More Vector Spaces with Atoms of Finite Lengths

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***Abstract*—We say an infinite structure is oligomorphic over a field if the following holds for each of its finite powers: in the corresponding free vector space, strict chains of equivariant subspaces are bounded in length. It has been shown that the countable pure set and the dense linear ordering without endpoints have this property. In this paper, we generalise these two results to a) reducts of smoothly approximable structures, provided the field has characteristic zero, and b) generaically ordered expansions of Fraïssé limits with free amalgamation, in languages with at most binary relations. As a special case, we prove the Rado graph is oligomorphic over any field using both methods.**

I. Introduction

II. Preliminaries

Definitions:

1) oligomorphic
2) homogeneous
3) smooth approximation by homogeneous substructures [3] (N.B. not the 'smooth approximation' from [4, Definitino 4])
4) *oligomorphic approximation* of a homogeneous structure by finite substructures with uniformly few orbits (i.e., types) that cover the age of $\mathbb{A}$

Definition II.1. An *interpretation in* $\mathbb{A}$ is a structure $\mathbb{A}' = (D/E; R_1, R_2, \dots)$, where

- D is an equivariant subset of $\mathbb{A}^n$ for some $n \geq 1$;
- E is an equivariant equivalence relation on D;
- D/E consists of equivalence classes $d/E \subseteq D$ for $d \in D$, which $\pi \in \mathrm{Aut}(\mathbb{A})$ acts on via $\pi \cdot d/E = (\pi \cdot d)/E$;
- every relation R_i of arity r_i is an equivariant subset of $(D/E)^{r_i}$.

We call $\mathbb{A}'$ a *reduct of* $\mathbb{A}$ if $D = \mathbb{A}$ and E is just equality.

Proposition II.2. *If* $\mathbb{A}$ *has the finite length property over* F*, then so does any interpretation* $\mathbb{A}'$*.*

Proof. Let $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l$ be a chain of $\mathrm{Aut}(\mathbb{A}')$-equivariant subspaces in $\mathrm{Lin}_{\mathsf{F}}(\mathbb{A}')^k = \mathrm{Lin}_{\mathsf{F}}(D/E)^k$. Each V_j is then $\mathrm{Aut}(\mathbb{A})$-equivariant: given $\pi \in \mathrm{Aut}(\mathbb{A})$, notice that $d/E \mapsto (\pi \cdot d)/E$ is an automorphism of $\mathbb{A}'$. Now l is bounded above by the $\mathrm{Aut}(\mathbb{A})$-length of $\mathrm{Lin}_{\mathsf{F}}(D/E)^k$, which is finite — $\mathrm{Lin}_{\mathsf{F}}(D/E)^k$ is isomorphic to the quotient of $\mathrm{Lin}_{\mathsf{F}} D^k$ by the span of

$$\left\{ \begin{array}{c} (d_1, \dots, d_k) \\ -(d'_1, \dots, d'_k) \end{array} \middle| d_1/E = d'_1/E, \dots, d_k/E = d'_k/E \right\}.$$

□

III. Rado graph, sans cogs

Theorem III.1. *Oligomorphically approximable homogeneous structures have the finite length property over any field of characteristic* 0*.*

Proof. Copy the 'bojań-trick' from §8.2 of Mikołaj's https://www.mimuw.edu.pl/~bojan/papers/notes-July3.pdf. □

Corollary III.2. *Also for* m *cliques of* n *vertices — interpretable in the equality atoms.*

A. Symplectic vector spaces

Throughout this subsection let f denote a finite field.

Definition III.3. A *symplectic vector space* is an f-vector space $\mathbb{W}$ equipped with a bilinear form $\omega : \mathbb{W} \times \mathbb{W} \to \mathsf{f}$ that is

1) alternating: $\omega(v, v) = 0$ for all v; and
2) non-degenerate: if $\omega(v, w) = 0$ for all w then $v = 0$.

Example III.4. Let $\mathbb{W}_n$ be the f-vector space with basis $e_1, \dots, e_n, f_1, \dots, f_n$. Define ω by bilinearly extending

$$\omega(e_i, f_i) = 1 = -\omega(f_i, e_i), \quad \omega(-, *) = 0 \text{ elsewhere}; \quad (\S)$$

one may straightforwardly check that ω is alternating and non-degenerate. Moreover, noticing that $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$, we obtain a countable-dimensional symplectic vector space $\mathbb{W}_\infty = \bigcup_n \mathbb{W}_n$.

We will refer to vectors satisfying (§) as a *symplectic basis*. Note such vectors must be linearly independent: if $v = \sum_i \lambda_i e_i + \mu_i f_i = 0$, then $\lambda_i = \omega(v, f_i) = 0$ and $\mu_i = \omega(e_i, v) = 0$ for each i. Such bases behave very much like the usual bases.

