The Short-Time Fourier Transform

Part 1

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The Short-Time Fourier Transform

- **The short-time Fourier transform (or windowed Fourier transform) is the widely used method for studying *transient* signals**

- **The concept behind it is simple and powerful**

- **Suppose we listen to a piece of music that lasts an hour where in the beginning there are violins and at the end drums**

- **If we Fourier analyze the whole hour, the Fourier analysis will show peaks at the frequencies corresponding to the violins and drums.**

- **That will tell us that there were violins and drums, but will not give us any indication of *when* the violins and drums were played**

The Short-Time Fourier Transform

- **The most straightforward thing to do is to break up the hour into segments (for example, the duration of each segment is five minutes) and Fourier analyze each segment**

- **Upon examining the Fourier transform of each segment we will see in which five minutes intervals the violins and drums occurred**

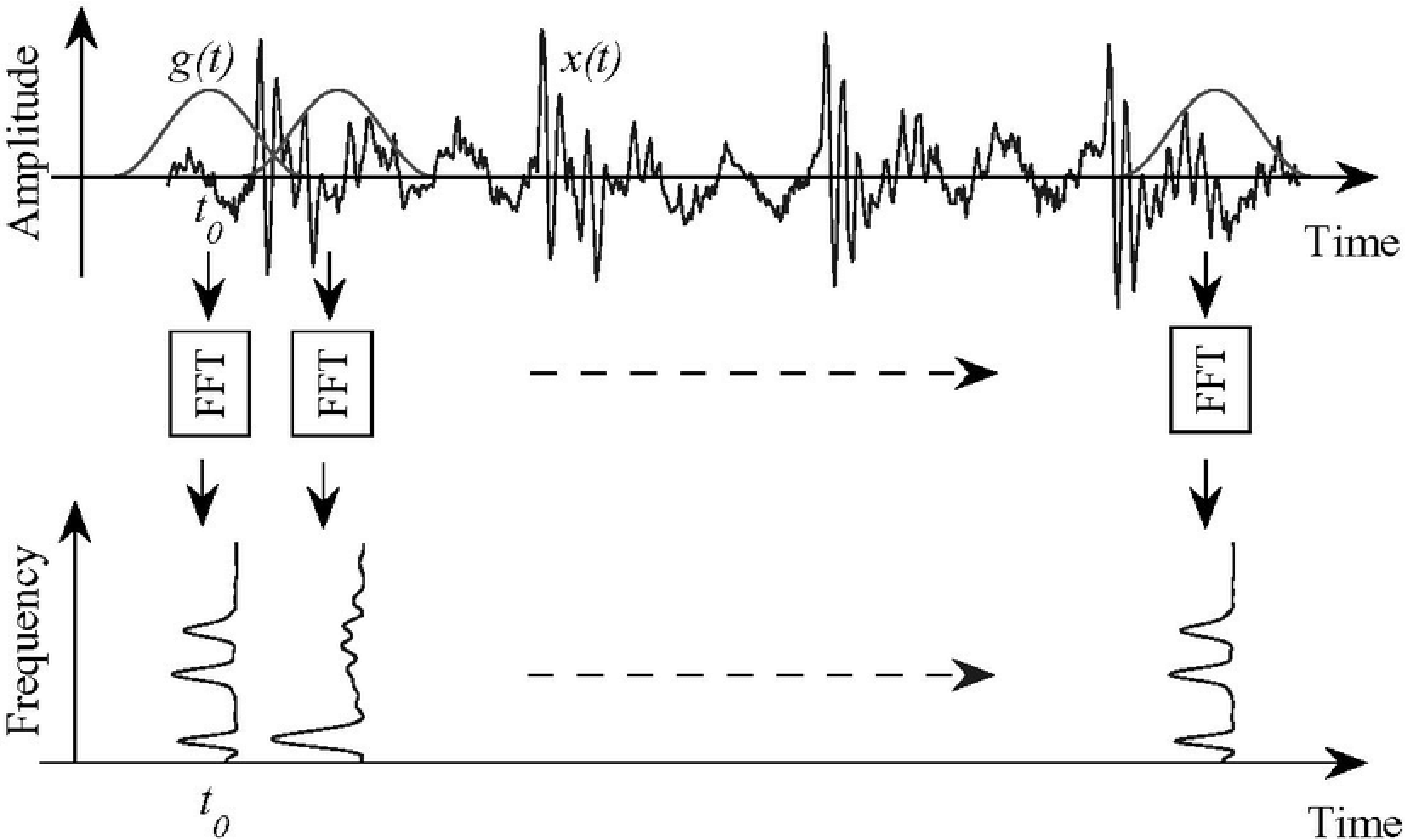
- **If we want to localize even better, we break up the hour into one minute segments or even smaller time intervals and Fourier analyze each segment**

