Characterization of B-Spline Digital Filters

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***Abstract*—Digital filters arising in the B-spline signal processing are characterized in a unified manner in this paper. The transfer functions of these filters are the z-transforms of the uniformly sampled central B-splines shifted by an arbitrary value. The transfer functions are cascades of an FIR kernel filter and a simple moving average FIR filter. For certain values of the shift parameter, the filters are identical to those referred to as the B-spline digital filters in the literature. The filters thus form a general family of B-spline digital filters. The kernel part of the B-spline filters may be used for transforming a discrete-time signal to a representation based on the B-spline coefficients. The B-spline filters may also be used to convert a sequence of B-spline coefficients to a discrete-time spline signal. The contributions of the paper are as follows. A unifying recurrence relation enabling the computation of the impulse response coefficients of the B-spline kernel filters is derived. An accompanying recurrence relation is also obtained for the entire transfer function of the kernel filters. The recurrences are valid for arbitrary values of the shift parameter. It is proved that the roots of the transfer functions of the kernel filters are distinct, negative and real. We also prove that the roots of the kernel filters of successive orders interlace. The results regarding the location of the zeros are also valid for arbitrary values of the shift parameter. The relation of the kernel filters to the Eulerain polynomials is discussed. It is shown that for certain choices of the parameters the kernel filters are equivalent to the classical Eulerian polynomials that frequently arise in combinatorics. An alternative closed-form expression for the kernel filters in the Bernstein form is also derived. Besides their importance in unifying the existing results on B-spline filters, the generalized family of B-spline filters studied in this paper find applications in fractional delay of B-spline signals.**

***Index Terms*—B-spline interpolation, B-spline signal processing, Bernstein polynomials, digital filters, Eulerain numbers, Eulerain polynomials, finite-impulse response (FIR) filters, fractional delay, recurrence relations, stability.**

I. Introduction

DISCRETE-TIME spline signals are sampled counterparts of the continuous splines. They can be expressed as linear combinations of the discrete B-splines. The discrete-time splines can be processed and interpolated by digital filtering. The filtering theory of B-spline signal processing was formally developed by Unser *et al.* [1]. The expository article by Unser [2] contains various examples of applications of discrete-time spline signal processing in image zooming and visualization, geometric transformations, image compression, multiscale processing and image registration, contour detection, snake and contour modeling, and analog–digital (A/D) conversion. Other authors have explored generalizations of the theory [3] and the conversion from the analytic B-splines to the discrete-time counterparts based on the bilinear transformation [4]. An attempt to derive an explicit formula for the z-transform of discrete polynomial splines with nonuniform knots may be found in [5]. Properties of a version of discrete splines that can be viewed as approximations of the discrete B-splines have been studied in [6].

The focus of this article is on the digital filters that arise in tasks such as the interpolation and differentiation of discrete-time signals using the B-spline bases. The digital filters are commonly referred to as the B-spline digital filters. The direct and indirect B-spline transforms are the two fundamental operations in the processing of discrete polynomial splines. The B-spline digital filters are employed in the both tasks. The indirect B-spline transform is a filtering process for transforming a discrete-time sequence of the B-spline coefficients to a discrete-time spline signal of order $n(\geq 0)$ of a sampling rate $m(\geq 1)$ times higher than that of the coefficient sequence. Unser *et al.* have shown that the indirect B-spline filter $B_m^n(z)$ combined with an m-fold upsampler can achieve this task [1]. They have also proved that the transfer function $B_m^n(z)$ may be decomposed into the cascade of an FIR *B-spline kernel* $B_1^n(z)$ and $n+1$ successive moving average filters. For $m=1$, the moving average part disappears and the filter consists of the remaining kernel part $B_1^n(z)$. The all-pole filter $1/B_1^n(z)$, on the other hand, may be used in the direct B-spline transform. The goal here is to transform an arbitrary discrete-time signal to a sequence of B-spline coefficients. Another type of B-spline filters arises in the differentiation of discrete spline signals and is denoted by $C_1^n(z)$ in [1]. The filter is obtained by shifting the B-spline basis functions of degree n by a half sample and taking the z-transform of the equidistant samples of the shifted function. As a result, another possible application of $C_1^n(z)$ is in delaying spline signals by a half sample. The coefficients of the B-spline kernels $B_1^n(z)$ and $C_1^n(z)$ have been characterized in [1] by two recurrence relations that mutually depend on each other. This is the only type of recurrence that has been reported in the B-spline signal processing literature for the computation of the kernel impulse response coefficients.

The first contribution of this paper is to derive a single recurrence that is valid for the impulse response coefficients of all types of B-spline kernels. To that end, we generalize the B-spline digital filters by defining a class of filters denoted by $H_{m,d}^n(z)$. We show that $H_{m,d}^n(z)$ is a cascade of a moving average filter $((1-z^{-m})/(1-z^{-1}))^n$ and a B-spline kernel $E_{m,d}^n(z)$. The filters $B_m^n(z)$ and $C_m^n(z)$ are then characterized as special cases corresponding to $d=0$ and $d=-1/2$, respectively. A recurrence relation characterizes the coefficients

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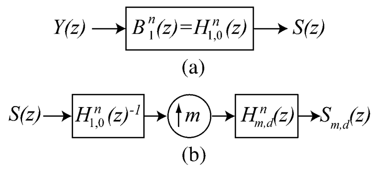

Fig. 1. (a) Block diagram of the indirect B-spline transform. (b) Generalized B-spline interpolator with upsampling factor m and fractional delay of d samples.

of $E_{m,d}^n(z)$. The generalized B-spline kernel $E_{m,d}^n(z)$ corresponds to the continuous central B-spline function $\beta^n(x)$ shifted by an arbitrary value d and sampled at the rate of m equidistant samples per unit interval. We also characterize $E_{m,d}^n(z)$ using a single recurrence relation in the z domain. Our results are complementary to the expressions and characterizations of Unser *et al.* in the theory of B-spline signal processing. Moreover, the filters $H_{m,d}^n(z)$ can be used to affect a fractional delay of d samples on a spline signal. This can be achieved, using the setup of Fig. 1, by first extracting the B-spline parameters of the signal and then constructing the delayed version of the signal.

