

More Vector Spaces with Atoms of Finite Lengths

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Abstract—*CRITICAL: Do Not Use Symbols, Special Characters, Footnotes, or Math in Paper Title or Abstract.

I. INTRODUCTION

II. RADO GRAPH, SANS COGS

\mathbb{A} is:

- 1) oligomorphic if, for $k = 0, 1, 2, \dots$, \mathbb{A}^k only has finitely many orbits;
- 2) \mathfrak{F} -oligomorphic if, for $k = 0, 1, 2, \dots$, $\text{Lin}_{\mathfrak{F}} \mathbb{A}^k$ only has finitely long chains.

Of note: with Stirling numbers of the second kind and Gaussian 2-binomial coefficients, the orbit counts are given by

$$\begin{aligned}\#\mathbb{N}^k &= \sum_{d=0}^k \left\{ \begin{matrix} k \\ d \end{matrix} \right\} \\ \#\mathbb{Q}^k &= \sum_{d=0}^k \left\{ \begin{matrix} k \\ d \end{matrix} \right\} d! \\ \#\mathbb{G}^k &= \sum_{d=0}^k \left\{ \begin{matrix} k \\ d \end{matrix} \right\} 2^{\binom{d}{2}} \\ \#\mathbb{V}_{\infty}^k &= \sum_{d=0}^k \left[\begin{matrix} k \\ d \end{matrix} \right]_2 \\ \#\mathbb{W}_{\infty}^k &= \sum_{d=0}^k \left[\begin{matrix} k \\ d \end{matrix} \right]_2 2^{\binom{d}{2}}\end{aligned}$$

To introduce:

- 1) *smooth approximation* by homogeneous substructures [3] (N.B. ‘smooth approximation’ from [4, Definition 4] seems to be entirely different)
- 2) *width-wise approximation* of a homogeneous structure by finite substructures with uniformly few orbits (i.e., types) that cover the age of \mathbb{A}
- 3) For the equality and ordered atoms, being supportively \mathfrak{F} -oligomorphic follows from being \mathfrak{F} -oligomorphic [1, Theorem 4.10]

A. Symplectic vector spaces

Throughout this subsection let \mathfrak{f} denote a finite field.

Definition II.1. A *symplectic vector space* is an \mathfrak{f} -vector space \mathbb{W} equipped with a bilinear form $\omega : \mathbb{W} \times \mathbb{W} \rightarrow \mathfrak{f}$ that is

- 1) alternating: $\omega(v, v) = 0$ for all v ; and
- 2) non-degenerate: if $\omega(v, w) = 0$ for all w then $v = 0$.

Example II.2. Let \mathbb{W}_n be the \mathfrak{f} -vector space with basis $e_1, \dots, e_n, f_1, \dots, f_n$. Define ω by bilinearly extending

$$\omega(e_i, f_i) = 1 = -\omega(f_i, e_i), \quad \omega(-, *) = 0 \text{ elsewhere; } (\S)$$

one may straightforwardly check that ω is alternating and non-degenerate. Moreover, noticing that $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \dots$, we obtain a countable-dimensional symplectic vector space $\mathbb{W}_{\infty} = \bigcup_n \mathbb{W}_n$.

We will refer to vectors satisfying (\S) as a *symplectic basis* — indeed, they must be linearly independent. Such bases behave very much like the usual bases.

Proposition II.3. Assume that \mathbb{W} is a symplectic vector space that is at most countable. Then any finite symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$ can be extended to a symplectic basis that spans the whole \mathbb{W} .

Proof. Suppose that $e_1, \dots, e_n, f_1, \dots, f_n$ does not already span \mathbb{W} ; take v to be a witness (that is least according to some fixed enumeration of \mathbb{W} in the case it is infinite). Put

$$e_{n+1} = v - \sum_{i=1}^n \omega(e_i, v) f_i + \sum_{i=1}^n \omega(f_i, v) e_i$$

so that $\omega(e_i, e_{n+1}) = 0 = \omega(f_i, e_{n+1})$. This cannot be the zero vector lest we contradict the choice of v . By the non-degeneracy of ω , there is — rescaling if necessary — some w such that $\omega(e_{n+1}, w) = 1$. Now define

$$f_{n+1} = w - \sum_{i=1}^n \omega(e_i, w) f_i + \sum_{i=1}^n \omega(f_i, w) e_i$$

in a similar manner, making $e_1, \dots, e_n, e_{n+1}, f_1, \dots, f_n, f_{n+1}$ a symplectic basis that spans v . We go through every element of \mathbb{W} by continuing this way. \square

In fact, we will also make use of the “symplectic basis and a half” variant below.

