The Wavelet Transform

Part 2

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Dependency between Scale and Frequency

- **How to convert the *scale* dependent wavelet transform to a *frequency* dependent wavelet transform?**

- **We have proved earlier that spectral components are *inversely* proportional to the scale**

Hence

$$f = \frac{f_{ch}}{a}$$

where f_{ch} **is a wavelet characteristic frequency**

Wavelet Characteristic Frequencies

- One of the commonly used characteristic frequencies is the ***passband centre*** of the wavelet's power spectral density

- The normalized second moment of the wavelet's power spectral density $|\Psi(f)|^2$ is used to define the ***passband centre*** as follows

$$f_c = \sqrt{\frac{\int_0^\infty f^2 |\Psi(f)|^2 df}{\int_0^\infty |\Psi(f)|^2 df}}$$

Wavelet Characteristic Frequencies

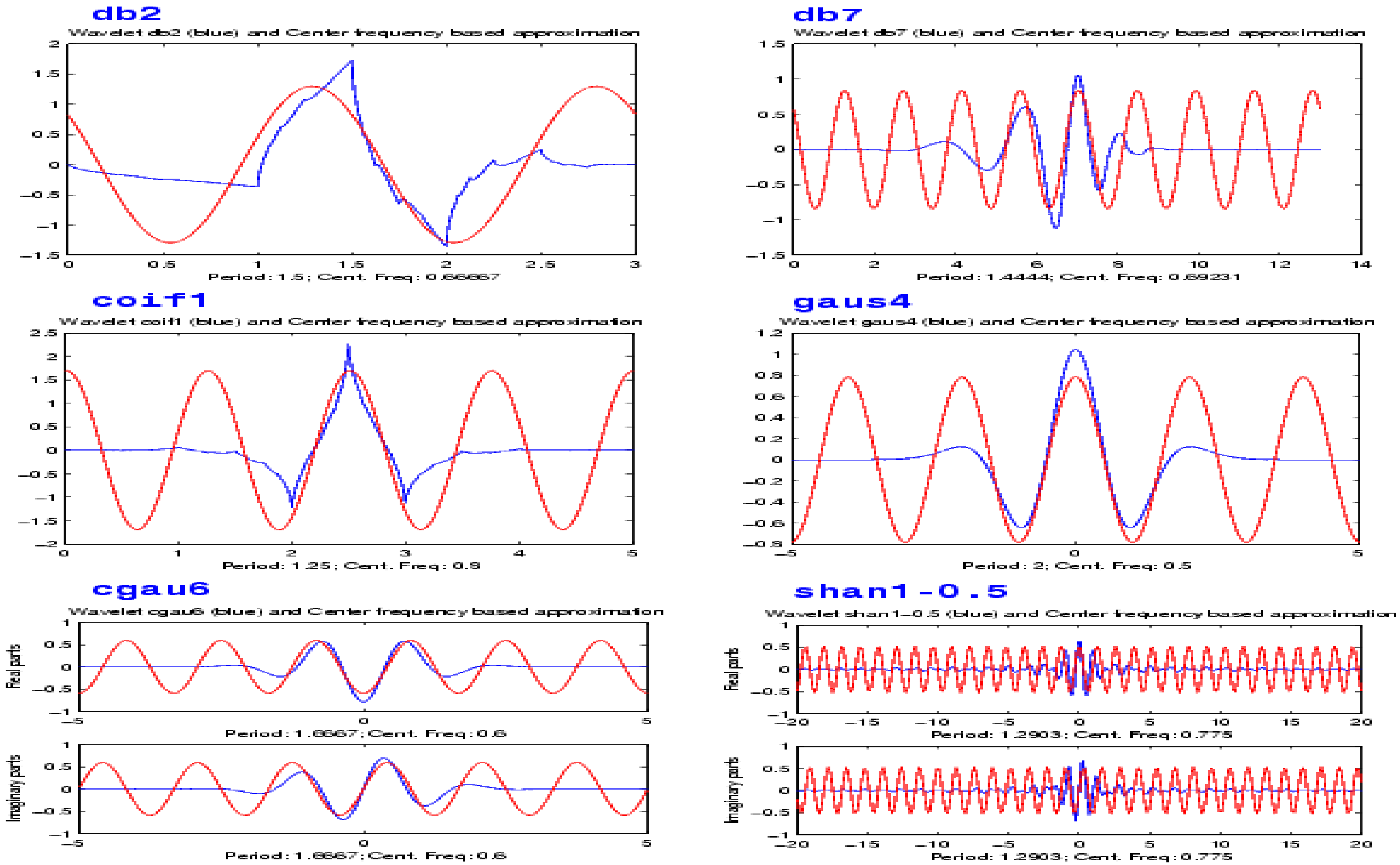
- **Another characteristic frequency of the wavelet such as *centre frequency*, i.e. the frequency maximizing the power spectral density of the wavelet is would be equally valid**