The next contribution of this paper is related to the stability issues of the direct B-spline filters with arbitrary values for the parameter d. Since the direct B-spline transform involves an all-pole filtering by $1/B_1^n(z)$, it is important to study the location of roots of $B_1^n(z)$. In our formulation, the stability problem is generalized to the location of the zeros of the B-spline kernel $E_{m,d}^n(z)$. Although researchers in this area are aware of the nature of the zeros of $B_1^n(z)$, there has been a very limited interest in providing rigorous mathematical proofs in this regard. For instance, we know that the zeros of $B_1^n(z)$ are all real and negative. However, no explicit proof of this fact is ever given in the signal processing literature. One exception is [7] where it is proved that the filters do not vanish on the unit circle. We prove what many researchers have mentioned and even more. We establish that the B-spline kernels of all types, including $B_1^n(z)$, $C_1^n(z)$ and the generalized shifted version of the form $E_{m,d}^n(z)$, have real, simple and negative zeros that interlace the zeros of the corresponding filters of one order lower.

The third contribution of this paper is the derivation of the Bernstein-form representation of $B_1^n(z)$. The final contribution of this paper is of academic nature. Our approach to the characterization of the filters is based on a generalized form of Worpitzky's identity. In its original form, the identity relates integer powers of a variable to the Eulerian numbers. Eulerian numbers have important combinatorial interpretations and form the coefficients of the Eulerian polynomials. The kth Eulerian number counts the number of those permutations of n integers that have k "ascents" or "rises" [12]. We show that depending on the values of n, m and d, the impulse response coefficients of $E_{m,d}^n(z)$ are identical to the Eulerian or the generalized pq-Eulerain numbers. This throws light on a deep relation between integer sequences of combinatorics and impulse response coefficients in a certain digital signal processing task. Schoenberg was aware of the relation between uniform samples of the B-splines and the coefficients of the Euler–Frobenius polynomials [8]. Our results are more general. They establish the relation between the generalized shifted B-spline filters and the Eulerian or the Euler–Frobenius polynomials.

The organization of this paper is as follows. In Section II, after providing a brief introduction to the B-spline filters, we derive the transfer function of an arbitrarily shifted B-spline filter in an explicit form. We then characterize the coefficients of the kernel filter in a unified manner using a single recurrence. The kernel filters are studied in detail in Section III where the location of their zeros are identified and a recurrence that relates the transfer function of the kernels of successive orders is developed. Section IV is concerned with the development of an alternative representation for the transfer function of the filters. The Bernstein form of the filters is derived from this representation. Conclusions are drawn in Section V.

II. Unified Formula for B-Spline Digital Filters

We provide a detailed derivation of an exact unified formula for the general family of the shifted B-spline digital filters.

A. B-Spline Preliminaries

This introductory part is mostly based on the theory developed in [1]. Let $s(x)$ be a continuous spline signal defined on equidistant knots. The spline signal $s(x)$ may be characterized by the B-spline expansion

$$s(x) = \sum_{i\in\mathbf{Z}} y(i)\beta^n(x-i) \tag{1}$$

where $\beta^n(x)$ is the central (symmetrical) B-spline of degree n given by [1]

$$\beta^n(x) = \frac{1}{n!}\sum_{j=0}^{n+1}(-1)^j \binom{n+1}{j} \times \left(x-j+\frac{(n+1)}{2}\right)^n \mu\left(x-j+\frac{(n+1)}{2}\right). \tag{2}$$

The unit step function $\mu(x)$ is defined by

$$\mu(x) = \begin{cases} 0, & x<0 \\ 1, & x\geq 0. \end{cases} \tag{3}$$

The coefficients $y(i)$ are called the B-spline coefficients. In the original notation employed by Schoenberg [8], the central B-spline function of (2) is denoted by $M_{n+1}(x)$. We follow the notation employed by Unser *et al.*.

The discrete-time spline signal $s(k)$, $k\in\mathbf{Z}$, may be manipulated by convolution techniques used in digital signal processing if we recognize that (1) implies

$$S(z) = Y(z)\,B_1^n(z) \tag{4}$$

where

$$\begin{aligned} S(z) &= \sum_{k\in\mathbf{Z}} s(k)z^{-k} \\ Y(z) &= \sum_{k\in\mathbf{Z}} y(k)\,z^{-k} \\ B_1^n(z) &= \sum_{k\in\mathbf{Z}} \beta^n(k)\,z^{-k}. \end{aligned} \tag{5}$$

We consider the following signal-processing scenario. We wish to upsample the discrete-time spline signal $s(k)$ by the sampling ratio m and then delay (shift) it by d samples. This is a basic task arising in a variety of applications. We need to evaluate the values of

$$s_{m,d}(k) = s\left(\frac{k}{m} - d\right) = \sum_{i\in\mathbf{Z}} y(i)\beta^n\left(\frac{k}{m} - d - i\right) \quad (6)$$

for $k \in \mathbf{Z}$. In order to put the above task within the framework of multirate digital signal processing, (6) should be expressed as a convolution. We introduce the signal

$$y'(i) = \begin{cases} y\left(\frac{i}{m}\right), & m|i \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

and write

$$s_{m,d}(k) = \sum_{i\in\mathbf{Z}} y'(i)\beta^n\left(\frac{(k-i)}{m} - d\right). \quad (8)$$

The above equation is in the form of a convolution and is thus the time-domain equivalent of

$$S_{m,d}(z) = Y(z^m)H^n_{m,d}(z) \quad (9)$$

where

$$H^n_{m,d}(z) = \sum_{k\in\mathbf{Z}} \beta^n\left(\frac{k}{m} - d\right) z^{-k}. \quad (10)$$