Proposition II.4. Now assume \mathbb{W} is a finite-dimensional symplectic vector space. Let

$$\begin{aligned}e_1, \dots, e_n, e_{n+1}, \dots, e_{n+k}, \\ f_1, \dots, f_n\end{aligned}$$

be linearly independent vectors satisfying (\S) . Then we can find the missing f_{n+1}, \dots, f_{n+k} to complete the symplectic basis.

Proof. We first need the following notion. Given a subspace $V \subseteq \mathbb{W}$, consider its orthogonal complement

$$V^\perp = \{w \in \mathbb{W} \mid \forall v \in V : \omega(v, w) = 0\}.$$

It is the kernel of the composite linear map

$$\begin{aligned} \mathbb{W} &\rightarrow (\mathbb{W} \xrightarrow{\text{lin.}} \mathfrak{f}) \rightarrow (V \xrightarrow{\text{lin.}} \mathfrak{f}) \\ w &\mapsto \omega(-, w) \mapsto \omega(-, w)|_V. \end{aligned}$$

Note this map is surjective: the first part is injective by non-degeneracy and hence surjective for dimension reasons, and the second part is surjective since we can extend a basis of V to one of \mathbb{W} . Therefore

$$\dim V^\perp = \dim \mathbb{W} - \dim V,$$

and in particular $V^{\perp\perp}$ is precisely equal to V .

Now suppose we have found f_{n+1}, \dots, f_{n+i} already. If e_{n+i+1} were to be spanned by

$$\begin{aligned} e_1, \dots, e_{n+i}, \overline{e_{n+i+1}}, e_{n+i+2}, \dots, e_{n+k}, \\ f_1, \dots, f_{n+i}, \end{aligned}$$

it would be spanned by $e_{n+i+2}, \dots, e_{n+k}$ alone because of (§); but this is impossible as we assumed linear independency. So

$$\begin{aligned} e_{n+i+1} &\notin \langle e_1, \dots, e_{n+i}, e_{n+i+2}, f_1, \dots, f_{n+i} \rangle \\ &= \langle e_1, \dots, e_{n+i}, e_{n+i+2}, f_1, \dots, f_{n+i} \rangle^{\perp\perp}, \end{aligned}$$

i.e., some $f_{n+i+1} \in \langle e_1, \dots, e_{n+i}, e_{n+i+2}, f_1, \dots, f_{n+i} \rangle^{\perp}$ satisfies $\omega(e_{n+i+1}, f_{n+i+1}) = 1$. \square

Given two symplectic vector spaces \mathbb{W} and \mathbb{W}' , we call a function α between $X \subseteq \mathbb{W}$ and $X' \subseteq \mathbb{W}'$ *isometric* if $\omega(\alpha(x_1), \alpha(x_2)) = \omega(x_1, x_2)$ for all $x_1, x_2 \in X$. We can make an easy observation:

Lemma II.5. *Let $\{e_i, f_i \mid i \in I\} \subseteq \mathbb{W}$, $\{e'_j, f'_j \mid j \in J\} \subseteq \mathbb{W}'$ be two symplectic bases and let $\alpha : I \rightarrow J$ be a bijection. Then*

$$e_i \mapsto e'_{\alpha(i)}, f_i \mapsto f'_{\alpha(i)}$$

defines an isometric linear isomorphism $\langle e_i, f_i \rangle \rightarrow \langle e'_j, f'_j \rangle$.

It then follows from Proposition II.3 that, up to isometric linear isomorphisms, $\mathbb{W}_0, \mathbb{W}_1, \mathbb{W}_2, \dots, \mathbb{W}_\infty$ are all the countable symplectic vector spaces. Whilst we may deduce that \mathbb{W}_∞ is oligomorphic by appealing to Ryll-Nardzewski, we will opt for a more direct proof that also establishes smooth approximation.