- **The MATLAB function *centfrq* can be used to compute the centre frequency and it allows the plotting of the wavelet with the associated approximation based on the centre frequency**

❑ **We show such plots for**

- **four real wavelets: Daubechies wavelets of order 2 and 7, coiflet of order 1, and the Gaussian derivative of order 4**
- **two complex wavelets: the complex Gaussian derivative of order 6 and a Shannon complex wavelet**

Wavelets and Centre Frequencies



Wavelets and Centre Frequencies

- **As you can see, the centre frequency-based approximation captures the main wavelet oscillations**

- **Thus, the centre frequency is a convenient and simple characterization of the leading dominant frequency of the wavelet**

Mother Wavelet: Properties

- **It is worthwhile to note that if we only want to analyse the signal and do *not want to recover* the original signal based upon the transformations, then the mother wavelet *could be any function we like***

- **However, when reconstruction is needed, the selection of the mother wavelet is restricted**

- **In order to perform a signal reconstruction, function must satisfy the following criteria:**

a wavelet must have a finite energy, i. e.

$$E = \int_{-\infty}^{\infty} |\psi(t)|^2 \, dt < \infty$$

Mother Wavelet: Admissibility

- **The following condition must hold**

$$C_g = \int_0^{\infty} \frac{|\Psi(f)|^2}{f} df < \infty$$

- **This implies that the wavelet has no zero frequency component, or, the wavelet must have a *zero mean***

- **This equation is known as the *admissibility condition* and C_g is called *admissibility constant*.**
- **The value of C_g depends on the chosen wavelet**

- **Wavelets satisfying the *admissibility condition* are in fact *bandpass filters (recall the wavelet transform as a convolution!)***

Mother Wavelet: Candidates?

- **What functions are candidates to be a wavelet?**

- **If a function:**

- **is continuous,**
- **has zero mean,**
- **decreases quickly towards 0 when time tends towards infinity, or is null outside a time segment,**

it is a likely candidate to become a wavelet

Wavelet Transform: Properties

- **The wavelet transform is a *linear* representation of a signal**
- **It means that the transform does not create any interferences**
- **This is convenient for the analysis of multicomponent signals**
- **The wavelet transform is *complex-valued* in general since the mother wavelet is generally complex-valued.**
- **By using complex wavelets, we can separate the phase and amplitude components within the signal**

Wavelet Transform: Conservation of Energy

- **The total energy of the signal can be expressed in terms of the wavelet transform:**

$$E_w = \frac{1}{C_g} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| C(a,b) \right|^2 \frac{da\,db}{a^2}$$

- **This allows to interpret the square of the modulus of the transform (scalogram) as a density of energy distribution over the time-frequency plane**

Wavelet Transform: Conservation of Energy

- **The relative contribution to the total energy contained within the signal at a specific scale is given by the scale dependent energy distribution:**

$$E(a) = \frac{1}{C_g} \int_{-\infty}^{\infty} \left| C(a,b) \right|^2 db$$

Peaks in $E(a)$ highlight the dominant energetic scales within the signal

Resolution Properties

- **The local resolutions of the wavelet transform in time and frequency is determined by the standard deviations of wavelet functions in time** Δt_{ψ} **and frequency** Δf_{ψ} **domains given by**

$$\Delta t = a \Delta t_{\psi}$$

$$\Delta f = \frac{\Delta f_{\psi}}{a}$$

❑ **It is seen that**

- **in the frequency domain the wavelet transform has good resolution a low frequencies (when** $a \geq 1$ **)**
- **in the time domain good resolution at high frequencies (when** $a \leq 1$ **), the latter being suitable for transient signal detection**

Resolution: Uncertainty Principle

- **Wavelets offer different time-frequency resolution compared with other time-frequency methods (e.g. short time Fourier transform, Wigner distribution)**

