Part 2 of PSD

The Spectral Leakage of the PSD

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The Spectral Leakage

- **Consider the power spectral density of a finite-length signal** $x_L[n]$ **, where** L **is the length of a signal**

- **It is frequently useful to interpret** $x_L[n]$ **as the result of multiplying an infinite signal,** $x[n]$**, by a finite-length rectangular window,** $w_R[n]$ **:**

$$x_L[n] = x[n] \cdot w_R[n]$$

The Spectral Leakage

- **The effect of the window and the spectral leakage are best understood for sinusoidal data**

- **Suppose that $x[n]$ is composed of a sum of M complex sinusoids:**

$$x[n] = \sum_{k=1}^{M} A_k e^{j\omega_k n}$$

where A_k **and** $\omega_k = 2\pi f_k$ **are sinusoid amplitude and circular frequency respectively**

The Spectral Leakage

- Its Fourier transform (for an infinite signal) is

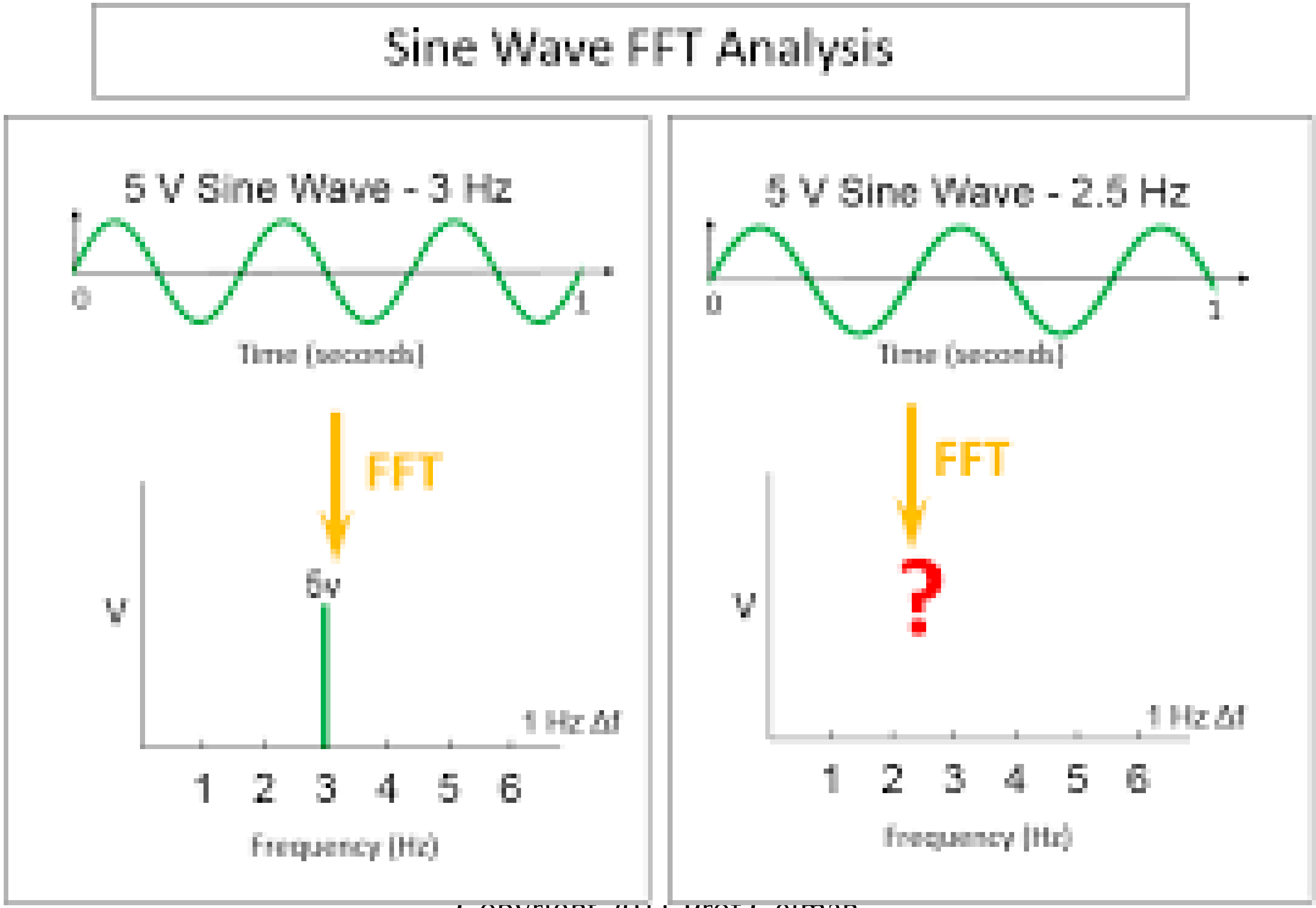
$$X(f) = f_s \sum_{k=1}^{M} A_k \delta(f - f_k)$$

