The Chirp-Wigner Transform

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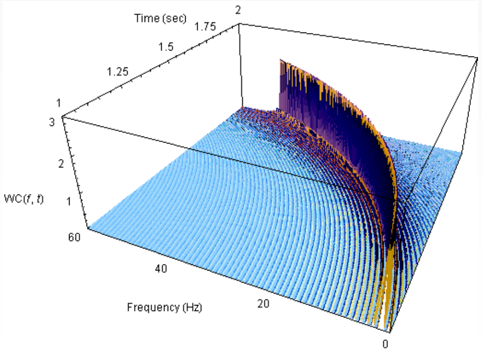
New Time-Frequency Transform for Non-Stationary Signals with Any Nonlinear Polynomial Variation of the Instantaneous Frequency



Background

- The Wigner-Ville distribution is an effective time-frequency transform for signals with only linear change of the instantaneous frequency (i.e. the chirps)
- It is not suitable for non-stationary signals with nonlinear change of the instantaneous frequency (i.e. the higher order chirps).
- An effective time-frequency transform is required to handle the higher order chirps.

Problem Statement

- **Normally, for transient vibro-acoustical signals from complex mechanical systems (e.g. turbines, gearboxes, etc.) time variations of the instantaneous frequency are *known* from independent synchronous measurements (e.g. from tachometer signals).**

- **Sometimes, these frequency variations even are known *a priori*.**

- **Therefore, the main task for these mechanical systems is *amplitude estimation* for transient signals (on the background of an interference) with *known* nonlinear time variation of the instantaneous frequency.**

Signals with the Polynomial Variation of the Instantaneous Frequency

Signal

$$x(t) = A\exp\left(j2\pi\theta(t)\right) = A\exp\left(j2\pi\left(f_0 t + \mathrm{V}(t)\right)\right)$$

$$V(t) = (c_{20}t^2/2) + (c_{30}t^3/3) + (c_{40}t^4/4) + \ldots + (c_{n0}t^n/n)$$

The instantaneous frequency

$$f_i(t) = \frac{d}{dt}\left(f_0 t + \mathrm{V}(t)\right) = f_0 + c_{20}t + c_{30}t^2 + c_{40}t^3 + \ldots + c_{n0}t^{n-1}$$

The initial frequency

The chirp rate

The frequency acceleration

The Chirp-Wigner Transform

The new transform based on the Wigner distribution is proposed by the author for signals with the polynomial variation of the instantaneous frequency

$$W_c(f,t) = \int_{-\infty}^{\infty} x^*(t-\frac{\tau}{2})x(t+\frac{\tau}{2})e^{-j2\pi[f\tau+\frac{c_2(t)}{2}\tau^2+\frac{c_3(t)}{3}\tau^3+\frac{c_4(t)}{4}\tau^4+\ldots+\frac{c_n(t)}{n}\tau^n]}d\tau$$

$x(t)$ – *signal*

$\frac{c_2(t)}{2}$ – *frequency speed*

$\frac{c_3(t)}{3}$ – *frequency.acceleration*

$\frac{c_N(t)}{n}$, *for* $n \geq 4$ – *higher order parameters*

The Chirp-Wigner Transform

The transform is real valued

Kernel coefficients:

matching conditions

$$c_{2k+1}(t) = \frac{V^{(2k+1)}(t)}{2^{2k}(2k)!} \quad , k = 1,\ldots,r$$

and $r = [(n-1)/2]$

$$c_3(t) = \frac{3c_{40}t + c_{30}}{4}, \; c_2(t) = c_4(t) = \ldots = c_n(t) = 0 \qquad n = 4$$

The Wigner Distribution and the Chirp-Wigner Transform

Let's assume that the higher order chirp is of order 4 and time variation of the instantaneous frequency is *known* from a frequency sensor

The Wigner distribution

$$W\ (t,f) = A^2 \int_{-\infty}^{\infty} \exp\{-j2\pi\tau[f - f_i(t)]\} \exp\left\{-j2\pi\left[(-\frac{c_{30}}{12} - \frac{c_{40}}{4}t)\tau^3\right]\right\} d\tau$$

The chirp-Wigner transform

$$W_c(t,f) = A^2 \int_{-\infty}^{\infty} \exp\{-j2\pi\tau[f - f_i(t)]\} \exp\left\{-j2\pi\left[\frac{c_2(t)}{2}\tau^2 + (\frac{c_3(t)}{3} - \frac{c_{30}}{12} - \frac{c_{40}}{4}t)\tau^3 + \frac{c_4(t)}{4}\tau^4 + \ldots + \frac{c_N(t)}{N}\tau^N\right]\right\} d\tau$$

The Wigner Transform

- **It can be seen that the traditional Wigner distribution of the signal under consideration has the *non-unity time-varying* multiplier defined by the second exponential function in the integral**

