Part 3 of the PSD

The Frequency Resolution of the PSD and PSD Estimations

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The Frequency Resolution of the PSD

The Frequency Resolution of the PSD

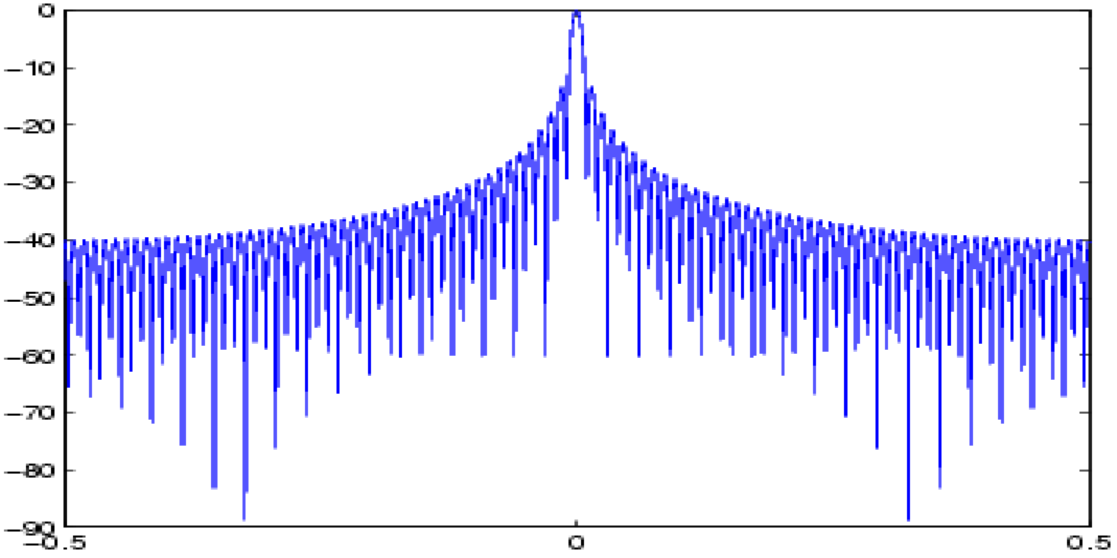
- **Frequency resolution refers to the ability to discriminate signals and is a key factor of spectral estimator performance**

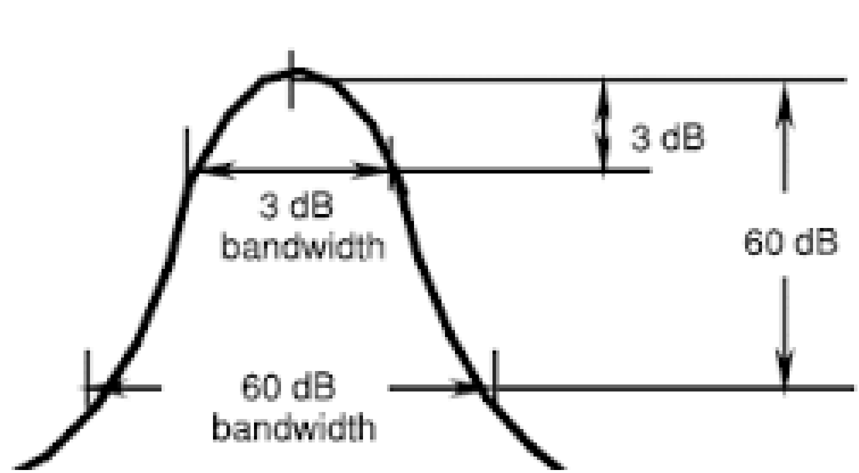
- **In order to resolve two sinusoids that are relatively close in frequency, it is necessary for the difference between the two signal frequencies to be equal or greater than the frequency width of the main lobe of the spectral analyzer**

- **The main lobe width is defined at the point where the power is half of the peak main lobe power (i.e. -3 dB width)**

