

A Few More Vector Spaces with Atoms of Finite Lengths

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Abstract

An infinite structure has, given a field, the finite length property if the following holds for each of its finite powers: in the corresponding free vector space, strict chains of equivariant subspaces are bounded in length. It has been shown that the countable pure set and the dense linear ordering without endpoints have this property. In this paper, we generalise these two results to a) interpretations in smoothly approximable structures, provided the field has characteristic zero, and b) generically ordered expansions of Fraïssé limits with free amalgamation over a graph language. As a special case, we prove the finite length property of the Rado graph using both methods.

CCS Concepts

- **Do Not Use This Code → Generate the Correct Terms for Your Paper;** *Generate the Correct Terms for Your Paper; Generate the Correct Terms for Your Paper; Generate the Correct Terms for Your Paper.*

Keywords

Do, Not, Use, This, Code, Put, the, Correct, Terms, for, Your, Paper

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1 Introduction

2 Preliminaries

Definitions:

- (1) oligomorphic
- (2) homogeneous
- (3) smooth approximation by homogeneous substructures [3]
(N.B. not the 'smooth approximation' from [4, Definitino 4])
- (4) *oligomorphic approximation* of a homogeneous structure by finite substructures with uniformly few orbits (i.e., types) that cover the age of \mathbb{A}

Definition 2.1. An interpretation in \mathbb{A} is a structure $\mathbb{A}' = (D/E; R_1, R_2, \dots)$, where

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- D is an equivariant subset of \mathbb{A}^n for some $n \geq 1$;
- E is an equivariant equivalence relation on D ;
- D/E consists of equivalence classes $d/E \subseteq D$ for $d \in D$, which $\pi \in \text{Aut}(\mathbb{A})$ acts on via $\pi \cdot d/E = (\pi \cdot d)/E$;
- every relation R_i of arity r_i is an equivariant subset of $(D/E)^{r_i}$.

We call \mathbb{A}' a *reduct* of \mathbb{A} if $D = \mathbb{A}$ and E is just equality.

PROPOSITION 2.2. If \mathbb{A} has the finite length property over F , then so does any interpretation \mathbb{A}' .

PROOF. Let $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_l$ be a chain of $\text{Aut}(\mathbb{A}')$ -equivariant subspaces in $\text{Lin}_{\mathsf{F}}(\mathbb{A}')^k = \text{Lin}_{\mathsf{F}}(D/E)^k$. Each V_j is then $\text{Aut}(\mathbb{A})$ -equivariant: given $\pi \in \text{Aut}(\mathbb{A})$, notice that $d/E \mapsto (\pi \cdot d)/E$ is an automorphism of \mathbb{A}' . Now l is bounded above by the $\text{Aut}(\mathbb{A})$ -length of $\text{Lin}_{\mathsf{F}}(D/E)^k$, which is finite – $\text{Lin}_{\mathsf{F}}(D/E)^k$ is isomorphic to the quotient of $\text{Lin}_{\mathsf{F}}D^k$ by the span of

$$\left\{ \begin{array}{l} (d_1, \dots, d_k) \\ -(d'_1, \dots, d'_k) \end{array} \middle| \begin{array}{l} d_1/E = d'_1/E, \dots, d_k/E = d'_k/E \end{array} \right\}.$$

□

3 Rado graph, sans cogs

In this section, we establish a method for proving the finite length property, assuming that the field has characteristic zero. Using this method, we will establish the finite length property for the Rado graph, and for vector spaces over a finite field, under the assumption of characteristic zero. These are new results. Also, we think that the proof itself, even when applied to get already known results, is of independent interest and arguably simpler than previous proofs.

Weak smooth approximation.

Definition 3.1 (Weak smooth approximation). We say that a structure \mathbb{A} has *weak smooth approximation* if for every $d \in \{1, 2, \dots\}$ there exists a family \mathcal{B} of finite substructures of \mathbb{A} such that:

- (1) every finite substructure of \mathbb{A} embeds into some $\mathbb{B} \in \mathcal{B}$; and
- (2) there is a common finite upper bound on the number of orbits in \mathbb{B}^d for $\mathbb{B} \in \mathcal{B}$.

The above definition is a relaxation of a stronger notion from model theory that is called *smooth approximation*. In the stronger notion, the family \mathcal{B} is independent of d , and there are other requirements as well [3, p. 440].

Example 3.2 (Equality atoms). The equality atoms are oligomorphically approximable. In this case, we can choose \mathcal{B} so that is independent of d , and this is simply all finite structures with equality only. For sufficiently large $\mathbb{B} \in \mathcal{B}$, the number of orbits in \mathbb{B}^d is the same as the number of orbits in \mathbb{A}^d .

Example 3.3 (Bit-vector atoms). Consider the bit-vector atoms. This is the vector space over the two-element field of countably infinite dimension, which is viewed as a structure by having a

binary function for addition of vectors. As in the previous example, we can choose \mathcal{B} independently of d , namely the family of vector spaces of finite dimension. If we changed the field to any finite field, the same argument would apply.

Example 3.4 (Non-example: rational numbers with order). A non-example is the rational number with the usual order. The finite substructures in this case are finite linear orders, and already for dimension $d = 1$, a finite linear order of size n will have n orbits. Hence, we cannot have a common finite upper bound on the number of orbits in \mathbb{B}^d for $\mathbb{B} \in \mathcal{B}$.

Example 3.5 (Rado graph). Consider the Rado graph. For every $n \in \{1, 2, \dots\}$ define a finite graph as follows. The set of vertices is 2^{2n} , which is seen as a vector space over the two-element field whose basis is

$$\{e_1, \dots, e_n, f_1, \dots, f_n\}.$$

Since the field has two elements, we can view vertices as subsets of the above basis. In this graph, there is an edge vw if and only if the sets

$$\begin{aligned} &\{i \in \{1, \dots, n\} \mid e_i \in v \text{ and } f_i \in w\} \\ &\{i \in \{1, \dots, n\} \mid f_i \in v \text{ and } e_i \in w\} \end{aligned}$$

have different sizes modulo two. This graph might seem a bit artificial, it is in fact a well-known construction, which is called the *symplectic graph* over the two-element field; see e.g. [BCP97]. These graphs are placed in more context in the appendix, where we also prove that they satisfy the necessary properties from Definition 3.1. actually

The main result of this section is the following theorem.

THEOREM 3.6. *If an oligomorphic structure \mathbb{A} has weak smooth approximation, then it has the finite length property over any field of characteristic 0.*

Before proving the theorem, let us apply it to prove some results – old and new – on the finite length property.

COROLLARY 3.7. *Over any field of characteristic 0, the following structures have the finite length property: (a) the equality atoms; (b) the bit-vector atoms; and (c) the Rado graph.*

PROOF. In Examples 3.2, 3.3, and 3.5 we have established that these structures satisfy the assumptions of Theorem 3.6. \square

As mentioned in the introduction, the finite length property was already known for the equality atoms, for arbitrary fields. The results for the bit-vector atoms and the Rado graph are new. The assumption on characteristic zero is important, at least in case of the bit-vector atoms, where the finite length property is known to fail over finite fields [1, Section 4.4]. Later on in this paper, we will prove the result for the Rado graph again using a different method that works for any field.

The rest of this section is devoted to proving Theorem 3.6.

PROOF. Copy the ‘bojań-trick’ from §8.2 of Mikołaj’s <https://www.mimuw.edu.pl/~bojan/papers/notes-July3.pdf>. \square

COROLLARY 3.8. *Also for m cliques of n vertices – interpretable in the equality atoms.*

3.1 Symplectic vector spaces

Throughout this subsection let \mathbf{f} denote a finite field.

Definition 3.9. A *symplectic vector space* is an \mathbf{f} -vector space \mathbb{W} equipped with a bilinear form $\omega : \mathbb{W} \times \mathbb{W} \rightarrow \mathbf{f}$ that is

- (1) alternating: $\omega(v, v) = 0$ for all v ; and
- (2) non-degenerate: if $\omega(v, w) = 0$ for all w then $v = 0$.

Example 3.10. Let \mathbb{W}_n be the \mathbf{f} -vector space with basis $e_1, \dots, e_n, f_1, \dots, f_n$. Define ω by bilinearly extending

$$\omega(e_i, f_i) = 1 = -\omega(f_i, e_i), \quad \omega(-, *) = 0 \text{ elsewhere}; \quad (\$)$$

one may straightforwardly check that ω is alternating and non-degenerate. Moreover, noticing that $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \dots$, we obtain a countable-dimensional symplectic vector space $\mathbb{W}_\infty = \bigcup_n \mathbb{W}_n$.

We will refer to vectors satisfying $(\$)$ as a *symplectic basis*. Note such vectors must be linearly independent: if $v = \sum_i \lambda_i e_i + \mu_i f_i = 0$, then $\lambda_i = \omega(v, f_i) = 0$ and $\mu_i = \omega(e_i, v) = 0$ for each i . Such bases behave very much like the usual bases.

PROPOSITION 3.11. *Assume that \mathbb{W} is a symplectic vector space that is at most countable. Then any finite symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$ can be extended to a symplectic basis that spans the whole \mathbb{W} .*

PROOF. Suppose that $e_1, \dots, e_n, f_1, \dots, f_n$ does not already span \mathbb{W} ; take v to be a witness (that is least according to some fixed enumeration of \mathbb{W} in the case it is infinite). Put

$$e_{n+1} = v - \sum_{i=1}^n \omega(e_i, v) f_i + \sum_{i=1}^n \omega(f_i, v) e_i$$

so that $\omega(e_i, e_{n+1}) = 0 = \omega(f_i, e_{n+1})$. This cannot be the zero vector lest we contradict the choice of v . By the non-degeneracy of ω , there is – rescaling if necessary – some w such that $\omega(e_{n+1}, w) = 1$. Now define

$$f_{n+1} = w - \sum_{i=1}^n \omega(e_i, w) f_i + \sum_{i=1}^n \omega(f_i, w) e_i$$

in a similar manner, making $e_1, \dots, e_n, e_{n+1}, f_1, \dots, f_n, f_{n+1}$ a symplectic basis that spans v . We go through every element of \mathbb{W} by continuing this way. \square

In fact, we will also make use of the “symplectic basis and a half” variant below.

PROPOSITION 3.12. *Now assume \mathbb{W} is a finite-dimensional symplectic vector space. Let*

$$\begin{aligned} &e_1, \dots, e_n, e_{n+1}, \dots, e_{n+k}, \\ &f_1, \dots, f_n, f_{n+1}, \dots, f_{n+i} \end{aligned}$$

be linearly independent vectors satisfying $(\$)$. Then we can find the missing f_{n+1}, \dots, f_{n+k} to complete the symplectic basis.

