The Wavelet Transform

Part 4

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The Daubechies Wavelets

- **The Daubechies wavelets can be easy implemented by the finite impulse response (FIR) filters**

- **The Daubechies filters have the property that they are *maximally flat* in passband**

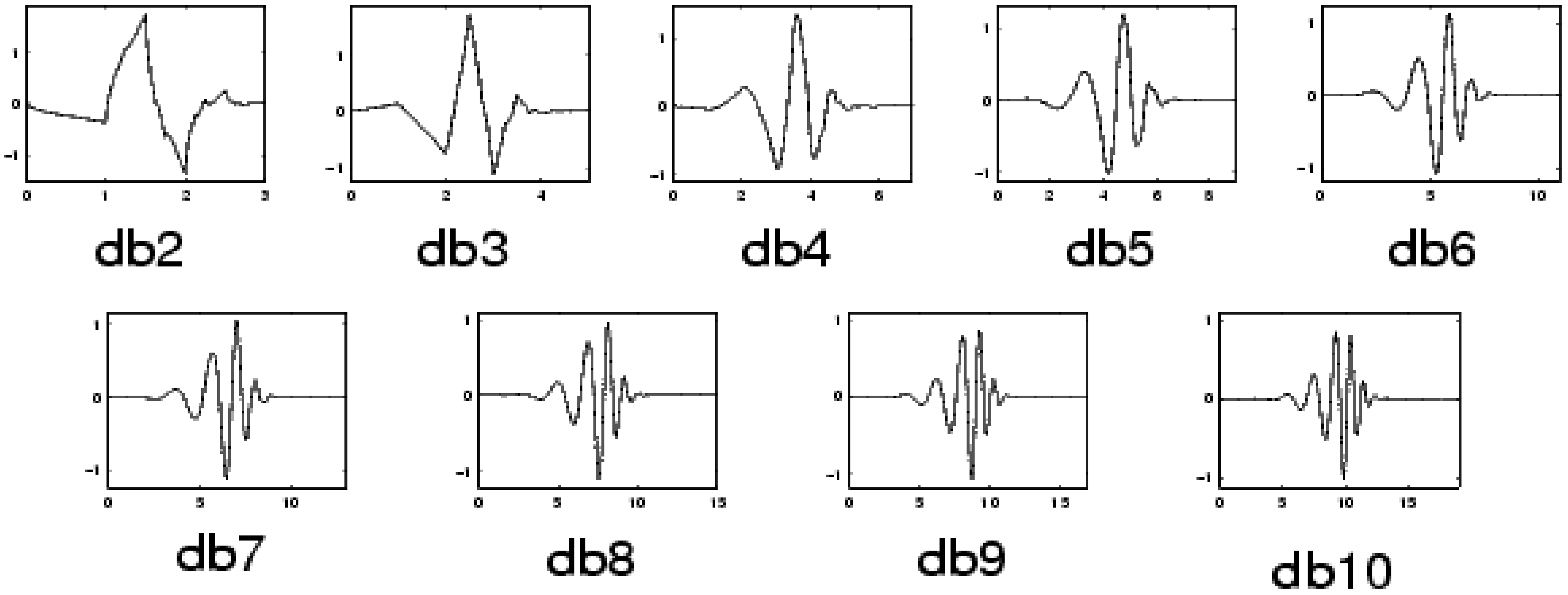
The Daubechies Wavelets

- **The names of the Daubechies family wavelets are written dbN, where N is the order, and “db” the "surname" of the wavelet.**

- **The db1 wavelet is the Haar wavelet.**

The Daubechies Wavelets

- **Here are the wavelet functions of the next nine members of the family**



- **As the order increases, their time support becomes *wider***

The Daubechies Wavelets

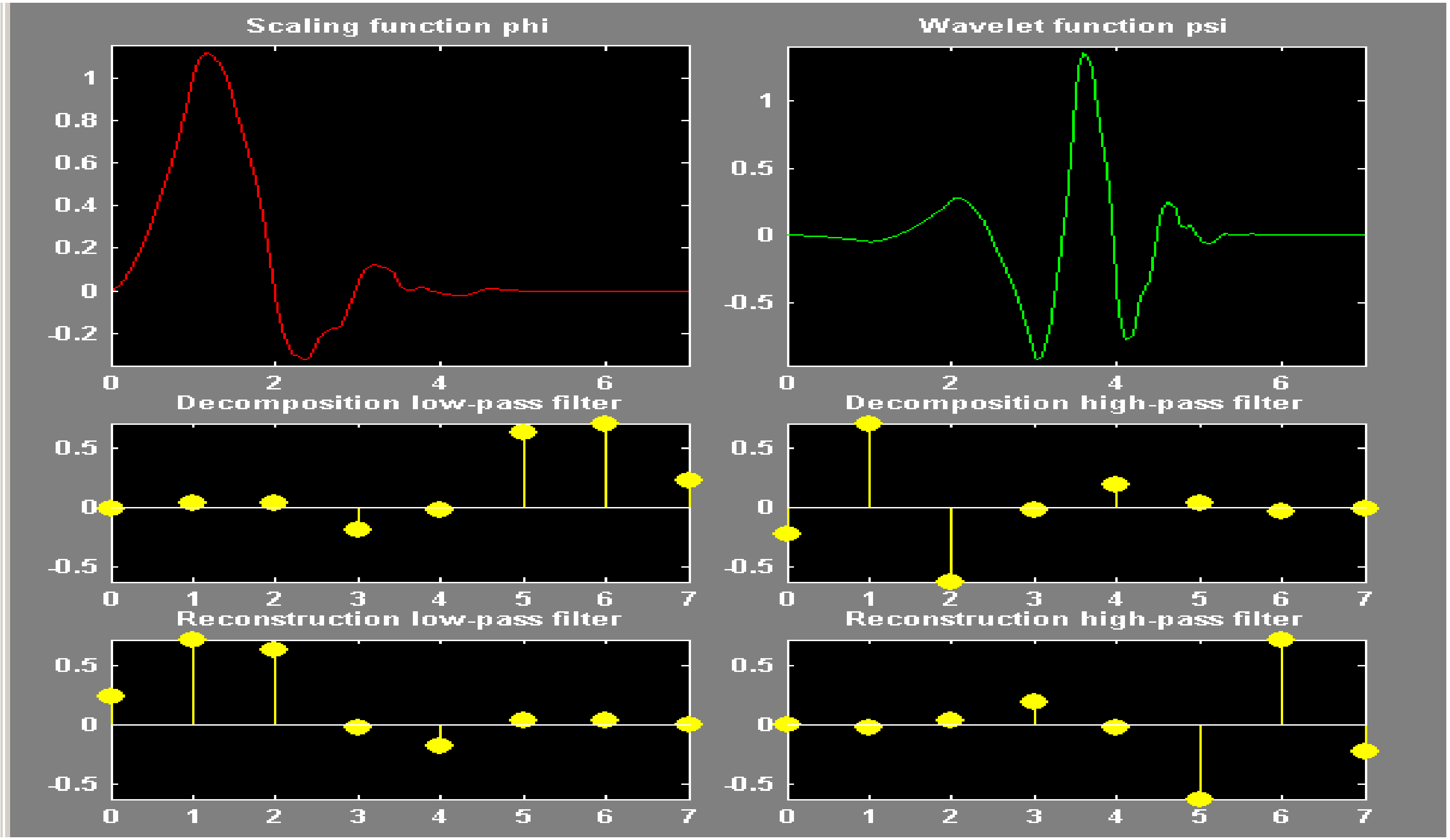
- **Daubechies wavelets have no explicit expression except for *db*1, which is the *Haar* wavelet**