- **This width is equal to** $\dfrac{f_s}{L}$

Frequency Resolution



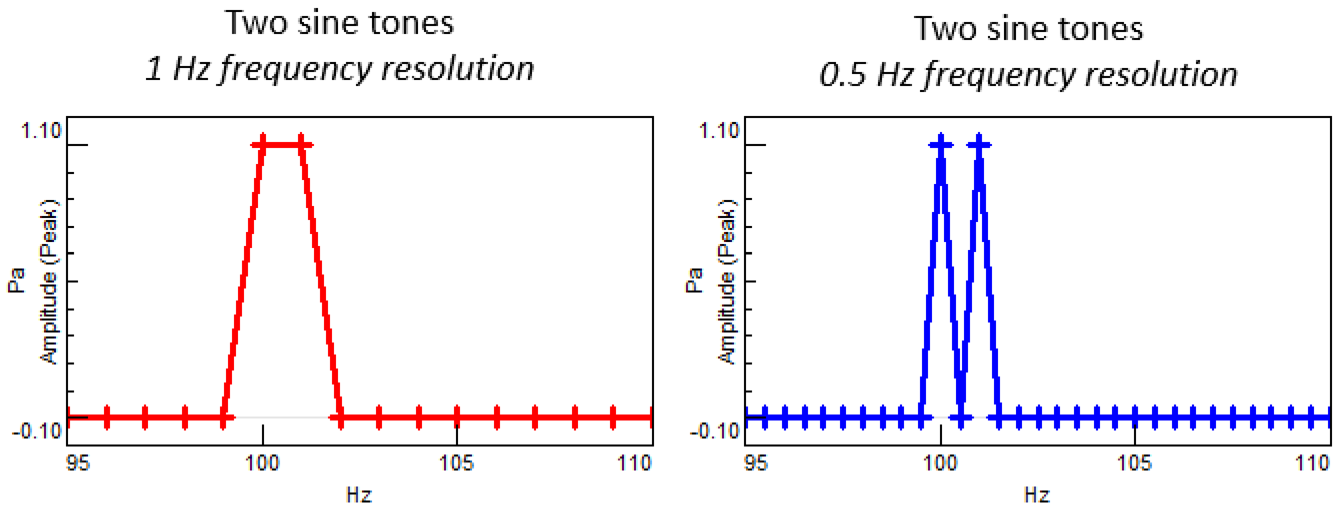


Frequency Resolution Condition

In other words, for two sinusoids, the resolution condition requires that

$$\left| f_1 - f_2 \right| \textit{more than } \frac{f_s}{L}$$

f2-f1=0.75Hz



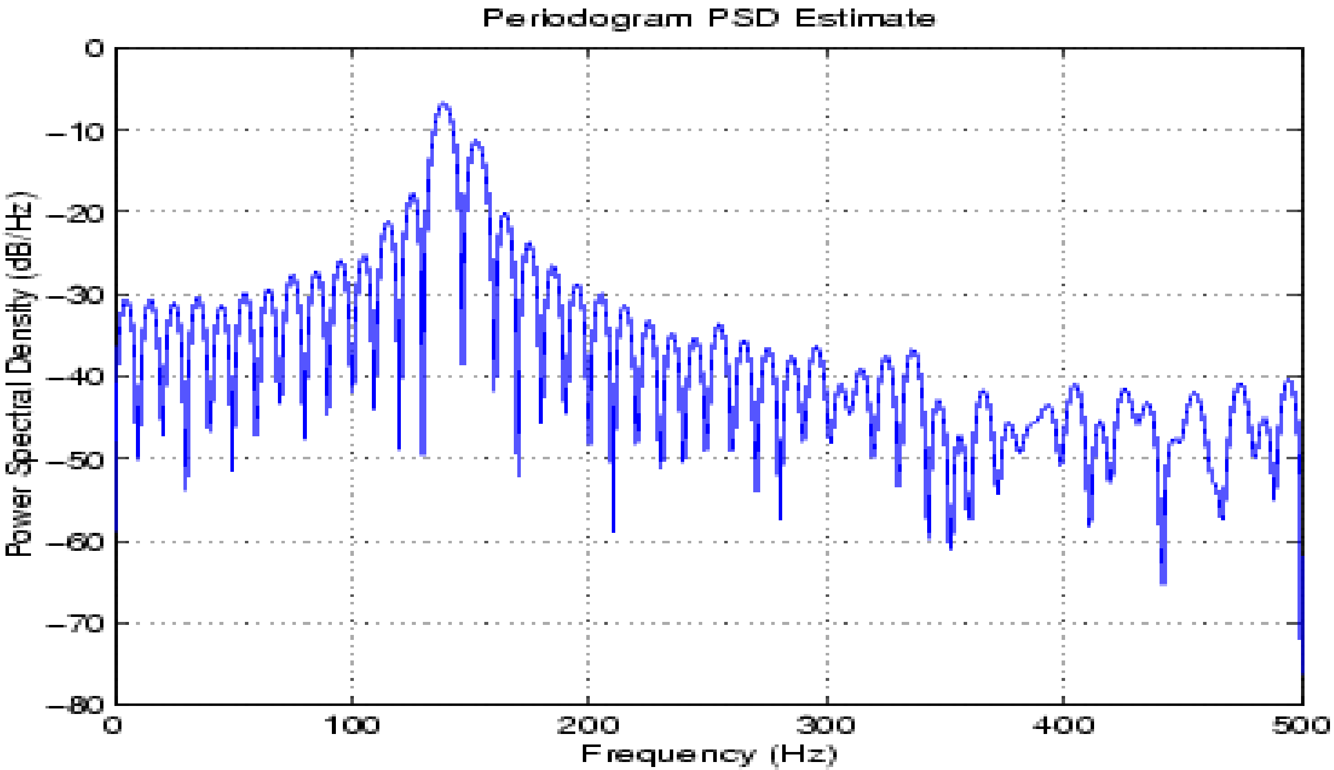
The Frequency Resolution: Case Study 1

- **We consider sum of two sinusoids and noise**

- **Sinusoid frequencies are 140 Hz and 150 Hz**

- **Sampling frequency is 1000Hz, number of samples is 100 (the signal duration is 0.1s), the frequency resolution is 10 Hz**

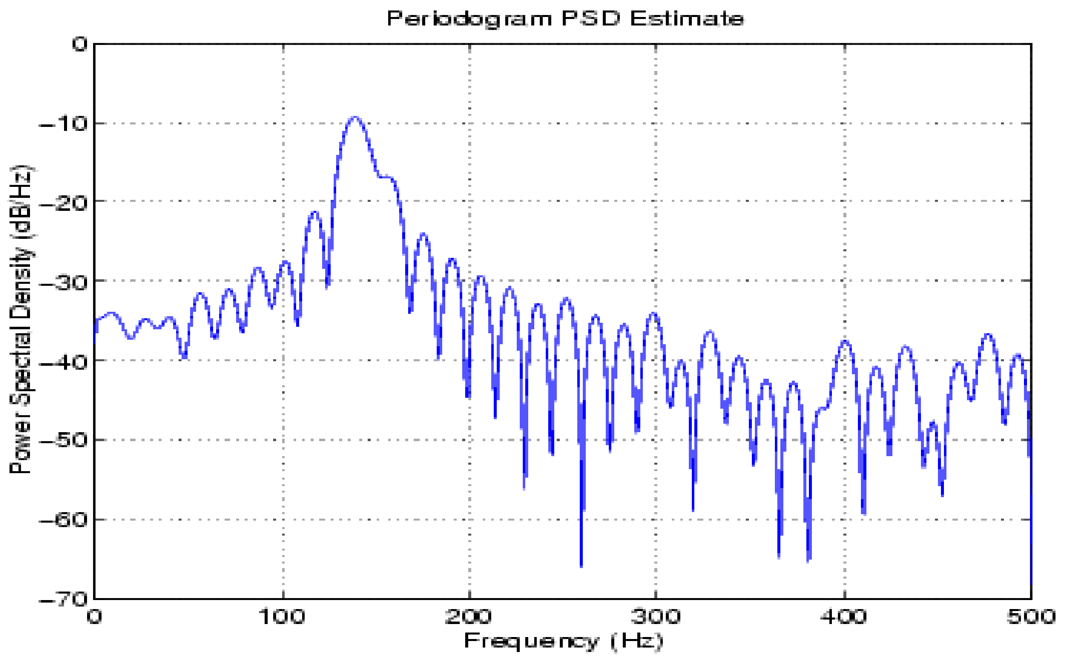
The Frequency Resolution: Case Study 1



In the example above, where two sinusoids are separated by resolution only 10Hz, the data record should be at least 100 samples (at sampling frequency 1000Hz) to allow resolution of two distinct sinusoids by a periodogram

The Frequency Resolution: Case Study 2

- **Consider a case below, where the resolution condition is not met, as for signal of 67 samples (the signal duration is 0.067s, data from the example 2, but frequency resolution is 15 Hz)**



- **In the example above, two sinusoids *are not separated* by resolution 15Hz**

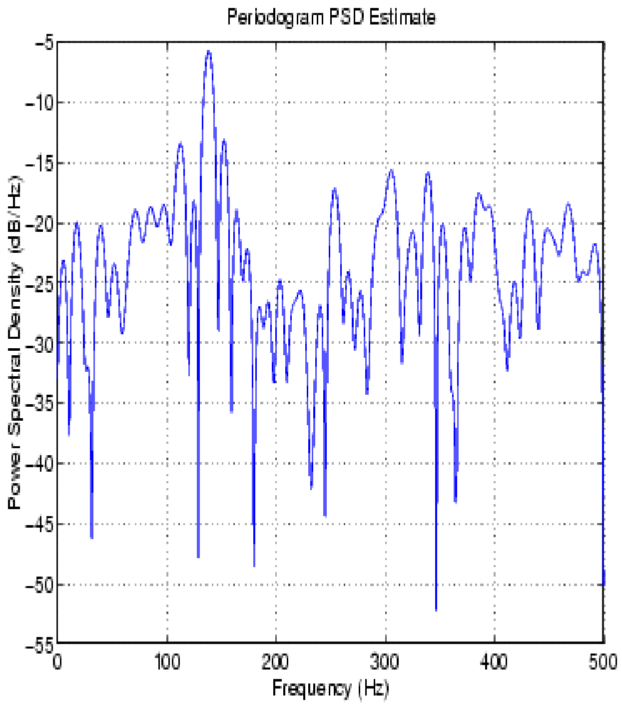
The Frequency Resolution: Case Study 3

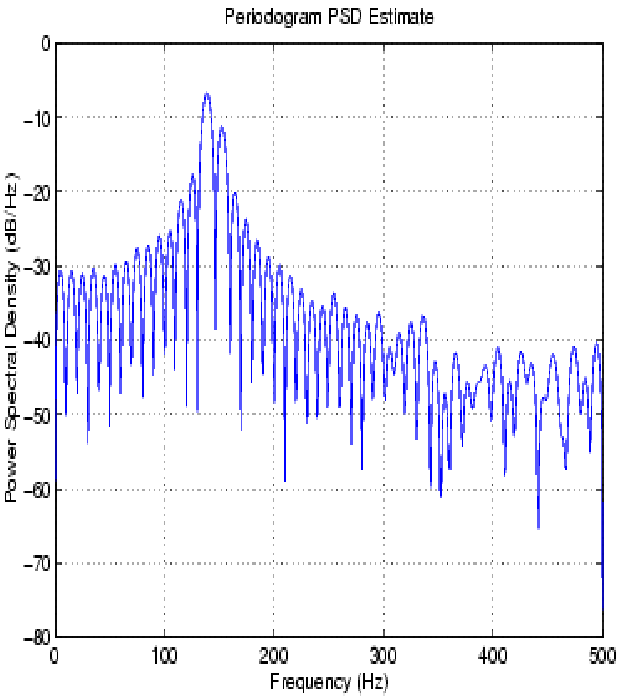
- **Case studies 1-2 did not consider the effects of noise since the signal-to-noise ratio has been relatively high**

- **When the signal-to-noise ratio is relatively low, true spectral features are much harder to distinguish, and noise artifacts appear in spectral estimates based on the periodogram**

- **In case study 3, we consider data from the case study 1, but signal-to-noise ratio is relatively low (20 times less than in the case study 1)**

The Frequency Resolution: Case Study 3





Case study 3 **Case study 1**

PSD Estimations

Periodogram Bias

- **The periodogram is a biased estimator of the PSD**
- **However, the periodogram is asymptotically unbiased, e.g. as the signal length tends to infinity, the frequency response of the rectangular window more closely approximates the Dirac delta function**

