The Wigner Distribution

Part 1

Prof. L. Gelman

The Wigner Distribution

- **As discussed in STFT chapters, the results obtained from short time Fourier transform, are subject to the selection of windowing functions**

- **To overcome those problems, a more general distribution is introducing here**

- **The Wigner distribution is a tool for time-frequency analysis, which has gained more and more importance due to essential characteristics**

The Wigner Distribution

- **The Wigner distribution can be derived by generalizing the relationship between the power spectral density and the autocorrelation function for non-stationary processes**

- **The physical interpretation of the Wigner distribution is that it represents the *instantaneous* power spectral density**

The Wigner Distribution

- **This generalized approach leads to the auto Wigner distribution defined as**

$$W(t, f) = \int_{-\infty}^{\infty} x^*(t - \frac{\tau}{2})\, x(t + \frac{\tau}{2})\, e^{-i 2\pi f \tau}\, d\tau$$

where $x(t)$ is the complex signal (time history) the Wigner distribution of which is to be calculated, f is frequency, τ is time delay, the superscript asterisk denotes complex conjugation

The Cross Wigner Distribution

- **The cross Wigner distribution is defined as**

$$W_{x,y}(t,f)=\int_{-\infty}^{\infty} x^*(t-\frac{\tau}{2})y\ (t+\frac{\tau}{2})e^{-i2\pi f\tau}d\tau$$

- **This representation may be interpreted as the Fourier transform of the instantaneous cross correlation function of a signal**

- **The Wigner distribution is *non-linear (bilinear)* because the signal enters twice in its calculation**

Properties

- **The distribution is a real-valued function, even if the signal is complex**

- **The distribution has a good resolution in both time and frequency**

- **If signal is non-zero in only a certain time (frequency) interval, then the Wigner distribution is also restricted to this time (frequency) interval**

- **The first moment of the distribution with respect to frequency locates the instantaneous frequency of the signal**

Properties

- **The integral of the distribution over frequency at a particular time yields the instantaneous power of the signal at that time**

- **The integral of the distribution over time at particular frequency yields the power spectral density at that frequency**

- **Integrating over time and frequency yields the signal energy**

- **A time shift in the signal has the same shift in the distribution**

- **Frequency modulation: Wigner distribution of the frequency-modulated signal is a frequency-shifted Wigner distribution of the un-modulated signal**

Properties

A simultaneous time shift and modulation lead to a time and frequency shift of the distribution

Case Studies: Sinusoid and Impulse

- **Sinusoid:** $x(t) = e^{i2\pi f_0 t}$

the Wigner distribution is $W(t, f) = \delta(f - f_0)$

- **For the sinusoid the Wigner distribution is totally concentrated along the frequency of the sinusoid**

- **Impulse:** $x(t) = \sqrt{2\pi}\,\delta(t - t_0)$

the Wigner distribution is $W(t, f) = \delta(t - t_0)$

- **For the impulse, the Wigner distribution is totally concentrated at the time of occurrence**

Case Study: Chirp

- ***Linear frequency sweep***

- **This example is presented in order to demonstrate that the Wigner distribution can accurately reveal the linear variation of the instantaneous frequency of a time-varying signal**

- **The analytical version of pure chirp is** $x(t) = e^{i\Phi(t)}$

where

$$\Phi(t) = 2\pi(\beta_1 \frac{t^2}{2} + f_0 t)$$

is the instantaneous phase, β_1 **is the rate of change of frequency and** f_0 **is the initial frequency**