PROOF. Suppose we have found f_{n+1}, \dots, f_{n+i} already such that

$$\begin{aligned} &e_1, \dots, e_n, e_{n+1}, \dots, e_{n+i}, e_{n+i+1}, e_{n+i+2}, \dots, e_{n+k}, \\ &f_1, \dots, f_n, f_{n+1}, \dots, f_{n+i} \end{aligned}$$

satisfy $(\$)$. Notice these vectors are linearly independent: in a linear combination that sums to 0, the coefficients of $e_1, f_1, \dots, e_{n+i}, f_{n+i}$ must be zero, and we assumed the linear independence of $e_{n+i+1}, \dots, e_{n+k}$.

By extending these to a basis B of \mathbb{W} , we may define a linear function

$$\psi : \mathbb{W} \rightarrow \mathbf{f}$$

which sends e_{n+i+1} to 1 but every other $b \in B$ to 0. Now apply Proposition 3.11 to obtain a symplectic basis $e'_1, f'_1, \dots, e'_m, f'_m$ of \mathbb{W} , and put

$$f_{n+i+1} = \sum_{j=1}^m \psi(e'_j) f'_j - \psi(f'_j) e'_j;$$

then $\omega(-, f_{n+i+1})$ agrees with ψ on this symplectic basis, so by linearity they must be the same function. In particular

$$\omega(e_{n+i+1}, f_{n+i+1}) = \psi(e_{n+i+1}) = 1,$$

whereas $\psi(e_1), \dots, \psi(e_{n+k}), \psi(f_1), \dots, \psi(f_{n+i})$ are all 0. Thus we have (§) as required. \square

Given two symplectic vector spaces \mathbb{W} and \mathbb{W}' , we call a function α between $X \subseteq \mathbb{W}$ and $X' \subseteq \mathbb{W}'$ *isometric* if $\omega(\alpha(x_1), \alpha(x_2)) = \omega(x_1, x_2)$ for all $x_1, x_2 \in X$. We can make an easy observation:

LEMMA 3.13. *Let $\{e_i, f_i \mid i \in I\} \subseteq \mathbb{W}$, $\{e'_j, f'_j \mid j \in J\} \subseteq \mathbb{W}'$ be two symplectic bases and let $\alpha : I \rightarrow J$ be a bijection. Then*

$$e_i \mapsto e'_{\alpha(i)}, f_i \mapsto f'_{\alpha(i)}$$

defines an isometric linear bijection $\langle e_i, f_i \rangle \rightarrow \langle e'_j, f'_j \rangle$.

It then follows from Proposition 3.11 that, up to isometric linear bijections, $\mathbb{W}_0, \mathbb{W}_1, \mathbb{W}_2, \dots, \mathbb{W}_\infty$ are all the countable symplectic vector spaces. Whilst we may deduce that \mathbb{W}_∞ is oligomorphic by appealing to Ryll-Nardzewski, we will opt for a more direct proof that also establishes smooth approximation.

PROPOSITION 3.14 (WITT EXTENSION). *Any isometric linear injection $\alpha : \langle X \rangle \subseteq \mathbb{W}_n \rightarrow \mathbb{W}_n$ can be extended to an automorphism of \mathbb{W}_n (i.e., an isometric linear bijection) and in turn to one of \mathbb{W}_∞ .*

PROOF. To begin with, find a basis x_1, \dots, x_k for the subspace $W = \{w \in \langle X \rangle \mid \forall x \in X : \omega(w, x) = 0\}$ and extend it to a basis $x_1, \dots, x_k, x_{k+1}, \dots, x_d$ for $\langle X \rangle$. Notice that

$$U = \langle x_{k+1}, \dots, x_d \rangle$$

must be a symplectic subspace: as it intersects with W trivially, given any non-zero vector $u \in U$ we must have $0 \neq \omega(u, w + u') = \omega(u, u')$ for some $w \in W$ and $u' \in U$. Hence use Proposition 3.11 to find a symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$ for U . Observe that

$$\begin{aligned} &e_1, \dots, e_n, x_1, \dots, x_k, \\ &f_1, \dots, f_n \end{aligned}$$

form a basis for $\langle X \rangle$ and satisfy (§). On the other hand,

$$\begin{aligned} &\alpha(e_1), \dots, \alpha(e_n), \alpha(x_1), \dots, \alpha(x_k), \\ &\alpha(f_1), \dots, \alpha(f_n) \end{aligned}$$

form a basis for $\alpha(\langle X \rangle)$ and also satisfy (§). Therefore apply Proposition 3.12 twice to find the missing y_1, \dots, y_k and y'_1, \dots, y'_k to complete the two symplectic bases – call them \mathcal{B} and \mathcal{B}' . They are of the same size.

Now, by using Proposition 3.11, extend \mathcal{B} and \mathcal{B}' to symplectic bases C and C' that span \mathbb{W}_n . These must both have size $2n$, so by Lemma 3.13 we obtain an isometric linear automorphism $\beta : \mathbb{W}_n \rightarrow \mathbb{W}_n$ extending α .

To finish, notice that $C, e_{n+1}, \dots, f_{n+1}, \dots$ as well as $C', e_{n+1}, \dots, f_{n+1}, \dots$ form a symplectic basis spanning \mathbb{W}_∞ . We obtain from Lemma 3.13 another time an isometric linear automorphism $\gamma : \mathbb{W}_\infty \rightarrow \mathbb{W}_\infty$ extending β that is the identity almost everywhere. \square

PROPOSITION 3.15. \mathbb{W}_∞^k has precisely $\sum_{d=0}^k \binom{k}{d}_q \cdot q^{\binom{d}{2}}$ orbits under $\text{Aut}(\mathbb{W}_\infty)$, where $q = |\mathbf{f}|$ and

$$\binom{k}{d}_q = \frac{(q^k - 1)(q^{k-1} - 1) \cdots (q^{k-d+1} - 1)}{(q^d - 1)(q^{d-1} - 1) \cdots (q^1 - 1)}$$

is the *q-binomial coefficient*.

Remark 3.16. To anticipate the next subsection, we note a similarity with the Rado graph: in \mathbb{G}^k there are $\sum_{d=0}^k \binom{k}{d} \cdot 2^{\binom{d}{2}}$ orbits – we may impose any edge relation on d vertices.

PROOF. To each $v_\bullet \in \mathbb{W}_\infty^k$ we associate a *type*, which comprises the following data:

- (1) pivot indices $I \subseteq \{1, \dots, k\}$ containing every i such that v_i is not spanned by v_1, \dots, v_{i-1} – so we inductively ensure that
- $$\{v_{i'} \mid i' \in I, i' \leq i\}$$

is a basis for $\langle v_1, \dots, v_i \rangle$;

- (2) for each $j \notin I$, an assignment $\Lambda_j : \{i \in I \mid i < j\} \rightarrow \mathbf{f}$ such that $v_j = \sum_{i \in I, i < j} \Lambda_j(i) v_i$;

- (3) a map $\Omega : \binom{I}{2} \rightarrow \mathbf{f}$ defined by $\Omega(\{i' < i\}) = \omega(v_{i'}, v_i)$.

If $\pi : \mathbb{W}_\infty \rightarrow \mathbb{W}_\infty$ is an isometric linear bijection, then $v_\bullet = (v_1, \dots, v_k)$ and $\pi \cdot v_\bullet = (\pi(v_1), \dots, \pi(v_k))$ evidently share the same type. Conversely, if w_\bullet has the type of v_\bullet , then

$$\begin{aligned} \alpha : \langle v_i \mid i \in I \rangle &\rightarrow \langle w_i \mid i \in I \rangle \subseteq \mathbb{W}_n \\ v_i &\mapsto w_i \end{aligned}$$

gives an isometric linear injection for some large enough n . Observe that α must send $v_j \mapsto w_j$ for $j \notin I$ too, and that it may be extended to an automorphism π of \mathbb{W}_∞ by Proposition 3.14. Furthermore we can find some v_\bullet that realises any given type $(I, \{\Lambda_j\}_j, \Omega)$: it suffices to put

$$v_i = e_i + \sum_{i' \in I, i' < i} \Omega(i', i) f_{i'}$$

for $i \in I$ and $v_j = \sum_{i \in I, i < j} \Lambda_j(i) v_i$ for $j \notin I$. Therefore the number of types is precisely the number of orbits in \mathbb{W}_∞^k .

Finally, we do some combinatorics. Fix $0 \leq d \leq k$ and count the number of types with $|I| = d$. There are $q^{\binom{d}{2}}$ choices for Ω and say $\#_{k,d}$ choices for the Λ_j 's; the two can be chosen independently. In total, this gives

$$\sum_{d=0}^k q^{\binom{d}{2}} \cdot \#_{k,d}$$

types for vectors in \mathbb{W}_∞^d . So focus on $\#_{k,d}$, the number of *linear types* – i.e., $(I, \{\Lambda_j\}_j)$, ignoring Ω – in \mathbb{W}_∞^k . (Incidentally $\sum_{d=0}^k \#_{k,d}$ is the number of orbits in \mathbb{W}_∞^k or, more generally, any countable-dimensional \mathbf{f} -vector space under linear automorphisms.) On the

small values we easily check that

$$\begin{aligned}\#_{0,0} &= 1 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right]_q, \\ \#_{1,0} &= 1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right]_q, \quad \#_{1,1} = 1 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right]_q.\end{aligned}$$

Given a linear type in \mathbb{W}_∞^k with $|I| = d$, we either have $1 \in I$ or $I \subseteq \{2, \dots, k\}$. In the first case, the linear type is specified by one of the $\#_{k-1,d-1}$ linear types in \mathbb{W}_∞^{k-1} together with how v_1 is involved in the span of the $(k-1)-(d-1)$ non-pivot vectors. In the second case, the linear type is simply one of the $\#_{k-1,d}$ linear types in \mathbb{W}_∞^{k-1} . Thus

$$\begin{aligned}\#_{k,d} &= q^{k-d} \cdot \#_{k-1,d-1} + \#_{k-1,d} \\ &= q^{k-d} \cdot \binom{k-1}{d-1}_q + \binom{k-1}{d}_q = \binom{k}{d}_q.\end{aligned}\quad \square$$

THEOREM 3.17. *The symplectic vector space \mathbb{W}_∞ is smoothly approximated by $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \dots$*

COROLLARY 3.18. *The symplectic f-vector space \mathbb{W}_∞ has the finite length property over any field of characteristic 0.*

3.2 Symplectic graphs

For this subsection let f be the two-element field f_2 .