- **However, the square modulus of the filter transfer function is explicit and fairly simple**

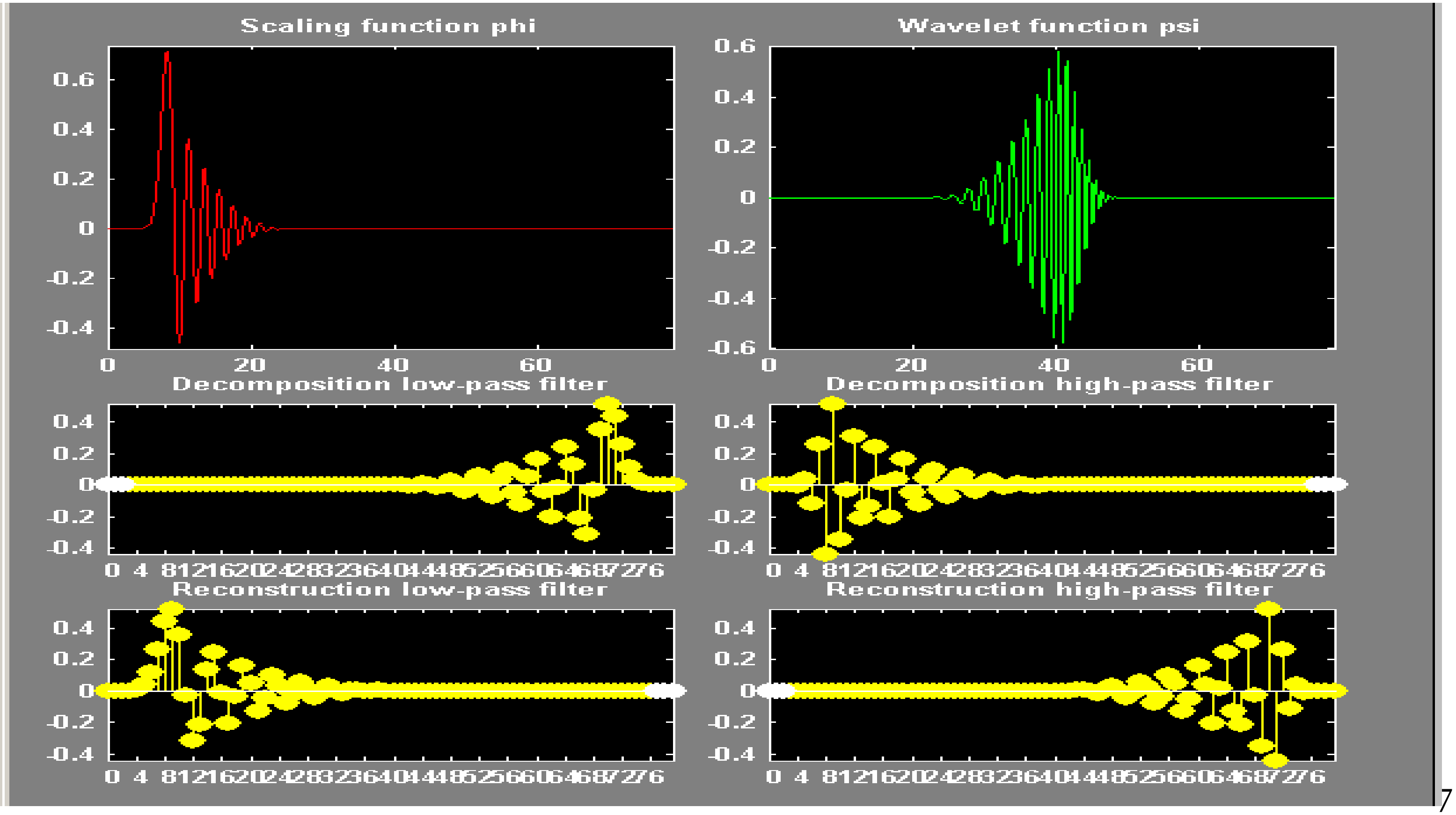
$$\left|H(z)\right|^2 = \sum_{k=0}^{N-1} C_k^{\,N-1+k} z^k$$

- **where** $C_k^{\,N-1+k}$ **denotes the binomial coefficients**

Daubechies Wavelet “db 4”: Scaling Function, Wavelet Function and Filters



Daubechies Wavelet "db 40": Scaling Function, Wavelet Function and Filters



The Daubechies Wavelets: Properties

- **The Daubechies wavelets turn the wavelet theory into a practical tool that can be programmed and used by scientist with a *minimum of mathematical training***

- **The Daubechies wavelets are very good at representing polynomial behaviour of signal and also very good tool for image processing**

- **The Daubechies filters *have no linear phase*, which is desirable in many applications**

The Biorthogonal Wavelets

- **This family exhibits the property of *linear phase,* which is needed for signal and image processing**

- **It is well known in the subband filtering community that *symmetry (e.g. linear phase) and exact reconstruction* are incompatible (except for the Haar wavelet) if the *same* FIR filters are used for reconstruction and decomposition**

- **Therefore, *two* wavelets, instead of just one, are introduced in the biorthogonal wavelets**

Biorthogonal Wavelets

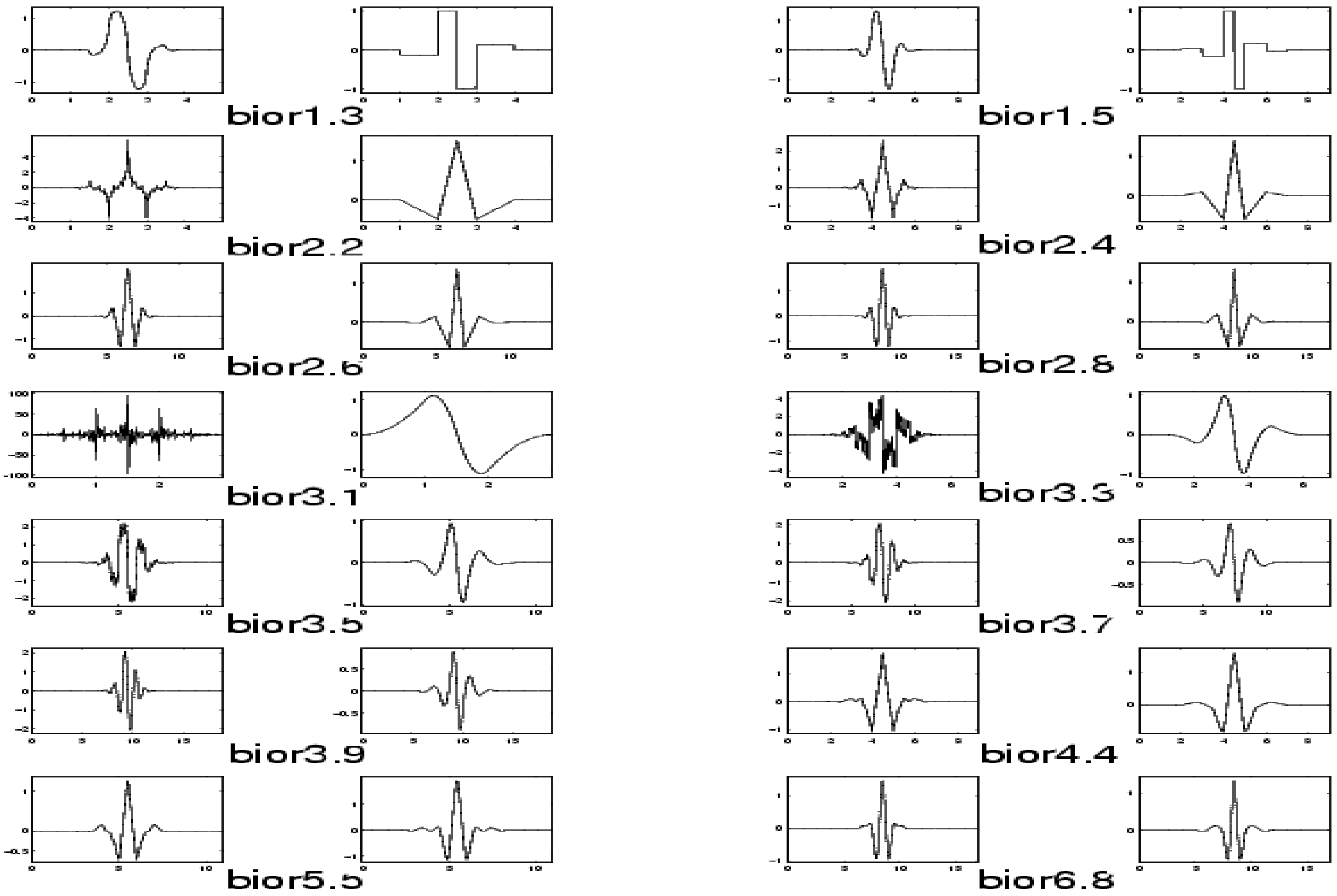
- **One is used in the analysis, and the coefficients of a signal are**

