The Wigner Distribution

Part 2

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The Wigner-Ville Distribution

- **Ville used the Wigner distribution in 1948 when he replaced the continuous complex signal with the analytical signal**

- **Here the analytical signal is defined as**

$$x_a(t) = x(t) + i\hat{x}(t)$$

where $\hat{x}(t)$ **is the Hilbert transform of** $x(t)$

- **In this case the Wigner distribution is termed the *Wigner-Ville distribution***

The Wigner-Ville Distribution

By using the Wigner-Ville distribution

- **the negative transform values which have no physical significance for a real signal are eliminated (it is well known that the spectrum of the analytic signal does not have any negative frequencies)**
- **consequently, the cross terms between the negative and positive parts are also eliminated**

- **Therefore, in order to reduce the cross-terms, we can use the Wigner distribution of analytical signal**

- **Thus, the Wigner-Ville distribution is very often used in practical application**

Smoothing in the Time Domain

- **In practical applications, the Wigner distribution requires some smoothing in order *to suppress the cross-terms between positive frequencies***

- **There are two methods to suppress the cross-terms**

- **The first method (*smoothing in the time domain*) is the application of a sliding window *in the time domain* before calculating the Wigner distribution**

- **The distribution obtained with a *time* window is often called the *pseudo Wigner distribution:***

$$W_p(t,f) = \int h(\tau)x^*(t-\frac{\tau}{2})x(t+\frac{\tau}{2})e^{-i2\pi f\tau}d\tau$$

The Pseudo Wigner-Distribution

- **The pseudo Wigner distribution is a *short time* version of the Wigner distribution using a time window**

- **This distribution corresponds only approximately to the Wigner distribution of the original signal**

- **The pseudo Wigner distribution gives a good time resolution at the expense of sacrificing the frequency resolution**

- **Unfortunately, this distribution destroys many of desirable properties of the Wigner distribution**

- **For example, the instantaneous frequency property no longer holds**

Pseudo Wigner-Distribution: Case Study

- **The sum of two sinusoids**
- **Take as an example a Gaussian window** $h(\tau) = e^{-at^2/2}$
- **The pseudo Wigner distribution can be calculated analytically**

$$W_p = \sqrt{\frac{2\pi}{a}}\left(A_1^{\,2} e^{-\pi(f-f_1)/a} + A_2^{\,2} e^{-\pi(f-f_2)/a}\right) + 2A_1A_2\sqrt{\frac{2\pi}{a}}\cos 2\pi(f_2 - f_1)t\, e^{-\pi(f-\frac{f_1+f_2}{2})^2/a}$$

- **If we choose a small** a **the cross-terms can be made small**
- **However, note that the auto terms get spread out**

Smoothing in Time and Frequency

- **A smoothing procedure can be defined as:**

$$W_s(t, f) = \iint W(t', f')S(t - t', f - f')dt'df'$$

where $S(t, f)$ **is a two-dimensional smoothing function**

- **An example of a smoothing function is a Gaussian function:**

$$S(t, f) = \frac{1}{2\pi\sigma_t\sigma_f} e^{-(\frac{t^2}{2\sigma_t^2} + \frac{(2\pi f)^2}{2\sigma_f^2})}$$

where σ_f **and** σ_t **are standard deviations in the frequency and time domains respectively**

Smoothing in Time and Frequency

- **It was shown that positive distribution values only (e.g. the smoothed Wigner distribution is positive) can always be assured by requiring that σ_f and σ_t both be greater than zero and that $\sigma_t^2 \sigma_f^2 \geq 0.25$**

  Otherwise, the smoothed Wigner distribution may contain negative values

- **To smooth the sampled Wigner distribution, the Gaussian window should be applied in the range $\pm 2\sigma_t$ and $\pm 2\sigma_f$**

- **A smoothing can substantially suppress the cross-terms However, smoothing will *reduce the resolution in the time and frequency domains.* A trade-off exists between the degree of smoothing and the resolution**

The Discrete Wigner-Ville Distribution

- **To calculate the Wigner distribution of a finite-length, sampled signal, a discrete approximation of Wigner distribution is required**

- **By letting** $u = \frac{\tau}{2}$ **and assuming that** Δt **is the sample interval, we have an approximation of the integral by**

$$W(t, f) = 2\Delta t \sum_{n} x(t + n\Delta t) x^{*}(t - n\Delta t) \exp(-i4\pi f n\Delta t)$$

- **In this form it is obvious that Wigner distribution is related to the discrete Fourier transform of the Wigner kernel and so may be evaluated using FFT algorithms**

The Discrete Wigner Distribution: Choice of Sampling Frequency

- **It is possible to prove that**

$$W(f + \frac{1}{2\Delta t}) = W(f)$$

- **Therefore, the period of Wigner distribution is twice less than the period of the Fourier transfrm**

- **As a consequence the sampling frequency of the signal should be *twice* as high as that used for the Fourier transform**

This is needed to avoid any aliasing in the distribution

The Discrete Wigner-Ville Distribution: Choice of Sampling Frequency

- **This sampling rate restriction can be eliminated if Wigner-Ville distribution (e.g. Wigner distribution of analytical signal) is used**

- **Thus the Wigner-Ville distribution requires the normal sampling frequency based on Nyquist theorem, e.g. the signal should be sampled at a frequency at least twice as high as the highest frequency content in the signal**

The Wigner Distribution: Frequency Resolution

- **The frequency resolution of Wigner distribution is different from that obtained by FFT of the original N point time history**

- **The frequency resolution of the Wigner transform is *one fourth* the resolution of the ordinary FFT**

The Wigner Distribution vs. The Fourier Transform

- **The Wigner distribution in some respects is better than any windowed Fourier transform**

- **One of the advantages of the Wigner distribution over the Fourier transform is that we do not have to bother with choosing the window**

- **The Wigner distribution gives a clear picture of the instantaneous frequency and time delay**

- **It is never true for the windowed Fourier transform**

The Wigner Distribution vs. The Fourier Transform

- **One of the advantages of the windowed Fourier transform is that it is positive; the results obtained from it can be interpreted, although they may be poor**

- **The Wigner distribution is never manifestly positive, which sometimes leads to results that cannot be interpreted**

The Wigner Distribution vs. The Fourier Transform

- Finally, the Wigner distribution and the Fourier transform allow us to ascertain whether a signal is multi-component

- But the Wigner distribution suffers from the fact that for multi-component signals we get cross-terms. On the other hand, the windowed Fourier transform often cannot resolve the components effectively

- The reason that makes the Wigner distribution so special is the Wigner distribution better characterizes a signal's frequency changes than any other schemes, such as the short time Fourier transform or wavelet transform

Wigner Distribution