which for a finite-length signal becomes

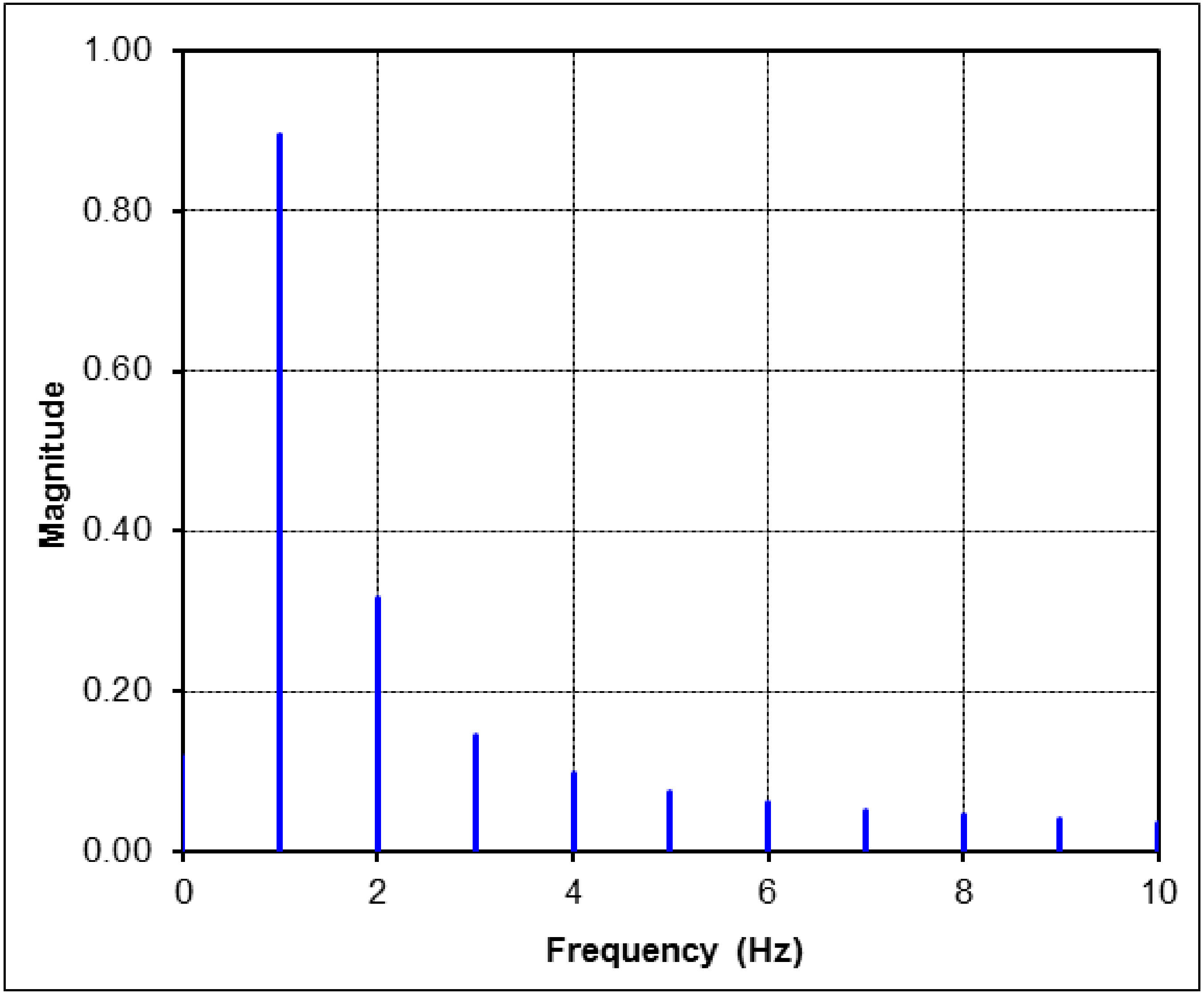
$$X_L(f) = \int_{-f_s/2}^{f_s/2} \left(\sum_{k=1}^{M} A_k \delta(\rho - f_k) \right) W_R(f - \rho) d\rho = \sum_{k=1}^{M} A_k W_R(f - f_k)$$

where f_s **is sampling frequency,** $\delta(t)$ **is the Dirac function**

The Fourier Transform for an Infinite Sine Signal



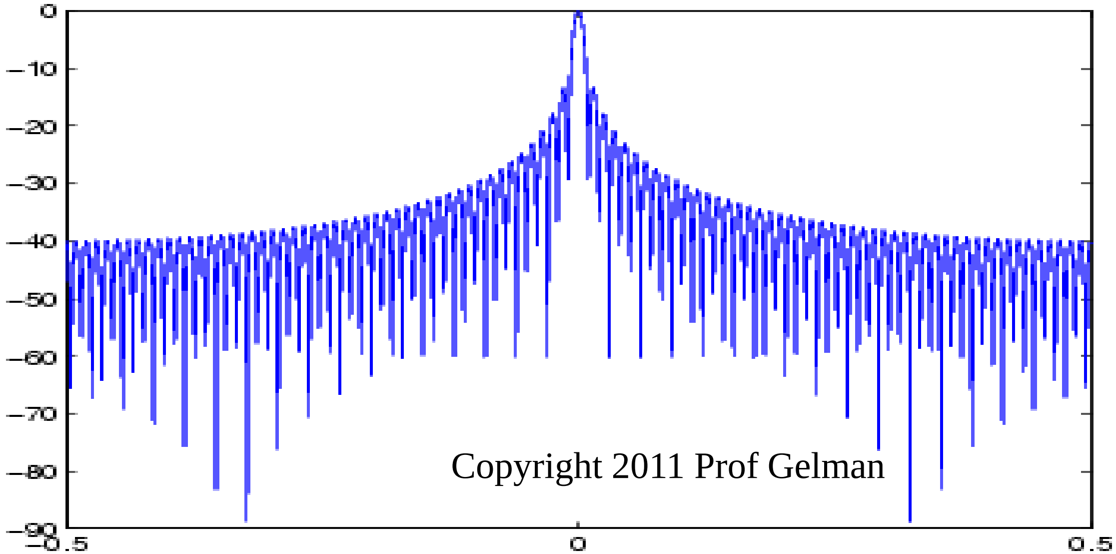
The Fourier Transform for an Infinite Sine Signal



