More Vector Spaces with Atoms of Finite Lengths

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***Abstract*—We say an infinite structure is oligomorphic over a field if the following holds for each of its finite powers: in the corresponding free vector space, strict chains of equivariant subspaces are bounded in length. It has been shown that the countable pure set and the dense linear ordering without endpoints have this property. In this paper, we generalise these two results to a) reducts of smoothly approximable structures, provided the field has characteristic zero, and b) generaically ordered expansions of Fraïssé limits with free amalgamation, in languages with at most binary relations. As a special case, we prove the Rado graph is oligomorphic over any field using both methods.**

I. Introduction

II. Preliminaries

Let $\mathbb{A}$ be a countable structure and $\mathrm{Aut}(\mathbb{A})$ its automorphism group.

III. Rado graph, sans cogs

1) $\mathbb{A}$ is oligomorphic if, for $k = 0, 1, 2, \cdots$, $\mathbb{A}^k$ only has finitely many orbits;
2) $\mathbb{A}$ has the finite length property over $\mathfrak{F}$ if, for $k = 0, 1, 2, \cdots$, $\mathrm{Lin}_{\mathfrak{F}}\, \mathbb{A}^k$ only has finitely long chains.

Of note: with Stirling numbers of the second kind and Gaussian 2-binomial coefficients, the orbit counts are given by

$$\#\mathbb{N}^k = \sum_{d=0}^{k} \left\{ {k \atop d} \right\}$$

$$\#\mathbb{Q}^k = \sum_{d=0}^{k} \left\{ {k \atop d} \right\} d!$$

$$\#\mathbb{G}^k = \sum_{d=0}^{k} \left\{ {k \atop d} \right\} 2^{\binom{d}{2}}$$

$$\#\mathbb{V}_\infty^k = \sum_{d=0}^{k} \begin{bmatrix} k \\ d \end{bmatrix}_2$$

$$\#\mathbb{W}_\infty^k = \sum_{d=0}^{k} \begin{bmatrix} k \\ d \end{bmatrix}_2 2^{\binom{d}{2}}$$

To introduce:

1) *smooth approximation* by *homogeneous substructures* [3] (N.B. 'smooth approximation' from [4, Definition 4] seems to be entirely different)
2) *oligomorphic approximation* of a homogeneous structure by finite substructures with uniformly few orbits (i.e., types) that cover the age of $\mathbb{A}$
3) For the equality and ordered atoms, being supportively $\mathfrak{F}$-oligomorphic follows from being $\mathfrak{F}$-oligomorphic [1, Theorem 4.10]

A. Symplectic vector spaces

Throughout this subsection let $\mathfrak{f}$ denote a finite field.

Definition III.1. A *symplectic vector space* is an $\mathfrak{f}$-vector space $\mathbb{W}$ equipped with a bilinear form $\omega : \mathbb{W} \times \mathbb{W} \to \mathfrak{f}$ that is

1) alternating: $\omega(v, v) = 0$ for all v; and
2) non-degenerate: if $\omega(v, w) = 0$ for all w then $v = 0$.

Example III.2. Let $\mathbb{W}_n$ be the $\mathfrak{f}$-vector space with basis $e_1, \dots, e_n, f_1, \dots, f_n$. Define ω by bilinearly extending

$$\omega(e_i, f_i) = 1 = -\omega(f_i, e_i), \quad \omega(-, *) = 0 \text{ elsewhere}; \qquad (\S)$$

one may straightforwardly check that ω is alternating and non-degenerate. Moreover, noticing that $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$, we obtain a countable-dimensional symplectic vector space $\mathbb{W}_\infty = \bigcup_n \mathbb{W}_n$.

We will refer to vectors satisfying (§) as a *symplectic basis*. Note such vectors must be linearly independent: if $v = \sum_i \lambda_i e_i + \mu_i f_i = 0$, then $\lambda_i = \omega(v, f_i) = 0$ and $\mu_i = \omega(e_i, v) = 0$ for each i. Such bases behave very much like the usual bases.

Proposition III.3. *Assume that $\mathbb{W}$ is a symplectic vector space that is at most countable. Then any finite symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$ can be extended to a symplectic basis that spans the whole $\mathbb{W}$.*

Proof. Suppose that $e_1, \dots, e_n, f_1, \dots, f_n$ does not already span $\mathbb{W}$; take v to be a witness (that is least according to some fixed enumeration of $\mathbb{W}$ in the case it is infinite). Put

$$e_{n+1} = v - \sum_{i=1}^{n} \omega(e_i, v) f_i + \sum_{i=1}^{n} \omega(f_i, v) e_i$$

so that $\omega(e_i, e_{n+1}) = 0 = \omega(f_i, e_{n+1})$. This cannot be the zero vector lest we contradict the choice of v. By the non-degeneracy of ω, there is — rescaling if necessary — some w such that $\omega(e_{n+1}, w) = 1$. Now define

$$f_{n+1} = w - \sum_{i=1^n} \omega(e_i, w) f_i + \sum_{i=1}^{n} \omega(f_i, w) e_i$$

in a similar manner, making $e_1, \dots, e_n, e_{n+1}, f_1, \dots, f_n, f_{n+1}$ a symplectic basis that spans v. We go through every element of $\mathbb{W}$ by continuing this way. $\square$

In fact, we will also make use of the "symplectic basis and a half" variant below.

Proposition III.4. *Now assume $\mathbb{W}$ is a finite-dimensional symplectic vector space. Let*

$$\begin{gathered} e_1, \dots, e_n, e_{n+1}, \dots, e_{n+k}, \\ f_1, \dots, f_n \end{gathered}$$

be linearly independent vectors satisfying (§). *Then we can find the missing $f_{n+1}, \dots, f_{n+k}$ to complete the symplectic basis.*

Proof. We first need the following notion. Given a subspace $V \subseteq \mathbb{W}$, consider its orthogonal complement

$$V^\perp = \{w \in \mathbb{W} \mid \forall v \in V : \omega(v, w) = 0\}.$$

It is the kernel of the composite linear map

$$\begin{gathered} \mathbb{W} \to (\mathbb{W} \xrightarrow{\text{lin.}} \mathfrak{f}) \to (V \xrightarrow{\text{lin.}} \mathfrak{f}) \\ w \mapsto \omega(-, w) \mapsto \omega(-, w)|_V. \end{gathered}$$

Note this map is surjective: the first part is injective by non-degeneracy and hence surjective for dimension reasons, and the second part is surjective since we can extend a basis of V to one of $\mathbb{W}$. Therefore

$$\dim V^\perp = \dim \mathbb{W} - \dim V,$$

and in particular $V^{\perp\perp}$ is precisely equal to V.

Now suppose we have found $f_{n+1}, \dots, f_{n+i}$ already. If e_{n+i+1} were to be spanned by

$$\begin{gathered} e_1, \dots, e_n, e_{n+1}, \dots e_{n+i}, \cancel{e_{n+i+1}}, e_{n+i+2}, \dots, e_{n+k}, \\ f_1, \dots, f_n, f_{n+1}, \dots, f_{n+i}, \end{gathered}$$

it would be spanned by $e_{n+i+2}, \dots, e_{n+k}$ alone because of (§); but this is impossible as we assumed linear independency. So

$$\begin{aligned} e_{n+i+1} &\notin \langle e_1, \dots, e_{n+i}, e_{n+i+2}, f_1, \dots, f_{n+i} \rangle \\ &= \langle e_1, \dots, e_{n+i}, e_{n+i+2}, f_1, \dots, f_{n+i} \rangle^{\perp\perp}, \end{aligned}$$

i.e., some $f_{n+i+1} \in \langle e_1, \dots, e_{n+i}, e_{n+i+2}, f_1, \dots, f_{n+i} \rangle^\perp$ satisfies $\omega(e_{n+i+1}, f_{n+i+1}) = 1$. ☐

Given two symplectic vector spaces $\mathbb{W}$ and $\mathbb{W}'$, we call a function α between $X \subseteq \mathbb{W}$ and $X' \subseteq \mathbb{W}'$ *isometric* if $\omega(\alpha(x_1), \alpha(x_2)) = \omega(x_1, x_2)$ for all $x_1, x_2 \in X$. We can make an easy observation:

Lemma III.5. *Let $\{e_i, f_i \mid i \in I\} \subseteq \mathbb{W}$, $\{e'_j, f'_j \mid j \in J\} \subseteq \mathbb{W}'$ be two symplectic bases and let $\alpha : I \to J$ be a bijection. Then*

$$e_i \mapsto e'_{\alpha(i)}, f_i \mapsto f'_{\alpha(i)}$$

defines an isometric linear isomorphism $\langle e_i, f_i \rangle \to \langle e'_j, f'_j \rangle$.

It then follows from Proposition III.3 that, up to isometric linear isomorphisms, $\mathbb{W}_0, \mathbb{W}_1, \mathbb{W}_2, \dots, \mathbb{W}_\infty$ are all the countable symplectic vector spaces. Whilst we may deduce that $\mathbb{W}_\infty$ is oligomorphic by appealing to Ryll-Nardzewski, we will opt for a more direct proof that also establishes smooth approximation.

Proposition III.6 (Witt Extension)**.** *Any isometric linear injection $\alpha : \langle X \rangle \subseteq \mathbb{W}_n \to \mathbb{W}_n$ can be extended to an isometric linear automorphism of $\mathbb{W}_n$ and in turn to one of $\mathbb{W}_\infty$.*

Proof. To begin with, find a basis $x_1, \dots, x_k$ for the subspace $W = \{w \in \langle X \rangle \mid \forall x \in X : \omega(w, x) = 0\}$ and extend it to a basis $x_1, \dots, x_k, x_{k+1}, \dots, x_d$ for $\langle X \rangle$. Notice that

$$U = \langle x_{k+1}, \dots, x_d \rangle$$

must be a symplectic subspace: as it intersects with W trivially, given any non-zero vector $u \in U$ we must have $0 \neq \omega(u, w + u') = \omega(u, u')$ for some $w \in W$ and $u' \in U$. Hence use Proposition III.3 to find a symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$ for U. Observe that

$$\begin{gathered} e_1, \dots, e_n, x_1, \dots, x_k, \\ f_1, \dots, f_n \end{gathered}$$

form a basis for $\langle X \rangle$ and satisfy (§). On the other hand,

$$\begin{gathered} \alpha(e_1), \dots, \alpha(e_n), \alpha(x_1), \dots, \alpha(x_k), \\ \alpha(f_1), \dots, \alpha(f_n) \end{gathered}$$

form a basis for $\alpha(\langle X \rangle)$ and also satisfy (§). Therefore apply Proposition III.4 twice to find the missing $y_1, \dots, y_k$ and $y'_1, \dots, y'_k$ to complete the two symplectic bases — call them $\mathcal{B}$ and $\mathcal{B}'$. They are of the same size.