Case Study: Chirp

- **The instantaneous frequency of a signal is defined as**

$$f_i = \frac{1}{2\pi}\frac{d\Phi}{dt} = \beta_1 t + f_0$$

- **The Wigner distribution of a chirp is**

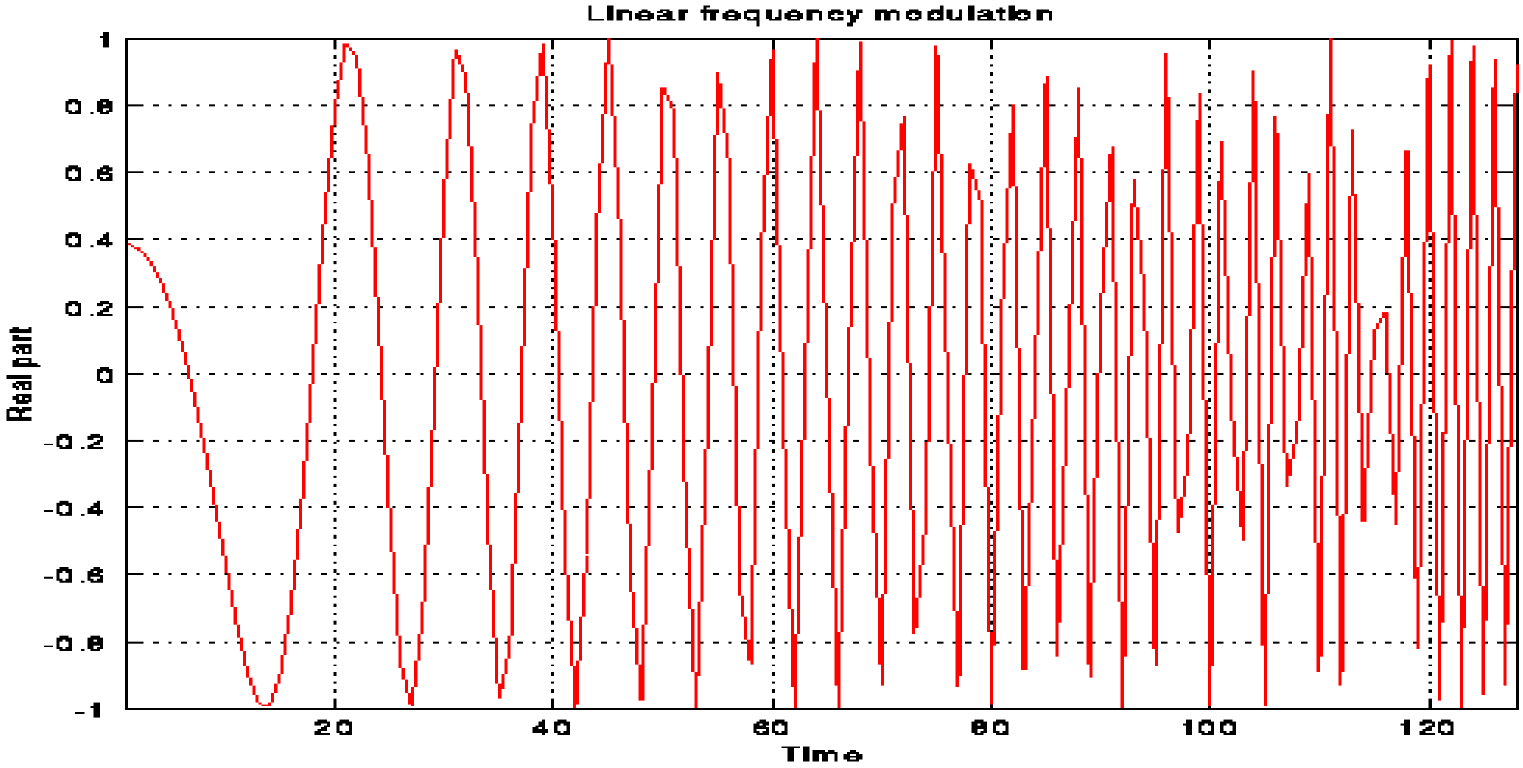
$$W(t, f) = \delta(f - f_i)$$

- **The Wigner distribution is *totally concentrated* along the instantaneous frequency**

- **In this case, neither the time history nor the Fourier transform give an accurate estimation of the variation of the signal frequency and amplitude with time**

