High Altitude Balloon Physics

1 Rate of Ascent

Question #1:

When launched, a balloon initially accelerates upward owing to the buoyant force of the helium gas filling the balloon. As the balloon rises, it experience a drag force given by

$$F_d = \frac{1}{2} C_D \rho_{\text{air}} A v^2, \tag{1}$$

where C_D is the drag coefficient (equal to 0.5 for a spherical object), ρ_{air} is the density of air, A is the cross sectional area of the balloon, and v is its vertical velocity. Since the drag force increases as the balloon speeds up, the system quickly reaches equilibrium (constant v).

Find the equilibrium value of v for a helium filled balloon launched at sea level having a volume of $V = 3$ m^3 and a total mass (balloon plus payload) of $m = 1.6$ kg.

2 Height of Atmosphere

To estimate atmospheric pressure as a function of altitude, we assume that the density of air molecules follows a Boltzmann distribution—i.e.,

$$N(h) = N_0 e^{-mgh/kT} \equiv N_0 e^{-h/H}, \tag{2}$$

where $m = 28 \times m_{\text{AMU}} = 4.6 \times 10^{-26}$ kg is the mass of an N_2 molecule, $g = 9.8$ is the acceleration due to gravity, h is the height in m, $k = 1.38 \times 10^{-23}$ is the Boltzmann constant, and $T = 250$ K is the mean temperature of the atomosphere.

Question #2:

What is the average height of the atmosphere? **Hint:** If the probability of finding a molecule at a height y is given by $p(y) = \lambda e^{-\lambda y}$, the average height of a randomly selected molecule is $\mu_y = 1/\lambda$.

3 Rate of Ascent vs. Height

As the balloon rises and the external pressure drops, it increases in radius. The density of the air and the helium in the balloon change accordingly. The drag force also changes, since the air is less dense.

Question #3:

Calculate the rate of ascent as a function of height. You may assume that the temperature is constant at $T = 250$ K.

Question # 4:

Make a plot of the rate of rise of the balloon as function of height, h. Compare this to the rate of rise based on the experimental data.

Question #5:

The result obtained above assumes that the pressure inside the balloon is the same as the pressure outside of the balloon. As the balloon expands, however, the surface tension in its wall compresses the gas in the balloon. As a result, the volume of the balloon is smaller than what one would estimate from the simple ideal gas model.

Calculating such an effect is not simple. A possible empirical model is to assume that the volume of the balloon is modified according to

$$V' = V \left[\frac{1}{2} \left(1 + e^{-h/H} \right) \right]^{\beta} \tag{3}$$

where the exponent β can be adjusted to fit the observed rate of rise as a function of height, h. As an initial guess, we can try $\beta = 2/3$. Compare this to the data.

Question #6:

Calculate the rate of descent after the balloon has burst. Compare this to the data. To calculate the rate of descent, start with the assumption that the falling balloon is in equilibrium—i.e., that its acceleration $a = 0$ (this is not exactly true, but as long as $a \ll g$ it is a reasonable assumption). In that case, the upward drag force will be equal to the downward force of gravity—i.e.,

$$F_D = \frac{1}{2} C_D \rho_{\text{air}} v^2 A = mg. \tag{4}$$

Although we don't know C_D and A for the falling gondola, it is reasonable to assume that they are constant. Since mg is also constant to a good approximation, we can write

$$\rho_{\text{air}}(h) v^2(h) = \frac{2mg}{C_D A} = k = \text{constant} \tag{5}$$

or

$$v(h) = \sqrt{\frac{k}{\rho_{\text{air}}(h)}}. \tag{6}$$

Since

$$\rho_{\text{air}} = \rho_0 e^{-h/H}, \tag{7}$$

it should be possible to describe the data using one free parameter (k from Eq. 6).

4 Cosmic Ray Count Rate

The primary instrument on the balloon is a particle detector, which is based on the MIT "Cosmic Watch" design (see `http://cosmicwatch.lns.mit.edu/detector`). The detector measures the number of cosmic ray particles passing through its sensitive volume per unit time. This quantity can be plotted as a function of height h and then compared to a simple model that will be developed here.

4.1 Production of Cosmic Rays

Cosmic ray particles are generated when high energy particles (mostly protons) that circulate in the galaxy strike the Earth's upper atmosphere and undergo violent interactions with the nuclei of air molecules. Each such interaction produces a "shower" of secondary reaction products. Most of the particles in these showers are a type of unstable particle called pions. These pions rapidly decay to muons and do not make it to the surface of the Earth. The muons, however, have a relatively long mean life of 2.2 μs and in many cases live long enough to make it to sea level. As a consequence, most of the particles that pass through the detector are cosmic ray muons.