Now, by using Proposition III.3, extend $\mathcal{B}$ and $\mathcal{B}'$ to symplectic bases $\mathcal{C}$ and $\mathcal{C}'$ that span $\mathbb{W}_n$. These must both have size $2n$, so by Lemma III.5 we obtain an isometric linear automorphism $\beta : \mathbb{W}_n \to \mathbb{W}_n$ extending α.

To finish, notice that $\mathcal{C}, e_{n+1}, \dots, f_{n+1}, \dots$ as well as $\mathcal{C}', e_{n+1}, \dots, f_{n+1}, \dots$ form a symplectic basis spanning $\mathbb{W}_\infty$. We obtain from Lemma III.5 another time an isometric linear automorphism $\gamma : \mathbb{W}_\infty \to \mathbb{W}_\infty$ extending β that is the identity almost everywhere. ☐

Proposition III.7. *$\mathbb{W}^k_\infty$ has precisely $\sum_{d=0}^{k} \begin{bmatrix} k \\ d \end{bmatrix}_q \cdot q^{\binom{d}{2}}$ orbits under isometric linear automorphisms, where $q = |\mathfrak{f}|$ and*

$$\begin{bmatrix} k \\ d \end{bmatrix}_q = \frac{(q^k - 1)(q^{k-1} - 1) \cdots (q^{k-d+1} - 1)}{(q^d - 1)(q^{d-1} - 1) \cdots (q^1 - 1)}$$

is the q-binomial coefficient.

Remark. To anticipate the next subsection, we note a similarity with the Rado graph: in $\mathbb{G}^k$ there are $\sum_{d=0}^{k} \binom{k}{d} \cdot 2^{\binom{d}{2}}$ orbits — we may impose any edge relation on d vertices.

Proof. To each $v_\bullet \in \mathbb{W}^k_\infty$ we associate a *type*, which comprises the following data:

1) pivot indices $I \subseteq \{1, \ldots, k\}$ containing every i such that v_i is not spanned by $v_1, \ldots, v_{i-1}$ — so we inductively ensure that
$$\{v_{i'} \mid i' \in I, i' \leq i\}$$
is a basis for $\langle v_1, \ldots, v_i \rangle$;
2) for each $j \notin I$, an assignment $\Lambda_j : \{i \in I \mid i < j\} \to \mathfrak{f}$ such that $v_j = \sum_{i \in I, i<j} \Lambda_j(i) v_i$;
3) a map $\Omega : \binom{I}{2} \to \mathfrak{f}$ defined by $\Omega(\{i' < i\}) = \omega(v_{i'}, v_i)$.

If $\pi : \mathbb{W}_\infty \to \mathbb{W}_\infty$ is an isometric linear automorphism, then $v_\bullet = (v_1, \ldots, v_k)$ and $\pi \cdot v_\bullet = (\pi(v_1), \ldots, \pi(v_k))$ evidently share the same type. Conversely, if $w_\bullet$ has the type of $v_\bullet$, then

$$\begin{aligned} \alpha : \langle v_i \mid i \in I \rangle &\to \langle w_i \mid i \in I \rangle \subseteq \mathbb{W}_n \\ v_i &\mapsto w_i \end{aligned}$$

gives an isometric linear injection for some large enough n. Observe that α must send $v_j \mapsto w_j$ for $j \notin I$ too, and that it may be extended to an isometric linear automorphism π of $\mathbb{W}_\infty$ by Propsoition III.6. Furthermore we can find some $v_\bullet$ that realises any given type $(I, \{\Lambda_j\}_j, \Omega)$: it suffices to put

$$v_i = e_i + \sum_{i' \in I, i' < i} \Omega(i', i) f_{i'}$$

for $i \in I$ and $v_j = \sum_{i \in I, i<j} \in \Lambda_j(i) v_i$ for $j \notin I$. Therefore the number of types is precisely the number of orbits in $\mathbb{W}_\infty^k$.

Finally, we do some combinatorics. Fix $0 \leq d \leq k$ and count the number of types with $|I| = d$. There are $q^{\binom{d}{2}}$ choices for Ω and say $\#_{k,d}$ choices for the Λ_j's; the two can be chosen independently. In total, this gives

$$\sum_{d=0}^{k} q^{\binom{d}{2}} \cdot \#_{k,d}$$

types for vectors in $\mathbb{W}_\infty^d$. So focus on $\#_{k,d}$, the number of *linear types* — i.e., $(I, \{\Lambda_j\}_j)$, ignoring Ω — in $\mathbb{W}_\infty^k$. (Incidentally $\sum_{d=0}^k \#_{k,d}$ is the number of orbits in $\mathbb{W}_\infty^k$ or, more generally, any countable-dimensional $\mathfrak{f}$-vector space under linear automorphisms.) On the small values we easily check that

$$\begin{aligned} &\#_{0,0} = 1 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_q, \\ &\#_{1,0} = 1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]_q, \qquad \#_{1,1} = 1 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]_q. \end{aligned}$$

Given a linear type in $\mathbb{W}_\infty^k$ with $|I| = d$, we either have $1 \in I$ or $I \subseteq \{2, \ldots, k\}$. In the first case, the linear type is specified by one of the $\#_{k-1,d-1}$ linear types in $\mathbb{W}_\infty^{k-1}$ together with how v_1 is involved in the span of the $(k-1) - (d-1)$ non-pivot vectors. In the second case, the linear type is simply one of the $\#_{k-1,d}$ linear types in $\mathbb{W}_\infty^{k-1}$. Thus

$$\begin{aligned} \#_{k,d} &= q^{k-d} \cdot \#_{k-1,d-1} + \#_{k-1,d} \\ &= q^{k-d} \cdot \begin{bmatrix} k-1 \\ d-1 \end{bmatrix}_q + \begin{bmatrix} k-1 \\ d \end{bmatrix}_q = \begin{bmatrix} k \\ d \end{bmatrix}_q. \end{aligned}$$ □

Theorem III.8. *The symplectic vector space $\mathbb{W}_\infty$ is smoothly approximated by $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$.*

Corollary III.9. *The symplectic $\mathfrak{f}$-vector space $\mathbb{W}_\infty$ has the finite length property over any field of characteristic 0.*

B. Symplectic graphs

For this subsection let $\mathfrak{f}$ be the two-element field $\mathfrak{f}_2$.

Definition III.10. For $n = 0, 1, 2, \ldots$, the *symplectic graph* $\widetilde{\mathbb{W}}_n$ has vertices $\mathbb{W}_n$ and edges

$$v_1 \sim v_2 \iff \omega(v_1, v_2) = 1.$$

This is indeed an undirected graph: as ω is alternating, we have $\omega(v_1, v_2) = -\omega(v_2, v_1) = \omega(v_2, v_1)$ over $\mathfrak{f}_2$.

Proposition III.11. $\mathrm{Aut}(\widetilde{\mathbb{W}}_n) = \mathrm{Aut}(\mathbb{W}_n)$.

Proof. Clearly any isometric linear automorphism of $\mathbb{W}_n$ is a graph automorphism of $\widetilde{\mathbb{W}}_n$. Conversely, any $f \in \widetilde{\mathbb{W}}_n$ is evidently isometric. To show that f is linear, take $\lambda_1, \lambda_2 \in \mathfrak{f}$ and $v_1, v_2 \in \mathbb{W}$. We calculate:

$$\begin{aligned} &\omega\Big(f(\sum_i \lambda_i v_i) - \sum_i \lambda_i f(v_i), f(w)\Big) \\ &= \omega\Big(f(\sum_i \lambda_i v_i), f(w)\Big) - \sum_i \lambda_i \omega\big(f(v_i), f(w)\big) \\ &= \omega\Big(\sum_i \lambda_i v_i, w\Big) - \sum_i \lambda_i \omega(v_i, w) \\ &= \omega(0, w) = 0 \end{aligned}$$

for all $f(w) \in f(\mathbb{W}_n) = \mathbb{W}_n$; since ω is non-degenerate, we conclude that $f(\sum_i \lambda_i v_i) = \sum_i \lambda_i f(v_i)$. □

So the number of orbits in $\widetilde{\mathbb{W}}_n^k$ is precisely equal to the number of orbits in $\mathbb{W}_n^k$ — in particular, it is bounded above by $\sum_{d=0}^k \left[\begin{smallmatrix} k \\ d \end{smallmatrix}\right]_2 \cdot 2^{\binom{d}{2}}$ independently of n by Proposition III.7.[1] It remains to show $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$ embeds all finite graphs:

Proposition III.12 ([2, Theorem 8.11.2])**.** *Every graph on at most $2n$ vertices embeds into $\widetilde{\mathbb{W}}_n$.*

Proof. Let G be a graph on at most $2n$ vertices. The conclusion is trivial when $n = 0$. Also, if G contains no edges, we can choose any $2n$ of the 2^n vectors in $\langle e_1, \ldots, e_n \rangle \subseteq \widetilde{\mathbb{W}}_n$.

