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Very Short Discussion of Relativisitic Kinematics

This note is designed to present some basic notions of relativistic kinematics using a minimum of words. To accomplish that goal, some expressions will be presented without proof or explanation. Readers wishing to learn more about Special Relativity should consult one of the many available sources.

1 β and γ

Here we introduce two important parameters used to describe the motion of a particle. The first is β, which is defined to be the ratio of the particle's speed to that of the speed of light—i.e.,

$$\beta \equiv \frac{v}{c}, \tag{1}$$

where v is the particle's speed and $c = 3 \times 10^8$ m/s is the speed of light. In the study of classical (Newtonian) mechanics, one generally assumes the $v \ll c$, in which case $\beta \ll 1$. Although this assumption is a good one for most situations involving macroscopic (large) bodies, it often fails when consider subatomic particles, which frequently move with speeds comparable to the speed of light.

A second parameter is the Lorentz factor, which is given by

$$\gamma = \sqrt{\frac{1}{1-\beta^2}}. \tag{2}$$

It is by this factor that fast moving objects appear to contract and by which clocks appear to run slower.

In Newtonian mechanics, the momentum of a particle is given by $p = mv$. To take relativistic effects into account, this is modified to read

$$p = \gamma m v \tag{3}$$

We can think of this as meaning that the mass m gets larger by a factor of γ as the speed of a particle approaches the speed of light. This allows particles to acquire arbitrarily large momenta without exceeding the speed of light. Note that if $v \ll c$, then Eq. 3 becomes

$$p = \gamma m v = \sqrt{\frac{1}{1-\beta^2}} m v \simeq m v, \tag{4}$$

as expected.

2 Mass Energy Equivalence

In Newtonian mechanics, the energy of a particle is given by

$$E = \frac{1}{2} m v^2. \tag{5}$$

In relativisitic mechanics the total energy of a particle is given by

$$E^2 = m^2 c^4 + p^2 c^2 \tag{6}$$

For small momentum values—i.e., for $p^2 c^2 \ll m^2 c^4$—this becomes

$$E = \sqrt{m^2 c^4 + p^2 c^2} = m c^2 \left(1 + \frac{p^2}{m^2 c^2}\right)^{\frac{1}{2}} \simeq m c^2 \left(1 + \frac{p^2}{2 m^2 c^2}\right) = m c^2 + \frac{p^2}{2m}. \tag{7}$$

The first term in this expression corresponds to Einstein's famous $E = mc^2$ and represents the energy corresponding to the "rest mass" of the particle. The second term is the kinetic energy associated with the particle's motion.

Returning to the relativistic case, and switching to units where the speed of light is defined to be one, we have

$$E^2 = m^2 + p^2. \tag{8}$$

If we write $p = \gamma m v = \gamma m \beta$, Eq. 8 becomes

$$E^2 = m^2 + \gamma^2 \beta^2 m^2 = m^2 \left(\gamma^2 \beta^2 + 1\right) = m^2 \gamma^2. \tag{9}$$

We also have

$$\frac{p}{E} = \frac{\sqrt{E^2 - m^2}}{E} = \sqrt{1 - \frac{m^2}{E^2}} = \sqrt{1 - \frac{1}{\gamma^2}} = \beta \tag{10}$$

Eqs. 9 and 10 can be summarized as

$$\gamma = \frac{E}{m} \quad \text{and} \quad \beta = \frac{p}{E}. \tag{11}$$

3 Ultra Relativistic Limit

It is frequently the case in subatomic particle physics that all particles involved are moving at speeds very close to the speed of light—i.e., $\beta \simeq 1$. In this case $p \gg m$ and

$$E^2 = m^2 + p^2 \simeq p^2 \implies E \simeq p. \tag{12}$$