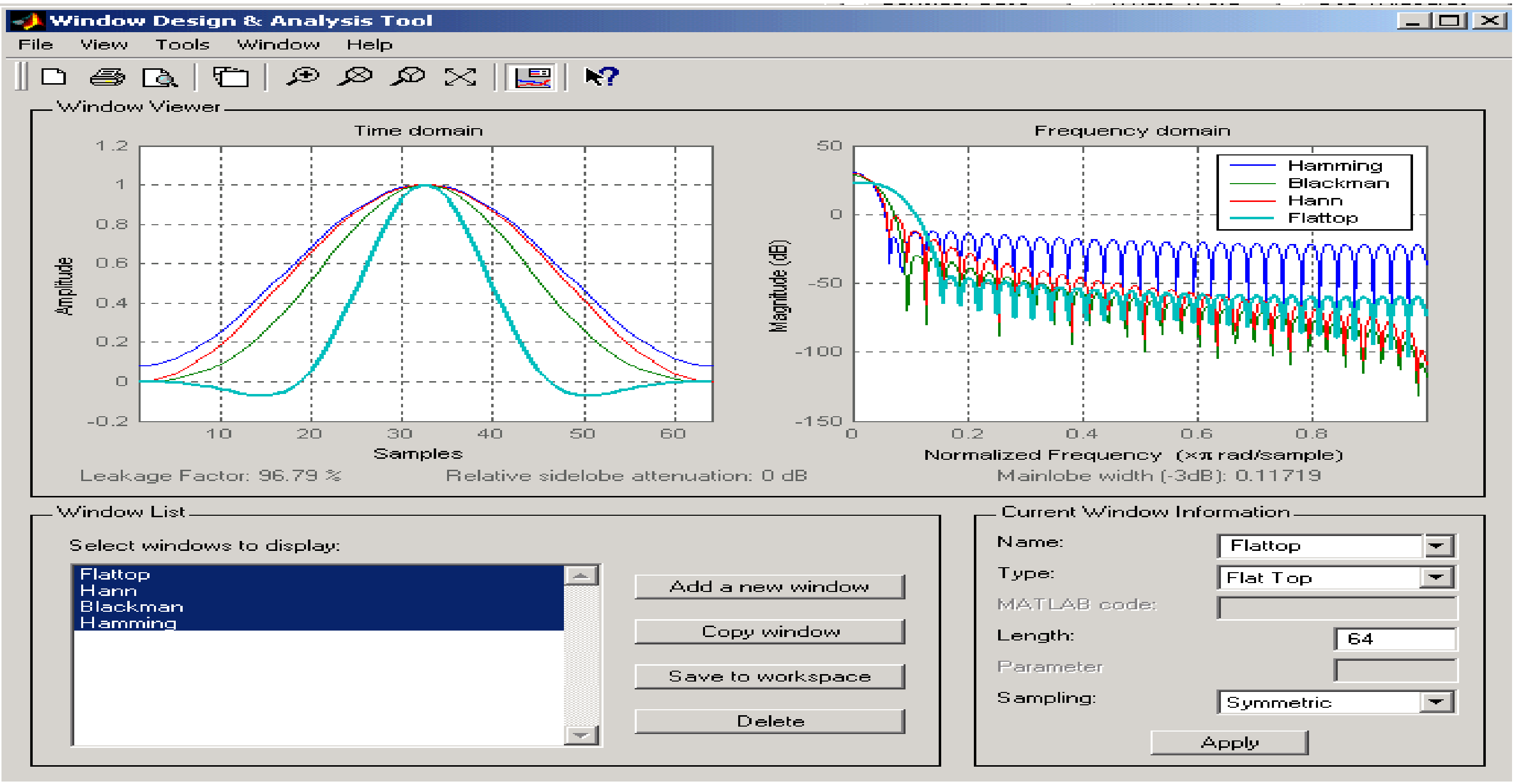
- **However, for the same window length, the width of the main lobe is also wider for these windows compared to the rectangular window, so, poor frequency resolution**

- **Consequently, these window functions provide more smoothing through convolution operation in the frequency domain**

- **To reduce the width, we can simply increase the length of the window, but we decrease time resolution**