Periodogram Variance

- **The periodogram is a poor estimator of the power spectral density even when the data record is long**

- **This is due to the variance of the periodogram**

- **The variance of the periodogram does not tend to zero as the data length L tends to infinity**

- **Nevertheless, the periodogram can be a *useful tool* for spectral estimation in situations where the signal-to-noise-ratio is high, and especially if the data record is long**

PSD Estimations

The various methods of power spectral density estimation available in Matlab Signal Processing Toolbox and in literature can be categorized as follows:

- **nonparametric methods (these methods make no assumption about how the data were generated)**
- **parametric methods (these methods make assumption about how the data were generated)**

Nonparametric Estimations

- **When trying to obtain power spectral density estimates of stochastic signals using a limited number of samples, poor periodogram estimates often result**

- **To get smoother power spectral density estimates, many independent periodograms have *to be averaged in the frequency domain***

Nonparametric Estimation: the Bartlett Method

The Bartlett's method for reducing the variance of the PSD involves three steps

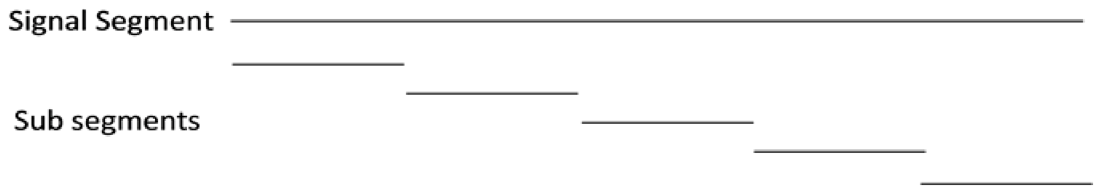
- **First, the N point sequence is subdivided into K non-overlapping segments, where each segment has length L**

- **This results in K data segments**

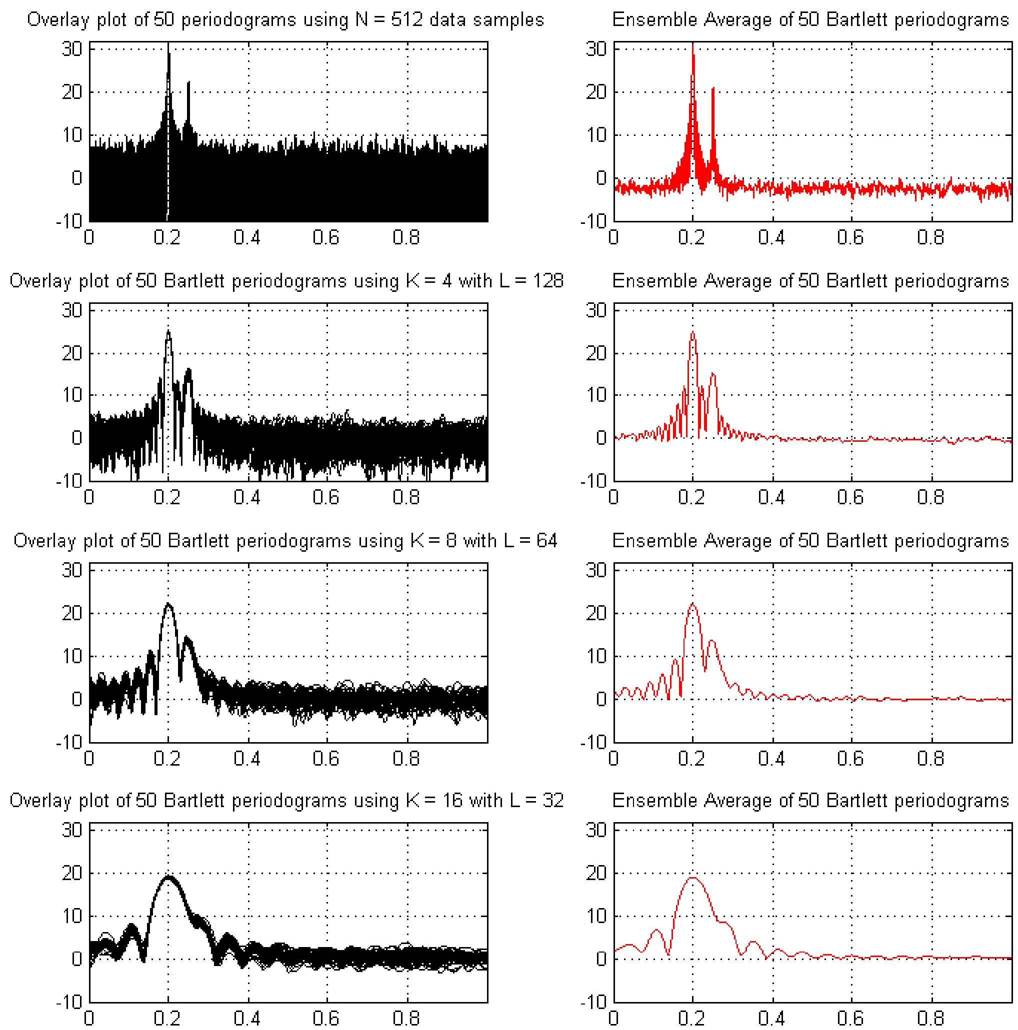
- **For each segment, the periodogram is computed**

- **Finally, the obtained periodograms of the segments are averaged to obtain the Bartlett's power spectral density estimate**

Nonparametric Estimation: the Bartlett Method



Nonparametric Estimation: the Bartlett Method



Nonparametric Estimation: the Bartlett Method

- **The effect of reducing the length of a segment from N points to L points results in a segment whose main spectral lobe has been increased by a factor of K**
- **Consequently, the frequency resolution has been increased by a factor K (i.e. poorer frequency resolution)**
- **In return for this poorer resolution, we have reduced the variance of the estimate by the factor K**

Nonparametric Estimation: the Welch Method

- **Welch made two basic modifications to the Bartlett method.**

- **First, he allowed the data segments *to overlap***

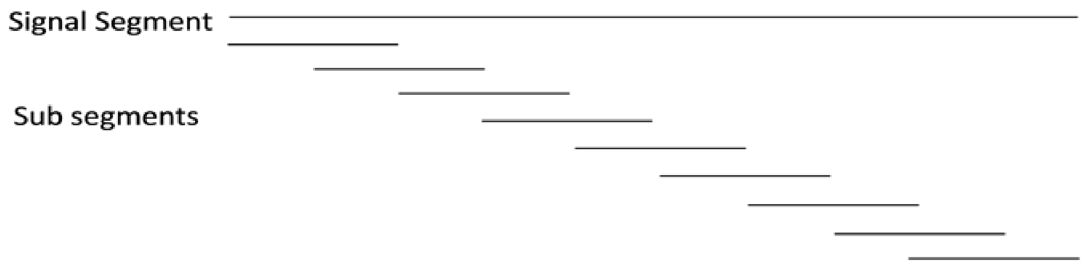
- **Thus, the data segments can be represented as**

