Exercise sheet 7

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1 Population dynamics

We study the system

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN(1 - N/K) - \frac{BN^2}{A^2 + N^2}$$

where r, K, A, B are real positive constants. First we transform the system into a dimensionless system. It is obvious that $[A] = [N]$, we choose $n = N/A$ as dimensionless version of N. Let $\tau = t/t_c$ be the dimensionless version of t. Substituting $N = nA$ and $t = \tau t_c$ into the equation we obtain

$$\begin{aligned} \frac{A}{t_c}\frac{\mathrm{d}n}{\mathrm{d}\tau} &= rAn(1 - An/K) - \frac{Bn^2A^2}{A^2 + n^2A} \\ \frac{\mathrm{d}n}{\mathrm{d}\tau} &= rt_c n(1 - An/K) - t_c\frac{B}{A}\frac{n^2}{1 + n^2} \end{aligned}$$

We choose $t_c = A/B \implies \tau = tB/A$ to remove the factor befor the n^2 term

$$\frac{\mathrm{d}n}{\mathrm{d}\tau} = r\frac{A}{B}n\left(1 - \frac{A}{K}n\right) - \frac{n^2}{1 + n^2}$$

This leaves two dimensionless factors, we define two dimensionless constants

$$\begin{aligned} \rho &= r\frac{A}{B} \\ \kappa &= \frac{A}{K} \\ \implies \frac{\mathrm{d}n}{\mathrm{d}\tau} &= \rho n(1 - \kappa n) - \frac{n^2}{1 + n^2} \end{aligned}$$

n is stationary, when $\partial_\tau n$ is zero

$$\begin{aligned} \implies \rho n(1 - \kappa n) - \frac{n^2}{1 + n^2} &= 0 \\ \rho n\left(1 + n^2\right)(1 - \kappa n) - n^2 &= 0 \\ n\left(\rho\left(1 + n^2\right)(1 - \kappa n) - n\right) &= 0 \\ \implies n = 0 \vee \rho\left(1 + n^2\right)(1 - \kappa n) - n &= 0 \end{aligned}$$

The solutions of the second equation can be calculated using by *Mathematica*.

```
Manipulate[Solve[x*(r*(x^2 + 1) (x/8 - 1) - x) == 0, x], {r, 0, 2, 0.01}]
```

Listing 1: mathematica code for calculating the solution of the second equation

Changing ρ shows that when ρ is ≤ 0.44 there are three, or rather four stationary points. And for $\rho \geq 0.45$ there is only one (two when counting $n = 0$) stationary point. Plotting the equation makes it easy to determine whether the stationary point is stable or unstable. If the derivative increases when n gets bigger by a small value, n will further increase and the stationary point is unstable. The same happens, if the derivative decreases when n gets smaller by a small value, then n will decrease further. The equation can be plotted easily using *Mathematica*.

```
Manipulate[Plot[x*(r*(x^2 + 1) (x/8 - 1) - x), {x, -20, 20}], {r, 0, 2, 0.01}]
```

Listing 2: mathematica code for calculating the solution of the second equation

Using this we can determine

- the smallest stationary point is stable (derivative decreases if n increases / increases if n decreases)
- the second smallest stationary point is unstable (derivative increases if n increases)
- the stationary point at $n = 0$ is stable (derivative decreases if n increases / increases if n decreases)
- the biggest stationary point is unstable (derivative increases if n increases)

(if $r \geq 0.45$ the smallest two stationary points are not real anymore, but the stability of the other two does not change)