Proposition III.5. *Assume that* $\mathbb{W}$ *is a symplectic vector space that is at most countable. Then any finite symplectic basis* $e_1, \dots, e_n, f_1, \dots, f_n$ *can be extended to a symplectic basis that spans the whole* $\mathbb{W}$*.*

Proof. Suppose that $e_1, \dots, e_n, f_1, \dots, f_n$ does not already span $\mathbb{W}$; take v to be a witness (that is least according to some fixed enumeration of $\mathbb{W}$ in the case it is infinite). Put

$$e_{n+1} = v - \sum_{i=1}^{n} \omega(e_i, v) f_i + \sum_{i=1}^{n} \omega(f_i, v) e_i$$

so that $\omega(e_i, e_{n+1}) = 0 = \omega(f_i, e_{n+1})$. This cannot be the zero vector lest we contradict the choice of v. By the non-degeneracy of ω, there is — rescaling if necessary — some w such that $\omega(e_{n+1}, w) = 1$. Now define

$$f_{n+1} = w - \sum_{i=1^n} \omega(e_i, w) f_i + \sum_{i=1}^{n} \omega(f_i, w) e_i$$

in a similar manner, making $e_1, \ldots, e_n, e_{n+1}, f_1, \ldots, f_n, f_{n+1}$ a symplectic basis that spans v. We go through every element of $\mathbb{W}$ by continuing this way. □

In fact, we will also make use of the "symplectic basis and a half" variant below.

Proposition III.6. *Now assume $\mathbb{W}$ is a finite-dimensional symplectic vector space. Let*

$$\begin{aligned}&e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+k},\\ &f_1, \ldots, f_n\end{aligned}$$

be linearly independent vectors satisfying (§). Then we can find the missing $f_{n+1}, \ldots, f_{n+k}$ to complete the symplectic basis.

Proof. Suppose we have found $f_{n+1}, \ldots, f_{n+i}$ already such that

$$\begin{aligned}&e_1, \ldots, e_n, e_{n+1}, \ldots e_{n+i}, e_{n+i+1}, e_{n+i+2}, \ldots, e_{n+k},\\ &f_1, \ldots, f_n, f_{n+1}, \ldots, f_{n+i}\end{aligned}$$

satisfy (§). Notice these vectors are linearly independent: in a linear combination that sums to 0, the coefficients of $e_1, f_1, \ldots, e_{n+i}, f_{n+i}$ must be zero, and we assumed the linear independence of $e_{n+i+1}, \ldots, e_{n+k}$. By extending these to a basis B of $\mathbb{W}$, we may define a linear function

$$\psi : \mathbb{W} \to \mathsf{f}$$

which sends e_{n+i+1} to 1 but every other $b \in B$ to 0. Now apply Proposition III.5 to obtain a symplectic basis $e'_1, f'_1, \ldots, e'_m, f'_m$ of $\mathbb{W}$, and put

$$f_{n+i+1} = \sum_{j=1}^{m} \psi(e'_j) f'_j - \psi(f'_j) e'_j;$$

then $\omega(-, f_{n+i+1})$ agrees with ψ on this symplectic basis, so by linearity they must be the same function. In particular

$$\omega(e_{n+i+1}, f_{n+i+1}) = \psi(e_{n+i+1}) = 1,$$

whereas $\psi(e_1), \ldots, \psi(e_{n+k}), \psi(f_1), \ldots, \psi(f_{n+i})$ are all 0 Thus we have (§) as required. □

Given two symplectic vector spaces $\mathbb{W}$ and $\mathbb{W}'$, we call a function α between $X \subseteq \mathbb{W}$ and $X' \subseteq \mathbb{W}'$ *isometric* if $\omega(\alpha(x_1), \alpha(x_2)) = \omega(x_1, x_2)$ for all $x_1, x_2 \in X$. We can make an easy observation:

Lemma III.7. *Let $\{e_i, f_i \mid i \in I\} \subseteq \mathbb{W}$, $\{e'_j, f'_j \mid j \in J\} \subseteq \mathbb{W}'$ be two symplectic bases and let $\alpha : I \to J$ be a bijection. Then*

$$e_i \mapsto e'_{\alpha(i)}, f_i \mapsto f'_{\alpha(i)}$$

defines an isometric linear bijection $\langle e_i, f_i \rangle \to \langle e'_j, f'_j \rangle$.