- **However, the products of the corresponding time and frequency resolutions remain constant**

$$\Delta f \Delta t = \Delta t_{\psi} \Delta f_{\psi}$$

which implies that the wavelets also obey the Heisenberg uncertainty principle

Wavelet: Constant Q Analysis

- **Unlike the short time Fourier transform, for the wavelets the ratio between frequency resolution and mean frequency is constant and independent of the scale factor, e.g.**

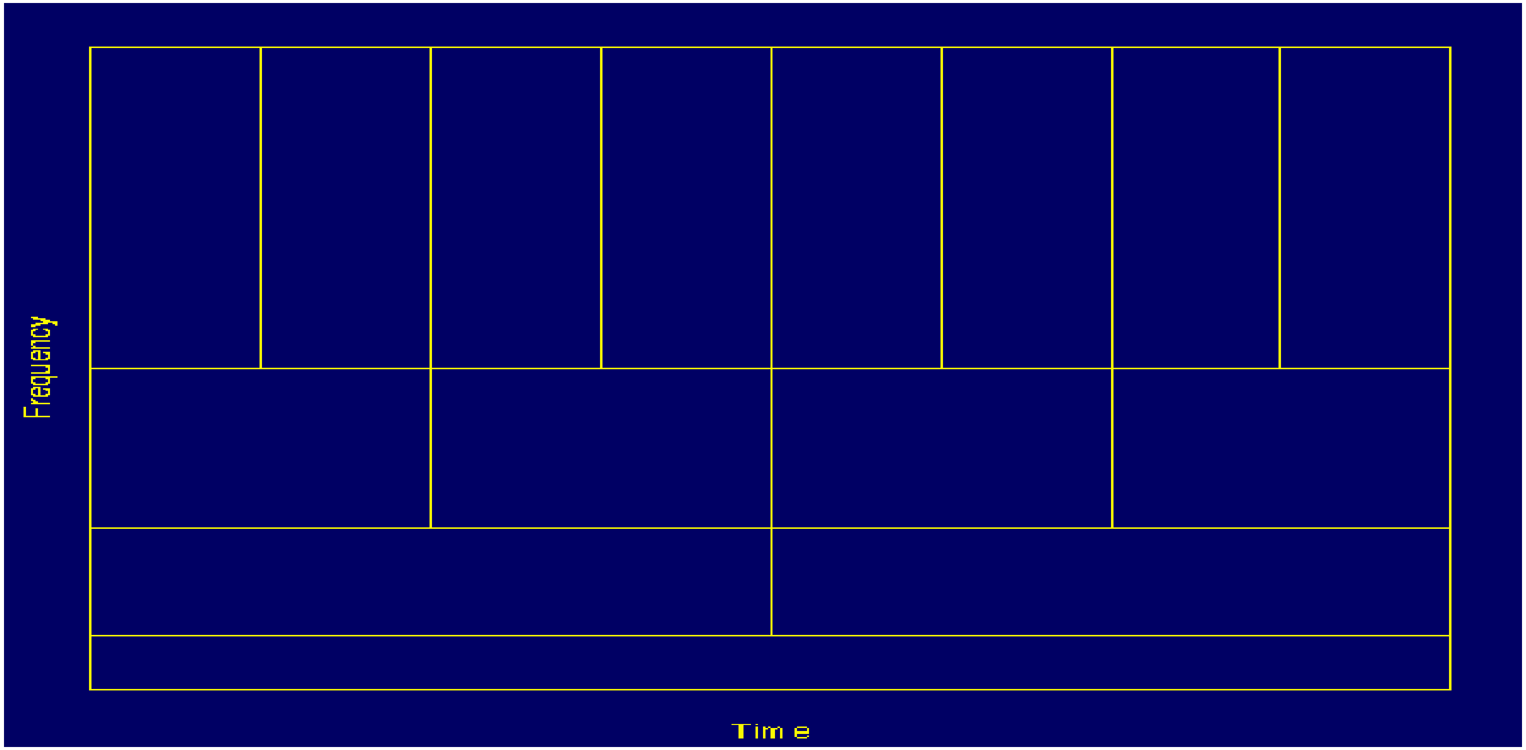
$$Q = \frac{\Delta f}{f_m} = \frac{\Delta f_\psi}{f_{ch}}$$

- **Thus, the wavelets can be considered to be a constant Q analysis**

- **The class of such bandpass filters are called constant Q filters**
- **For certain signals such analysis is more appropriate than fixed bandwidth analysis offered by the short time Fourier transform**
- **Indeed, the human ear performs an analysis which is approximately constant Q**

