More Vector Spaces with Atoms of Finite Lengths

Jingjie Yang
University of Oxford

Mikołaj Bojańczyk
University of Warsaw

Bartek Klin
University of Oxford

***Abstract*—We say an infinite structure is oligomorphic over a field if the following holds for each of its finite powers: in the corresponding free vector space, strict chains of equivariant subspaces are bounded in length. It has been shown that the countable pure set and the dense linear ordering without endpoints have this property. In this paper, we generalise these two results to a) reducts of smoothly approximable structures, provided the field has characteristic zero, and b) generaically ordered expansions of Fraïssé limits with free amalgamation, in languages with at most binary relations. As a special case, we prove the Rado graph is oligomorphic over any field using both methods.**

I. Introduction

II. Rado graph, sans cogs

$\mathbb{A}$ is:

1) oligomorphic if, for $k = 0, 1, 2, \cdots$, $\mathbb{A}^k$ only has finitely many orbits;
2) $\mathfrak{F}$-oligomorphic if, for $k = 0, 1, 2, \cdots$, $\mathrm{Lin}_{\mathfrak{F}}\, \mathbb{A}^k$ only has finitely long chains.

Of note: with Stirling numbers of the second kind and Gaussian 2-binomial coefficients, the orbit counts are given by

$$\#\mathbb{N}^k = \sum_{d=0}^{k} \left\{ {k \atop d} \right\}$$

$$\#\mathbb{Q}^k = \sum_{d=0}^{k} \left\{ {k \atop d} \right\} d!$$

$$\#\mathbb{G}^k = \sum_{d=0}^{k} \left\{ {k \atop d} \right\} 2^{\binom{d}{2}}$$

$$\#\mathbb{V}_\infty^k = \sum_{d=0}^{k} \begin{bmatrix} k \\ d \end{bmatrix}_2$$

$$\#\mathbb{W}_\infty^k = \sum_{d=0}^{k} \begin{bmatrix} k \\ d \end{bmatrix}_2 2^{\binom{d}{2}}$$

To introduce:

1) *smooth approximation* by *homogeneous substructures* [3] (N.B. 'smooth approximation' from [4, Definition 4] seems to be entirely different)
2) *width-wise approximation* of a homogeneous structure by finite substructures with uniformly few orbits (i.e., types) that cover the age of $\mathbb{A}$
3) For the equality and ordered atoms, being supportively $\mathfrak{F}$-oligomorphic follows from being $\mathfrak{F}$-oligomorphic [1, Theorem 4.10]

A. Symplectic vector spaces

Throughout this subsection let $\mathfrak{f}$ denote a finite field.

Definition II.1. A *symplectic vector space* is an $\mathfrak{f}$-vector space $\mathbb{W}$ equipped with a bilinear form $\omega : \mathbb{W} \times \mathbb{W} \to \mathfrak{f}$ that is

1) alternating: $\omega(v, v) = 0$ for all v; and
2) non-degenerate: if $\omega(v, w) = 0$ for all w then $v = 0$.

Example II.2. Let $\mathbb{W}_n$ be the $\mathfrak{f}$-vector space with basis $e_1, \ldots, e_n, f_1, \ldots, f_n$. Define ω by bilinearly extending

$$\omega(e_i, f_i) = 1 = -\omega(f_i, e_i), \quad \omega(-, *) = 0 \text{ elsewhere}; \quad (\S)$$

one may straightforwardly check that ω is alternating and non-degenerate. Moreover, noticing that $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$, we obtain a countable-dimensional symplectic vector space $\mathbb{W}_\infty = \bigcup_n \mathbb{W}_n$.

We will refer to vectors satisfying (§) as a *symplectic basis* — indeed, they must be linearly independent. Such bases behave very much like the usual bases.

Proposition II.3. *Assume that $\mathbb{W}$ is a symplectic vector space that is at most countable. Then any finite symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ can be extended to a symplectic basis that spans the whole $\mathbb{W}$.*

Proof. Suppose that $e_1, \ldots, e_n, f_1, \ldots, f_n$ does not already span $\mathbb{W}$; take v to be a witness (that is least according to some fixed enumeration of $\mathbb{W}$ in the case it is infinite). Put

$$e_{n+1} = v - \sum_{i=1}^{n} \omega(e_i, v) f_i + \sum_{i=1}^{n} \omega(f_i, v) e_i$$

so that $\omega(e_i, e_{n+1}) = 0 = \omega(f_i, e_{n+1})$. This cannot be the zero vector lest we contradict the choice of v. By the non-degeneracy of ω, there is — rescaling if necessary — some w such that $\omega(e_{n+1}, w) = 1$. Now define

$$f_{n+1} = w - \sum_{i=1^n} \omega(e_i, w) f_i + \sum_{i=1}^{n} \omega(f_i, w) e_i$$

in a similar manner, making $e_1, \ldots, e_n, e_{n+1}, f_1, \ldots, f_n, f_{n+1}$ a symplectic basis that spans v. We go through every element of $\mathbb{W}$ by continuing this way. ☐

In fact, we will also make use of the "symplectic basis and a half" variant below.