It then follows from Proposition III.5 that, up to isometric linear bijections, $\mathbb{W}_0, \mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_\infty$ are all the countable symplectic vector spaces. Whilst we may deduce that $\mathbb{W}_\infty$ is oligomorphic by appealing to Ryll-Nardzewski, we will opt for a more direct proof that also establishes smooth approximation.

Proposition III.8 (Witt Extension)**.** *Any isometric linear injection $\alpha : \langle X \rangle \subseteq \mathbb{W}_n \to \mathbb{W}_n$ can be extended to an automorphism of $\mathbb{W}_n$ (i.e., an isometric linear bijection) and in turn to one of $\mathbb{W}_\infty$.*

Proof. To begin with, find a basis $x_1, \ldots, x_k$ for the subspace $W = \{w \in \langle X \rangle \mid \forall x \in X : \omega(w, x) = 0\}$ and extend it to a basis $x_1, \ldots, x_k, x_{k+1}, \ldots, x_d$ for $\langle X \rangle$. Notice that

$$U = \langle x_{k+1}, \ldots, x_d \rangle$$

must be a symplectic subspace: as it intersects with W trivially, given any non-zero vector $u \in U$ we must have $0 \neq \omega(u, w + u') = \omega(u, u')$ for some $w \in W$ and $u' \in U$. Hence use Proposition III.5 to find a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ for U. Observe that

$$\begin{aligned}&e_1, \ldots, e_n, x_1, \ldots, x_k,\\ &f_1, \ldots, f_n\end{aligned}$$

form a basis for $\langle X \rangle$ and satisfy (§). On the other hand,

$$\begin{aligned}&\alpha(e_1), \ldots, \alpha(e_n), \alpha(x_1), \ldots, \alpha(x_k),\\ &\alpha(f_1), \ldots, \alpha(f_n)\end{aligned}$$

form a basis for $\alpha(\langle X \rangle)$ and also satisfy (§). Therefore apply Proposition III.6 twice to find the missing $y_1, \ldots, y_k$ and $y'_1, \ldots, y'_k$ to complete the two symplectic bases — call them $\mathcal{B}$ and $\mathcal{B}'$. They are of the same size.

Now, by using Proposition III.5, extend $\mathcal{B}$ and $\mathcal{B}'$ to symplectic bases $\mathcal{C}$ and $\mathcal{C}'$ that span $\mathbb{W}_n$. These must both have size $2n$, so by Lemma III.7 we obtain an isometric linear automorphism $\beta : \mathbb{W}_n \to \mathbb{W}_n$ extending α.

To finish, notice that $\mathcal{C}, e_{n+1}, \ldots, f_{n+1}, \ldots$ as well as $\mathcal{C}', e_{n+1}, \ldots, f_{n+1}, \ldots$ form a symplectic basis spanning $\mathbb{W}_\infty$. We obtain from Lemma III.7 another time an isometric linear automorphism $\gamma : \mathbb{W}_\infty \to \mathbb{W}_\infty$ extending β that is the identity almost everywhere. □

Proposition III.9. *$\mathbb{W}_\infty^k$ has precisely $\sum_{d=0}^{k} \begin{bmatrix} k \\ d \end{bmatrix}_q \cdot q^{\binom{d}{2}}$ orbits under $\mathrm{Aut}(\mathbb{W}_\infty)$, where $q = |\mathsf{f}|$ and*

$$\begin{bmatrix} k \\ d \end{bmatrix}_q = \frac{(q^k - 1)(q^{k-1} - 1) \cdots (q^{k-d+1} - 1)}{(q^d - 1)(q^{d-1} - 1) \cdots (q^1 - 1)}$$

is the q-binomial coefficient.

Remark III.10. To anticipate the next subsection, we note a similarity with the Rado graph: in $\mathbb{G}^k$ there are $\sum_{d=0}^{k} \begin{Bmatrix} k \\ d \end{Bmatrix} \cdot 2^{\binom{d}{2}}$ orbits — we may impose any edge relation on d vertices.

Proof. To each $v_\bullet \in \mathbb{W}_\infty^k$ we associate a *type*, which comprises the following data:

1) pivot indices $I \subseteq \{1,\dots,k\}$ containing every i such that v_i is not spanned by $v_1,\dots,v_{i-1}$ — so we inductively ensure that
$$\{v_{i'} \mid i' \in I, i' \leq i\}$$
is a basis for $\langle v_1,\dots,v_i\rangle$;
2) for each $j \notin I$, an assignment $\Lambda_j : \{i \in I \mid i < j\} \to \mathsf{f}$ such that $v_j = \sum_{i \in I, i<j} \Lambda_j(i) v_i$;
3) a map $\Omega : \binom{I}{2} \to \mathsf{f}$ defined by $\Omega(\{i' < i\}) = \omega(v_{i'}, v_i)$.