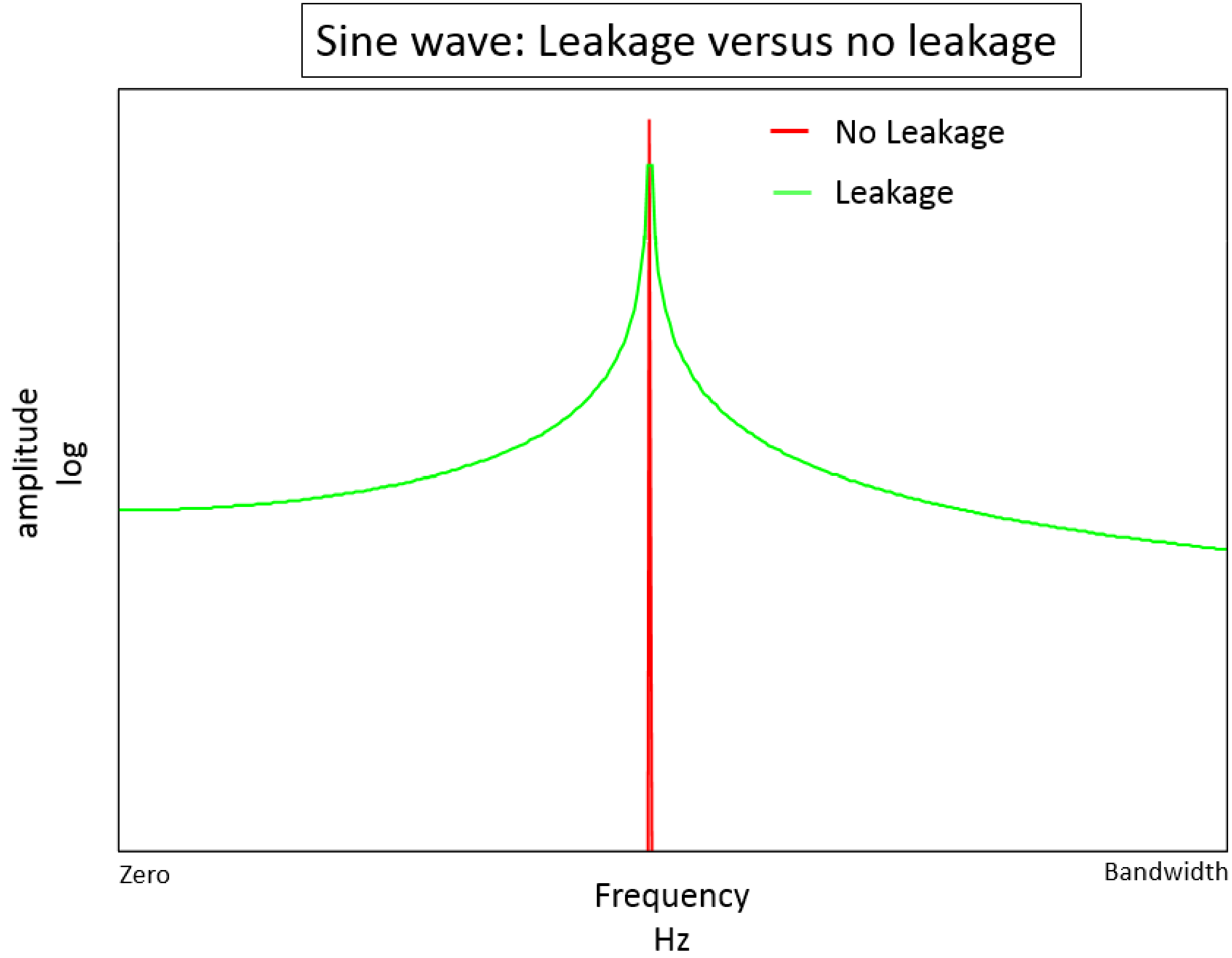
The Spectral Leakage

- So, in the Fourier transform of the finite-length signal, the Dirac functions *have been replaced* by terms of the form $W_R(f - f_k)$, which corresponds to the Fourier transform of a rectangular window

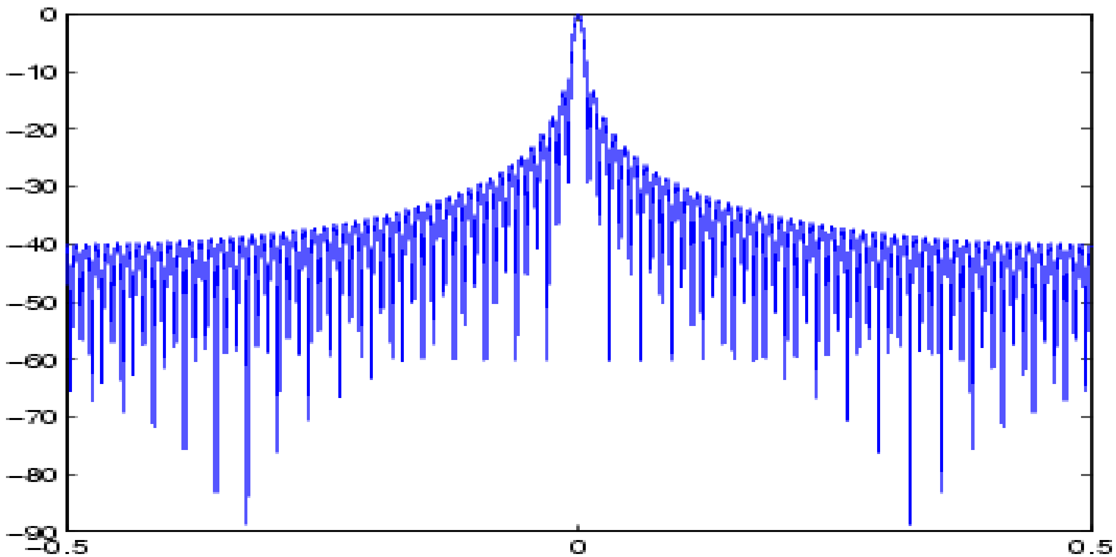
- The frequency response of a rectangular window has the shape of a sinc signal, as shown below



The Fourier Transform for an Infinite Sine Signal



The Spectral Leakage



•**The plot displays the main lobe and several side lobes, the largest of which is 13.5dB below the main lobe peak**

•**These lobes account for the effect known as the *spectral leakage* (the Gibbs phenomenon).**

The Spectral Leakage

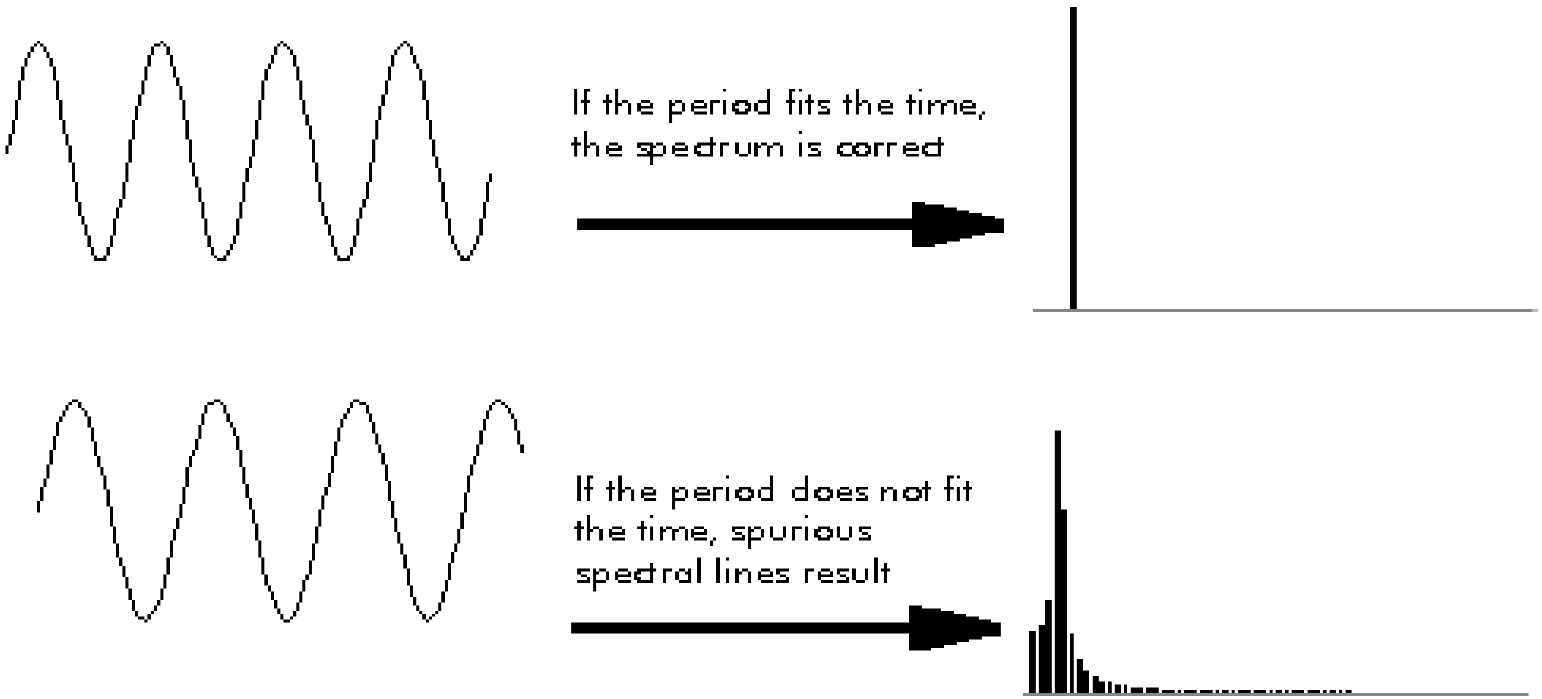
- **While the *infinite-length signal* has its power spectral density concentrated *exactly* at the discrete frequency points, the windowed (or truncated) signal has a continuum of power "leaked" around the discrete frequency points**