Definition 3.19. For $n = 0, 1, 2, \dots$, the *symplectic graph* $\tilde{\mathbb{W}}_n$ has vertices \mathbb{W}_n and edges

$$v_1 \sim v_2 \iff \omega(v_1, v_2) = 1.$$

This is indeed an undirected graph: as ω is alternating, we have $\omega(v_1, v_2) = -\omega(v_2, v_1) = \omega(v_2, v_1)$ over f_2 .

PROPOSITION 3.20. $\text{Aut}(\tilde{\mathbb{W}}_n) = \text{Aut}(\mathbb{W}_n)$.

PROOF. Clearly any isometric linear automorphism of \mathbb{W}_n is a graph automorphism of $\tilde{\mathbb{W}}_n$. Conversely, any $f \in \tilde{\mathbb{W}}_n$ is evidently isometric. To show that f is linear, take $\lambda_1, \lambda_2 \in f$ and $v_1, v_2 \in \mathbb{W}_n$. We calculate:

$$\begin{aligned}&\omega\left(f\left(\sum_i \lambda_i v_i\right) - \sum_i \lambda_i f(v_i), f(w)\right) \\ &= \omega\left(f\left(\sum_i \lambda_i v_i\right), f(w)\right) - \sum_i \lambda_i \omega(f(v_i), f(w)) \\ &= \omega\left(\sum_i \lambda_i v_i, w\right) - \sum_i \lambda_i \omega(v_i, w) \\ &= \omega(0, w) = 0\end{aligned}$$

for all $f(w) \in f(\mathbb{W}_n) = \mathbb{W}_n$; since ω is non-degenerate, we conclude that $f(\sum_i \lambda_i v_i) = \sum_i \lambda_i f(v_i)$. \square

So the number of orbits in $\tilde{\mathbb{W}}_n^k$ is precisely equal to the number of orbits in \mathbb{W}_n^k – in particular, it is bounded above by $\sum_{d=0}^k \binom{k}{d}_2 \cdot 2^{\binom{d}{2}}$ independently of n by Proposition 3.15.¹ It remains to show $\tilde{\mathbb{W}}_0 \subseteq \tilde{\mathbb{W}}_1 \subseteq \tilde{\mathbb{W}}_2 \subseteq \dots$ embeds all finite graphs:

PROPOSITION 3.21 ([2, THEOREM 8.11.2]). *Every graph on at most $2n$ vertices embeds into $\tilde{\mathbb{W}}_n$.*

¹This is the k th term in the OEIS sequence A028631.

PROOF. Let G be a graph on at most $2n$ vertices. The conclusion is trivial when $n = 0$. Also, if G contains no edges, we can choose any $2n$ of the 2^n vectors in $\langle e_1, \dots, e_n \rangle \subseteq \tilde{\mathbb{W}}_n$.

So suppose $n \geq 1$ and G has an edge $s \sim t$. Let $G_{s,t}$ be the graph on vertices $G \setminus \{s, t\}$ with edges which we will specify later. By induction, some embedding $f : G_{s,t} \rightarrow \tilde{\mathbb{W}}_{n-1}$ exists. Define $f' : G \rightarrow \tilde{\mathbb{W}}_n$ by

$$\begin{aligned}x \in G_{s,t} &\mapsto f(x) - [\![x \sim s]\!] f_n + [\![x \sim t]\!] e_n \\ s &\mapsto e_n \\ t &\mapsto f_n\end{aligned}$$

where $[\![\phi]\!]$ is 1 if ϕ holds and 0 otherwise. Then we have $\omega(f'(x), f'(s)) = [\![x \sim s]\!]$ and $\omega(f'(x), f'(t)) = [\![x \sim t]\!]$ as desired, on one hand. On the other,

$$\begin{aligned}\omega(f'(x_1), f'(x_2)) &= [\![x_1 \sim x_2]\!] + [\![x_1 \sim s]\!] [\![x_2 \sim t]\!] \\ &\quad + [\![x_1 \sim t]\!] [\![x_2 \sim s]\!]\end{aligned}$$

tells us how we should define the edge relation in $G_{s,t}$ for f' to be an embedding of graphs. \square

THEOREM 3.22. *The Rado graph is oligomorphically approximated by $\tilde{\mathbb{W}}_0 \subseteq \tilde{\mathbb{W}}_1 \subseteq \tilde{\mathbb{W}}_2 \subseteq \dots$*

COROLLARY 3.23. *The Rado graph has the finite length property over any field of characteristic 0.*

This proof of finite length also applies to *oriented graphs* (i.e., $x \rightarrow y \implies y \not\rightarrow x$ but unlike in a tournament, it may occur that $x \not\rightarrow y \wedge y \not\rightarrow x$) – use the three-element field instead of f_2 .

4 Rado graph, with cogs

In this section we work with the following setting:

- \mathcal{L}_0 is a (**should we just assume finite?**) relational language consisting of unary and binary symbols;
- C_0 is a free amalgamation class of finite \mathcal{L}_0 -structures, where every $R \in \mathcal{L}_0$ is interpreted irreflexively.²
- \mathcal{L} consists of \mathcal{L}_0 together with a new binary symbol $<$;
- C consists of \mathcal{L} -structures obtained from C_0 by expanding with all possible linear orderings – this is still an amalgamation class;
- \mathbb{A}_0 and \mathbb{A} are the respective Fraïssé limits of C_0 and C , where without loss of generality (**because of the extension property**) we assume \mathbb{A}_0 and \mathbb{A} share the same domain so that $\text{Aut}(\mathbb{A}_0) \supseteq \text{Aut}(\mathbb{A})$.

Example 4.1. Take \mathcal{L}_0 to consist of $=$ only and C_0 to be all finite sets. Then \mathbb{A}_0 is isomorphic to the pure set \mathbb{N} , whereas \mathbb{A} is isomorphic to \mathbb{Q} with the usual order.

Example 4.2. Let \mathcal{L}_0 consist of $=$ together with a single binary symbol \sim and let C_0 consist of all finite undirected graphs not embedding the complete graph K_n , where $3 \leq n (\leq \infty)$. Then \mathbb{A}_0 is the K_n -free Henson graph (or the Rado graph when $n = \infty$), and \mathbb{A} is its generically ordered counterpart. (Allowing $n = 2$ makes these degenerate to \mathbb{N} and \mathbb{Q} above).

²We may assume irreflexivity with no loss of generality: see [5, beginning of §2.4].

Free amalgamation in C_0 allows us to free atoms in \mathbb{A} from undesired relations. Let us make this more precise. Say that atoms $a, b \in \mathbb{A}$ are *related* if $a = b$ or $\mathbb{A} \models R(a, b) \vee R(b, a)$ for some $R \in \mathcal{L}_0$ (they are certainly related by $<$, but we disregard it here).

LEMMA 4.3. *Let $X, Y, \{z\} \subseteq \mathbb{A}$ be disjoint and finite. Then there is some automorphism $\tau \in \text{Aut}(\mathbb{A})$ such that*

- (1) τ fixes every $x \in X$;
- (2) $\tau(z)$ is unrelated to all $y \in Y$ and to z ;
- (3) $\tau(z) > z$.

PROOF. In \mathbb{A}_0 , form the free amalgam

$$\begin{array}{ccc} & X \cup Y \cup \{z\} & \\ & \nearrow \quad \searrow & \swarrow \quad \nwarrow \\ X & & X \cup Y \cup \{z, z'\} \\ & \searrow \quad \nearrow & \\ & X \cup \{z\} & \end{array}$$

$x \in X \mapsto x, z \mapsto z'$

so that no element of $Y \cup \{z\}$ is related to z' . Now we make $X \cup Y \cup \{z, z'\}$ an \mathcal{L} -structure: inherit the order on $X \cup Y \cup \{z\}$ from \mathbb{A} , and declare that $z < z'$ as well as $z' < a$ if a , the next element of $X \cup Y$ larger than z , exists at all. Observe that

$$x \in X \mapsto x, z \mapsto z'$$

is still an embedding in presence of the order. By homogeneity, we may embed $X \cup Y \cup \{z, z'\}$ into \mathbb{A} via some f which is the identity on $X \cup Y \cup \{z\}$; again by homogeneity, we may extend the embedding

$$f(x) = x \in X \mapsto f(x), f(z) \mapsto f(z')$$

to some automorphism τ which makes 1), 2), and 3) true. \square

On the other hand, an \mathcal{L} -structure fails to embed into \mathbb{A} precisely when it embeds some forbidden structure, in which every two distinct elements are related:

LEMMA 4.4. *Let \mathcal{F}_0 consist of minimal (with respect to \subseteq) \mathcal{L}_0 -structures which do not appear in C_0 . Then*

- (1) C_0 consists of every \mathcal{L}_0 -structure that does not embed any $F \in \mathcal{F}_0$.
- (2) C consists of every \mathcal{L} -structure whose \mathcal{L}_0 -reduct does not embed any $F \in \mathcal{F}_0$.
- (3) In any $F \in \mathcal{F}_0$, every two distinct elements $x, y \in F$ are related by some $R \in \mathcal{L}_0$.

PROOF. As C_0 is closed under substructures, its complement is closed under superstructures and thus is — since there are no infinite strictly descending chain of embedded substructures — determined by its minimal structures. 2) follows because an \mathcal{L} -structure is in C precisely when its \mathcal{L}_0 -reduct is in C_0 . For 3), notice that $F \setminus \{x\}, F \setminus \{y\}$ are in C_0 by minimality; therefore so is their free amalgam over $F \setminus \{x, y\}$, which then cannot agree with F . \square

In what follows, we will juggle with Lemma 4.3 just enough so that we avoid the forbidden structures described in Lemma 4.4. A main result will be:

THEOREM 4.5. *\mathbb{A} has the finite length property even with finitely many constants fixed, provided that \mathbb{A} is oligomorphic (for instance if \mathcal{L}_0 is finite).*

And a corollary will be that \mathbb{A} from Examples 4.1 and 4.2 has the finite length property; as is its reduct \mathbb{A}_0 .