$$\tilde{C}(m,n) = \int_{-\infty}^{\infty} s(t)\tilde{\psi}(m,n,t)\,dt$$

- **The other is used in the synthesis**

$$s(t) = \sum_{m}\sum_{n}\tilde{C}(m,n)\psi(m,n,t)$$

Biorthogonal Wavelets

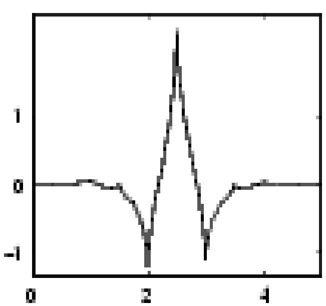


For decomposition on the left side and for reconstruction on the right side

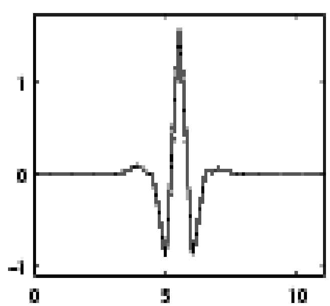
Coiflets

- **The Coiflets are *nearly symmetrical;* however, much more symmetrical than the Daubechies wavelets**

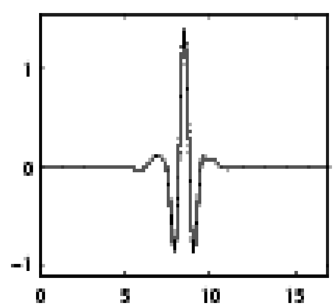
- **Unfortunately, true symmetry (or anti-symmetry) cannot be achieved for wavelets with compact support with one exception: the Haar wavelet which is antisymmetric**



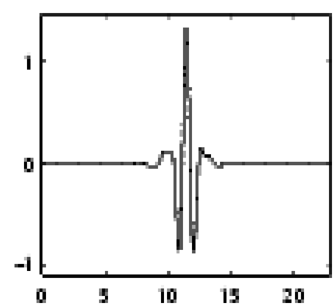
coif1



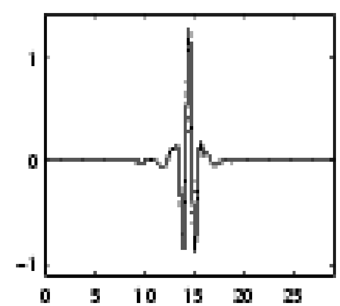
coif2



coif3



coif4

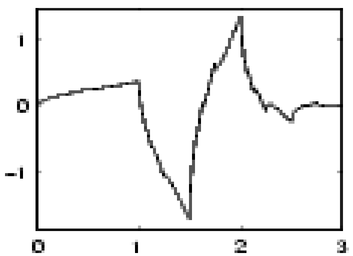


coif5

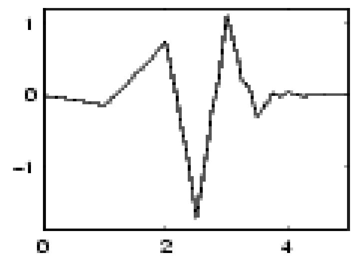
Symlets

- **The symlets are *nearly* symmetrical wavelets proposed by Daubechies as modifications to her family while retaining simplicity**

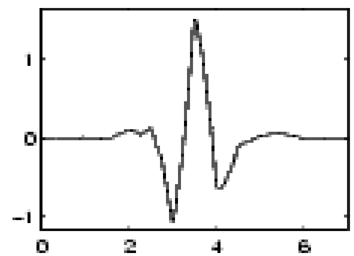
- **The properties of the two wavelet families are similar**