- **Let us consider conditions for the leakage appearance**

The Spectral Leakage

- **If a signal is periodic, two main cases arise:**
- **an integral number of periods fit into the total duration of a signal**
- **an integral number of periods does not fit into the total duration of a signal; therefore, signal has little “glitch”**
- **The “glitch” is a short signal**

Spectral Leakage: Two Conditions



The Spectral Leakage

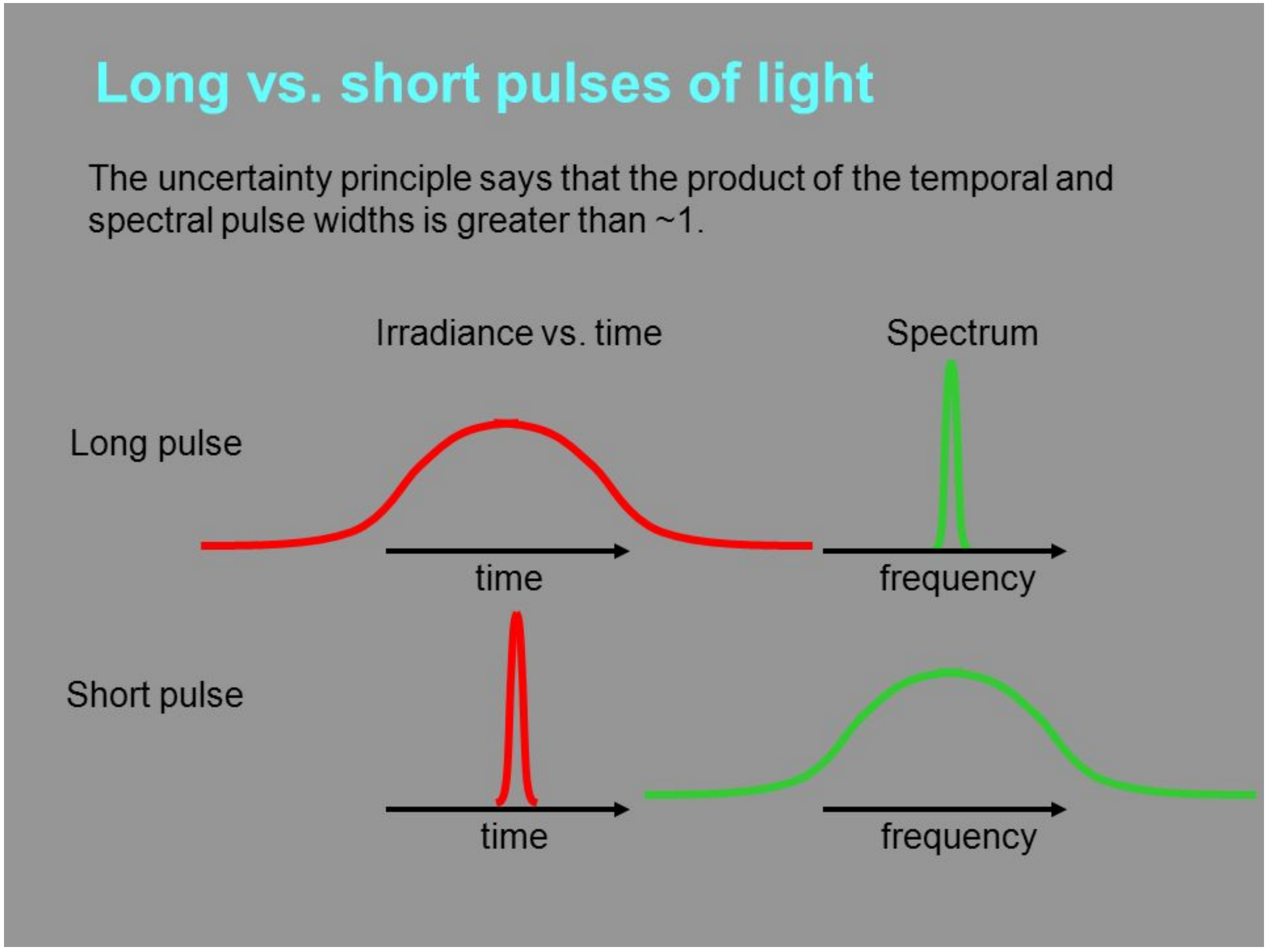
- There is a direct relation between a signal's duration in time and the width of its power spectral density:

- short signals have a broad power spectral density
- long signals have a narrow power spectral density

- So, short glitches have a broad power spectral density.

- This broadening is superimposed on the power spectral density of the actual signal

Long vs. Short Pulses



The Spectral Leakage: Two Conditions

- **if an integral number of periods exactly fits a signal duration, the power spectral density is represented by one single line**

- **if an integral number of periods does not match a signal duration, the power spectral density is broadened**

The Spectral Leakage

- **For example, a sine wave should have a power spectral density which consists of one single line**

- **But in practice, if measured by a spectrum analyzer, the power spectral density will be a broad line - with the side lobes flapping up and down**

- ***When we see a perfect (single line) power spectral density, this has in fact been obtained by tuning the signal frequency carefully so that an integral number of periods exactly fits the measurement time and the power spectral density is the best obtainable***