JY: I changed \mathbf{o}_\bullet 's to \mathbf{o} in subsections A and C

4.1 Two reductions: orbits and projections

To start with, let us view \mathbb{A}^d as $\mathbb{A}^{\{1, \dots, d\}}$ and more generally consider \mathbb{A}^I for a finite totally ordered indexing set $I \subseteq \mathbb{Q}$. Fix a finite support $S \subseteq \mathbb{A}$. If \mathbb{A} is oligomorphic, the tuples in \mathbb{A}^I split into finitely many $\text{Aut}(\mathbb{A})(S)$ -equivariant orbits. Let $O = \text{Aut}(\mathbb{A})(S) \cdot o$ be one such orbit. We shall call O (S -)ordered if $o_i \notin S$ and if $o_i < o_j$ whenever $i < j$. By removing the entries in o that repeat or come from S and reordering the rest, we can always find an $\text{Aut}(\mathbb{A})$ -equivariant bijection to an S -ordered orbit. An easy observation is that we may focus on a single ordered orbit at a time:

PROPOSITION 4.6. *The following are equivalent:*

- (1) For $d = 0, 1, 2, \dots$ and any finite $S \subseteq \mathbb{A}$, chains of $\text{Aut}(\mathbb{A})(S)$ -equivariant subspaces in $\text{Lin}_{\mathbb{F}} \mathbb{A}^d$ are bounded in length;
- (2) \mathbb{A} is oligomorphic, and $\text{Lin}_{\mathbb{F}} O$ has finite length for any ordered orbit O .

PROOF. We have $\text{len}(\text{Lin}_{\mathbb{F}}(\bigcup_i O_i)) = \text{len}(\bigoplus_i \text{Lin}_{\mathbb{F}} O_i) = \sum_i \text{len}(\text{Lin}_{\mathbb{F}} O_i)$. \square

So fix an ordered orbit $O = \text{Aut}(\mathbb{A})(S) \cdot o \subseteq \mathbb{A}^I$. From here we take an inductive approach. By $o|J$ we mean the restriction of $o : I \rightarrow \mathbb{A}$ to $J \subseteq I$; we will often write $o|^{-i}$ instead of $o|^{I \setminus \{i\}}$. Note the image $O|I$ of O under this projection agrees with $\text{Aut}(\mathbb{A})(S) \cdot o|I$ and is still ordered.

To anticipate more general statements later, Let E be a finite-dimensional \mathbb{F} -vector space — for instance, \mathbb{F} itself. Things become more interesting when we lift $(-)|^J$ to a linear $\text{Aut}(\mathbb{A})(S)$ -equivariant map

$$\begin{aligned} (-)|^J : \text{Lin}_{\mathbb{F}} O &\rightarrow \text{Lin}_{\mathbb{F}} O|^J \\ v &\mapsto v|J. \end{aligned}$$

Many cancellations can occur under $(-)|^J$; the *projection kernel* is the $\text{Aut}(\mathbb{A})(S)$ -equivariant subspace

$$\text{Ker}_{\mathbb{F}} O = \bigcap_{i \in I} \ker(-)|^{-i}$$

of $\text{Lin}_{\mathbb{F}} O$.

PROPOSITION 4.7. *The following are equivalent:*

- (1) $\text{Lin}_{\mathbb{F}} O$ has finite length for every ordered orbit O ;
- (2) $\text{Ker}_{\mathbb{F}} O$ has finite length for every ordered orbit O .

PROOF. That 1) implies 2) is clear as $\text{Ker}_{\mathbb{F}} O \subseteq \text{Lin}_{\mathbb{F}} O$.

To prove the other implication, assume 2) and let $O \subseteq \mathbb{A}^I$. We proceed by induction on $|I|$. If $I = \emptyset$, then O must be the entire singleton $\mathbb{A}^\emptyset = \{()\}$; as $\text{Lin}_{\mathbb{F}} O$ has no nontrivial subspaces (let alone finitely supported ones), it has length 1. Now if $|I| \geq 1$, assemble all $|I|$ projection maps into a single map

$$\begin{aligned} \text{Lin}_{\mathbb{F}} O &\rightarrow \bigoplus_{i \in I} O|^{-i} \\ v &\mapsto (v|^{-i})_{i \in I} \end{aligned}$$

whose kernel is precisely $\text{Ker}_F O$. We have

$$\text{len}(\text{Lin}_F O) - \text{len}(\text{Ker}_F O) \leq \sum_{i \in I} \text{len}(\text{Lin}_F O|^{-i})$$

which shows that $\text{len}(\text{Lin}_F O)$ is finite from the assumptions. \square

We call a vector from the projection kernel *balanced*. As we will see in the next subsection, cogs are a prominent example.

4.2 Cogs

Definition 4.8. Let $O \subseteq \mathbb{A}^I$ be an S -ordered orbit. An O -duo $a_\bullet \parallel b_\bullet$ consists of tuples $a_\bullet, b_\bullet \in O$ such that:

- (1) $a_i < b_i$ for all $i \in I$;
- (2) $b_i < a_j$ for all $i < j \in I$;
- (3) for any binary R in \mathcal{L}_0 (except for $=$) and $i, j \in I$:

$$R(a_i, b_j) \iff R(b_i, a_j) \iff R(a_i, a_j).$$

Remark 4.9. Conditions (1) and (2) specify a total order on the $2|I|$ atoms in a duo. Moreover, thanks to irreflexivity, each a_i is unrelated to its counterpart b_i . Further, given any $J \subseteq I$, the combined tuple $a|_\bullet^{I \setminus J}; b|_\bullet^J$ satisfies the same relations as $a_\bullet; b_\bullet$, so it lies in O . In particular, taking $J = \{i\}$, there is an automorphism π_i that sends a_i to b_i and fixes all the other elements of a_\bullet, b_\bullet and S . Finally, all O -duos are in the same $\text{Aut}(\mathbb{A})(S)$ -equivariant orbit.

Definition 4.10. Given $\lambda \in E$ and an O -duo $a_\bullet \parallel b_\bullet$, the corresponding O -cog with coefficient λ is the vector

$$\lambda \cdot a_\bullet \between b_\bullet = \sum_{J \subseteq I} (-1)^{|J|} \lambda \cdot a|_\bullet^{I \setminus J}; b|_\bullet^J$$

in $\text{Lin}_E O$. The linear span of all O -cogs with coefficients from E is denoted by $\text{Cog}_E O$.

As remarked above, given any two O -duos there is some $\pi \in \text{Aut}(\mathbb{A})(S)$ such that $\pi \cdot (a_\bullet \parallel b_\bullet) = a'_\bullet \parallel b'_\bullet$ and thus $\pi \cdot (\lambda \cdot a_\bullet \between b_\bullet) = \lambda \cdot a'_\bullet \between b'_\bullet$. Hence $\text{Cog}_E O$ is an $\text{Aut}(\mathbb{A})(S)$ -equivariant subspace of $\text{Lin}_E O$ and it is generated by cogs based on a single duo.

PROPOSITION 4.11. $\text{Cog}_E O$ is contained in $\text{Ker}_E O$.

PROOF. Let $O \subseteq \mathbb{A}^I$, let $a_\bullet \parallel b_\bullet$ be an O -duo, and let $i \in I$. The subsets of I come in pairs of J and $J \cup \{i\}$, where J is a subset of $I \setminus \{i\}$. The two tuples $a|_\bullet^J; b|_\bullet^{I \setminus J}$ and $a|_\bullet^{I \setminus \{i\}}; b|_\bullet^{\{i\}}$ differ only on the i th entry. But this difference gets erased under $(-)|^{-i}$, so the two corresponding terms in $\lambda \cdot a_\bullet \between b_\bullet$ will cancel out and hence $(\lambda \cdot a_\bullet \between b_\bullet)|^{-i} = 0$ overall. \square

In fact, cogs arise anywhere.

LEMMA 4.12. Suppose $a_\bullet \parallel b_\bullet$ is an O -duo, where $O \subseteq \mathbb{A}^I$ is S -ordered. Given $z \in S$,

- write $S' = S \setminus \{z\}$;
- let $j \notin I$ be such that $O' = \text{Aut}(\mathbb{A})(S') \cdot (a_\bullet; z) \subseteq \mathbb{A}^{I \cup \{j\}}$ is ordered,
- let $X \subseteq \mathbb{A}$ be a finite set containing $\{a_i, b_i \mid i \in I\} \cup S'$ but not z ;
- let $Y \subseteq \mathbb{A}$ be any finite set disjoint from $X \cup \{z\}$;

then the $\tau \in \text{Aut}(\mathbb{A})(X)$ afforded by Lemma 4.3 gives us an O' -duo $(a_\bullet; z) \parallel (b_\bullet; \tau(z))$.

PROOF. First, notice that $b_\bullet; \tau(z) \in O'$ and that we have the required order relations with z and $\tau(z)$. The remaining condition of Def. 4.8, for any R in \mathcal{L}_0 , splits into the following cases (and their symmetric versions):

- $R(a_i, b_j) \iff R(a_i, a_j)$ since $a_\bullet \parallel b_\bullet$ is an O -duo;
- $R(a_i, \tau(z)) \iff R(a_i, z)$ since τ is an automorphism that fixes all a_i ;
- $R(a_i, z) \iff R(b_i, z)$ since $a_\bullet, b_\bullet \in O$ and $z \in S$;
- $R(z, \tau(z))$ and $R(z, z)$ are both false: $\tau(z)$ is unrelated to z by Lemma 4.3, and R is irreflexive.

\square

Starting from an empty duo, we may apply the previous lemma inductively.

PROPOSITION 4.13. Let $O \subseteq \mathbb{A}^I$ be an S -ordered orbit. Then any $a_\bullet \in O$ can be extended to an O -duo $a_\bullet \parallel b_\bullet$.

PROOF. Enumerate the indices of I as i_1, \dots, i_d . Suppose that we have found b_{i_1}, \dots, b_{i_k} such that

$$a|_\bullet^{\{i_1, \dots, i_k\}} \parallel (i_1 \mapsto b_{i_1}, \dots, i_k \mapsto b_{i_k})$$

is a duo for $O_k = \text{Aut}(\mathbb{A})(S \cup \{a_{i_{k+1}}, \dots, a_{i_d}\}) \cdot a|_\bullet^{\{i_1, \dots, i_k\}}$ — note that $() \parallel ()$ is certainly a duo for O_0 at the start. If $k < d$, with $z = a_{i_{k+1}}$, $X = \{a_{i_1}, b_{i_1}, \dots, a_{i_k}, b_{i_k}\} \cup S \cup \{a_{i_{k+2}}, \dots, a_{i_d}\}$, and $Y = \emptyset$, a straightforward application of Lemma 4.12 yields an atom $b_{i_{k+1}}$ that makes

$$a|_\bullet^{\{i_1, \dots, i_k, i_{k+1}\}} \parallel (i_1 \mapsto b_{i_1}, \dots, i_k \mapsto b_{i_k}, i_{k+1} \mapsto b_{i_{k+1}})$$

a duo for O_{k+1} . For $k = d$ we thus obtain the desired duo for $O_d = O$. \square

The result below substantiates the slogan that cogs are found everywhere.