If the B-spline coefficients $y(k)$ are available, the filter $H^n_{m,d}(z)$ can be used to interpolate (upsample) $s(k)$ by a factor of m and delay it by d samples to yield the signal $s_{m,d}(k)$. Obviously

$$B^n_1(z) = H^n_{1,0}(z). \quad (11)$$

The filter $1/H^n_{1,0}(z)$ may be used to transform the signal $S(z)$ to its B-spline representation $Y(z)$ and is known as the *direct B-spline filter*, whereas the role of $H^n_{m,d}(z)$ is to transform the B-spline coefficients to an m-fold interpolation of the signal with an extra delay of d samples. For $d = 0$, this process has been referred to as the *indirect B-spline transform* [1] and the resulting transfer function is denoted by $B^n_m(z)$. Clearly

$$B^n_m(z) = H^n_{m,0}(z). \quad (12)$$

Fig. 1 illustrates the concept by means of block diagrams. A related transfer function, $C^n_1(z)$, arising in the differentiation of discrete splines, is specified by

$$C^n_1(z) = H^n_{1,-1/2}(z). \quad (13)$$

B. Derivation of $H^n_{m,d}(z)$

The goal of this section is to derive a single closed-form formula for $H^n_{m,d}(z)$ that is valid for all values of m and n regardless of their parity, and for all real values of d. From (2) and (10), we have

$$\begin{aligned} H^n_{m,d}(z) = \sum_{k\in\mathbf{Z}} \frac{1}{n!}\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j} \\ \times\left(\frac{k - m(j+d) + \frac{m(n+1)}{2}}{m}\right)^n \\ \times\,\mu\left(k - m(j+d) + \frac{m(n+1)}{2}\right) z^{-k} \end{aligned} \quad (14)$$

where we have used the fact that $m \geq 1$ to remove the denominator of the argument of the unit step function. Let us concentrate on the sum with respect to k in the double-sum expression for $H^n_{m,d}(z)$. Bundling the terms in the summand that are dependent on the summation index k, we obtain the sum

$$\begin{aligned} \sigma = \sum_{k\in\mathbf{Z}}\left(\frac{k - m(j+d) + \frac{m(n+1)}{2}}{m}\right)^n \\ \times\mu\left(k - m(j+d) + \frac{m(n+1)}{2}\right) z^{-k}. \end{aligned} \quad (15)$$

Define

$$f(n,m,d) \stackrel{\text{def}}{=} m(j+d) - \frac{m(n+1)}{2}. \quad (16)$$

We can always write

$$f(n,m,d) = \lceil f(n,m,d)\rceil - \{f(n,m,d)\} \quad (17)$$

where $\{\cdot\}$ denotes the nonnegative fractional part of its argument, and $\lceil\cdot\rceil$ is the ceiling function. Note that

$$\lceil f(n,m,d)\rceil = m\,j + \left\lceil md - \frac{m(n+1)}{2}\right\rceil \quad (18)$$

is j dependent, whereas

$$\{f(n,m,d)\} = \left\{md - \frac{m(n+1)}{2}\right\} \quad (19)$$

is j free. This observation is crucial in manipulation of the sums. By using $f(n,m,d)$, (15) becomes

$$\begin{aligned} \sigma = \frac{1}{m^n}\sum_{k\in\mathbf{Z}}(k - \lceil f(n,m,d)\rceil + \{f(n,m,d)\})^n \\ \times\mu(k - \lceil f(n,m,d)\rceil)z^{-k}. \end{aligned} \quad (20)$$

We can remove the unit step function in the summand and revise the summation range to obtain

$$\sigma = \frac{1}{m^n}\sum_{k\geq\lceil f(n,m,d)\rceil}(k - \lceil f(n,m,d)\rceil + \{f(n,m,d)\})^n z^{-k}. \quad (21)$$

The sum σ may be further simplified as

$$\sigma = \frac{1}{m^n}\sum_{k\geq 0}(k + \{f(n,m,d)\})^n\, z^{-k-\lceil f(n,m,d)\rceil}. \quad (22)$$

To obtain a closed-form expression for σ and subsequently S, we make a brief diversion.

C. Worpitzky's Identity

Consider the following problem.

Problem: Characterize the sequence $h(n,l)_{p,q} = \left\langle {n \atop l} \right\rangle_{p,q}$ so that the identity

$$(p(x+1) + qx)^n = \sum_{0\leq l\leq n}\left\langle {n \atop l} \right\rangle_{p,q}\binom{x+l}{n} \quad (23)$$

holds for all nonnegative integers n and numbers p and q.

The sequence $\left\langle {n \atop l} \right\rangle_{p,q}$ is related to some counting problems in combinatorics [11]. For the especial case of $p=0$ and $q=1$, (23) is known as the Worpitzky's identity and the numbers $\left\langle {n \atop l} \right\rangle$ as the Eulerian numbers [12]. In this paper, we deal with a general form (23) of the Worpitzky's identity that is valid for all values of p and q. Our goal is to characterize the sequence $\left\langle {n \atop l} \right\rangle_{p,q}$. We take an approach based on induction on n.

Let us consider the identity for $n = 0$. In this case, (23) becomes

$$1 = \left\langle {0 \atop 0} \right\rangle_{p,q} \binom{x}{0}.$$

This is true as an identity if and only if

$$\left\langle {0 \atop 0} \right\rangle_{p,q} = 1. \tag{24}$$

Now, let us assume that (23) is true for $n = n' - 1$. Then

$$(p(x+1)+qx)^{n'-1} = \sum_{0 \le l \le n'-1} \left\langle {n'-1 \atop l} \right\rangle_{p,q} \binom{x+l}{n'-1} \tag{25}$$

holds as an identity. For $n = n'$, we can write

$$(p(x+1)+qx)^{n'} = \sum_{0 \le l \le n'-1} \left\langle {n'-1 \atop l} \right\rangle_{p,q} \times (p(x+1)+qx) \binom{x+l}{n'-1}. \tag{26}$$