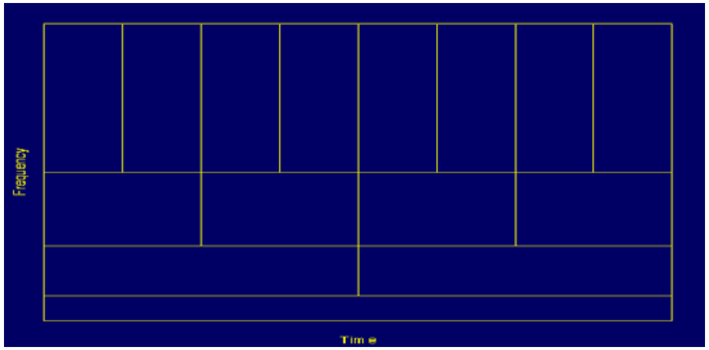
Time-Frequency Resolution

- **This illustration is commonly used to explain how time and frequency resolutions should be interpreted**



- **Each box (Heisenberg box) corresponds to a value of the wavelet transform in the time-frequency plane**

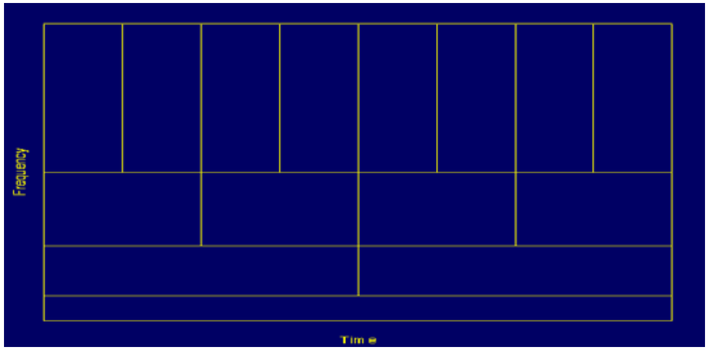
Time-Frequency Resolution



- **Although the widths and heights of boxes change, the area is *constant* (uncertainty principle)**

- **Each box represents an equal portion of the time-frequency plane, but giving *different proportions* to time and frequency**

Time-Frequency Resolution

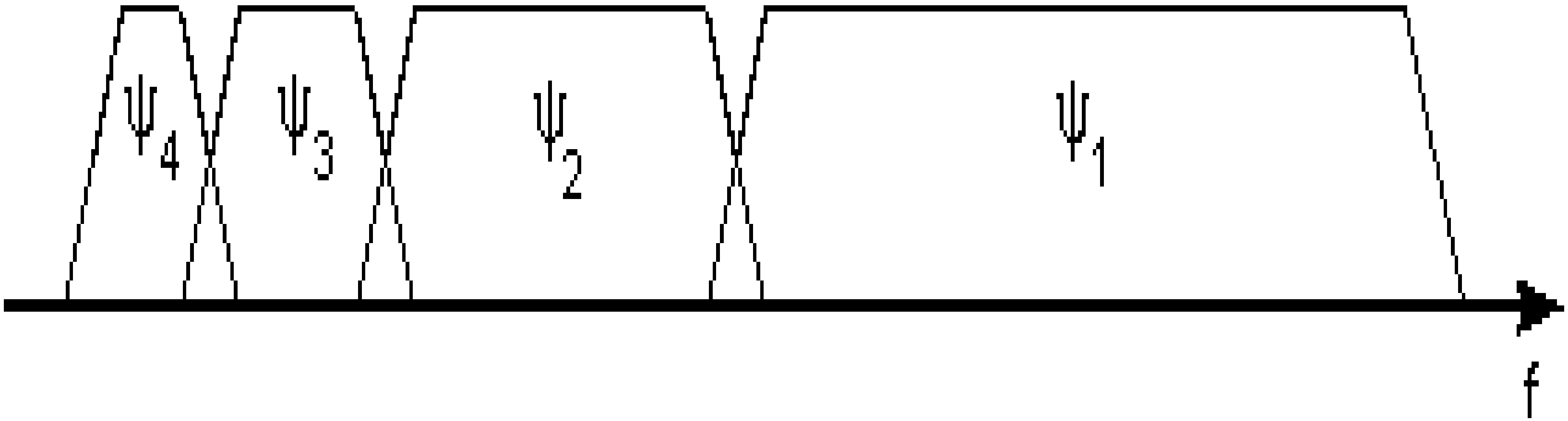


- At lower frequencies, the height of the boxes are shorter (which corresponds to better frequency resolutions), but their widths are longer (which correspond to poor time resolution)

