The Higher Order Spectra: the Bispectrum and the Trispectrum

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The Classical Higher Order Spectra: the Bispectrum and the Trispectrum

The Classical Higher Order Spectra

The classical nonlinear Fourier-based higher order spectra (HOS) of order *n* for *(n-1)* components can be written as follows:

$$H_{s_i s_j \dots s_k}(i,\ j,\ \dots,\ k,\ T) = \frac{1}{M}\sum_{m=1}^{M} X_m(f_{mi})X_m(f_{mj})\dots X_m(f_{mk})X_m^{\ *}(f_{mi}+f_{mj}+\dots+f_{mk})$$

where *Xm (f)* is the discrete Fourier transform of the *m*th segment of a signal, * is a symbol of the complex conjugate

The Classical Bispectrum

The classical bispectrum depends on two frequencies and defined by the Fourier transforms of a signal at three frequencies:

$$Bisp\ (k_1, k_2) = \frac{1}{M} \sum_{m=1}^{M} X_m(k_1) X_m(k_2) X_m^{*}(k_1 + k_2)$$

where X_m (k) is the discrete Fourier transform of the mth segment of a signal at discrete frequency k

The Classical Bicoherence and Skewness

The bicoherence and the skewness are the normalized bispectrums:

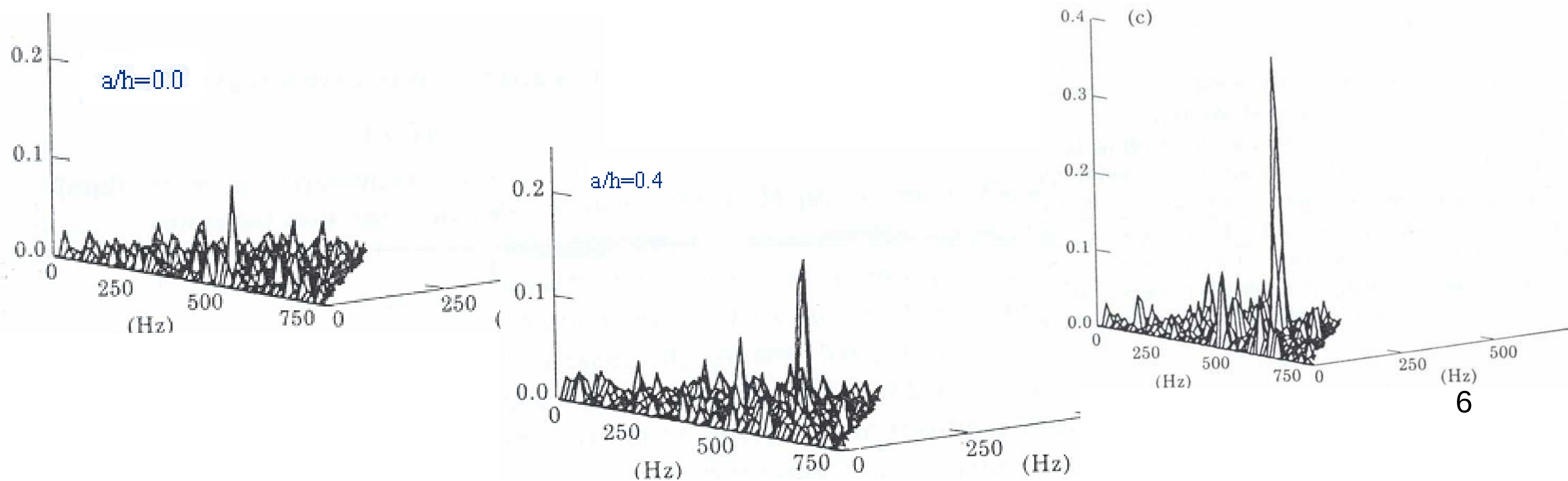
$$b(k_1,k_2)=\frac{\sum_{m=1}^{M} X_m(k_1)X_m(k_2)X_m(k_1+k_2)^*}{\sqrt{\sum_{m=1}^{M}\left|X_m(k_1)_m X(k_2)\right|^2 \sum_{m=1}^{M}\left|X_m(k_1+k_2)\right|^2}}$$

$$sk(k_1,k_2)=\frac{\sum_{m=1}^{M} X_m(k_1)X_m(k_2)X_m(k_1+k_2)^*}{\sqrt{\sum_{m=1}^{M}\left|X_m(k_1)\right|^2 \cdot \sum_{m=1}^{M}\left|X_m(k_2)\right|^2 \cdot \sum_{m=1}^{M}\left|X_m(k_1+k_2)\right|^2}}$$

The Bicoherence: Nonlinearity Detection

The bicoherence and the skewness are very sensitive to the presence of nonlinearity

Bicoherence changes during nonlinearity change



Nonlinearity Detection Based on the Bicoherence

- **The proposed** detection is **based on the magnitude of the bicoherence at frequencies that are integer multiples of the characteristic frequency of an object:**