4.2 Count Rate versus Height

The proton interactions typically occur at a height of $h \simeq 20$ km. A muon traveling at the speed of light at this altitude takes roughly

$$t_{\text{flight}} = \frac{h}{c} = \frac{2 \times 10^4 \text{ m}}{3 \times 10^8 \text{ m/s}} = 66\ \mu\text{s} \tag{8}$$

to reach ground level. Since this is much longer than the mean life of the muon, one would expect nearly all of the muons to decay away before reaching the ground. Conversely, one would expect a dramatic increase in muon count rate as the balloon gains altitude. According to the theory of special relativity, however, the muons, which are moving at close to the speed of light, experience a "time dilation" effect, which makes them live much longer in the laboratory (in this case the atmosphere).

For a typical muon energy of 6 GeV, the Lorentz factor is $\gamma = E_\gamma/m_\mu \simeq 60$, which means that on average such a muon would travel 38 km before decaying. Such a muon would lose about 2/3 of its energy to ionization before reaching sea level, but would still be readily detectable. The important point here is that the time dilation effect has a dramatic effect on the count rate versus height curve.

4.3 Model for Muon Rate versus Height

Much of the following will be based on the paper of M. Gardener *et al.* [1].

4.3.1 Penetration Depth

As shown in the note entitled "Cross Sections and Luminosity," the probability of a particle moving in the z direction undergoing an interaction as it passes through a thin layer of material is

$$dp = \sigma n(z) dz, \tag{9}$$

where σ is the cross section and $n(z)$ is the number density of the targets as a function of z. For a material comprising a single atomic species, the relationship between $n(z)$ and the density is

$$n(z) = \frac{\rho(z)}{A_W} N_A, \tag{10}$$

[1] M. Gardener et al., 1962 Proc. Phys. Soc. 80 697

where $\rho(z)$ is the mass density (in gr/cm^3) and A_W is the atomic weight, and $N_A = 6.02\times10^{23}$ is Avogadro's number. This suggests the introduction of variable known as the "penetration depth," which is defined to be

$$t \equiv \int \rho(z)dz. \tag{11}$$

The penetration depth t has units of mass/area and is a measure of the thickness of a layer of material weighted by its density. The full penetration depth of the atmosphere is given by

$$t_{\text{atm}} = \int_0^\infty \rho(h)dh = \rho_0 \int_0^\infty \exp\left(-h/H\right)dh = \rho_0 H \simeq 10^4 \text{ kg/m}^2 \tag{12}$$

where $\rho_0 = 1.23$ kg/m^3 is the density of air at sea level ($h = 0$) and $H = 7570$ m is the mean height of the atmosphere.

Since the cosmic rays penetrate the atmosphere from the outside, the relation between depth of penetration t and height h is found by integrating from the outside in—i.e.,

$$t = \int_\infty^h \rho(h')dh' = \rho_0 H \left[e^{-h'/H}\right|_\infty^h = \rho_0 H e^{-h/H}. \tag{13}$$

Note that $t = 0$ corresponds to $h = \infty$ (no penetration) and $h = 0$ corresponds to the full penetration depth of Eq. 12.

The mean free path for primary cosmic ray interactions is $\lambda_{\text{int}} = 1200$ kg/m^2, and the probability density distribution of penetration depths is given by

$$p(t) = \frac{1}{\lambda_{\text{int}}} e^{-t/\lambda_{\text{int}}}. \tag{14}$$

Also, the height corresponding to the mean penetration depth is

$$h = H \log \frac{\rho_0 H}{\lambda_{\text{int}}} = 15.5 \text{ km}. \tag{15}$$

4.3.2 Muon Production

Each interaction gives rise to some number of charged pions, which subsequently decay to muons. The momentum spectrum of the pions that are produced is given by Gardener *et al.*. Following that paper, we assume that the muons that are produced from the decaying pions have momenta given by $p_\mu = 0.787 p_\pi$. The pion momentum spectrum is given in Table 4 of Gardener *et al.*, which is reproduced here in Fig. 1. A central task will be to reproduce the distribution of pion momenta in simulation. This can most easily be done by coding a function that represents the pion momentum spectrum and using a trial and error method.