- At higher frequencies the width of the boxes decreases, i.e., the better time resolution, and the heights of the boxes increase, i.e., poorer frequency resolution

Frequency Resolution: Filter Bank

- **A series of scaled wavelets can be seen as *a bandpass filter bank* One can see that the frequency bandwidth of wavelets increases as frequencies increase**



The Fourier Transform of Mother Wavelet

- **The Fourier transform of the normalized wavelet function is given by**

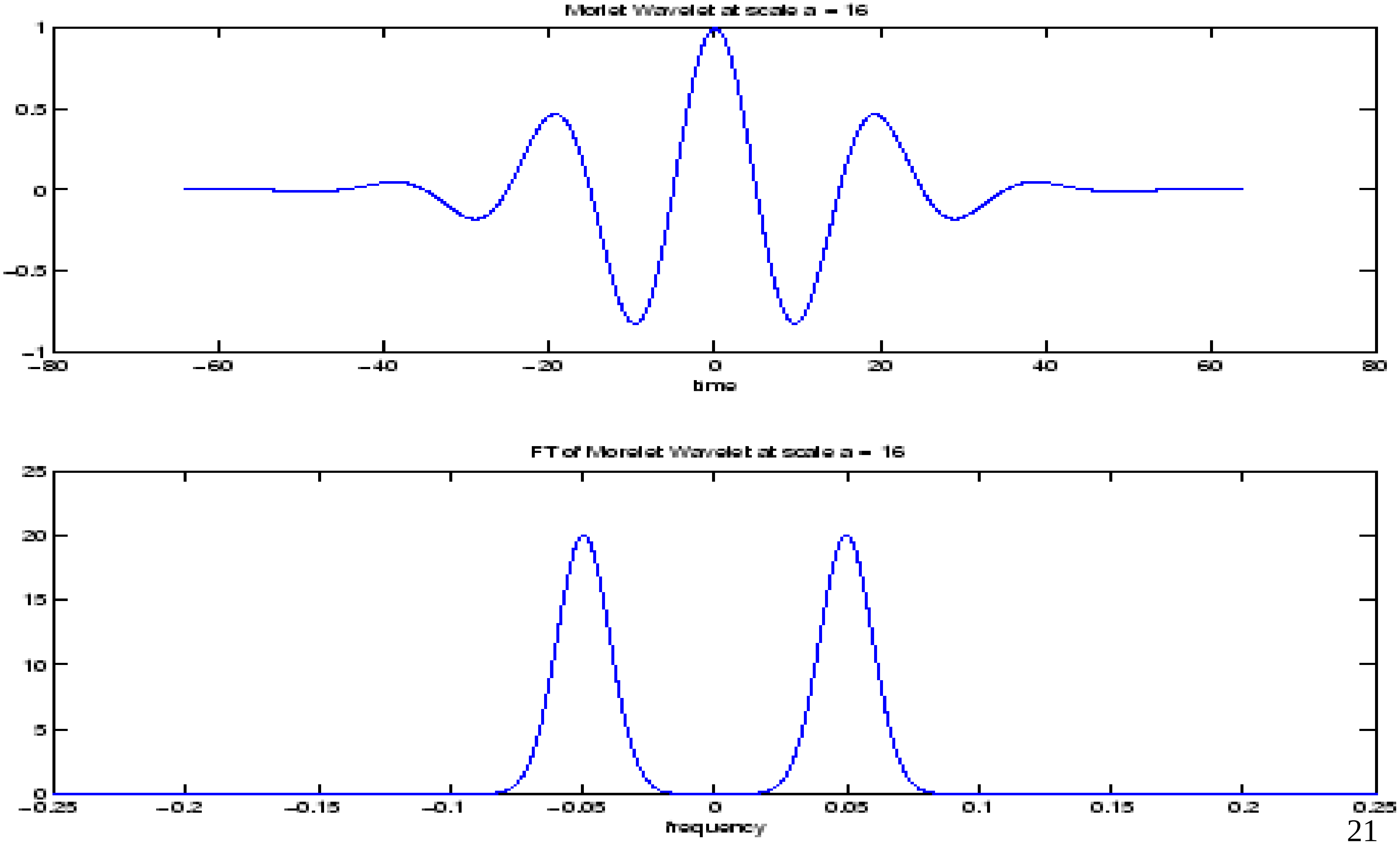
$$\Psi_{a,b}(f) = \frac{a}{\sqrt{|a|}} e^{ib2\pi f} \Psi(a2\pi f)$$