Proposition II.4. *Now assume $\mathbb{W}$ is a finite-dimensional symplectic vector space. Let*

$$\begin{aligned}&e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+k},\\ &f_1, \ldots, f_n\end{aligned}$$

be linearly independent vectors satisfying (§). *Then we can find the missing $f_{n+1}, \ldots, f_{n+k}$ to complete the symplectic basis.*

Proof. We first need the following notion. Given a subspace $V \subseteq \mathbb{W}$, consider its orthogonal complement

$$V^{\perp} = \{w \in \mathbb{W} \mid \forall v \in V : \omega(v, w) = 0\}.$$

It is the kernel of the composite linear map

$$\begin{aligned}\mathbb{W} &\to (\mathbb{W} \xrightarrow{\text{lin.}} \mathfrak{f}) \to (V \xrightarrow{\text{lin.}} \mathfrak{f})\\ w &\mapsto \omega(-, w) \mapsto \omega(-, w)|_V.\end{aligned}$$

Note this map is surjective: the first part is injective by non-degeneracy and hence surjective for dimension reasons, and the second part is surjective since we can extend a basis of V to one of $\mathbb{W}$. Therefore

$$\dim V^{\perp} = \dim \mathbb{W} - \dim V,$$

and in particular $V^{\perp\perp}$ is precisely equal to V.

Now suppose we have found $f_{n+1}, \ldots, f_{n+i}$ already. If e_{n+i+1} were to be spanned by

$$\begin{aligned}&e_1, \ldots, e_{n+i}, \cancel{e_{n+i+1}}, e_{n+i+2}, \ldots, e_{n+k},\\ &f_1, \ldots, f_{n+i},\end{aligned}$$

it would be spanned by $e_{n+i+2}, \ldots, e_{n+k}$ alone because of (§); but this is impossible as we assumed linear independency. So

$$\begin{aligned}e_{n+i+1} &\notin \langle e_1, \ldots, e_{n+i}, e_{n+i+2}, f_1, \ldots, f_{n+i}\rangle\\ &= \langle e_1, \ldots, e_{n+i}, e_{n+i+2}, f_1, \ldots, f_{n+i}\rangle^{\perp\perp},\end{aligned}$$

i.e., some $f_{n+i+1} \in \langle e_1, \ldots, e_{n+i}, e_{n+i+2}, f_1, \ldots, f_{n+i}\rangle^{\perp}$ satisfies $\omega(e_{n+i+1}, f_{n+i+1}) = 1$. ☐

Given two symplectic vector spaces $\mathbb{W}$ and $\mathbb{W}'$, we call a function α between $X \subseteq \mathbb{W}$ and $X' \subseteq \mathbb{W}'$ *isometric* if $\omega(\alpha(x_1), \alpha(x_2)) = \omega(x_1, x_2)$ for all $x_1, x_2 \in X$. We can make an easy observation:

Lemma II.5. *Let $\{e_i, f_i \mid i \in I\} \subseteq \mathbb{W}$, $\{e'_j, f'_j \mid j \in J\} \subseteq \mathbb{W}'$ be two symplectic bases and let $\alpha : I \to J$ be a bijection. Then*

$$e_i \mapsto e'_{\alpha(i)}, f_i \mapsto f'_{\alpha(i)}$$

defines an isometric linear isomorphism $\langle e_i, f_i\rangle \to \langle e'_j, f'_j\rangle$.

It then follows from Proposition II.3 that, up to isometric linear isomorphisms, $\mathbb{W}_0, \mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_\infty$ are all the countable symplectic vector spaces. Whilst we may deduce that $\mathbb{W}_\infty$ is oligomorphic by appealing to Ryll-Nardzewski, we will opt for a more direct proof that also establishes smooth approximation.

Proposition II.6 (Witt Extension)**.** *Any isometric linear injection $\alpha : \langle X\rangle \subseteq \mathbb{W}_n \to \mathbb{W}_n$ can be extended to an isometric linear automorphism of $\mathbb{W}_n$ and in turn to one of $\mathbb{W}_\infty$.*

Proof. To begin with, find a basis $x_1, \ldots, x_k$ for $\langle X\rangle^{\perp} = \{w \in W \mid \forall x \in X : \omega(w, x) = 0\}$ and extend it to a basis $x_1, \ldots, x_k, x_{k+1}, \ldots, x_d$ for $\langle X\rangle$. Notice that

$$U = \langle x_{k+1}, \ldots, x_d\rangle$$

must be a symplectic subspace: as it intersects with $\langle X\rangle^{\perp}$ trivially, given any non-zero vector $u \in U$ we must have $0 \neq \omega(u, x + u') = \omega(u, u')$ for some $x \in \langle X\rangle^{\perp}$ and $u' \in U$. Hence use Proposition II.3 to find a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ for U. Observe that

$$\begin{aligned}&e_1, \ldots, e_n, x_1, \ldots, x_k,\\ &f_1, \ldots, f_n\end{aligned}$$

form a basis for $\langle X\rangle$ and satisfy (§). On the other hand,

$$\begin{aligned}&\alpha(e_1), \ldots, \alpha(e_n), \alpha(x_1), \ldots, \alpha(x_k),\\ &\alpha(f_1), \ldots, \alpha(f_n)\end{aligned}$$

form a basis for $\alpha(\langle X\rangle)$ and also satisfy (§). Therefore apply Proposition II.4 twice to find the missing $y_1, \ldots, y_k$ and $y'_1, \ldots, y'_k$ to complete the two symplectic bases — call them $\mathcal{B}$ and $\mathcal{B}'$. They are of the same size.

Now, by using Proposition II.3, extend $\mathcal{B}$ and $\mathcal{B}'$ to symplectic bases $\mathcal{C}$ and $\mathcal{C}'$ that span $\mathbb{W}_n$. These must both have size $2n$, so by Lemma II.5 we obtain an isometric linear automorphism $\beta : \mathbb{W}_n \to \mathbb{W}_n$ extending α.