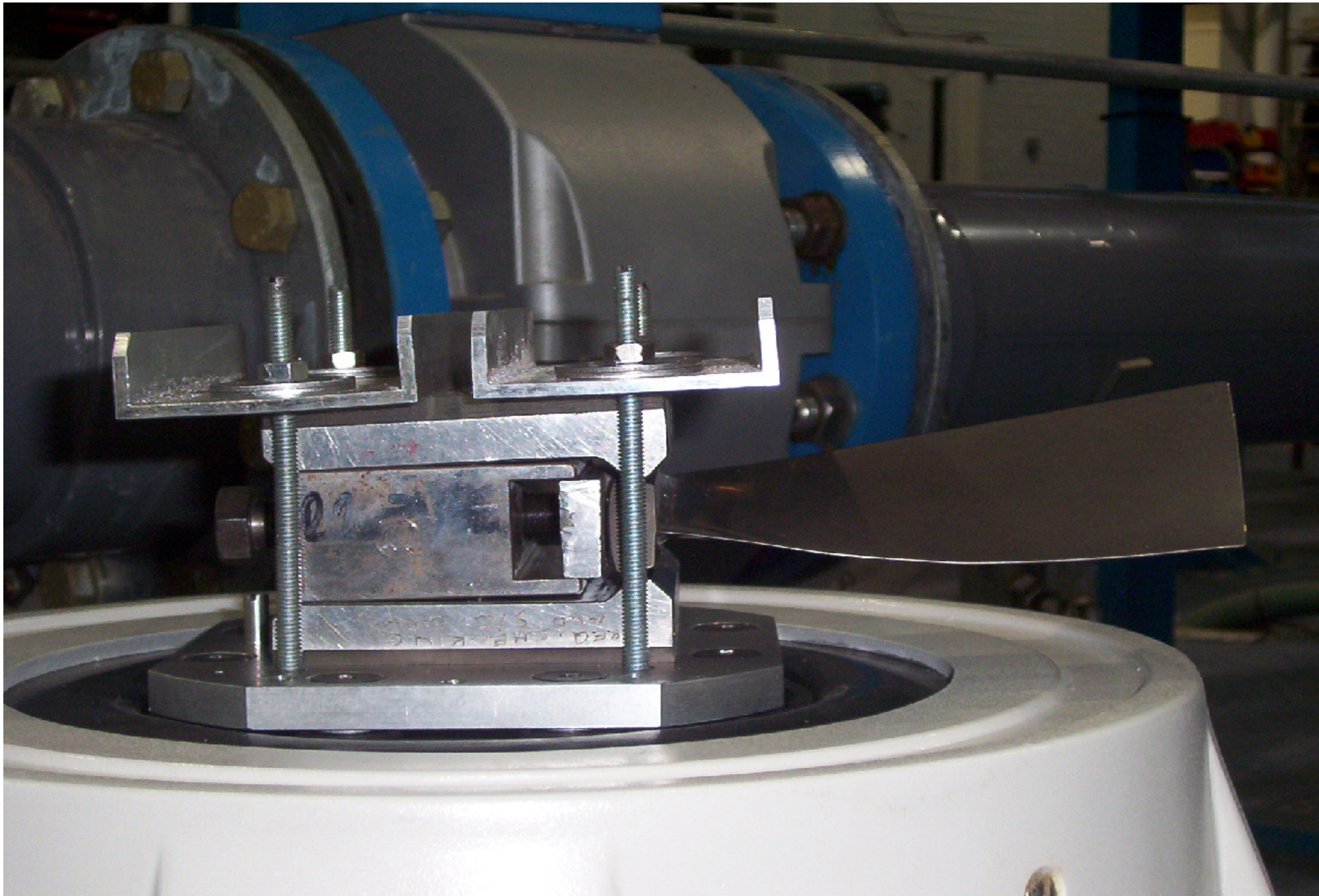
$$b(pk_0, qk_0)$$

where k_0 **is the characteristic frequency and (*p*, *q*) is an integer pair**

- **Normally, the bicoherence between the fundamental and the second harmonics of characteristic frequency is used for nonlinearity detection**