Time Windowing for Decrease of the Spectral Leakage

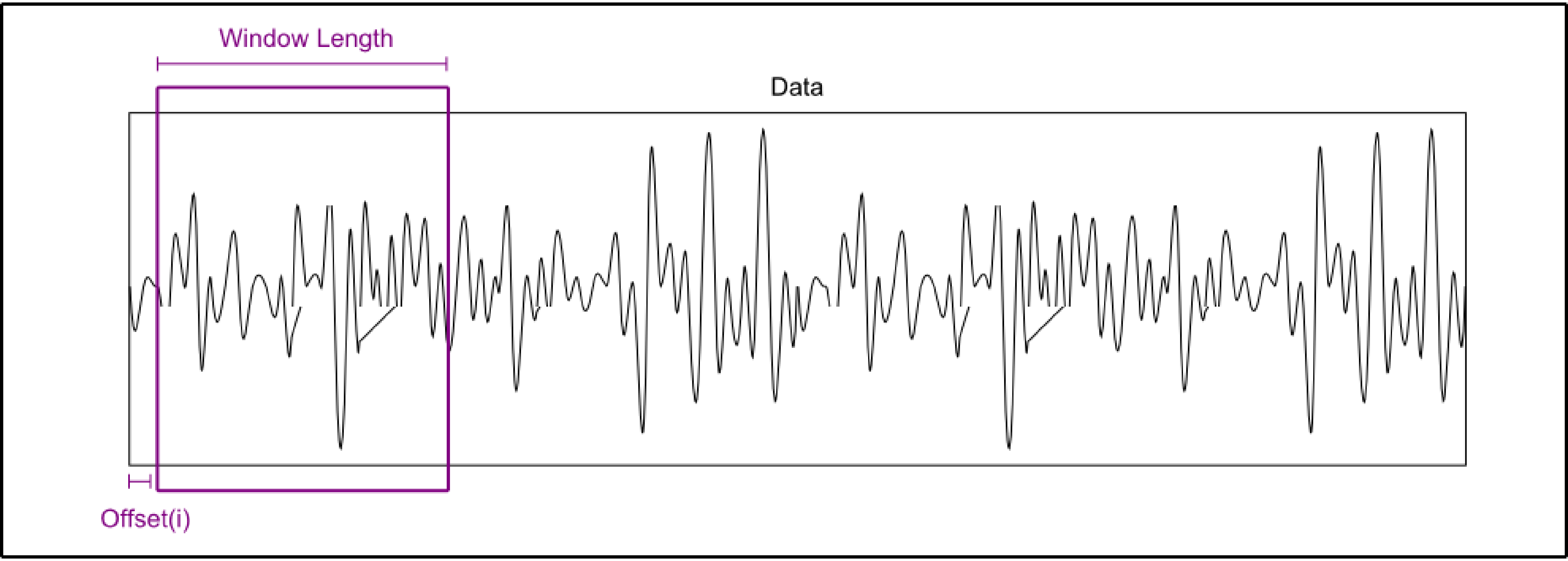
- **Non-rectangular signal window decreases the spectral leakage**

- **The choice of the *analysis window* is important, since it directly affects the spectral leakage (i.e. side-lobe attenuation)**

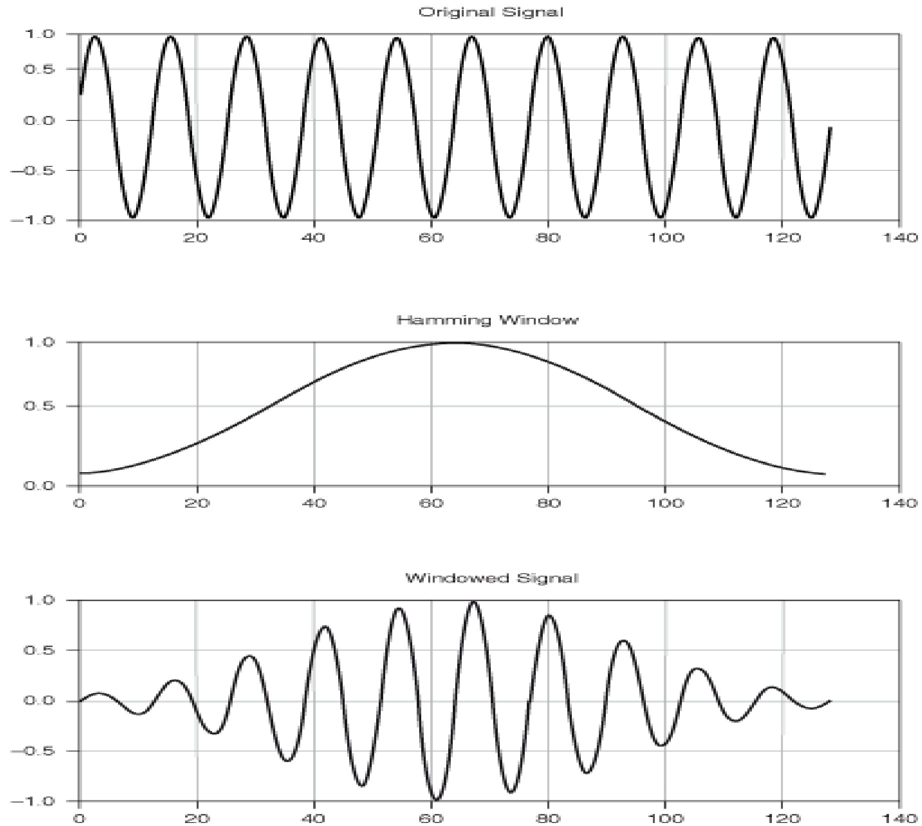
- **To understand the effect of the window, let us consider its effect on a complex sinusoidal signal**

- **It is well-known that the Fourier transform of a windowed sinusoid is the Fourier transform of the window function shifted to be centred at the frequency of the sinusoid**

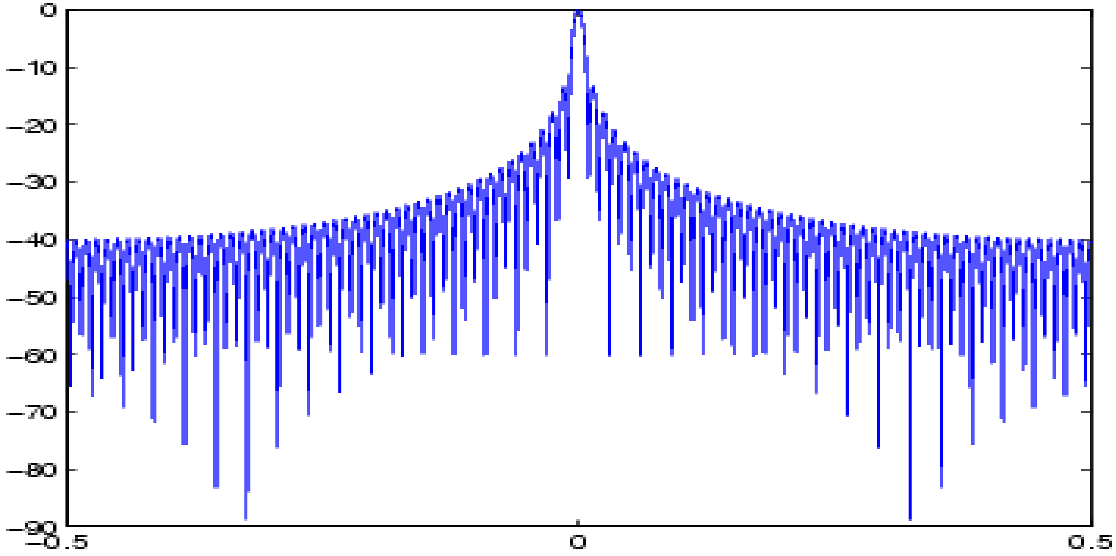
Time Windowing for Decrease of the Spectral Leakage



