Cogs span the projection kernel, version $n + 1$

Jingjie Yang

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1 Notations

Let $\mathcal{O} \subseteq \mathbb{A}^I$ be an S-ordered orbit. Given a vector $v \in \mathrm{Lin}_{\mathsf{E}}\, \mathcal{O}$, write $\underline{v}$ for its set-theoretic support, i.e., the finite subset $v^{-1}(\mathsf{E}^*) \subseteq \mathcal{O}$. More generally, given any finite subset $\sigma \subseteq \mathcal{O}$, write

$$\overline{\sigma} = \{(i, a_i) \mid i \in I, a \in \sigma\}$$

and define two binary relations

$$\begin{aligned}(i, a_i)?(j, b_j) &\iff a_i = b_j \text{ but } i \neq j,\\ (i, a_i)!(j, b_j) &\iff a_i, b_j \text{ are related but not in the same way as } a_i, a_j\end{aligned}$$

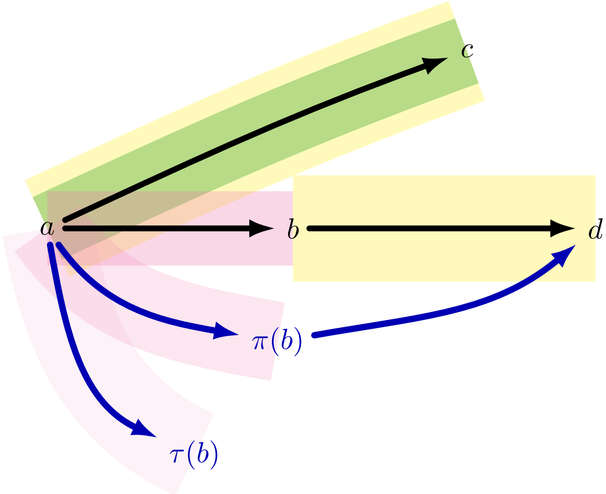
called *ambiguities* and *obstructions*. (Recall that a_i and b_j are *not related* if they are freely amalgamated in the reduct $\mathbb{A}_0$ of $\mathbb{A}$ — i.e., they are not equal, and for no $R \in \mathcal{L}_0$ does $R(a_i, b_j) \vee R(b_j, a_i)$ hold; in that case, we write $a_i \mathop{\smile\hspace{-0.9em}\downarrow} b_j$). Both relations are symmetric and $? \subseteq !$; denote their images by $?\overline{\sigma} \subseteq !\overline{\sigma} \subseteq \overline{\sigma}$. Lastly, refer to the atoms that appear in σ by $\sqrt{\sigma} \subseteq \mathbb{A}$.

> **Remark 1** Assume $?\overline{\sigma} = \emptyset$. Given any $a_i \in \sqrt{\sigma}$, we have $(i, a_i) \in \overline{\sigma}$.

The prototypical example of an unobstructed family is $\overline{\underline{\lambda \cdot a^+ \parr a^-}} = \overline{\{a^+, a^-\}} = \overline{\{a^\pm\}}$, where $a^+ \parallel a^-$ is an $\mathcal{O}$-dipole.

2 Lemmas

Let $v \in \mathrm{Ker}_{\mathsf{E}}\, \mathcal{O}$ and let $v^{i:a_i}$ be a subvector. In the cog decomposition results we are about to prove, we will work with the assumption that $V = \underline{v}$ is unobstructed (resp., unambiguous); then so is $V' = \underline{v^{i:a_i}} \subseteq \underline{v}$. We will be able to write $v^{i:a_i} = \sum_{a^\pm \in A^\pm} (\lambda_{a^\pm} \cdot a^+ \parr a^-) a_i$ where the union of V' and $K = A^\pm a_i = \{a^+ a_i, a^- a_i \mid a^\pm \in A^\pm\}$ is unobstructed (resp., unambiguous). But we want to make $V \cup K$ unobstructed (resp., unambiguous); we can do so by choosing K more carefully.