The Short-Time Fourier Transform

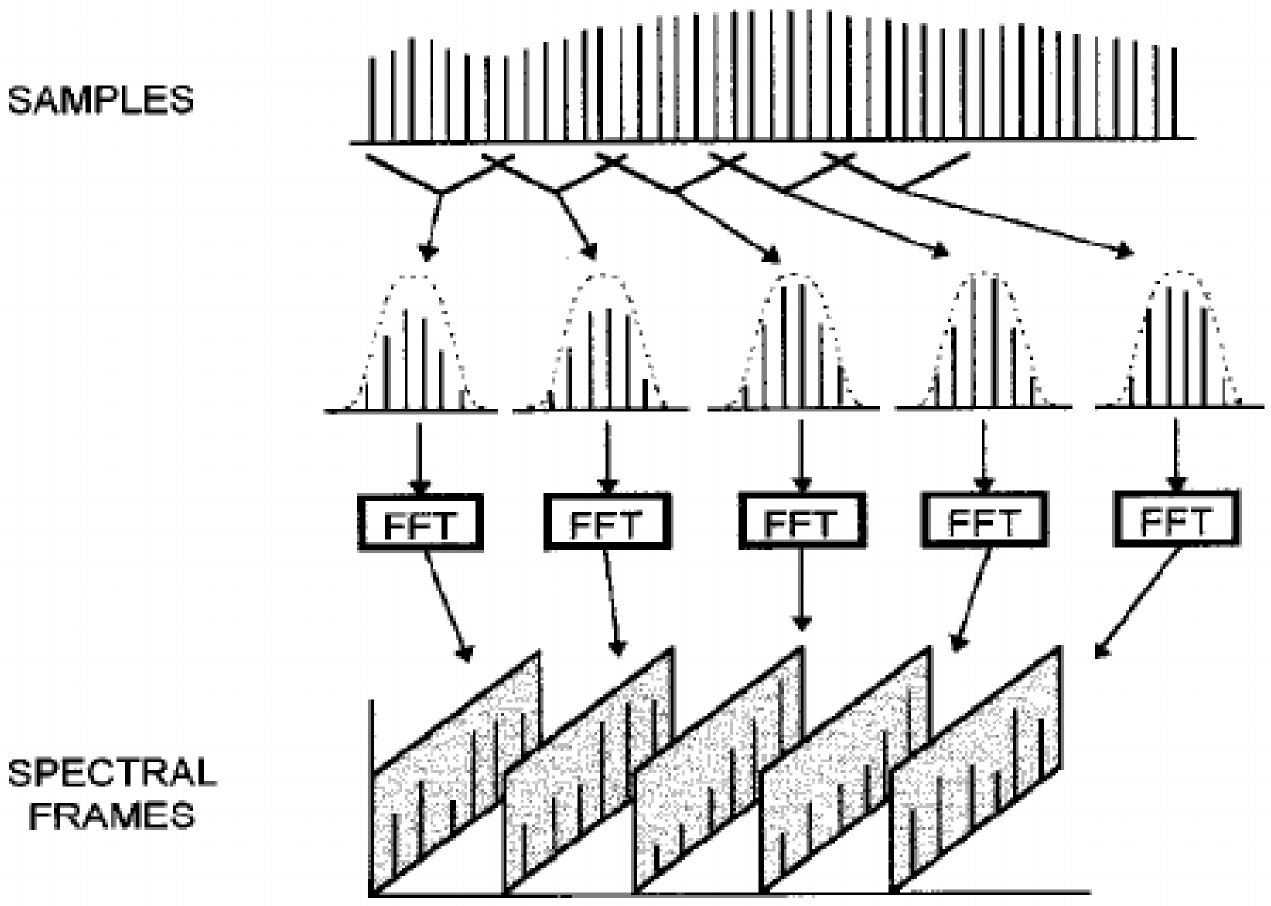
- **That is the basic idea of the short-time Fourier transform: break up the signal into small time segments by using a window and Fourier analyze each time segment to ascertain the signals that existed in that segment**

- **The *totality of transforms* indicates how the Fourier transform is varying in time**

