Exercise sheet 8

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February 1, 2019

1 Many Species Population Dynamics

We study 6 populations, 3 predator- (P_i) and 3 prey-species (N_i). Their dynamic is given by

$$\begin{aligned}\frac{\mathrm{d}N_i}{\mathrm{d}t} &= N_i\left(a_i - N_i - \sum_j b_{ij}P_j\right)\\ \frac{\mathrm{d}P_i}{\mathrm{d}t} &= P_i\left(\sum_j c_{ij}N_j - d_i\right)\end{aligned}$$

Where

$$\begin{aligned}a_i &= (56, 12, 35)_i\\ d_i &= (85, 9, 35)_i\\ b_{ij} &= \begin{pmatrix}20 & 30 & 5\\ 1 & 3 & 7\\ 4 & 10 & 20\end{pmatrix}\\ c_{ij} &= \begin{pmatrix}20 & 30 & 35\\ 3 & 3 & 3\\ 7 & 8 & 20\end{pmatrix}\end{aligned}$$

We are searching for the fixed points of the system, so

$$\begin{aligned}\frac{\mathrm{d}N_i}{\mathrm{d}t} &= 0\\ \frac{\mathrm{d}P_i}{\mathrm{d}t} &= 0\end{aligned}$$

The latter equation only depends on one set of the populations and can be solved easily

$$P_i\left(\sum_j c_{ij}N_j - d_i\right) = 0 \implies P_i = 0 \vee \sum_j c_{ij}N_j - d_i = 0$$

Let us first solve the system for $P_i \neq 0 \implies$. The resulting set of equations can be written as matrix equation

$$\begin{pmatrix}20 & 30 & 35\\ 3 & 3 & 3\\ 7 & 8 & 20\end{pmatrix}\begin{pmatrix}N_1\\ N_2\\ N_3\end{pmatrix} = \begin{pmatrix}85\\ 9\\ 35\end{pmatrix}$$

This can be solved using gauss elimination but the solution can also be quite easily be seen directy from the equation:

$$(N_1, N_2, N_3) = (1, 1, 1)$$

This can be substituted back in the first set of equations

$$N_i\left(a_i - N_i - \sum_j b_{ij}P_j\right) = 0$$

$$\rightarrow \begin{pmatrix}56\\12\\35\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}20 & 30 & 5\\1 & 3 & 7\\4 & 10 & 20\end{pmatrix}\begin{pmatrix}P_1\\P_2\\P_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

the solution of this equation can also be easily guessed:

$$(P_1, P_2, P_3) = (1, 1, 1)$$

We obtain the fixed point

$$(N_1, N_2, N_3, P_1, P_2, P_3) = (1, 1, 1, 1, 1, 1)$$

Now let us study the equation for $P_i = 0$. Substituting this back into the first equation leads to

$$N_i(a_i - N_i) = 0$$

The solution for this is

$$\begin{aligned}
N_1 &= 0 \vee N_1 = 56\\
N_2 &= 0 \vee N_2 = 12\\
N_3 &= 0 \vee N_3 = 35
\end{aligned}$$

The N_i are independent from each other so every combination of the solution each N_i is a fixed point. We obtain

$$\begin{aligned}
(N_1, N_2, N_3, P_1, P_2, P_3) &= (0, 0, 0, 0, 0, 0)\\
(N_1, N_2, N_3, P_1, P_2, P_3) &= (0, 0, 35, 0, 0, 0)\\
(N_1, N_2, N_3, P_1, P_2, P_3) &= (0, 12, 0, 0, 0, 0)\\
(N_1, N_2, N_3, P_1, P_2, P_3) &= (0, 12, 35, 0, 0, 0)\\
(N_1, N_2, N_3, P_1, P_2, P_3) &= (56, 0, 0, 0, 0, 0)\\
(N_1, N_2, N_3, P_1, P_2, P_3) &= (56, 0, 35, 0, 0, 0)\\
(N_1, N_2, N_3, P_1, P_2, P_3) &= (56, 12, 0, 0, 0, 0)\\
(N_1, N_2, N_3, P_1, P_2, P_3) &= (56, 12, 35, 0, 0, 0)
\end{aligned}$$