Table 4

The Production Spectrum of Pions in the Atmosphere, $I(p_\pi) = Ap_\pi^{-\gamma}$

p_π (Gev/c)	A	γ	$I(p_\pi)$ (cm^{-2} sec^{-1} sterad^{-1} (Gev/c)$^{-1}$)
1·00	$7{\cdot}0 . 10^{-2}$	1·93	$7{\cdot}01 . 10^{-2}$
1·50	$7{\cdot}0 . 10^{-2}$	1·93	$3{\cdot}21 . 10^{-2}$
2·00	$7{\cdot}0 . 10^{-2}$	1·93	$1{\cdot}84 . 10^{-2}$
3·00	$8{\cdot}0 . 10^{-2}$	2·05	$8{\cdot}43 . 10^{-3}$
4·00	$1{\cdot}02 . 10^{-1}$	2·25	$4{\cdot}50 . 10^{-3}$
5·00	$1{\cdot}50 . 10^{-1}$	2·50	$2{\cdot}69 . 10^{-3}$
6·00			$1{\cdot}66 . 10^{-3}$
7·00			$1{\cdot}10 . 10^{-3}$
8·00	$1{\cdot}93 . 10^{-1}$	2·65	$7{\cdot}76 . 10^{-4}$
9·00			$5{\cdot}69 . 10^{-4}$
10·00			$4{\cdot}31 . 10^{-4}$
12·00			$2{\cdot}66 . 10^{-4}$

Figure 1: Table 4 from Gardener *et al.*, which gives the momentum spectrum for pions produced in cosmic ray collisions.

Once a pion momentum is selected, the corresponding muon momentum is determined using $p_\mu = 0.787 p_\pi$. The resulting muon will either decay in flight or will "range out" as a result of energy loss. The mean free distance for decay in flight is given by

$$\lambda_{\text{decay}} = \gamma\beta c\tau \tag{16}$$

where $\tau = 2.2$ μs is the muon's mean lifetime, β is the ratio of the muon's velocity relative to that of the speed of light $c = 3 \times 10^8$ m/s, and γ is the relativistic time dilation factor. Since $\beta = p/E$ and $\gamma = E/m$, Eq. 16 becomes

$$\lambda_{\text{decay}}(p) = \gamma\beta c\tau = \frac{p}{m} c\tau, \tag{17}$$

where $m = 0.107$ GeV is the muon mass. The probabilty that a muon of momentum p would decay in passing through a height Δh is therefore

$$P_{\text{decay}} = 1 - e^{-\Delta h/\lambda_{\text{decay}}(p)}. \tag{18}$$

4.3.3 Muon Energy Loss

Muons lose energy to ionization as they propagate through the atomosphere. The amount of energy loss in a step size of Δh is roughly given by

$$\Delta E = \left(\frac{dE}{dx}\right)_0 \rho(h)\Delta h, \tag{19}$$

where $(dE/dx)_0 = 2 \times 10^{-4}$ GeV $\cdot$ m^2/kg, $\rho(h)$ is the atmospheric density as a function of altitude, expressed in units of kg/m^3, and Δh is specified in meters.

4.4 Simulation Algorithm

The measured data take the form of a plot of relative rate for cosmic rays as a function of altitude h. The goal of the simulation is to describe that data with a curve based on the theory outlined above. This will be done by simulating a large number of events. For each event, the simulation program will track whether or not a particle passed through a given altitude range. The result of the simulation should therefore be an array `nCrossing[i]`, where each element corresponds to a 100-m-wide height range. If we assume $h_{\text{max}} = 10^5$ m, this array will have 1000 elements, where the i^{th} elements corresponds to a height of $h_i = 10^2 \times i$.

The simulation of each cosmic ray event proceeds in steps, as outlined below.

1. Choose a penetration depth by choosing a random number in the range $0 < r < 1$ and computing

$$t = -\lambda_{\text{int}} \ln r,$$

   where λ_{int} is the mean free path for interactions (see Section 4.3.1).

2. Convert the penetration depth, t, found above to a height h by inverting Eq. 13—i.e.,

$$h = -H \ln \frac{t}{\rho_0 H}. \tag{20}$$

3. Choose a pion momentum according to $f(p_\pi)$, the distribution given in Table 1. To do this:

   (a) Choose a pion momentum p_π from a uniform distribution over the range $0.5 < p_\pi < 10$ GeV.

   (b) Choose a y value from a uniform range $0. < y < y_{\text{max}}$, where y_{max} is the largest value that the function from Table 1 can take on ($y_{\text{max}} = 0.3$ is a good choice).

   (c) If $y > f(p_\pi)$, repeat steps (a) and (b).

   (d) If $y < f(p_\pi)$ proceed to the next step.

4. Compute $p_\mu = 0.787 p_\pi$

5. Track the muon as it travels down to the surface of the Earth in a series of steps, each of size $\Delta h = 100$ m.

   (a) $p_\mu \longrightarrow p_\mu - \Delta E$, where $\Delta E \simeq \Delta p$ is given in Eq. 19.

   (b) Calculate the probability that the muon will decay in a step of size Δh using Eq. 18.

   (c) Generate a random number $0 < r < 1$.

   (d) If $p_\mu > 0$ and $r > P_{\text{decay}}$ increment the crossing count array—i.e., `nCrossing[i] += 1`, where $i = h/100$, and repeat steps (a) through (c) above. Otherwise exit the loop and start the full list of steps again.

   (e) Continue until the desired number of events has been generated.