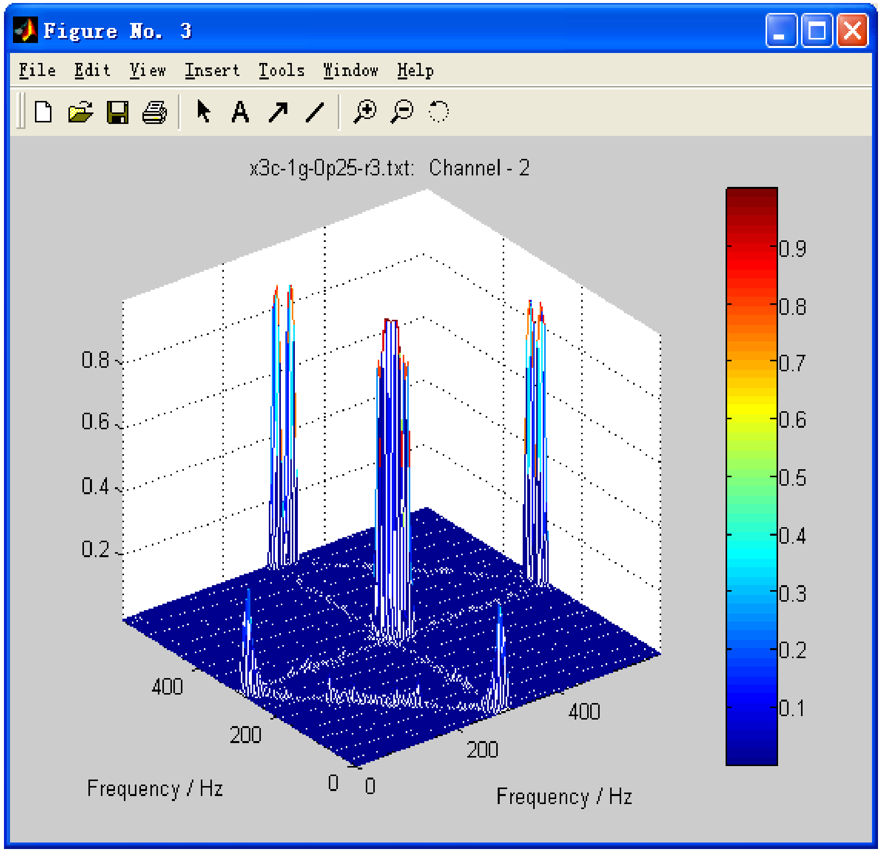
Case Study: Experiments with Engine Blades

Experiments with Engine Blades

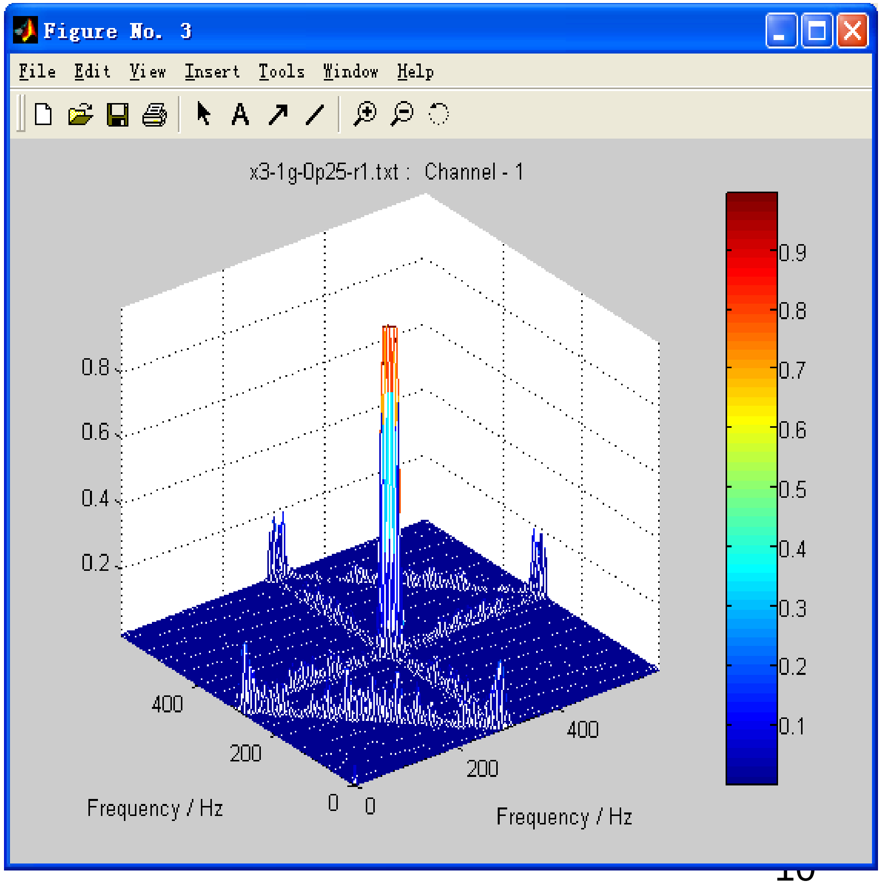


The Bicoherence of Damaged and Undamaged Blades

Damaged blade, 8% crack size



Undamaged blade



The Classical Trispectrum, the Kurtosis and the Tricoherence

- The trispectrum is a function of three frequencies and defined by the Fourier transforms of a signal at four frequencies

$$Trisp\,(k_1,k_2,k_3)=\frac{1}{M}\sum_{m=1}^{M}X_m(k_1)X_m(k_2)X(k_3)X_m^{\,*}(k_1+k_2+k_3)$$

- The tricoherence is a normalized trispectrum:

$$t\,(k_1,k_2,k_3)=\frac{Trisp(k_1,k_2,k_3)}{\sqrt{E\left[\left|\mathbf{X}(k_1)\mathbf{X}(k_2)\mathbf{X}(k_3)\right|^2\right]E\left[\left|\mathbf{X}(k_1+k_2+k_3)\right|^2\right]}}$$