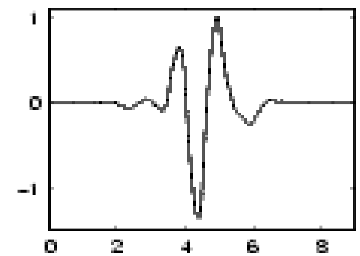
sym2



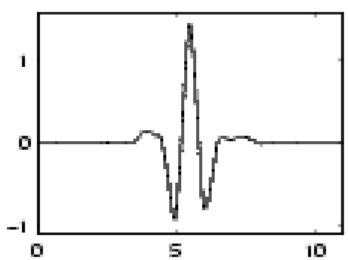
sym3



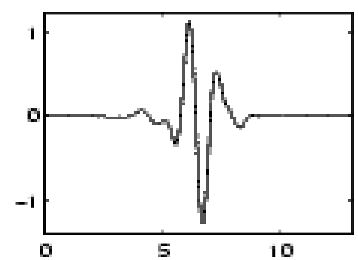
sym4



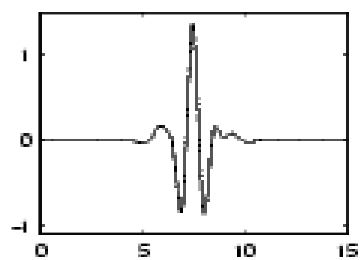
sym5



sym6



sym7

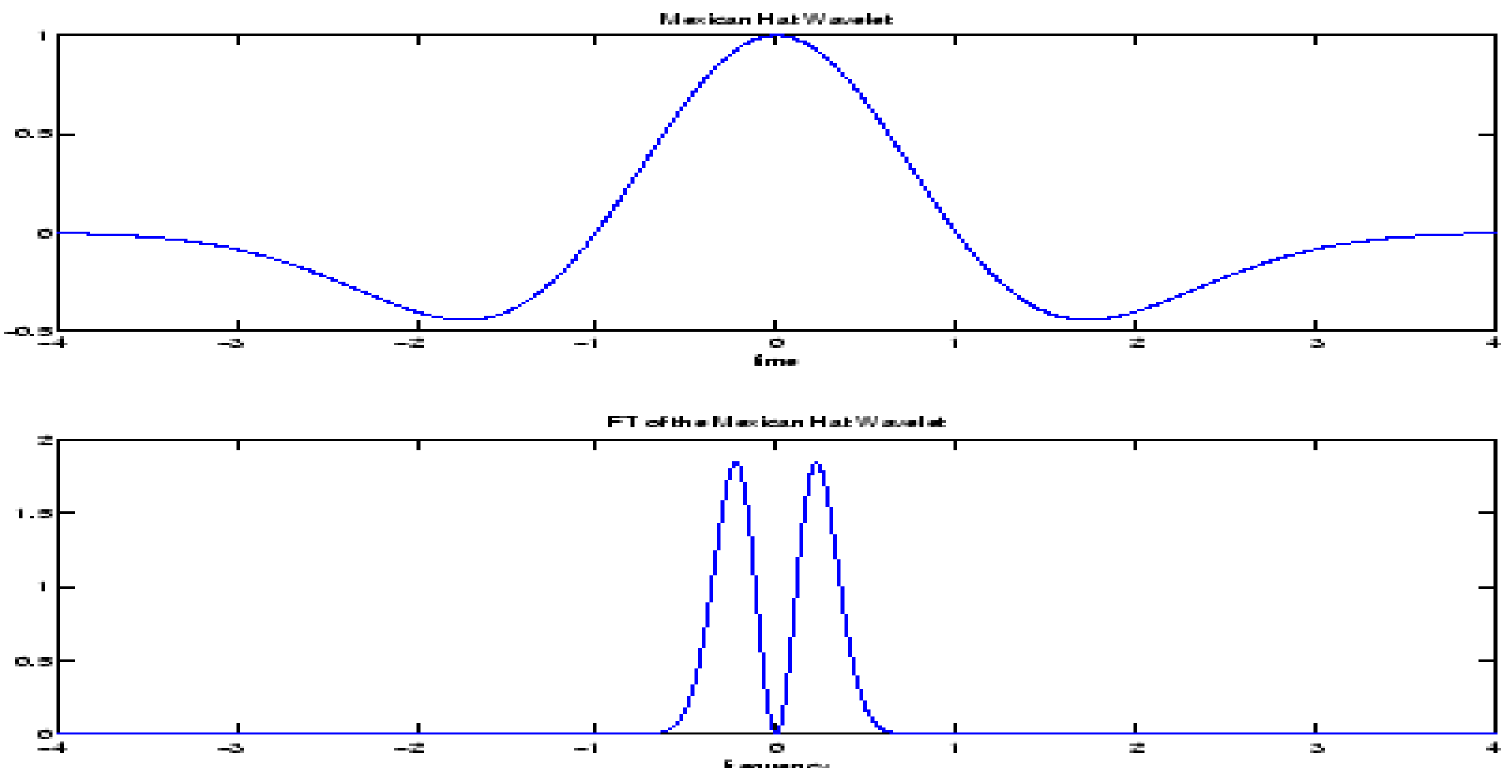


sym8

Mexican Hat

- **This wavelet is the negative second derivative of the Gaussian probability density function (** $\exp(-t^2/2)$

- **The Mexican hat wavelet and its power spectral density are shown below**



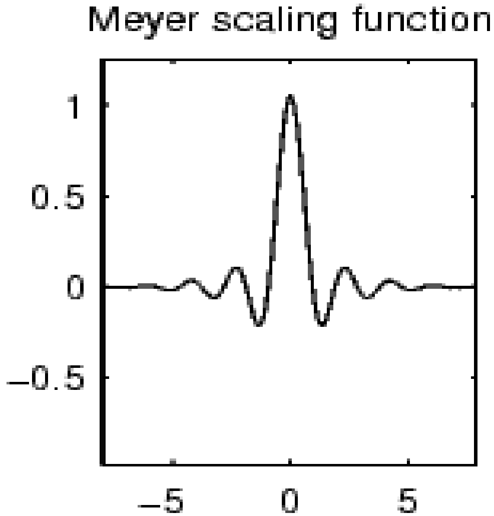
Mexican Hat

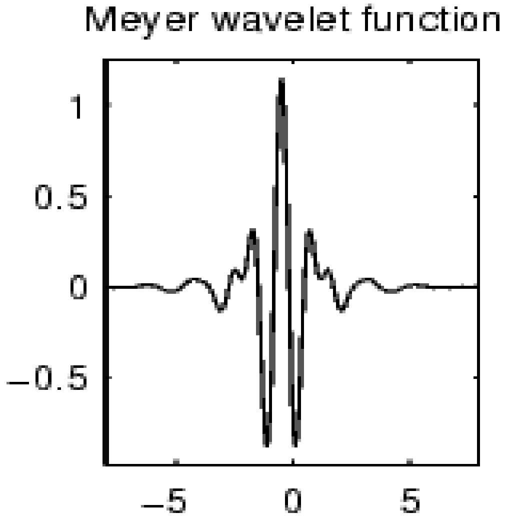
- **All derivatives of the Gaussian function may be employed as wavelet**
- **Which wavelet is the most appropriate one to use depends on the application.**
- **The first and second derivatives are most often used in practice**
- **The passband centre of this wavelet is 0.25 Hz**

The Meyer Wavelet

- **Because Meyer wavelets are compact in the frequency domain, the wavelet function does not have compact support in time domain, but it decreases to 0 when $t \to \infty$, faster than any inverse polynomial**

- **IIR filters can be constructed for this wavelet.**





The Meyer Wavelet

Although the Meyer wavelet is not compactly supported, there exists a good approximation leading to FIR filters, and then allowing DWT, e.g. *FIR based approximation of the Meyer wavelet*

Complex Wavelets

- **The real wavelets *do not deliver phase information***

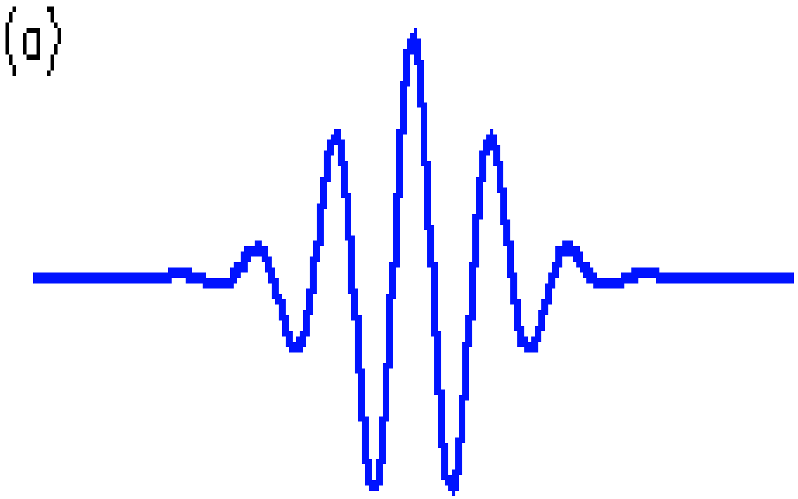
- **The complex wavelets do yield phase information**