THEOREM 4.14. Any $\text{Aut}(\mathbb{A})(S)$ -equivariant subspace V of $\text{Lin}_E O$ contains $\text{Cog}_E(V) O$, where $O \subseteq \mathbb{A}^I$ is S -ordered and $E(V)$ is the subspace spanned by $\{v(a_\bullet) \mid v \in V, a_\bullet \in O\}$ of E .

PROOF. Pick any $v \in V$ and $a_\bullet \in O$; it is enough to show that V contains $v(a_\bullet) \cdot a_\bullet \between b_\bullet$ for some O -duo $a_\bullet \parallel b_\bullet$. Actually, write

$$S' = S \cup \{c_i \mid v(c_\bullet) \neq 0, i \in I\} \setminus \{a_i \mid i \in I\} \supseteq S$$

and put $O' = \text{Aut}(\mathbb{A})(S') \cdot a_\bullet \subseteq O$ — then O' is S' -ordered. By Proposition 4.13, we can find $b_\bullet \in O'$ such that $a_\bullet \parallel b_\bullet$ is an O' -duo and *a fortiori* an O -duo. Take the automorphisms $\pi_{i_1}, \dots, \pi_{i_d}$ from Remark 4.9, where i_1, \dots, i_d enumerate I . Now define $v^{(0)} = v$ and

$$v^{(k)} = v^{(k-1)} - \pi_{i_k} \cdot v^{(k-1)}.$$

We can check inductively that for $k = 0, 1, \dots, d$, with $O^{(k)} = \{c_\bullet \mid v(c_\bullet) \neq 0, \{c_{i_1}, \dots, c_{i_k}, \dots, c_{i_d}\} \supseteq \{a_{i_1}, \dots, a_{i_k}\}\}$ we have

$$v^{(k)} = \sum_{c_\bullet \in O^{(k)}} \sum_{J \subseteq \{i_1, \dots, i_k\}} (-1)^{|J|} v(c_\bullet) \prod_{j \in J} \pi_j \cdot c_\bullet.$$

But $\{c_{i_1}, \dots, c_{i_d}\} \supseteq \{a_{i_1}, \dots, a_{i_d}\}$ means that $c_\bullet = a_\bullet$, so at the end $v^{(d)}$ is the desired O -cog. \square

COROLLARY 4.15. $\text{Cog}_F O$ has length 1.

PROOF. Let $V \subseteq \text{Cog}_F O$ be a non-zero $\text{Aut}(\mathbb{A})_{(S)}$ -equivariant subspace. Then $\{0\} \subsetneq E(V) \subseteq E = F$ so E must be the entire field F , and by above V must be $\text{Cog}_F O$ itself. \square

In light of Propositions 4.6 and 4.7, we will be able to prove the finite length property for an oligomorphic structure with free amalgamation over any field and support if we know $\text{Ker}_F O = \text{Cog}_F O$. Let us now attempt to show that.

4.3 Subvectors

This is a good time to recall a view we have tacitly taken: with O as a standard basis, a vector $v \in \text{Lin}_E O$ is just a finite set of pairs in $E \times O$. A *subvector* of v is a subset of these pairs.

Now suppose as usual that $O \subseteq \mathbb{A}^I$ is S -ordered. Given $i \in I$ and $a \in O$, we write

$$O^{i:a_i} = \{b \in O \mid b_i = a_i\};$$

this is an $\text{Aut}(\mathbb{A})_{Sa_i}$ -orbit, and its projection $O^{i:a_i}|^{-i} = \text{Aut}(\mathbb{A})_{Sa_i} \cdot a|^{-i}$ is ordered. For a vector $v \in \text{Lin}_E O$, by

$$v^{i:a_i} \in \text{Lin}_E O^{i:a_i}$$

we mean the subvector consisting of all pairs in $E \times O^{i:a_i}$.

LEMMA 4.16. *Let $v \in \text{Lin}_E O$ be balanced. Then any projected subvector $v^{i:a_i}|^{-i} \in \text{Lin}_E O^{i:a_i}|^{-i}$ is also balanced.*

PROOF. Let $j \in I \setminus \{i\}$. By assumption we have

$$0 = v|^{-j} = \sum_a v^{i:a_i}|^{-j}$$

in $\text{Lin}_E \mathbb{A}^{I \setminus \{j\}}$, so by looking at i th entries we see that each $v^{i:a_i}|^{-j}$ must be the zero vector. Hence so is $v^{i:a_i}|^{-j}|^{-i} = v^{i:a_i}|^{-i}|^{-j}$, which shows that $v^{i:a_i}|^{-i}$ is in the projection kernel. \square

So we can try to prove $\text{Ker}_E O \subseteq \text{Cog}_E O$ for any ordered $O \subseteq \mathbb{A}^I$ by inducting on $|I|$.

4.4 Special case: unobstructed vectors

We will begin by showing any $v \in \text{Ker}_E O$ lies in $\text{Cog}_E O$ provided that v satisfies an additional condition which, as we will explain in the subsection, may be assumed without loss of generality. We motivate and introduce this condition now.

For a vector $v \in \text{Lin}_E O$ where O is S -ordered, define:

$$[v] = \{a \in O \mid v(a) \neq 0\}.$$

Take any atom a_i for $a \in [v]$, and let $P, R \in \mathcal{L}_0$ be unary and binary respectively. Then whether $P(a_i), R(a_i, s), R(s, a_i)$ hold in \mathbb{A} for some $s \in S$ is determined by O and i , and so it does not depend on the choice of a .

Now take another atom b_j for some $b \in [v]$. What can be said about $R(a_i, b_j)$? Unless $a = b$, not much. The index j may not even be unique: we may well have $a_i = b_j$ even if $i \neq j$. We want to avoid such confusions:

Definition 4.17. Let O be an S -ordered orbit in \mathbb{A}^I , and fix some representative $o \in O$. Call a finite family $V \subseteq O$

- (1) *unambiguous* if the assignment $\sqrt{-} : a_i \mapsto o_i$ is a well-defined function on the atoms present in V , i.e., if for any $a, b \in V$, if $a_i = b_j$ then $i = j$;

- (2) *unobstructed* if it is unambiguous and for any $a, b \in V$ and $i, j \in I$, either a_i and b_j are unrelated or

$$R(a_i, b_j) \iff R(o_i, o_j)$$

for every binary relation $R \in \mathcal{L}_0$.

(Obviously these properties do not depend on the choice of $o \in O$.) Furthermore, we call a vector $v \in \text{Lin}_E O$ unambiguous or unobstructed if the set $[v]$ is so.

For example, it is easy to see that cogs are unobstructed.

THEOREM 4.18. *Let $v \in \text{Ker}_E O$ be unobstructed. Then we have*

$$v = \sum_{k \in K} v(b^{(k)}) \cdot x^{(k)} \between y^{(k)}$$

for some $b^{(k)}, x^{(k)}, y^{(k)} \in O$, where moreover

$$[v] \cup \{x^{(k)}, y^{(k)} \mid k \in K\}$$

is unobstructed.

Note that the last part of the theorem does not trivially follow from the first: $[x^{(k)} \between y^{(k)}]$ may contain some tuples which are absent from $[v]$ due to cancellations with other cogs.

We spend the rest of this subsection giving the proof, which proceeds by induction on $|I|$ for S -ordered orbits $O \subseteq \mathbb{A}^I$ for every S at once.

The base case $I = \emptyset$ is immediate: as O is the singleton $\{\}\}$, any vector $v \in \text{Ker}_E O = \text{Lin}_E O = \text{Cog}_E O$ is a cog already; so the theorem says nothing more than $v = v$.

Now suppose that some $i^* \in I$ exists – in fact, let i^* be maximal. Consider an unobstructed vector $v \in \text{Ker}_E O$. We can decompose v into finitely many subvectors to write

$$v = v^{i^*:a_1} + \dots + v^{i^*:a_m}.$$

Then each projection $v^{i^*:a_j}|^{-i^*}$ lies in $\text{Ker}_E O^{i^*:a_j}|^{-i^*}$ by Lemma 4.16, and we may straightforwardly check that it is unambiguous and unobstructed. It follows from the inductive hypothesis of Theorem 4.18 that

$$v^{i^*:a_j}|^{-i^*} = \sum_{k \in K_j} v^{i^*:a_j}|^{-i^*}(b^{(a_j, k)}) \cdot x^{(a_j, k)} \between y^{(a_j, k)}$$

for some tuples $b^{(a_j, k)}, x^{(a_j, k)}, y^{(a_j, k)} \in O^{i^*:a_j}|^{-i^*}$ such that $[v^{i^*:a_j}|^{-i^*}] \cup \{x^{(a_j, k)}, y^{(a_j, k)} \mid k \in K_j\}$ is unobstructed. We can return to O by adding a_j back as the i^* th term to every tuple: we get

$$v^{i^*:a_j} = \sum_{k \in K_j} (v(b^{(a_j, k)}; a_j) \cdot x^{(a_j, k)} \between y^{(a_j, k)}); a_j,$$

where the family

$$U_j = [v^{i^*:a_j}] \cup \{(x^{(a_j, k)}; a_j), (y^{(a_j, k)}; a_j) \mid k \in K_j\}$$

is easily seen to be unobstructed. We can ask for more:

CLAIM 4.19. *We may choose $x^{(a_j, k)}$'s, and $y^{(a_j, k)}$'s so that*

$$V_0 = [v], \quad V_j = V_{j-1} \cup U_j$$

is unobstructed for $j = 1, 2, \dots, m$.

PROOF. Let A_j denote the atoms that appear in $[v^{i^*:a_j}]$. Given any automorphism $\pi \in \text{Aut}(\mathbb{A})$ that fixes S as well as A_j , using the tuples $\pi \cdot x^{(a_j,k)}$ and $\pi \cdot y^{(a_j,k)}$ gives us another decomposition of $v^{i^*:a_j} \mid -^{i^*}$ that is unobstructed. We will show that composing such automorphisms — which will be provided by Lemma 4.3 — suffices to make the claim hold.

Assume that V_{j-1} is unobstructed. Suppose that V_j is not even unambiguous. Then for some $b \in V_j \setminus V_{j-1} \subseteq U_j$ and $c \in V_{j-1}$ have $b_i = c_{i'}$ with $i \neq i'$. Now if we were to have $b_i \in A_j$ (e.g., if $b_i = a_j$), then $b_i = d_{i''}$ for some $d \in [v^{i^*:a_j}] \subseteq [v]$. Since U_j is unambiguous, $i = i''$; and since V_{j-1} is unambiguous, $i'' = i'$. This contradicts the assumption that $i \neq i'$.

