Assignment 8

Due: 5 December, 2025 11:55 PM

Instructions

- Ensure that all answers include complete and detailed workings for full credit.
- Collaboration between students is permitted, but copying answers is strictly prohibited.
- Any assistance received for the completion of this assignment must be clearly indicated.
- Assignments must be submitted on LMS prior to the specified deadline.
- Late submissions will not be accepted under any circumstances.
- Submit **both** the .pdf and the .tex file of your solution in a zipped folder named according to the convention `CS210_A8_RollNumber`.

Graded Problems

Question 1 [10 marks]

For each recurrence below, indicate whether it is linear, homogeneous (if yes, mention the order too) and has constant coefficients.

(a) $f_n = 4f_{n-1} - 3f_{n-2}$ (2)

(b) $f_n = nf_{n-1} - 2$ (2)

(c) $f_n = 2f_{n-1} + 5$ (2)

(d) $f_n = f_{n-1}^2 + 3f_{n-2}$ (2)

(e) $f_n = 7f_{n-3}$ (2)

Question 2 [10 marks]

(a) Imagine you are forming a research team of size k from a total pool of n scientists. You decide to fix one specific scientist. Explain what the following two terms $\binom{n-1}{k}$ and $\binom{n-1}{k-1}$ represent in this scenario.
Hint: *Think in terms of the scientist's inclusion or exclusion from this team.* (2)

(b) Consider the term x^3y^2 in the expansion of $(x+y)^5$. Using the *Product of Factors* logic from the slides, explain why the coefficient of this specific term is calculated as $\binom{5}{2}$ (or $\binom{5}{3}$). Specifically, what "choice" are you making regarding the 5 factors? (3)

(c) Find the closed-form formula for the following recurrence using the Substitution Method. Assume $n = 2^k$.

$$T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + 1, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases}$$

(5)

Question 3 [5 marks]

Draw the following graphs, or explain why they cannot exist:

(a) A graph on n vertices with a vertex of degree 0 and another of degree $n - 1$. (1)

(b) A cubic graph of order 5. (2)

(c) A bipartite graph of order 5 and size 7. (2)

Question 4 [10 marks]

Consider the graph G with vertex set $V(G) = \{A, B, C, D, E, F\}$ and adjacency list:

$$\begin{aligned} A &: B, C, D \\ B &: A, C, D \\ C &: A, B, E \\ D &: A, B, E \\ E &: C, D, F \\ F &: E \end{aligned}$$

(a) Draw the graph G based on the adjacency list. (2)

(b) Identify and draw a subgraph H of G that is a tree with 5 vertices. (2)

(c) Find a walk from A to F. Is this walk a path? Explain. (2)

(d) Determine if G contains any cycles. (2)

(e) Find the complement graph $\overline{G}$ and determine if it is connected. (2)

Question 5 [5 marks]

The directed graph G has vertices $V(G) = \{A, B, C, D, E\}$. Its adjacency list (outgoing edges) is given below.

$$\begin{aligned} A &: B, C \\ B &: C, D \\ C &: D \\ D &: E \\ E &: A \end{aligned}$$

Write the adjacency matrix of G.

Question 6 [10 marks]

A region has 6 islands. Its king wants to connect the islands with bridges, requiring every island to be reachable from every other island. Considering the costs it would incur, he wants to build **as few bridges as possible**.

(a) What is the minimum number of bridges he must have made? If he tries to build fewer bridges, explain why it will not be possible for all islands to remain connected. (3)

Afterwards, the king creates two councils: a House of Nobles with 4 nobles, and a House of Commons with 7 citizens. A communication network is to be set up so that messages can only be exchanged **between a noble and a citizen**—never within the same group.

(b) If the king wants every noble to be able to communicate with every citizen, how many communication links must be established? (1)

(c) Prove that the resulting network is bipartite and explain why it cannot contain any odd cycle. (6)

Ungraded Problems

Question 1

There are 25 fields in a farming village. The villagers want to dig water channels so that each field is connected to exactly 7 other fields. Is it possible to design such a network? Explain why or why not. You may ignore any larger body of water supplying the channels.

Question 2

Prove that a connected graph G with at least two vertices has an Euler circuit if and only if every vertex of G has even degree.

Question 3

Prove that a connected graph has an Euler path if and only if it has exactly two vertices of odd degree.

Question 4

Every tree $T = (V, E)$, ($|V| = n$, $|E| = m$) satisfies the following properties:

1. $m = n - 1$
2. There is a unique path between every pair of vertices
3. T is an edge-maximal acyclic graph (i.e. Adding any edge to T creates a cycle)
4. Every edge in T is a cut-edge
5. T is an edge-minimal connected graph (i.e. Removing any edge disconnects T)
6. T has at least two leaves.