To simplify the summand on the right-hand side, we need to determine $a(l)$ and $b(l)$ so that

$$p(x+1)+qx = a(l)\frac{x+l+1}{n'} + b(l)\frac{x+l-(n'-1)}{n'} \tag{27}$$

holds as an identity. Solving the resulting system of linear equations, the values of $a(l)$ and $b(l)$ turn out to be

$$\begin{aligned} a(l) &= (n'-l)(p+q)-p \\ b(l) &= (l+1)(p+q)-p. \end{aligned}$$

Now, we can write

$$(p(x+1)+qx)\binom{x+l}{n'-1} = a(l)\binom{x+l+1}{n'} + b(l)\binom{x+l}{n'}.$$

Substitution of the above into (26) results in

$$(p(x+1)+qx)^{n'} = \sum_{0 \le l \le n'-1} a(l) \left\langle {n'-1 \atop l} \right\rangle_{p,q} \binom{x+l+1}{n'} + \sum_{0 \le l \le n'-1} b(l) \left\langle {n'-1 \atop l} \right\rangle_{p,q} \binom{x+l}{n'}. \tag{28}$$

It follows that

$$(p(x+1)+qx)^{n'} = \sum_{1 \le l \le n'} a(l-1) \left\langle {n'-1 \atop l-1} \right\rangle_{p,q} \binom{x+l}{n'} + \sum_{0 \le l \le n'} b(l+1) \left\langle {n'-1 \atop l} \right\rangle_{p,q} \binom{x+l}{n'} \tag{29}$$

where we have assumed that

$$\left\langle {n'-1 \atop n'} \right\rangle_{p,q} = 0. \tag{30}$$

We can thus combine the two sums on the right-hand side of (29) and obtain

$$(p(x+1)+qx)^{n'} = \sum_{0 \le l \le n'} \left\langle {n' \atop l} \right\rangle_{p,q} \binom{x+l}{n'} \tag{31}$$

where we have assumed that

$$\left\langle {n'-1 \atop -1} \right\rangle_{p,q} = 0. \tag{32}$$

This completes the final step of the induction. From the above argument, the sequence $\left\langle {n \atop l} \right\rangle p, q$ is characterized by the recurrence

$$\left\langle {n \atop l} \right\rangle_{p,q} = (p(n-l+1)+q(n-l)) \left\langle {n-1 \atop l-1} \right\rangle_{p,q} + (pl+q(l+1)) \left\langle {n-1 \atop l} \right\rangle_{p,q} \tag{33}$$

and the boundary conditions

$$\left\langle {0 \atop 0} \right\rangle_{p,q} = 1 \quad \left\langle {n \atop -1} \right\rangle_{p,q} = 0 \quad \left\langle {n+1 \atop n} \right\rangle_{p,q} = 0. \tag{34}$$

D. Explicit Formula for $H_{m,d}^n(z)$

Now, we are in a position to obtain an expression for $(k + \{f(n,m,d)\})^n$ in terms of consecutive binomial coefficients. From (23), it can be readily verified that

$$(k+\{f(n,m,d)\})^n = \sum_{0 \le l \le n} \left\langle {n \atop l} \right\rangle_{\{f(n,m,d)\},1-\{f(n,m,d)\}} \binom{k+l}{n} \tag{35}$$

and thus

$$\sigma = \frac{z^{-\lceil f(n,m,d) \rceil}}{m^n} \sum_{k \ge 0} \sum_{0 \le l \le n} \left\langle {n \atop l} \right\rangle_{\{f(n,m,d)\},1-\{f(n,m,d)\}} \times \binom{k+l}{n} z^{-k}. \tag{36}$$

We replace k with $k+n-l$ on the right-hand side and rearrange the sums to obtain

$$\sigma = \frac{z^{-n-\lceil f(n,m,d) \rceil}}{m^n} \sum_{0 \le l \le n} \left\langle {n \atop l} \right\rangle_{\{f(n,m,d)\},1-\{f(n,m,d)\}} z^l \times \sum_{k \ge l-n} \binom{k+n}{n} z^{-k}. \tag{37}$$

The range for the second sum in (37) may be simplified to $k \ge 0$ because the binomial coefficient in the summand vanishes for negative values of k. Also define

$$E_{m,d}^n(z) = \sum_{0 \le l \le n} \left\langle {n \atop l} \right\rangle_{\{f(n,m,d)\},1-\{f(n,m,d)\}} z^l. \tag{38}$$

Then, by invoking the binomial theorem, (37) can be written as

$$\sigma = \frac{z^{-n-\lceil f(n,m,d)\rceil}}{m^n} E^n_{m,d}(z) \frac{1}{(1-z^{-1})^{n+1}}. \quad (39)$$

The above explicit expression for σ may be substituted into (14) to yield

$$H^n_{m,d}(z) = \frac{1}{n!m^n} E^n_{m,d}(z) \frac{1}{(1-z^{-1})^{n+1}} \times \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} z^{-n-\lceil f(n,m,d)\rceil}. \quad (40)$$

From (18), we can write

$$H^n_{m,d}(z) = \frac{1}{n!m^n} z^{-n-\lceil md-m(n+1)/2\rceil} E^n_{m,d}(z) \times \frac{1}{(1-z^{-1})^{n+1}} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} z^{-mj}. \quad (41)$$

Again, by the binomial theorem, we conclude that

$$H^n_{m,d}(z) = \frac{1}{n!m^n} z^{-n-\lceil md-m(n+1)/2\rceil} E^n_{m,d}(z) \left(\frac{1-z^{-m}}{1-z^{-1}}\right)^{n+1}. \quad (42)$$

This is a unified expression valid for the transfer functions of all types of B-spline digital filters.