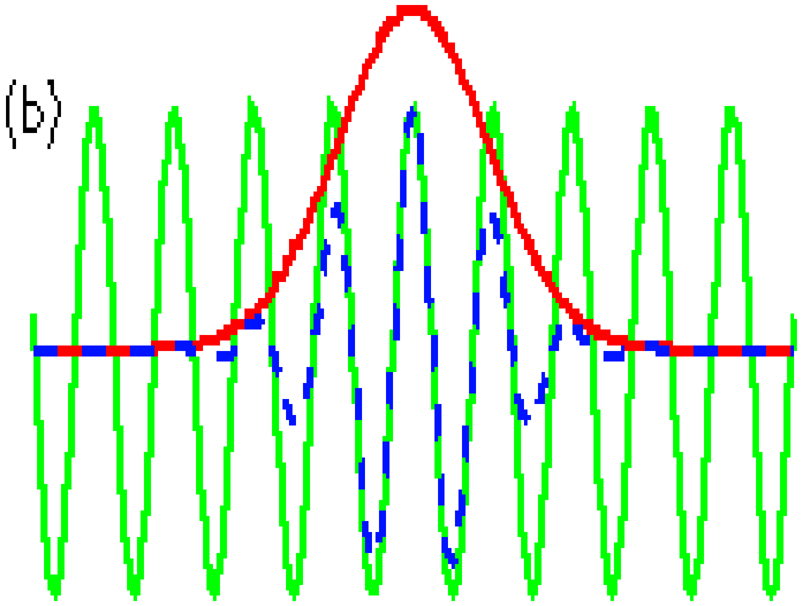
- **It is even possible to retrieve an instantaneous phase using the complex wavelets**

- **The structure of the discrete wavelet transform via filter bank, using complex wavelets, is the same, except that filters have complex coefficients and generate complex output samples**

The Morlet Wavelets

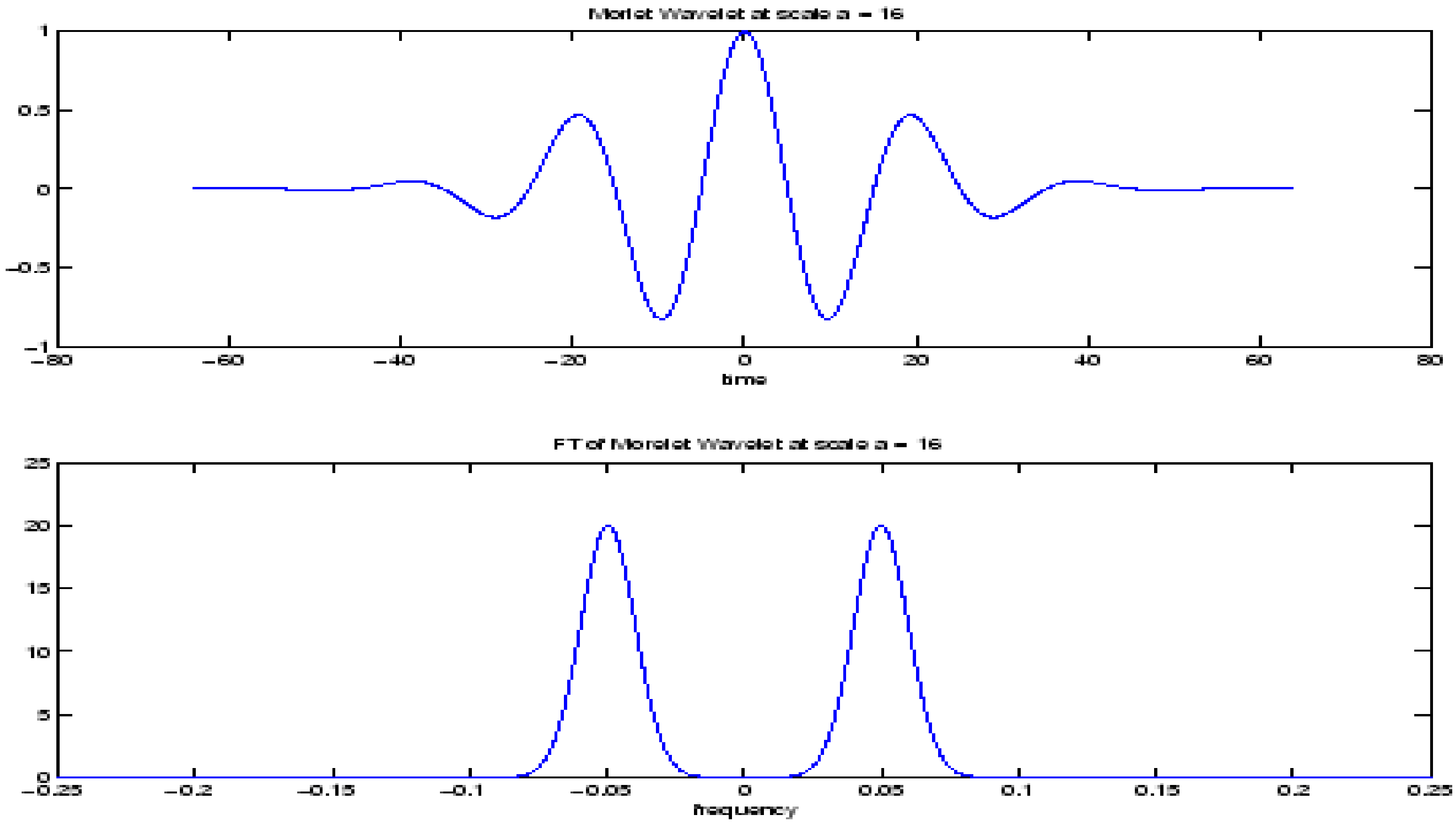
- **The complex Morlet wavelet, is a symmetric wavelet that results from the multiplication of a complex exponent and a Gaussian envelope**
- **The real Morlet wavelet is also widely used in applications**





The Morlet Wavelets

The **real** **Morlet wavelet and its power spectral density are shown below**



The Morlet Wavelet

- **Morlet wavelet can be written as**

$$\psi(t) = \left(\frac{1}{f_b \pi}\right)^{1/2} [\exp(i2\pi f_0 t) - \exp(-(2\pi f_0)^2/2)]\exp(-t^2/f_b)$$

where f_0 is the central frequency of the mother wavelet, f_b is a bandwidth parameter

- **The second term in the square brackets is known as the *correction term***

- **Without correction term, the Morlet wavelet has a *non-zero* mean, e.g. the zero frequency term of its corresponding power spectral density is non-zero; therefore, it does not satisfy the admissibility condition**

The Morlet Wavelet

- **However, for f_0 large enough (typically $5 \le 2\pi f_0 \le 6$), this correction term is negligible**

- **This central frequency f_0 is generally chosen to be the characteristic frequency of the Morlet wavelet rather than the passband centre frequency**

- **We recall that the Gabor short time transform is the Gaussian windowed Fourier transform. Thus, the Morlet wavelet has a form very similar to the Gabor transform.**

  The main difference is obvious:

the internal frequency is allowed to vary within a Gaussian window of a *fixed width* in the Gabor transform, whereas in the Morlet wavelet transform we scale the window and enclosed sinusoid *together*

The Morlet Wavelet: Applications

- **Because of its smoothness and periodicity, the Morlet wavelet is a good choice for data that is varying continuously in time and is periodic or quasi-periodic**