Non-Rectangular Windows

- The Blackman, Flat Top, Hamming, Hann (Hanning), and rectangular windows are all special cases of the *generalized cosine window*



The Chebyshev Window

- **Interesting property of this window is that for a given window length it has the smallest main lobe width compared to other windows**

- **The ratio of the main lobe to the side lobe is relatively high, i. e. 40 dB**

Window Summary

- **Table summarizes important frequency features of the various window functions**

window	main lobe width	main lobe/side lobe, dB
Rectangular	$4\pi / M$	13.5
Bartlett	$8\pi / M$	27
Hanning	$8\pi / M$	32
Hamming	$8\pi / M$	43
Blackman	$12\pi / M$	58

M **is the window length (segment size)**

Window Summary

- **Hence, as M increases, the main lobe becomes narrower**
- **However, the ratio of the main lobe to the largest side lobe remains unaffected by an increase in M**
- **Thus, first we need select window to give the required ratio of the main lobe to the largest side lobe**

Window Size Selection

- **The frequency and time resolution are set mainly by the size of the window.**

- **The selection of the window size is very application dependent.**

- **If the signal under analysis has rapid transients, then short analysis window is required, so as not to smooth out the transients.**

- **The cost for this approach is that: the resulting data set will be very large and frequency resolution is poor**

Window Size Selection

- On the other hand, if it is desired to represent only the long term signal evolution, then it may be sufficient to perform an analysis using long analysis window

- There is main difficulty with spectrogram that for a particular signal a particular window may be more appropriate (gives better resolution) than another

- But what if we have a signal which consists of few signals; each signal is requiring its own window for best results

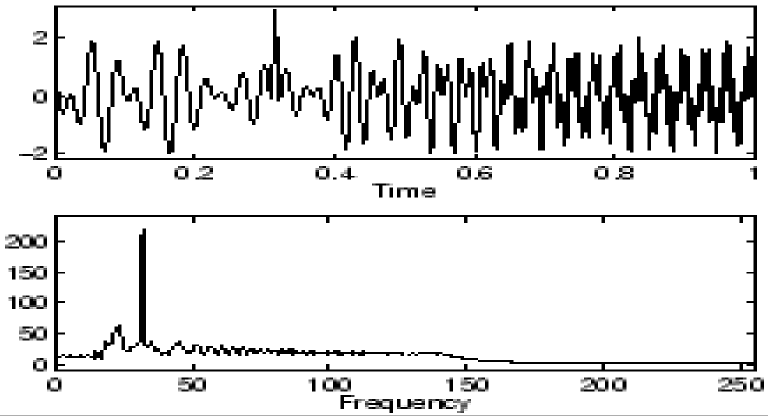
- Clearly that one window will not do the job
- Usage of multiple windows

Window Size Selection: Case Study

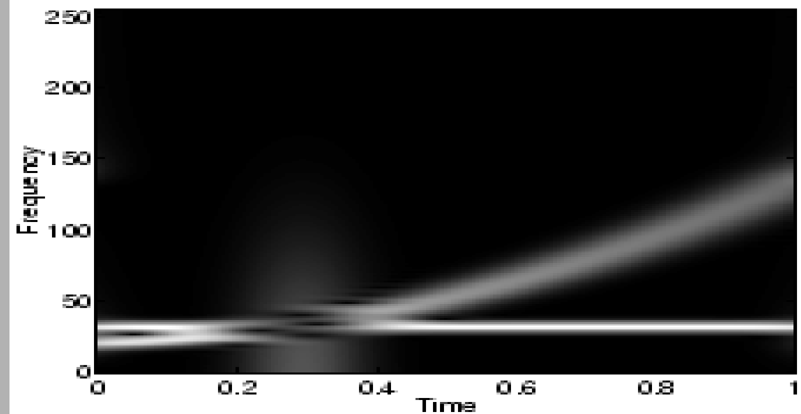
- **The signal itself consists of a constant sine wave (with 35 Hz), a quadratic chirp (starting at time 0 with 25 Hz and ending after one second at 140 Hz) and a short pulse (appearing after 0.3 s)**