- **Since the function is assumed to be zero mean, we obtain**

$$\Psi_{a,b}(f=0) = \int_{-\infty}^{\infty} \psi\left(\frac{t-b}{a}\right) dt = 0$$

- **e.g. wavelet is *bandpass filter***

The Fourier Transform of Morlet Wavelet



Wavelet Transform Through Fourier Transform

- **The wavelet transform can be expressed through the Fourier transform:**

$$C(a,b) = \sqrt{|a|}\int_{-\infty}^{\infty} S(f)\Psi^*(af)e^{i2\pi fb}\,df$$

where $S(f)$ **is the Fourier transform of the signal**

- **The last equation has the form of an inverse Fourier transform**

- **This is useful result for discretized approximation of the continuous wavelet transform, as the fast Fourier transform (FFT) may be employed (for the DFT and IDFT) for *rapid calculation* of the wavelet transform**

Wavelet Transform Through Fast Fourier Transform

$$C(a,b) = \sqrt{|a|}\int_{-\infty}^{\infty} S(f)\Psi^*(af)e^{i2\pi fb}\,df$$

- **The Fourier transform of the wavelet function is usually known in analytic form and hence need not be computed using an FFT**

- **Only an FFT of the original signal is required**

- **Then, to get the wavelet transform, we take the inverse FFT of the product of the signal Fourier transform and the wavelet Fourier transform for each required scale and multiply the result by** $\sqrt{|a|}$

Wavelet Transform: a Tool for Local Analysis

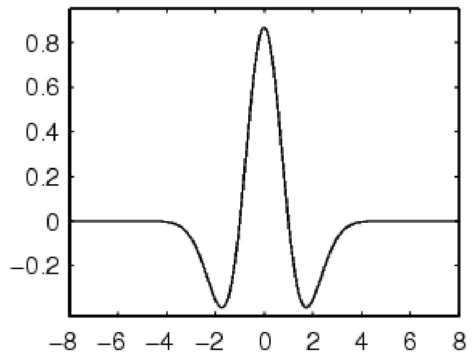
- **Major advantage afforded by wavelets is the ability to perform *local analysis* -- that is, to analyze a localized area of a signal**

- **Wavelet analysis is capable of revealing aspects of data that other techniques miss; aspects like trends, breakdown points, discontinuities in the signal and in its higher derivatives, and self-similarity**

Detection of Abrupt Changes (Edges)

- **A simple example of a discontinuity when a constant positive signal suddenly drops to a constant negative signa**

- **To see how the wavelet picks out such a discontinuity we follow a real and symmetric (for simplicity) wavelet, Mexican hat of arbitrary scale as it traverses the signal discontinuity**



Wavelet function psi

- **The effect of wavelet location on the transform is discussed for different locations on the signal**

Detection of Abrupt Changes (Edges)

- ***Location A* (*much earlier than the discontinuity*)**

- **The wavelet and the constant signal combine to give near-zero values of the integral for** $C(a,b)$

- **As it is a localized function, the wavelet becomes approximately zero at relatively short distances from its centre**

- **Hence the wavelet transform effectively becomes a convolution of the wavelet with a constant valued signal producing a zero value**

Detection of Abrupt Changes (Edges)

- ***Location B (the wavelet is just beginning to traverse the discontinuity)***

- **The left-hand lobe of the wavelet produces a negative contribution to the integral**

- **The right-hand lobe produces an equal positive contribution**

- **Leaving the central bump of the wavelet to produce a significant positive value for the wavelet integral at this location**

Detection of Abrupt Changes (Edges)

- ***Location C (the signal discontinuity coincides with the wavelet centre)***

- **The right and left halves of the wavelet contribute to a *zero* value of the wavelet integral**

Detection of Abrupt Changes (Edges)

- ***Location D (this location is similar to location B).***

- **As the wavelet traverses the discontinuity further, the left hand lobe of the signal produces a negative contribution to the wavelet integral**

- **The right hand lobe of the signal produces an equal positive contribution to the wavelet integral**

- **This time, however, the central bump of the wavelet coincides with the negative constant signal and hence the wavelet integral produces a significant negative value at this location**

Detection of Abrupt Changes (Edges)

- ***Location E (this location is similar to location A)*. The wavelet and signal combine to give near-zero values of the wavelet integral**

- **Hence, as the wavelet traverses the discontinuity there are first positive then negative values returned by the wavelet integral**