Lemma 2 *Let K, V', V be finite subsets of $\mathcal{O}$ such that $?\overline{V' \cup K} = \emptyset = ?\overline{V} \supseteq ?\overline{V'}$. Then there exists $\pi \in \mathrm{Aut}(\mathbb{A}/S \cup \sqrt{V'})$ that satisfies*

$$?\overline{V \cup \pi(K)} = \emptyset.$$

Proof. Fix V', V and induct on the size of $?\overline{V \cup K}$.

Let $(i, a_i)?(j, b_j)$; without loss of generality we may assume $(j, b_j) \in \overline{V}$ and $(i, a_i) \in \overline{K} \setminus \overline{V'}$. Since $?\overline{V'} = \emptyset$, by Remark 1 we see that $a_i \notin \sqrt{V'}$; also $a_i \notin S$, as $\mathcal{O}$ is S-ordered. With strong amalgamation, we may find some $\pi \in \mathrm{Aut}(\mathbb{A}/S \cup \sqrt{V'})$ such that $\pi(a_i) \notin \sqrt{V \cup K}$. Now it is straightforward to check that

$$?\overline{V \cup \pi(K)} \subseteq ?\overline{V \cup K} \setminus \{(i, a_i)\}.$$

Because $?\overline{V' \cup \pi(K)} = ?\overline{\pi(V') \cup \pi(K)} = \pi(\emptyset) = \emptyset$ still, the inductive hypothesis gives us some $\pi' \in \mathrm{Aut}(\mathbb{A}/S \cup \sqrt{V'})$ such that $?\overline{V \cup \pi'\pi(K)} = \emptyset$. ■

Lemma 3 *Let K, V', V be finite subsets of $\mathcal{O}$ such that $!\overline{V' \cup K} = \emptyset = !\overline{V} \supseteq !\overline{V'}$. Then*

$$!\overline{V \cup \pi(K)} = \emptyset$$

for some $\pi \in \mathrm{Aut}(\mathbb{A}/S \cup \sqrt{V'})$.

Proof. By Lemma 2 we may assume that $?\overline{V \cup K} = \emptyset$ already. As before, fix V', V and proceed by induction on the size of $!\overline{V \cup K}$.

Let $(i, a_i)!(j, b_j)$; without loss of generality we may assume $(j, b_j) \in \overline{V}$ and $(i, a_i) \in \overline{K}$. Now $(i, a_i) \notin \overline{V'}$, so $a_i \notin \sqrt{V'}$ by Remark 1; further, whenever $(i, a_i)!(k, c_k)$ we observe that $c_k \notin S \cup \{a_i\} \cup \sqrt{K \cup V'}$. Let Y consist of all such c_k and put

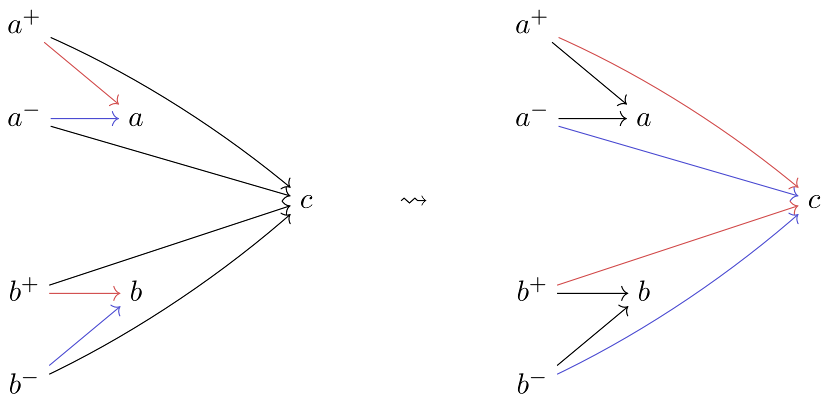
$$X = S \cup \sqrt{K \cup V} \setminus (Y \cup \{a_i\}).$$