- The kurtosis is also a normalized trispectrum:

$$k\left(k_1,k_2,k_3\right)=\frac{\sum_{m=1}^{M}X_m\left(k_1\right)X_m\left(k_2\right)X_m\left(k_1+k_2\right)^*}{\sqrt{\sum_{m=1}^{M}\left|X_m\left(k_1\right)\right|^2\cdot\sum_{m=1}^{M}\left|X_m\left(k_2\right)\right|^2\cdot\sum_{m=1}^{M}\left|X_m(k_3)\right|^2\sum_{m=1}^{M}\left|X_m\left(k_1+k_2+k_3\right)\right|^2}}$$

The Bicoherence and the Tricoherence

- **The bicoherence and the tricoherence are varying between 0 and 1**

- **The bicoherence and the tricoherence contain both amplitude and phase information between the fundamental and higher harmonics of signals**

Nonlinearity Detection Based on the Tricoherence

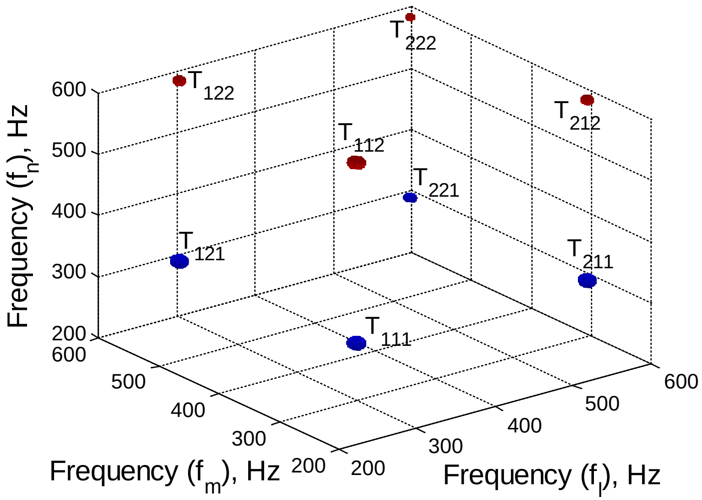
- **The proposed detection is based on the magnitude of the tricoherence at frequencies that are integer multiples of the characteristic frequency of an object:**

$$b(pk_0, qk_0, rk_0)$$

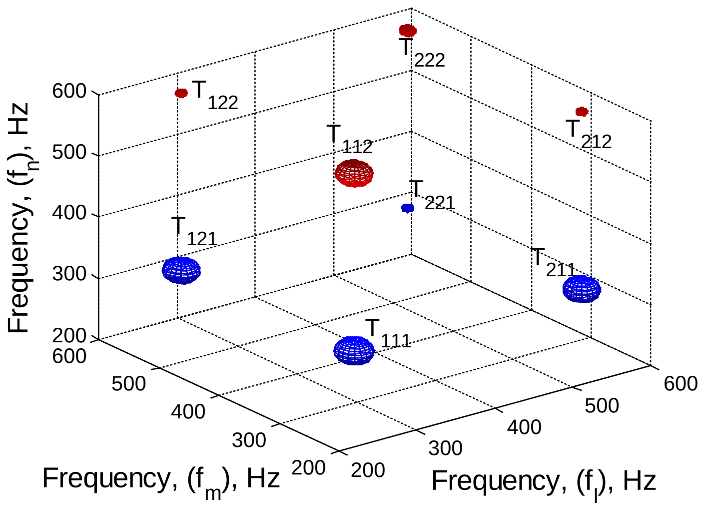
where k_0 **is the characteristic frequency**

- **Normally, the tricoherence between the fundamental and the higher harmonics of the characteristic frequency is used for nonlinearity detection**