Proposition II.6 (Witt Extension). *Any isometric linear injection $\alpha : \langle X \rangle \subseteq \mathbb{W}_n \rightarrow \mathbb{W}_n$ can be extended to an isometric linear automorphism of \mathbb{W}_n and in turn to one of \mathbb{W}_∞ .*

Proof. To begin with, find a basis x_1, \dots, x_k for $\langle X \rangle^\perp = \{w \in \mathbb{W} \mid \forall x \in X : \omega(w, x) = 0\}$ and extend it to a basis $x_1, \dots, x_k, x_{k+1}, \dots, x_d$ for $\langle X \rangle$. Notice that

$$U = \langle x_{k+1}, \dots, x_d \rangle$$

must be a symplectic subspace: as it intersects with $\langle X \rangle^\perp$ trivially, given any non-zero vector $u \in U$ we must have $0 \neq \omega(u, x + u') = \omega(u, u')$ for some $x \in \langle X \rangle^\perp$ and $u' \in U$. Hence use Proposition II.3 to find a symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$ for U . Observe that

$$\begin{aligned} e_1, \dots, e_n, x_1, \dots, x_k, \\ f_1, \dots, f_n \end{aligned}$$

form a basis for $\langle X \rangle$ and satisfy (§). On the other hand,

$$\begin{aligned} \alpha(e_1), \dots, \alpha(e_n), \alpha(x_1), \dots, \alpha(x_k), \\ \alpha(f_1), \dots, \alpha(f_n) \end{aligned}$$

form a basis for $\alpha(\langle X \rangle)$ and also satisfy (§). Therefore apply Proposition II.4 twice to find the missing y_1, \dots, y_k and y'_1, \dots, y'_k to complete the two symplectic bases — call them \mathcal{B} and \mathcal{B}' . They are of the same size.

Now, by using Proposition II.3, extend \mathcal{B} and \mathcal{B}' to symplectic bases \mathcal{C} and \mathcal{C}' that span \mathbb{W}_n . These must both have size $2n$, so by Lemma II.5 we obtain an isometric linear automorphism $\beta : \mathbb{W}_n \rightarrow \mathbb{W}_n$ extending α .

To finish, notice that $\mathcal{C}, e_{n+1}, \dots, f_{n+1}, \dots$ as well as $\mathcal{C}', e_{n+1}, \dots, f_{n+1}, \dots$ form a symplectic basis spanning \mathbb{W}_∞ . We obtain from Lemma II.5 another time an isometric linear automorphism $\gamma : \mathbb{W}_\infty \rightarrow \mathbb{W}_\infty$ extending β that is the identity almost everywhere. \square

Proposition II.7. *\mathbb{W}_∞^k has precisely $\sum_{d=0}^k \begin{bmatrix} k \\ d \end{bmatrix}_q \cdot q^{\binom{d}{2}}$ orbits under isometric linear automorphisms, where $q = |\mathfrak{f}|$ and*

$$\begin{bmatrix} k \\ d \end{bmatrix}_q = \frac{(q^k - 1)(q^{k-1} - 1) \cdots (q^{k-d+1} - 1)}{(q^d - 1)(q^{d-1} - 1) \cdots (q^1 - 1)}$$

is the q -binomial coefficient.

Remark. To anticipate the next subsection, we note a similarity with the Rado graph: in \mathbb{G}^k there are $\sum_{d=0}^k \binom{k}{d} \cdot 2^{\binom{d}{2}}$ orbits — we may impose any edge relation on d vertices.

Proof. To each $v_\bullet \in \mathbb{W}_\infty^k$ we associate a *type*, which comprises the following data:

- 1) pivot indices $I \subseteq \{1, \dots, k\}$ containing every i such that v_i is not spanned by v_1, \dots, v_{i-1} — so we inductively ensure that

$$\{v_{i'} \mid i' \in I, i' \leq i\}$$

is a basis for $\langle v_1, \dots, v_i \rangle$;

- 2) for each $j \notin I$, an assignment $\Lambda_j : \{i \in I \mid i < j\} \rightarrow \mathfrak{f}$ such that $v_j = \sum_{i \in I, i < j} \Lambda_j(i) v_i$;

- 3) a map $\Omega : \binom{I}{2} \rightarrow \mathfrak{f}$ defined by $\Omega(\{i' < i\}) = \omega(v_{i'}, v_i)$.