Then $X, Y, \{a_i\}$ are pairwise disjoint, and we see X contains $S \cup \sqrt{V'}$ as well as $\sqrt{K} \setminus \{a_i\}$. With free amalgamation (in $\mathbb{A}_0$) this time, we may find some $\tau \in \operatorname{Aut}(\mathbb{A}/S \cup \sqrt{V'})$ such that $\tau(a_i) \notin X \cup Y \cup \{a_i\}$ and $\tau \perp Y$. Again, we can check that

$$!\overline{V \cup \tau(K)} \subseteq !\overline{V \cup K} \setminus (i, a_i)$$

so the conclusion follows straightforwardly from the inductive hypothesis. ∎

3 Unobstructed vector



Proposition 4 *Let $v \in \operatorname{Ker}_{\mathsf{E}} \mathcal{O}$ and suppose that $!\underline{v} = \emptyset$. Then we can write*

$$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ \between a^-$$

with $!\overline{\underline{v} \cup A^\pm} = \emptyset$ and $\lambda_{a^\pm} \in v(\mathcal{O})$.

This is the same statement as in §IV.D; here I use a slightly different proof.

We proceed by induction on the dimension $|I|$, noting that when $I = \emptyset$ we just have $v = v() \cdot () = v() \cdot (\between)$ without any possible obstructions.

So suppose I is non-empty; let $d \in I$ be the greatest. Group the terms in v by their greatest atom so that $v = v^1 + v^2 + \cdots + v^k$. We now induct on k. If $k < 2$, we are done: as $v_{-d} = 0$ we must have $v = 0$, so the empty sum will do. Otherwise

$$v = v^{d:a_d} + v^{d:b_d} + v'.$$

By the outer inductive hypothesis, we get

$$v^{d:a_d} = v^{d:a_d}_{-d} a_d = \sum_{A^\pm} (\lambda_{a^\pm} \cdot a^+ \between a^-) a_d$$

where we only know $!\overline{\underline{v^{d:a_d}_{-d}} \cup A^{\pm}} = \emptyset$ so that $!\overline{\underline{v^{d:a_d}} \cup A^{\pm}a_d} = \emptyset$. But any $\pi \in \mathrm{Aut}(\mathbb{A}/S \cup \sqrt{\underline{v^{d:a_d}}})$ satisfies

$$v^{d:a_d} = \pi(v^{d:a_d}) = \sum_{a^{\pm} \in A^{\pm}} \lambda_{a^{\pm}} \cdot \pi a^{+} \between \pi a^{-},$$

so by Lemma 3 we may assume without loss of generality that

$$!\overline{\underline{v} \cup A^{\pm}a_d} = \emptyset.$$

Similarly, we can write

$$v^{d:a_d} = \sum_{B^{\pm}} (\lambda_{b^{\pm}} \cdot b^{+} \between b^{-}) b_d$$

where, in turn, we may upgrade the assumption that $!\overline{\underline{v^{d:b_d}} \cup B^{\pm}b_d} = \emptyset$ to

$$!\overline{\underline{v} \cup A^{\pm}a_d \cup B^{\pm}b_d} = \emptyset.$$

The key is that we may now invent a new element z, on which we impose the following relations with $S \cup \sqrt{A^{\pm}a_d \cup B^{\pm}b_d} \subseteq \mathbb{A}$:

1. $a_d, b_d < z$, and $z < s$ if $a_d, b_d < s$ for some $s \in S$ (enough to let s be the least such);
2. for any unary relation $P \in \mathcal{L}_0$:

$$P(z) \;:\Longleftrightarrow\; P(a_d) \iff P(b_d)$$

   — recall that $a, b \in \mathcal{O}$;
3. for any binary relation $R \in \mathcal{L}_0$ and $s \in S$, $a^{\pm} \in A^{\pm}$, $b^{\pm} \in B^{\pm}$, $i \in I \setminus \{d\}$:
   (a) $R(z, s) \;:\Longleftrightarrow\; R(a_d, s) \iff R(b_d, s)$,
   (b) $R(z, a_i^{\pm}) \;:\Longleftrightarrow\; R(a_d, a_i^{\pm})$,
   (c) $R(z, a_d) \;:\Longleftrightarrow\; \perp$,
   (d) $R(z, b_i^{\pm}) \;:\Longleftrightarrow\; R(b_d, b_i^{\pm})$;
   (e) $R(z, b_d) \;:\Longleftrightarrow\; \perp$,
   (f) and symmetrically for $R(-, z)$.
   These are well-defined because $a^{\pm}a_d, b^{\pm}b_d \in \mathcal{O}$ and $i = j$ whenever $a_i^{\pm} = b_j^{\pm}$.