Together with the first fixed point found ($N_i = P_i = 1$) these are all possible fixed points. We can combine the two sets of equations into one, doing this we get

$$\begin{aligned}
\vec{v} &= (N_1, N_2, N_3, P_1, P_2, P_3)\\
\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} &= \vec{f}(\vec{v})\\
\vec{f}(\vec{v}) &= \begin{pmatrix}(N_1(56 - N_1 - 20P_1 - 30P_2 - 5P_3))\\(N_2(12 - N_2 - 1P_1 - 3P_2 - 7P_3))\\(N_3(35 - N_3 - 4P_1 - 10P_2 - 20P_3))\\(P_1(20N_1 + 30N_2 + 35N_3 - 85))\\(P_2(3N_1 + 3N_2 + 3N_3 - 9))\\(P_3(7N_1 + 8N_2 + 20N_3 - 35))\end{pmatrix}
\end{aligned}$$

We then can calculate the Jacobi matrix of $\vec{f}$:

$$Df = \begin{pmatrix} 56-2N_1-20P_1-30P_2-5P_3 & 0 & 0 & -20N_1 & -30N_1 & -5N_1 \\ 0 & 12-2N_2-1P_1-3P_2-7P_3 & 0 & -1N_2 & -3N_2 & -7N_2 \\ 0 & 0 & 35-3N_3-4P_1-10P_2-20P_3 & -4N_3 & -10N_3 & -20N_3 \\ 20P_1 & 30P_1 & 35P_1 & (20N_1+30N_2+35N_3-85) & 0 & 0 \\ 3P_2 & 3P_2 & 3P_2 & 0 & (3N_1+3N_2+3N_3-9) & 0 \\ 7P_3 & 8P_3 & 20P_3 & 0 & 0 & (7N_1+8N_2+20N_3-35) \end{pmatrix}$$

Setting $N_i = P_i = 1$ we obtain A

$$\begin{pmatrix} -1 & 0 & 0 & -20 & -30 & -5 \\ 0 & -1 & 0 & -1 & -3 & -7 \\ 0 & 0 & -1 & -4 & -10 & -20 \\ 20 & 30 & 35 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 7 & 8 & 20 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues and eigenvectors can be determined using *Mathematica*:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} = \begin{pmatrix} -0.5+33.6256i \\ -0.5-33.6256i \\ -0.5+7.67949i \\ -0.5-7.67949i \\ -1.13602 \\ 0.136024 \end{pmatrix}$$

$$\begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \\ \vec{v}_5 \\ \vec{v}_6 \end{pmatrix} = \begin{pmatrix} -0.0260729+1.75344i & -0.00446549+0.300309i & -0.0140883+0.947453i & 2.29703+0.i & 0.26776+\text{3.774436480993636*\^{}-16}i & 1. \\ -0.0260729-1.75344i & -0.00446549-0.300309i & -0.0140883-0.947453i & 2.29703+0.i & 0.26776-\text{3.774436480993636*\^{}-16}i & 1. \\ 0.234576-3.60286i & -0.037271+0.572445i & -0.0921933+1.416i & -0.693262-\text{9.43609120248409*\^{}-17}i & -0.630672+\text{1.1795114003105113*\^{}-17}i & 1. \\ 0.234576+3.60286i & -0.037271-0.572445i & -0.0921933-1.416i & -0.693262+\text{9.43609120248409*\^{}-17}i & -0.630672-\text{1.1795114003105113*\^{}-17}i & 1. \\ 5.85892 & -3.21185 & -0.822684 & 7.01658 & -4.81782 & 1. \\ -0.701528 & 0.384576 & 0.0985054 & 7.01658 & -4.81782 & 1. \end{pmatrix}$$