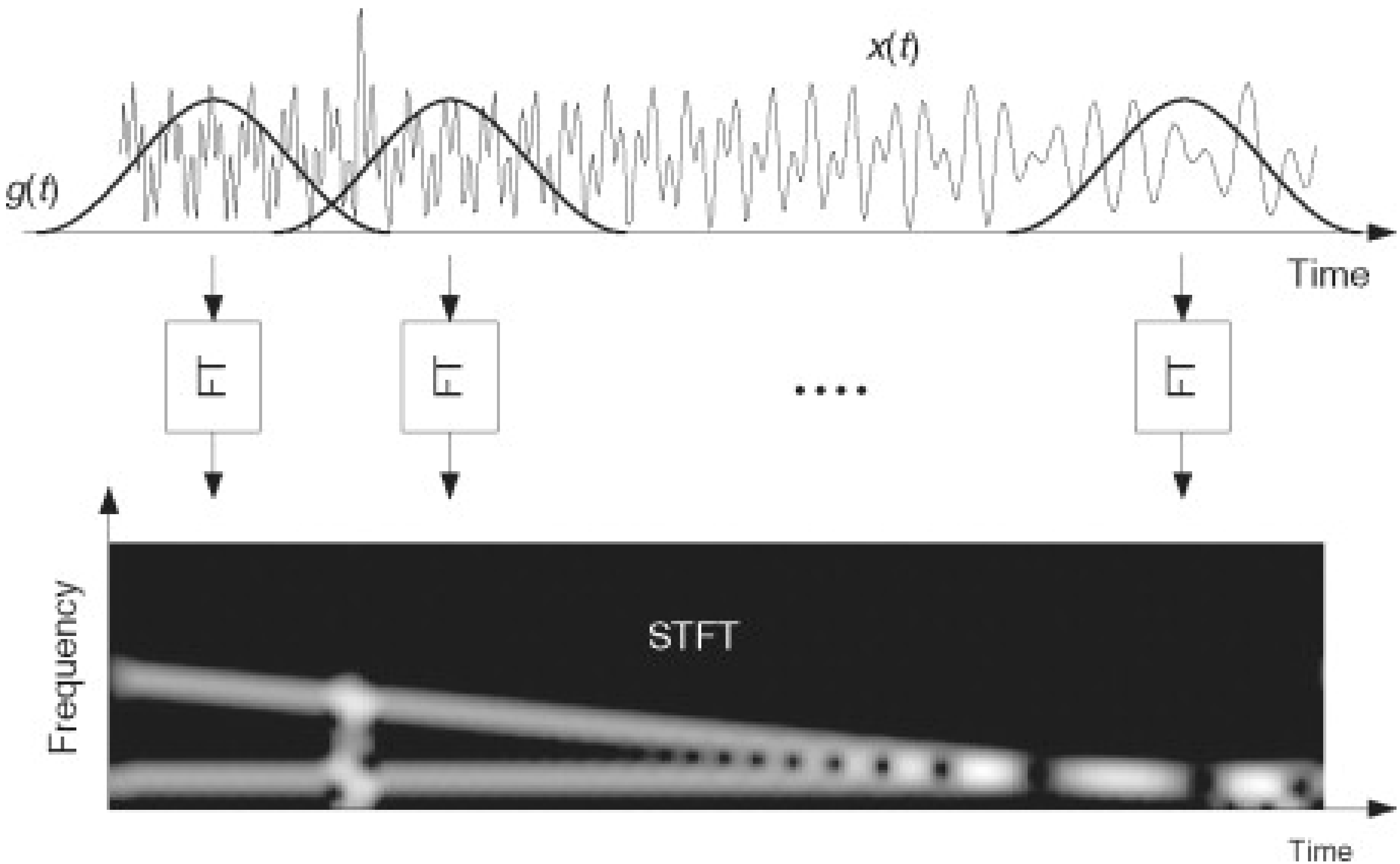
The Short-Time Fourier Transform



The Short-Time Fourier Transform



The Short-Time Fourier Transform



The Short-Time Fourier Transform

- **Can this process be continued to achieve finer and finer time localization? Can we make the intervals as short as we want?**

- **The answer is "no", because after a certain narrowing the answers we get for the Fourier transform become meaningless and show no relation to the Fourier transform of the signal**

The Short-Time Fourier Transform

- The reason is that short duration signals have inherently large bandwidths, and the Fourier transform of such short duration signals have very little to do with the properties of the original signal

- It is the well-known uncertainty principle as applied to the small time intervals that we have created for the purpose of analysis

- We should always keep in mind that in the short-time Fourier transform the properties of the signal are scrambled with the properties of the window function; unscrambling is required for proper interpretation and estimation of the original signal

The Short-Time Fourier Transform

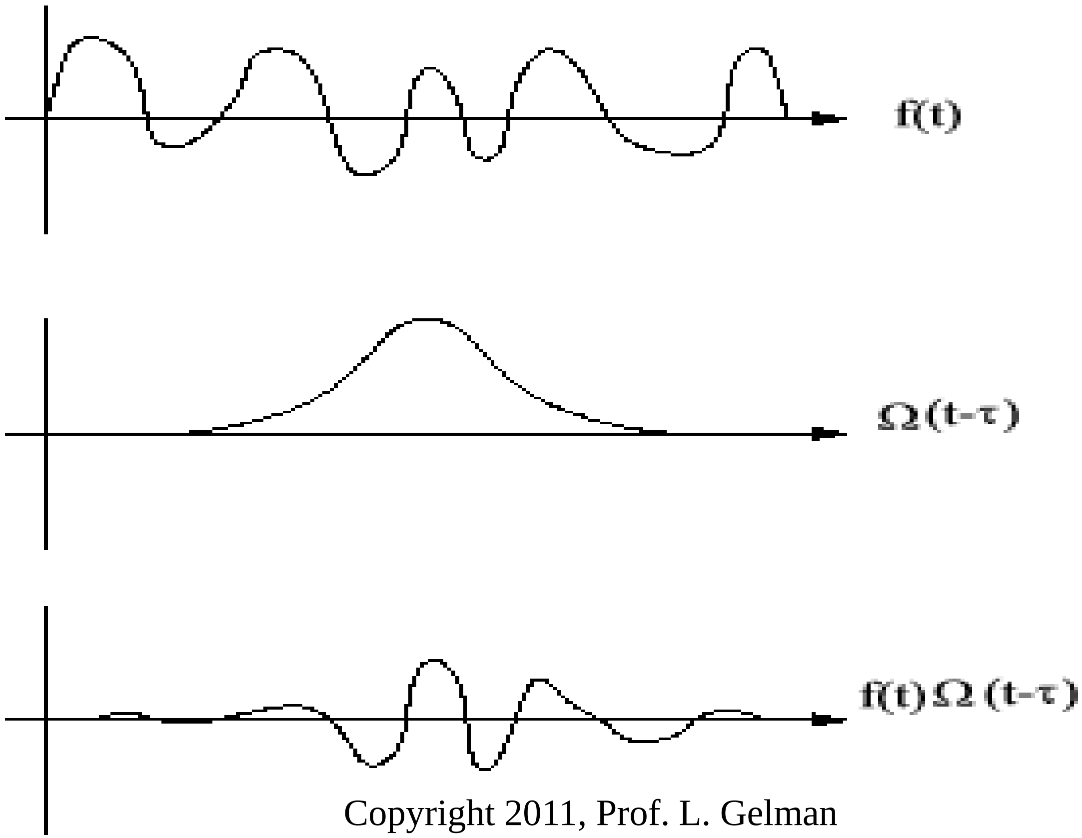
- **The above difficulty notwithstanding; the short-time Fourier transform is very good in many respects**