$$x_i(n) = x(n+iD) \quad n = 0,1,\ldots,L-1 \quad i = 0,1,\ldots,M-1$$

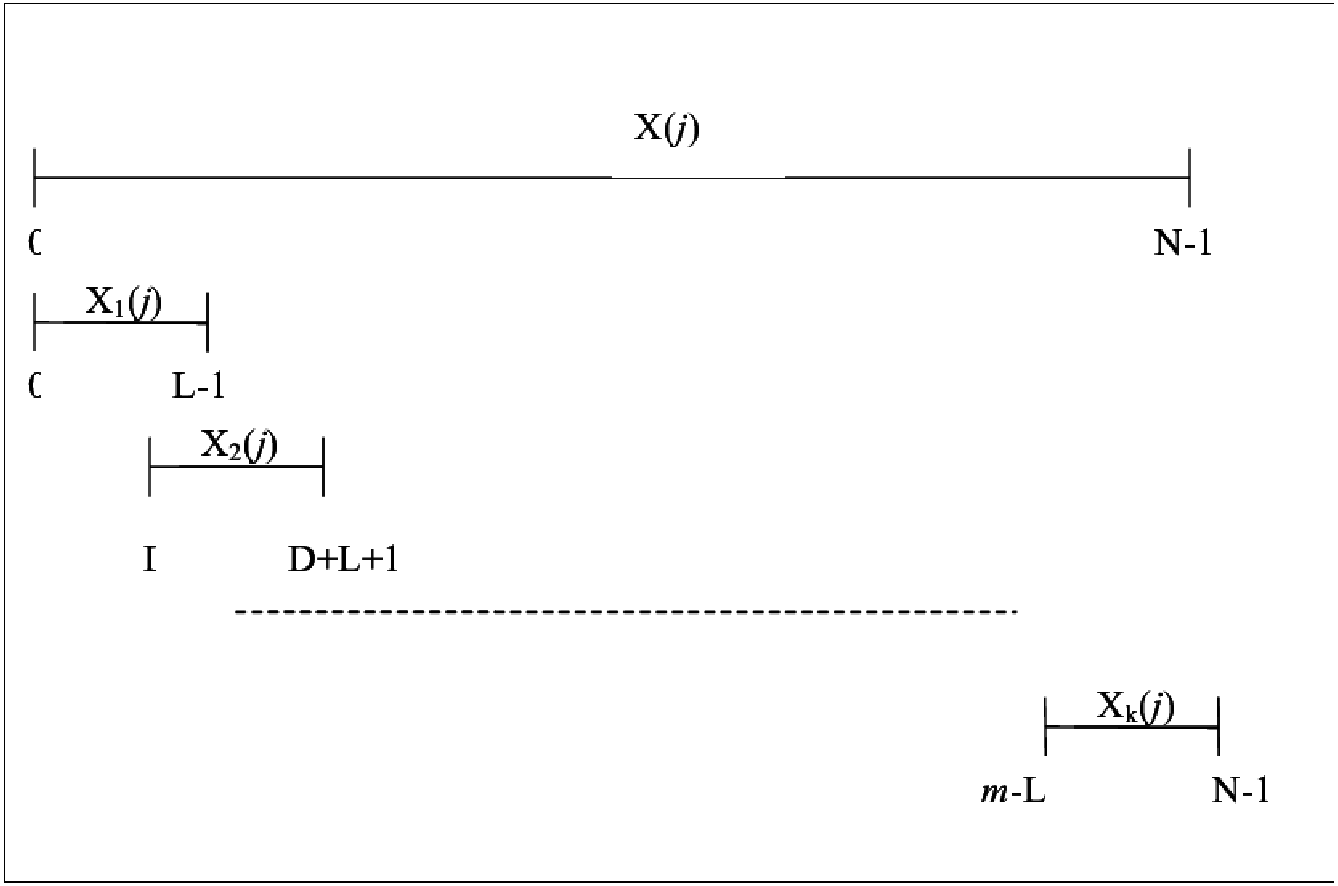
where iD **is the starting point for the** ith **sequence**

- **Observe that if** $D = L$ **, the segments do not overlap and the number of data segments is identical to the number in the Bartlett method**

Nonparametric Estimation: the Welch Method



Nonparametric Estimation: the Welch Method



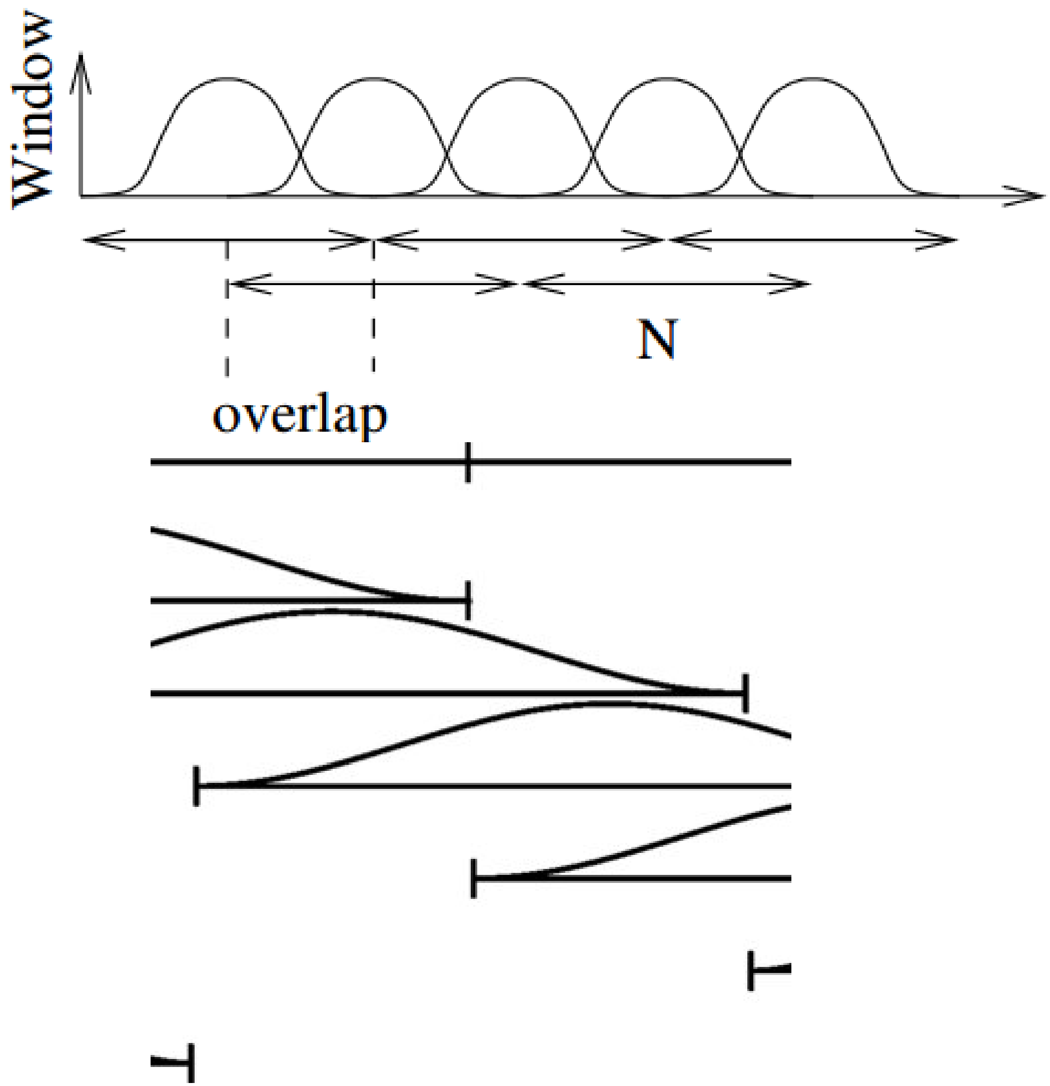
Nonparametric Estimation: the Welch Method

- **However, if $D = \frac{L}{2}$, there is 50% overlap between successive data segments and $M = 2K$ segments are obtained**

- **The second modification made by Welch to the Bartlett method is to *window* the data segments prior to computing the periodogram**