If $\pi : \mathbb{W}_\infty \rightarrow \mathbb{W}_\infty$ is an isometric linear automorphism, then $v_\bullet = (v_1, \dots, v_k)$ and $\pi \cdot v_\bullet = (\pi(v_1), \dots, \pi(v_k))$ evidently share the same type. Conversely, if w_\bullet has the type of v_\bullet , then

$$\begin{aligned} \alpha : \langle v_i \mid i \in I \rangle &\rightarrow \langle w_i \mid i \in I \rangle \subseteq \mathbb{W}_n \\ v_i &\mapsto w_i \end{aligned}$$

gives an isometric linear injection for some large enough n . Observe that α must send $v_j \mapsto w_j$ for $j \notin I$ too, and that it may be extended to an isometric linear automorphism π of \mathbb{W}_∞ by Proposition II.6. Furthermore we can find some v_\bullet that realises any given type $(I, \{\Lambda_j\}_j, \Omega)$: it suffices to put

$$v_i = e_i + \sum_{i' \in I, i' < i} \Omega(i', i) f_{i'}$$

for $i \in I$ and $v_j = \sum_{i \in I, i < j} \Lambda_j(i) v_i$ for $j \notin I$. Therefore the number of types is precisely the number of orbits in \mathbb{W}_∞^k .

Finally, we do some combinatorics. Fix $0 \leq d \leq k$ and count the number of types with $|I| = d$. There are $q^{\binom{d}{2}}$ choices for Ω and say $\#_{k,d}$ choices for the Λ_j 's; the two can be chosen separately. In total, this gives

$$\sum_{d=0}^k q^{\binom{d}{2}} \cdot \#_{k,d}$$

types for vectors in \mathbb{W}_∞^d . So focus on $\#_{k,d}$, the number of *linear types* — i.e., $(I, \{\Lambda_j\}_j)$, ignoring Ω — in \mathbb{W}_∞^k . (Incidentally $\sum_{d=0}^k \#_{k,d}$ is the number of orbits in \mathbb{W}_∞^k or, more generally, any countable-dimensional \mathfrak{f} -vector space under linear automorphisms.) On the small values we easily check that

$$\begin{aligned} \#_{0,0} &= 1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q, \\ \#_{1,0} &= 1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q, & \#_{1,1} &= 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q. \end{aligned}$$

Given a linear type in \mathbb{W}_∞^k with $|I| = d$, we either have $1 \in I$ or $I \subseteq \{2, \dots, k\}$. In the first case, the linear type is specified by one of the $\#_{k-1,d-1}$ linear types in \mathbb{W}_∞^{k-1} together with how v_1 is involved in the span of the $(k-1) - (d-1)$ non-pivot vectors. In the second case, the linear type is simply one of the $\#_{k-1,d}$ linear types in \mathbb{W}_∞^{k-1} . Thus

$$\begin{aligned} \#_{k,d} &= q^{k-d} \cdot \#_{k-1,d-1} + \#_{k-1,d} \\ &= q^{k-d} \cdot \begin{bmatrix} k-1 \\ d-1 \end{bmatrix}_q + \begin{bmatrix} k-1 \\ d \end{bmatrix}_q = \begin{bmatrix} k \\ d \end{bmatrix}_q. \quad \square \end{aligned}$$

Theorem II.8. *The symplectic vector space \mathbb{W}_∞ is smoothly approximated by $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \dots$.*

Corollary II.9. *Provided \mathfrak{F} is of characteristic 0, the symplectic \mathfrak{f} -vector space \mathbb{W}_∞ is \mathfrak{F} -oligomorphic.*

B. Symplectic graphs

For this subsection let \mathfrak{f} be the two-element field.

Definition II.10. For $n = 0, 1, 2, \dots$, the *symplectic graph* $\widetilde{\mathbb{W}}_n$ has vertices \mathbb{W}_n and edges

$$v_1 \sim v_2 \iff \omega(v_1, v_2) = 1.$$

This is indeed an undirected graph: as ω is alternating, we have $\omega(v_1, v_2) = -\omega(v_2, v_1) = \omega(v_2, v_1)$.