The following comments are in order. There is a strong connection between $H^n_{m,d}(z)$ and the filter $B^n_{m,\Delta}(z)$ in [17] defined as

$$B^n_{m,\Delta}(z) = \sum_{k\in\mathbf{Z}} \beta^n\left(\frac{k-\Delta}{m}\right) z^{-k}. \quad (43)$$

The filter $B^n_{m,\Delta}(z)$ arises in the post-filter operation in the construction of least squares pyramids using shifted polynomial spline basis functions [17]. Our generalized B-spline filter is related to $B^n_{m,\Delta}(z)$ by

$$H^n_{m,d}(z) = B^n_{m,md}(z). \quad (44)$$

The closed-form expressions obtained in this paper provide an efficient method to design the shifted B-spline filters of [17]. The derivation of (42) in its factored form may be simplified by working in the Fourier domain. Our approach, however, provides a recursive characterization of the coefficients which is more difficult to achieve in the Fourier domain.[1]

E. Examples

We provide concrete examples to clarify the results.

Example 1: Design the B-Spline filter $H^4_{1,0}(z)$.

Solution. The parameters are $n = 4$, $m = 1$, and $d = 0$. Hence, the filter is given by

$$H^4_{1,0}(z) = \frac{1}{24} z^{-2} E^4_{1,0}(z). \quad (45)$$

The filter is thus equivalent to $B^4_1(z)$ in [1]. First, we have $\{f(4,1,0)\} = \{-5/2\} = 1/2$. Therefore, the impulse response coefficients of $E^4_{1,0}(z)$ are given by $\left\langle {4 \atop l} \right\rangle_{1/2,1/2}$ for $l = 0,\ldots,4$. The result of execution of the recurrence (33) for j running from 1 to 4 is

$$\begin{array}{cccccc} \left\langle {n \atop l} \right\rangle_{1/2,1/2} & l=0 & l=1 & l=2 & l=3 & l=4 \\ n=0 & 1 & & & & \\ n=1 & \frac{1}{2} & \frac{1}{2} & & & \\ n=2 & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & \\ n=3 & \frac{1}{8} & \frac{23}{8} & \frac{23}{8} & \frac{1}{8} & \\ n=4 & \frac{1}{16} & \frac{19}{4} & \frac{115}{8} & \frac{19}{4} & \frac{1}{16}. \end{array} \quad (46)$$

The transfer function becomes

$$H^4_{1,0}(z) = \frac{1}{24} z^{-2} \times \left(\frac{1}{16} + \frac{19}{4}z + \frac{115}{8}z^2 + \frac{19}{4}z^3 + \frac{1}{16}z^4\right).$$

Example 2: Design the B-Spline filter $H^3_{2,1/8}(z)$.

*Solution.*The parameters are $n = 3$, $m = 2$, and $d = 1/8$. First, we have $\{f(3,2,1/8)\} = \{-15/4\} = 3/4$. Hence, the impulse response coefficients of $E^3_{2,1/8}(z)$ are $\left\langle {3 \atop l} \right\rangle_{3/4,1/4}$ for $l = 0,\ldots,3$. The result of execution of the recurrence (33) for j running from 1 to 3 is

$$\begin{array}{ccccc} \left\langle {n \atop l} \right\rangle_{3/4,1/4} & l=0 & l=1 & l=2 & l=3 \\ n=0 & 1 & & & \\ n=1 & \frac{1}{4} & \frac{3}{4} & & \\ n=2 & \frac{1}{16} & \frac{11}{8} & \frac{9}{16} & \\ n=3 & \frac{1}{64} & \frac{121}{64} & \frac{235}{64} & \frac{27}{64}. \end{array} \quad (47)$$

The transfer function becomes

$$H^3_{2,1/8}(z) = \frac{1}{48}\left(\frac{1}{64} + \frac{121}{64}z + \frac{235}{64}z^2 + \frac{27}{64}z^3\right)(1+z^{-1})^4.$$

III. Properties of B-Spline Kernel $E^n_{m,d}(z)$

The purpose of this section is to study in detail the properties of $E^n_{m,d}(z)$, the B-spline kernel filter.

A. Symmetries in Impulse Response Coefficients

As shown in Section II, the impulse response coefficients of the anticausal filter $E^n_{m,d}(z)$ are specified by the sequence $\left\langle {n \atop l} \right\rangle_{p,q}$ for $0 \le l \le n$. Thus, the filter is of length $n+1$ in general.

Proposition: For all values of p and q

$$\left\langle {n \atop l} \right\rangle_{p,q} = \left\langle {n \atop n-l} \right\rangle_{q,p} \quad (48)$$

holds.

Identity (48) can be proved by induction on n. An interesting corollary of (48) is that for $p = q$, the sequence $\left\langle {n \atop l} \right\rangle_{p,q}$ is symmetric. By applying this result to the filter $H^n_{m,d}(z)$, it follows

[1]We thank an anonymous reviewer for bringing the connection and the alternative approach to our attention.

that for $\{f(n,m,d)\} = 1/2$, the filter possesses symmetric impulse response coefficients and has a linear phase.

The other important example of a symmetric impulse response is the case $\{f(n,m,d)\} = 0$. This case involves the sequence $\left\langle \begin{matrix} n \\ l \end{matrix} \right\rangle_{0,1}$ and the following can be proved by induction on n.

Proposition: For all values of n, we have

$$\left\langle \begin{matrix} n \\ n \end{matrix} \right\rangle_{0,1} = 0 \tag{49}$$

and

$$\left\langle \begin{matrix} n \\ l \end{matrix} \right\rangle_{0,1} = \left\langle \begin{matrix} n \\ n-l-1 \end{matrix} \right\rangle_{0,1}. \tag{50}$$

In this case, the filter $E^n_{m,d}(z)$ degenerates to a filter of length n.