So suppose $n \geq 1$ and G has an edge $s \sim t$. Let $G_{s,t}$ be the graph on vertices $G \setminus \{s, t\}$ with edges which we will specify later. By induction, some embedding $f : G_{s,t} \to \widetilde{\mathbb{W}}_{n-1}$ exists. Define $f' : G \to \widetilde{\mathbb{W}}_n$ by

$$\begin{aligned} x \in G_{s,t} &\mapsto f(x) - [\![x \sim s]\!] f_n + [\![x \sim t]\!] e_n \\ s &\mapsto e_n \\ t &\mapsto f_n \end{aligned}$$

[1]This is the kth term in the OEIS sequence A028631.

where $[\![\phi]\!]$ is 1 if ϕ holds and 0 otherwise. Then we have $\omega(f'(x), f'(s)) = [\![x \sim s]\!]$ and $\omega(f'(x), f'(t)) = [\![x \sim t]\!]$ as desired, on one hand. On the other,

$$\begin{aligned}\omega(f'(x_1), f'(x_2)) = [\![x_1 \sim x_2]\!] + [\![x_1 \sim s]\!][\![x_2 \sim t]\!] \\ + [\![x_1 \sim t]\!][\![x_2 \sim s]\!]\end{aligned}$$

tells us how we should define the edge relation in $G_{s,t}$ for f' to be an embedding of graphs. □

Theorem III.13. *The Rado graph is oligomorphically approximated by $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$.*

Corollary III.14. *The Rado graph has the finite length property over any field of characteristic* 0*.*

This proof of finite length also applies to *oriented graphs* (i.e., $x \to y \implies y \not\to x$ but unlike in a tournament, it may occur that $x \not\to y \wedge y \not\to x$) — use the three-element field instead of $\mathfrak{f}_2$.

IV. Rado graph, with cogs

In this section we work with the following setting:

- $\mathcal{L}_0$ is a (possibly infinite) relational language containing a binary symbol $=$;
- $\mathcal{C}_0$ is a free amalgamation class of $\mathcal{L}_0$-structures where $=$ is interpreted as true equality, but every other $R \in \mathcal{L}_0$ is interpreted irreflexively.[2]
- $\mathcal{L}$ consists of $\mathcal{L}_0$ together with a new binary symbol $<$;
- $\mathcal{C}$ consists of $\mathcal{L}$-structures obtained from $\mathcal{C}_0$ by expanding with all possible linear orderings;
- $\mathbb{A}_0$ and $\mathbb{A}$ are the respective Fraïssé limits of $\mathcal{C}_0$ and $\mathcal{C}$, where without loss of generality we assume $\mathbb{A}_0$ and $\mathbb{A}$ share the same domain so that $\mathrm{Aut}(\mathbb{A}_0) \supseteq \mathrm{Aut}(\mathbb{A})$.

Example IV.1. Take $\mathcal{L}_0$ to consist of $=$ only and $\mathcal{C}_0$ to be all finite sets. Then $\mathbb{A}_0$ is isomorphic to the pure set $\mathbb{N}$, whereas $\mathbb{A}$ is isomorphic to $\mathbb{Q}$ with the usual order.

Example IV.2. Let $\mathcal{L}_0$ consist of $=$ together with a single binary symbol $\sim$ and let $\mathcal{C}_0$ consist of all finite undirected graphs not embedding the complete graph K_n, where $3 \leq n$ ($\leq \infty$). Then $\mathbb{A}_0$ is the K_n-free Henson graph (or the Rado graph when $n = \infty$), and $\mathbb{A}$ is its generically ordered counterpart. (Allowing $n = 2$ makes these degenerate to $\mathbb{N}$ and $\mathbb{Q}$ above).

Free amalgamation in $\mathcal{C}_0$ allows us to free atoms in $\mathbb{A}$ from undesired relations. Let us make this more precise. Given atoms $z_i, z_j \in Z \subseteq \mathbb{A}$, we say z_i is *related* to z_j in Z if z_i, z_j appear together in some tuple $z_\bullet \in Z^*$ such that $\mathbb{A} \models R(a_\bullet)$ for some $R \in \mathcal{L}_0$ (since z_i and z_j are certainly related by $<$). Otherwise we say z_i is *unrelated* to z_j in Z — notice it is still possible that z_i becomes related to z_j in some bigger Z unless every relation in $\mathcal{L}_0$ is at most binary.

[2]We may assume irreflexivity with no loss of generality: see [5, beginning of §2.4].

Lemma IV.3. *Let $X, Y, \{z\} \subseteq \mathbb{A}$ be disjoint and finite. Then there is some automorphism $\tau \in \mathrm{Aut}(\mathbb{A})$ such that*

1) *τ fixes every $x \in X$;*
2) *$\tau(z)$ is unrelated to any $y \in Y$ and to z in $X \cup Y \cup \{z, \tau(z)\}$;*
3) *$\tau(z) > z$.*

Proof. In $\mathbb{A}_0$, form the free amalgam

$$\begin{tikzcd} & X \cup Y \cup \{z\} \arrow[dr, dashed, hook, "\subseteq"] & \\ X \arrow[ur, hook] \arrow[dr, hook] & & X \cup Y \cup \{z, z'\} \\ & X \cup \{z\} \arrow[ur, dashed, hook, "x \in X \mapsto x, z \mapsto z'"'] & \end{tikzcd}$$

so that no element of $Y \cup \{z\}$ is related to z' in $X \cup Y \cup \{z, z'\}$. Now we make $X \cup Y \cup \{z, z'\}$ an $\mathcal{L}$-structure: inherit the order on $X \cup Y \cup \{z\}$ from $\mathbb{A}$, and declare that $z < z'$ as well as $z' < a$ if a, the next element of $X \cup Y$ larger than z, exists at all. Observe that

$$x \in X \mapsto x, z \mapsto z'$$

is still an embedding in presence of the order. By homogeneity, we may embed $X \cup Y \cup \{z, z'\}$ into $\mathbb{A}$ via some f which is the identity on $X \cup Y \cup \{z\}$; again by homogeneity, we may extend the embedding

$$f(x) = x \in X \mapsto f(x), f(z) \mapsto f(z')$$

to some automorphism τ which makes 1), 2), and 3) true. □

On the other hand, an $\mathcal{L}$-structure fails to embed into $\mathbb{A}$ precisely when it embeds some forbidden structure, in which every two distinct elements are related:

Lemma IV.4. *Let $\mathcal{F}_0$ consist of minimal (with respect to $\subseteq$) $\mathcal{L}_0$-structures which do not appear in $\mathcal{C}_0$. Then*

1) *$\mathcal{C}_0$ consists of every $\mathcal{L}_0$-structure that does not embed any $F \in \mathcal{F}_0$.*
2) *$\mathcal{C}$ consists of every $\mathcal{L}$-structure whose $\mathcal{L}_0$-reduct does not embed any $F \in \mathcal{F}_0$.*
3) *In any $F \in \mathcal{F}_0$, every two distinct elements $x, y \in F$ are related by some $R \in \mathcal{L}_0$.*

Proof. As $\mathcal{C}_0$ is closed under substructures, its complement is closed under superstructures and thus is — since there are no infinite strictly descending chain of embedded substructures — determined by its minimal structures. 2) follows because an $\mathcal{L}$-structure is in $\mathcal{C}$ precisely when its $\mathcal{L}_0$-reduct is in $\mathcal{C}_0$. For 3), notice that $F \setminus \{x\}$, $F \setminus \{y\}$ are in $\mathcal{C}_0$ by minimality; therefore so is their free amalgam over $F \setminus \{x, y\}$, which then cannot agree with F. □

In what follows, we will juggle with Lemma IV.3 just enough so that we avoid the forbidden structures described in Lemma IV.4. A main result will be:

Theorem IV.5. *Assume each $R \in \mathcal{L}_0$ has arity at most two. Then $\mathbb{A}$ is $\mathfrak{F}$-oligomorphic for any field $\mathfrak{F}$ even with finitely many constants fixed, provided that $\mathbb{A}$ is oligomorphic (for instance if $\mathcal{L}_0$ is finite).*

And a corollary will be that $\mathbb{A}$ from Examples IV.1 and IV.2 is $\mathfrak{F}$-oligomorphic; as is its reduct $\mathbb{A}_0$.

A. Two reductions: orbits and projections

To start with, let us view $\mathbb{A}^d$ as $\mathbb{A}^{\{1,\dots,d\}}$ and more generally consider $\mathbb{A}^I$ for a finite indexing set $I \subseteq \mathbb{N}$. Fix a finite support $S \subseteq \mathbb{A}$. If $\mathbb{A}$ is oligomorphic, the tuples in $\mathbb{A}^I$ split into finitely many $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant orbits. Let $\mathcal{O} = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot o_\bullet$ be one such orbit. We shall call $\mathcal{O}$ *(S-)rigid* if $o_i \notin S$ and if $o_i < o_j$ whenever $i < j$. By removing the entries in $o_\bullet$ that repeat or come from S and reordering the rest, we can always find an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant bijection to an S-rigid orbit. An easy observation is that we may focus on a single rigid orbit at a time:

Proposition IV.6. *The following are equivalent:*

1) *For $d = 0, 1, 2, \dots$ and any finite $S \subseteq \mathbb{A}$, chains of $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces in $\mathrm{Lin}_{\mathfrak{F}} \mathbb{A}^d$ are bounded in length;*
2) *$\mathbb{A}$ is oligomorphic, and $\mathrm{Lin}_{\mathfrak{F}} \mathcal{O}$ has fintie length for any rigid orbit $\mathcal{O}$.*

Proof. We have $\mathrm{len}(\mathrm{Lin}_{\mathfrak{F}}(\biguplus_i \mathcal{O}_i)) = \mathrm{len}(\bigoplus_i \mathrm{Lin}_{\mathfrak{F}} \mathcal{O}_i) = \sum_i \mathrm{len}(\mathrm{Lin}_{\mathfrak{F}} \mathcal{O}_i)$. □

So fix an rigid orbit $\mathcal{O} = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot o_\bullet \subseteq \mathbb{A}^I$. From here we take an inductive approach. By $o|^J_\bullet$ we mean the restriction of $o_\bullet : I \to \mathbb{A}$ to $J \subseteq I$; we will often write $o|^{-i}_\bullet$ instead of $o|^{I\setminus\{i\}}_\bullet$. Note the image $\mathcal{O}|^I$ of $\mathcal{O}$ under this projection agrees with $\mathrm{Aut}(\mathbb{A})_{(S)} \cdot o|^J_\bullet$ and is still rigid.

To anticipate more general statements later, Let $\mathfrak{E}$ be a finite-dimensional $\mathfrak{F}$-vector space — for instance, $\mathfrak{F}$ itself. Things become more interesting when we lift $(-)|^J$ to a linear $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant map

$$\begin{aligned}(-)|^J : \mathrm{Lin}_{\mathfrak{E}} \mathcal{O} &\to \mathrm{Lin}_{\mathfrak{E}} \mathcal{O}|^J\\ v &\mapsto v|^J.\end{aligned}$$

Many cancellations can occur under $(-)|^J$; the *projection kernel* is the $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace

$$\mathrm{Ker}_{\mathfrak{E}} \mathcal{O} = \bigcap_{i \in I} \ker (-)|^{-i}$$

of $\mathrm{Lin}_{\mathfrak{E}} \mathcal{O}$.