- **It is well defined, based on reasonable physical principles, and for many signals it gives an excellent time-frequency structure consistent with our intuition**

- **However, for certain situations it may not be the best method in the sense that it does not always give us the clearest possible picture of frequency content**

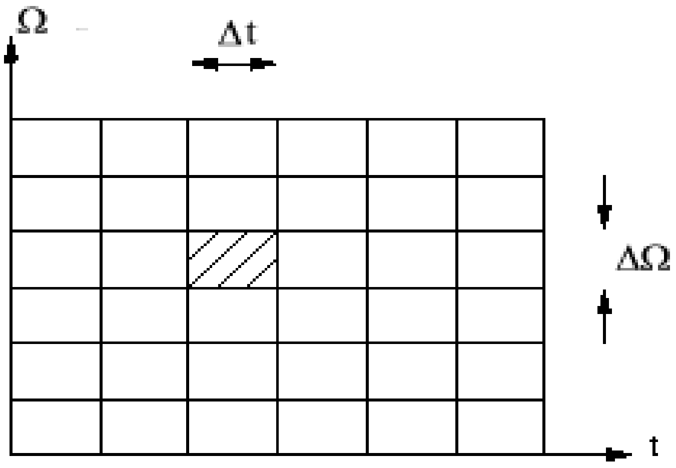
The Short-Time Fourier Transform

The short-time Fourier transform can be interpreted as a "sliding window Fourier transform": to slide the time center of window in time, window the input signal, and compute the Fourier transform of the result



The Short-Time Fourier Transform

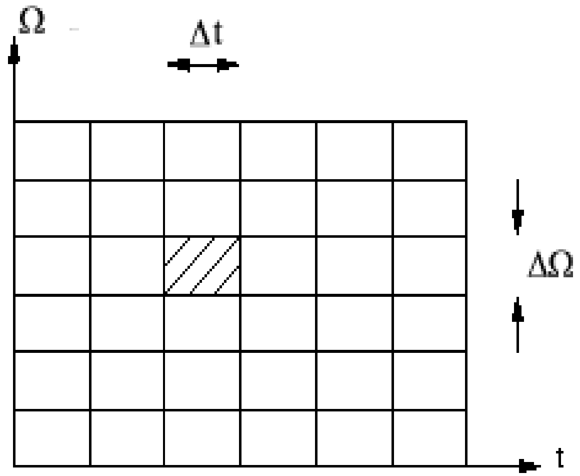
- **The idea is to isolate the signal in the vicinity of time, then perform the Fourier transform in order to estimate the "local" frequency content at time.**
- **This can be understood as time and frequency shifts of the window function. The short-time Fourier transform basis is often illustrated by a tiling of the time-frequency plane, where each tile represents a particular basis element**



The Short-Time Fourier Transform

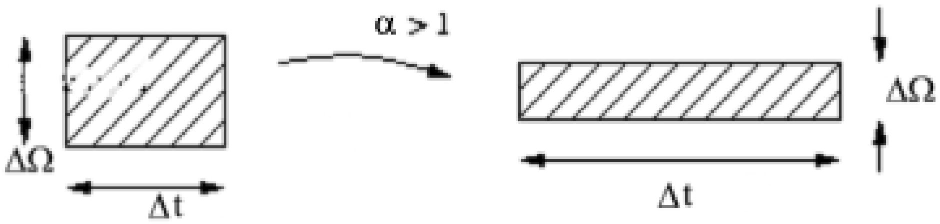
- **The height and width of a tile represent the frequency and temporal widths of the basis element, respectively, and the position of a tile represents the spectral and temporal centers of the basis element**

- **Note that, while the tiling diagram suggests that the short-time Fourier transform uses a discrete set of time/frequency shifts, the short-time Fourier transform basis is really constructed from a continuum of time/frequency shifts**



Uncertainty Principle

- **Note that we can decrease spectral width at the cost of increased temporal width by stretching basis waveforms in time, although the time-bandwidth product (i.e., the area of each tile) will remain constant**



Uncertainty Principle

- **Unfortunately both spectral and time widths cannot be made arbitrary narrow; hence there is an inherent trade-off between time and frequency resolutions for a particular window (uncertainty principle), that**