If $\pi : \mathbb{W}_\infty \to \mathbb{W}_\infty$ is an isometric linear bijection, then $v_\bullet = (v_1,\dots,v_k)$ and $\pi \cdot v_\bullet = (\pi(v_1),\dots,\pi(v_k))$ evidently share the same type. Conversely, if $w_\bullet$ has the type of $v_\bullet$, then

$$\begin{aligned}\alpha : \langle v_i \mid i \in I\rangle &\to \langle w_i \mid i \in I\rangle \subseteq \mathbb{W}_n \\ v_i &\mapsto w_i\end{aligned}$$

gives an isometric linear injection for some large enough n. Observe that α must send $v_j \mapsto w_j$ for $j \notin I$ too, and that it may be extended to an automorphism π of $\mathbb{W}_\infty$ by Propsoition III.8. Furthermore we can find some $v_\bullet$ that realises any given type $(I, \{\Lambda_j\}_j, \Omega)$: it suffices to put

$$v_i = e_i + \sum_{i' \in I, i' < i} \Omega(i', i) f_{i'}$$

for $i \in I$ and $v_j = \sum_{i \in I, i<j} \in \Lambda_j(i) v_i$ for $j \notin I$. Therefore the number of types is precisely the number of orbits in $\mathbb{W}_\infty^k$.

Finally, we do some combinatorics. Fix $0 \leq d \leq k$ and count the number of types with $|I| = d$. There are $q^{\binom{d}{2}}$ choices for Ω and say $\#_{k,d}$ choices for the Λ_j's; the two can be chosen independently. In total, this gives

$$\sum_{d=0}^{k} q^{\binom{d}{2}} \cdot \#_{k,d}$$

types for vectors in $\mathbb{W}_\infty^d$. So focus on $\#_{k,d}$, the number of *linear types* — i.e., $(I, \{\Lambda_j\}_j)$, ignoring Ω — in $\mathbb{W}_\infty^k$. (Incidentally $\sum_{d=0}^k \#_{k,d}$ is the number of orbits in $\mathbb{W}_\infty^k$ or, more generally, any countable-dimensional f-vector space under linear automorphisms.) On the small values we easily check that

$$\begin{aligned}&\#_{0,0} = 1 = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]_q, \\ &\#_{1,0} = 1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_q, \qquad \#_{1,1} = 1 = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]_q.\end{aligned}$$

Given a linear type in $\mathbb{W}_\infty^k$ with $|I| = d$, we either have $1 \in I$ or $I \subseteq \{2,\dots,k\}$. In the first case, the linear type is specified by one of the $\#_{k-1,d-1}$ linear types in $\mathbb{W}_\infty^{k-1}$ together with how v_1 is involved in the span of the $(k-1)-(d-1)$ non-pivot vectors. In the second case, the linear type is simply one of the $\#_{k-1,d}$ linear types in $\mathbb{W}_\infty^{k-1}$. Thus

$$\begin{aligned}\#_{k,d} &= q^{k-d} \cdot \#_{k-1,d-1} + \#_{k-1,d} \\ &= q^{k-d} \cdot \begin{bmatrix}k-1\\d-1\end{bmatrix}_q + \begin{bmatrix}k-1\\d\end{bmatrix}_q = \begin{bmatrix}k\\d\end{bmatrix}_q. \qquad \square\end{aligned}$$

Theorem III.11. *The symplectic vector space $\mathbb{W}_\infty$ is smoothly approximated by $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$.*

Corollary III.12. *The symplectic f-vector space $\mathbb{W}_\infty$ has the finite length property over any field of characteristic 0.*

B. Symplectic graphs

For this subsection let f be the two-element field f_2.

Definition III.13. For $n = 0,1,2,\dots$, the *symplectic graph* $\widetilde{\mathbb{W}}_n$ has vertices $\mathbb{W}_n$ and edges

$$v_1 \sim v_2 \iff \omega(v_1, v_2) = 1.$$

This is indeed an undirected graph: as ω is alternating, we have $\omega(v_1,v_2) = -\omega(v_2,v_1) = \omega(v_2,v_1)$ over f_2.

Proposition III.14. $\mathrm{Aut}(\widetilde{\mathbb{W}}_n) = \mathrm{Aut}(\mathbb{W}_n)$.