- **This multiplier depends on time, the frequency acceleration and the higher order parameter of the signal.**

- **One can find that**

$$W(t, f) \neq A^2 \int_{-\infty}^{\infty} \exp\{-j2\pi\tau[f - f_i(t)]\} d\tau = A^2 \delta[f - f_i(t)]$$

The Chirp-Wigner Transform (CWT)

- It can be seen from equation of the CWT that the proposed transform also has the time-varying multiplier defined by the second exponential function in the integral
- However, one can find from expression of the CWT that if the following adaptation conditions

$$c_3(t) = \frac{3c_{40}t + c_{30}}{4}, \; c_2(t) = c_4(t) = \ldots = c_N(t) = 0$$

apply for the chirp-Wigner transform, then this multiplier is equal to unity at any time and parameters and, hence, the new transform with adaptation conditions can be written as follows:

$$W_c(t, f) = A^2 \delta\left[f - f_i(t)\right]$$

Property of the New Transform

It can be seen that the chirp-Wigner transform with the *adaptation conditions* ideally follows the nonlinear polynomial frequency variation of the signal without amplitude errors.

Properties of the New Transform

- The transform kernel is matched to the instantaneous phase of the signal

- The transform is adaptive

- The transform is quadratic in the signal, and, hence non-linear.

- Kernel parameters are time dependent

Adaptation of the New Transform

- The new transform is adaptive; this is the major advantage over the Wigner-Ville distribution.

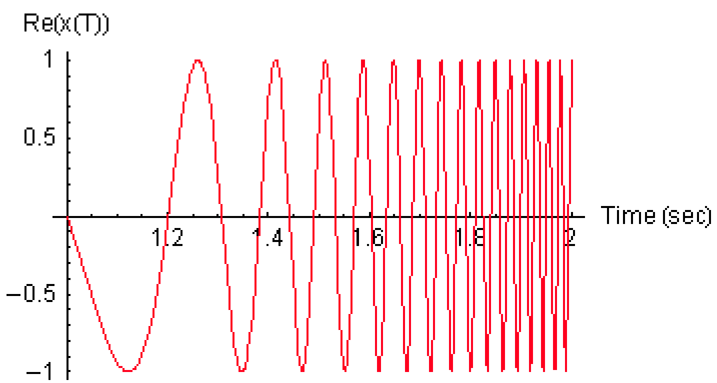
- Adaptive amplitude estimation of the higher-order chirp could be carried out using a two-step procedure:

  - independent estimation of the time dependency of the instantaneous frequency
  - use of the chirp-Wigner transform with the adapted kernel coefficients

