Advanced Linear Algebra Week 9 Day 1

2018/11/12 – Jonathan Hayase, updated by Prof. Weiqing GU

1 A Big Picture of Linear ALgebra

Suppose you have $A\boldsymbol{x} = \boldsymbol{b}$. Then let $\overline{A} = \left[A \mid \boldsymbol{b}\right]$. Then there is a solution if and only if $\operatorname{rank} A = \operatorname{rank} \overline{A}$.

On the other hand, if $\det A = 0$ then there is a nontrivial solution to $A\boldsymbol{x} = \boldsymbol{0}$. But if $\det A \neq 0$ then we have the trivial solution only to $A\boldsymbol{x} = \boldsymbol{0}$.

Things like rank, det, tr are the invariants of A.

Suppose we have a torus and a sphere. We can always slide a rubber band off a sphere, but that is not so for a torus, so they are not equivalent topologically.

Similarly, if we care about geometry, then an ellipsoid and sphere are not the same.

1.1 Differential Geometry

Given a differentiable manifold, we can always find a tangent plane, which lets us push the problem into linear algebra.

Inverse function on manifold: If $\left[\mathrm{d}\phi\right]$ is invertibe, then ϕ is locally invertible.

1.2 Multivariate Taylor

$$f(\boldsymbol{x}) = \underbrace{f(\boldsymbol{x}_0) + \nabla f(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)}_{\text{tangent plane}} + \frac{1}{2!}(\boldsymbol{x} - \boldsymbol{x}_0)^T \nabla^2 f(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)$$

If $\boldsymbol{x}_0$ is a ctirical point, then $\nabla f(\boldsymbol{x}_0) = \boldsymbol{0}$.

$$\begin{aligned} f(\boldsymbol{x}) - f(\boldsymbol{x}_0) &= \frac{1}{2!}(\boldsymbol{x} - \boldsymbol{x}_0)^T \nabla^2 f(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0) \\ &= \frac{1}{2!}\langle \boldsymbol{x} - \boldsymbol{x}_0, \boldsymbol{x} - \boldsymbol{x}_0 \rangle_{\nabla^2 f(\boldsymbol{x}_0)} \end{aligned}$$

Where $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_M = \boldsymbol{x}^T M \boldsymbol{y}$ as long as M is positive definite.

1.3 Current Research Area in Linear Algebra

Research on approximation algorithms for matrices which cannot fit in memory and/or are sparsly represented.

Topological data analysis can only tell us how many "holes" a piece of data has, which is why it "hasn't gone very far".

Example: Suppose given $A \in M^{n\times k}$ and $B \in M^{k\times n}$ which cannot fit into (your) memory. But we would still like to find AB. Suppose we went for some subset C of A and D of B such that the result of multiplying the two is still $n \times m$.

Then suppose our goal was to stochastically achieve $\|AB - CD\| < \epsilon$ with probability 99%.

Prof Gu says: 左行右列

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$$

This means we can do things like

$$\overbrace{\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 & \boldsymbol{v}_5 \end{bmatrix}}^{A} \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & 1 & \dots \\ \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & \dots \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_3 \end{bmatrix}$$

Also want to solve $A_{n\times k}x_{k\times 1} = b$. Then we might calculate $SAx = Sb$, where S means we will select "important rows" of A, such that $\left\{\tilde{x}^* \mid SAx = Sb\right\}$ and $\left\{x^* \mid Ax = b\right\}$ where $\tilde{x}^*$ and x^* are close to each other with probability 0.99 (or more generally $1 - \delta$).

Another problem: If A is large, then we can't fit into (your) computer's memory. But we want to find the eigenvalues and vectors of A.

1.4 Applications

Suppose you have a UAV flying along a curve, we can attach a frame to every point along the curve representing the local coordinate system of the UAV.

Consider the Lie group (Lie groups are both manifolds and groups)

$$SO(3) = \left\{A \mid A^T A = I, \det A = 1\right\}.$$

That is all.

Now, suppose you have a compact manifold. It's really $\mathbb{R}P^3$, the real projective plane, whose double cover is S^3. However, once we go into the tangent plane, then this becomes linear algebra. Tangent plane at I is called the Lie algebra of the Lie group.

Suppose we have a curve $A(t)$ on the manifold starting at I, then $A^T(t)A(t) = I$ and $A(0) = I$. Suppose we want

$$\begin{aligned} \left[A^T(t)\right]' A(t) + A^T(t)A'(t) &= 0 \\ \left[A^T(0)\right]' A(0) + A^T(0)A'(0) &= 0 && \text{plugging in } t = 0 \left[A^T(0)\right]' + A'(0) = 0 \\ \left[A'(0)\right]^T + A'(0) &= 0 \end{aligned}$$

So $A'(0)$ is a skew-symmetric 3×3 matrix. In fact the Lie algebra is the set of skew symmetric matrices.

A Lie algebra is a vector space equipped with the commutator bracket:

$$[A, B] = AB - BA$$

Satisfying the Jacobi identity:

$$\left[[a, B], C\right] + \left[[B, C], A\right] + \left[[C, A], B\right] = 0.$$

2 Convex optimization

Suppose f is twice differentiable. Then f is convex if and only if $\mathrm{H}\, f \succ 0$.