- **These values are localized *in the vicinity of discontinuity***

- **These values become more localized as the scale parameter reduces**

Detection of Abrupt Changes (Edges)

- **In summary, the main feature of the discontinuities in the signal is:**

- ***there is a localised increase in the scalogram for small values of the scale parameter around the time point where the discontinuity exists in the signal***

Sliding of the Haar Wavelet via Abrupt Change (Edge)

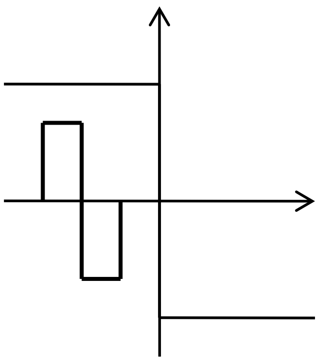


Figure 1

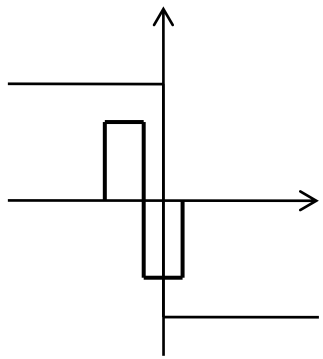


Figure 2

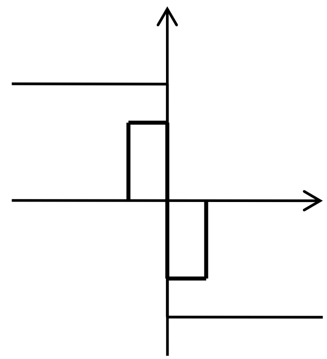


Figure 3

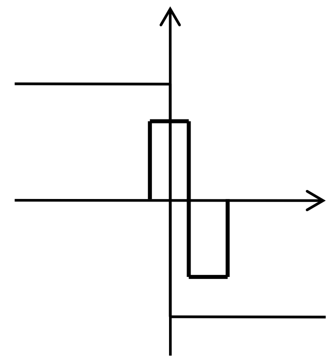


Figure 4

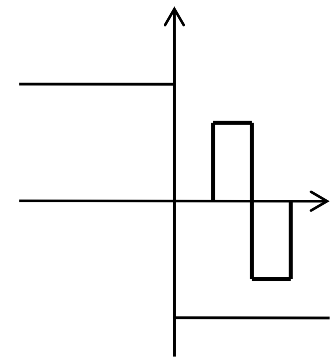


Figure 5

The Two-Dimensional Wavelet Transform

- **The two-dimensional wavelet transform is using two-dimensional mother wavelet:**

$$C(a,\vec{b}) = \frac{1}{a}\int_{-\infty}^{\infty} s(\vec{t})\psi^{*}\left(\frac{\vec{t}-\vec{b}}{a}\right)d\vec{t}$$

where $\vec{t}$ **is the coordinate vector** (t_1, t_2) **and** $\vec{b}$ **is time shift vector** (b_1, b_2)

- **The coordinate vector is normally used to specify two spatial coordinates, where the signal could be image, topographic feature, etc.**

The Two-Dimensional Wavelet Transform

- **The most common arrangement is to use the same mother wavelet in the horizontal and vertical directions, i. e.**

$$\psi\left(\frac{\vec{t}-\vec{b}}{a}\right)=\psi\left(\frac{t_1-b_1}{a}\right)\psi\left(\frac{t_2-b_2}{a}\right)$$

- **The corresponding inverse wavelet transform is:**

$$s(\vec{t})=\frac{1}{C_g}\int_a\int_{b_1}\int_{b_2}\psi\left(\frac{\vec{t}-\vec{b}}{a}\right)C(a,\vec{b})\frac{dad\vec{b}}{a^4}$$