Case Study: The Tricoherence of Undamaged and Damaged Blades

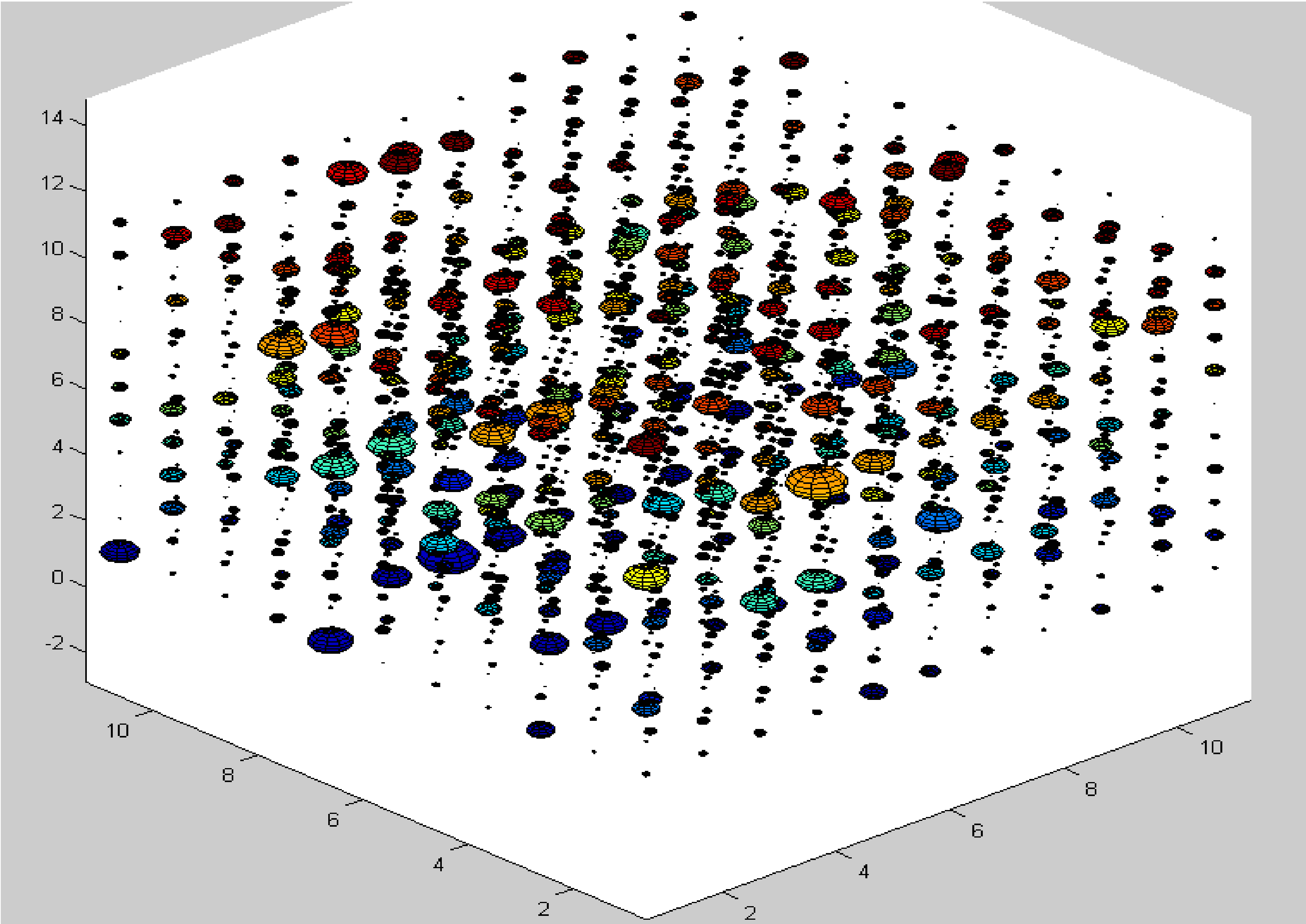


Undamaged blade

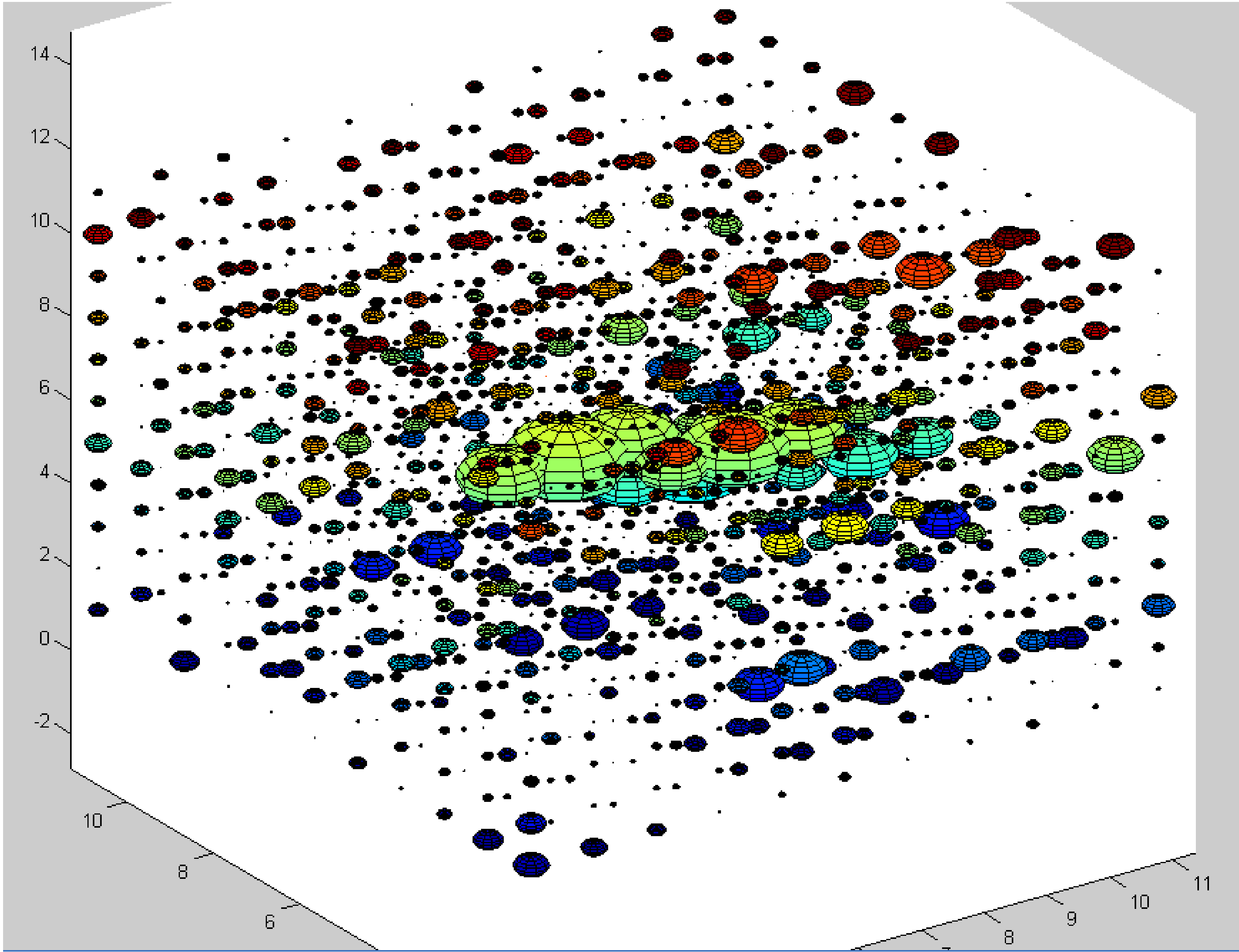


Damaged blade, 8% crack size

The Tricoherence: a Low Nonlinearity



The Tricoherence: a High Nonlinearity



New Technique, the Bispectrum and the Trispectrum Based on the Chirp-Fourier Transform for Transient Signals

The Bispectrum and the Trispectrum Based on the Chirp-Fourier Transform

- **The classical bispectrum and trispectrum are not suitable for transient signals with instantaneous frequency variation because the stationary kernel of the Fourier transform is not matched with these signals**

- **It is proposed the new techniques for transient signals with linear and nonlinear polynomial variation of instantaneous frequency:**

the bispectrum and trispectrum based on the short time higher order chirp-Fourier transforms

The Bispectrum and the Trispectrum Based on the Chirp-Fourier Transform

- **The proposed bispectrum (the trispectrum) depend on two (three) frequencies and defined by the chirp-Fourier transforms at these frequencies:**

$$Bisp\ (k_1,k_2)=\frac{1}{M}\sum_{m=1}^{M}X_m(k_1)X_m(k_2)X_m{}^*(k_1+k_2)$$

$$Trisp\ (k_1,k_2,k_3)=\frac{1}{M}\sum_{m=1}^{M}X_m(k_1)X_m(k_2)X(k_3)X_m{}^*(k_1+k_2+k_3)$$

where *Xm* (*k*) is the discrete chirp-Fourier transform of the *m*th segment of a signal at discrete frequency *k*

The Bispectrum and Trispectrum Based on the Chirp-Fourier Transform

• **The proposed bispectrum (trispectrum) is** $(N+1)$ **or** $(N+2)$ **dimensional transform, i. e. two (three) frequencies and** $(N-1)$ **adaptive parameters of the higher order chirp-Fourier transform**

• **Physical significance of the proposed techniques is that these techniques use transient (not stationary) polynomial kernel which is suitable for transient signals**

The Bispectrum and the Trispectrum Based on the Chirp-Fourier Transform

- **The proposed technique is suitable for the transient signals with polynomial variation of the instantaneous frequency because the non-traditional transient kernel of the higher order chirp-Fourier transform is matched with these signals**

