## Exercise 7 (1)

Complete the above proof. Prove, in particular that for any $x \in X$, $c_P(x)$ is indeed a distribution; that $P_c$ is a stochastic matrix; and that $P_{cP} = P$ and $c_{P_c} = c$.

**Solution** - part 1
The previous proof has already provided that $c_P(x_i)(x_j) = P(i,j)$ and $P$ is a stochastic matrix (because it is a transition matrix), which implies that $c_P(x)$ is a distribution $\forall x$ because the elements belonging to the same row sum to 1.

- part 2
Recalling the functional definition we have that $c : X \mapsto D(X)$, thus all the values of $P_c$ are in the closed interval between 0 and 1, furthermore each row of $P_c$ sums to 1 as they represent distributions in $D(X)$ i.e. the set of all the possible distribution over $X$.

- part 3 and 4
These two parts can be easily verified by definition, indeed we have that:

$$P_{c_P}(i,j) = c_P(x_i)(x_j) = P(i,j) \ \forall i,j$$

and that:

$$c_{P_c}(x_i)(x_j) = P_c(i)c(j) = c(x_i)(x_j) \ \forall i,j.$$

## Exercise 7 (2)

Prove that $c(x) = c^*(\delta_x)$.

**Solution** By just the use of our intuition we can see that the identity above is equivalent to say that the multiplication of a row vector made of all 0's with the exception of the $i^{th}$ entry by a proper sized matrix results in a vector made by just the $i^{th}$ row of the matrix.
In order to address this point we first write the definition of the $i^{th}$ entry of $c^*(\psi)$:

$$c^*(\psi)_i = \textstyle\sum_{j=1}^n \psi(x_j) \cdot c(x_j)(y_i)$$

And then we observe that if $\psi$ is a distribution vector of the type we mentioned before, then we have:

$$\textstyle\sum_{j=1}^n \psi(x_j) \cdot c(x_j)(y_i) = c(x)(y_i)$$

where $i$ is the index of the only 1 in the vector.
$c^*(\delta_x)$ is just a diagonal matrix made of only that kind of vector, thus we finally have: $c^*(\delta_x)_i = c(x)(y_i) = c(x)_i$ i.e. $c^*(\delta_x) = c(x)$.

## Exercise 7 (3)

Prove that $c^*(\psi) = \psi P_c$.

**Solution** Following the definition of dot product between the 1 x $n$ row vector $/phi$ and the $n$ x $n$ transition matrix $P_c$ we have that:

$$\psi P_c(i) = \textstyle\sum_{j=1}^{n} \psi(x_j) \cdot c(x_j)(x_i) = c^*(\psi)(x_i)\ \forall i \in \{1...n\}.$$

## Exercise 7 (4)

Prove that if $\psi$ satisfied $DBC$, then $\psi$ is stationary for $P$.

**Solution** Let $P$ be the transition matrix, we write the $DBC$ hypothesis:

$$\psi(x) \cdot P(x,y) = \psi(y) \cdot P(y,x)\ \forall x, y.$$

By considering a single row x column product we can write the following:

$$\textstyle\sum_{j=1}^{n} \psi(x_j) \cdot P(x_j, x_i) = \sum_{j=1}^{n} \psi(x_i) \cdot P(x_i, x_j)$$

Of course this holds $\forall i$ i.e. for every row x columns product.
Then we do the following:

$$\textstyle\sum_{j=1}^{n} \psi(x_j) \cdot P(x_j, x_i) = \psi(x_i) \sum_{j=1}^{n} P(x_i, x_j)\ \forall i$$

and using the definition of probability distribution:

$$\textstyle\sum_{j=1}^{n} \psi(x_j) \cdot P(x_j, x_i) = \psi(x_i)\text{x}1\ \forall i$$

To sum up: every component of the $\psi P$ vector is equal to the corresponding component of the distribution vector $\psi$, thus they are the same.

$$\psi P = \psi$$

So we have just proven what was required as that is the definition for $\psi$ to be a stationary distribution for $P$.