To see that the $\mathcal{L}$-structure $S \cup \sqrt{A^{\pm}a_d \cup B^{\pm}b_d} \cup \{z\}$ still embeds into $\mathbb{A}$, suppose towards a contradiction that it contains a forbidden $\mathcal{L}_0$-substructure F. Then F must contain z. Since any two elements in F are necessarily related, we must have $a_d, b_d \notin F$. Similarly, whenever F contains x_i where $x \in A^{\pm} \cup B^{\pm}, i \in I \setminus \{d\}$ it does not contain a distinct atom of the form x'_i. It follows that

$$s \mapsto s, \quad x_i \mapsto a_i, \quad z \mapsto a_d$$

defines an injective function $\phi : F \to \mathbb{A}_0$, which is furthermore an embedding (we only need to check this for pairs!) because $!\overline{A^{\pm} \cup B^{\pm}} = \emptyset$ and any $x_i, x'_{i'}$ for $i \neq i'$ are related. This is impossible — therefore assume $z \in \mathbb{A}$.

It is now routine to check that $a^+a_d \parallel a^-z$ and $b^+a_d \parallel b^-z$ are $\mathcal{O}$-dipoles for $a^\pm \in A^\pm, b^\pm \in B^\pm$ and that $!\overline{A^+a_d \cup A^-z \cup B^+b_d \cup B^-z} = \emptyset$. By Lemma 3 we may assume that $!\overline{\underline{v} \cup A^+a_d \cup A^-z \cup B^+b_d \cup B^-z} = \emptyset$. (Alternatively, we could have explicitly ensured this when defining z). Then

$$\begin{aligned} v'' &= v - \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+a_d \between a^-z - \sum_{B^\pm} \lambda_{b^\pm} \cdot b^+b_d \between b^-z \\ &= v_{-d}^{d:a_d} z + v_{-d}^{d:b_d} z + v', \end{aligned}$$

when grouped into subvectors by the largest atom in each term, has at least one fewer component than v. By the inner inductive hypothesis, we may write we may write

$$v'' = \sum_{C^\pm} \lambda_{c^\pm} \cdot c^+ \between c^-$$

where $!\overline{\underline{v''} \cup C^\pm} = \emptyset$. But $\overline{\underline{v''}} \subseteq \overline{\underline{v} \cup A^+a_d \cup A^-z \cup B^+b_d \cup B^-z}$, so one last application of Lemma 3 allows us to assume that

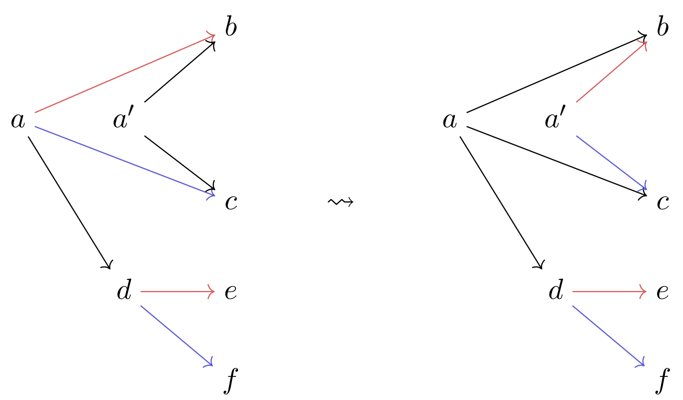
$$!\overline{\underline{v} \cup A^+a_d \cup A^-z \cup B^+b_d \cup B^-z \cup C^\pm} = \emptyset.$$

We conclude that

$$v = \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+a_d \between a^-z + \sum_{B^\pm} \lambda_{b^\pm} \cdot b^+b_d \between b^-z + \sum_{C^\pm} \lambda_{c^\pm} \cdot c^+ \between c^-;$$

in other words, we have decomposed an unobstructed vector into an unobstructed family of cogs. (The notation $\lambda_{a^\pm}, \lambda_{b^\pm}$ is a bit sloppy ...)