B. Periodicity

An interesting property of $E^n_{m,d}(z)$ is that it is periodic with respect to d. The period is equal to unity. This follows from the periodicity of the fractional part function $\{\cdot\}$. It is simple to verify that

$$\{f(n,m,d)\} = \{f(n,m,d+1)\}$$

and, thus, it can be readily shown that

$$E^n_{m,d}(z) = E^n_{m,d\pm 1}(z) = E^n_{m,\{d\}}(z). \tag{51}$$

C. Location of Roots

We are also interested in the location of the zeros of $E^n_{m,d}(z)$. In the literature of B-spline signal processing, it is mentioned that the roots of $E^n_{1,0}(z) = B^n_1(z)$ are real and negative [9]. A proof is provided in [7] that shows that $B^n_1(z)$ has no roots on the unit circle. We present a very general and deep result on the roots of $E^n_{m,d}(z)$ that is valid for all values of n, m and d. Note that, $E^n_{m,d}(z)$ is an FIR filter and thus a polynomial in z of degree at most n. For a fixed n, it has, at most n roots. To investigate the location of those roots, we need to characterize the polynomial rigorously. Although we have already characterized the impulse response coefficients $\left\langle \begin{matrix} n \\ l \end{matrix} \right\rangle_{\{f(n,m,d)\},1-\{f(n,m,d)\}}$ of $E^n_{m,d}(z)$ by the recurrence (33), it is not simple to extract information about the location of its roots from that recurrence. We will see that it is more helpful to work with a recurrence for the entire transfer function rather than one for the coefficients.

Let us invoke the recurrence (33) and use (38) to write

$$\begin{aligned} E^n_{m,d}(z) = \sum_{0\le l\le n} \Bigg(&\Big(p\,(n-l+1) + q\,(n-l)\Big) \left\langle \begin{matrix} n-1 \\ l-1 \end{matrix} \right\rangle_{p,q} \\ &+ (pl + q(l+1)) \left\langle \begin{matrix} n-1 \\ l \end{matrix} \right\rangle_{p,q} \Bigg) z^l \end{aligned} \tag{52}$$

where

$$p = \{f(n,m,d)\} \text{ and } q = 1 - \{f(n,m,d)\}. \tag{53}$$

The sum on the right-hand side of (52) may be decomposed as

$$\begin{aligned} E^n_{m,d}(z) = &(p+q) \sum_{1\le l\le n} n \left\langle \begin{matrix} n-1 \\ l-1 \end{matrix} \right\rangle_{p,q} z^l \\ &+ (p+q) \sum_{1\le l\le n} l \left\langle \begin{matrix} n-1 \\ l \end{matrix} \right\rangle_{p,q} z^l \\ &- (p+q) \sum_{1\le l\le n} l \left\langle \begin{matrix} n-1 \\ l-1 \end{matrix} \right\rangle_{p,q} z^l \\ &+ p \sum_{1\le l\le n} \left\langle \begin{matrix} n-1 \\ l-1 \end{matrix} \right\rangle_{p,q} z^l \\ &+ q \sum_{0\le l\le n-1} \left\langle \begin{matrix} n-1 \\ l \end{matrix} \right\rangle_{p,q} z^l. \end{aligned} \tag{54}$$

This can be simplified as

$$\begin{aligned} E^n_{m,d}(z) = &n(p+q)zE^{n-1}_{m,d}(z) + (p+q)z\frac{d}{dz}E^{n-1}_{m,d}(z) \\ &- (p+q)z\frac{d}{dz} \sum_{1\le l\le n} \left\langle \begin{matrix} n-1 \\ l-1 \end{matrix} \right\rangle_{p,q} z^l \\ &+ pzE^{n-1}_{m,d}(z) + qE^{n-1}_{m,d}(z). \end{aligned} \tag{55}$$

It follows that $E^n_{m,d}(z)$ satisfies

$$\begin{aligned} E^n_{m,d}(z) = &\left(\left((n-1)(p+q)+p\right)z + q\right) E^{n-1}_{m,d}(z) \\ &+ (p+q)z(1-z)\frac{d}{dz}E^{n-1}_{m,d}(z). \end{aligned} \tag{56}$$

Substituting the values of p and q from (53) into the above recurrence yields

$$\begin{aligned} E^n_{m,d}(z) = &\left(\left((n-1) + \{f(n,m,d)\}\right)z + 1 - \{f(n,m,d)\}\right) \\ &\times E^{n-1}_{m,d}(z) + z(1-z)\frac{d}{dz}E^{n-1}_{m,d}(z). \end{aligned} \tag{57}$$

The boundary condition is specified by

$$E^0_{m,d}(z) = 1. \tag{58}$$

We conclude this section by proving the following proposition.

Proposition: The transfer function $E^n_{m,d}(z)$ has real, negative and distinct zeros. Moreover, the roots of $E^n_{m,d}(z)$ and those of $E^{n-1}_{m,d}(z)$ mutually interlace.

Proof: We prove by induction on n. Let us consider the case $n = 1$. From (57), $E^1_{m,d}(z)$ is computed as

$$E^1_{m,d}(z) = \{f(1,m,d)\}z + (1 - \{f(m,1,d)\}). \tag{59}$$

The zero of this linear function is given by

$$\lambda = -\frac{1 - \{f(1,m,d)\}}{\{f(1,m,d)\}}$$

that is a negative real number because either $0 < \{f(1,m,d)\} < 1$ or $f(1,m,d) = 0$. The case $\{f(1,m,d\} = 0$

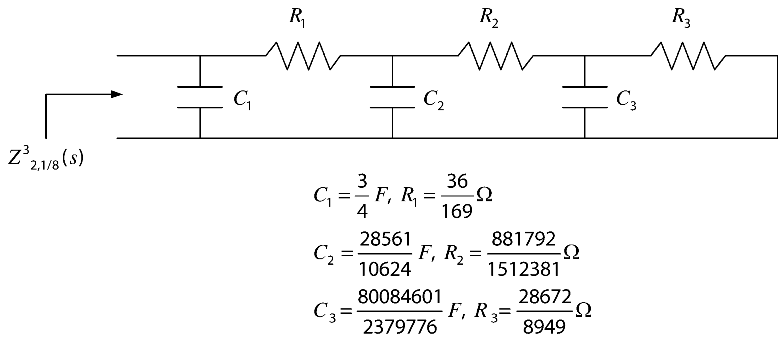

Fig. 2. *RC* network synthesis of driving point impedance $Z^3_{2,1/8}$.