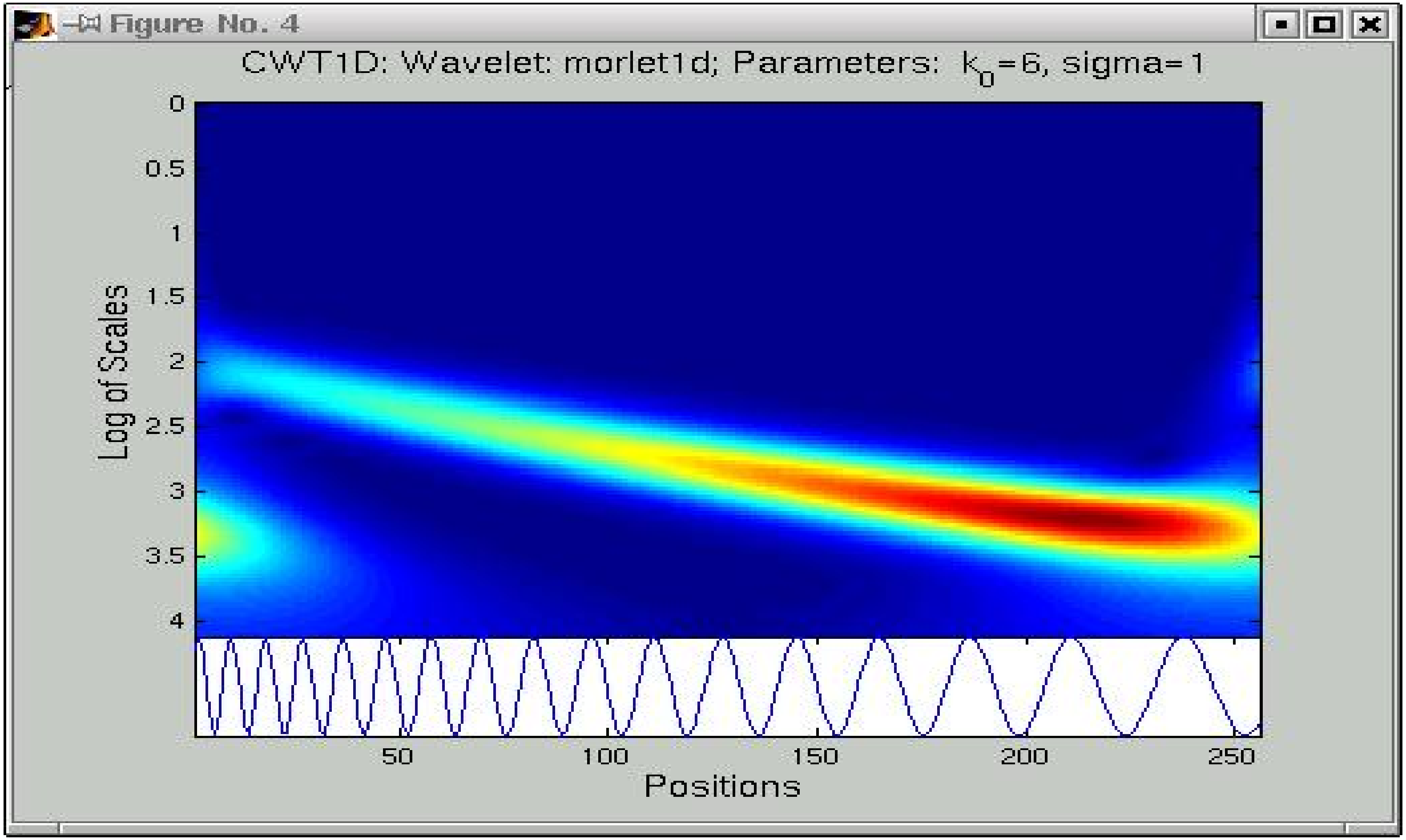
- **The complex Morlet wavelet transform is especially convenient for analyzing signals with:**

➢ **a wide range of dominant frequencies which are localized in different time intervals**