4 Unambiguous vector



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Proposition 5 *Let $v \in \mathrm{Ker}_{\mathsf{E}}\, \mathcal{O}$ and suppose that $?\overline{v} = \emptyset$. Then we can write*

$$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ \between a^-$$

with $?\overline{\underline{v} \cup A^\pm} = \emptyset$ and $\lambda_{a^\pm} \in v(\mathcal{O})$.

This is an amended version of Proposition IV.25 (the one with $\acute{\check{v}}$). The proof there had a gap, which is filled by the additional conclusion about unambiguity here.

We proceed again by induction, first on the dimension $|I|$ then on the cardinality of $!\overline{v}$. The outer base case $I = \emptyset$ is trivial — we have $v = v() \cdot (\between)$, and no ambiguities may arise — whilst the inner base case is just Proposition 4.

So suppose that $!\overline{v}$ contains some (i, a_i). Since $\overline{\underline{v_{-i}^{i:a_i}}} \subseteq \overline{\underline{v^{i:a_i}}} \subseteq \overline{v}$, we know that $v_{-i}^{i:a_i}$ is unambiguous. Applying the outer inductive hypothesis, we may write

$$v^{i:a_i} = v_{-i}^{i:a_i} a_i = \sum_{A^\pm} (\lambda_{a^\pm} \cdot a^+ \between a^-) a_i$$

where $\emptyset = ?\overline{(\underline{v_{-i}^{i:a_i}} \cup A^\pm) a_i} = ?\overline{\underline{v^{i:a_i}} \cup A^\pm a_i}$. Moreover, we may assume by Lemma 2 that

$$?\overline{\underline{v} \cup A^\pm a_i} = \emptyset.$$

Now we can show that whenever $(j, b_j)!(i, a_i)$ in $\overline{v}$ we have $b_j \notin \sqrt{A^\pm a_i}$. Already $b_j = a_i$ would imply $j = i$ as $?\overline{v} = 0$, which is impossible. So suppose to the contrary that $b_j = a_k^\pm$ for some $a^\pm \in A^\pm$ and $k \in I \setminus \{i\}$. By the assumption above, we must have $j = k$; this is a contradiction: note that $a^\pm a_i \in \mathcal{O}$.

Put $Y = \{b_j \mid (j, b_j) \in \overline{v}, (j, b_j)!(i, a_i)\}$. It follows that $X = S \cup \sqrt{\underline{v} \cup A^\pm} \setminus Y \setminus \{a_i\}$ contains $S \cup \sqrt{A^\pm}$ and that $X, Y, \{a_i\}$ are pairwise disjoint. Using free amalgamation in $\mathbb{A}_0$, we may find $\tau \in \mathrm{Aut}(\mathbb{A}/X)$ such that $\tau(a_i) \notin X \cup Y \cup \{a_i\}$, is greater than a_i, and is not related to any of $Y \cup \{a_i\}$. Then, given any $a^\pm \in A^\pm$, we can straightforwardly check that $a^+ a_i \parallel a^- \tau(a_i)$ is an $\mathcal{O}$-dipole, that $?\overline{\underline{v} \cup A^+ a_i \cup A^- \tau(a_i)} = \emptyset$, and that

$$\begin{aligned} v - \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \between a^- \tau(a_i) &= v - v^{i:a_i} + v^{i:\tau(a_i)} \\ &= v^{i:a_i \mapsto \tau(a_i)} \end{aligned}$$

satisfies $!\overline{\underline{v^{i:a_i \mapsto \tau(a_i)}}} \subseteq\, !\overline{v} \setminus \{(i, a_i)\}$. The inner inductive hypothesis tells us that $v^{i:a_i \mapsto \tau(a_i)} = \sum_{B^\pm} \lambda_{B^\pm} \cdot b^+ \between b^-$ with

$$?\overline{\underline{v^{i:a_i \mapsto \tau(a_i)}} \cup B^\pm} = \emptyset.$$

But $\overline{\underline{v^{i:a_i \mapsto \tau(a_i)}}} \subseteq \overline{\underline{v}} \subseteq \overline{A^+ a_i \cup A^- \tau(a_i)}$, so Lemma 2 allows us to assume that