Prove each of these properties.

Question 5

Let G be a simple graph. Show that an edge e is a **cut edge** if and only if e is not part of any cycle in G.

Question 6

Consider the directed graph G with vertex set $V(G) = \{A, B, C, D, E, F\}$ and adjacency list:

$$\begin{aligned} A &: B, C, D \\ B &: C, D \\ C &: E \\ D &: E \\ E &: F \\ F &: \end{aligned}$$

(a) Draw the directed graph G based on the adjacency list.

(b) Identify a subgraph H of G that is a **directed tree** containing exactly 5 vertices. Draw H.

(c) Find a directed walk from vertex A to vertex F. Is this walk a directed path? Explain why or why not.

(d) Determine if G contains any directed cycles.

(e) Find the transpose of G, G^T, and draw it.

Question 7

The graph G has vertices $V(G) = \{A, B, C, D, E\}$. Its adjacency list is given below:

$$\begin{aligned} A &: B, C \\ B &: A, C, D \\ C &: A, B, D \\ D &: B, C, E \\ E &: D \end{aligned}$$

Write the adjacency matrix of G.

Question 8

Let G be a connected bipartite planar graph with v vertices and e edges. Prove that

$$e \leq 2v - 4.$$

Question 9

A wizard has n potion bottles arranged in a row. Each day he can pick either 1 or 2 bottles. Let P_n be the number of ways he can select bottles. Write a recurrence for P_n.

Question 10

A sequence $y_1, y_2, y_3, y_4, \ldots$ is given by

$$y_{n+1} = 4y_n - 3, \quad y_1 = 2.$$

(a) Find the values of y_2, y_3, y_4, and y_5.

(b) It is further given that $y_{10} = 262145$. Calculate the value of y_9.

Question 11

A sequence $u_1, u_2, u_3, u_4, \ldots$ is given by the recurrence relation:

$$u_{n+2} = 2u_{n+1} - u_n + 2, \quad u_1 = 10.$$

Find the values of u_2, u_3, u_4, and u_5.

Question 12

A robot moves on a $m \times n$ grid. It starts at the top-left corner and can move only right or down. Let $R_{i,j}$ denote the number of ways the robot can reach cell (i, j).

(a) Write a recurrence relation for $R_{i,j}$.

(b) Compute $R_{3,3}$ using your recurrence.

Question 13

A staircase has n steps. A person can climb either 1, 2, or 3 steps at a time. Let S_n denote the number of ways to reach the top.

(a) Write a recurrence relation for S_n.

(b) Compute S_5.

Question 14

A bank account grows according to the recurrence

$$A_n = 1.05A_{n-1} + 100, \quad A_0 = 1000.$$

(a) Find A_1, A_2, and A_3.

(b) Determine if the recurrence is linear, homogeneous, and/or has constant coefficients.

Question 15

A gamer earns points according to the recurrence

$$P_n = 5P_{n-1} + 100, \quad P_0 = 50.$$

(a) Compute P_1, P_2, and P_3.

(b) Classify the recurrence as linear/non-linear, homogeneous/non-homogeneous, and constant/non-constant coefficients.

Question 16

A small tree grows according to the model

$$T_n = 1.2T_{n-1} + 10, \quad T_0 = 5.$$

(a) Compute T_1, T_2, and T_3.

(b) Determine if the recurrence is linear, homogeneous, and/or has constant coefficients.

Question 17

A population of fish in a pond increases according to

$$F_n = F_{n-1} + 2n^2, \quad F_0 = 5.$$

(a) Compute F_1, F_2, and F_3.

(b) Determine if the recurrence is linear, homogeneous, and/or has constant coefficients.

Question 18

A sequence $\{t_n\}$ is defined for $n \geq 1$ by the recurrence below, where a and b are non-zero constants.

$$t_{n+1} = at_n + b, \quad t_1 = 2,$$

It is further given that

$$t_2 = 3 \quad \text{and} \quad \sum_{r=1}^{3} t_r = 12.$$

Determine the possible values of a and b.

Question 19

Consider the recurrence relation for the sequence s_n:

$$s_n = \begin{cases} s_{n-1} + (2n-1), & \text{if } n \geq 1, \\ 0, & \text{if } n = 0. \end{cases}$$

The proposed closed-form solution for this sequence is $s_n = n^2$. Comment on the correctness of this formula using induction.