Case Study: Chirp

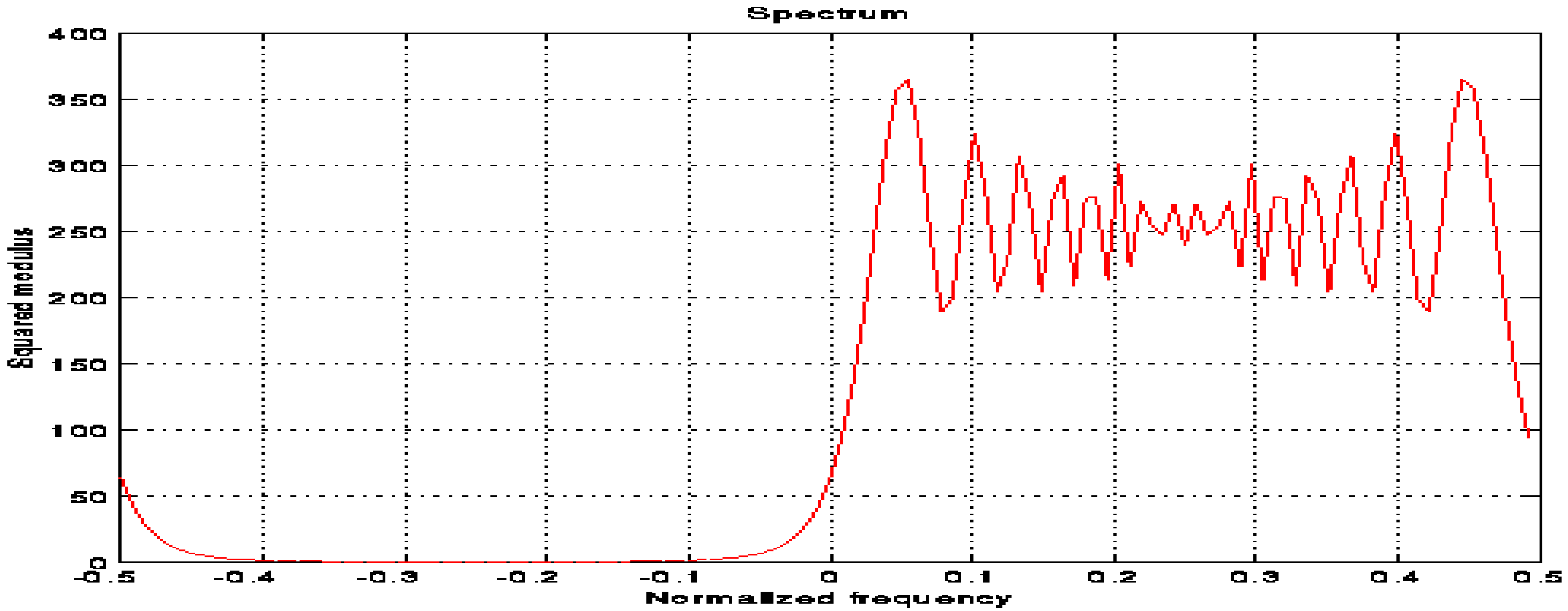
- **Let's consider an analytic chirp, whose normalized frequency is changing from 0 to 0.5**



- **From this time-domain representation, it is difficult to say what kind of modulation is contained in this signal**

Case Study: Chirp

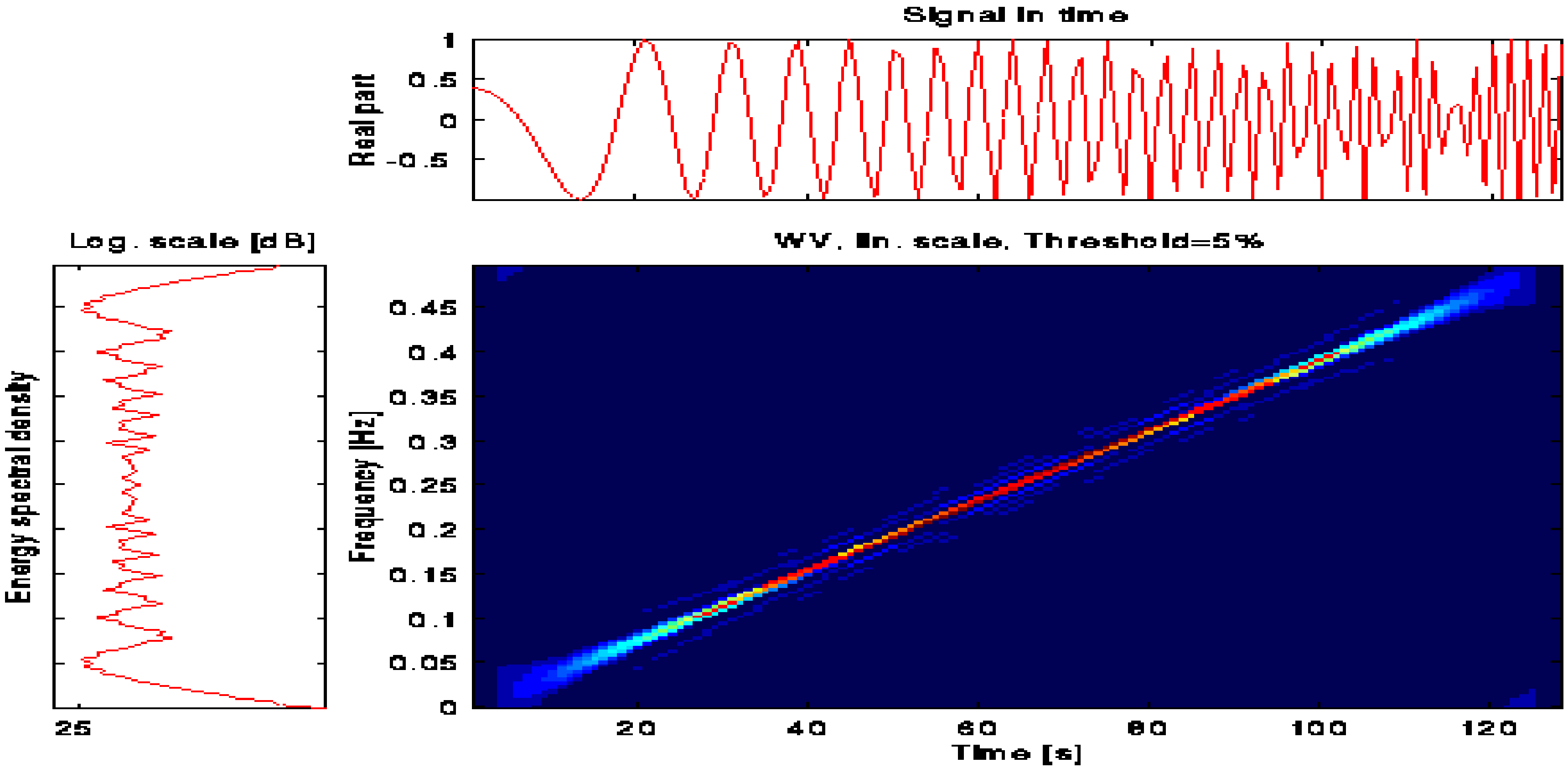
- **Let's consider the power spectral density of the chirp**