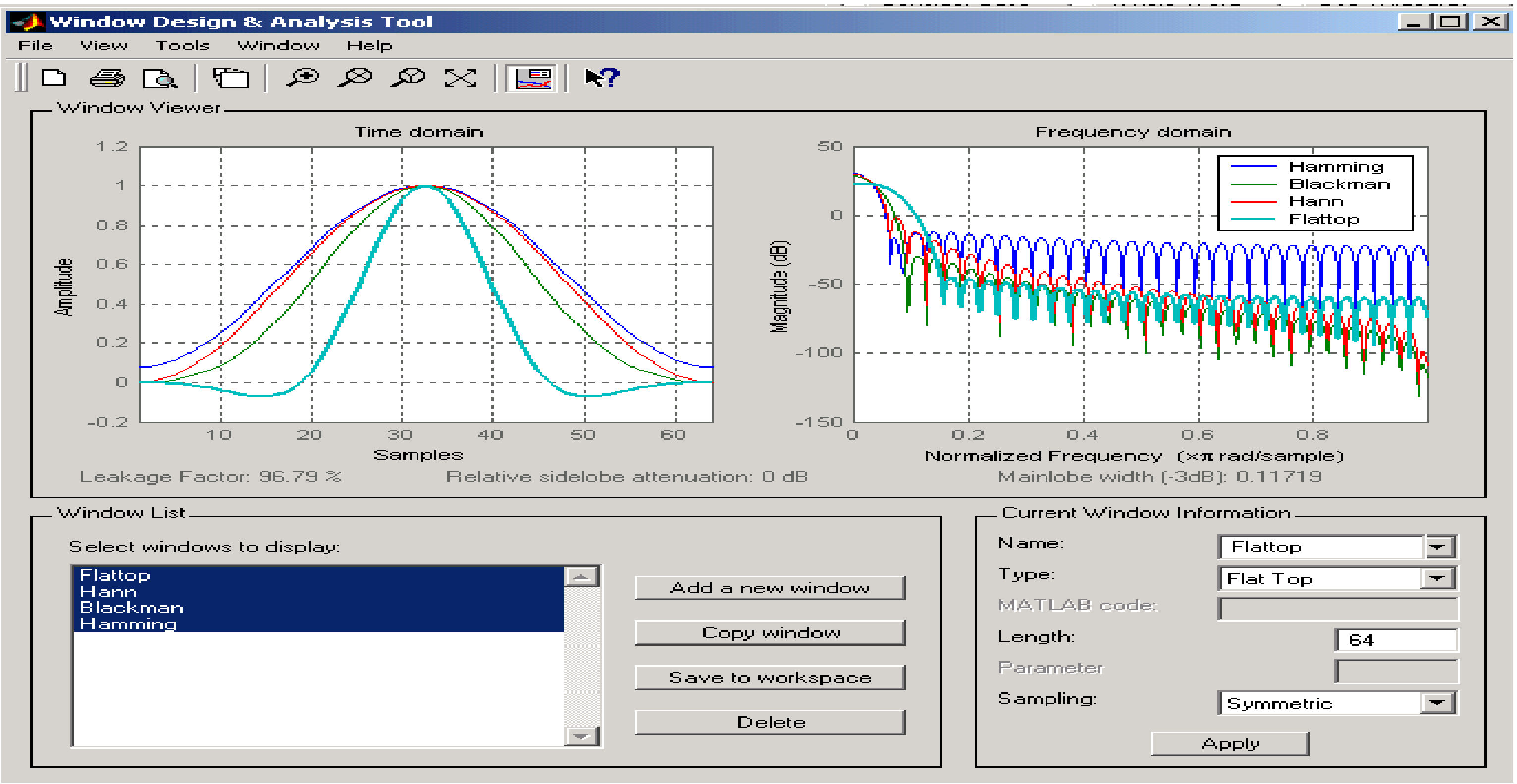
- **The result is a "modified" periodogram**

Nonparametric Estimation: the Welch Method



Non-Rectangular Windows

Blackman, Flat Top, Hamming, Hann (Hanning), and rectangular windows are all special cases of the *generalized* *cosine window*



Nonparametric Estimation: the Welch Method

Welch's method is implemented in the MATLAB Signal Processing Toolbox

Nonparametric Estimation: the Welch Method

- **The averaging of the modified periodograms decreases the variance of the estimate relative to a single periodogram estimate**

- **Although overlap between segments introduces redundant information, this effect *is diminished* by the use of a *nonrectangular* window, which reduces the importance or weight given to the end samples of segments (i.e. the samples that overlapped)**

- **However, as mentioned above, the combined use of shorter segment data and nonrectangular windows results in poorer frequency resolution of a PSD estimator**

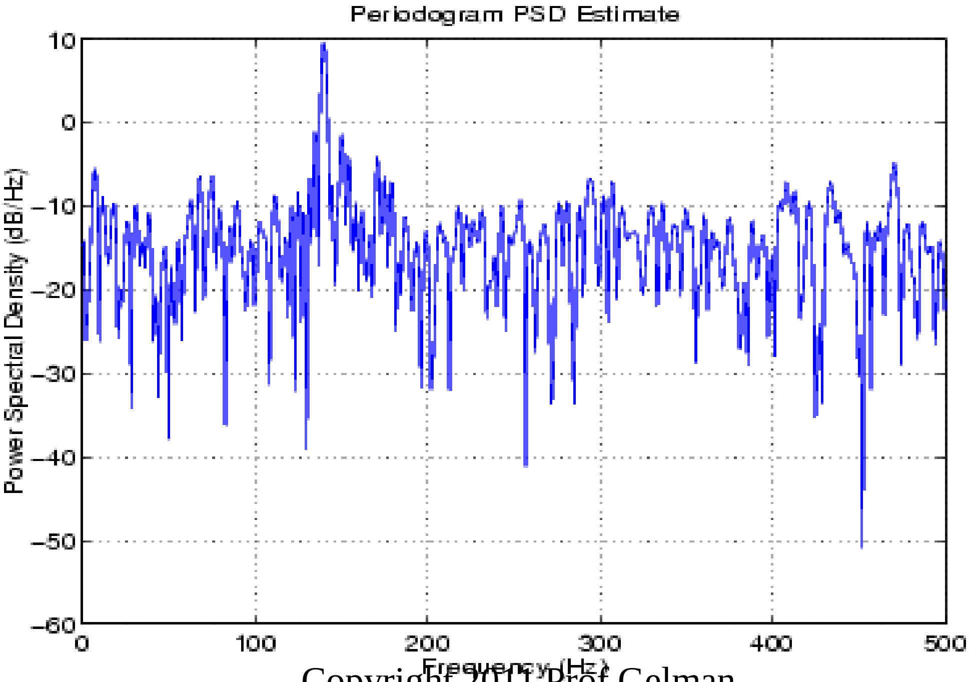
Nonparametric Estimation: Welch Method

- **In summary, there is a tradeoff between variance reduction and frequency resolution change**

- **One can manipulate the parameters in Welch's method to obtain improved estimates relative to the periodogram, especially when the signal/noise ratio is low**

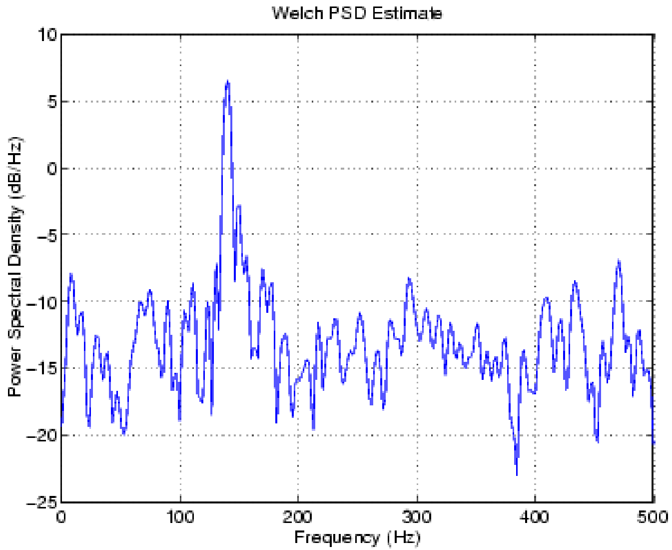
The Welch Method: Case Study

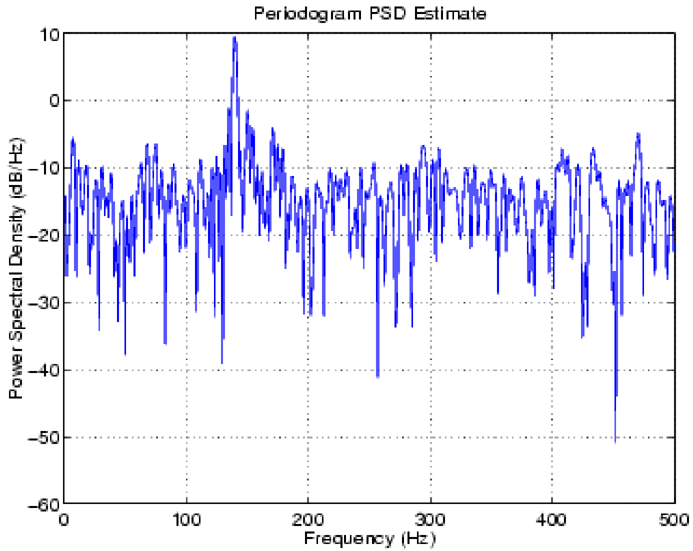
- **We consider sum of two sinusoids and noise**
- **Sinusoid frequencies are 140 Hz and 150 Hz**
- **The *periodogram* of this signal is below, frequency resolution is 3.3 Hz**