To finish, notice that $\mathcal{C}, e_{n+1}, \ldots, f_{n+1}, \ldots$ as well as $\mathcal{C}', e_{n+1}, \ldots, f_{n+1}, \ldots$ form a symplectic basis spanning $\mathbb{W}_\infty$. We obtain from Lemma II.5 another time an isometric linear automorphism $\gamma : \mathbb{W}_\infty \to \mathbb{W}_\infty$ extending β that is the identity almost everywhere. ☐

Proposition II.7. *$\mathbb{W}^k_\infty$ has precisely $\sum_{d=0}^{k} \left[\begin{smallmatrix} k \\ d \end{smallmatrix}\right]_q \cdot q^{\binom{d}{2}}$ orbits under isometric linear automorphisms, where $q = |\mathfrak{f}|$ and*

$$\begin{bmatrix} k \\ d \end{bmatrix}_q = \frac{(q^k - 1)(q^{k-1} - 1) \cdots (q^{k-d+1} - 1)}{(q^d - 1)(q^{d-1} - 1) \cdots (q^1 - 1)}$$

is the q-binomial coefficient.

Remark. To anticipate the next subsection, we note a similarity with the Rado graph: in $\mathbb{G}^k$ there are $\sum_{d=0}^{k} \binom{k}{d} \cdot 2^{\binom{d}{2}}$ orbits — we may impose any edge relation on d vertices.

Proof. To each $v_\bullet \in \mathbb{W}^k_\infty$ we associate a *type*, which comprises the following data:

1) pivot indices $I \subseteq \{1, \ldots, k\}$ containing every i such that v_i is not spanned by $v_1, \ldots, v_{i-1}$ — so we inductively ensure that
$$\{v_{i'} \mid i' \in I, i' \leq i\}$$
is a basis for $\langle v_1, \ldots, v_i\rangle$;
2) for each $j \notin I$, an assignment $\Lambda_j : \{i \in I \mid i < j\} \to \mathfrak{f}$ such that $v_j = \sum_{i \in I, i<j} \Lambda_j(i) v_i$;

3) a map $\Omega : \binom{I}{2} \to \mathfrak{f}$ defined by $\Omega(\{i' < i\}) = \omega(v_{i'}, v_i)$.

If $\pi : \mathbb{W}_\infty \to \mathbb{W}_\infty$ is an isometric linear automorphism, then $v_\bullet = (v_1, \dots, v_k)$ and $\pi \cdot v_\bullet = (\pi(v_1), \dots, \pi(v_k))$ evidently share the same type. Conversely, if $w_\bullet$ has the type of $v_\bullet$, then

$$\alpha : \langle v_i \mid i \in I\rangle \to \langle w_i \mid i \in I\rangle \subseteq \mathbb{W}_n$$
$$v_i \mapsto w_i$$

gives an isometric linear injection for some large enough n. Observe that α must send $v_j \mapsto w_j$ for $j \notin I$ too, and that it may be extended to an isometric linear automorphism π of $\mathbb{W}_\infty$ by Propsoition II.6. Furthermore we can find some $v_\bullet$ that realises any given type $(I, \{\Lambda_j\}_j, \Omega)$: it suffices to put

$$v_i = e_i + \sum_{i' \in I, i' < i} \Omega(i', i) f_{i'}$$

for $i \in I$ and $v_j = \sum_{i \in I, i<j} \in \Lambda_j(i) v_i$ for $j \notin I$. Therefore the number of types is precisely the number of orbits in $\mathbb{W}_\infty^k$.

Finally, we do some combinatorics. Fix $0 \le d \le k$ and count the number of types with $|I| = d$. There are $q^{\binom{d}{2}}$ choices for Ω and say $\#_{k,d}$ choices for the Λ_j's; the two can be chosen separately. In total, this gives

$$\sum_{d=0}^{k} q^{\binom{d}{2}} \cdot \#_{k,d}$$

types for vectors in $\mathbb{W}_\infty^d$. So focus on $\#_{k,d}$, the number of *linear types* — i.e., $(I, \{\Lambda_j\}_j)$, ignoring Ω — in $\mathbb{W}_\infty^k$. (Incidentally $\sum_{d=0}^k \#_{k,d}$ is the number of orbits in $\mathbb{W}_\infty^k$ or, more generally, any countable-dimensional $\mathfrak{f}$-vector space under linear automorphisms.) On the small values we easily check that

$$\#_{0,0} = 1 = \begin{bmatrix}0\\0\end{bmatrix}_q,$$
$$\#_{1,0} = 1 = \begin{bmatrix}1\\0\end{bmatrix}_q, \qquad \#_{1,1} = 1 = \begin{bmatrix}1\\1\end{bmatrix}_q.$$

Given a linear type in $\mathbb{W}_\infty^k$ with $|I| = d$, we either have $1 \in I$ or $I \subseteq \{2, \dots, k\}$. In the first case, the linear type is specified by one of the $\#_{k-1,d-1}$ linear types in $\mathbb{W}_\infty^{k-1}$ together with how v_1 is involved in the span of the $(k-1)-(d-1)$ non-pivot vectors. In the second case, the linear type is simply one of the $\#_{k-1,d}$ linear types in $\mathbb{W}_\infty^{k-1}$. Thus

$$\begin{aligned}\#_{k,d} &= q^{k-d} \cdot \#_{k-1,d-1} + \#_{k-1,d} \\ &= q^{k-d} \cdot \begin{bmatrix}k-1\\d-1\end{bmatrix}_q + \begin{bmatrix}k-1\\d\end{bmatrix}_q = \begin{bmatrix}k\\d\end{bmatrix}_q.\end{aligned}$$ □

Theorem II.8. *The symplectic vector space $\mathbb{W}_\infty$ is smoothly approximated by $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$.*

Corollary II.9. *Provided $\mathfrak{F}$ is of characteristic 0, the symplectic $\mathfrak{f}$-vector space $\mathbb{W}_\infty$ is $\mathfrak{F}$-oligomorphic.*

B. Symplectic graphs

For this subsection let $\mathfrak{f}$ be the two-element field.