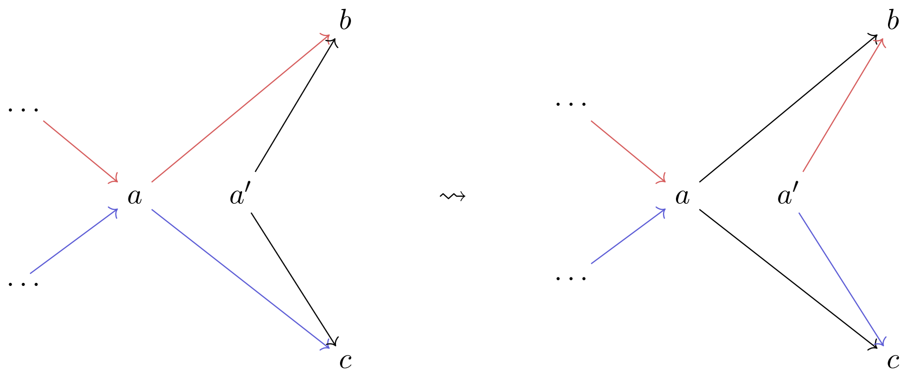
$$?\overline{\underline{v} \cup A^+ a_i \cup A^- \tau(a_i) \cup B^\pm} = \emptyset.$$

We conclude that

$$v = \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \between a^- \tau(a_i) + \sum_{B^\pm} \lambda_{b^\pm} \cdot b^+ \between b^-$$

— in other words, we have decomposed an unambiguous vector into an unambiguous family of cogs.

5 General vector



> **Proposition 6** *Let $v \in \mathrm{Ker}_{\mathsf{E}}\, \mathcal{O}$. Then we can write*
>
> $$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ \between a^-$$
>
> *with $\lambda_{a^\pm} \in v(\mathcal{O})$.*
>
> This is the same statement and proof as Proposition IV.24 (the one with $\ddot{v}$).

This is an easier induction on $|I|$ then on $?\overline{\underline{v}}$. If $I = \emptyset$, the decomposition is trivial; if v is unambiguous already, the decomposition comes from Proposition 5.

Now suppose $?\overline{\underline{v}}$ contains some (i, a_i), and use the outer inductive hypothesis to write

$$v^{i:a_i} = v^{i:a_i}_{-i} a_i = \sum_{A^\pm} (\lambda_{a^\pm} \cdot a^+ \between a^-) a_i.$$

Then neither S nor $\sqrt{A^\pm}$ contains a_i, so $X = S \cup \sqrt{\underline{v} \cup A^\pm} \setminus \{a_i\}$ contains $S \cup \sqrt{A^\pm}$. Using free amalgamation in $\mathbb{A}_0$ and the generic order in $\mathbb{A}$, we may find some $\pi \in \mathrm{Aut}(\mathbb{A}/X)$ such that $\pi(a_i) \notin X$, $\pi(a_i) > a_i$, $\pi(a_i) \perp a_i$. We can check that

1. $a^+ a_i \parallel a^- \pi(a_i)$ is an $\mathcal{O}$-dipole, given any $a^\pm \in A^\pm$;

2. $v - \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \parr a^- \pi(a_i) = v - v^{i:a_i} + v^{i:\pi(a_i)} = v^{i:a_i \mapsto \pi(a_i)}$ satisfies

$$?\underline{\overline{v^{i:a_i \mapsto \pi(a_i)}}} \subseteq ?\underline{\overline{v}} \setminus \{(i, a_i)\}.$$

It follows from the inner inductive hypothesis that

$$\begin{aligned} v &= \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \parr a^- \pi(a_i) + v^{i:a_i \mapsto \pi(a_i)} \\ &= \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \parr a^- \pi(a_i) + \sum_{b^\pm \in B^\pm} \lambda_{b^\pm} \cdot b^+ \parr b^-. \end{aligned}$$