Proof. Clearly any isometric linear automorphism of $\mathbb{W}_n$ is a graph automorphism of $\widetilde{\mathbb{W}}_n$. Conversely, any $f \in \widetilde{\mathbb{W}}_n$ is evidently isometric. To show that f is linear, take $\lambda_1, \lambda_2 \in \mathsf{f}$ and $v_1, v_2 \in \mathbb{W}$. We calculate:

$$\begin{aligned}&\omega\Big(f(\sum_i \lambda_i v_i) - \sum_i \lambda_i f(v_i), f(w)\Big) \\ &= \omega\Big(f(\sum_i \lambda_i v_i), f(w)\Big) - \sum_i \lambda_i \omega\big(f(v_i), f(w)\big) \\ &= \omega\Big(\sum_i \lambda_i v_i, w\Big) - \sum_i \lambda_i \omega(v_i, w) \\ &= \omega(0, w) = 0\end{aligned}$$

for all $f(w) \in f(\mathbb{W}_n) = \mathbb{W}_n$; since ω is non-degenerate, we conclude that $f(\sum_i \lambda_i v_i) = \sum_i \lambda_i f(v_i)$. $\square$

So the number of orbits in $\widetilde{\mathbb{W}}_n^k$ is precisely equal to the number of orbits in $\mathbb{W}_n^k$ — in particular, it is bounded above by $\sum_{d=0}^k \left[\begin{smallmatrix}k\\d\end{smallmatrix}\right]_2 \cdot 2^{\binom{d}{2}}$ independently of n by Proposition III.9.[1] It remains to show $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$ embeds all finite graphs:

Proposition III.15 ([2, Theorem 8.11.2])**.** *Every graph on at most $2n$ vertices embeds into $\widetilde{\mathbb{W}}_n$.*

Proof. Let G be a graph on at most $2n$ vertices. The conclusion is trivial when $n = 0$. Also, if G contains no edges, we can choose any $2n$ of the 2^n vectors in $\langle e_1,\dots,e_n\rangle \subseteq \widetilde{\mathbb{W}}_n$.

So suppose $n \geq 1$ and G has an edge $s \sim t$. Let $G_{s,t}$ be the graph on vertices $G \setminus \{s,t\}$ with edges which we will specify later. By induction, some embedding $f : G_{s,t} \to \widetilde{\mathbb{W}}_{n-1}$ exists. Define $f' : G \to \widetilde{\mathbb{W}}_n$ by

$$\begin{aligned}x \in G_{s,t} &\mapsto f(x) - [\![x \sim s]\!] f_n + [\![x \sim t]\!] e_n \\ s &\mapsto e_n \\ t &\mapsto f_n\end{aligned}$$

[1]This is the kth term in the OEIS sequence A028631.

where $[\![\phi]\!]$ is 1 if ϕ holds and 0 otherwise. Then we have $\omega(f'(x), f'(s)) = [\![x \sim s]\!]$ and $\omega(f'(x), f'(t)) = [\![x \sim t]\!]$ as desired, on one hand. On the other,

$$\begin{aligned}\omega(f'(x_1), f'(x_2)) = [\![x_1 \sim x_2]\!] + [\![x_1 \sim s]\!][\![x_2 \sim t]\!] \\ + [\![x_1 \sim t]\!][\![x_2 \sim s]\!]\end{aligned}$$

tells us how we should define the edge relation in $G_{s,t}$ for f' to be an embedding of graphs. $\square$

Theorem III.16. *The Rado graph is oligomorphically approximated by $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$.*

Corollary III.17. *The Rado graph has the finite length property over any field of characteristic* 0*.*

This proof of finite length also applies to *oriented graphs* (i.e., $x \to y \implies y \not\to x$ but unlike in a tournament, it may occur that $x \not\to y \wedge y \not\to x$) — use the three-element field instead of f_2.

IV. Rado graph, with cogs

In this section we work with the following setting:

- $\mathcal{L}_0$ is a (should we just assume finite?) relational language consisting of unary and binary symbols;
- $\mathcal{C}_0$ is a free amalgamation class of finite $\mathcal{L}_0$-structures, where every $R \in \mathcal{L}_0$ is interpreted irreflexively.[2]
- $\mathcal{L}$ consists of $\mathcal{L}_0$ together with a new binary symbol $<$;
- $\mathcal{C}$ consists of $\mathcal{L}$-structures obtained from $\mathcal{C}_0$ by expanding with all possible linear orderings — this is still an amalgamation class;
- $\mathbb{A}_0$ and $\mathbb{A}$ are the respective Fraïssé limits of $\mathcal{C}_0$ and $\mathcal{C}$, where without loss of generality (because of the extension property) we assume $\mathbb{A}_0$ and $\mathbb{A}$ share the same domain so that $\mathrm{Aut}(\mathbb{A}_0) \supseteq \mathrm{Aut}(\mathbb{A})$.