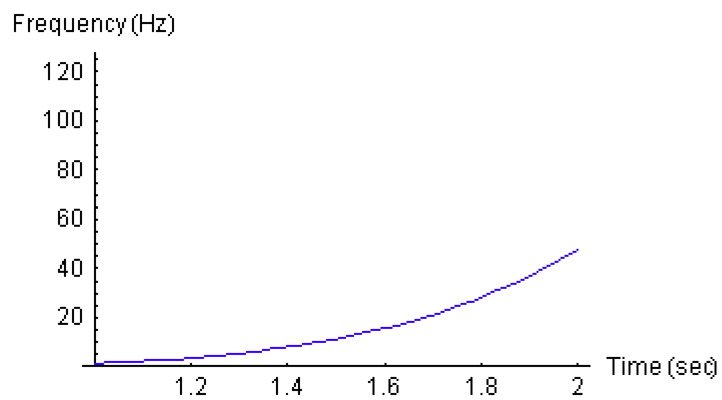
Case Study

Signal $x(t) = \exp\left(j2\pi t^6/4\right)$

The instantaneous frequency $f_i(t) = 6t^5/4$

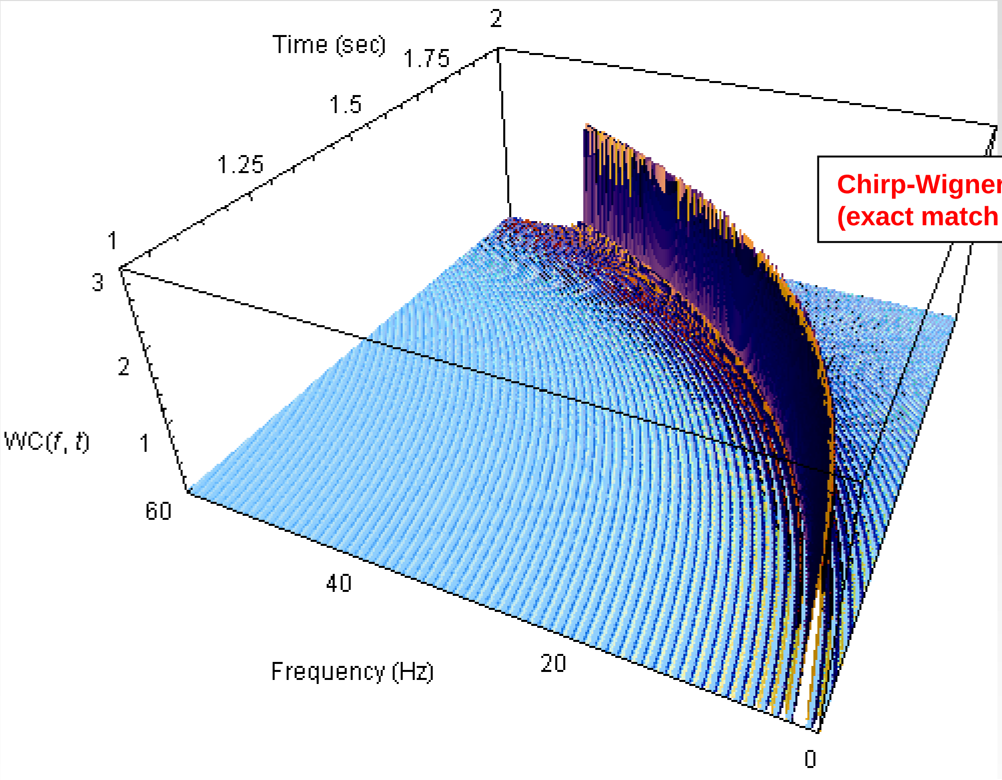


Signal



The instantaneous frequency

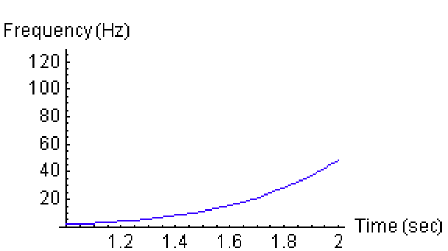
The Chirp-Wigner Transform



Chirp-Wigner transform (exact match of kernel)

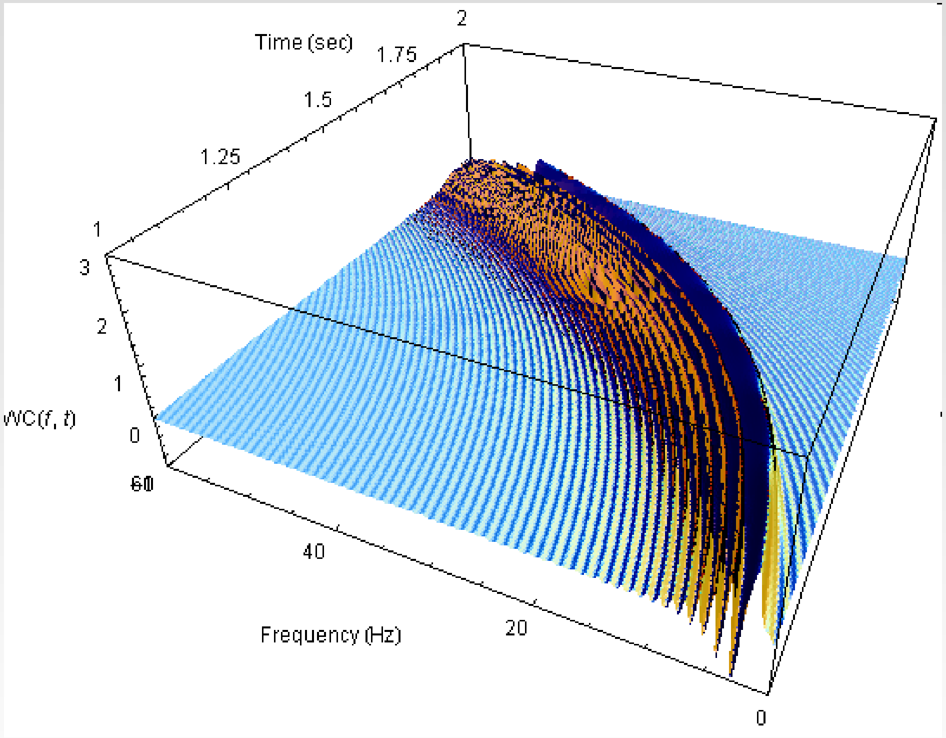
Signal $x(t) = \exp\left(j2\pi t^6/4\right)$

The instantaneous frequency of the signal



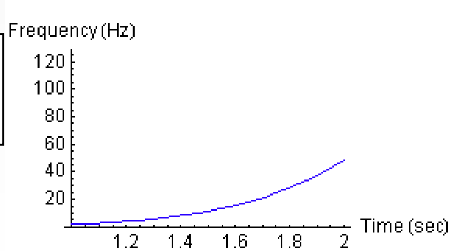
The chirp-Wigner transform ideally follows the instantaneous frequency without amplitude errors