is an exception where $E^1_{m,d}(z)$ degenerates to a constant. Thus, the proposition holds for $n = 1$. Now, define

$$P_n(z) \stackrel{\text{def}}{=} z^q \, \frac{E^n_{m,d}(z)}{(z-1)^{n+1}}, \tag{60}$$

where $q = 1 - \{f(n,m,d)\}$ as before. The finite zeros of the function $P_n(z)$ are those of $E^n_{m,d}(z)$ plus a zero at $z = 0$. We can obtain a recurrence for $P_n(z)$ by substituting (60) into (57). After a series of algebraic manipulations we have

$$\begin{aligned} P_0(z) &= \frac{z^q}{z-1} \\ P_n(z) &= -z\frac{d}{dz}P_{n-1}(z). \end{aligned} \tag{61}$$

Assume that the proposition is true for $n = n' - 1$. Hence, all finite zeros of $P_{n'-1}(z)$ are real, negative (except for the one at $z = 0$), and distinct. Then, by Rolle's theorem $(d/dz)P_{n'-1}(z)$ has a real, negative zero between every two real, negative zeros of $P_{n'-1}(z)$. Two additional zeros also exist. One is between the smallest zero of $(d/dz)P_{n'-1}(z)$ and $-\infty$. This is true because $\lim_{z\to-\infty} P_{n'-1}(z) = 0$. The other is between $z = 0$ and the largest zero of $(d/dz)P_{n'-1}(z)$. Hence, from (61), $P_{n'}(z)$ and $(d/dz)P_{n'-1}(z)$ share those zeros. We conclude that $E^{n'}_{m,d}(z)$ also vanishes only at those real, distinct and negative zeros. The interlacing properties of the zeros is evident from the argument on the relation between the zeros of $P_{n'-1}(z)$ and those of $(d/dz)P_{n'-1}(z)$.

D. B-Spline Kernel and Circuit Theory

The preceding results on the location of the zeros of the successive spline kernels may be viewed from a point of view that is very familiar to electrical engineers. Let us replace the indeterminate z with the variable s and write the kernel as $E^n_{m,d}(s)$. Assume that s is a complex variable and may take on any given value on the complex plane. It is known that a rational function $Z(s)$ is realizable as the driving point impedance function of an *RC* network if its poles and zeros are simple and alternate on the negative real axis with the nearest to the origin being a pole [13], [14]. Evidently, from the results of this section, a function of the form

$$Z^n_{m,d}(s) = \frac{E^{n-1}_{m,d}(s)}{E^n_{m,d}(s)} \tag{62}$$

can always be realized as the driving point function of an *RC* network. As an example, consider the synthesis for the rational function

$$Z^3_{2,1/8}(s) = \frac{E^2_{2,1/8}(s)}{E^3_{2,1/8}(s)} = \frac{\frac{9}{16}\,s^2 + \frac{11}{8}\,s + \frac{1}{16}}{\frac{27}{64}\,s^3 + \frac{235}{64}\,s^2 + \frac{121}{64}\,s + \frac{1}{64}}. \tag{63}$$

The coefficients of the spline kernels above have been read off (47). This is possible because the coefficients of both kernels are determined by the same values of p and q, i. e.,

$$\left\{f\left(3,2,\frac{1}{8}\right)\right\} = \left\{f\left(2,2,\frac{1}{8}\right)\right\} = \frac{3}{4} = p, \quad q = 1-p = \frac{1}{4}. \tag{64}$$

This function can be expressed by a continued fraction expansion. The result is given by

$$Z^3_{2,1/8}(s) = \cfrac{1}{\frac{3}{4}s + \cfrac{1}{\frac{36}{169} + \cfrac{1}{\frac{28\,561}{10624}s + \cfrac{1}{\frac{881\,792}{1\,512\,381} + \cfrac{1}{\frac{80\,084\,601}{2\,379\,776}s + \frac{1}{\frac{28\,672}{8949}}}}}}}. \tag{65}$$

The corresponding realization is shown in Fig. 2. Hence, the zeros of $E^2_{2,1/8}(s)$ and $E^3_{2,1/8}(s)$ are real, negative, and alternate.

E. Special Values

We can use the recurrence (57) to study the frequency response of $E^n_{m,d}(z)$ at the zero (dc) frequency and its value at $z = 0$. From (57), we have

$$E^n_{m,d}(1) = nE^{n-1}_{m,d}(1). \tag{66}$$

This combined with the boundary condition (58), reveals that

$$E^n_{m,d}(1) = n!. \tag{67}$$

Interestingly, the right-hand side does not depend on m or d. By using the L'Hospital's rule, for the overall transfer function, we have

$$H^n_{m,d}(1) = \frac{1}{n!m^n}E^n_{m,d}(1)m^{n+1} = m. \tag{68}$$

Thus, the dc response of the filter depends solely on the value of m and may be normalized to 1 easily.

In the same way, we can use (57) to show that

$$E_{m,d}^n(0) = (1 - \{f(n,m,d)\})E_{m,d}^{n-1}(0). \tag{69}$$

Then, from (58) we conclude that

$$E_{m,d}^n(0) = (1 - \{f(n,m,d)\})^n \tag{70}$$

that is the explicit value of the constant term of the polynomial $E_{m,d}^n(z)$.

