PDE Solution

1. 试求解下列偏微分方程

$$\pi^2\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},(0\le x\le 1,0\le t\le 2)$$

$$\begin{cases} IC: u(x,0)=\sin(\pi x) \\ BC1: u(0,t)=0 \\ BC2: \pi e^{-t}+\dfrac{\partial u(1,t)}{\partial x}=0 \end{cases}$$

tips：*IC denotes initial condition*；*BC denotes boundary condition*.

本问题的解析解为$u(x,t)=e^{-t}\sin(\pi x)$

1.1 有限差分法求解

将$x\times t$平面区域分割为$n\times m$的网格区域，索引分别采用i,j，根据微分的定义得到如下差分方程：

$$\begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{u_i^{j+1}-u_i^j}{\Delta t} \\ \dfrac{\partial u}{\partial x}=\dfrac{u_{i+1}^j-u_i^j}{\Delta x}\rightarrow\dfrac{\partial^2 u}{\partial x^2}=\dfrac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{\Delta x^2} \end{cases} \tag{1-1}$$

$$\frac{\partial u(1,t)}{\partial x}=\frac{u_{x+\Delta x}^t-u_i^t}{\Delta x}\bigg|_{x=1}=\frac{u_n^t-u_{n-1}^t}{\Delta x} \tag{1-2}$$

结合式 1-2 得到如下初值条件和边界条件的差分形式：

$$\begin{cases} IC\text{: } u_i^0=\sin(\pi x(i)) \\ BC1: u_0^t=0 \\ BC2: \dfrac{u_n^t-u_{n-1}^t}{\Delta x}=-\pi e^{-t} \end{cases} \tag{1-3}$$

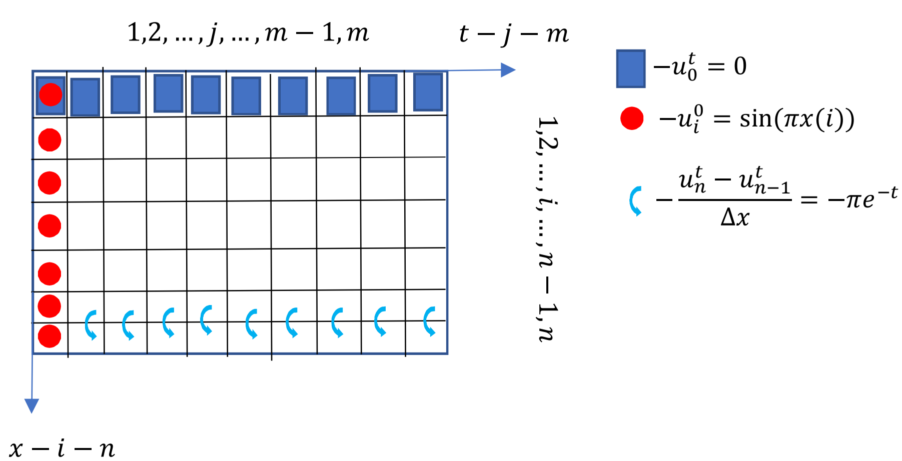
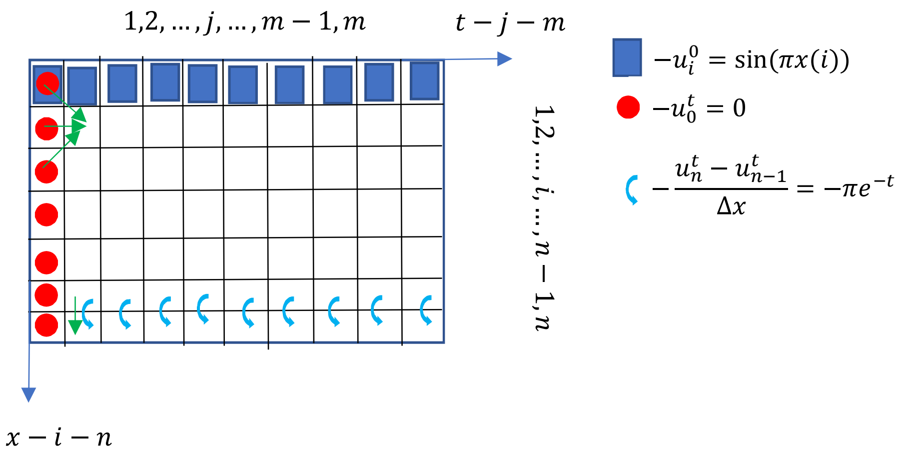


图 1-1 网格化

原偏微分方程结合式 1-1 可得递推公式：

$$\begin{cases} u_i^{j+1} = u_i^j + \Delta t \dfrac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\pi \Delta x)^2} \quad (i = 1, \dots, n-1) \\ u_n^t = u_{n-1}^t - \pi e^{-t} \Delta x \end{cases} \qquad (1-4)$$

注意网格比$\boldsymbol{r} = \dfrac{\Delta t}{(\pi \Delta x)^2}$必须满足$\boldsymbol{r} \le \dfrac{1}{2}$，否则获得的解是不准确的！



故可根据递推式 1-4 通过迭代循环一遍完成差分网格计算.