The Welch Method: Case Study

Welch's spectral estimate is shown below (left) for averaging 3 segments with 50% overlap, rectangular window and the same level of the signal-to-noise-ratio





The Welch Method: Case Study

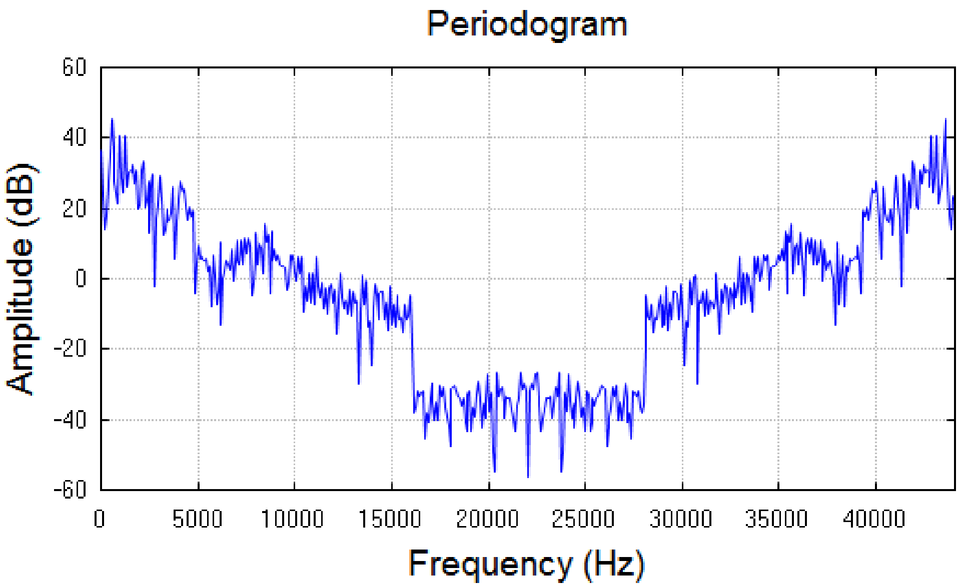
- **In the periodogram in the previous slide, noise and the leakage make one of the sinusoids essentially indistinguishable from the artificial peaks**

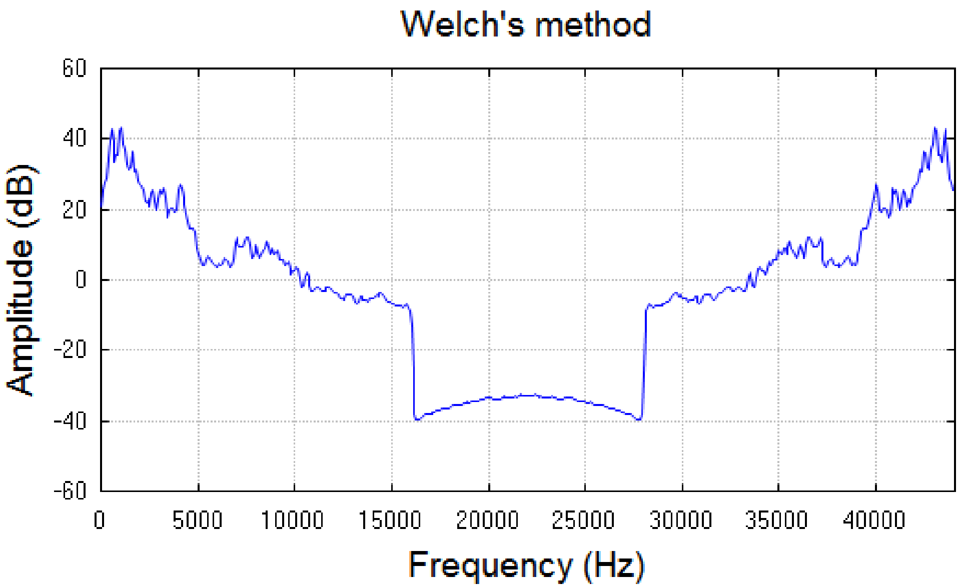
- **In contrast, although the PSD produced by Welch's method has wider peaks (e.g. frequency resolution is poorer), you can still distinguish the two sinusoids, which stand out from the "noise floor"**

The Welch Method

- **Welch's method yields a biased estimator of the power spectral density**
- **The variance of Welch's estimator is difficult to compute because it depends on both the window type used and the amount of overlap between segments**
- **However, the variance is inversely proportional to the number of segments whose modified periodograms are being averaged**

The Welch Method





Performance Characteristics of Nonparametric Methods

- **The quality of the Bartlett and Welch methods is compared below**

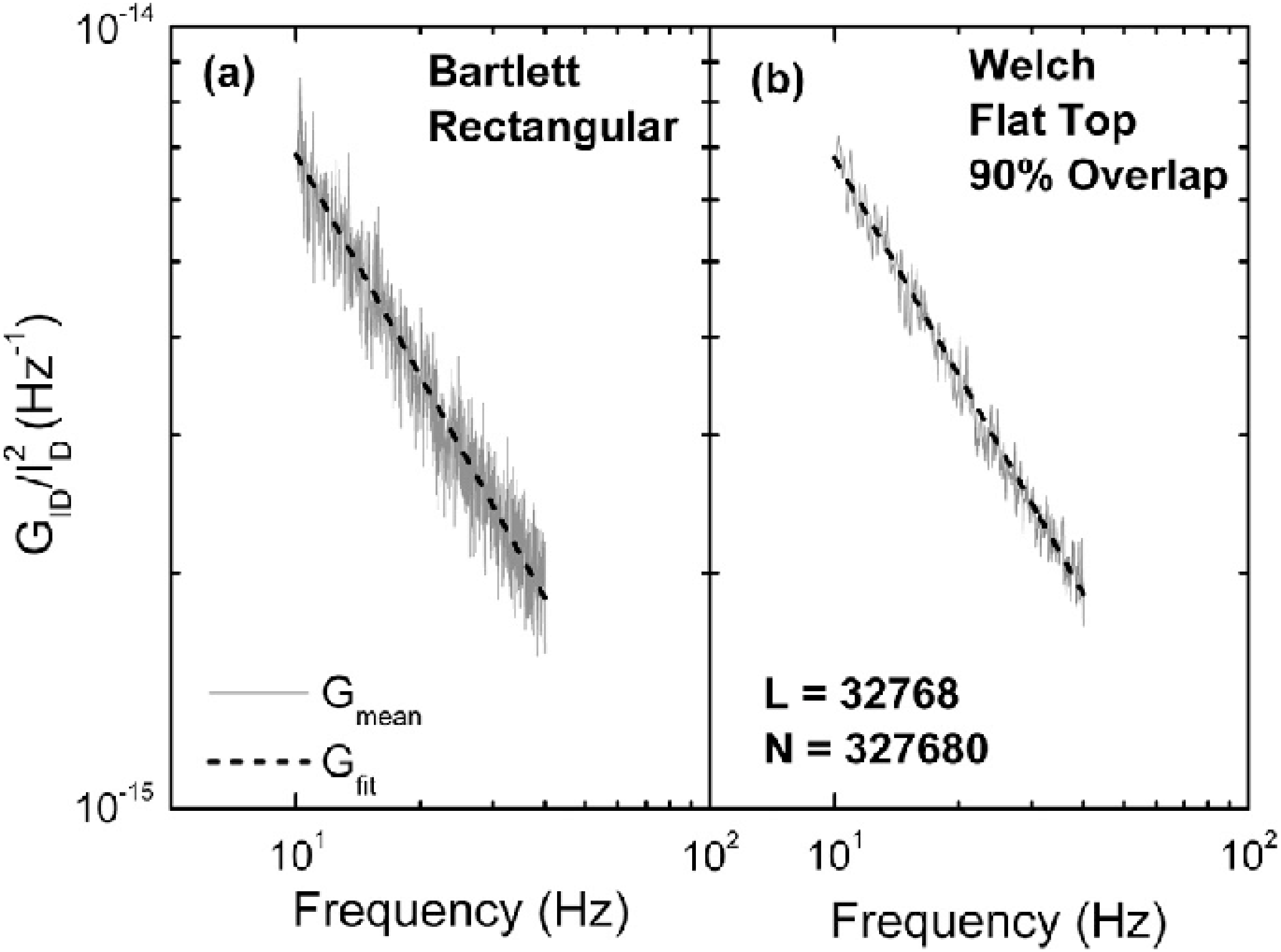
- **As a measure of quality, we use we use the ratio of the square of the estimate mean to the estimate variance**