So the ambiguous atom b_i does not belong to A_j . In other words, the set X of all atoms that appear in V_j except b_i contains A_j (and a_j). Apply Lemma 4.3 to get an automorphism τ which fixes $S \cup X$ but sends b_i to a fresh atom. Then, in the new family

$$V_{j-1} \cup \{(\tau \cdot x^{(a_j,k)}; a_j), (\tau \cdot y^{(a_j,k)}; a_j) \mid k \in K_j\},$$

the ambiguous atoms are precisely the ones in the old family minus b_i — these are all fixed by τ . We continue this way until all ambiguous atoms are freshened.

Hence assume V_j is unambiguous. Suppose b_i and $c_{i'}$, for some $b, c \in V_j$, are the reason why V_j fails to be unobstructed. To have b_i in A_j is impossible: otherwise $b_i = d_{i''}$ for some $d \in [v^{i^*:a_j}] \subseteq [v]$ and since V_j is unambiguous, we get $i = i''$. But d_i (hence b_i) cannot be part of an obstruction in V_j because both U_j and V_{j-1} are unobstructed. Symmetrically, we see that $c_{i'} \notin A_j$.

Since V_{j-1} is unobstructed, b (or, symmetrically, c) must be one of the $x^{(a_j,k)}; a_j$ or $y^{(a_j,k)}; a_j$. This time, split all atoms that appear in V_j except b_i into two sets X and Y : put any atom $c_{i'}$ which makes b_i an obstruction in Y , and put everything else in X . Then X contains A_j as discussed above; it also contains $\{x_i^{(a_j,k)}, y_i^{(a_j,k)} \mid k \in K_j, i \in I\}$. Use Lemma 4.3 to get $\tau \in \text{Aut}(\mathbb{A})_{(S \cup X)}$ which sends b_i to a fresh atom disjoint from $S \cup X \cup Y \cup \{b_i\}$ that is unrelated to everything in Y . Observe that the new family

$$V_{j-1} \cup \{(\tau \cdot x^{(a_j,k)}; a_j), (\tau \cdot y^{(a_j,k)}; a_j) \mid k \in K_j\},$$

remains unambiguous. Moreover, the obstructions here are precisely the ones from V_j except the ones that involve b_i , which are all fixed by τ . So we may repeat this process until all obstructions are removed. \square

Having chosen the fresh atoms carefully, let X consist of all the atoms appearing in the unobstructed family V_m . It will be convenient to fix a representative $o \in O$ and recall the function

$$\sqrt{-}: X \rightarrow \{o_1, \dots, o_d\}$$

introduced in Definition 4.17. We now add a new element b^* to the finite \mathcal{L} -structure $S \cup X \subseteq \mathbb{A}$ so that $(x^{(a_j,k)}; a_j) \parallel (y^{(a_j,k)}; b^*)$ becomes a well-formed O -duo for each j . To this end we define b^* to be greater (wrt. $>$) than every atom in X , and the remaining

relations are defined to make b^* mimic the greatest atom in o :

$$\begin{aligned} b^* &> c \quad \text{for } c \in X \\ b^* &> s \iff \mathbb{A} \models o_{i^*} > s \\ P(b^*) &\iff \mathbb{A} \models P(o_{i^*}) \\ R(b^*, s) &\iff \mathbb{A} \models R(o_{i^*}, s) \\ R(s, b^*) &\iff \mathbb{A} \models R(s, o_{i^*}) \\ R(b^*, c) &\iff \mathbb{A} \models R(o_{i^*}, \sqrt{c}) \text{ for } c \in X \\ R(c, b^*) &\iff \mathbb{A} \models R(\sqrt{c}, o_{i^*}) \text{ for } c \in X \end{aligned}$$

for every unary predicate P and binary relation R in \mathcal{L}_0 . The last two clauses are well defined (i.e. they do not depend on the choice of $a \in V_m$ that contains the atom c) since V_m is unambiguous. Moreover, since V_m is unobstructed, our definition does not introduce any forbidden substructures:

CLAIM 4.20. $S \cup X \cup \{b^*\}$ embeds into \mathbb{A} .

PROOF. If not, by Lemma 4.4 there is a forbidden \mathcal{L}_0 -structure F of pairwise related elements which embeds into the \mathcal{L}_0 -reduct of $S \cup X \cup \{b^*\}$, via ϕ say. Extend $\sqrt{-}$ from above to:

$$\sqrt{-}: X \cup S \cup \{b^*\} \rightarrow \{o_1, \dots, o_d\} \cup S$$

by putting $\sqrt{b^*} = o_{i^*}$ and $\sqrt{s} = s$ for $s \in S$. We will check that $\sqrt{-} \circ \phi$ then embeds F into the \mathcal{L}_0 -reduct of \mathbb{A} , namely \mathbb{A}_0 , and reach a contradiction.

So let $P \in \mathcal{L}_0$ be unary and let $R \in \mathcal{L}_0$ be binary. It is immediate that $\sqrt{-}$ preserves and reflects P for all atoms in $X \cup S$ and for b^* — we do not even need to know they are in the image of ϕ . We can also see that $\sqrt{-}$ preserves and reflects R whenever at least one of the arguments is from $S \cup \{b^*\}$. Now consider the remaining case of $\phi(f), \phi(f') \in X$. As f and f' are related in F , their images under ϕ must be related in $\phi(F)$. But the family V_m that gave rise to X is unobstructed, which forces $\sqrt{-}$ to preserve and reflect R here as well. Since \mathcal{L}_0 only has unary and binary relations, this means that $\sqrt{\phi(F)} \subseteq \mathbb{A}_0$ is in C_0 yet embeds F ; this is a contradiction. \square

Using homogeneity we may assume that $S \cup X \cup \{b^*\} \subseteq \mathbb{A}$.

CLAIM 4.21. $x^{(a_j,k)}; a_j \parallel y^{(a_j,k)}; b^*$ forms an O -duo for $1 \leq j \leq m$ and $k \in K_j$. Furthermore, the family

$$[v] \cup \{(x^{(a_j,k)}; a_j), (y^{(a_j,k)}; b^*) \mid k \in K_j, 1 \leq j \leq m\}$$

is unobstructed.

PROOF. Recall that $x^{(a_j,k)} \parallel y^{(a_j,k)}$ is a duo for $O^{i^*:a_j} \mid -^{i^*}$. Since $x^{(a_j,k)}, y^{(a_j,k)} \in X$, the remaining conditions for $x^{(a_j,k)}; a_j \parallel y^{(a_j,k)}; b^*$ being an O -duo follow directly from our definition of relations on b^* .

The family in the claim is unambiguous since V_m is unambiguous and b^* is a fresh atom. Similarly, it is unobstructed since V_m is unobstructed, and the additional conditions that involve b^* follow from the relations that we imposed on b^* .

BK: This argument feels a bit shaky; I will think about it some more. \square

CLAIM 4.22. We have $v = \sum_{j=1}^m \sum_{k \in K_j} v(b^{(a_j,k)}; a_j) \cdot (x^{(a_j,k)}; a_j) \parallel (y^{(a_j,k)}; b^*)$.

PROOF. Observe that, by definition of cogs, for any O -duo $a; c \parallel b; d$ with $c, d \in \mathbb{A}$ we have

$$(a; c) \between (b; d) = (a \between b); c - (a \between b); d.$$

Using this, calculate:

$$\begin{aligned} & \sum_{j=1}^m \sum_{k \in K_j} v(b^{(a_j, k)}) \cdot (x^{(a_j, k)}; a_j \between y^{(a_j, k)}; b^*) \\ &= \sum_{j=1}^m \sum_{k \in K_j} \left(v(b^{(a_j, k)}) \cdot x^{(a_j, k)} \between y^{(a_j, k)} \right); a_j \\ &\quad - \sum_{j=1}^m \sum_{k \in K_j} \left(v(b^{(a_j, k)}) \cdot x^{(a_j, k)} \between y^{(a_j, k)} \right); b^* \\ &= \sum_{j=1}^m v^{i^*: a_j} - \sum_{j=1}^m v^{i^*: a_j | -i^*}; b^* \\ &= v - \sum_{j=1}^m v^{i^*: a_j | -i^*}; b^*, \end{aligned}$$

and it suffices to show the last sum vanishes. Recall from Proposition 4.11 that cogs are balanced. By projecting away the i^* th entry, we obtain

$$0 = 0 - \left(\sum_{j=1}^m v^{i^*: a_j | -i^*}; b^* \right)^{-i^*}.$$

But we can simply add back b^* as the i^* th entry, yielding $0 = \sum_{j=1}^m v^{i^*: a_j | -i^*}; b^*$ as needed — replacing the i^* th entry by a common atom achieves the same effect of projecting it away. \square

This completes the proof of Theorem 4.18: we have shown any balanced vector v is spanned by cogs as long as $[v]$ is unobstructed. Let us lift this restriction.

4.5 Removing ambiguities and obstructions

THEOREM 4.23. *Let $v \in \text{Ker}_{\mathbb{E}} O$ be arbitrary. Then (assuming as before that \mathcal{L}_0 is at most binary) we have*

$$v = \sum_{k \in K} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

for some $b_\bullet^{(k)}, x_\bullet^{(k)}, y_\bullet^{(k)} \in O$.

As a reminder, $O \subseteq \mathbb{A}^I$ is an S -ordered orbit and \mathbb{E} is a finite-dimensional F -vector space. We induct on $|I|$, noting that when $I = \emptyset$ we are just saying $v = v() \cdot (\between)$. Hereafter assume $I \neq \emptyset$.

PROPOSITION 4.24. *Let $v \in \text{Ker}_{\mathbb{E}} O$. Then we can find $b_\bullet^{(k)}, x_\bullet^{(k)}, y_\bullet^{(k)} \in O$ so that*

$$\ddot{v} = v - \sum_{k \in K_1} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

satisfies $\ddot{v}(O) = v(O) \subseteq \mathbb{E}$ and $O(\ddot{v})$ is unambiguous.