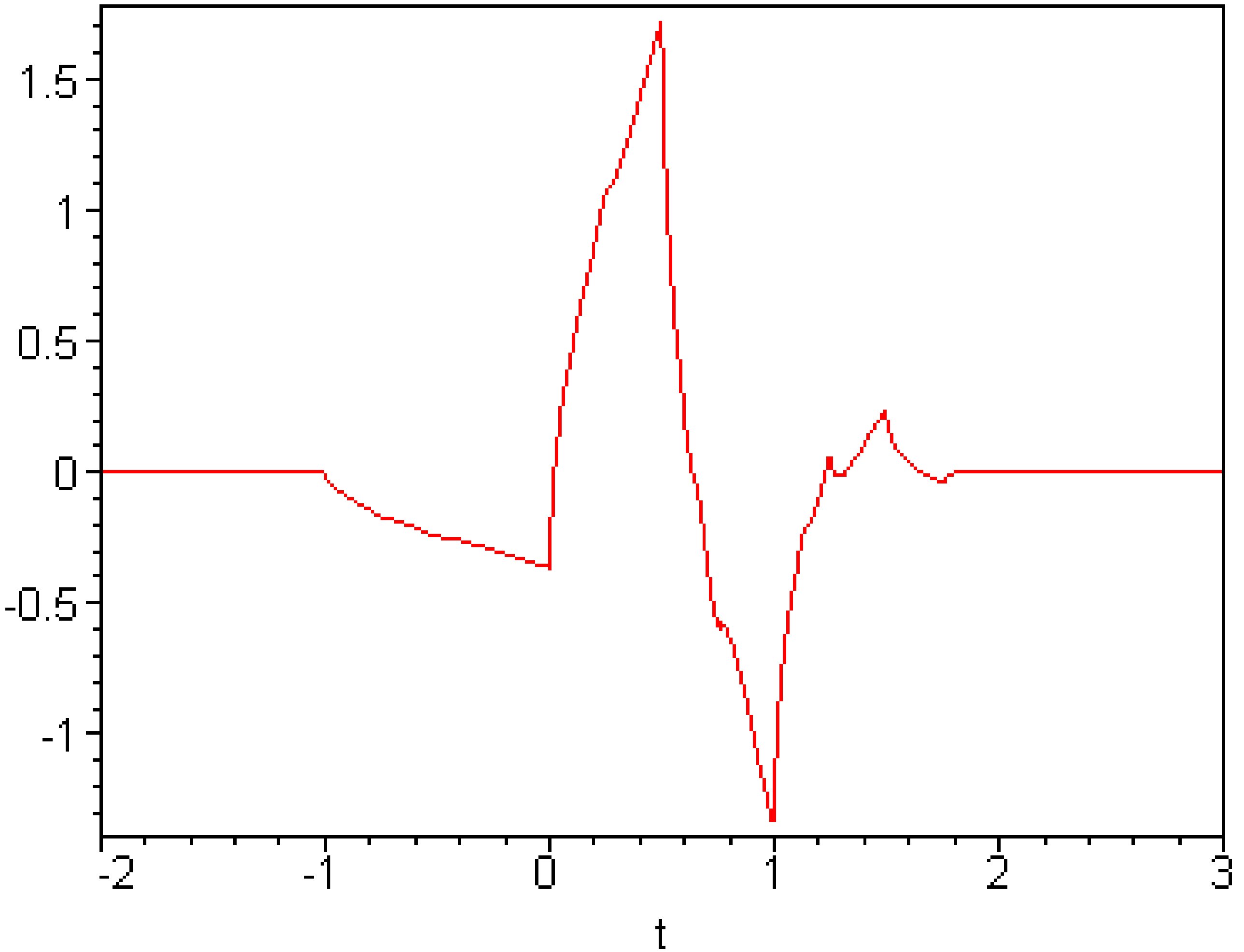
Wavelet Functions of Compact Support

- **Wavelets of a *compact support in time domain are as follows:* mother wavelets *are non-zero only on a finite interval***

- **Most of wavelet functions are of a compact support**

- **The compact support allows the wavelet transform to efficiently represent signals which have localized features**

Wavelet of Compact Support: db2 Wavelet Function

- 

Wavelets of Compact Support: Applications

- **The compact support is important in applications such as signal compression, signal detection and de-noising**

- **The common principle is that a structured component of a signal is well represented by a relative few of the wavelet coefficients, whereas, an unstructured component of a signal (e.g. noise) projects almost equally onto all wavelet coefficients**

- **The structured and unstructured parts of a signal are easily separated in the wavelet transform domain**

Wavelets of Compact Support

- **If a wavelet function is compactly supported in time domain, it is not band-limited in frequency domain.**

- **However, most compactly supported wavelets are designed to have a rapid fall-off; so, they can be considered as band-limited**

- **The larger wavelet compact support makes it less localized in the time domain and, hence, less able to detect local features in signal, including edge effects**

Wavelets of Compact Support

A short compact support allows:

- **to use *short* digital filters and to reduce computation complexity of the wavelet transform**
- **better time resolution**
- **poorer frequency resolution**