-

Estimate	Quality
Periodogram	1
Bartlett	$1.1 N \Delta f$
Welch (50%overlap)	$1.4 N \Delta f$

- **where** N **is data length,** Δf **is frequency resolution**

Nonparametric Estimation: Welch Method vs. Bartlett Method



Performance Characteristics of Nonparametric Methods

- **It is apparent from the results that the Welch's estimate is better than the Bartlett's estimate**

- **The main point is that the quality factor increases with an increase in the length of the data**

- **The quality factor depends on the product of the data length and frequency resolution.**

- **For a desired level of quality, the length of the data can be increased (at constant frequency resolution)**

Nonparametric Methods: Brief Summary

- **The nonparametric estimations of the power spectral density are relatively simple, well understood because of their age and nature, and easy to compute using FFT algorithm**

- **However, these methods require the availability of long data records with good signal-to-noise ratio in order to obtain the necessary frequency resolution**

- **Furthermore, these methods suffer from spectral leakage effects, due to time windowing**

- **Often, the spectral leakage masks weak signals**

Parametric Estimation Methods of the PSD

- **In these methods, a model for the signal generation can be constructed with a number of parameters that can be estimated from a signal**

- **From the constructed model and the estimated parameters, we can compute the power spectral density implied by the model**

Parametric Estimation Methods of the PSD: Advantages

- **The modeling approach eliminates the need for window functions**

- **As a consequence, parametric methods avoid the problem of the spectral leakage and provide better frequency resolution than the FFT-based nonparametric methods.**

- **This is especially important in applications where short time signals are available due to transient phenomena**

Parametric Estimation Methods

- **The power spectral density of the output data from model filter is**

$$S_{xx}(f) = \left|H(f)\right|^2 S_{ii}(f)$$

where $S_{ii}(f)$ **is the power spectral density of the input signal and** $H(f)$ **is the frequency response of the model filter**

Parametric Estimation Methods

- **Since our objective is to estimate the power spectral density $S_{xx}(f)$, it is convenient to assume that the input signal is white noise with mean value zero and variance σ_i^2**

- **The output power spectral density will then simplify to**

$$S_{xx}(f) = \left|H(f)\right|^2 \sigma_i^2$$

- **Hence, the power spectral density can be obtained if the model filter transfer function and white noise variance are known**

Parametric Estimation Methods

- **The parametric analysis consists of three steps:**

- **selecting the appropriate model of the filter**
- **estimating the filter coefficients of the model filter, and the white noise variance from sampled signal data**
- **evaluating the power spectral density according to equation**

$$S_{xx}(f) = \left|H(f)\right|^2 \; \sigma_i^{\;2}$$

Zero-Padding

- **When the DFT and PSD are implemented, it is useful to add a certain number of zeros after the time signal sequence in order**

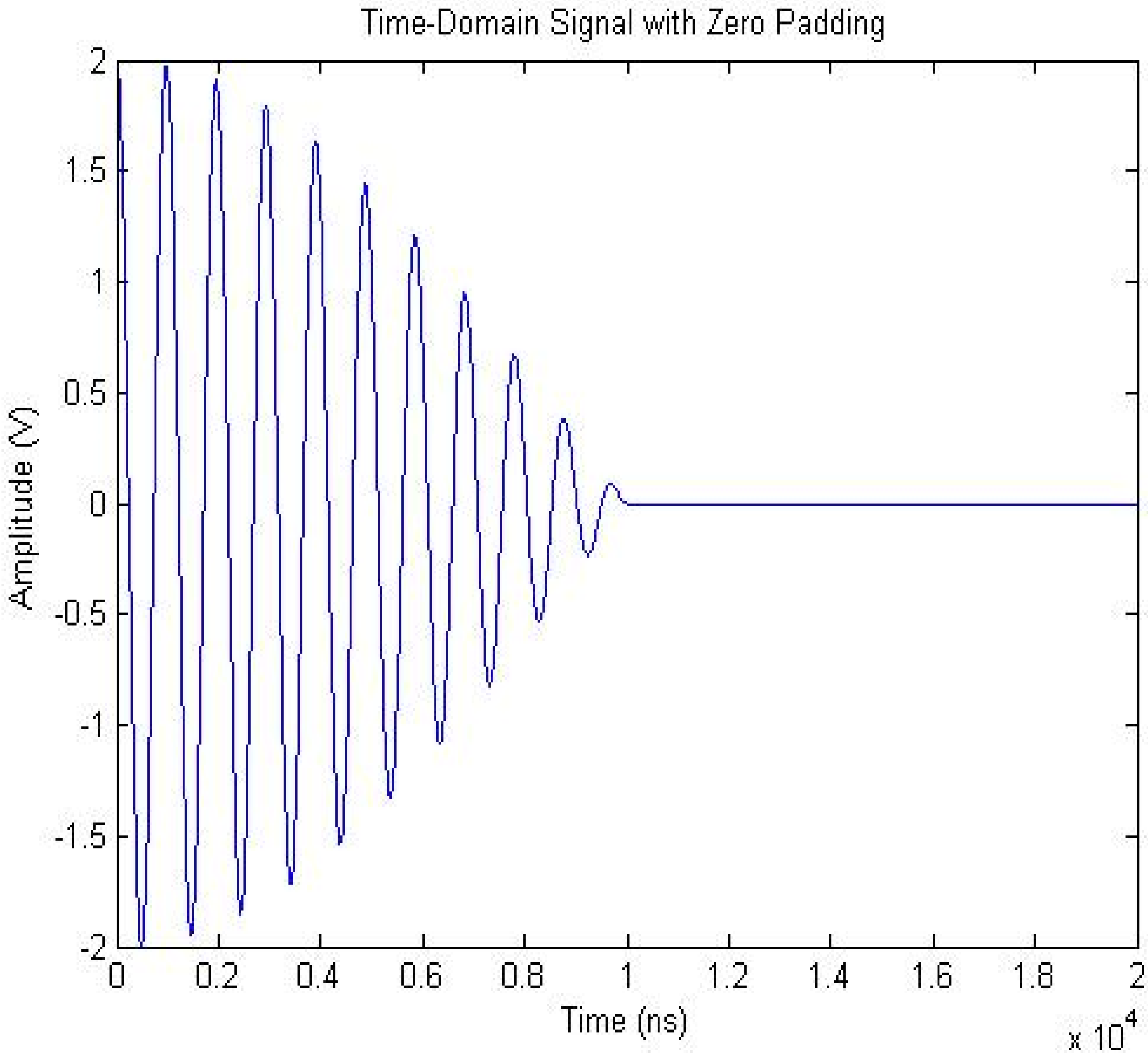
- **to improve the quality of the DFT and PSD presentations**
- **to increase sequence to at least the next higher power of 2 for FFT implementation**

- **MATLAB function**

fft (x, N **)**

automatically zero-pads the sequence if the sequence length and DFT length are different

Zero-Padding



Zero-Padding

- ***Zero padding*** **consists of appending zeros to a signal**
- **It maps a** N **length signal to a length** $M > N$ **signal by**

$$ZeroPad\ \ x(n) = \begin{cases} x(n), & 0 \le n \le N-1 \\ 0 & N \le n \le M-1 \end{cases}$$