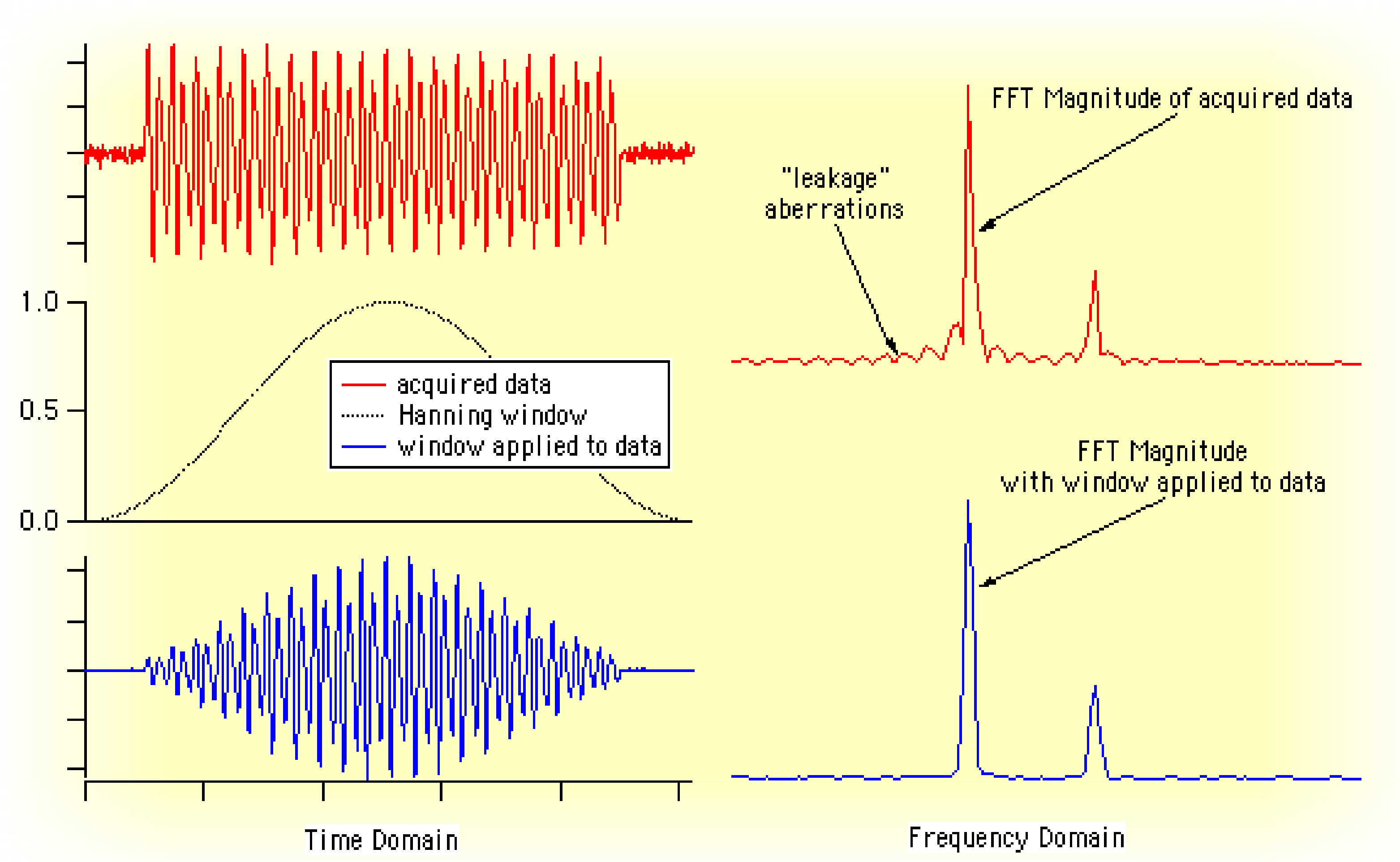
Time Windowing for Decrease of the Spectral Leakage



Case Study: the Windowed Sinusoid



Time Windowing for Decrease of the Spectral Leakage

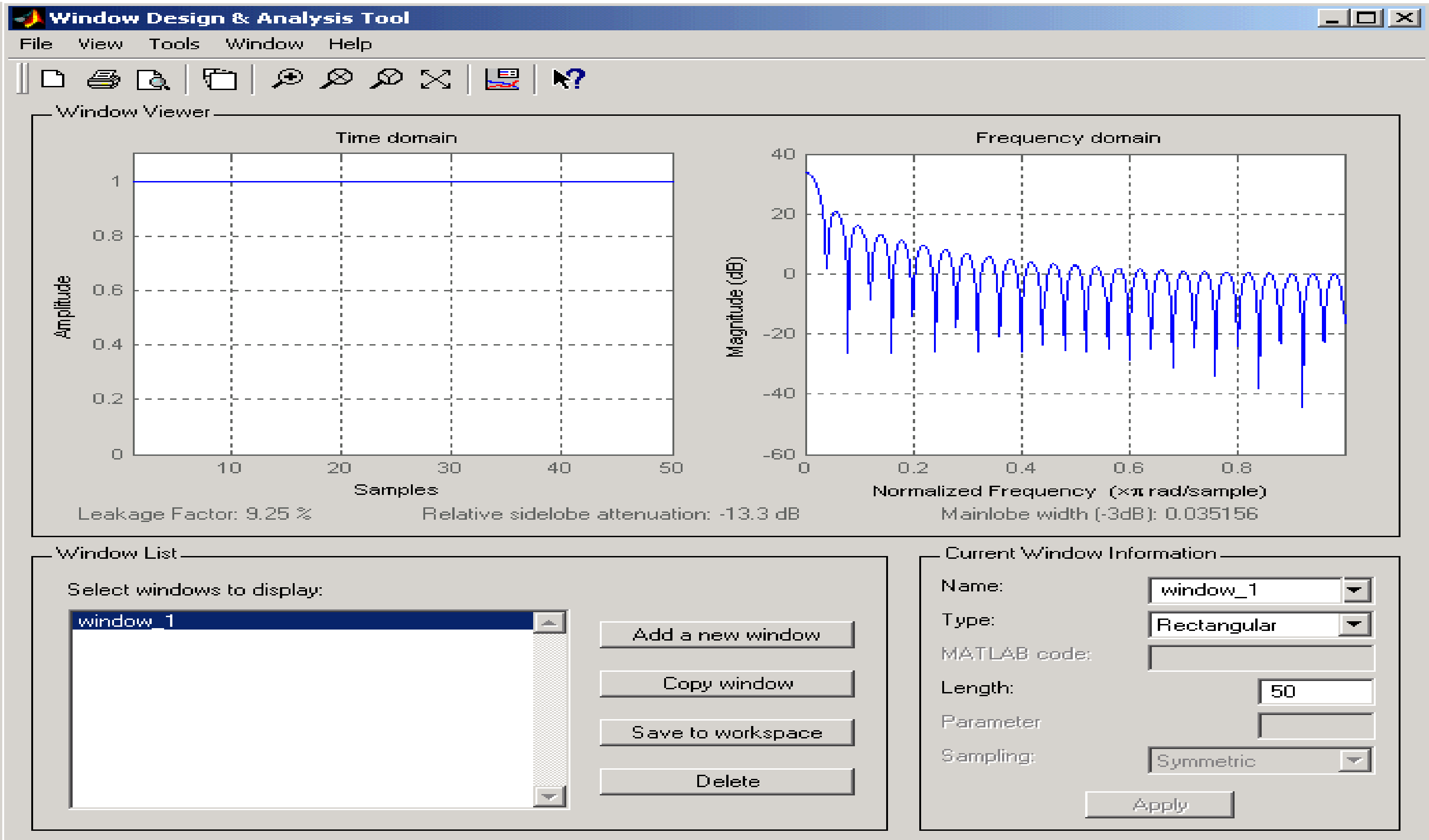


Window Selection

- **A good window function must present a magnitude response (i.e. magnitude of the Fourier transform of the window function) characterized by the ratio of the main-lobe amplitude to the largest side-lobe amplitude**