➢ **amplitude and frequency modulated spectral components**

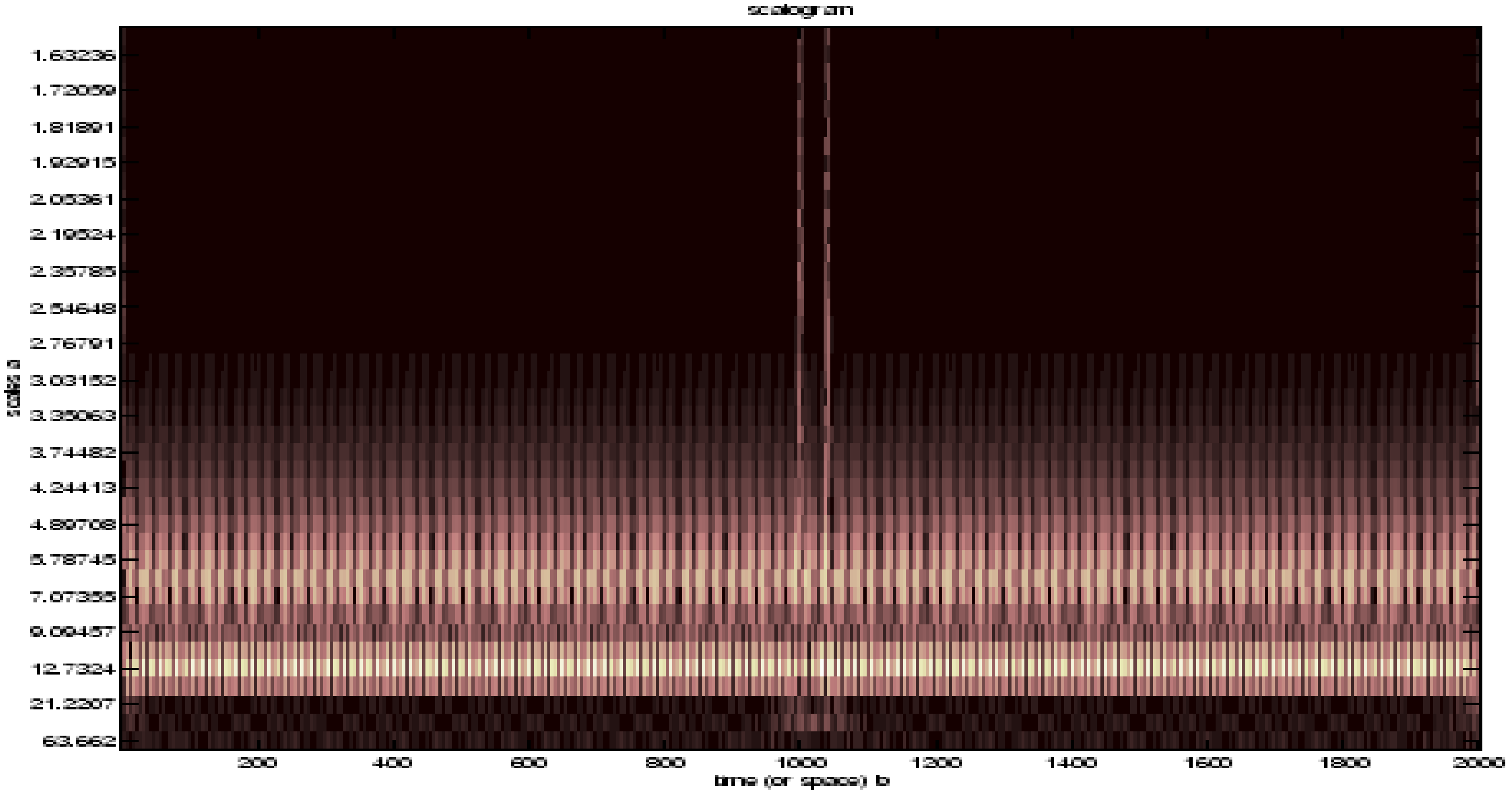
The Morlet Wavelet: Case Study 1

- **The Morlet wavelet transform of the transient chirp signal is shown below**



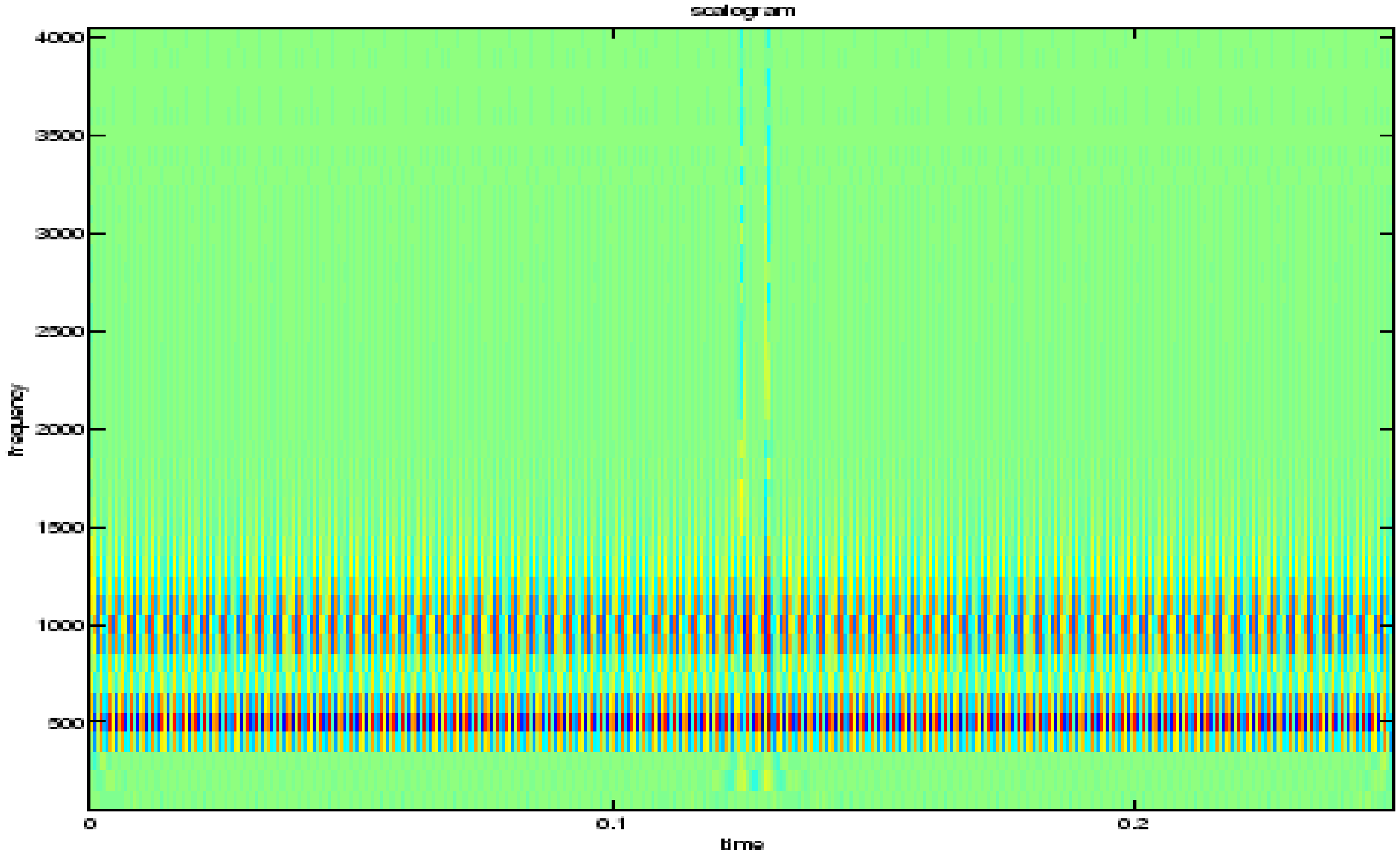
The Morlet Wavelet: Case Study 2

- **Signal is the sum of two sinusoids of frequencies 500Hz and 1000Hz and two impulses at times 125 ms and 130 ms**



The Morlet Wavelet: Case Study 2

Converting the scale to the frequency, we obtain:



Wavelets as Demodulation Technique

If a general amplitude and phase modulated signal is considered

$$s(t) = k(t)\cos\left[\varphi(t)t\right]$$

where $k(t)$ **and** $\varphi(t)$ **are time-varying envelope and phase,**

the Morlet wavelet transform of the signal has the following expression:

$$C(a,b) = \sqrt{a}k(b)e^{-(a-\varphi(b)-\omega_0)^2}e^{i\varphi(b)b}$$

Wavelets as Demodulation Technique

- **For a fixed frequency, we obtain the scalogram and phase of the wavelet transform:**

$$\left|C(a_i,b)\right|^2 = a_i k^2(b) e^{-2(a_i-\varphi(b)-\omega_0)^2}$$

$$\angle C(a_i,b) = \varphi(b) b$$

- **Using these equations, we obtain:**

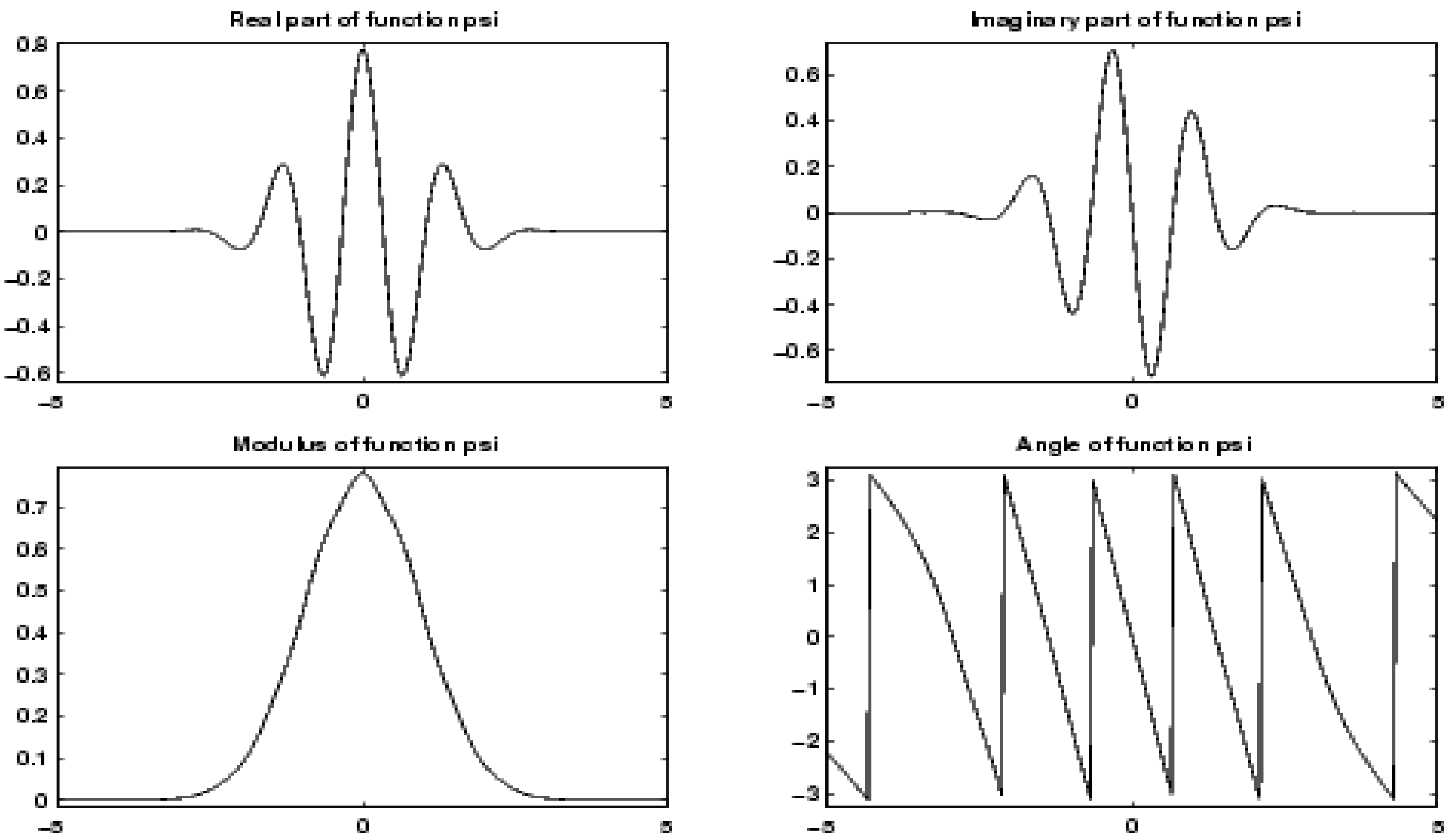
$$\varphi(b) = \frac{\angle C(a_i,b)}{b} \qquad k(b) = \frac{\left|C(a_i,b)\right|}{\sqrt{a_i} e^{-(a_i-\varphi(b)-\omega_0)^2}}$$