Proposition IV.7. *The following are equivalent:*

1) *$\mathrm{Lin}_{\mathfrak{F}} \mathcal{O}$ has finite length for every rigid orbit $\mathcal{O}$;*
2) *$\mathrm{Ker}_{\mathfrak{F}} \mathcal{O}$ has finite length for every rigid orbit $\mathcal{O}$.*

Proof. That 1) implies 2) is clear as $\mathrm{Ker}_{\mathfrak{F}} \mathcal{O} \subseteq \mathrm{Lin}_{\mathfrak{F}} \mathcal{O}$.

To prove the other implication, assume 2) and let $\mathcal{O} \subseteq \mathbb{A}^I$. We proceed by induction on $|I|$. If $I = \emptyset$, then $\mathcal{O}$ must be the entire singleton $\mathbb{A}^\emptyset = \{()\}$; as $\mathrm{Lin}_{\mathfrak{F}} \mathcal{O}$ has no nontrivial subspaces (let alone finitely supported ones), it has length 1. Now if $|I| \geq 1$, assemble all $|I|$ projection maps into a single map

$$\begin{aligned}\mathrm{Lin}_{\mathfrak{F}} \mathcal{O} &\to \bigoplus_{i\in I} \mathcal{O}|^{-i}\\ v &\mapsto (v|^{-i})_{i \in I}\end{aligned}$$

whose kernel is precisely $\mathrm{Ker}_{\mathfrak{F}} \mathcal{O}$. We have

$$\mathrm{len}(\mathrm{Lin}_{\mathfrak{F}} \mathcal{O}) - \mathrm{len}(\mathrm{Ker}_{\mathfrak{F}} \mathcal{O}) \leq \sum_{i \in I} \mathrm{len}(\mathrm{Lin}_{\mathfrak{F}} \mathcal{O}|^{-i})$$

which shows that $\mathrm{len}(\mathrm{Lin}_{\mathfrak{F}} \mathcal{O})$ is finite from the assumptions. □

We call a vector from the projection kernel *balanced*. As we will see in the next subsection, cogs are a prominent example.

B. Cogs

Definition IV.8. Let $\mathcal{O} = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot o_\bullet \subseteq \mathbb{A}^I$ be an S-rigid orbit. An *$\mathcal{O}$-cog duo* $a_\bullet \parallel b_\bullet$ consists of $2 \cdot |I|$ atoms in $\mathbb{A}$ with the following $\mathcal{L}$-structure on $\{a_i, b_i \mid i \in I\} \cup S$:

1) $a_{i_1} < b_{i_1} < a_{i_2} < b_{i_2} < \cdots < a_{i_d} < b_{i_d}$ where I consists of the indices $i_1 < i_2 < \cdots < i_d$;
2) $a_i, b_i < s$ if and only if $o_i < s$;
3) any relation in $\mathcal{L}_0$ (in particular $=$) holds for a tuple $c_\bullet$ with entries in $a_I \cup b_I \cup S$ if and only if it holds for $c_\bullet$ with each entry equal to a_i, b_i replaced by o_i.

Three remarks are in order. First, each a_i is unrelated to its counterpart b_i in $a_I \cup b_I \cup S$. Second, given any $J \subseteq I$, the combined tuple $a|^J_\bullet ; b|^{I\setminus J}_\bullet$ lies in $\mathcal{O}$ by homogeneity: observe

$$\begin{aligned}a_j &\mapsto o_j, j \in J;\\ b_i &\mapsto o_i, i \in I \setminus J;\\ s &\mapsto s, s \in S\end{aligned}$$

defines an embedding. Third, by homogeneity still, any two cog duos $a_\bullet \parallel b_\bullet$ and $a'_\bullet \parallel b'_\bullet$ belong to the same $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant orbit.

Definition IV.9. Given $\lambda \in \mathfrak{E}$ and an $\mathcal{O}$-cog duo $a_\bullet \parallel b_\bullet$, the corresponding *$\mathcal{O}$-cog* with coefficient λ is the vector

$$\lambda \cdot a_\bullet \between b_\bullet = \sum_{J \subseteq I} (-1)^{|J|} \lambda \cdot a|^{I\setminus J}_\bullet ; b^J_\bullet$$

in $\mathrm{Lin}_{\mathfrak{E}} \mathcal{O}$. The linear span of all $\mathcal{O}$-cogs with coefficients from $\mathfrak{E}$ is denoted by $\mathrm{Cog}_{\mathfrak{E}} \mathcal{O}$.

As remarked above, given any two $\mathcal{O}$-cog duos there is some $\pi \in \mathrm{Aut}(\mathbb{A})_{(S)}$ such that $\pi \cdot (a_\bullet \parallel b_\bullet) = a'_\bullet \parallel b'_\bullet$ and thus $\pi \cdot (\lambda \cdot a_\bullet \between b_\bullet) = \lambda \cdot a'_\bullet \between b'_\bullet$. Hence $\mathrm{Cog}_{\mathfrak{E}} \mathcal{O}$ is an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace of $\mathrm{Lin}_{\mathfrak{E}} \mathcal{O}$ and it is generated by cogs based on a single duo.

Proposition IV.10. *$\mathrm{Cog}_{\mathfrak{E}} \mathcal{O}$ is contained in $\mathrm{Ker}_{\mathfrak{E}} \mathcal{O}$.*

Proof. Let $\mathcal{O} \subseteq \mathbb{A}^I$, let $a_\bullet \parallel b_\bullet$ be an $\mathcal{O}$-cog duo, and let $i \in I$. The subsets of I come in pairs of J and $J \cup \{i\}$, where J is a subset of $I \setminus \{i\}$. The two tuples $a|_\bullet^J; b|_\bullet^{I\setminus J}$ and $a|_\bullet^{J\cup\{i\}}; b|_\bullet^{I\setminus(J\cup\{i\})}$ differ only on the ith entry. But this difference gets erased under $(-)|^{-i}$, so the two corresponding terms in $\lambda \cdot a_\bullet \between b_\bullet$ will cancel out and hence $(\lambda \cdot a_\bullet \between b_\bullet)|^{-i} = 0$ overall. ◻

In fact, cogs arise anywhere.

Lemma IV.11. *Suppose $a_\bullet \parallel b_\bullet$ is an $\mathcal{O}$-cog duo, where $\mathcal{O} \subseteq \mathbb{A}^I$ is S-rigid. Given $z \in S$,*

- *write $S' = S \setminus \{z\}$;*
- *let $j \notin I$ be such that $\mathcal{O}' = \mathrm{Aut}(\mathbb{A})_{(S')} \cdot (a_\bullet; z) \subseteq \mathbb{A}^{I\cup\{j\}}$ is rigid,*
- *let $X \subseteq \mathbb{A}$ be a finite set containing $\{a_i, b_i \mid i \in I\} \cup S'$ but not z;*
- *let $Y \subseteq \mathbb{A}$ be any finite set disjoint from $X \cup \{z\}$;*

then the $\tau \in \mathrm{Aut}(\mathbb{A})_{(X)}$ afforded by Lemma IV.3 gives us an $\mathcal{O}'$-cog duo $(a_\bullet; z) \parallel (b_\bullet; \tau \cdot z)$.

Proof. To start with, notice that we have the required order relations with z and $\tau(z)$. Now let $R \in \mathcal{L}_0$ and let $c_\bullet$ be a tuple with entries in $a_{I\cup\{j\}} \cup b_{I\cup\{j\}} \cup S' = a_I \cup b_I \cup S \cup \{\tau \cdot z\}$; we want to show that

$$\mathbb{A} \models R(c_\bullet) \text{ if and only if } \mathbb{A} \models R(d_\bullet), \tag{¶}$$

where $d_\bullet$ is obtained from $c_\bullet$ by replacing every b_i with a_i and $\tau \cdot z$ with z. We split into three cases.

1) Suppose z and $\tau \cdot z$ both appear in $c_\bullet$. Then $\mathbb{A} \not\models R(c_\bullet)$ because $\tau \cdot z$ is unrelated to z by design, and $\mathbb{A} \not\models R(d_\bullet)$ because R is irreflexive and z appears twice.
2) So if z appears in $c_\bullet$, we may assume that $\tau \cdot z$ does not — that is, $c_\bullet$ has entries in $a_I \cup b_I \cup S$. In this case (¶) follows from the assumption that $a_\bullet \parallel b_\bullet$ is a cog duo in $\mathcal{O} = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot a_\bullet$.
3) Finally, suppose that $\tau \cdot z$ appears in $c_\bullet$ but not z. Then only z appears in $\tau^{-1} \cdot c_\bullet$, and $d_\bullet$ is obtained by replacing every b_i with a_i in $\tau^{-1} \cdot c_\bullet$. But $\mathbb{A} \models R(c_\bullet)$ if and only if $\mathbb{A} \models R(\tau^{-1} \cdot c_\bullet)$, and now (¶) holds as we discussed for the case above. ◻

Starting from an empty cog duo, we may apply the previous lemma inductively.