- **This ratio must be as large as possible**

The Rectangular Window



The main problem associated with the rectangular window is the relatively low level (13.5dB) of the ratio of the main lobe to the largest side lobe

The Rectangular and Non-Rectangular Windows

- **Such a problem is due to the inherent discontinuity of the rectangular window in the time domain.**

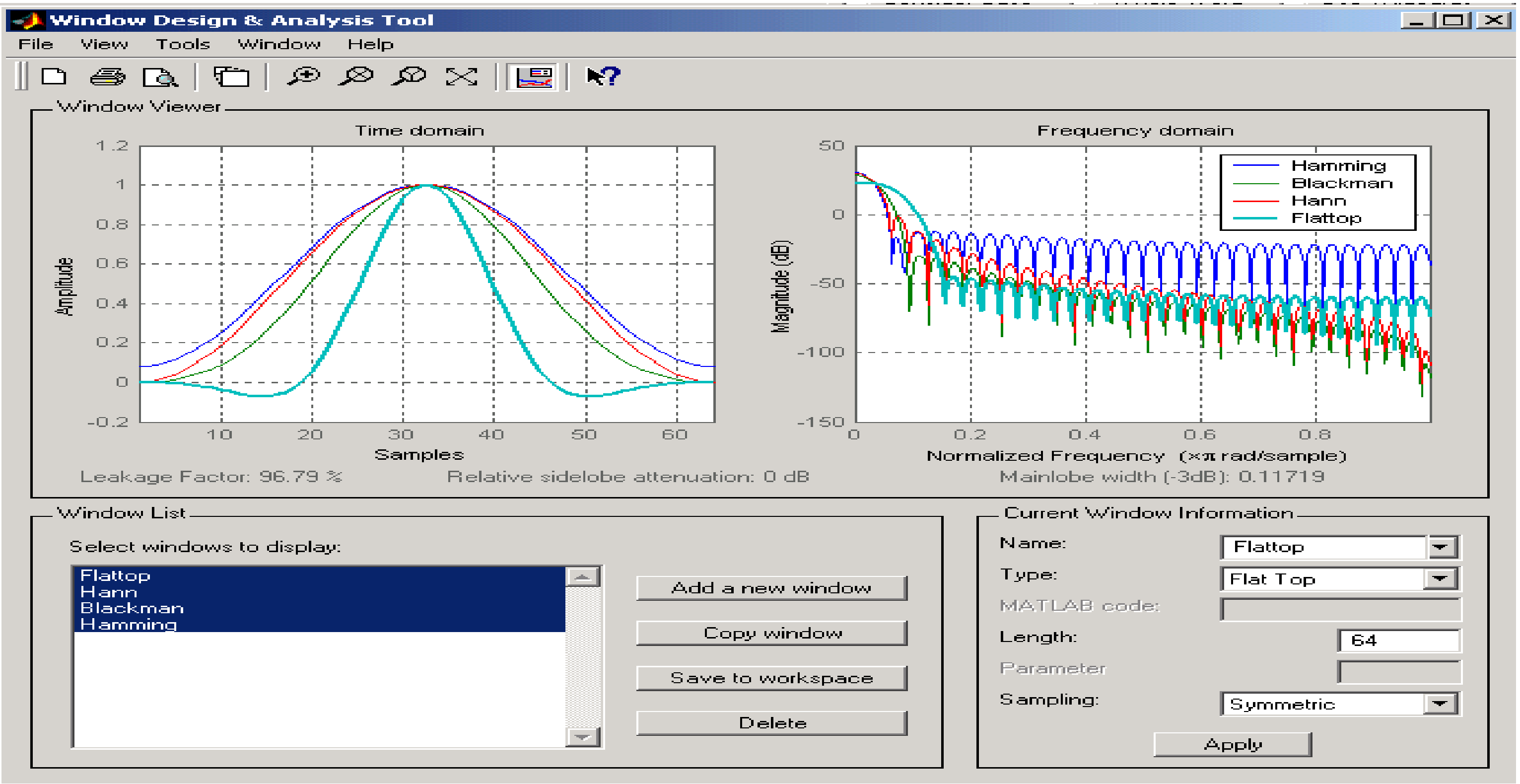
- **One way to reduce such a discontinuity is to employ windows, which contain a taper and decays toward zero gradually, instead of abruptly, and, therefore, present only small discontinuities near its edges**

- **Literature lists several window functions that possess desirable magnitude responses: Bartlett (triangular), Blackman, Hamming, Hanning, Flat Top, Kaizer**

- **All of these functions have significantly increase ratio of the main-lobe amplitude to the largest side-lobe compared with the rectangular window**