PROOF. Assume there is an ambiguity in $O(v)$, i.e., there are $a_\bullet, a'_\bullet \in O(v)$ such that $a_i = a'_{i'}$ but $i \neq i'$ (so $|I| \geq 2$). Let us prevent the atom-index pair (a_i, i) from causing ambiguities. By Lemma 4.16,

the projected subvector $v^{i:a_i | -i}$ belongs to $\text{Ker}_{\mathbb{E}} O^{i:a_i | -i}$; by the inductive hypothesis of Theorem 4.23 at hand, we have

$$v^{i:a_i} = v^{i:a_i | -i}; a_i = \sum_{k \in K} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); a_i.$$

We can invoke Lemma 4.3 to find an automorphism τ which fixes all of the atoms except a_i that appear in $x_\bullet^{(k)}, y_\bullet^{(k)}$, $k \in K$ and in $b_\bullet \in O(v)$. It follows from the appropriate modification of Lemma 4.12 that we get O -duos $x_\bullet^{(k)}; a_i \parallel y_\bullet^{(k)}; \tau \cdot a_i$ for $k \in K$. Observe that

$$\begin{aligned} & v^{i:a_i} - \sum_{k \in K} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); \tau \cdot a_i \\ &= v^{i:a_i} - \sum_{k \in K} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); a_i \\ &\quad + \sum_{k \in K_1} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); \tau \cdot a_i \end{aligned}$$

is equal to $v^{i:a_i | -i}; \tau \cdot a_i$, so

$$\begin{aligned} v' &= v - \sum_{k \in K} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)}; \tau \cdot a_i \between y_\bullet^{(k)}; a_i) \\ &= v - v^{i:a_i} + v^{i:a_i | -i}; \tau \cdot a_i \end{aligned}$$

is the vector obtained by changing the i th entry in every tuple of v from a_i to $\tau \cdot a_i$, a fresh atom disjoint from all atoms present. Then $v'(O) = v(O)$. Moreover, we can directly check that an ambiguous atom-index pair (b_j, j) in $O(v')$ cannot be $(\tau \cdot a_i, i)$, and hence it must already be one in $O(v)$ except it cannot be (a_i, i) . Therefore we may remove all ambiguities by iterating this process. \square

Next we tackle the obstructions.

PROPOSITION 4.25. *Let $\ddot{v} \in \text{Ker}_{\mathbb{E}} O$ be such that $O(\ddot{v})$ is unambiguous. Then we can find $b_\bullet^{(k)}, x_\bullet^{(k)}, y_\bullet^{(k)} \in O$ so that*

$$\dot{v} = \ddot{v} - \sum_{k \in K_2} \ddot{v}(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

satisfies $\dot{v}(O) = \ddot{v}(O)$ and $O(\dot{v})$ is unobstructed.

PROOF. We follow the same strategy: let the atom a_i be an obstruction with some other $a'_{i'}$ in $O(\ddot{v})$. Consider the projected subvector $\ddot{v}^{i:a_i | -i}$. By Lemma 4.16 and the inductive hypothesis of Theorem 4.23, we can write

$$\ddot{v}^{i:a_i} = \sum_{k \in K} \ddot{v}(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); a_i.$$

Leveraging no algebraicity like we did at the beginning of the proof of Claim 4.19, we may assume that $O(\ddot{v}) \cup \{(x_\bullet^{(k)}; a_i), (y_\bullet^{(k)}; a_i) \mid k \in K\}$ is unambiguous.

Split the atoms except a_i appearing in that family into two parts X and Y , where Y consists of all the atoms equal to some $a'_{i'}$ that make a_i an obstruction. Then $x_{i''}^{(k)}, y_{i''}^{(k)}$ cannot belong to Y for any $i'' \in I$. Indeed, if say $x_{i''}^{(k)}$ is equal to an obstructive atom $a'_{i'}$, we must have $i'' = i'$ by unambiguity; we now have a contradiction: $a'_{i'} = x_{i''}^{(k)}$ and a_i necessarily satisfy the right binary relations, since $x_\bullet^{(k)}; a_i \in O$.

Next, we invoke the Lemma 4.3 to find an automorphism $\tau \cdot \text{Aut}(\mathbb{A})$ making $\tau \cdot a_i$ greater than a_i , distinct from any $x \in X$,

and unrelated to any $y \in Y$. By Lemma 4.12, we have O -duos $x_\bullet^{(k)}; a_i \parallel y_\bullet^{(k)}; \tau \cdot a_i$ for $k \in K$. The refined vector

$$\begin{aligned}\ddot{v}' &= \ddot{v} - \sum_{k \in K} \ddot{v}(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)}; \tau \cdot a_i \between y_\bullet^{(k)}; a_i) \\ &= \ddot{v} - \ddot{v}^{i;a_i} + \ddot{v}^{i:\tau \cdot a_i}\end{aligned}$$

satisfies $\ddot{v}'(O) = \ddot{v}(O)$. As we are just changing a_i to the fresh atom $\tau \cdot a_i$ in the i th entry of every tuple appearing in \ddot{v} , the family $O(\ddot{v}')$ remains unambiguous; note also that a_i can only appear in the i th entry as $O(\ddot{v})$ is unambiguous, so we have completely removed a_i from \ddot{v}' . But $\tau \cdot a_i$ by design cannot cause an obstruction in $O(\ddot{v}')$, and consequently any atom causing obstructions in $O(\ddot{v}')$ must already do so in $O(\ddot{v})$ except that it cannot be a_i . Hence we can repeat this procedure until all obstructions are excised. \square

We are at last in a position to prove the inductive step of Theorem 4.23. Given $v \in \text{Ker}_F O$, we apply Propositions 4.24 and 4.25 to obtain

$$\dot{v} = v - \sum_{k \in K_1 \cup K_2} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

where $\dot{v}(O) = \ddot{v}(O) = v(O)$ and $O(\dot{v})$ is unobstructed. Recalling Proposition 4.11, we see that $\dot{v} \in \text{Ker}_F O$. The special Theorem 4.18 tells us that \dot{v} is a sum of O -cogs with coefficients from $\dot{v}(O) = v(O)$. It follows that so is

$$v = \dot{v} + \sum_{k \in K_1 \cup K_2} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)},$$

which establishes the general Theorem 4.23.

4.6 All those equivariant subspaces

We finish this section with an important corollary of Theorem 4.23. Let $O_1 \subseteq \mathbb{A}^{I_1}, \dots, O_n \subseteq \mathbb{A}^{I_n}$ all be S -ordered orbits. Then $\text{len}(\text{Lin}_F(O_1 \cup \dots \cup O_n)) = 2^{|I_1|} + \dots + 2^{|I_n|}$; in fact, we know and can characterise all the $\text{Aut}(\mathbb{A})(S)$ -equivariant subspaces of $\text{Lin}_F(O_1 \cup \dots \cup O_n) \simeq \text{Lin}_F(O_1) \oplus \dots \oplus \text{Lin}_F(O_n)$.

Local coefficients. First we set up some notations. Consider the $\sum_k 2^{|I_k|}$ projected S -ordered orbits $O_k|J$ for $1 \leq k \leq n, J \subseteq I_k$. Suppose

$$f : O_k|J \rightarrow O_{k'}|J'$$

is an $\text{Aut}(\mathbb{A})(S)$ -equivariant bijection. Take any $o_\bullet \in O_k|J$, and enumerate its entries as $o_1 < \dots < o_{|J|}$. Similarly, enumerate the entries of $f(o_\bullet)$ as $o'_1 < \dots < o'_{|J'|}$. Then $\{o_1, \dots, o_{|J|}\} = \{o'_1, \dots, o'_{|J'|}\}$ because \mathbb{A} has no algebraicity; since the orbits are ordered, we must have $|J| = |J'|$ and $o_1 = o'_1, \dots, o_{|J|} = o'_{|J'|}$. That is, f must be the obvious function that reindexes a J -tuple to a J' -tuple — hence we will write $o_\bullet^{J'}$ instead of $f(o_\bullet)$, leaving f implicit.

Now, let $Q_1 = O_{k_1}|J_1, \dots, Q_t = O_{k_t}|J_t$ be the distinct S -ordered orbits up to $\text{Aut}(\mathbb{A})(S)$ -equivariant bijections, which we enumerate in such a way that $|J_1| \geq |J_2| \geq \dots \geq |J_t| = 0$.

Definition 4.26. For $i = 1, \dots, t$, let P_i consist of pairs (k, J) such that $O_k|J$ is $\text{Aut}(\mathbb{A})(S)$ -equivariantly isomorphic to Q_i . Assemble all $|P_i|$ projections into a single map

$$(-)\restriction_i : \text{Lin}_F(O_1 \cup \dots \cup O_n) \rightarrow \text{Lin}_{F^{P_i}} Q_i.$$

More precisely $(v_1, \dots, v_n)|_i(a_\bullet)$ is the P_i -tuple whose entry at (k, J) is $v_k|_J(a_\bullet^{J/J}) \in F$. It is straightforward to check that $(-)|_i$ is $\text{Aut}(\mathbb{A})(S)$ -equivariant and linear.

Let $W \subseteq \text{Lin}_F(O_1 \cup \dots \cup O_n)$ be an $\text{Aut}(\mathbb{A})(S)$ -equivariant subspace. Using the t finite-dimensional vector spaces $W|_1(Q_1) \subseteq F^{P_1}, \dots, W|_t(Q_t) \subseteq F^{P_t}$ we define \widetilde{W} , which consists of all vectors $v \in \text{Lin}_F(O_1 \cup \dots \cup O_n)$ such that

$$v|_1(Q_1) \subseteq W|_1(Q_1), \dots, v|_t(Q_t) \subseteq W|_t(Q_t).$$

Then \widetilde{W} is an $\text{Aut}(\mathbb{A})(S)$ -equivariant subspace that contains W . It turns out these two are equal:

LEMMA 4.27.

$$\begin{aligned}\widetilde{W} \cap \ker(\restriction_{i+1}) \cap \dots \cap \ker(\restriction_t) \\ \subseteq W \cap \ker(\restriction_{i+1}) \cap \dots \cap \ker(\restriction_t).\end{aligned}$$

In particular $\widetilde{W} \subseteq W$ when $i = t$.

Proof of the lemma. by induction on i . When $i = 0$, this containment is trivial:

CLAIM 4.28. $\ker(\restriction_1) \cap \ker(\restriction_2) \cap \dots \cap \ker(\restriction_t) = \{0\}$.

PROOF. Let $v = (v_1, \dots, v_n) \in \ker(\restriction_1) \cap \ker(\restriction_2) \cap \dots \cap \ker(\restriction_t)$. Each (k, I_k) belongs to some P_i ; that $v|_i = 0$ implies $0 = v_k|_{I_k} = v_k$. \square

To prove the containment for $i+1$, we allow $v|_{i+1}$ to be non-zero. But $v|_{i+1}$ satisfies the next best property:

CLAIM 4.29. *The image of*

$$\widetilde{W} \cap \ker(\restriction_{i+2}) \cap \dots \cap \ker(\restriction_t)$$

under \restriction_{i+1} *is contained in* $\text{Ker}_{W|_{i+1}(Q_{i+1})} Q_{i+1}$.