- **This proposition also includes (as an important particular case) the bispectrum and the trispectrum based on the chirp-Fourier transform for transient signals with the *linear* variation of the instantaneous frequency.**

The Bispectrum and the Trispectrum Based on the Chirp-Fourier Transform

- **The proposed technique should be estimated by the following steps:**

- **select an external time window and slide the time center of the window with an overlapping via a transient signal**

- **window a transient signal into overlapping time blocks and divide each time block into overlapping segments using an internal window**

- **estimate for each segment the higher order chirp-Fourier transforms**

- **estimate the product of these higher order chirp-Fourier transforms**

- **average these products over each time block**

The Bispectrum and the Trispectrum Based on the Chirp-Fourier Transform

The proposed technique is a generalization of the classical bicoherence and the tricoherence based on the Fourier transform for the case of transient signals with the linear and *nonlinear* polynomial variation of the instantaneous frequency.

The Adaptive Bispectrum Based on the Chirp-Fourier Transform vs. Non-Adaptive Classical Fourier Based Bispectrum

Nonlinear Model

- **To compare the proposed and the classical techniques, an input transient random sinusoidal signal with constant amplitude, random initial phase and linearly or quadratic changed frequency in time has been passed via the following nonlinear model**

- $$\begin{cases} \ddot{x} + 2h\dot{x} + \omega_S x = A\cos[\omega(t)t + \varphi], \;\; x \geq 0, \\ \\ \ddot{x} + 2h\dot{x} + \omega_C x = A\cos[\omega(t)t + \varphi], \;\; x < 0, \end{cases}$$

- **This model is widely used for nonlinearity simulation**

Nonlinearity Detection by the Bispectrum Based on the Chirp-Fourier Transform

Nonlinearity is detected by the proposed bispectrum based on the

- **the chirp-Fourier transform in the case of the linear frequency variation of the excitation**

- **the higher order chirp-Fourier transform in the case of the quadratic frequency variation of the excitation**

Simulation Parameters

Parameters used for simulation are as follows:

- **the frequency speed of the linear excitation is in the range (0-18Hz/s)**

- **the frequency acceleration of the quadratic excitation is in the range (0-6Hz/s^2)**

- **variation of the excitation frequency is in the range (1400-1560) Hz**

- **the sampling frequency is 20kHz**

- **the relative damage sizes are 0 (undamaged specimen) and 0.1 (damaged specimen)**

- **the resonance frequencies of the undamaged and damaged specimens are 1500 Hz and 1460.5 Hz respectively; damping is 250.**

Simulation Parameters

- **The Gaussian white noise was added to the output signal to hinder the nonlinearity detection and more closely mimic data from the early stage of nonlinearity**

- **The signal/noise ratio is 40dB.**

Nonlinearity Detection by the Bispectrum

- **The fully adaptive chirp-Fourier bicoherence at the fundamental and the second harmonics is used for nonlinearity detection.**

- **Parameters used for bicoherence estimation are as follows:**

- **segment size is 0. 4s, i.e. the frequency resolution is 2.5 Hz**

- **signal duration is 8s**

- **overlapping of the internal Hamming window is 50%**

- **the external window is rectangular in the vicinity of specimen resonance**

- **adaptive bicoherence parameters are matched with the frequency speed or frequency acceleration of the linear or quadratic excitation respectively**

Bispectrum Normalization

Two normalizations of the proposed chirp-Fourier bispectrum were investigated:

- **normalization I based on the chirp-Fourier transform**

- **normalization II based on the classical Fourier transform**

Bispectrum Normalization

The higher order Chirp-Fourier transform or the classical Fourier transform could be used for bispectrum normalization

$$X_m\left(f_j, c_2, \ldots, c_N\right) = \int_{-\infty}^{\infty} x_m(t) \cdot e^{-i\cdot 2\pi \cdot \left(f_j \cdot t + \frac{c_2}{2} t^2 + \ldots + \frac{c_N}{N} t^N\right)} dt$$

$$b\left(f_1, f_2\right) = \frac{X\left(f_1\right) \cdot X\left(f_2\right) \cdot X^*\left(f_1 + f_2\right)}{\sqrt{\left|X\left(f_1\right) \cdot X\left(f_2\right)\right|^2 \cdot \left|X\left(f_1 + f_2\right)\right|^2}}$$

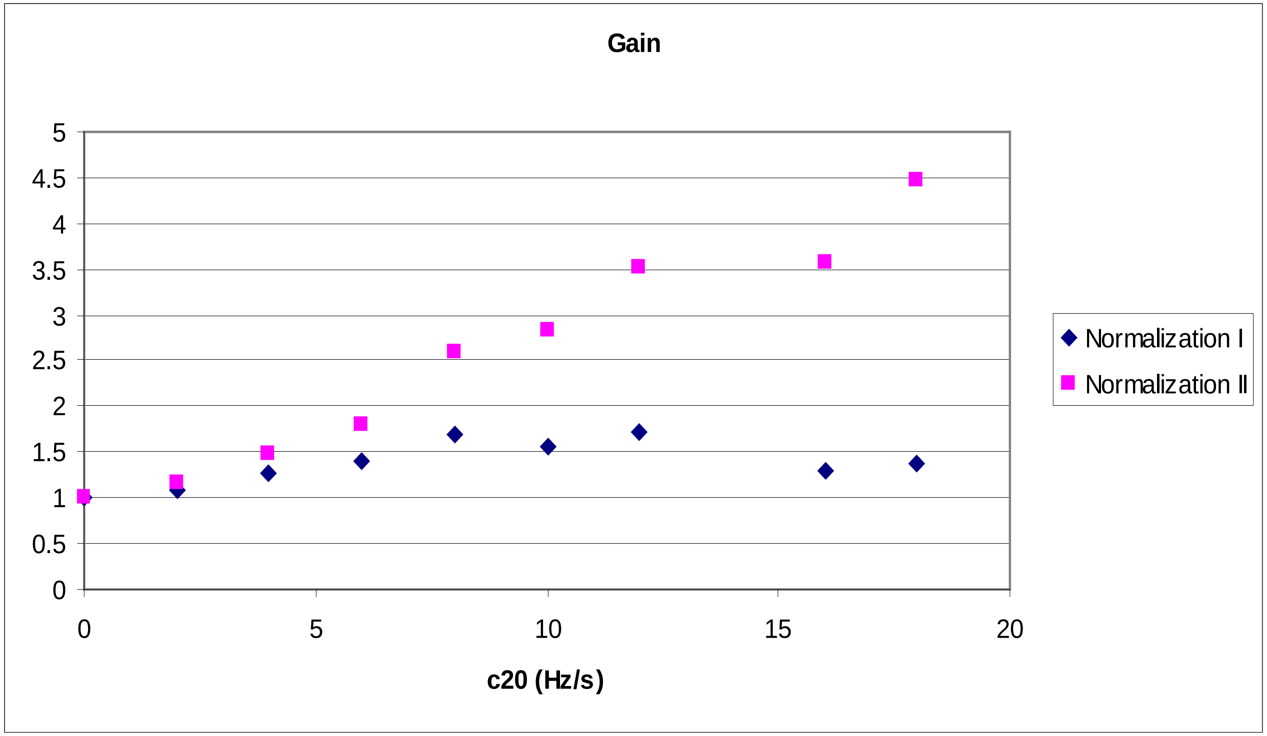
$$X_m\left(f_j, c_2, \ldots, c_N\right) = \int_{-\infty}^{\infty} x_m(t) \cdot e^{-i\cdot 2\pi \cdot \left(f_j \cdot t + \frac{c_2}{2} t^2 + \ldots + \frac{c_N}{N} t^N\right)} dt$$