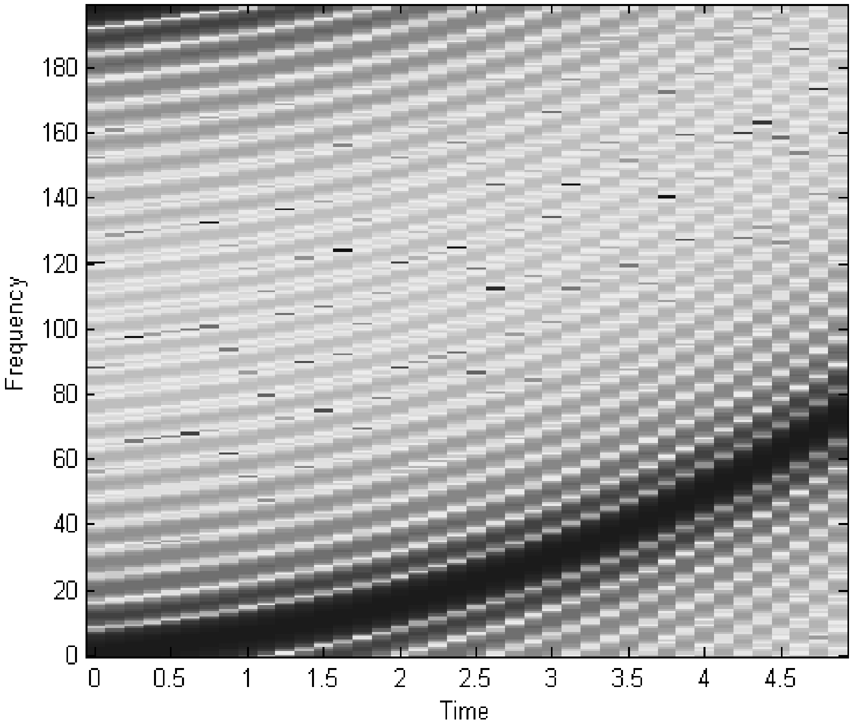
$$\Delta_t \Delta_f \geq a$$

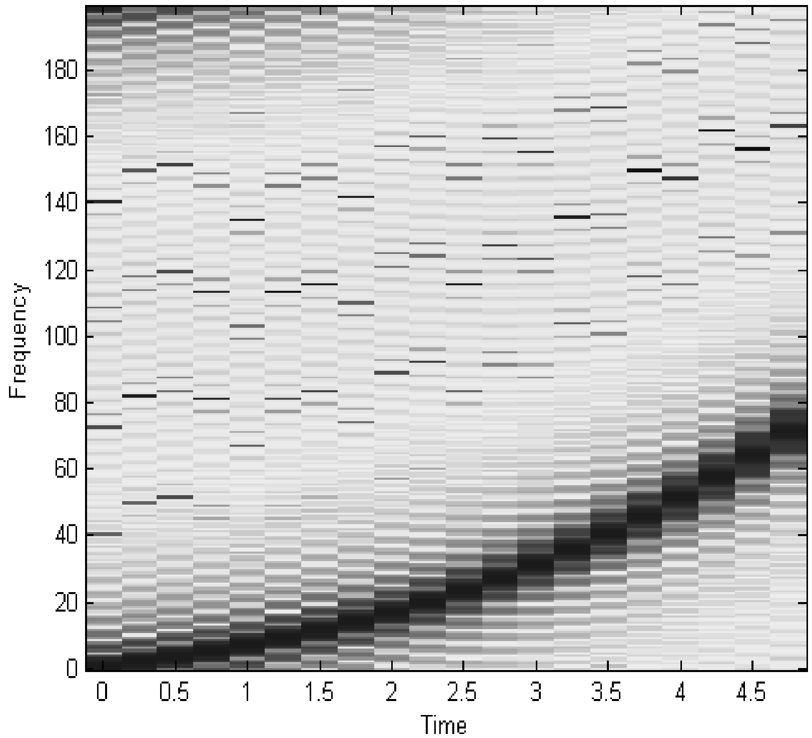
- **where** Δ_t **is duration of time window, is** Δ_f **bandwidth of frequency window**

- **The Gaussian window function** $h(\tau) = e^{-at^2/2}$ **is *optimal* in terms of the uncertainty principle; i. e.** $a = 0.5$

- **The short-time Fourier transform using the Gaussian window is known as the Gabor transform**

The Short-Time Fourier Transform





The Short-Time Fourier Transform: Definition

- **In the conventional Fourier transform, the signal is compared with complex sinusoidal functions**

- **Because sinusoidal functions are spread over the entire time domain and are not concentrated in time, the Fourier transform does not explicitly indicate how a signal's frequency contents evolve in time**

- **To study the properties of the signal at time , one emphasizes the signal *at that time* and suppresses the signal at other times**

The Short-Time Fourier Transform

- **This is achieved by multiplying the signal by a window function, centered at time t :**

$$x_t(\tau) = x(\tau)h(\tau - t)$$

- **The modified signal is a function of two times, the fixed time we are interested in, t , and the running time, τ**

- **The window function is chosen to leave the signal more or less unaltered around the time of interest t but to suppress the signal for times distant from the time of interest**

The Short-Time Fourier Transform

- **Since the modified signal emphasizes the signal around the time t , the short-time Fourier transform will reflect the distribution of frequency around that time:**

$$X(t, f) = \int x(\tau)h(\tau - t)e^{-i2\pi f\tau} d\tau$$

- **The power spectral density at that time is therefore:**

$$S_{xx}(t, f) = \left| X(t, f) \right|^2$$

- **The magnitude of the short-time Fourier transform is called the *spectrogram***