- **For example:**

$$\textsc{ZeroPad}_{10}([1,2,3,4,5]) = [1,2,3,4,5,0,0,0,0,0]$$

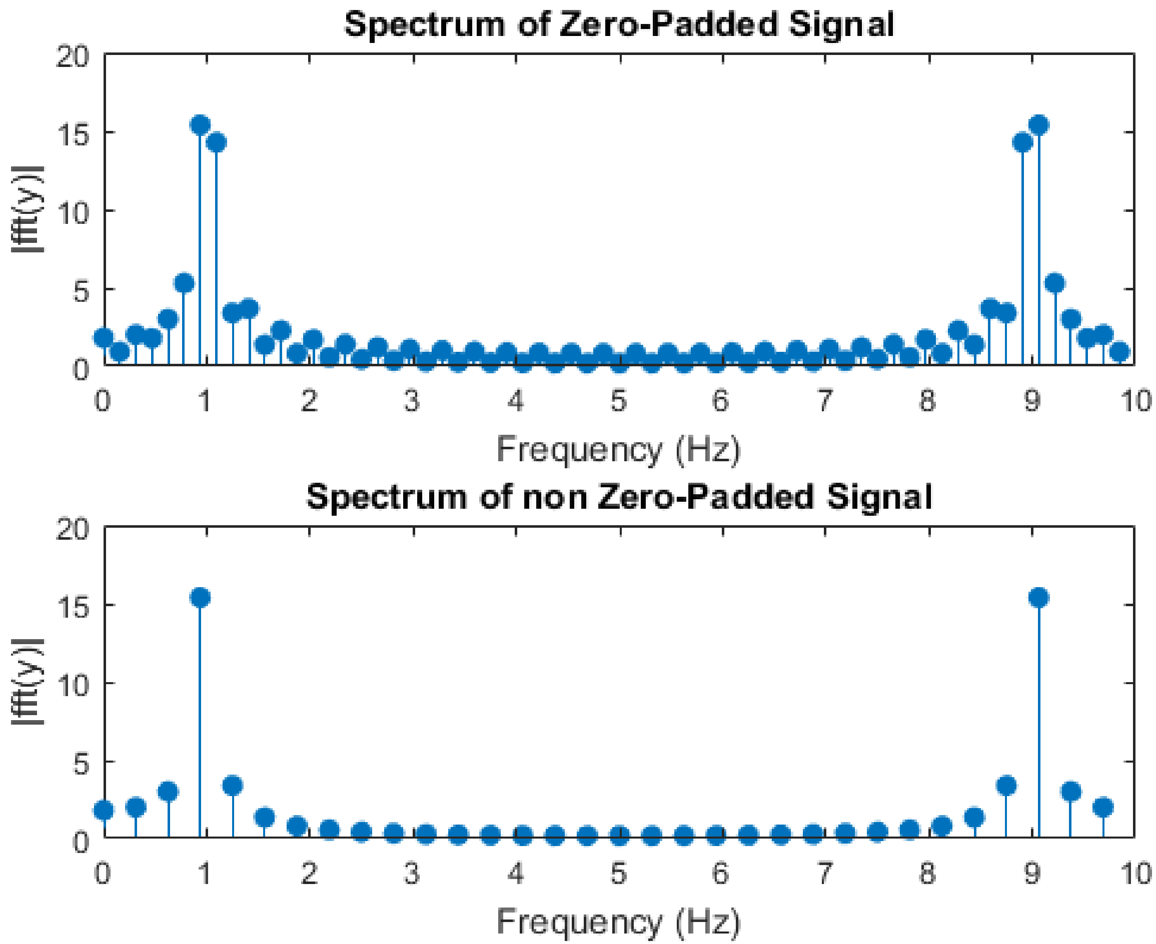
- **Zero padding is not increasing the frequency resolution of the power spectral density of the signal and does not provide any additional information about this power spectral density**

- **Zero padding provides faster FFT calculation**

Zero-Padding

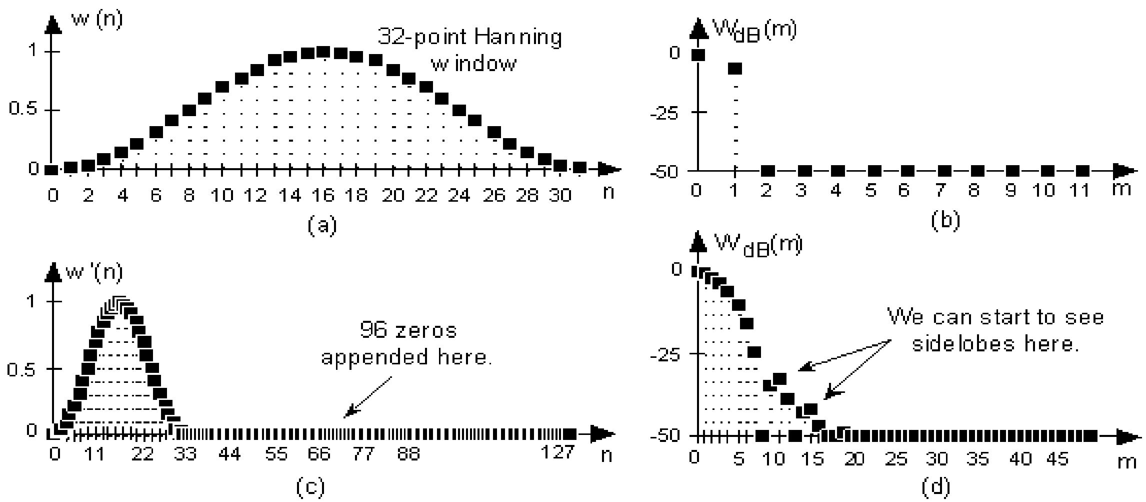
Zero padding can be considered as interpolation of original $X(k)$ to make the appearance of $X(k)$ more smooth

Effect of Zero-Padding



Zero-Padding: Case Study

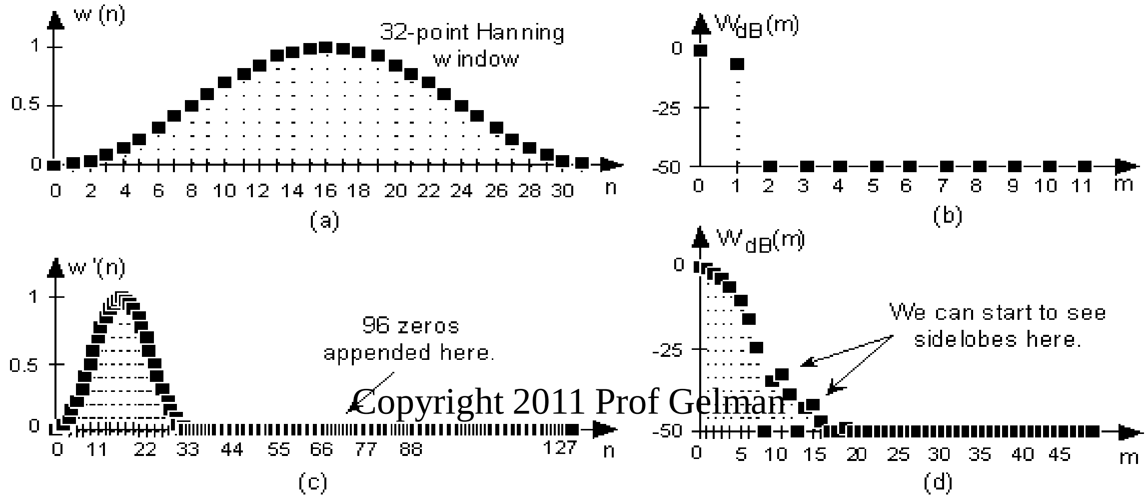
- **Figure *a* shows a Hanning window sequence defined by 32 time samples.**
- **If we take a 32-point DFT and just look at the magnitudes, we get the power spectral density plot shown in Figure *b***



- **If, on the other hand, we zero-pad 96 zero-valued samples to the end of time sequence, we'll have the 128-sample sequence shown in Figure *c***

Zero-Padding: Case Study

- The spectral magnitude plot of a 128-point DFT is shown in Figure *d*, where we can see the more detailed structure of the Fourier transform of a Hanning window.

- We can say that zero padding in the time domain results in an increased sampling rate in the frequency domain
- Here the zero padding increased our frequency-domain sampling by a factor of four
- 

Zero-Padding: Case Study

- **For each sample in Figure *b*, we have four samples in Figure *d***
- **Thus we achieve a better presentation of the periodogram**
- **The amount of zero padding used depends on the complexity allowed, because the larger the amount of zero padding the greater computational and storage requirements.**