or

$$X\left(f\right) = \int_{-\infty}^{\infty} x(t) e^{-2\pi f t} dt$$

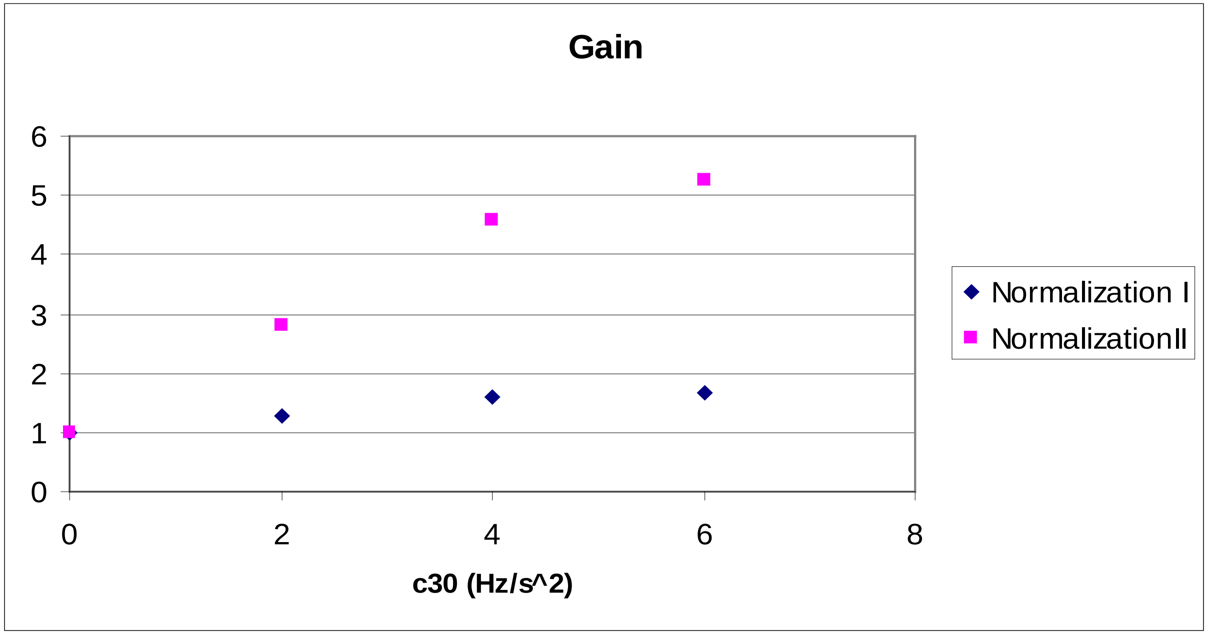
The Proposed Bispectrum vs. the Classical Bispectrum

- **800 simulation tests were performed for the linear case**
- **The gains, the ratios of the effectiveness criterion (i.e. the Fisher criterion) for the proposed and the classical techniques, are shown below**



The Proposed Bispectrum vs. the Classical Bispectrum

- **800 simulation tests were performed the quadratic case**
- **The gains, the ratios of the Fisher criteria for the proposed and the classical techniques, are shown below**



The Proposed Bispectrum vs. the Classical Bispectrum

- **The gains for the best bispectrum normalization II are in the ranges (1.25-4.5) and (3-5) in the linear and quadratic cases respectively**

- **The gain increases with increment of the frequency speed and frequency acceleration**

- **Thus, the proposed technique provides higher estimates of the Fisher criterion in comparison with the classical technique and is, therefore, more effective for nonlinearity detection in transient conditions**

The Proposed Trispectrum vs. the Fourier Based Trispectrum: Case Study

- **The trispectrum value at the fundamental, the fundamental and the second harmonics (T112) are often used for nonlinearity detection.**

- **In practice, the normalized trispectrums, i. e. the tricoherence or the kurtosis are employed for nonlinearity detection.**

The Proposed Trispectrum vs. the Classical Trispectrum

- **800 simulation tests were performed for the simulation of a nonlinearity, using the linear variation of the instantaneous frequency**

- **The values of the effectiveness indicator, the Fisher criterion, for the classical kurtosis and the kurtosis based on the chirp-Fourier transform at the fundamental, the fundamental and the second harmonics (K112) are 245.7 and 483.9 respectively**

The Proposed Trispectrum vs. the Fourier Based Trispectrum

- **It can be seen that the proposed technique provides higher estimates of the Fisher criterion in comparison with the classical technique and is, therefore, more effective for nonlinearity detection**

- **The gain, the ratio of the Fisher criteria for the proposed and the classical techniques, is 1.97**