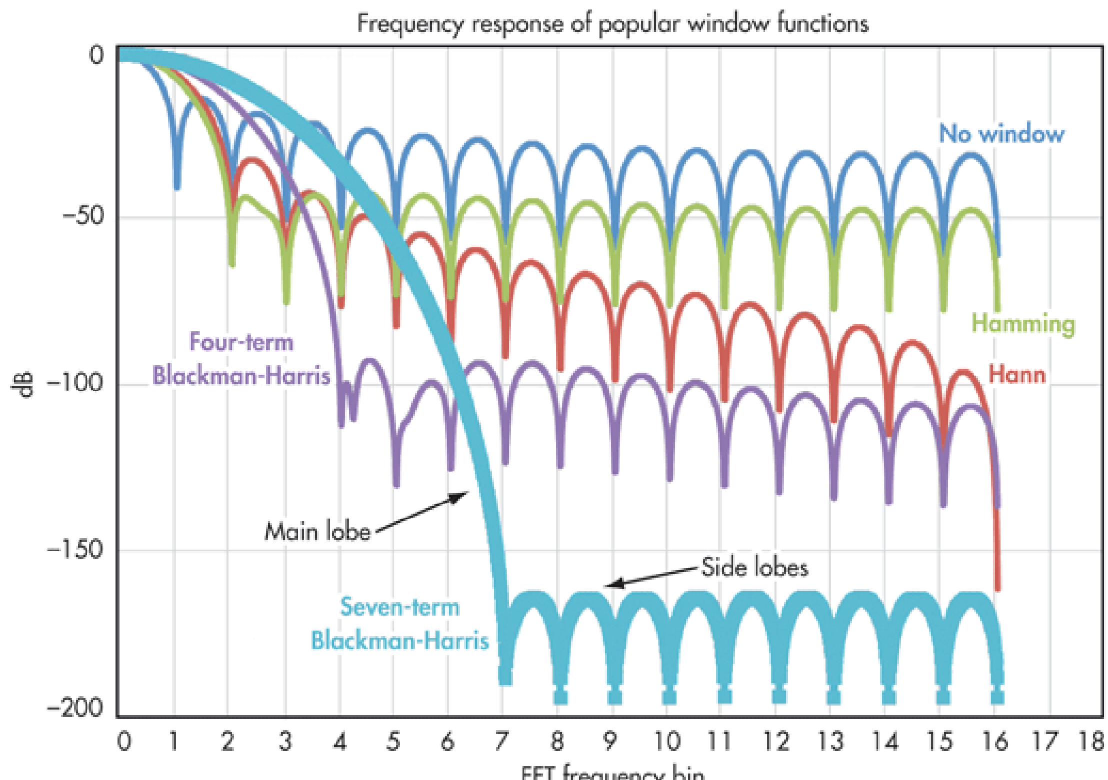
Non-Rectangular Windows

The Blackman, Flat Top, Hamming, Hann (Hanning), and rectangular windows are all special cases of the *generalized* *cosine window*



Non-Rectangular Windows

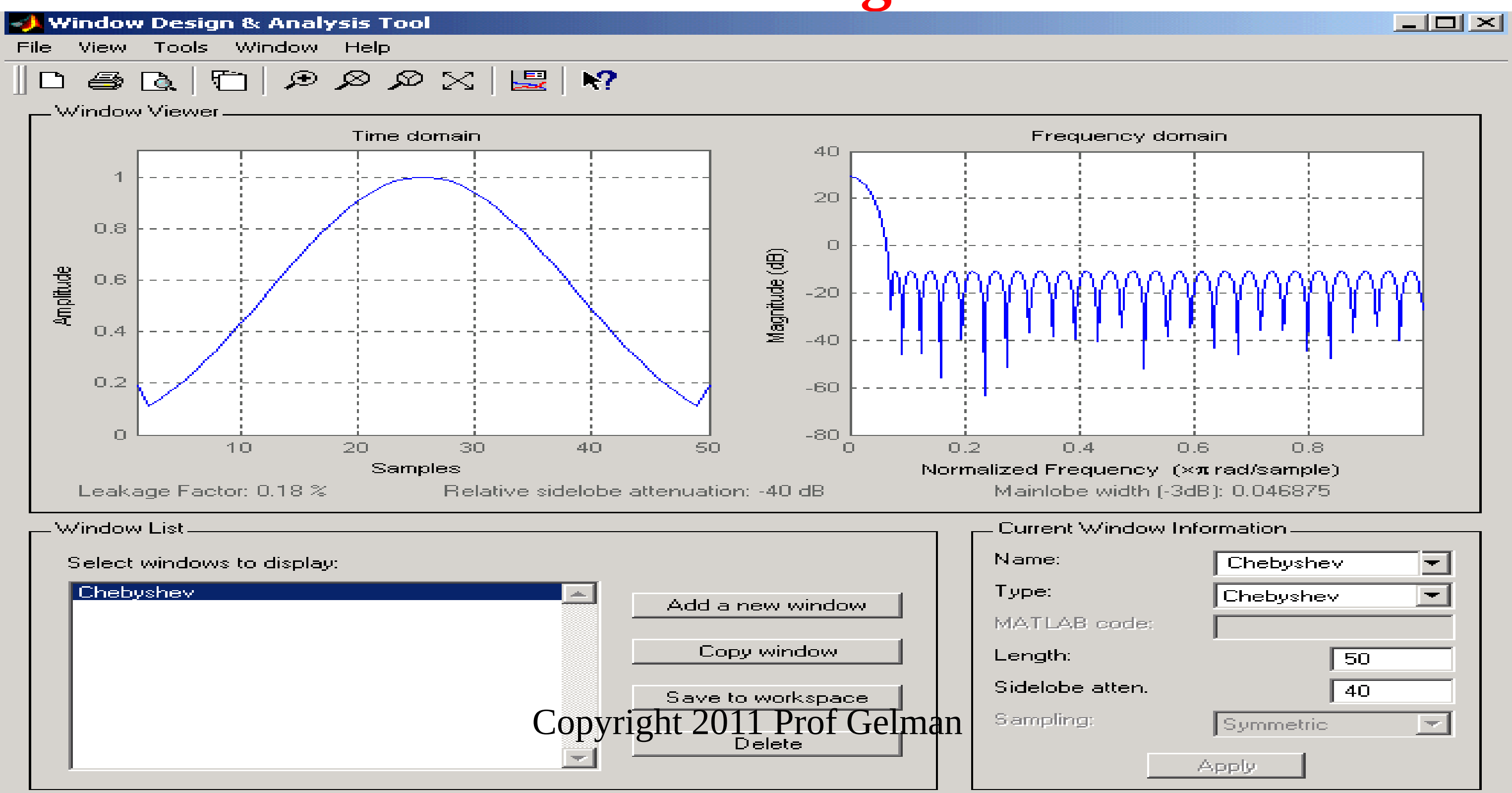
The Blackman-Harris, Hamming, Hann, and rectangular windows



The Chebyshev Window

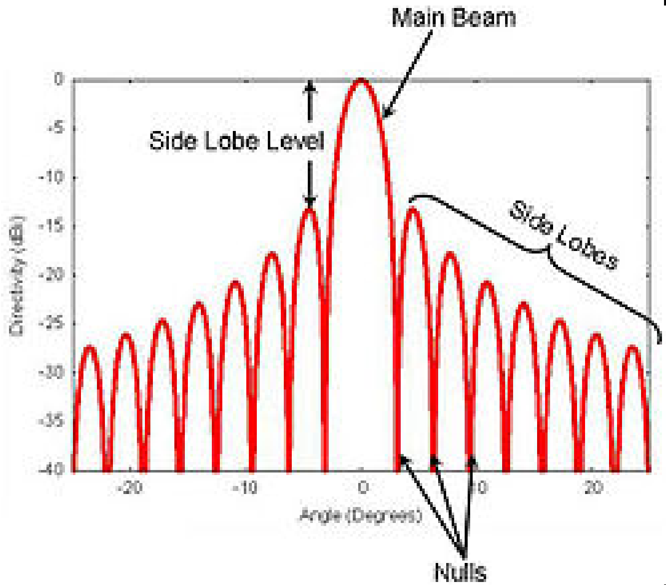
- **The Chebyshev window also maximizes the main lobe-side lobe ratio; The ratio of the main lobe to the side lobe is relatively high, i. e. 40 dB**

- **Its all side lobes have the same height**



Window Summary

Table summarizes main lobe-side lobe ratios of the various window functions



Time window	Main lobe/side lobe ratio, dB
Rectangular	13.5
Bartlett	27
Hanning	32
Hamming	43
Blackman	58