Complex Numbers

Complex Numbers

- There are many applications of complex numbers in science and engineering, in particular in electrical alternating current theory and in mechanical vector analysis.
- There are two main forms of complex number – Cartesian (or rectangular) form and polar form.

Complex Numbers

Consider the quadratic equation $x^2 - 4x + 13 = 0$.

If we try to solve it using the quadratic equation $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ we obtain $x = 2 \pm 3\sqrt{-1}$.

It is not possible to find $\sqrt{-1}$ in real terms but if an operator j is defined as $j = \sqrt{-1}$ then the solution may be expressed as $x = 2 \pm j3$.

Complex Numbers

- $2 + j3$ and $2 - j3$ are called **complex numbers**.
- Both solutions are of the form $a + jb$, ‘a’ is called the **real** part and ‘b’ is called the **imaginary** part.
- Complex numbers in the form $a + jb$ are called a **Cartesian complex number.**

Complex Numbers

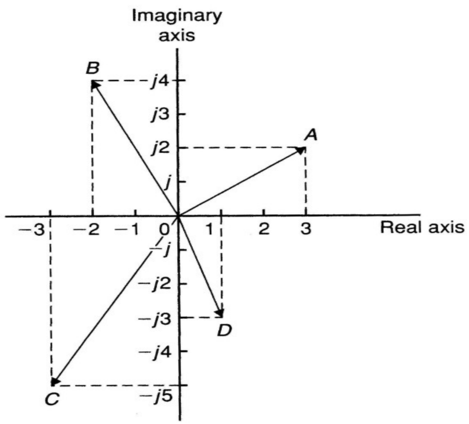
- In pure mathematics the symbol i is used (i being the first letter of the word imaginary) but in engineering the symbol i is the symbol for electric current so the letter j (the next letter in the alphabet) is used to avoid possible confusion.

Complex Numbers

Examples

1. Solve the quadratic equation $x^2 + 9 = 0.$
2. Solve the quadratic equation $2x^2 + 3x + 5 = 0.$
3. Evaluate j^2, j^3, j^4 and $j^5.$

The Argand Diagram



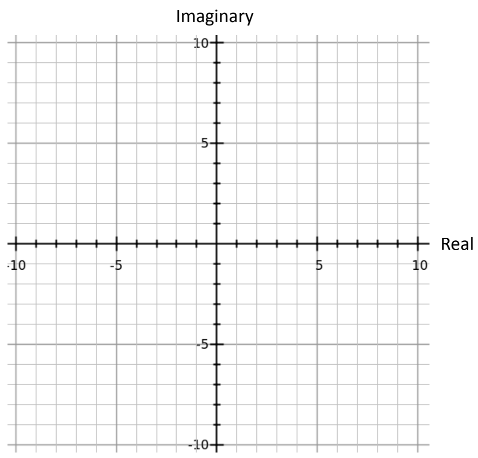
Addition and subtraction of complex numbers in Cartesian form

Combine the real terms and the imaginary terms

Example 4

If $z_1 = 3 + j2$
and $z_2 = 5 - j6$
find
(a) $z_1 + z_2$ and
(b) $z_1 - z_2$

Represent these calculations on an Argand diagram.



Multiplication of complex numbers in Cartesian form

Expand the brackets remembering that $j^2 = -1$

Example 5

Find the product $(3 + j2)(4 + j)$.

The complex conjugate

The complex conjugate of a complex number is obtained by changing the sign of the imaginary part. So the complex conjugate of $a + jb$ is $a - jb$.

The product of a complex number with its complex conjugate is always a real number.

Example 6

Show that the product of $3 - j4$ and its conjugate is a real number.

Division of complex numbers in Cartesian form

This is done by multiplying both the numerator and denominator by the complex conjugate of the denominator.

Example 7

Evaluate $\frac{4+j5}{1-j}$

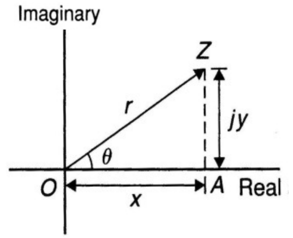
Practice Questions

If $z_1 = 1 - j3$,
$z_2 = -2 + j5$ and
$z_3 = -3 - j4$ find
the following in
$a + jb$ form:

(a) $z_1 z_2$
(b) $z_1 \div z_3$
(c) $\frac{z_1 z_2}{z_1 + z_2}$

The polar form of a complex number

$$\begin{aligned} z &= x + jy \\ &= r\cos\theta + jr\sin\theta \\ &= r(\cos\theta + j\sin\theta) \\ &= r\angle\theta \end{aligned}$$



r is called the modulus or magnitude of z

θ is called the argument or amplitude of z

$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\frac{y}{x}$

Changing from Cartesian form to polar form and vice versa

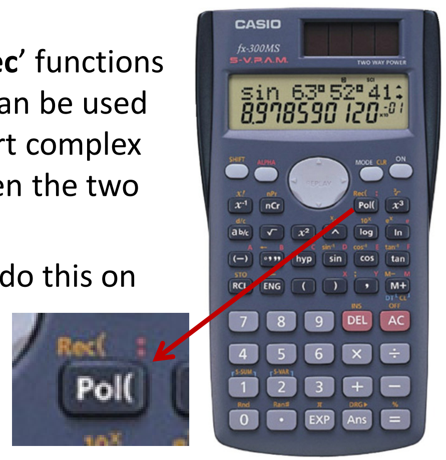
Examples

8. Convert $z = 2 + j3$ into polar form.
9. Convert $z = -3 - j4$ into polar form.
10. Convert $z = 5\angle 150°$ into Cartesian form.
11. Convert $z = 3\angle\frac{\pi}{2}$ into Cartesian form.

Using a calculator

The ‘**Pol**’ and ‘**Rec**’ functions on a calculator can be used to quickly convert complex numbers between the two forms.

Find out how to do this on your calculator.



Multiplication and division of complex numbers in polar form

If $z_1 = r_1\angle\theta_1$ and $z_2 = r_2\angle\theta_2$ then

$$z_1 z_2 = r_1 r_2\angle(\theta_1 + \theta_2)$$

Multiply the moduli and add the arguments

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

Add the moduli and subtract the arguments

Note: It is not possible to add or subtract complex numbers in polar form, they must first be converted into Cartesian form.

Examples

Examples

12. Find
    $6\angle 17° \times 3\angle 35°$
13. Find
    $16\angle 74° \div 2\angle 23°$

The square root of a complex number

If $z = r\angle\theta$ then $\sqrt{z} = \sqrt{r}\angle\frac{\theta}{2}$

Root the moduli and half the arguments

To find the square root of a complex number in Cartesian form it must first be converted to polar form.

Example 14

Find $\sqrt{3 + j2}$
in Cartesian form.