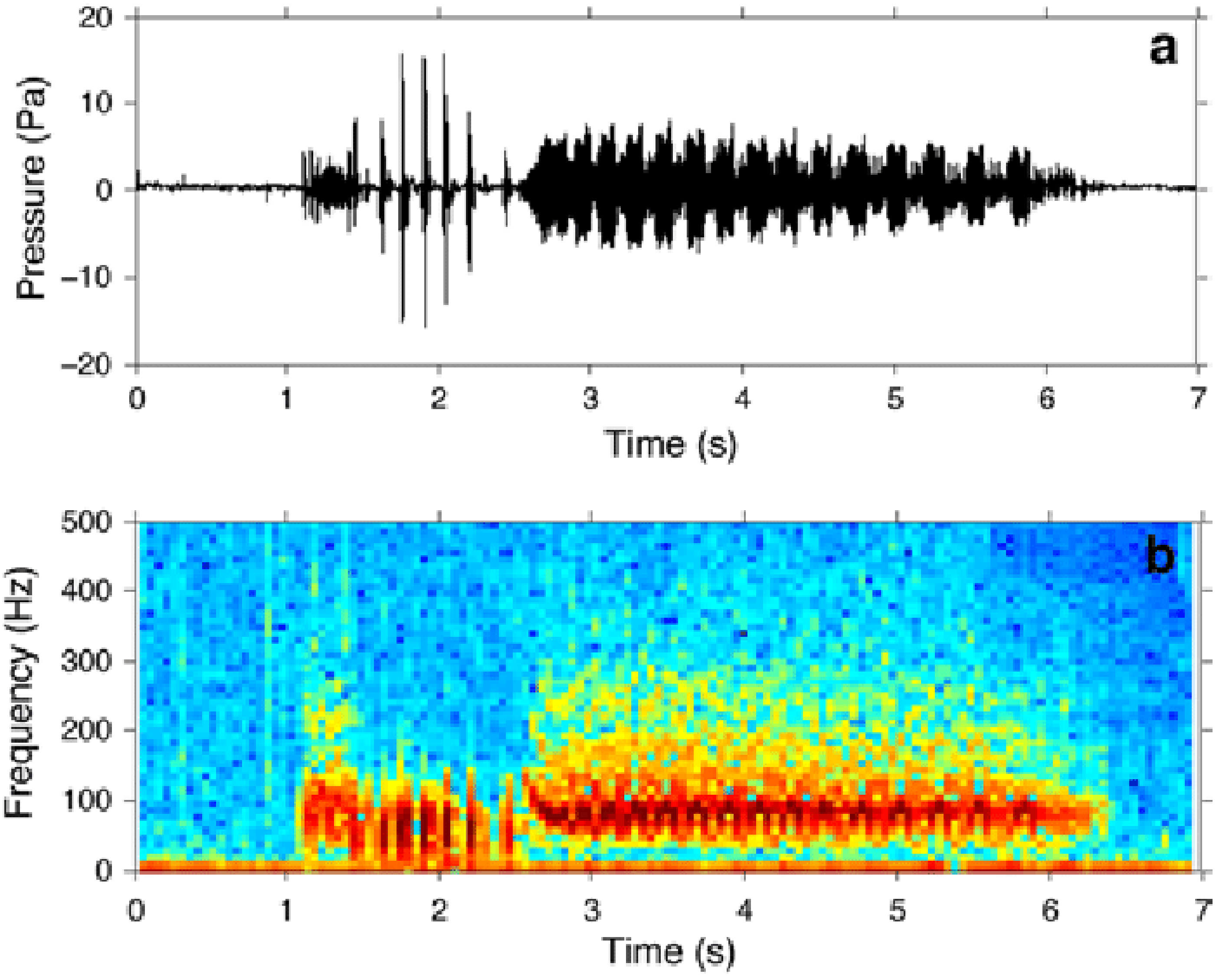
- **For each different time we get a different spectrogram and the *totality* of these spectrograms is the *time-frequency distribution***

The Short-Time Fourier Transform

- **These spectrograms can be plotted one behind the other in the “waterfall” diagram**

- **Since we are interested in analyzing the signal around the time t, we presumably have chosen a narrow window that is peaked around that time**