Proposition IV.12. *Let $\mathcal{O} \subseteq \mathbb{A}^I$ be an S-rigid orbit. Then for any $a_\bullet \in \mathcal{O}$, there is:*

1) *some $b_\bullet \in \mathcal{O}$ such that $a_\bullet \parallel b_\bullet$ is an $\mathcal{O}$-cog duo; and*
2) *for each $i \in I$, some $\pi_i \in \mathrm{Aut}(\mathbb{A})_{(S\cup\{a_j, b_j | j \in I, j \neq i\})}$ sending $a_i \mapsto b_i$.*

Proof. Enumerate the indices of I as $i_1, \ldots, i_d$. Suppose that we have found $b_{i_1}, \ldots, b_{i_k}$ such that

$$a|_\bullet^{\{i_1,\ldots,i_k\}} \parallel (i_1 \mapsto b_{i_1}, \ldots, i_k \mapsto b_{i_k})$$

is a cog duo in $\mathcal{O}_k = \mathrm{Aut}(\mathbb{A})_{(S\cup\{a_{i_{k+1}},\ldots,a_{i_d}\})} \cdot a|_\bullet^{\{i_1,\ldots,i_k\}}$ — note that $() \parallel ()$ is certainly a cog duo in $\mathcal{O}_0$ at the start. If $k < d$, with $z = a_{i_{k+1}}$, $X = \{a_{i_1}, b_{i_1}, \ldots, a_{i_k}, b_{i_k}\} \cup S \cup \{a_{i_{k+2}}, \ldots, a_{i_d}\}$, and $Y = \emptyset$, a straightforward application of Lemma IV.11 yields an atom $b_{i_{k+1}}$ that makes

$$a|_\bullet^{\{i_1,\ldots,i_k,i_{k+1}\}} \parallel (i_1 \mapsto b_{i_1}, \ldots, i_k \mapsto b_{i_k}, i_{k+1} \mapsto b_{i_{k+1}})$$

a cog duo in $\mathcal{O}_{k+1}$. We thus obtain the desired cog duo in $\mathcal{O}_d = \mathcal{O}$ as we reach $k = d$. The automorphisms π_{i_k} now come directly from homogeneity and the definition of an $\mathcal{O}$-cog duo: the map

$$\begin{aligned} &a_{i_1} \mapsto a_{i_1}, \ldots, a_{i_k} \mapsto b_{i_k}, \ldots, a_{i_d} \mapsto a_{i_d} \\ &b_{i_1} \mapsto b_{i_1}, \ldots, \qquad\qquad\quad \ldots, b_{i_d} \mapsto b_{i_d}, s \in S \mapsto s \end{aligned}$$

is an embedding. ◻

The result below substantiates the slogan that cogs are found everywhere.

Theorem IV.13. *Any $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace V of $\mathrm{Lin}_{\mathfrak{E}}\, \mathcal{O}$ contains $\mathrm{Cog}_{\mathfrak{E}(V)}\, \mathcal{O}$, where $\mathcal{O} \subseteq \mathbb{A}^I$ is S-rigid and $\mathfrak{E}(V)$ is the subspace spanned by $\{v(a_\bullet) \mid v \in V, a_\bullet \in \mathcal{O}\}$ of $\mathfrak{E}$.*

Proof. Pick any $v \in V$ and $a_\bullet \in \mathcal{O}$; it is enough to show that V contains $v(a_\bullet) \cdot a_\bullet \between b_\bullet$ for some $\mathcal{O}$-cog duo $a_\bullet \parallel b_\bullet$. Actually, write

$$S' = S \cup \{c_i \mid v(c_\bullet) \neq 0, i \in I\} \setminus \{a_i \mid i \in I\} \supseteq S$$

and put $\mathcal{O}' = \mathrm{Aut}(\mathbb{A})_{(S')} \cdot a_\bullet \subseteq \mathcal{O}$ — then $\mathcal{O}'$ is S'-rigid. By Proposition IV.12, we can find $b_\bullet \in \mathcal{O}'$ such that $a_\bullet \parallel b_\bullet$ is a cog duo in $\mathcal{O}'$ and *a fortiori* a cog duo in $\mathcal{O}$. Also take the automorphisms $\pi_{i_1}, \ldots, \pi_{i_d}$ from there, where $i_1, \ldots, i_d$ enumerate I. Now define $v_0 = v$ and

$$v_k = v_{k-1} - \pi_{i_k} \cdot v_{k-1}.$$

We can check inductively that for $k = 0, 1, \ldots, d$, with $\mathcal{O}_k = \{c_\bullet \mid v(c_\bullet) \neq 0, \{c_{i_1}, \ldots, c_{i_k}, \ldots, c_{i_d}\} \supseteq \{a_{i_1}, \ldots, a_{i_k}\}\}$ we have

$$v_k = \sum_{c_\bullet \in \mathcal{O}_k} \sum_{J \subseteq \{i_1,\ldots,i_k\}} (-1)^{|J|} v(c_\bullet) \cdot c|_\bullet^J; b|_\bullet^{I\setminus J}.$$

But $\{c_{i_1}, \ldots, c_{i_d}\} \supseteq \{a_{i_1}, \ldots, a_{i_d}\}$ means that $c_\bullet = a_\bullet$, so at the end v_d is the desired $\mathcal{O}$-cog. ◻

Corollary IV.14. $\mathrm{Cog}_{\mathfrak{F}}\, \mathcal{O}$ *has length* 1.

Proof. Let $V \subseteq \mathrm{Cog}_{\mathfrak{F}}\, \mathcal{O}$ be a non-zero $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace. Then $\{0\} \subsetneq \mathfrak{E}(V) \subseteq \mathfrak{E} = \mathfrak{F}$ so $\mathfrak{E}$ must be the entire field $\mathfrak{F}$, and by above V must be $\mathrm{Cog}_{\mathfrak{F}}\, \mathcal{O}$ itself. ◻

In light of Propositions IV.6 and IV.7, we will be able to prove the finite length property for an oligomorphic structure with free amalgamation over any field and support if we know $\mathrm{Ker}_{\mathfrak{F}}\, \mathcal{O} = \mathrm{Cog}_{\mathfrak{F}}\, \mathcal{O}$. Let us now attempt to show that.

C. Subvectors

This is a good time to recall a view we have tacitly taken: with $\mathcal{O}$ as a standard basis, a vector $v \in \mathrm{Lin}_{\mathfrak{E}}\, \mathcal{O}$ is just a finite set of pairs in $\mathfrak{E} \times \mathcal{O}$. A *subvector* of v is a subset of these pairs.

Now suppose as usual that $\mathcal{O} \subseteq \mathbb{A}^I$ is S-rigid. Given $i \in I$ and $a = a_i \in \mathbb{A}$ for some $a_\bullet \in \mathcal{O}$, we write

$$\mathcal{O}^{i:a} = \{b_\bullet \in \mathcal{O} \mid b_i = a\};$$

this is an $\mathrm{Aut}(\mathbb{A})_{(S\cup\{a\})}$-orbit, and its projection $\mathcal{O}^{i:a}|^{-i} = \mathrm{Aut}(\mathbb{A})_{(S\cup\{a\})} \cdot a|^{-i}_{\bullet}$ is rigid. For a vector $v \in \mathrm{Lin}_{\mathfrak{E}}\, \mathcal{O}$, by

$$v^{i:a} \in \mathrm{Lin}_{\mathfrak{E}}\, \mathcal{O}^{i:a}$$

we mean the subvector consisting of all pairs in $\mathfrak{E} \times \mathcal{O}^{i:a}$.

Lemma IV.15. *Let $v \in \mathrm{Lin}_{\mathfrak{E}}\, \mathcal{O}$ be balanced. Then any projected subvector $v^{i:a}|^{-i} \in \mathrm{Lin}_{\mathfrak{E}}\, \mathcal{O}^{i:a}|^{-i}$ is also balanced.*

Proof. Let $j \in I \setminus \{i\}$. By assumption we have

$$0 = v|^{-j} = \sum_a v^{i:a}|^{-j}$$

in $\mathrm{Lin}_{\mathfrak{E}}\, \mathbb{A}^{I\setminus\{j\}}$, so by looking at ith entries we see that each $v^{i:a}|^{-j}$ must be the zero vector. Hence so is $v^{i:a}|^{-j}|^{-i} = v^{i:a}|^{-i}|^{-j}$, which shows that $v^{i:a}|^{-i}$ is in the projection kernel. □

So we can try to prove $\mathrm{Ker}_{\mathfrak{E}}\, \mathcal{O} \subseteq \mathrm{Cog}_{\mathfrak{E}}\, \mathcal{O}$ for any rigid $\mathcal{O} \subseteq \mathbb{A}^I$ by inducting on $|I|$; we just need to reassemble the various cogs in $\mathcal{O}^{i:a}|^{-i}$ back into $\mathcal{O}$-cogs. Unfortunately we were only able to do so under the hypothesis that $\mathcal{L}_0$ is an at most binary language, which we henceforth assume.

D. Special case: unobstructed vectors

We will begin by showing any $v \in \mathrm{Ker}_{\mathfrak{E}}\, \mathcal{O}$ lies in $\mathrm{Cog}_{\mathfrak{E}}\, \mathcal{O}$ provided that v satisfies an additional condition which, as we will explain in the subsection, may be assumed without loss of generality. We motivate and introduce this condition now.

Consider the atoms in the tuples

$$\mathcal{O}(v) = \{b_\bullet \in \mathcal{O} \mid v(b_\bullet) \neq 0\}.$$

that appear in a vector $v \in \mathrm{Lin}_{\mathfrak{E}}\, \mathcal{O}$, with $\mathcal{O}$ being S-rigid. Take $a = b_i$ to be one such atom, and let $R_1, R_2 \in \mathcal{L}_0$ be unary and binary respectively. Then whether $R_1(a)$, $R_2(a, s)$, $R_2(s, a)$ hold in $\mathbb{A}$ given $s \in S$ is determined by $\mathcal{O}$ — more preciesly, by whether $R_1(o_i)$, $R_2(o_i, s)$, $R_2(s, o_i)$ hold in $\mathbb{A}$ for any $o_\bullet \in \mathcal{O}$.

Now take another atom $a' = b'_j$ occurring in $\mathcal{O}(v)$. What can be said about $R_2(a, a')$? Except in those happy dispositions where $b'_\bullet = b_\bullet$, not much — whether $\mathbb{A} \models R_2(o_i, o_j)$ need not be a constraint for a and a'. The index j may not even be unique: we may well have $b'_j = b''_k$ for $j \neq k$. We want to avoid such confusions:

Definition IV.16. Let $\mathcal{O}$ be S-rigid in $\mathbb{A}^I$, and take any $o_\bullet \in \mathcal{O}$. We say a finite family $\{b^{(k)}_\bullet \in \mathcal{O} \mid k \in K\}$ is

1) *unambiguous* if $\sqrt{-} : b^{(k)}_i \mapsto o_i$ for $k \in K$ is a function — i.e., if $i = i'$ whenever $b^{(k)}_i = b^{(k')}_{i'}$;
2) *unobstructed* if it is well-indexed and for any $b^{(k)}_i, b^{(k')}_{i'}$, either they are unrelated (in $\mathbb{A}$) or

$$\begin{aligned}\mathbb{A} \models R(b^{(k)}_i, b^{(k')}_{i'}) &\leftrightarrow R(\sqrt{b^{(k)}_i}, \sqrt{b^{(k')}_{i'}}) \\ \wedge\, R(b^{(k')}_{i'}, b^{(k)}_i) &\leftrightarrow R(\sqrt{b^{(k')}_{i'}}, \sqrt{b^{(k)}_i})\end{aligned}$$

for every binary relation $R \in \mathcal{L}_0$.[3]

We can now state the precise result.