- **Last equations show how general time-varying envelope and phase can be obtained using the scalogram and the wavelet transform phase**

Complex Gaussian Wavelets

- **This family is built starting from the complex Gaussian function:** $\psi(t) = C_p e^{-it} e^{-t^2}$ **by taking the** pth **derivative**

The integer p is the parameter of this family



Complex Frequency B-spline Wavelets

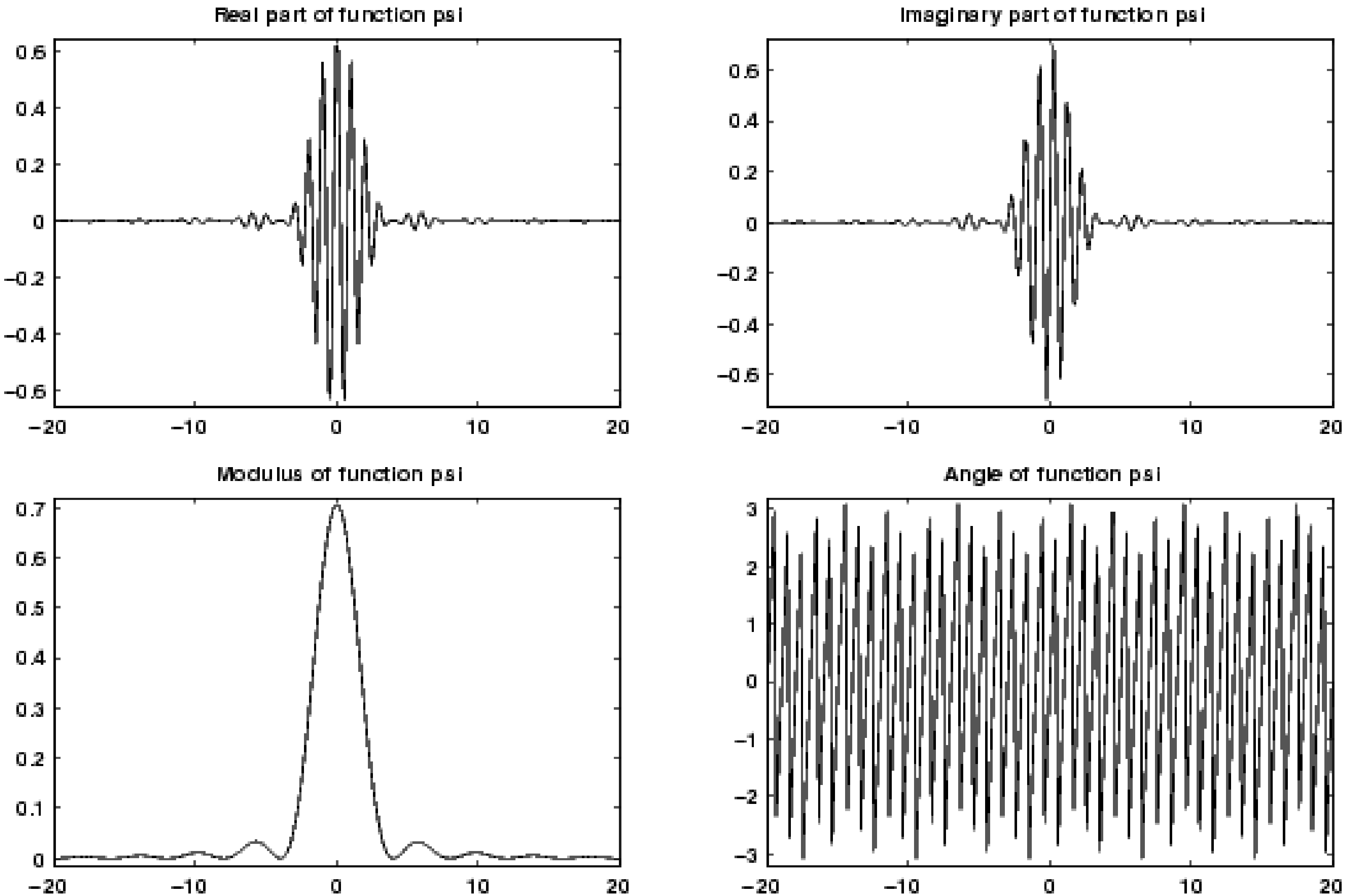
- **A complex frequency B-spline wavelet is defined by**

$$\psi(t) = \sqrt{f_b}\left[\sin c\left(\frac{f_b t}{m}\right)\right]^m e^{i\omega_0 t}$$

depending on three parameters:

- m **is an integer order parameter,** $m \geq 1$
- f_b **is a bandwidth parameter**
- ω_0 **is a wavelet centre frequency**

Complex Frequency B-spline Wavelets



Wavelet Choice

- **Advantage of the wavelet analysis is the multiple choice of a mother wavelet function**
- ***The most effective procedure for selecting a proper mother wavelet is through trial and error***
- **However, there are several factors, which should be taken into account, for mother wavelet selection**

Complex or Real Wavelets ?

- **The complex wavelets will give information about both amplitude and phase and are better adapted for oscillatory signals**

- **The real wavelets can be used to isolate peaks or discontinuities**

Wavelet Shape?

- **The wavelet function should reflect the type of features present in a signal**

- **For signals with sharp jumps or steps, one would choose sharp and jumped wavelets, while for smoothly varying signals one would choose a smooth wavelet**

Wavelet Width?

- **The resolution of the wavelet transform is determined by the balance between the width in time domain and frequency domain**

- **A narrow (in time) wavelet function will have good time resolution but poor frequency resolution, and visa versa**