1.2 求解

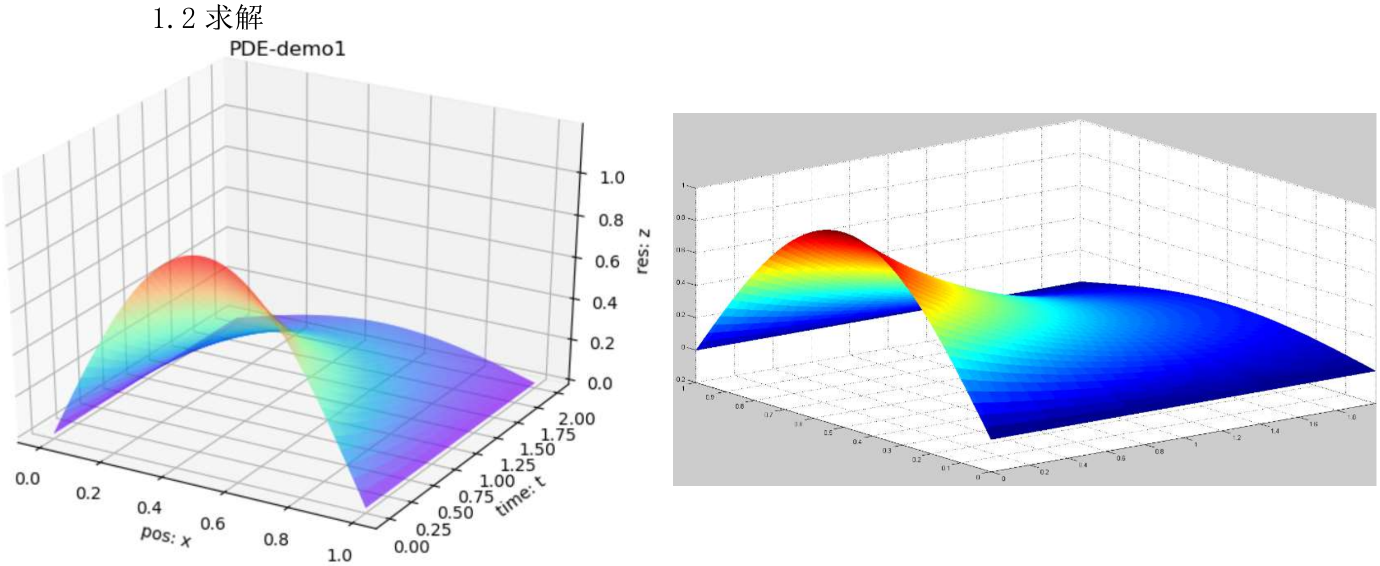


图 1-2 Python and MATLAB’s results

以下提供 Python 和 MATLAB 实现的代码

1.2.1 Python Code

```
import numpy as np
from numpy import exp, sin, pi
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

n, m = 51, 2001
x, t = np.linspace(0, 1, n), np.linspace(0, 2, m)
u = np.zeros(shape=(n, m))
dx, dt = x[1] - x[0], t[1] - t[0]
for i in range(n): u[i][0] = sin(pi * x[i])
for j in range(m): u[0][j] = 0


for j in range(0, m - 1):
    for i in range(1, n):
        if i == n - 1:
            u[i][j + 1] = u[i - 1][j + 1] - pi * exp(-(t[j + 1])) * dx
        else:
            u[i][j + 1] = u[i][j] + dt * (u[i + 1][j] - 2 * u[i][j] + u[i -
1][j]) / ((pi * dx) ** 2)
            u[i][j + 1] = u[i][j] + dt * (u[i + 1][j] - 2 * u[i][j] + u[i -
1][j]) / ((pi * dx) ** 2)


def plot3d(x,y,z):
    ''' surface plot '''
    fig = plt.figure()
    ax = fig.gca(projection='3d')
    ax.plot_surface(x, y, z, rstride=1, cstride=1, cmap='rainbow')
    ax.set_zlim(0, 1.2)
    ax.set_xlabel('pos: x')
    ax.set_ylabel('time: t')
    ax.set_zlabel('res: z')
    ax.set_title('PDE-demo1')
    plt.show()

x, t = np.meshgrid(x, t)
u = np.transpose(np.array(u))
print(x.shape, t.shape, u.shape)
plot3d(x, t, u)

r = dt / ((pi * dx) ** 2)
print("r = %f" % (r))
```

1.2.2 MATLAB Code

```
clear, clc;
n = 50;
m = 2000;
x = linspace(0, 1, n);        % 位移x方向分割
t = linspace(0, 2, m);        % 时间t方向分割
u = zeros(n, m);
dx = x(2) - x(1);             % 单位步长
dt = t(2) - t(1);             % 单位时间
for i = 1:n                   % 初值条件 u(x, 0) = sin(pi*x)
    u(i, 1) = sin(pi * x(i));
end
for j = 1:m                   % 边界条件1: u(0, t) = 0
    u(1, j) = 0;              % 左边界
end

for j = 1: m-1
    for i = 2:n
        if i == n             % 边界条件2: dudx(1,t) = -pi*exp(-t)
            u(i,j+1) = u(i-1,j+1) - pi * exp(-(t(j+1))) * dx;
        else
            u(i,j+1) = u(i,j) + dt * (u(i+1,j) - 2 * u(i, j) + u(i-1,
j)) / ((pi * dx) ^ 2);
        end
    end
end
mesh(t, x, u);
```

1.3 MATLAB PDE 工具箱求解

参考《数学建模算法与应用》第 20 章 偏微分方程的数值解