Theorem IV.17. *Let $v \in \mathrm{Ker}_{\mathfrak{E}}\, \mathcal{O}$ be such that $\mathcal{O}(v)$ is unobstructed. Then (assuming that $\mathcal{L}_0$ is at most binary) we have*

$$v = \sum_{k\in K} v(b^{(k)}_\bullet) \cdot x^{(k)}_\bullet \between y^{(k)}_\bullet$$

for some $b^{(k)}_\bullet$'s, $x^{(k)}_\bullet$'s, and $y^{(k)}_\bullet$'s, where moreover

$$\mathcal{O}(v) \cup \{x^{(k)}_\bullet, y^{(k)}_\bullet \mid k \in K\}$$

is also unobstructed.

We spend the rest of this subsection giving the proof, which proceeds by induction on $|I|$ for S-rigid orbits $\mathcal{O} \subseteq \mathbb{A}^I$ for every S at once.

The base case $I = \emptyset$ is immediate: as $\mathcal{O}$ is the singleton $\{()\}$, any vector $v \in \mathrm{Ker}_{\mathfrak{E}}\, \mathcal{O} = \mathrm{Lin}_{\mathfrak{E}}\, \mathcal{O} = \mathrm{Cog}_{\mathfrak{E}}\, \mathcal{O}$ is a cog already, where $\mathcal{O}(v)$ is vacuously unobstructed; so the theorem says nothing more than $v = v$.

Now suppose that some $i^* \in I$ exists — in fact, let i^* be maximal. Consider a vector $v \in \mathrm{Ker}_{\mathfrak{E}}\, \mathcal{O}$ with $\mathcal{O}(v)$ being unobstructed. We can decompose v into finitely many subvectors to write

$$v = v^{i^*:a_1} + \cdots + v^{i^*:a_m}.$$

Then each projection $v^{i^*:a_j}|^{-i^*}$ lies in $\mathrm{Ker}_{\mathfrak{E}}\, \mathcal{O}^{i^*:a_j}|^{-i^*}$ by Lemma IV.15, and we may straightforwardly check that *a fortiori* $\mathcal{O}^{i^*:a_j}|^{-i^*}(v^{i^*:a_j}|^{-i^*})$ is unambiguous and unobstructed. It follows from the inductive hypothesis of Theorem IV.17 that

$$v^{i^*:a_j}|^{-i^*} = \sum_{k\in K_j} v^{i^*:a_j}|^{-i^*}(b^{(a_j,k)}_\bullet) \cdot x^{(a_j,k)}_\bullet \between y^{(a_j,k)}_\bullet$$

for some tuples $b^{(a_j,k)}_\bullet$'s, $x^{(a_j,k)}_\bullet$'s, and $y^{(a_j,k)}_\bullet$'s such that $\mathcal{O}^{i^*:a_j}|^{-i^*}(v^{i^*:a_j}|^{-i^*}) \cup \{x^{(a_j,k)}_\bullet, y^{(a_j,k)}_\bullet \mid k \in K_j\}$ is unobstructed. We can return to $\mathcal{O}$ by adding a_j back as the i^*th term to every tuple: we get

$$v^{i^*:a_j} = \sum_{k\in K_j} \Big(v(b^{(a_j,k)}_\bullet; a_j) \cdot x^{(a_j,k)}_\bullet \between y^{(a_j,k)}_\bullet\Big); a_j,$$

where the family

$$\mathcal{O}(v^{i^*:a_j}) \cup \{(x^{(a_j,k)}_\bullet; a_j), (y^{(a_j,k)}_\bullet; a_j) \mid k \in K_j\}$$

is easily seen to be unobstructed. We can ask for more:

[3]Of course, whether the family is unambiguous or unobstructed does not depend on the choice of the specific representative $o_\bullet \in \mathcal{O}$!

Claim IV.18. *We may choose $x_\bullet^{(a_j,k)}$'s, and $y_\bullet^{(a_j,k)}$'s so that*

$$\mathcal{O}(v) \cup \{(x_\bullet^{(a_{j'},k)}; a_{j'}), (y_\bullet^{(a_{j'},k)}; a_{j'}) \mid k \in K_{j'}, 1 \le j' \le j\}$$

is unobstructed for $j = 1, 2, \ldots, m$.

Proof. Let A_j denote the atoms that appear in $\mathcal{O}(v^{i^*:a_j})$. Given any automorphism $\pi \in \mathrm{Aut}(\mathbb{A})$ that fixes S as well as A_j, using the tuples $\pi \cdot x_\bullet^{(a_j,k)}$ and $\pi \cdot y_\bullet^{(a_j,k)}$ gives us another decomposition of $v^{i^*:a_j}|^{-i^*}$ that is unobstructed. We will show that composing such automorphisms — which will be provided by Lemma IV.3 — suffices to make the claim hold for j.

Assume the claim for $j-1$. Suppose that the family in the claim for j is not even unambiguous. We must have $b_i^{(k)} = b_{i'}^{(k')}$ with $i \ne i'$ where $b_\bullet^{(k)}$, say, is some $x_\bullet^{(a_j,k)}; a_j$ or $y_\bullet^{(a_j,k)}; a_j$. Now if we were to have $b_i^{(k)} \in A_j$ — e.g., if $b_i^{(k)} = a_j$ — then $b_i^{(k)} = b_{i''}^{(k'')}$ for some $b_\bullet^{(k'')} \in \mathcal{O}(v^{i^*:a_j}) \subseteq \mathcal{O}(v)$. This is impossible: we know two families that are unambiguous which imply $i = i''$ and $i'' = i'$.

So the ambiguous atom $b_i^{(k)}$ does not belong to A_j. In other words, the set X of all atoms except $b_i^{(k)}$ that appear in $\mathcal{O}(v) \cup \{(x_\bullet^{(a_{j'},k)}; a_{j'}), (y_\bullet^{(a_{j'},k)}; a_{j'}) \mid k \in K_{j'}, 1 \le j' \le j\}$ contains A_j and a_j. Apply Lemma IV.3 — particularly, use the fact that $\mathbb{A}$ has no algebraicity — to get an automorphism τ which fixes $S \cup X$ but sends $b_i^{(k)}$ to a fresh atom. Then, in the new family

$$\mathcal{O}(v) \cup \{(x_\bullet^{(a_{j'},k)}; a_{j'}), (y_\bullet^{(a_{j'},k)}; a_{j'}) \mid k \in K_{j'}, 1 \le j' < j\}$$
$$\cup \{(\tau \cdot x_\bullet^{(a_j,k)}; a_j), (\tau \cdot y_\bullet^{(a_j,k)}; a_j) \mid k \in K_j\},$$

the ambiguous atoms are precisely the ones in the old family minus $b_i^{(k)}$ — they are all fixed by τ. We thus continue this way until all ambiguous atoms are freshened.

Hence assume that the family in the claim for j is unambiguous. Suppose $b_i^{(k)}$ and some $b_{i'}^{(k')}$ are the reason why the family fails to be unobstructed. To have the obstruction $b_i^{(k)}$ in A_j is impossible: otherwise $b_i^{(k)} = b_{i''}^{(k'')}$ for some $b_\bullet^{(k'')} \in \mathcal{O}(v^{i^*:a_j}) \subseteq \mathcal{O}(v)$; then $i = i''$ by the unambiguity assumption, yet $b_{i''}^{(k'')}$ — hence $b_i^{(k)}$ — cannot be an obstruction because we have two subfamilies that are unobstructed. Symmetrically, we see that $b_{i'}^{(k')} \notin A_j$.

Finally, notice that by the claim for $j-1$, an obstruction $b_i^{(k)}$ must again come from $b_\bullet^{(k)}$ being some $x_\bullet^{(a_j,k)}; a_j$ or $y_\bullet^{(a_j,k)}; a_j$. This time, split all atoms except $b_i^{(k)}$ which appear in $\mathcal{O}(v) \cup \{(x_\bullet^{(a_{j'},k)}; a_{j'}), (y_\bullet^{(a_{j'},k)}; a_{j'}) \mid k \in K_{j'}, 1 \le j' \le j\}$ into two sets X and Y: put any atom $b_{i'}^{(k')}$ which makes $b_i^{(k)}$ an obstruction in Y, and put everything else in X. Then X contains A_j as discussed above; in fact, X contains $\{x_i^{(a_j,k)}, y_i^{(a_j,k)} \mid k \in K_j, i \in I\}$. Use Lemma IV.3 to get $\tau \in \mathrm{Aut}(\mathbb{A})_{(S \cup X)}$ which sends $b_i^{(k)}$ to a fresh atom disjoint from $S \cup X \cup Y \cup \{b_i^{(k)}\}$ that is unrelated to everything in Y. Observe that the new family

$$\mathcal{O}(v) \cup \{(x_\bullet^{(a_{j'},k)}; a_{j'}), (y_\bullet^{(a_{j'},k)}; a_{j'}) \mid k \in K_{j'}, 1 \le j' < j\}$$
$$\cup \{(\tau \cdot x_\bullet^{(a_j,k)}; a_j), (\tau \cdot y_\bullet^{(a_j,k)}; a_j) \mid k \in K_j\},$$

remains unambiguous. Moreover, the obstructions here are precisely the ones from the old family minus $b_k^{(i)}$, which are all fixed by τ. So we may repeat this process until all obstructions are removed. □

Having chosen the fresh atoms carefully, let X consist of all the atoms appearing in the unobstructed family $\mathcal{O}(v) \cup \{(x_\bullet^{(a_{j'},k)}; a_{j'}), (y^{(a_{j'},k)\bullet}; a_{j'}) \mid k \in K_{j'}, 1 \le j' \le m\}$. The trick now is to add a new element b^* to the finite $\mathcal{L}$-structure $S \cup X \subseteq \mathbb{A}$. We define the new relations in $S \cup X \cup \{b^*\}$ by

$$\begin{cases} \cdots \models R_1(b^*) \iff \mathbb{A} \models R_1(\sqrt{b^*}) \\ \cdots \models R_2(b^*, s) \iff \mathbb{A} \models R_2(\sqrt{b^*}, \sqrt{s}) \\ \cdots \models R_2(s, b^*) \iff \mathbb{A} \models R_2(\sqrt{s}, \sqrt{b^*}) \\ \cdots \models R_2(b^*, b_i^{(k)}) \iff \mathbb{A} \models R_2(\sqrt{b^*}, \sqrt{b_i^{(k)}}) \\ \cdots \models R_2(b_i^{(k)}, b^*) \iff \mathbb{A} \models R_2(\sqrt{b_i^{(k)}}, \sqrt{b^*}) \\ \cdots \models b^* > s \iff \mathbb{A} \models \sqrt{b^*} > \sqrt{s} \\ \cdots \models b^* > b_i^{(k)} \end{cases}$$

for every unary relation R_1 and binary relation R_2 in $\mathcal{L}_0$, where we extend $\sqrt{-} : X \to \{o_1, \ldots, o_d\}$ to

$$\sqrt{-} : S \cup X \cup \{b^*\} \to \{o_1, \ldots, o_d\} \cup S$$

by letting $\sqrt{b_i^{(k)}} = o_i$ as before and $\sqrt{s} = s$, $\sqrt{b^*} = o_{i^*}$ in addition.