Proposition II.11. $\text{Aut}(\widetilde{\mathbb{W}}_n) = \text{Aut}(\mathbb{W}_n)$.

Proof. Clearly any isometric linear automorphism of \mathbb{W}_n is a graph automorphism of $\widetilde{\mathbb{W}}_n$. Conversely, any $f \in \widetilde{\mathbb{W}}_n$ is evidently isometric. To show that f is linear, take $\lambda_1, \lambda_2 \in \mathfrak{f}$ and $v_1, v_2 \in \mathbb{W}$. We calculate:

$$\begin{aligned} & \omega\left(f\left(\sum_i \lambda_i v_i\right) - \sum_i \lambda_i f(v_i), f(w)\right) \\ &= \omega\left(f\left(\sum_i \lambda_i v_i\right), f(w)\right) - \sum_i \lambda_i \omega(f(v_i), f(w)) \\ &= \omega\left(\sum_i \lambda_i v_i, w\right) - \sum_i \lambda_i \omega(v_i, w) \\ &= \omega(0, w) = 0 \end{aligned}$$

for all $f(w) \in f(\mathbb{W}_n) = \mathbb{W}_n$; since ω is non-degenerate, we conclude that $f(\sum_i \lambda_i v_i) = \sum_i \lambda_i f(v_i)$. \square

So the number of orbits in $\widetilde{\mathbb{W}}_n^k$ is precisely equal to the number of orbits in \mathbb{W}_n^k — in particular, it is bounded above by $\sum_{d=0}^k \begin{bmatrix} k \\ d \end{bmatrix}_2 \cdot 2^{\binom{d}{2}}$ independently of n by Proposition II.7.¹ It remains to show $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \dots$ embeds all finite graphs:

Proposition II.12 ([2, Theorem 8.11.2]). *Every graph on at most $2n$ vertices embeds into $\widetilde{\mathbb{W}}_n$.*

Proof. Let G be a graph on at most $2n$ vertices. The conclusion is trivial when $n = 0$. Also, if G contains no edges, we can choose any $2n$ of the 2^n vectors in $\langle e_1, \dots, e_n \rangle \subseteq \widetilde{\mathbb{W}}_n$.

So suppose $n \geq 1$ and G has an edge $s \sim t$. Let $G_{s,t}$ be the graph on vertices $G \setminus \{s, t\}$ with edges which we will specify later. By induction, some embedding $f : G_{s,t} \rightarrow \widetilde{\mathbb{W}}_{n-1}$ exists. Define $f' : G \rightarrow \widetilde{\mathbb{W}}_n$ by

$$\begin{aligned} x \in G_{s,t} &\mapsto f(x) - \llbracket x \sim s \rrbracket f_n + \llbracket x \sim t \rrbracket e_n \\ s &\mapsto e_n \\ t &\mapsto f_n \end{aligned}$$

where $\llbracket \phi \rrbracket$ is 1 if ϕ holds and 0 otherwise. Then we have $\omega(f'(x), f'(s)) = \llbracket x \sim s \rrbracket$ and $\omega(f'(x), f'(t)) = \llbracket x \sim t \rrbracket$ as desired, on one hand. On the other,

$$\begin{aligned} \omega(f'(x_1), f'(x_2)) &= \llbracket x_1 \sim x_2 \rrbracket + \llbracket x_1 \sim s \rrbracket \llbracket x_2 \sim t \rrbracket \\ &\quad + \llbracket x_1 \sim t \rrbracket \llbracket x_2 \sim s \rrbracket \end{aligned}$$

tells us how we should define the edge relation in $G_{s,t}$ for f' to be an embedding of graphs. \square

Theorem II.13. *The Rado graph is width-wise approximated by $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \dots$.*

Corollary II.14. *Provided \mathfrak{F} is of characteristic 0, the Rado graph is \mathfrak{F} -oligomorphic.*

This proof of finite length also applies to *oriented graphs* (i.e., $x \rightarrow y \implies y \not\rightarrow x$ but unlike in a tournament, it may occur that $x \not\rightarrow y \wedge y \not\rightarrow x$) — use the three-element field as \mathfrak{f} instead.

¹This is the k th term in the OEIS sequence A028631.