IV. Bernstein Form of $E_{m,d}^n(z)$

We have completely characterized the power-form representation of $E_{m,d}^n(z)$ in Section II. A finite-impulse response (FIR) transfer function has infinitely many representations corresponding to particular selections of the basis functions adopted for its representation. For a transfer function of degree n, besides the set $\{1, z, \ldots, z^n\}$ of $n+1$ power-form basis functions, we may adopt $n+1$ Bernstein basis functions

$$\left\{\binom{n}{0}(1-z)^n, \binom{n}{1}z(1-z)^{n-1}, \ldots, \binom{n}{n-1}z^{n-1}(1-z), \binom{n}{n}z^n\right\}.$$

This set is of particular interest in approximation theory [15]. The scaled form of the Bernstein basis functions given by

$$\{(1-z)^n, z(1-z)^{n-1}, \ldots, z^{n-1}(1-z), z^n\}. \tag{71}$$

In digital signal processing, such representations may simplify the implementation of the resulting system by developing structures based on linear combination of signals obtained by shifting (z^i) and differencing $((1-z)^i)$ the input signal successively. In the following, we derive an expression for $E_{m,d}^n(z)$ based on (71). Besides an academic interest in obtaining a Bernstein-form representation, our results may also be used to study implementation issues. From the recurrence (61) derived in Section III for the function $P_n(z)$ defined by (60), we may write

$$P_n(z) = \left(-z\frac{d}{dz}\right)^n P_0(z) = \left(-z\frac{d}{dz}\right)^n \frac{z^q}{z-1}. \tag{72}$$

Thus, the succinct expression

$$E_{m,d}^n(z) = z^{-q}(1-z)^{n+1}\vartheta^n \frac{z^q}{1-z} \tag{73}$$

is obtained. The operator

$$\vartheta = z\,D = z\,\frac{d}{dz} \tag{74}$$

is a differential operator that acts on a function by differentiating it and then multiplying by z. The expression (73) is a Rodrigues-type representation (operator representation) of $E_{m,d}^n(z)$. We can use (73) to obtain an expression for $E_{m,d}^n(z)$ of the Bernstein form. Here, a Bernstein-form representation of the polynomial $E_{m,d}^n(z)$ is meant to be an expression using the basis functions given by (71) combined in the form

$$E_{m,d}^n(z) = \sum_{0\le l\le n} b_l z^l (1-z)^{n-l} \tag{75}$$

where b_l are the coefficients. Our objective is to characterize the coefficients b_l. For results on general methods for converting a polynomial from the power form to the Bernstein form see [16].

The n-fold operator ϑ^n may be expanded as [12]

$$\vartheta^n = \sum_{0\le k\le n} \left\{ {n \atop k} \right\} z^k D^k \tag{76}$$

where $\left\{ {n \atop k} \right\}$ are the Stirling numbers of the second kind [12]. Also recall from the elementary differential calculus that

$$D^k(uv) = \sum_{0\le l\le k} \binom{k}{l} D^l u D^{k-l} v \tag{77}$$

where u and v are functions of z. Thus, from (73), (76), and (77), and after some routine algebraic manipulations, the Rodrigues-type formula (73) reveals itself as an expression of the form

$$E_{m,d}^n(z) = \sum_{0\le l\le n}\sum_{l\le k\le n} \left\{ {n \atop k} \right\} \binom{k}{l} l!\, q^{\underline{k-l}} z^l (1-z)^{n-l}. \tag{78}$$

Thus, the coefficients of the Bernstein-form representation of the transfer function become

$$b_l = \sum_{l\le k\le n} \left\{ {n \atop k} \right\} \binom{k}{l} l!\, q^{\underline{k-l}}, \quad l = 0, 1, \ldots, n. \tag{79}$$

The quantity $q^{\underline{k-l}}$ is called "q to the $k-l$ falling." The underline signifies a falling factorial power defined by

$$x^{\underline{i}} = x(x-1)\cdots(x-(i-1)). \tag{80}$$

From Section III, we already know that the values of b_l at the two end points of the Bernstein-form representation, i. e., $z=0$ and $z=1$, are given by

$$b_0 = E_{m,d}^n(0) = q^n, \quad b_n = E_{m,d}^n(1) = n!. \tag{81}$$

V. Conclusion and Remarks

The digital filtering approach to the problem of B-spline signal interpolation was advanced by Unser *et al.* [10]. This paper provides a comprehensive study of the B-spline filters by unifying and refining the results of [1]. The two B-spline filters $B_1^n(z)$ and $C_1^n(z)$ that are the z-transforms of the symmetrical B-spline $\beta^n(k)$ and its shifted version $\beta^n(k+1/2)$ have been treated by a single transfer function $H_{m,d}^n(z)$ that is the z-transform of $\beta^n(k/m-d)$. The filter $H_{m,d}^n(z)$ is a cascade of a kernel part $E_{m,d}^n(z)$ and a moving average part. The impulse response coefficients of $E_{m,d}^n(z)$ have been characterized by a single recurrence relation. By using this recurrence, we can avoid moving between two mutually dependent recurrences to compute the values of the coefficients of $B_1^n(z)$ or $C_1^n(z)$, as is the case for the relations given in [1]. A recurrence has also been derived for the entire kernel transfer function as well. These recurrences enable us to readily design B-spline digital filters for arbitrary values of d. Such filters may be used as fractional delay filters to delay a discrete spline signal by a fractional number of samples. For instance, in order to delay the spline signal $s(k)$ of order n by d samples and then upsample the result by a factor of m, we can use the digital filter pairs $H_{m,d}^n(z)$ and $H_{1,0}^n(z)$ as shown in Fig. 1(b). We have also studied the location of the zeros of $E_{m,d}^n(z)$. It has been proved that the zeros are all real, simple, negative and interlaced by those of $E_{m,d}^{n+1}(z)$. We have shown that the kernel filters whose parameters satisfy $\{f(n,m,d)\} = 0$ or $\{f(n,m,d)\} = 1/2$ possess symmetric impulse response coefficients and are thus guaranteed to have zeros both outside and inside the unit circle. It follows that the corresponding reciprocal infinite-impulse response filters are not stable. However, if the input signal is of finite length, as is the case in digital image processing, there is no stability problem and the filters may be implemented by two successive passes of stable filters [2], [10].

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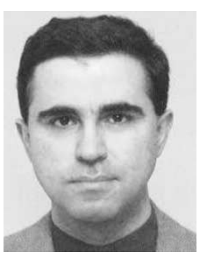

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