Definition II.10. For $n = 0, 1, 2, \dots$, the *symplectic graph* $\widetilde{\mathbb{W}}_n$ has vertices $\mathbb{W}_n$ and edges

$$v_1 \sim v_2 \iff \omega(v_1, v_2) = 1.$$

This is indeed an undirected graph: as ω is alternating, we have $\omega(v_1, v_2) = -\omega(v_2, v_1) = \omega(v_2, v_1)$.

Proposition II.11. $\mathrm{Aut}(\widetilde{\mathbb{W}}_n) = \mathrm{Aut}(\mathbb{W}_n)$.

Proof. Clearly any isometric linear automorphism of $\mathbb{W}_n$ is a graph automorphism of $\widetilde{\mathbb{W}}_n$. Conversely, any $f \in \widetilde{\mathbb{W}}_n$ is evidently isometric. To show that f is linear, take $\lambda_1, \lambda_2 \in \mathfrak{f}$ and $v_1, v_2 \in \mathbb{W}$. We calculate:

$$\begin{aligned}&\omega\Big(f(\sum_i \lambda_i v_i) - \sum_i \lambda_i f(v_i), f(w)\Big) \\ &= \omega\Big(f(\sum_i \lambda_i v_i), f(w)\Big) - \sum_i \lambda_i \omega\big(f(v_i), f(w)\big) \\ &= \omega\Big(\sum_i \lambda_i v_i, w\Big) - \sum_i \lambda_i \omega(v_i, w) \\ &= \omega(0, w) = 0\end{aligned}$$

for all $f(w) \in f(\mathbb{W}_n) = \mathbb{W}_n$; since ω is non-degenerate, we conclude that $f(\sum_i \lambda_i v_i) = \sum_i \lambda_i f(v_i)$. □

So the number of orbits in $\widetilde{\mathbb{W}}_n^k$ is precisely equal to the number of orbits in $\mathbb{W}_n^k$ — in particular, it is bounded above by $\sum_{d=0}^k \begin{bmatrix}k\\d\end{bmatrix}_2 \cdot 2^{\binom{d}{2}}$ independently of n by Proposition II.7.[1] It remains to show $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$ embeds all finite graphs:

Proposition II.12 ([2, Theorem 8.11.2])**.** *Every graph on at most $2n$ vertices embeds into $\widetilde{\mathbb{W}}_n$.*

Proof. Let G be a graph on at most $2n$ vertices. The conclusion is trivial when $n = 0$. Also, if G contains no edges, we can choose any $2n$ of the 2^n vectors in $\langle e_1, \dots, e_n\rangle \subseteq \widetilde{\mathbb{W}}_n$.

So suppose $n \ge 1$ and G has an edge $s \sim t$. Let $G_{s,t}$ be the graph on vertices $G \setminus \{s, t\}$ with edges which we will specify later. By induction, some embedding $f : G_{s,t} \to \widetilde{\mathbb{W}}_{n-1}$ exists. Define $f' : G \to \widetilde{\mathbb{W}}_n$ by

$$\begin{aligned}x \in G_{s,t} &\mapsto f(x) - [\![x \sim s]\!] f_n + [\![x \sim t]\!] e_n \\ s &\mapsto e_n \\ t &\mapsto f_n\end{aligned}$$

where $[\![\phi]\!]$ is 1 if ϕ holds and 0 otherwise. Then we have $\omega(f'(x), f'(s)) = [\![x \sim s]\!]$ and $\omega(f'(x), f'(t)) = [\![x \sim t]\!]$ as desired, on one hand. On the other,

$$\begin{aligned}\omega(f'(x_1), f'(x_2)) = [\![x_1 \sim x_2]\!] &+ [\![x_1 \sim s]\!][\![x_2 \sim t]\!] \\ &+ [\![x_1 \sim t]\!][\![x_2 \sim s]\!]\end{aligned}$$

tells us how we should define the edge relation in $G_{s,t}$ for f' to be an embedding of graphs. □

Theorem II.13. *The Rado graph is width-wise approximated by $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$.*

Corollary II.14. *Provided $\mathfrak{F}$ is of characteristic 0, the Rado graph is $\mathfrak{F}$-oligomorphic.*

[1] This is the kth term in the OEIS sequence A028631.

This proof of finite length also applies to *oriented graphs* (i.e., $x \to y \implies y \not\to x$ but unlike in a tournament, it may occur that $x \not\to y \wedge y \not\to x$) — use the three-element field as $\mathfrak{f}$ instead.

III. Rado graph, with cogs

In this section we work with the following setting:

- $\mathcal{L}_0$ is a (possibly infinite) relational language containing a binary symbol $=$;
- $\mathcal{C}_0$ is a free amalgamation class of $\mathcal{L}_0$-structures where $=$ is interpreted as true equality, but every other $R \in \mathcal{L}_0$ is interpreted irreflexively.[2]
- $\mathcal{L}$ consists of $\mathcal{L}_0$ together with a new binary symbol $<$;
- $\mathcal{C}$ consists of $\mathcal{L}$-structures obtained from $\mathcal{C}_0$ by expanding with all possible linear orderings;
- $\mathbb{A}_0$ and $\mathbb{A}$ are the respective Fraïssé limits of $\mathcal{C}_0$ and $\mathcal{C}$, where without loss of generality we assume $\mathbb{A}_0$ and $\mathbb{A}$ share the same domain so that $\mathrm{Aut}(\mathbb{A}_0) \supseteq \mathrm{Aut}(\mathbb{A})$.