The Short-Time Fourier Transform



The Short-Time Fourier Transform: Waterfall Plots

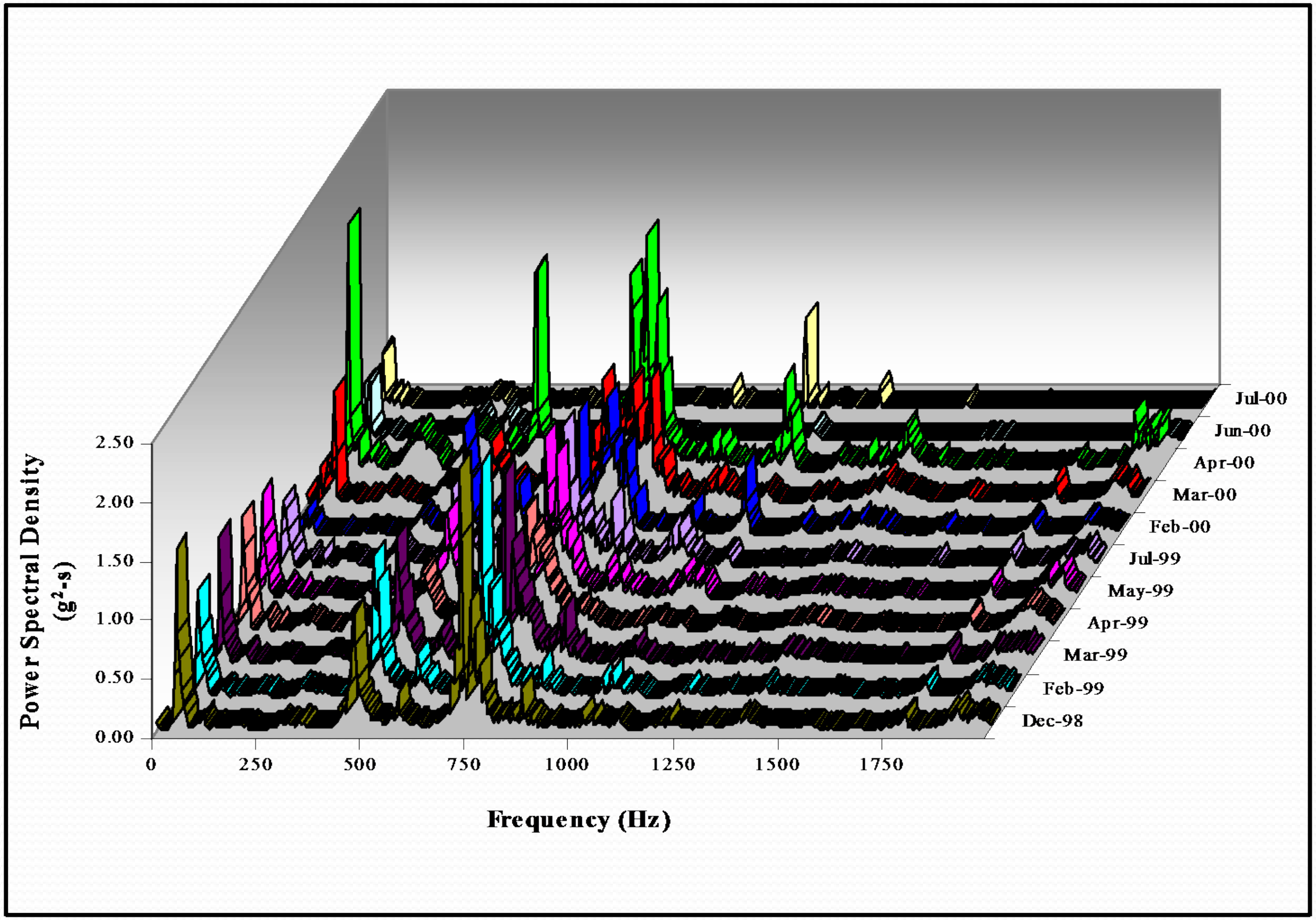
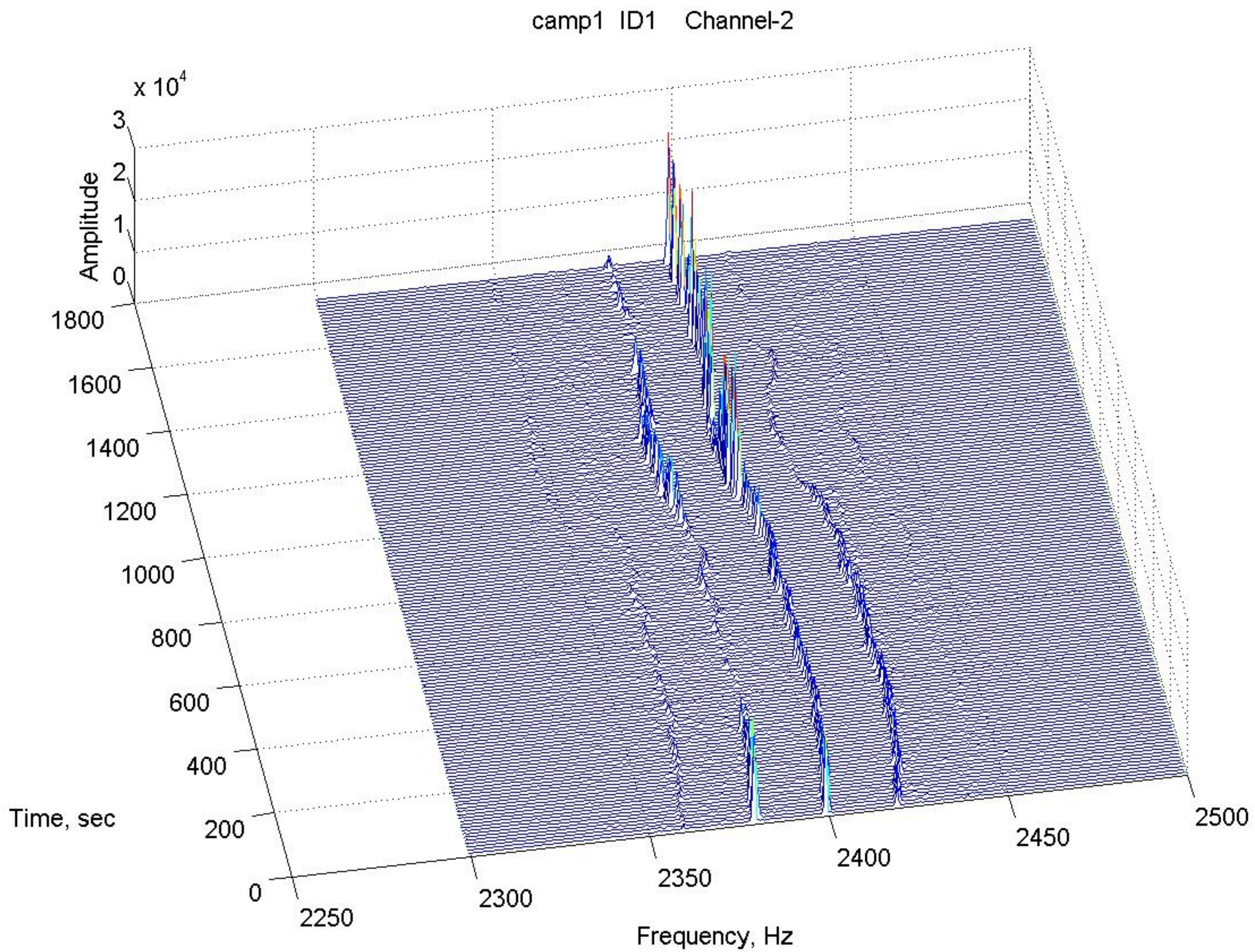


Figure 12.15 Waterfall plot of the signals in the radial direction for the bottom bearing on the pump