Example IV.1. Take $\mathcal{L}_0$ to consist of $=$ only and $\mathcal{C}_0$ to be all finite sets. Then $\mathbb{A}_0$ is isomorphic to the pure set $\mathbb{N}$, whereas $\mathbb{A}$ is isomorphic to $\mathbb{Q}$ with the usual order.

Example IV.2. Let $\mathcal{L}_0$ consist of $=$ together with a single binary symbol $\sim$ and let $\mathcal{C}_0$ consist of all finite undirected graphs not embedding the complete graph K_n, where $3 \leq n$ $(\leq \infty)$. Then $\mathbb{A}_0$ is the K_n-free Henson graph (or the Rado graph when $n = \infty$), and $\mathbb{A}$ is its generically ordered counterpart. (Allowing $n = 2$ makes these degenerate to $\mathbb{N}$ and $\mathbb{Q}$ above).

Free amalgamation in $\mathcal{C}_0$ allows us to free atoms in $\mathbb{A}$ from undesired relations. Let us make this more precise. Say that atoms $a, b \in \mathbb{A}$ are *related* if $a = b$ or $\mathbb{A} \models R(a, b) \vee R(b, a)$ for some $R \in \mathcal{L}_0$ (they are certainly related by $<$, but we disregard it here).

Lemma IV.3. *Let $X, Y, \{z\} \subseteq \mathbb{A}$ be disjoint and finite. Then there is some automorphism $\tau \in \mathrm{Aut}(\mathbb{A})$ such that*

1) *τ fixes every $x \in X$;*
2) *$\tau(z)$ is unrelated to all $y \in Y$ and to z;*
3) *$\tau(z) > z$.*

Proof. In $\mathbb{A}_0$, form the free amalgam

$$\begin{tikzcd} & X \cup Y \cup \{z\} \arrow[dr, dashed, "\subseteq"] & \\ X \arrow[ur] \arrow[dr] & & X \cup Y \cup \{z, z'\} \\ & X \cup \{z\} \arrow[ur, dashed, "x \in X \mapsto x, z \mapsto z'"'] & \end{tikzcd}$$

so that no element of $Y \cup \{z\}$ is related to z'. Now we make $X \cup Y \cup \{z, z'\}$ an $\mathcal{L}$-structure: inherit the order on $X \cup Y \cup \{z\}$ from $\mathbb{A}$, and declare that $z < z'$ as well as $z' < a$ if a, the next element of $X \cup Y$ larger than z, exists at all. Observe that

$$x \in X \mapsto x, z \mapsto z'$$

is still an embedding in presence of the order. By homogeneity, we may embed $X \cup Y \cup \{z, z'\}$ into $\mathbb{A}$ via some f which is the identity on $X \cup Y \cup \{z\}$; again by homogeneity, we may extend the embedding

$$f(x) = x \in X \mapsto f(x), f(z) \mapsto f(z')$$

to some automorphism τ which makes 1), 2), and 3) true. $\square$

On the other hand, an $\mathcal{L}$-structure fails to embed into $\mathbb{A}$ precisely when it embeds some forbidden structure, in which every two distinct elements are related:

Lemma IV.4. *Let $\mathcal{F}_0$ consist of minimal (with respect to $\subseteq$) $\mathcal{L}_0$-structures which do not appear in $\mathcal{C}_0$. Then*

1) *$\mathcal{C}_0$ consists of every $\mathcal{L}_0$-structure that does not embed any $F \in \mathcal{F}_0$.*
2) *$\mathcal{C}$ consists of every $\mathcal{L}$-structure whose $\mathcal{L}_0$-reduct does not embed any $F \in \mathcal{F}_0$.*
3) *In any $F \in \mathcal{F}_0$, every two distinct elements $x, y \in F$ are related by some $R \in \mathcal{L}_0$.*

Proof. As $\mathcal{C}_0$ is closed under substructures, its complement is closed under superstructures and thus is — since there are no infinite strictly descending chain of embedded substructures — determined by its minimal structures. 2) follows because an $\mathcal{L}$-structure is in $\mathcal{C}$ precisely when its $\mathcal{L}_0$-reduct is in $\mathcal{C}_0$. For 3), notice that $F \setminus \{x\}$, $F \setminus \{y\}$ are in $\mathcal{C}_0$ by minimality; therefore so is their free amalgam over $F \setminus \{x, y\}$, which then cannot agree with F. $\square$

In what follows, we will juggle with Lemma IV.3 just enough so that we avoid the forbidden structures described in Lemma IV.4. A main result will be:

Theorem IV.5. *$\mathbb{A}$ has the finite length property even with finitely many constants fixed, provided that $\mathbb{A}$ is oligomorphic (for instance if $\mathcal{L}_0$ is finite).*

And a corollary will be that $\mathbb{A}$ from Examples IV.1 and IV.2 has the finite length property; as is its reduct $\mathbb{A}_0$.