Example III.1. Take $\mathcal{L}_0$ consist of $=$ only and let $\mathcal{C}_0$ to be all finite sets. Then $\mathbb{A}_0$ is isomorphic to the pure set $\mathbb{N}$, whereas $\mathbb{A}$ is isomorphic to $\mathbb{Q}$ with the usual order.

Example III.2. Let $\mathcal{L}_0$ consist of $=$ together with a single binary symbol $\sim$ and let $\mathcal{C}_0$ consist of all finite undirected graphs not embedding the complete graph K_n, where $3 \le n$ $(\le \infty)$. Then $\mathbb{A}_0$ is the K_n-free Henson graph (or the Rado graph when $n = \infty$), and $\mathbb{A}$ is its generically ordered counterpart. (Allowing $n = 2$ makes these degenerate to $\mathbb{N}$ and $\mathbb{Q}$ above).

We note two technicalities and a triviality.

Lemma III.3. *Let $\mathcal{F}_0$ consist of minimal $\mathcal{L}_0$-structures which do not appear in $\mathcal{C}_0$. Then*

1) *$\mathcal{C}_0$ consists of every $\mathcal{L}_0$-structure that does not embed any $F \in \mathcal{F}_0$.*
2) *$\mathcal{C}$ consists of every $\mathcal{L}$-structure whose $\mathcal{L}_0$-reduct does not embed any $F \in \mathcal{F}_0$.*
3) *In any $F \in \mathcal{F}_0$, every two distinct elements $x, y \in F$ are related by some $R \in \mathcal{L}_0$.*

Proof. As $\mathcal{C}_0$ is closed under substructures, its complement is closed under superstructures and thus — since there are no infinite strictly descending chain of embedded substructures — determined by its minimal structures. 2) follows because an $\mathcal{L}$-structure is in $\mathcal{C}$ precisely when its $\mathcal{L}_0$-reduct is in $\mathcal{C}_0$. For 3), notice that $F \setminus \{x\}$, $F \setminus \{y\}$ are in $\mathcal{C}_0$ by minimality; therefore so is their free amalgam over $F \setminus \{x, y\}$, which then cannot agree with F. □

[2] We can enforce irreflexivity by considering a language $\mathcal{L}_0'$ which consists, for each $R \in \mathcal{L}_0 \setminus \{=\}$ of arity r and each partition ¶ of r into k parts, of a k-ary relation symbol $R_{¶}$. Then $\mathcal{L}_0$-structures may be viewed as $\mathcal{L}_0'$-structures and vice versa, without changing the meaning of embeddings. In this way, we get a free amalgamation class $\mathcal{C}_0'$ with a Fraïssé limit which, viewed as an $\mathcal{L}_0$-structrure, is isomorphic to $\mathbb{A}_0$.

Lemma III.4. *Let $X, Y, \{z\} \subseteq \mathbb{A}$ be disjoint and finite. Then there is some automorphism $\tau \in \mathrm{Aut}(\mathbb{A})$ such that*

1) *τ fixes every $x \in X$;*
2) *$\tau(z)$ does not appear together with any $y \in Y$ or with z in any tuple $a_\bullet \in (X \cup Y \cup \{z, \tau(z)\})^*$ such that $\mathbb{A} \models R(a_\bullet)$ for some $R \in \mathcal{L}_0$;*
3) *$\tau(z) > z$.*

Proof. In $\mathbb{A}_0$, form the free amalgam

$$\begin{tikzcd}
 & X \cup Y \cup \{z\} \arrow[dr, dashed, "\subseteq"] & \\
X \arrow[ur] \arrow[dr] & & X \cup Y \cup \{z, z'\} \\
 & X \cup \{z\} \arrow[ur, dashed, "x \in X \mapsto x, z \mapsto z'"'] &
\end{tikzcd}$$

so that no element of $Y \cup \{z\}$ is related with z' by any $R \in \mathcal{L}_0$. Now we make $X \cup Y \cup \{z, z'\}$ an $\mathcal{L}$-structure: inherit the order on $X \cup Y \cup \{z\}$ from $\mathbb{A}$, and declare that $z < z'$ as well as $z' < a$ if a, the next element of $X \cup Y$ larger than z, exists at all. Notice that

$$x \in X \mapsto x, z \mapsto z'$$

is still an embedding in presence of the order. By homogeneity, we may embed $X \cup Y \cup \{z, z'\}$ into $\mathbb{A}$ via some f which is the identity on $X \cup Y \cup \{z\}$; again by homogeneity, we may extend the embedding

$$f(x) = x \in X \mapsto f(x), f(z) \mapsto f(z')$$

to some automorphism τ which makes 1), 2), and 3) true. □

Proposition III.5. *The S-supported length of $\mathrm{Lin}\, {\mathbb{A}_0}^d$ is at most that of $\mathrm{Lin}\, \mathbb{A}^d$ for any finite $S \subseteq \mathbb{A}_0 = \mathbb{A}$.*

Proof. Any chain of subspaces in $\mathrm{Lin}\, {\mathbb{A}_0}^d = \mathrm{Lin}\, \mathbb{A}^d$ that are invariant under $\mathrm{Aut}(\mathbb{A}_0)_{(S)}$ must also be invariant under the subgroup $\mathrm{Aut}(\mathbb{A})_{(S)}$. □

A. Cogs in an orbit

An inconvenience of $\mathbb{A}^d$ is that it may have many orbits.