PROOF. Take any $v = (v_1, \dots, v_n)$ satisfying

$$v|_{i+2} = 0, \dots, v|_t = 0$$

from \widetilde{W} . That $v|_{i+1} \in \text{Ker}_{W|_{i+1}(Q_{i+1})} Q_{i+1}$ is clear from the definition of \widetilde{W} . Recall that $Q_{i+1} = O_{k_{i+1}}|J_{i+1}$. Given $j \in J_{i+1}$, we need to prove that $v|_{i+1}|^{-j} = 0$.

To do so, take $(k, J) \in P_{i+1}$; the unique monotone bijection between J_{i+1} and J restricts to one between $J_{i+1} \setminus \{j\}$ and $J \setminus \{j'\}$. Now $(k, J \setminus \{j'\})$ belongs to some $P_{i'}$ with $i' > i+1$, so $v|_{i'} = 0$. We calculate that $v|_{i+1}|^{-j}(a_\bullet)_{k,J}$ is equal to

$$\sum_{b_\bullet \in O_k, b_\bullet|J \setminus \{j'\} = a_\bullet|J \setminus \{j'\}} v_k(b_\bullet),$$

and that so is $0 = v|_{i'}(a_\bullet|J \setminus \{j'\})_{k,J \setminus \{j'\}}$. \square

Now Theorem 4.23 tells us that $\text{Ker}_{\widetilde{W}|_{i+1}(Q_{i+1})} Q_{i+1} \subseteq \text{Cog}_{\widetilde{W}|_{i+1}(Q_{i+1})} Q_{i+1}$, which is good news:

CLAIM 4.30. $\text{Cog}_{W|_{i+1}(Q_{i+1})} Q_{i+1}$ *is contained in the image of*

$$W \cap \ker(\restriction_{i+2}) \cap \dots \cap \ker(\restriction_t)$$

under \restriction_{i+1} .

PROOF. Let $w \upharpoonright_{i+1}(a_\bullet) \in W \upharpoonright_{i+1}(Q_{i+1})$. Let S' consist of S together with every atom appearing in w but not in a_\bullet . We generalise the proof of Theorem 4.14.

Start by applying Proposition 4.13 and Remark 4.9 to get automorphisms π_j for $j \in J_{i+1}$ such that $a_\bullet \parallel \prod_{j \in J_{i+1}} \pi_j \cdot a_\bullet$ is an Q_{i+1} -duo, where π_j fixes S' and $a_{j'}, \pi_{j'} \cdot a_{j'}$ for $j' \in J \setminus \{j\}$. Put

$$w' = \prod_{j \in J_{i+1}} (1 - \pi_j) \cdot w \in W.$$

Given $1 \leq i' \leq t$, observe that as no more atoms can appear in $w \upharpoonright_{i'}$ than in w , we have

$$\begin{aligned} w' \upharpoonright_{i'} &= \prod_{j \in J_{i+1}} (1 - \pi_j) \cdot w \upharpoonright_{i'} \\ &= \sum_{b_\bullet \in Q_{i'}, \{b_j | j\} \supseteq \{a_j | j\}} \sum_{J' \subseteq J} (-1)^{|J'|} w \upharpoonright_{i'}(b_\bullet) \prod_{j \in J'} \pi_j \cdot b_\bullet. \end{aligned}$$

Suppose $\{b_j \mid j \in J_{i'}\} \supseteq \{a_j \mid j \in J_{i+1}\}$. Then $i' \leq i+1$; if $i' = i+1$, we must have $b_\bullet = a_\bullet$. We therefore have

$$w' \upharpoonright_{i+1} = w \upharpoonright_{i+1}(a_\bullet) \cdot a_\bullet \asymp \prod_{j \in J_{i+1}} \pi_j \cdot a_\bullet$$

and $w' \upharpoonright_{i+2} = 0, \dots, w' \upharpoonright_t = 0$. This proves that $(W \cap \ker(\upharpoonright_{i+2}) \cap \dots \cap \ker(\upharpoonright_t)) \upharpoonright_{i+1}$ contains $\text{Cog}_{W \upharpoonright_{i+1}(Q_{i+1})} Q_{i+1}$. \square

This is enough to establish Lemma 4.27 for $i+1$ assuming the result for i . Indeed, given $v \in \widetilde{W} \cap \ker(\upharpoonright_{i+2}) \cap \dots \cap \ker(\upharpoonright_t)$, we can find $w \in W \cap \ker(\upharpoonright_{i+2}) \cap \dots \cap \ker(\upharpoonright_t) \subseteq \widetilde{W}$ such that

$$v \upharpoonright_{i+1} = w \upharpoonright_{i+1}$$

by the preceding claims. But $(v - w) \upharpoonright_{i+1} = 0$ — that is, $v - w$ lies in $\ker(\upharpoonright_{i+1})$ as well as $\widetilde{W} \cap \ker(\upharpoonright_{i+2}) \cap \dots \cap \ker(\upharpoonright_t)$. It follows from the inductive hypothesis that

$$v - w \in W \cap \ker(\upharpoonright_{i+2}) \cap \dots \cap \ker(\upharpoonright_t),$$

so $v = (v - w) + w$ is a member of $W \cap \ker(\upharpoonright_{i+2}) \cap \dots \cap \ker(\upharpoonright_t)$ as well.

Lengths. Let W, W' be two $\text{Aut}(\mathbb{A})_{(S)}$ -equivariant subspaces of $\text{Lin}_F(O_1 \uplus \dots \uplus O_n)$. If we have $W \upharpoonright_1(Q_1) = W' \upharpoonright_1(Q_1), \dots, W \upharpoonright_t(Q_t) = W' \upharpoonright_t(Q_t)$, then $W = \widetilde{W} = \widetilde{W}' = W'$ by Lemma 4.27. An immediate consequence is:

PROPOSITION 4.31. *Let $W_0 \subsetneq W_1 \subsetneq \dots \subsetneq W_l$ be a chain of $\text{Aut}(\mathbb{A})_{(S)}$ -equivariant subspaces in $\text{Lin}_F(O_1 \uplus \dots \uplus O_n)$. Then $l \leq 2^{|I_1|} + \dots + 2^{|I_n|}$.*

PROOF. We obtain t chains

$$W_0 \upharpoonright_1(Q_1) \subseteq W_1 \upharpoonright_1(Q_1) \subseteq \dots \subseteq W_l \upharpoonright_1(Q_1) \subseteq F^{P_1},$$

$$W_0 \upharpoonright_2(Q_2) \subseteq W_1 \upharpoonright_2(Q_2) \subseteq \dots \subseteq W_l \upharpoonright_2(Q_2) \subseteq F^{P_2},$$

$$\vdots$$

$$W_0 \upharpoonright_t(Q_t) \subseteq W_1 \upharpoonright_t(Q_t) \subseteq \dots \subseteq W_l \upharpoonright_t(Q_t) \subseteq F^{P_t}.$$

At each of the t steps, one of the t containments must be strict. Hence $l \leq |P_1| + |P_2| + \dots + |P_t| = 2^{|I_1|} + \dots + 2^{|I_t|}$. \square

It follows that any $\text{Aut}(\mathbb{A})_{(S)}$ -equivariant subspace of $\text{Lin}_F(O_1 \uplus \dots \uplus O_n)$ is finitely generated. We can compute the local coefficients of such subspaces easily:

Remark 4.32. For $v \in \text{Lin}_F(O_1 \uplus \dots \uplus O_n)$, let $\langle v \rangle$ denote the $\text{Aut}(\mathbb{A})_{(S)}$ -equivariant subspace it generates. Then:

- (1) $\langle v \rangle \upharpoonright_i(Q_i)$ is the subspace of F^{P_i} generated by vectors of the form $v \upharpoonright_i(a_\bullet)$, which is zero unless every atom appearing in a_\bullet appears in v — there are only finitely many such a_\bullet 's;
- (2) $\langle v, v' \rangle \upharpoonright_i(Q_i) = \langle v \rangle \upharpoonright_i(Q_i) + \langle v' \rangle \upharpoonright_i(Q_i)$.

We may now exhibit a chain of $\text{Aut}(\mathbb{A})_{(S)}$ -equivariant subspaces whose length is precisely $\sum_{i=1}^t 2^{|P_i|}$, generalising [1, Corollary 4.12]. Take any $(k, J) \in P_i$. Pick some $a_\bullet \in O_k$, and let $\pi_j, j \in I_k$ be the automorphisms from Proposition 4.13. Define a vector

$$\begin{aligned} v_{k,J}^i &\in \text{Lin}_F(O_1 \uplus \dots \uplus O_n) \\ &\simeq \text{Lin}_F(O_1) \oplus \dots \oplus \text{Lin}_F(O_n) \end{aligned}$$

with $\prod_{j \in J} (1 - \pi_j) \cdot a_\bullet$ as its k th component and zero everywhere else. Then

$$\langle v_{k,J}^i \rangle \upharpoonright_{i'}(Q_{i'})_{k',J'} = \begin{cases} F & \text{if } k = k' \text{ and } J \subseteq J', \\ \{0\} & \text{otherwise.} \end{cases}$$

Enumerating each P_i as $(k_1^i, J_1^i), (k_2^i, J_2^i), \dots, (k_{|P_i|}^i, J_{|P_i|}^i)$, we obtain a chain

$$\begin{aligned} &\langle \rangle \\ &\subsetneq \langle v_{k_1^t, J_1^t}^t \rangle \\ &\subsetneq \langle v_{k_1^t, J_1^t}, v_{k_2^t, J_2^t}^t \rangle \\ &\subsetneq \dots \\ &\subsetneq \langle v_{k_1^t, J_1^t}, v_{k_2^t, J_2^t}, \dots, v_{k_{|P_t|}^t, J_{|P_t|}^t}^t \rangle \\ &\subsetneq \langle v_{k_1^t, J_1^t}, v_{k_2^t, J_2^t}, \dots, v_{k_{|P_t|}^t, J_{|P_t|}^t}, v_{k_1^{t-1}, J_1^{t-1}}^{t-1} \rangle \\ &\subsetneq \dots \end{aligned}$$

of length $|P_t| + |P_{t-1}| + \dots + |P_1| = 2^{|I_1|} + \dots + 2^{|I_n|}$. With the upper bound in Proposition 4.31, we conclude:

THEOREM 4.33. $\text{len}(\text{Lin}_F(O_1 \uplus \dots \uplus O_n)) = 2^{|I_1|} + \dots + 2^{|I_n|}$.

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