- **From this plot, we can not say anything about the evolution in time of the instantaneous frequency**

Case Study: Chirp

- **In order to have a more informative description of such a signal, let's employ the Wigner distribution of this signal**



- **We can see that the linear change of the frequency with time, from 0 to 0.5, is clearly shown**

Wigner Distribution: Disadvantages

Despite the desirable properties of the distribution, it has two major disadvantages:

- **it is not necessarily non-negative**
- **it is a bilinear function producing interferences (cross-terms) for multi-component signals**

Wigner Distribution: Cross-Terms

- The Wigner distribution of the simple multi-component signal, the sum of two signals $x_1(t) + x_2(t)$, is

$$W_{x_1+x_2} = W_{x_1} + W_{x_2} + W_{x_1 x_2} + W_{x_2 x_1}$$

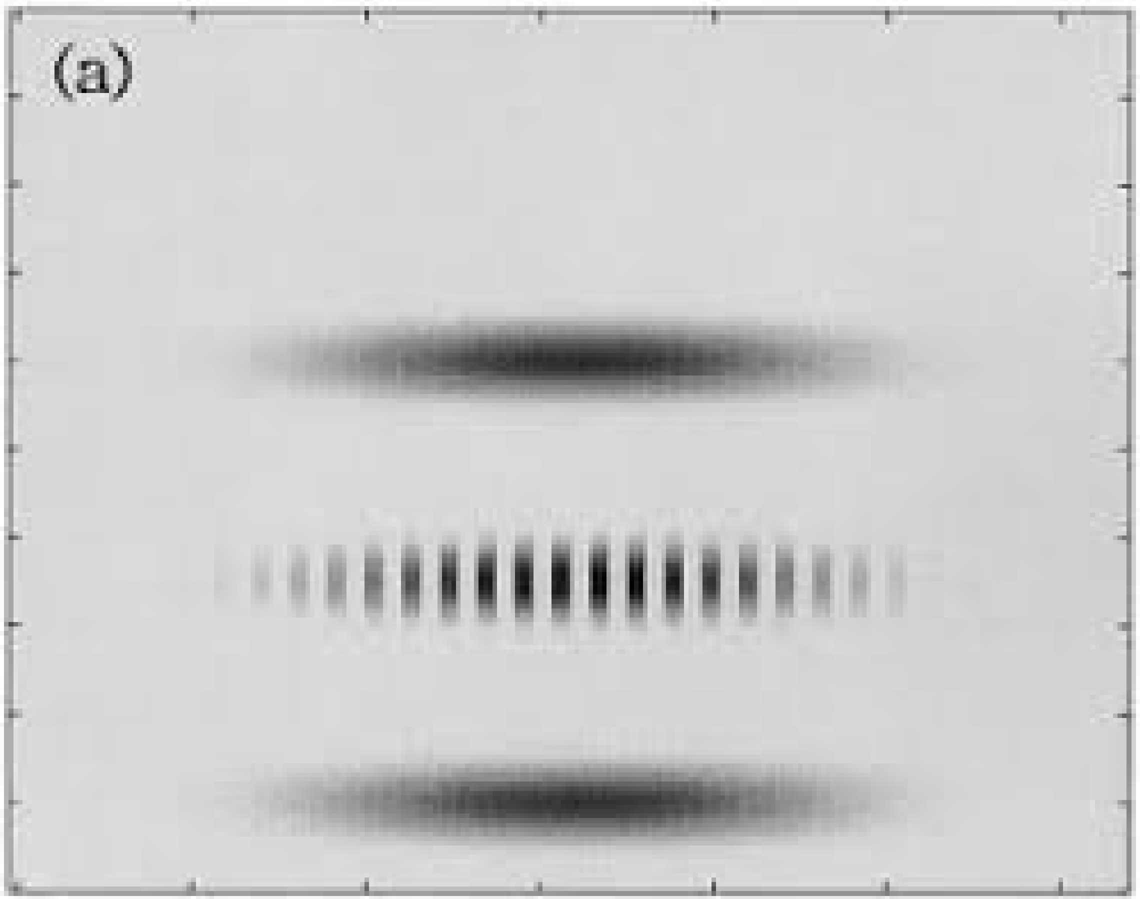
where W_{x_i} is the Wigner distribution of the signal x_i $W_{x_1 x_2}$ and $W_{x_2 x_1}$ are the cross Wigner distributions (i.e. the cross-terms of the signals)