JY: I changed $o_\bullet$'s to o in subsections A and C

[2] We may assume irreflexivity with no loss of generality: see [5, beginning of §2.4].

A. Two reductions: orbits and projections

To start with, let us view $\mathbb{A}^d$ as $\mathbb{A}^{\{1,\dots,d\}}$ and more generally consider $\mathbb{A}^I$ for a finite totally ordered indexing set $I \subseteq \mathbb{Q}$. Fix a finite support $S \subseteq \mathbb{A}$. If $\mathbb{A}$ is oligomorphic, the tuples in $\mathbb{A}^I$ split into finitely many $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant orbits. Let $\mathcal{O} = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot o$ be one such orbit. We shall call $\mathcal{O}$ *(S-)ordered* if $o_i \notin S$ and if $o_i < o_j$ whenever $i < j$. By removing the entries in o that repeat or come from S and reordering the rest, we can always find an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant bijection to an S-ordered orbit. An easy observation is that we may focus on a single ordered orbit at a time:

Proposition IV.6. *The following are equivalent:*

1) *For $d = 0, 1, 2, \dots$ and any finite $S \subseteq \mathbb{A}$, chains of $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces in $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^d$ are bounded in length;*
2) *$\mathbb{A}$ is oligomorphic, and $\mathrm{Lin}_{\mathsf{F}}\, \mathcal{O}$ has finite length for any ordered orbit $\mathcal{O}$.*

Proof. We have $\mathrm{len}(\mathrm{Lin}_{\mathsf{F}}(\biguplus_i \mathcal{O}_i)) = \mathrm{len}(\bigoplus_i \mathrm{Lin}_{\mathsf{F}}\, \mathcal{O}_i) = \sum_i \mathrm{len}(\mathrm{Lin}_{\mathsf{F}}\, \mathcal{O}_i)$. □

So fix an ordered orbit $\mathcal{O} = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot o \subseteq \mathbb{A}^I$. From here we take an inductive approach. By $o|^J$ we mean the restriction of $o : I \to \mathbb{A}$ to $J \subseteq I$; we will often write $o|^{-i}$ instead of $o|^{I\setminus\{i\}}$. Note the image $\mathcal{O}|^I$ of $\mathcal{O}$ under this projection agrees with $\mathrm{Aut}(\mathbb{A})_{(S)} \cdot o|^J$ and is still ordered.

To anticipate more general statements later, Let E be a finite-dimensional F-vector space — for instance, F itself. Things become more interesting when we lift $(-)|^J$ to a linear $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant map

$$\begin{aligned}(-)|^J : \mathrm{Lin}_{\mathsf{E}}\, \mathcal{O} &\to \mathrm{Lin}_{\mathsf{E}}\, \mathcal{O}|^J \\ v &\mapsto v|^J.\end{aligned}$$

Many cancellations can occur under $(-)|^J$; the *projection kernel* is the $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace

$$\mathrm{Ker}_{\mathsf{E}}\, \mathcal{O} = \bigcap_{i\in I} \ker{(-)|^{-i}}$$

of $\mathrm{Lin}_{\mathsf{E}}\, \mathcal{O}$.

Proposition IV.7. *The following are equivalent:*

1) *$\mathrm{Lin}_{\mathsf{F}}\, \mathcal{O}$ has finite length for every ordered orbit $\mathcal{O}$;*
2) *$\mathrm{Ker}_{\mathsf{F}}\, \mathcal{O}$ has finite length for every ordered orbit $\mathcal{O}$.*

Proof. That 1) implies 2) is clear as $\mathrm{Ker}_{\mathsf{F}}\, \mathcal{O} \subseteq \mathrm{Lin}_{\mathsf{F}}\, \mathcal{O}$.

To prove the other implication, assume 2) and let $\mathcal{O} \subseteq \mathbb{A}^I$. We proceed by induction on $|I|$. If $I = \emptyset$, then $\mathcal{O}$ must be the entire singleton $\mathbb{A}^\emptyset = \{()\}$; as $\mathrm{Lin}_{\mathsf{F}}\, \mathcal{O}$ has no nontrivial subspaces (let alone finitely supported ones), it has length 1. Now if $|I| \geq 1$, assemble all $|I|$ projection maps into a single map

$$\begin{aligned}\mathrm{Lin}_{\mathsf{F}}\, \mathcal{O} &\to \bigoplus_{i\in I} \mathcal{O}|^{-i} \\ v &\mapsto (v|^{-i})_{i\in I}\end{aligned}$$

whose kernel is precisely $\mathrm{Ker}_{\mathsf{F}}\, \mathcal{O}$. We have

$$\mathrm{len}(\mathrm{Lin}_{\mathsf{F}}\, \mathcal{O}) - \mathrm{len}(\mathrm{Ker}_{\mathsf{F}}\, \mathcal{O}) \leq \sum_{i\in I} \mathrm{len}(\mathrm{Lin}_{\mathsf{F}}\, \mathcal{O}|^{-i})$$

which shows that $\mathrm{len}(\mathrm{Lin}_{\mathsf{F}}\, \mathcal{O})$ is finite from the assumptions. □