We can then obtain the time evolution of the populations using

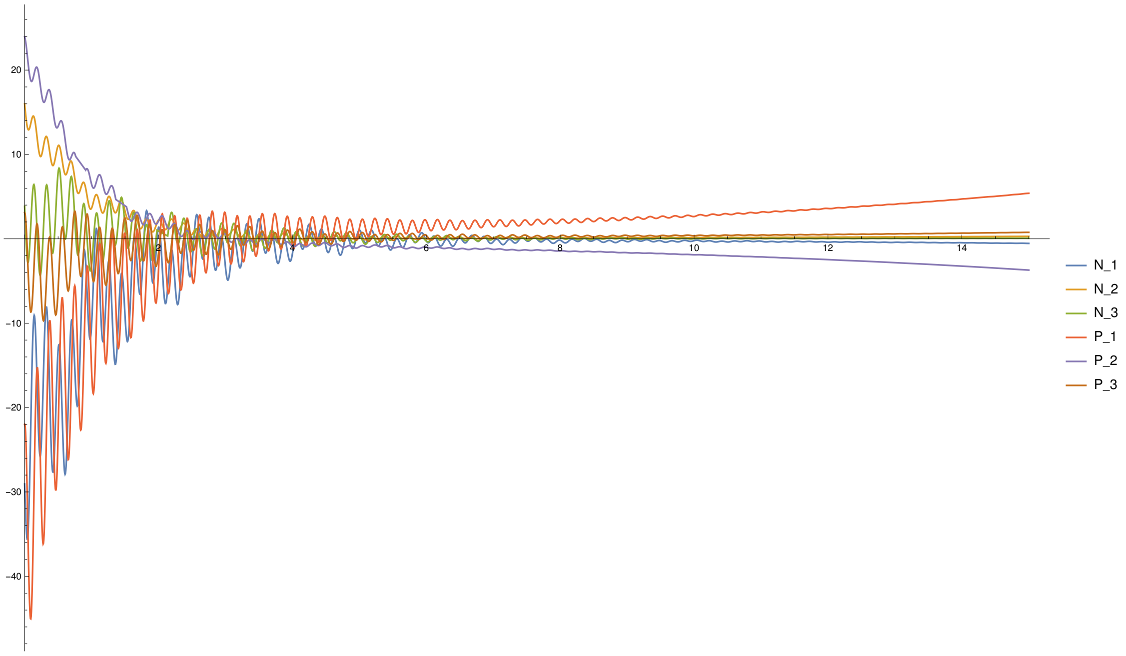

Figure 1: Plot of the time evolution of the populations

All populations are decreasing exponentially in the beginning, but are oscillating. Over time the oscillation decreases and the populations are also increasing again. For $t \longrightarrow \infty$ they all increase exponentially.

```
  a = {56, 12, 35};
d = {85, 9, 35};
b = {{20, 30, 5}, {1, 3, 7}, {4, 10, 20}};
c = {{20, 30, 35}, {3, 3, 3}, {7, 8, 20}};
P = {P1, P2, P3};
Ns = {N1, N2, N3};
v = Join[Ns, P ];
dn = Ns*(a - Ns - b.P);
dp = P*(c.Ns - d);
dv = Join[dn, dp];
dv // MatrixForm
J = D[dv, {v}] ;
J // MatrixForm
A = J /. {N1 -> 1, N2 -> 1, N3 -> 1, P1 -> 1, P2 -> 1, P3 -> 1} ;
A // MatrixForm
ews = A // Eigenvalues;
evs = A // Eigenvectors;
ews // N // MatrixForm // TeXForm
evs // N // MatrixForm
cs = {3, 3, 1, 1, -5, .1};
p = Show[Plot[
   Evaluate[Table[Total[Exp[ews*t]*cs*evs][[i]], {i, 6}]], {t, 0, 10},
   PlotRange -> Full,
   PlotLegends -> {"N_1", "N_2", "N_3", "P_1", "P_2", "P_3"},
   ImageSize -> 1000]]
   Export["plot.pdf", p];
```

Listing 1: mathematica code for calculating the eigenvalues / eigenvectors and the time evolution