PHYS6017 Project A2

COMPUTER TECHNIQUES IN PHYSICS

Structure of White Dwarf Stars

White dwarf stars are one possible end result of the conventional nuclear processes that build elements by binding nucleons into nuclei. Usually they are composed of Fe^{56} nuclei and their electrons but nucleosynthesis may terminate early and then C^{12}, for example, might predominate. The structure is determined by the balance between gravity, which acts to compress the star, and the degeneracy pressure of the electrons.

We assume that the star is spherically symmetric and not rotating. Consider a small cube of matter at a distance r from the centre with height δh and area δA. The gravitational force acting on it will be

$$F = -\frac{Gm(r)}{r^2}\rho(r)\,\delta h\,\delta A$$

where G is the gravitational constant, $\rho(r)$ is the density at radius r, and $m(r)$ is the mass contained within radius r. This force must equal the difference in pressure δP times δA, so that

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)}{r^2}\,\rho(r).$$

We can write

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}\rho}{\mathrm{d}r}\frac{\mathrm{d}P}{\mathrm{d}\rho}$$

and the second factor in this equation depends only on the properties of the material of which the star is composed: its ‘equation of state’. Combining the equations we therefore have

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)^{-1}\frac{Gm}{r^2}\rho$$

and by considering a thin spherical shell within the star we have a second differential equation

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2\rho$$

connecting m and ρ: This pair of equations can be solved numerically using the Runge-Kutta method for any equation of state.

A good approximation to the equation of state for matter in a white dwarf star is the relativistic free Fermi gas. You should give a reference to the derivation of the result. Let Y_e be the number of electrons per nucleon, so if the star is pure C^{12} $Y_\mathrm{e} = 0.5$, while if it is Fe^{56} then $Y_\mathrm{e} = 26/56 = 0.464$. Then it can be shown that

$$\frac{\mathrm{d}P}{\mathrm{d}\rho} = Y_\mathrm{e}\frac{m_e c^2}{m_p}\gamma\left(\frac{\rho}{\rho_0}\right)$$

where m_e and m_p are the masses of the electron and proton, $\rho_0 = m_p m_e^3 c^3 / 3\pi^2 \hbar^3 Y_e = 9.79 \times 10^8 / Y_e \, \mathrm{kg\,m^{-3}}$, and $\gamma(y)$ is the function

$$\gamma(y) = \frac{y^{2/3}}{3(1+y^{2/3})^{1/2}}.$$

Before starting numerical work it is essential to scale the equations since most variables are otherwise very large. Some of the physical constants needed are assigned symbolic names in the file "constant.nfb" which can be included in your program. This is in the directory for the unit and you can delete the lines you do not want. It can also be used to create `#define` statements for a C program.

Introduce a unit of length R_0 and write $r = R_0 x$ in the equation. ρ_o provides a natural unit for density and $4\pi\rho_0 R_0^3/3$ is a natural unit for mass; so express the density and mass in terms of these and choose R_0 to get rid of all the dimensional constants in the equations. Alternatively express the variables in solar masses and radii, as below. If you do this remember to express ρ_0 in these units as well.

The equations can be integrated outwards from $r = 0$. The boundary conditions are $m = 0$ and $\rho = \rho_c$, the central density. The surface of the star is the point at which $\rho = 0$ and you should determine the radius R at which this happens. The total mass of the star M is the corresponding value of $m(r)$. For each value of ρ_c therefore you determine a mass and radius for the star.

Plot a graph (by hand, at least initially) of the variation of M with R. You should find that as the density at the centre ρ_c becomes very large the mass tends to a constant value and the radius becomes very small.

Parameters of three white dwarf stars are given in the following table

	Mass	Radius
Sirius B	1.053 ± 0.028	0.0074 ± 0.0006
40 Eri B	0.48 ± 0.02	0.0124 ± 0.0005
Stein 2051	0.50 ± 0.05	0.0115 ± 0.0012

The units are solar masses $M_\odot = 1.98 \times 10^{30}\,\mathrm{kg}$ and solar radii $R_\odot = 6.95 \times 10^8\,\mathrm{m}$. Verify that these observations are consistent with your predictions.

Further information can be found in:

S. Chandrasekhar, An Introduction to the Study of Stellar Structure. (Dover, New York 1957).

S. Chandrasekhar, Reviews of Modern Physics, vol 56, page 137 (1984)

S.L.Shapiro and S.A.Teukolsky, Black Holes, White Dwarfs, and Neutron Stars, (J Wiley and Sons, Inc., New York 1983) Chapter 3.