So far we only relied on the unambiguity of the family; next we use the full assumption that it is unobstructed:

Claim IV.19. *$S \cup X \cup \{b^*\}$ embeds into $\mathbb{A}$.*

Proof. If not, by Lemma IV.4 there is a forbidden $\mathcal{L}_0$-structure F of pairwise related elements which embeds into the $\mathcal{L}_0$-reduct of $S \cup X \cup \{b^*\}$, via ϕ say. We will check that $\sqrt{-} \circ \phi$ then embeds F into the $\mathcal{L}_0$-reduct of $\mathbb{A}$, namely $\mathbb{A}_0$, and reach a contradiction.

So let $R_1 \in \mathcal{L}_0$ be unary and let $R_2 \in \mathcal{L}_0$ be binary. It is immediate that $\sqrt{-}$ preserves and reflects R_1 for $s \in S$, for $b_i^{(k)} \in X$, and for b^* — we do not even need to know they are in the image of ϕ. We can also see that $\sqrt{-}$ preserves and reflects R_2 whenever at least one of the arguments is from $S \cup \{b^*\}$. Now consider the remaining case where $b_i^{(k)} = \phi(f)$ and $b_{i'}^{(k')} = \phi(f')$. As f and f' are related in F, their images $b_i^{(k)}$ and $b_{i'}^{(k')}$ must be related in $\phi(F)$. But we ensured that the family giving rise to X is unobstructed, which forces $\sqrt{-}$ to preserve and reflect R_2 here as well. Since $\mathcal{L}_0$ only has unary and binary relations, this means that $\sqrt{\phi(F)} \subseteq \mathbb{A}_0$ is in $\mathcal{C}_0$ yet embeds F; this is obviously impossible. □

Using homogeneity we may assume that $S \cup X \cup \{b^*\} \subseteq \mathbb{A}$.

Claim IV.20. *$x_\bullet^{(a_j,k)}; a_j \parallel y_\bullet^{(a_j,k)}; b^*$ forms an $\mathcal{O}$-cog duo for $1 \le j \le m$ and $k \in K_j$. Furthermore, the family*

$$\mathcal{O}(v) \cup \{(x_\bullet^{(a_j,k)}; a_j), (y_\bullet^{(a_j,k)}; b^*) \mid k \in K_j, 1 \le j \le m\}$$

is unobstructed.

Proof. Recall that $x_\bullet^{(a_j,k)} \parallel y_\bullet^{(a_j,k)}$ is a cog duo in $\mathcal{O}^{i^*:a_j}|^{-i^*}$. Given any $o_\bullet \in \mathcal{O}$, it remains to show that

$$\mathbb{A} \models R_1(b^*) \leftrightarrow R_1(o_{i^*})$$

for any unary $R_1 \in \mathcal{L}_0$ and that

$$\begin{aligned} \mathbb{A} \models & R_2(b^*, s) \leftrightarrow R_2(o_{i^*}, s) \\ & \wedge R_2(s, b^*) \leftrightarrow R_2(s, o_{i^*}) \\ & \wedge R_2(b^*, b_i^{(k)}) \leftrightarrow R_2(o_{i^*}, o_i) \\ & \wedge R_2(b_i^{(k)}, b^*) \leftrightarrow R_2(o_i, o_{i^*}) \end{aligned}$$

for any binary $R_2 \in \mathcal{L}_0$, where $b_\bullet^{(k)} \in \{x_\bullet^{(a_j,k)}, y_\bullet^{(a_j,k)}\}$ and $s \in S$ are arbitrary.

Similarly, to show that the family in the claim is unambiguous and unobstructed, it is enough to check pairs of the form $b^*, b_i^{(k)}$ with $b_i^{(k)} \in X$. And indeed $b^* \mapsto o_{i^*}, b_i^{(k)} \mapsto o_i$ is always a function, since we chose b^* to be a new element. This is furthermore an embedding if $i \neq i^*$ because of the relations we imposed on b^*; otherwise b^* is unrelated to $b_i^{(k)}$ by irreflexivity, as required. □

Claim IV.21. *We have $v = \sum_{j=1}^m \sum_{k \in K_j} v(b_\bullet^{(a_j,k)}; a_j) \cdot (x_\bullet^{(a_j,k)}; a_j \between y_\bullet^{(a_j,k)}; b^*)$.*

Proof. Observe that

$$\begin{aligned} & \sum_{j=1}^m \sum_{k \in K_j} v(b_\bullet^{(a_j,k)}) \cdot (x_\bullet^{(a_j,k)}; a_j \between y_\bullet^{(a_j,k)}; b^*) \\ &= \sum_{j=1}^m \sum_{k \in K_j} \left(v(b_\bullet^{(a_j,k)}) \cdot x_\bullet^{(a_j,k)} \between y_\bullet^{(a_j,k)} \right); a_j \\ & \quad - \sum_{j=1}^m \sum_{k \in K_j} \left(v(b_\bullet^{(a_j,k)}) \cdot x_\bullet^{(a_j,k)} \between y_\bullet^{(a_j,k)} \right); b^* \\ &= \sum_{j=1}^m v^{i^*:a_j} - \sum_{j=1}^m v^{i^*:a_j}|^{-i^*}; b^* \\ &= v - \sum_{j=1}^m v^{i^*:a_j}|^{-i^*}; b^*, \end{aligned}$$

so it suffices to show the last sum vanishes. Recall from Proposition IV.10 that cogs are balanced. By projecting away the i^*th entry, we obtain

$$0 = 0 - \left(\sum_{j=1}^m v^{i^*:a_j}|^{-i^*}; b^* \right) \Bigg|^{-i^*}.$$

But we can simply add back b^* as the i^*th entry, yielding $0 = \sum_{j=1}^m v^{i^*:a_j}|^{-i^*}; b^*$ as needed — replacing the i^*th entry by a common atom achieves the same effect of projecting it away. □

This completes the proof of Theorem IV.17: we have shown any balanced vector v is spanned by cogs as long as $\mathcal{O}(v)$ is unambiguous and unobstructed. Let us lift this restriction.

E. Removing ambiguities and obstructions

Theorem IV.22. *Let $v \in \mathrm{Ker}_{\mathfrak{E}} \mathcal{O}$ be arbitrary. Then (assuming as before that $\mathcal{L}_0$ is at most binary) we have*

$$v = \sum_{k \in K} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

for some $b_\bullet^{(k)}, x_\bullet^{(k)}, y_\bullet^{(k)} \in \mathcal{O}$.

As a reminder, $\mathcal{O} \subseteq \mathbb{A}^I$ is an S-rigid orbit and $\mathfrak{E}$ is a finite-dimensional $\mathfrak{F}$-vector space. We induct on $|I|$, noting that when $I = \emptyset$ we are just saying $v = v() \cdot (\between)$. Hereafter assume $I \neq \emptyset$.

Proposition IV.23. *Let $v \in \mathrm{Ker}_{\mathfrak{E}} \mathcal{O}$. Then we can find $b_\bullet^{(k)}, x_\bullet^{(k)}, y_\bullet^{(k)} \in \mathcal{O}$ so that*

$$\ddot{v} = v - \sum_{k \in K_1} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

satisfies $\ddot{v}(\mathcal{O}) = v(\mathcal{O}) \subseteq \mathfrak{E}$ and $\mathcal{O}(\ddot{v})$ is unambiguous.

Proof. Assume there is an ambiguity in $\mathcal{O}(v)$, i.e., there are $a_\bullet, a'_\bullet \in \mathcal{O}(v)$ such that $a_i = a'_{i'}$ but $i \neq i'$ (so $|I| \ge 2$). Let us prevent the atom-index pair (a_i, i) from causing ambiguities. By Lemma IV.15, the projected subvector $v^{i:a_i}|^{-i}$ belongs to $\mathrm{Ker}_{\mathfrak{E}} \mathcal{O}^{i:a_i}|^{-i}$; by the inductive hypothesis of Theorem IV.22 at hand, we have

$$v^{i:a_i} = v^{i:a_i}|^{-i}; a_i = \sum_{k \in K} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); a_i.$$

We can invoke Lemma IV.3 to find an automorphism τ which fixes all of the atoms except a_i that appear in $x_\bullet^{(k)}, y_\bullet^{(k)}, k \in K$ and in $b_\bullet \in \mathcal{O}(v)$. It follows from the appropriate modification of Lemma IV.11 that we get $\mathcal{O}$-cog duos $x_\bullet^{(k)}; a_i \parallel y_\bullet^{(k)}; \tau \cdot a_i$ for $k \in K$. Observe that

$$\begin{aligned} & v^{i:a_i} - \sum_{k \in K} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)}; a_i \between y_\bullet^{(k)}; \tau \cdot a_i) \\ &= v^{i:a_i} - \sum_{k \in K} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); a_i \\ & \quad + \sum_{k \in K_1} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); \tau \cdot a_i \end{aligned}$$

is equal to $v^{i:a_i}|^{-i}; \tau \cdot a_i$, so

$$\begin{aligned} v' &= v - \sum_{k \in K} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)}; \tau \cdot a_i \between y_\bullet^{(k)}; a_i) \\ &= v - v^{i:a_i} + v^{i:a_i}|^{-i}; \tau \cdot a_i \end{aligned}$$

is the vector obtained by changing the ith entry in every tuple of v from a_i to $\tau \cdot a_i$, a fresh atom disjoint from all atoms present. Then $v'(\mathcal{O}) = v(\mathcal{O})$. Moreover, we can directly check that an ambiguous atom-index pair (b_j, j) in $\mathcal{O}(v')$ cannot be $(\tau \cdot a_i, i)$, and hence it must already be one in $\mathcal{O}(v)$ except it cannot be (a_i, i). Therefore we may remove all ambiguities by iterating this process. $\square$

Next we tackle the obstructions.