The Wigner Transform



Signal $x(t) = \exp\left(j 2\pi t^6/4\right)$

The instantaneous frequency of the signal



The Wigner transform follows the instantaneous frequency with essential amplitude errors

The Discrete Chirp-Wigner Transform: Finite Signal

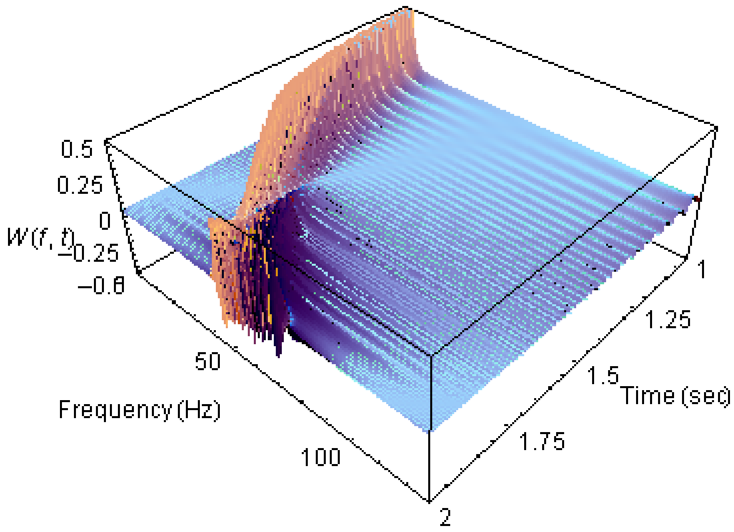
$$W_{cd}(n,r) = \frac{2}{N}\sum_{k=0}^{N-1} x^*(n-k)x(n+k)\exp\left(-j2\pi\left[\frac{2rk}{N}+\frac{c_3(n)}{3}(2k)^3+\ldots+\frac{c_{2r+1}(n)}{2r+1}(2k)^{2r+1}\right]\right)$$

$$W_{cd}(n,r/2) = DFT_N\left(2x^*(n-k)x(n+k)\exp\left(-j2\pi\left[\frac{c_3(n)}{3}(2k)^3+\ldots+\frac{c_{2r+1}(n)}{2r+1}(2k)^{2r+1}\right]\right)\right)$$

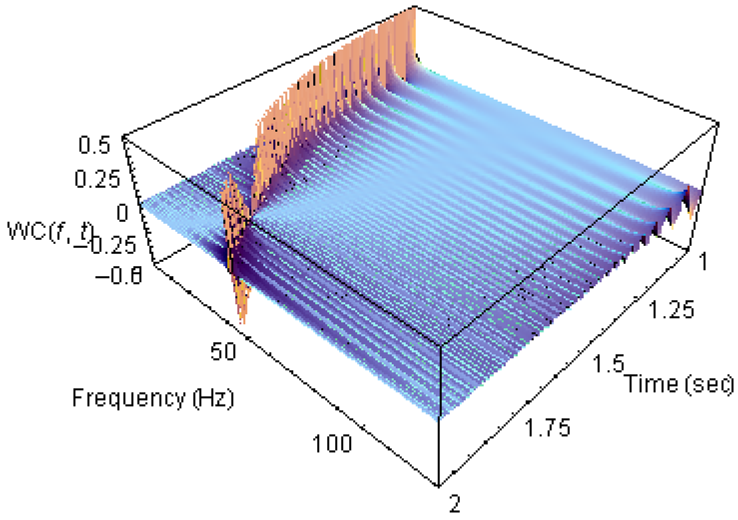
The DFT/FFT could be employed for estimation of the discrete chirp-Wigner transform.

The Chirp-Wigner Transform vs. the Wigner Transform

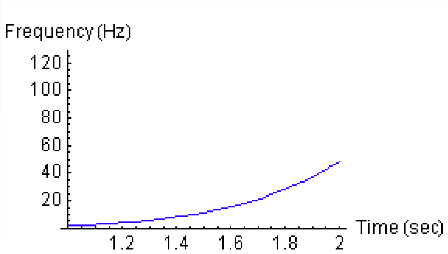
Discrete Wigner-Ville



Discrete chirp-Wigner



Signal $x(t) = \exp\left(j 2 \pi t^{6} / 4\right)$



The Chirp-Wigner Transform vs. the Wigner Transform

- **For signals with nonlinear variation of instantaneous frequency, the chirp-Wigner transform tracks the instantaneous amplitude better than the Wigner distribution.**

- **The new transform is a radical improvement over the traditional Wigner distribution.**