The Wigner Distribution: Cross-Terms

Thus, the Wigner distribution of the sum of two signals does not equal the sum of their respective Wigner distributions, e.g. it has the cross-terms in addition to the two auto terms:

$$W_{x_1+x_2} = W_{x_1} + W_{x_2} + W_{x_1 x_2} + W_{x_2 x_1}$$

Cross-Terms



Cross-Terms: Case Study

- **Multi-component signal: sum of two sinusoids:**

$$x(t) = A_1 e^{i2\pi f_1 t} + A_2 e^{i2\pi f_2 t}$$

- **The Wigner distribution of this signal is**

$$W(t,f) = A_1^2 \delta(f - f_1) + A_2^2 \delta(f - f_2) + 2A_1 A_2 \delta(f - \frac{1}{2}(f_1 + f_2)) \cos 2\pi (f_2 - f_1) t$$

- **This is an illustration of the cross-terms. Besides the concentration of the distribution at f_1 and f_2, we also have nonzero *oscillating* values at the frequency**

$$\frac{1}{2}(f_1 + f_2)$$

Cross-Terms

- **Cross-terms lie between signal components and are time-dependent and oscillatory**

- **They can have a peak value as high as the auto terms and make the interpretation of the distribution very difficult**

- **Each pair of auto terms creates one cross-term**

- **For real signals, the cross-terms, which usually overlap with auto terms, could be more complicated, and desired time-frequency distribution could be confusing**

Cross-Terms

- **The cross-term in fact reflects the *non-zero cross-correlation* of the corresponding signals**

- **Its location and the rate of oscillations are determined by centers (in time-frequency plane) of the auto terms**

- **It is known that the farther apart the auto terms are, the less energy the cross-term contains**

Cross-Terms of Gaussian Signals

- **When cross-term is created by two Gaussian signals whose time-frequency centers are *far apart*, the cross-term highly oscillates, which has limited influence on Wigner distribution. In this case the cross-term is indeed not important and could be removed**

- **When the cross-term is created by two Gaussian signals whose time-frequency centers *are close*, the cross-term will oscillate less and, therefore, has a larger average**

- **If such a cross-term is discarded, the resulting presentation will leave significant signal energy out. In this case, we can expect to lose much of the Wigner distribution properties.**
- **Thus, discarding all cross-terms is not always a right approach**

Cross-Terms

However, in case of multi-component signals, they are interferences.

Signal Reconstruction

- **By an inverse Fourier transform of Wigner distribution with respect to frequency we obtain the Wigner kernel (the instantaneous autocorrelation function)**

$$\phi(t,\tau) = x^*(t - \frac{\tau}{2})\,x(t + \frac{\tau}{2})$$

- **Along the time** $t = \dfrac{\tau}{2}$ **we get the estimate of reconstructed signal:**

$$\hat{x}(\tau) = \phi(\frac{\tau}{2}, \tau) = x^*(0)x(\tau)$$

- **This means that any signal can be perfectly reconstructed from its Wigner distribution except for the pre-factor** $x^*(0)$