The Short-Time Fourier Transform



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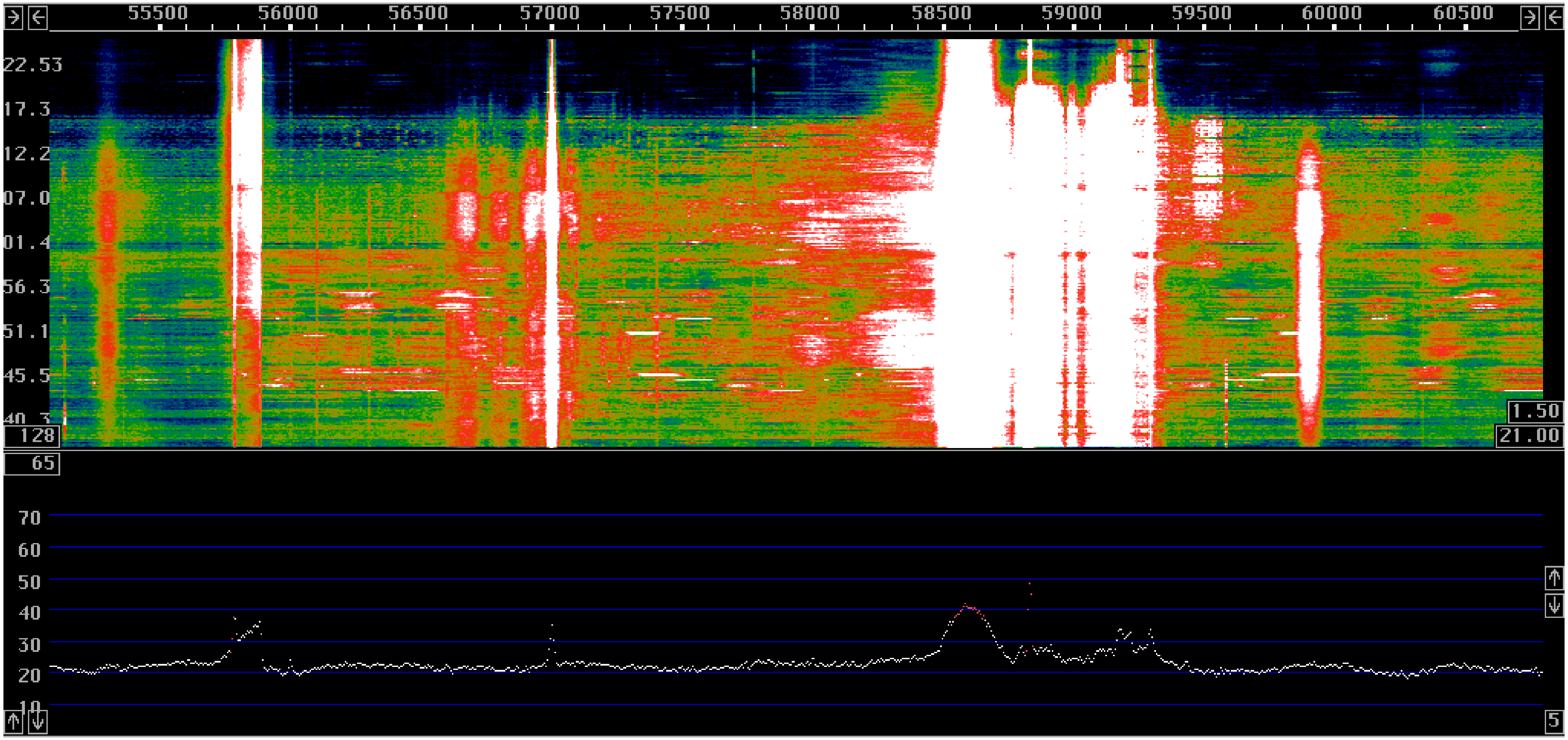

Properties

- **The spectrogram is a real-valued function, even if the signal is complex**

- **A time shift in the signal has the same time shift in the short-time Fourier transform**

Case Study 1

- **Impulse and application of the Gaussian window for STFT**

$$x(t) = \sqrt{2\pi}\,\delta(t - t_0) \qquad h(t) = \left(\frac{a}{\pi}\right)^{0.25} e^{-at^2/2}$$

- **The short-time Fourier transform is:**

$$X(t, f) = \left(\frac{a}{\pi}\right)^{0.25} e^{-i2\pi t_0} \exp\left[-\frac{a(t - t_0)^2}{2}\right]$$

which yields the following spectrogram (modulus):

$$S_{xx}(t, f) = \left(\frac{a}{\pi}\right)^{0.5} \exp\left[-a(t - t_0)^2\right]$$

Case Study 2

- **The sum of a sinusoid and impulse; application of the Gaussian window for STFT**
- **Using previous results we obtain the short-time Fourier transform (spectrogram):**

$$S_{xx}(t, f) = \frac{1}{(a\pi)^{0.5}} \exp\left[-\frac{4\pi^2 (f - f_0)^2}{a}\right] + \left(\frac{a}{\pi}\right)^{0.5} \exp\left[-a(t - t_0)^2\right]$$

$$+ \frac{2}{\sqrt{\pi}} e^{-\frac{4\pi^2 (f - f_0)^2}{a - a(t - t_0)^2}} \cos 2\pi (f(t - t_0) - f_0 t)$$

- **This example illustrates one of the fundamental difficulties with the spectrograms, i. e. for one window we cannot have high resolution in time and frequency**

The Short-Frequency Time Transform

- **We may wish to study time properties at a *particular frequency***

- **We then window the Fourier transform with a frequency window, $H(f)$, and take the time transform, which, of course, is the inverse Fourier transform:**

$$x(t, f) = \int X(f')H(f - f')e^{-i2\pi ft}df'$$