Proposition IV.24. *Let $\ddot{v} \in \operatorname{Ker}_{\mathfrak{E}} \mathcal{O}$ be such that $\mathcal{O}(\ddot{v})$ is unambiguous. Then we can find $b_\bullet^{(k)}, x_\bullet^{(k)}, y_\bullet^{(k)} \in \mathcal{O}$ so that*

$$\acute{\ddot{v}} = \ddot{v} - \sum_{k \in K_2} \ddot{v}(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

satisfies $\acute{\ddot{v}}(\mathcal{O}) = \ddot{v}(\mathcal{O})$ and $\mathcal{O}(\acute{\ddot{v}})$ is unobstructed.

Proof. We follow the same strategy: let the atom a_i be an obstruction with some other $a'_{i'}$ in $\mathcal{O}(\ddot{v})$. Consider the projected subvector $\ddot{v}^{i:a_i}|^{-i}$. By Lemma IV.15 and the inductive hypothesis of Theorem IV.22, we can write

$$\ddot{v}^{i:a_i} = \sum_{k \in K} \ddot{v}(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); a_i.$$

Leveraging no algebraicity like we did at the beginning of the proof of Claim IV.18, we may assume that $\mathcal{O}(\ddot{v}) \cup \{(x_\bullet^{(k)}; a_i), (y_\bullet^{(k)}; a_i) \mid k \in K\}$ is unambiguous.

Split the atoms except a_i appearing in that family into two parts X and Y, where Y consists of all the atoms equal to some $a'_{i'}$ that make a_i an obstruction. Then $x_{i''}^{(k)}$, $y_{i''}^{(k)}$ cannot belong to Y for any $i'' \in I$. Indeed, if say $x_{i''}^{(k)}$ is equal to an obstructive atom $a'_{i'}$, we must have $i'' = i'$ by unambiguity; we now have a contradiction: $a'_{i'} = x_{i''}^{(k)}$ and a_i necessarily satisfy the right binary relations, since $x_\bullet^{(k)}; a_i \in \mathcal{O}$.

Next, we invoke the Lemma IV.3 to find an automorphism $\tau \cdot \operatorname{Aut}(\mathbb{A})$ making $\tau \cdot a_i$ greater than a_i, distinct from any $x \in X$, and unrelated to any $y \in Y$. By Lemma IV.11, we have $\mathcal{O}$-cog duos $x_\bullet^{(k)}; a_i \parallel y_\bullet^{(k)}; \tau \cdot a_i$ for $k \in K$. The refined vector

$$\begin{aligned}\ddot{v}' &= \ddot{v} - \sum_{k \in K} \ddot{v}(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)}; \tau \cdot a_i \between y_\bullet^{(k)}; a_i) \\ &= \ddot{v} - \ddot{v}^{i:a_i} + \ddot{v}^{i:\tau \cdot a_i}\end{aligned}$$

satisfies $\ddot{v}'(\mathcal{O}) = \ddot{v}(\mathcal{O})$. As we are just changing a_i to the fresh atom $\tau \cdot a_i$ in the ith entry of every tuple appearing in $\ddot{v}$, the family $\mathcal{O}(\ddot{v}')$ remains unambiguous; note also that a_i can only appear in the ith entry as $\mathcal{O}(\ddot{v})$ is unambiguous, so we have completely removed a_i from $\ddot{v}'$. But $\tau \cdot a_i$ by design cannot cause an obstruction in $\mathcal{O}(\ddot{v}')$, and consequently any atom causing obstructions in $\mathcal{O}(\ddot{v}')$ must already do so in $\mathcal{O}(\ddot{v})$ except that it cannot be a_i. Hence we can repeat this procedure until all obstructions are excised. $\square$

We are at last in a position to prove the inductive step of Theorem IV.22. Given $v \in \operatorname{Ker}_{\mathfrak{E}} \mathcal{O}$, we apply Propositions IV.23 and IV.24 to obtain

$$\acute{\ddot{v}} = v - \sum_{k \in K_1 \cup K_2} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

where $\acute{\ddot{v}}(\mathcal{O}) = \ddot{v}(\mathcal{O}) = v(\mathcal{O})$ and $\mathcal{O}(\acute{\ddot{v}})$ is unobstructed. Recalling Proposition IV.10, we see that $\acute{\ddot{v}} \in \operatorname{Ker}_{\mathfrak{E}} \mathcal{O}$. The special Theorem IV.17 tells us that $\acute{\ddot{v}}$ is a sum of $\mathcal{O}$-cogs with coefficients from $\acute{\ddot{v}}(\mathcal{O}) = v(\mathcal{O})$. It follows that so is

$$v = \acute{\ddot{v}} + \sum_{k \in K_1 \cup K_2} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)},$$

which establishes the general Theorem IV.22.

F. All those equivariant subspaces

We finish this section with an important corollary of Theorem IV.22. Let $\mathcal{O}_1 \subseteq \mathbb{A}^{I_1}, \ldots, \mathcal{O}_m \subseteq \mathbb{A}^{I_n}$ all be S-rigid orbits. Then $\operatorname{len}(\operatorname{Lin}_{\mathfrak{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)) = 2^{|I_1|} + \cdots + 2^{|I_n|}$; in fact, we know and can characterise all the $\operatorname{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces of $\operatorname{Lin}_{\mathfrak{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)$.

First we set up some notations. Consider the $\sum_k 2^{|I_k|}$ projected S-rigid orbits $\mathcal{O}_k|^J$ for $1 \leq k \leq n, J \subseteq I_k$. Suppose

$$f : \mathcal{O}_k|^J \to \mathcal{O}_{k'}|^{J'}$$

is an $\operatorname{Aut}(\mathbb{A})_{(S)}$-equivariant bijection. Take any $o_\bullet \in \mathcal{O}_k|^J$, and enumerate its entries as $o_1 < \cdots < o_{|J|}$. Similarly, enumerate the entries of $f(o_\bullet)$ as $o'_1 < \cdots < o'_{|J'|}$. Then $\{o_1, \ldots, o_{|J|}\} = \{o'_1, \ldots, o'_{|J'|}\}$ because $\mathbb{A}$ has no algebraicity; since the orbits are rigid, we must have $|J| = |J'|$ and $o_1 = o'_1, \ldots, o_{|J|} = o'_{|J'|}$. That is, f must be the obvious function that reindexes a J-tuple to a J'-tuple — and hence we will just write $\breve{o}_\bullet$ for $f(o_\bullet)$, making f implicit.

Now, let $\mathcal{Q}_1 = \mathcal{O}_{k_1}|^{J_1}, \ldots, \mathcal{Q}_t = \mathcal{O}_{k_t}|^{J_t}$ be the distinct S-rigid orbits up to $\operatorname{Aut}(\mathbb{A})_{(S)}$-equivariant bijections, which we enumerate in such a way that $|J_1| \geq |J_2| \geq \cdots \geq |J_t|$.

Definition IV.25. For $i = 1, \ldots, t$, let P_i consist of pairs (k, J) such that $\mathcal{O}_k|^J$ is $\operatorname{Aut}(\mathbb{A})_{(S)}$-equivariantly isomorphic to $\mathcal{Q}_i$. Assemble all $|P_i|$ projections into a single map

$$(-)\restriction_i : \operatorname{Lin}_{\mathfrak{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n) \to \operatorname{Lin}_{\mathfrak{F}^{P_i}} \mathcal{Q}_i.$$

More precisely, $(v_1, \ldots, v_n)\restriction_i(a_\bullet)$ is the P_i-tuple whose entry at (k, J) is $v_k|^J(\breve{a}_\bullet) \in \mathfrak{F}$. It is straightforward to check that $(-)\restriction_i$ is $\operatorname{Aut}(\mathbb{A})_{(S)}$-equivariant and linear.

Let $W \subseteq \operatorname{Lin}_{\mathfrak{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)$ be an $\operatorname{Aut}(\mathbb{A})_{(S)}$-equivariant subspace. Using the t finite-dimensional vector spaces $W\restriction_1(\mathcal{Q}_1) \subseteq \mathfrak{F}^{P_1}, \ldots, W\restriction_t(\mathcal{Q}_t) \subseteq \mathfrak{F}^{P_t}$ we define $\widetilde{W}$, which consists of all vectors $v \in \operatorname{Lin}_{\mathfrak{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)$ such that

$$v\restriction_1(\mathcal{Q}_1) \subseteq W\restriction_1(\mathcal{Q}_1), \ldots, v\restriction_t(\mathcal{Q}_t) \subseteq W\restriction_t(\mathcal{Q}_t).$$

Then $\widetilde{W}$ is an $\operatorname{Aut}(\mathbb{A})_{(S)}$-equivariant subspace that contains W. It turns out these two are equal:

Theorem IV.26. *$\widetilde{W} \subseteq W$.*

The key is to consider the following refinement. Let $K_i \subseteq \operatorname{Lin}_{\mathfrak{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)$ consist of vectors v where

$$v\restriction_{i+1} = 0, \ldots, v\restriction_t = 0.$$

Then K_0 is the zero space, whilst K_t is the whole space.

Claim IV.27. $\widetilde{W} \cap K_0 \subseteq W \cap K_0$.

Claim IV.28. $(\widetilde{W} \cap K_{i+1})\restriction_{i+1} \subseteq \mathrm{Ker}_{W\restriction_{i+1}(\mathcal{Q}_{i+1})}\, \mathcal{Q}_{i+1}$.

$v \in \widetilde{W}$ but $v\restriction_1 = 0, \ldots, v\restriction_i = 0$

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