III. RADO GRAPH, WITH COGS

In this section we work with the following setting:

- \mathcal{L}_0 is a (possibly infinite) relational language containing a binary symbol $=$;
- \mathcal{C}_0 is a monotone, free amalgamation class of \mathcal{L}_0 -structures where each $R \in \mathcal{L}_0$ is interpreted irreflexively, except $=$ being interpreted as true equality;
- \mathcal{L} consists of \mathcal{L}_0 together with a new binary symbol $<$;
- \mathcal{C} consists of \mathcal{L} -structures obtained from \mathcal{C}_0 by expanding with all possible linear orderings;
- \mathbb{A}_0 and \mathbb{A} are the respective Fraïssé limits of \mathcal{C}_0 and \mathcal{C} , where without loss of generality we assume \mathbb{A}_0 and \mathbb{A} share the same domain so that $\text{Aut}(\mathbb{A}_0) \supseteq \text{Aut}(\mathbb{A})$.

Example III.1. Take \mathcal{L}_0 to be the empty language and \mathcal{C}_0 to be all finite sets. Then \mathbb{A}_0 is isomorphic to the pure set \mathbb{N} , whereas \mathbb{A} is isomorphic to \mathbb{Q} with the usual order.

Example III.2. Let \mathcal{L}_0 consist of a single binary symbol \sim and let \mathcal{C}_0 consist of all finite undirected graphs not embedding the complete graph K_n , where $3 \leq n \leq \infty$. Then \mathbb{A}_0 is the K_n -free Henson graph (or the Rado graph when $n = \infty$), and \mathbb{A} is its generically ordered counterpart. (Allowing $n = 2$ makes these degenerate to \mathbb{N} and \mathbb{Q} above).

We note three technicalities and a triviality.

Lemma III.3. 1) *forbidden structures are link structures*
2) *free amalgamation with order*

Proposition III.4. *The S -supported length of $\text{Lin } \mathbb{A}_0^d$ is at most that of $\text{Lin } \mathbb{A}^d$ for any finite $S \subseteq \mathbb{A}_0 = \mathbb{A}$.*

Proof. Any chain of subspaces in $\text{Lin } \mathbb{A}_0^d = \text{Lin } \mathbb{A}^d$ that are invariant under $\text{Aut}(\mathbb{A}_0)_{(S)}$ must also be invariant under the subgroup $\text{Aut}(\mathbb{A})_{(S)}$. \square

A. Cogs in an orbit

One problem with \mathbb{A}^d is that it can have many orbits.

Definition III.5. An orbit $\mathcal{O} = \text{Aut}(\mathbb{A})_{(S)} \cdot o_\bullet \subseteq \mathbb{A}^d$ is *strict* if the atoms in o_\bullet satisfy $o_1 < \dots < o_d$ and are not from the finite set S .

Any orbit $\text{Aut}(\mathbb{A})_{(S)} \cdot a_\bullet$ is in an $\text{Aut}(\mathbb{A})_{(S)}$ -equivariant bijection with a strict one — it suffices to remove entries in a_\bullet that repeat or come from S and reorder the rest.

Proposition III.6. \mathbb{A} is *supportively* \mathfrak{F} -oligomorphic if and only if \mathbb{A} is oligomorphic and $\text{Lin}_{\mathfrak{F}} \mathcal{O}$ has finite length for every strict orbit \mathcal{O} .

Proof. Indeed, we have $\text{len}(\text{Lin}_{\mathfrak{F}} \mathbb{A}^d) = \text{len}(\text{Lin}_{\mathfrak{F}} (\biguplus_i \mathcal{O}_i)) = \text{len}(\bigoplus_i \text{Lin}_{\mathfrak{F}} \mathcal{O}_i) = \sum_i \text{len}(\text{Lin}_{\mathfrak{F}} \mathcal{O}_i)$ given a finite support. \square

Thus fix a strict orbit \mathcal{O} with support S .

Remark. As \mathbb{A} has no algebraicity, the strict orbit \mathcal{O} uniquely determines its support S : we have $\mathbb{A} \setminus S = \{a_1 \mid a_\bullet \in \mathcal{O}\}$.