Definition III.6. Let $S \subseteq \mathbb{A}$ be finite. We say an orbit $\mathcal{O} \subseteq \mathbb{A}^d$ is S-orderly if $\mathcal{O} = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot o_\bullet$ for some/any $o_\bullet \in \mathcal{O}$ where $o_1 < \cdots < o_d$ and $o_1, \ldots, o_d \notin S$.

By removing entries of $o_\bullet$ that repeat or come from S and reordering the rest, we can always find an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant bijection to an S-orderly orbit. Moreover, we may focus on a single S-orderly orbit at a time:

Proposition III.7. *The following are equivalent for any finite $S \subseteq \mathbb{A}$:*

1) *$\mathbb{A}_S$ (that is, $\mathbb{A}$ with constants from S fixed) is $\mathfrak{F}$-oligomorphic;*
2) *$\mathbb{A}$ is oligomorphic and for every S-orderly orbit $\mathcal{O}$, the S-supported length of $\mathrm{Lin}_{\mathfrak{F}}\, \mathcal{O}$ is finite.*

Proof. Indeed we have $\operatorname{len}(\operatorname{Lin}_{\mathfrak{F}} \mathbb{A}^d_S) = \operatorname{len}(\operatorname{Lin}_{\mathfrak{F}}(\biguplus_i \mathcal{O}_i)) = \operatorname{len}(\bigoplus_i \operatorname{Lin}_{\mathfrak{F}} \mathcal{O}_i) = \sum_i \operatorname{len}(\operatorname{Lin}_{\mathfrak{F}} \mathcal{O}_i)$, where the $\mathcal{O}_i$'s are the S-orderly counterparts of the $\operatorname{Aut}(\mathbb{A})_{(S)}$-orbits in $\mathbb{A}^d$. □

We now introduce the workhorse for understanding $\operatorname{Lin}_{\mathfrak{F}} \mathcal{O}$.

Definition III.8. Let $\mathcal{O} \subseteq \mathbb{A}^d$ be an S-orderly orbit. An *$\mathcal{O}$-cog duo* $a_\bullet \parallel b_\bullet$ consists of atoms $a_1 < b_1 < a_2 < b_2 < \cdots < a_d < b_d$ satisfying the following: for some/any $o_\bullet \in \mathcal{O}$, for every relation $R \in \mathcal{L}$ of arity r, and for each r-tuple $x_\bullet$ with entries in $\{a_1, \ldots, a_d, b_1, \ldots, b_d\} \cup S$, we have

$$\mathbb{A} \models R(x_\bullet) \leftrightarrow R(x_\bullet[a_i \mapsto o_i, b_i \mapsto o_i \mid 1 \leq i \leq d]).$$

(By $x_\bullet[a \mapsto b, c \mapsto d]$ we mean the r-tuple where each entry x_i equal to a is replaced by b, and each entry equal to c — assumed to be distinct from a — is replaced by d).

In particular, for all $I \subseteq \{1, \ldots, d\}$ we may take $x_\bullet$ above to have entries in $\{a_i \mid i \in I\} \cup \{b_j \mid j \notin I\} \cup S$, showing that

$$\begin{cases} a_i \mapsto o_i, & i \in I; \\ b_j \mapsto o_j, & j \notin I; \\ s \mapsto s & s \in S \end{cases}$$

defines an embedding. It follows from homogeneity that $a_\bullet[a_i \mapsto b_i \mid i \in I]$ lies in the orbit $\mathcal{O} = \operatorname{Aut}(\mathbb{A})_{(S)} \cdot o_\bullet$.

Definition III.9. The $\mathcal{O}$-cog corresponding to an $\mathcal{O}$-cog duo $a_\bullet \parallel b_\bullet$ is the vector

$$a_\bullet \between b_\bullet = \sum_{I \subseteq \{1, \ldots, d\}} (-1)^{|I|} a_\bullet[a_i \mapsto b_i \mid i \in I] \in \operatorname{Lin}_{\mathfrak{F}} \mathcal{O}.$$

The linear span of all $\mathcal{O}$-cogs is denoted by $\operatorname{Cog}_{\mathfrak{F}} \mathcal{O}$.

Proposition III.10. *Let $\mathcal{O}$ be S-orderly. Then $\operatorname{Cog}_{\mathfrak{F}} \mathcal{O}$ is an $\operatorname{Aut}(\mathbb{A})_{(S)}$-equivariant subspace of $\operatorname{Lin}_{\mathfrak{F}} \mathcal{O}$ generated by any single $\mathcal{O}$-cog.*

Proof. Suppose $a_\bullet \parallel b_\bullet$ is an $\mathcal{O}$-cog duo. The definition completely specifies the $\mathcal{L}$-structure on $\{a_1, b_1, \ldots, a_d, b_d\} \cup S$ and says that

$$a_i \mapsto a'_i, b_i \mapsto b'_i, s \mapsto s$$

is an isomorphism given another $\mathcal{O}$-cog duo $a'_\bullet \parallel b'_\bullet$. Homogeneity then yields an automorphism $\pi \in \operatorname{Aut}(\mathbb{A})_{(S)}$ satisfying $\pi \cdot (a_\bullet \between b_\bullet) = a'_\bullet \between b'_\bullet$. □

Though the definitions were a mouthful, the example below should explain how cogs arise.