- **A signal, its Fourier transform and short time Fourier transforms with windows of different sizes are shown in figures below**

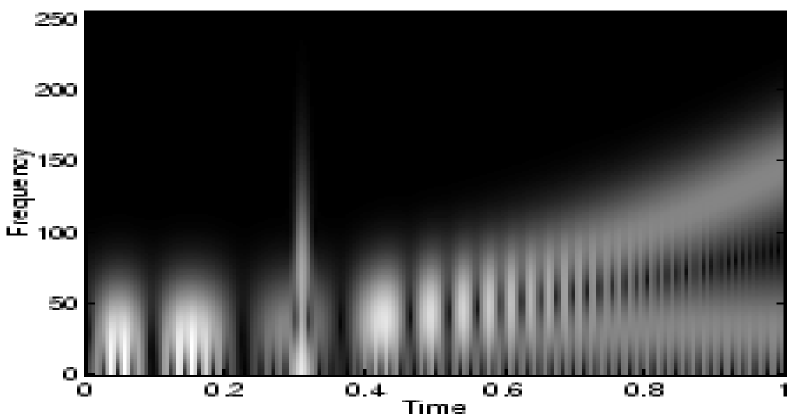
Window Size Selection: Case Study



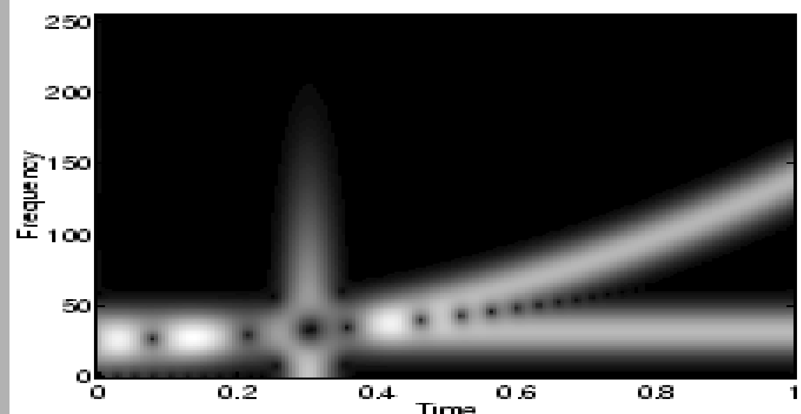
(a) Signal and its Fourier transform



(b) STFT with wide window



(c) STFT with narrow window



(d) STFT with medium window

Window Size Selection: Case Study

- ***Figure 4a.*** **Using the classical Fourier transform, only the constant frequency term can be clearly seen**
- ***Figure 4b.*** **Using a wide window leads to good frequency resolution. The constant frequency term can be clearly seen, also the quadratic chirp. However, the short pulse is hardly visible**
- ***Figure 4c*. Using a narrow window gives good time resolution, clearly localizing the short pulse at 0.3 sec., but the constant harmonic and the chirp get very un-sharp**
- ***Figure 4d.*** **A window of medium width yields a satisfactory resolution both in time and frequency, clearly localizing the three signals**

The STFT: Optimization

Optimizing the STFT usually involves:

- **finding an appropriate window size**
- **zero-padding the FFT for small segment sizes to better render spectral maximums**
- **choosing an appropriate window shape**
- **choosing an appropriate overlapping of windows**

Signal Reconstruction

- **The short-time Fourier transform can be considered as the Fourier transform of the product** $x(\tau)h(\tau - t)$

- **Hence, given short-time Fourier transform and window function, we can recover the original signal simply by performing the inverse Fourier transform (the 1/N normalization is not considered) , i. e.**

$$x(\tau)h(\tau - t) = \int_{-\infty}^{\infty} X_h(t, f)e^{i2\pi f\tau}df$$

where the subscript of X **emphasizes that short-time transform is computed by a definite window function**

The Digital Short-Time Fourier Transform

- **It is necessary to extend the short-time Fourier transform to discrete-time signals**

- **For practical implementation, each Fourier transform in the short-time Fourier transform has to be replaced by the discrete Fourier transform; the resulting short-time Fourier transform is discrete in both time and frequency and thus is suitable for digital implementation**