Also, since \mathbb{A} is homogeneous, a tuple $a_\bullet \in \mathbb{A}^d$ belongs to \mathcal{O} if and only if

$$R(x_\bullet) \iff R(x_\bullet[o_i/a_i \mid i])$$

holds in \mathbb{A} for every symbol R in \mathcal{L} of arity r and every $x_\bullet \in (\{a_1, \dots, a_d\} \cup S)^r$.

We now introduce the workhorse we use to understand $\text{Lin } \mathcal{O} \subseteq \text{Lin } \mathbb{A}^d$.

Definition III.7. An \mathcal{O} -cog $a_\bullet | b_\bullet$ consists of atoms $a_1 < b_1 < a_2 < b_2 < \dots < a_d < b_d$ such that for any symbol R in \mathcal{L} of arity r , given any $x_\bullet \in (\{a_1, b_1, \dots, a_d, b_d\} \cup S)^r$ we have

$$R(x_\bullet) \iff R(x_\bullet[o_i/a_i, o_i/b_i \mid i])$$

in \mathbb{A} . In particular, for each $I \subseteq \{1, \dots, d\}$ we see that $a_\bullet[b_i/a_i \mid i \in I]$ is in the strict orbit \mathcal{O} from the remark above; we define the corresponding \mathcal{O} -cogwheel to be the vector

$$a_\bullet \wr b_\bullet = \sum_{I \subseteq \{1, \dots, d\}} (-1)^{|I|} a_\bullet[b_i/a_i \mid i \in I]$$

in $\text{Lin } \mathcal{O}$.

It is actually easier to find cogs than to define them.

Example III.8. Let $\mathbb{A} = \mathbb{Q}$ as described in Example III.1; there is a unique equivariant strict orbit \mathcal{O} in \mathbb{A}^2 . Consider the vector

$$v = (0, 5) + (5, 6) - (5, 10) - (0, 10)$$

in $\text{Lin } \mathcal{O}$. We can find $5 < 5 + \varepsilon < 6 < 6 + \delta < 10$ in \mathbb{A} together with monotone bijections $\pi_1, \pi_2 \in \text{Aut}(\mathbb{A})$ such that

$$\pi_1 : \begin{cases} 0 \mapsto 0, \\ 5 \mapsto 5 + \varepsilon, \\ 6 \mapsto 6, \\ 10 \mapsto 10; \end{cases} \quad \pi_2 : \begin{cases} 0 \mapsto 0, \\ 5 \mapsto 5, \\ 6 \mapsto 6 + \delta, \\ 10 \mapsto 10 \end{cases}$$

by interpolating linearly for example. Then

$$v_1 = v - \pi_1 \cdot v = (0, 5) + (5, 6) - (5, 10) - (0, 5 + \varepsilon) - (5 + \varepsilon, 6) + (5 + \varepsilon, 10)$$

duplicates the tuples with 5 in it but kills the one without it. Similarly

$$v_{1,2} = v_1 - \pi_2 \cdot v_1 = (5, 6) - (5, 6 + \delta) - (5 + \varepsilon, 6) + (5 + \varepsilon, 6 + \delta)$$

only leaves and duplicates the tuples with 6 in it. Here $v_{1,2}$ is the cogwheel for the cog $(5, 6 | 5 + \varepsilon, 6 + \delta)$ in \mathcal{O} as well as the smaller strict orbit $\mathcal{O}' = \text{Aut}(\mathbb{A})_{(-2, 10)} \cdot (5, 6) \subseteq \mathcal{O}$.

In general, we may build cogs and cogwheels as follows.

Lemma III.9. *Take any (a_1, \dots, a_d) from a strict orbit \mathcal{O}' with support S' . Then:*

- 1) *we can find (b_1, \dots, b_d) so that $a_\bullet | b_\bullet$ forms an \mathcal{O}' -cog;*
- 2) *for $i = 1, \dots, d$ there is some $\pi_i \in \text{Aut}(\mathbb{A})_{(S')}$ that sends $a_i \mapsto b_i$ but fixes $a_{i'}$ for $i' \neq i$.*

Proof. We find b_1, b_2, \dots inductively. Write

$$X_i = \{a_1, \dots, a_d, b_1, \dots, b_{i-1}\} \cup S,$$

and form the \mathcal{L}_0 -amalgam □

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Evans

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