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Cross Sections and Luminosity

1 Geometric Interpretation

As its name implies, the cross section is the area that a target presents to a probe (beam) particle. We start with a highly idealized example of target holder of area A that is host to a single target of area σ, as shown in Fig. 1. If we fire a beam of particles at the target such they impinge at right angles and are uniformly distributed over the area A, the probability that a given beam particle will strike the target σ is given by

$$p_1 = \frac{\sigma}{A}. \tag{1}$$

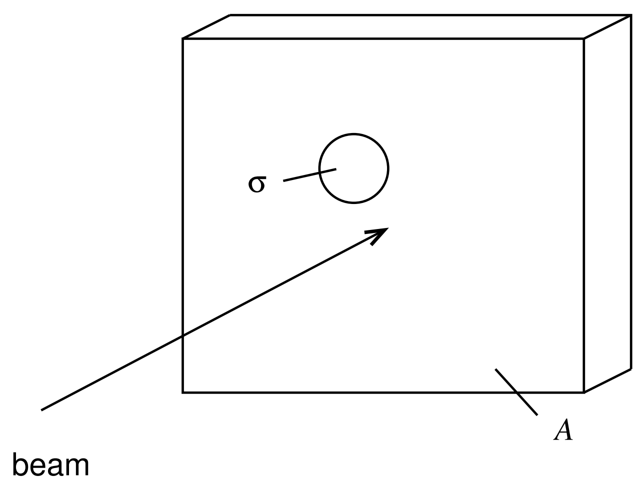

Figure 1: A target holder of area A houses one target of area a.

If instead of a single target of area σ we have N targets each of area σ randomly distributed over the area A, the probability becomes

$$p_N = \frac{N\sigma}{A}. \tag{2}$$

We hasten to add that this result will only be valid in the limit $N\sigma \ll A$, since if $N\sigma \sim A$, the small targets will tend to overlap and p_N will be smaller $N\sigma/A$.

In a practical experiment, A will be macroscopic area and σ will typically be microscopic. Moreover, the target holder will be a three-dimensional object of thickness Δz, as shown in Fig. 2.

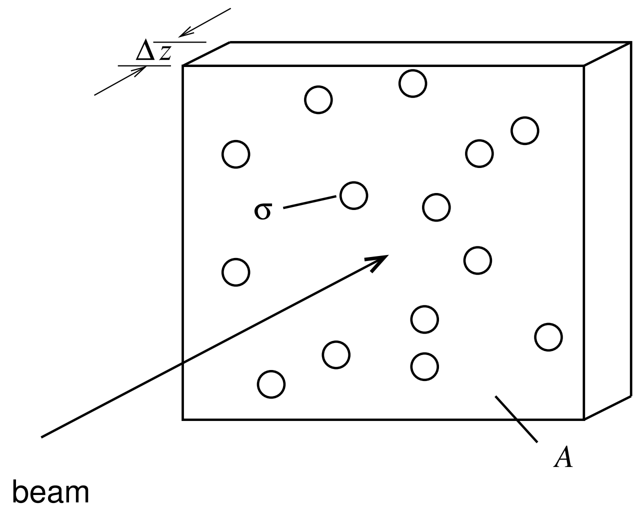

Figure 2: A target holder of area A and thickness z holds N targets of area σ.

In this case the number of targets $N = nV = nA\Delta z$, where n is the number density of the targets. Thus Eq. 2 becomes

$$p = n\sigma\Delta z. \tag{3}$$

If we take the infinitesimal limit $\Delta z \to dz$, this becomes

$$dp = n\sigma dz. \tag{4}$$

The number of beam particles that interact is given by $N_b dp$. If we assume that they are removed from the beam, this becomes

$$dN_b = -n\sigma N_b dz \implies \frac{dN_b(z)}{dz} = -n\sigma N_b(z) \tag{5}$$

where the minus sign reflects the decrease in beam intensity. The solution to Eq. 5 is

$$N_b(z) = N_B(0)e^{-z/\lambda}, \tag{6}$$

where $\lambda = 1/n\sigma$ is the mean free path for collisions.

2 Probabilistic Interpretation

In a classical (geometric) interpretation described above, the meaning of "striking" a target is clear. For the microscopic targets of interest, the geometric picture may or may not be relevant. The cross section for scattering a high energy proton from a heavy nucleus will be close to the $\sigma = \pi r_{\rm nuc}^2$ value that one would expect in the geometric model ($r_{\rm nuc}$ is the nuclear radius). However, for the scattering of low energy neutrons, which have a long de Broglie wavelength, the cross sections can be significantly larger. What remains valid, however, is the probabilistic interpretation embodied in Eq. 4. In other words, if we measure the probability p that a collision occurs, we can use Eq. 4 to determine the cross section σ for the process. The number of beam particles N_b and the number density of the targets depend on experimental details, whereas the cross section σ is intrinsic to the reaction under study and can be computed by theorists without the need to know experimental details. Quoting reaction strengths in terms of cross sections also allows one to compare experimental data taken under different conditions.