Example III.11. Let $\mathbb{A} = \mathbb{Q}$ as described in Example III.1; there is a unique $\{\}$-orderly orbit $\mathcal{O}$ in $\mathbb{A}^2$. Consider the vector

$$v = (0, 4) + (4, 9) - (9, 10) - (0, 10)$$

in $\operatorname{Lin} \mathcal{O}$. We can find $4 < 4 + \varepsilon < 9 < 9 + \delta < 10$ in $\mathbb{A}$ together with monotone bijections $\pi_1, \pi_2 \in \operatorname{Aut}(\mathbb{A})$ such that

$$\pi_1 : \begin{cases} 0 \mapsto 0, \\ 4 \mapsto 4 + \varepsilon, \\ 9 \mapsto 9, \\ 10 \mapsto 10; \end{cases} \quad \pi_2 : \begin{cases} 0 \mapsto 0, \\ 4 \mapsto 4, \\ 9 \mapsto 9 + \delta, \\ 10 \mapsto 10 \end{cases}$$

by interpolating linearly for example. Then

$$\begin{aligned} v_1 = v - \pi_1 \cdot v = (0, 4) + (4, 9) - (4, 10) \\ - (0, 4 + \varepsilon) - (4 + \varepsilon, 9) + (4 + \varepsilon, 10) \end{aligned}$$

duplicates the tuples with 4 in it but kills the one without it. Similarly

$$\begin{aligned} v_{1,2} = v_1 - \pi_2 \cdot v_1 = (4, 9) - (4, 9 + \delta) \\ - (4 + \varepsilon, 9) + (4 + \varepsilon, 9 + \delta) \end{aligned}$$

only leaves and duplicates the tuples with 9 in it. Here $v_{1,2}$ is the cog for the duo $(4, 9 \parallel 4 + \varepsilon, 9 + \delta)$ in $\mathcal{O}$ as well as in the smaller $\{0, 10\}$-orderly orbit $\mathcal{O}' = \operatorname{Aut}(\mathbb{A})_{(0,10)} \cdot (4, 9) \subseteq \mathcal{O}$.

To find cog duos in general, we iterate the following procedure.

Lemma III.12. *Let $a_\bullet \parallel b_\bullet$ be an $\mathcal{O}$-cog duo, where $\mathcal{O} \subseteq \mathbb{A}^d$ is S-orderly. Given $s \in S$ with $a_{j-1} < s < a_j$ (where we treat a_0 and a_{d+1} as $\pm\infty$), we write $S' = S \setminus \{s\}$ and let $a_\bullet ;_j s \in \mathbb{A}^{d+1}$ be the tuple obtained by inserting s in $a_\bullet$ as the jth entry. Then*

$$\mathcal{O}' = \operatorname{Aut}(\mathbb{A})_{(S')} \cdot a_\bullet ;_j s \subseteq \mathbb{A}^{d+1}$$

is S'-orderly.

Apply Lemma III.4 with $X ::= \{a_1, b_1, \ldots, a_d, b_d\} \cup S'$, $z ::= s$, and any finite $Y \subseteq \mathbb{A}$ disjoint from $X \cup \{z\}$ to obtain $\tau \in \operatorname{Aut}(\mathbb{A})_{(X)}$ and $\tau(z) =:: s'$. Then $a_\bullet ;_j s \parallel b_\bullet ;_j s'$ is an $\mathcal{O}'$-cog duo.

Proof. Already $a_{j-1} < b_{j-1} < s$ (if $j > 1$), $s < s'$, and $s' < a_j < b_j$ (if $j \leq d$). Now pick any relation $R \in \mathcal{L}_0$ of arity r and take any $x_\bullet \in (\{a_1, b_1, \ldots, a_d, b_d, s, s'\} \cup S')^r = (X \cup \{s, s'\} \cup Y)^r$. We split into three cases.

If s and s' both appear in $x_\bullet$, then we have

$$\mathbb{A} \models R(x_\bullet[b_i \mapsto a_i, s' \mapsto s \mid 1 \leq i \leq d]) \leftrightarrow R(x_\bullet). \qquad (¶)$$

Indeed, the left is false because R is irreflexive; so is the right by the design of s'.

Now if s appears in $x_\bullet$ we may assume s' does not. This time around we have (¶) as $x_\bullet \in (\{a_1, b_1, \ldots, a_d, b_d\} \cup S)^r$ — so we can ignore the $[s' \mapsto s]$ substitution — and $a_\bullet \parallel b_\bullet$ is a cog duo in $\mathcal{O} = \operatorname{Aut}(\mathbb{A})_{(S)} \cdot a_\bullet$.

Finally, suppose s' appears in $x_\bullet$ and s does not. Then $x_\bullet[s' \mapsto s] = \tau^{-1} \cdot x_\bullet$ and

$$\begin{aligned} x_\bullet[b_i \mapsto a_i, s' \mapsto s \mid 1 \leq i \leq d] \\ = (\tau^{-1} \cdot x_\bullet)[b_i \mapsto a_i \mid 1 \leq i \leq d] \end{aligned}$$

where $\tau^{-1} \cdot x_\bullet \in (\{a_1, b_1, \ldots, a_d, b_d\} \cup S)^r$. On the one hand, as discussed in the case above we get

$$\mathbb{A} \models R((\tau^{-1} \cdot x_\bullet)[b_i \mapsto a_i \mid 1 \leq i \leq d]) \leftrightarrow R(\tau^{-1} \cdot x_\bullet).$$

On the other hand, we certainly have

$$\mathbb{A} \models R(\tau^{-1} \cdot x_\bullet) \leftrightarrow R(x_\bullet)$$

since τ is an automorphism. This establishes (¶) again, showing that $a_\bullet s \parallel b_\bullet s'$ is an $\mathcal{O}'$-cog duo. □

Proposition III.13. *Let $\mathcal{O} \subseteq \mathbb{A}^d$ be S-orderly. Then, given $a_\bullet \in \mathcal{O}$, there exists $b_\bullet \in \mathcal{O}$ such that $a_\bullet \parallel b_\bullet$ is an $\mathcal{O}$-cog duo. Moreover, for each $i = 1, \ldots, d$ there exists an automorphism $\pi_i \in \mathrm{Aut}(\mathbb{A})_{(\{a_1,b_1,\ldots,a_{i-1},b_{i-1},a_{i+1},b_{i+1},\ldots,a_d,b_d\}\cup S)}$ sending $a_i \mapsto b_i$.*

Proof. For $1 \le i \le d+1$, let $S^{(i)} = S \cup \{a_i, \ldots, a_d\}$ and let $\mathcal{O}^{(i)} = \mathrm{Aut}(\mathbb{A})_{(S^{(i)})} \cdot (a_1, \ldots, a_{i-1})$; then $\mathcal{O}^{(i)}$ is $S^{(i)}$-orderly.

Suppose we have found $b_1, \ldots, b_{i-1}$ so that

$$(a_1, \ldots, a_{i-1}) \parallel (b_1, \ldots, b_{i-1})$$

is an $\mathcal{O}^{(i)}$-cog duo — note that $() \parallel ()$ is trivially a cog duo in $\mathcal{O}^{(1)} = \{()\}$. As $S^{(i+1)} = S^{(i)} \setminus \{a_i\}$, a straightforward application of Lemma III.12 with $Y ::= \{\}$ gives us an atom b_i such that

$$(a_1, \ldots, a_{i-1}, a_i) \parallel (b_1, \ldots, b_{i-1}, b_i)$$

is a cog duo in $\mathcal{O}^{(i+1)}$. We are done when we reach $S^{(d+1)} = S$ and $\mathcal{O}^{(d+1)} = \mathcal{O}$.

The automorphisms $\pi_1, \ldots, \pi_d$ now come directly from homogeneity and the definition of an $\mathcal{O}$-cog duo: the map

$$\begin{aligned} &a_1 \mapsto a_1, \ldots, a_i \mapsto b_i, \ldots, a_d \mapsto a_d \\ &b_1 \mapsto b_1, \ldots, \qquad\qquad \ldots, b_d \mapsto b_d, s \in S \mapsto s \end{aligned}$$

is an embedding. □

Theorem III.14. *Given an S-orderly orbit $\mathcal{O}$, any non-zero $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace of $\mathrm{Lin}_{\mathfrak{F}}\,\mathcal{O}$ contains $\mathrm{Cog}_{\mathfrak{F}}\,\mathcal{O}$.*

Proof. Let $V \subseteq \mathrm{Lin}_{\mathfrak{F}}\,\mathcal{O}$ be a non-zero $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace and let $v \in V$ be a non-zero vector; then $v(a_\bullet) \neq 0$ for some $a_\bullet \in \mathcal{O}$. Put

$$\begin{aligned} \mathcal{O}_v &= \{o_\bullet \in \mathcal{O} \mid v(o_\bullet) \neq 0\}, \\ S' &= S \cup \{o_i \mid o_\bullet \in \mathcal{O}_v, 1 \le i \le d\} \setminus \{a_1, \ldots, a_d\}, \\ \mathcal{O}' &= \mathrm{Aut}(\mathbb{A})_{(S')} \cdot a_\bullet \subseteq \mathcal{O}' \end{aligned}$$

and use Proposition III.13 to find the $\mathcal{O}'$-cog duo $a_\bullet \parallel b_\bullet$ — which is *a fortiori* an $\mathcal{O}$-cog duo — and the automorphisms $\pi_1, \ldots, \pi_d$. Now define $v_0 = v$ and

$$v_{i+1} = v_i - \pi_{i+1} \cdot v_i.$$

We can check inductively that for $i = 1, \ldots, d$, with $\mathcal{O}_v^{(i)} = \{o_\bullet \in \mathcal{O}_v \mid \{o_1, \ldots, o_d\} \supseteq \{a_1, \ldots, a_i\}\}$ we have

$$v_i = \sum_{o_\bullet \in \mathcal{O}_v^{(i)}} \sum_{J \subseteq \{1,\ldots,i\}} (-1)^{|J|} v(o_\bullet) \cdot o_\bullet[a_j \mapsto b_j \mid j \in J].$$

But $\{o_1, \ldots, o_d\} \supseteq \{a_1, \ldots, a_d\}$ means that $o_\bullet = a_\bullet$, so at the end we get the $\mathcal{O}$-cog $v_d = a_\bullet \between b_\bullet$. We conclude by Proposition III.10 that V contains $\mathrm{Cog}_{\mathfrak{F}}\,\mathcal{O}$. □

Corollary III.15. $\mathrm{Cog}_{\mathfrak{F}}\,\mathcal{O}$ *has length* 1.

Proof. By Theorem III.14, an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace $V \subseteq \mathrm{Cog}_{\mathfrak{F}}\,\mathcal{O}$ is either $\{0\}$ or $\mathrm{Cog}_{\mathfrak{F}}\,\mathcal{O}$ itself. □

B. *Projecting down*

$$(-)|_{I\setminus\{i\}} = (-)|_{-i}$$

C. *Building up*

$$a_{i_1}^{(1)}, a_{i_2}^{(2)}, \ldots, a_{i_n}^{(n)}, b_*, s_1, \ldots, s_m$$

$$o_{i_1}, o_{i_2}, \ldots, o_{i_n}, o_N, s_1, \ldots, s_m$$

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