Thus far, we have been vague about the nature of the microscopic interaction between the beam and the target—i.e., we haven't really said what we mean by "striking" the target. Two possibilities immediately come to mind: elastic and inelastic interactions. In elastic scattering, the beam particle is deflected from the target (which recoils in such a way as to conserver momentum), but both the beam and the target continue on their merry ways without a change in their nature. For inelastic scattering, the target (or the beam) may be placed in an excited state, which removes kinetic energy from the system. In this simple scheme we define three cross sections, one for elastic scattering, one for inelastic scattering, and a third, called the total cross section, which is the sum of the two—i.e.,

$$\sigma_{\rm total} = \sigma_{\rm elastic} + \sigma_{\rm inelastic}. \tag{7}$$

Since most targets have a large number of excited states, $\sigma_{\rm inelastic}$ could be further subdivided into corresponding cross sections, i.e.,

$$\sigma_{\rm inelastic} = \sum_i \sigma^i_{\rm inelastic}, \tag{8}$$

where $\sigma^i_{\rm inelastic}$ is the cross section for the $i^{\rm th}$ inelastic process. The division in Eq. 8 is discrete. A continuous division is also possible. Imagine, for example, inelastic interactions that leave the target in a positive energy (unbound) final state of excitation energy E. In that case, E can take on a continuous set of values and the cross section might be written $d\sigma/dE$, where

$$\sigma(E, \Delta E)_{\rm inelastic} = \int_E^{E+\Delta E} \frac{d\sigma(E)}{dE} dE, \tag{9}$$

where $\sigma(E, \Delta E)_{\text{inelastic}}$ is the cross section for excitation energies between E and $E + \Delta E$.

Elastic scattering can also be divided up into smaller cross sections according to the angle of deflection of the beam particle (see Fig. 3). In this case one refers to the differential cross section denoted by $d\sigma(\theta)/d\Omega$, where

$$\sigma(\theta, \Delta\Omega) = \frac{d\sigma(\theta)}{d\Omega}\Delta\Omega, \tag{10}$$

where $\Delta\Omega$ represents a finite solid angle (for a discussion of solid angle see `https://en.wikipedia.org/wiki/Solid_angle`).

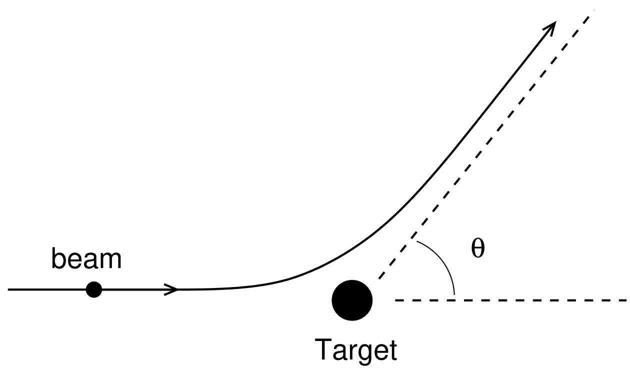

Figure 3: Elastic scattering through an angle θ.

3 Luminosity

For a "fixed target" experiment, the relationship between cross section and event rate is

$$N_{\text{events}} = N_b \sigma n \Delta z, \tag{11}$$

where N_b is the number of beam particles, σ is the cross section for the process of interest, n is the number density of the targets, and Δz is the thickness of the target. We can rewrite this as

$$N_{\text{events}} \equiv \mathcal{L}\sigma, \tag{12}$$

where we have defined the luminosity $\mathcal{L} \equiv N_b n \Delta z$. Note that $\mathcal{L}$ combines the experimental details into a single number.

Most modern day measurements in particle physics are carried out at colliding beam accelerators, which offer much higher center-of-mass energy for a given beam energy. There is no target *per se*,[1] but the notion of luminosity expressed in Eq. 12 nonetheless applies.

An additional detail has to do with the notions of acceptance and efficiency. There is an implicit assumption built in to Eq. 12 that all events that are produced will be detected. This is rarely true in practice. Most detectors are subject to inefficiencies and have limited coverage in momentum. Moreover most detector systems are limited in the number of events that they can log in a given time. If two events occur in rapid succession, the second may be lost because the system is busy processing the first, an effect known as deadtime. As a result, Eq. 12 is often modified to read

$$N_{\text{events}} = \varepsilon \mathcal{L} \sigma, \tag{13}$$

where ε represents the experimental detection efficiency losses discussed above.

4 Units: the "Barn"

Cross sections are areas and thus have units of m^2 or cm^2. Microscopic cross sections expressed in such units would lead to inconveniently large exponents, so it is natural to adopt a new unit representing a microscopic area. That unit is whimsically called the "barn." This word was coined by workers in the Manhattan project, who for reasons of secrecy chose something without obvious connection to nuclear physics. The barn, which is defined to be 10^{-28} m^2 = 10^{-24} cm^2, is roughly the same as the cross section for scattering from uranium, which was said to be "as big as a barn." Although the barn (abbreviated by "b") is much closer to the typical cross section encountered in particle physics, there are many processes that have cross sections that are significantly smaller than one barn. One consequently encounters millibarns (mb), microbarns (μb), nanobarns (nb), and picobarns (pb). Indeed, many of the processes under study at the Large Hadron Collider have cross sections in the femtobarn (1 fb = 10^{-15} b) range.

Luminosities are expressed in related units. In recent years, the CMS detector has collected samples on the order of 50 fb^{-1}. That means that for a cross section of (say) 10 fb, a total of 500 events would be expected. Note that 1 fb^{-1} = 1000 pb^{-1} = 10^6 nb^{-1} etc.

[1] By "target" in this context, I mean the macroscopic object that holds numerous microscopic targets and is placed in the way of the beam.