We call a vector from the projection kernel *balanced*. As we will see in the next subsection, cogs are a prominent example.

B. Cogs

Definition IV.8. Let $\mathcal{O} \subseteq \mathbb{A}^I$ be an S-ordered orbit. An *$\mathcal{O}$-duo* $a_\bullet \parallel b_\bullet$ consists of tuples $a_\bullet, b_\bullet \in \mathcal{O}$ such that:

1) $a_i < b_i$ for all $i \in I$;
2) $b_i < a_j$ for all $i < j \in I$;
3) for any binary R in $\mathcal{L}_0$ (except for $=$) and $i, j \in I$:

$$R(a_i, b_j) \iff R(b_i, a_j) \iff R(a_i, a_j).$$

Remark IV.9. Conditions (1) and (2) specify a total order on the $2|I|$ atoms in a duo. Moreover, thanks to irreflexivity, each a_i is unrelated to its counterpart b_i. Further, given any $J \subseteq I$, the combined tuple $a|_\bullet^{I\setminus J}; b|_\bullet^J$ satisfies the same relations as $a_\bullet; b_\bullet$, so it lies in $\mathcal{O}$. In particular, taking $J = \{i\}$, there is an automorphism π_i that sends a_i to b_i and fixes all the other elements of $a_\bullet$, $b_\bullet$ and S. Finally, all $\mathcal{O}$-duos are in the same $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant orbit.

Definition IV.10. Given $\lambda \in \mathsf{E}$ and an $\mathcal{O}$-duo $a_\bullet \parallel b_\bullet$, the corresponding *$\mathcal{O}$-cog* with coefficient λ is the vector

$$\lambda \cdot a_\bullet \between b_\bullet = \sum_{J\subseteq I} (-1)^{|J|} \lambda \cdot a|_\bullet^{I\setminus J}; b_\bullet^J$$

in $\mathrm{Lin}_{\mathsf{E}}\, \mathcal{O}$. The linear span of all $\mathcal{O}$-cogs with coefficients from E is denoted by $\mathrm{Cog}_{\mathsf{E}}\, \mathcal{O}$.

As remarked above, given any two $\mathcal{O}$-duos there is some $\pi \in \mathrm{Aut}(\mathbb{A})_{(S)}$ such that $\pi \cdot (a_\bullet \parallel b_\bullet) = a'_\bullet \parallel b'_\bullet$ and thus $\pi \cdot (\lambda \cdot a_\bullet \between b_\bullet) = \lambda \cdot a'_\bullet \between b'_\bullet$. Hence $\mathrm{Cog}_{\mathsf{E}}\, \mathcal{O}$ is an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace of $\mathrm{Lin}_{\mathsf{E}}\, \mathcal{O}$ and it is generated by cogs based on a single duo.

Proposition IV.11. $\mathrm{Cog}_{\mathsf{E}}\, \mathcal{O}$ *is contained in* $\mathrm{Ker}_{\mathsf{E}}\, \mathcal{O}$.

Proof. Let $\mathcal{O} \subseteq \mathbb{A}^I$, let $a_\bullet \parallel b_\bullet$ be an $\mathcal{O}$-duo, and let $i \in I$. The subsets of I come in pairs of J and $J \cup \{i\}$, where J is a subset of $I \setminus \{i\}$. The two tuples $a|_\bullet^J; b|_\bullet^{I\setminus J}$ and $a|_\bullet^{J\cup\{i\}}; b|_\bullet^{I\setminus(J\cup\{i\})}$ differ only on the ith entry. But this difference gets erased under $(-)|^{-i}$, so the two corresponding terms in $\lambda \cdot a_\bullet \between b_\bullet$ will cancel out and hence $(\lambda \cdot a_\bullet \between b_\bullet)|^{-i} = 0$ overall. □

In fact, cogs arise anywhere.

Lemma IV.12. *Suppose $a_\bullet \parallel b_\bullet$ is an $\mathcal{O}$-duo, where $\mathcal{O} \subseteq \mathbb{A}^I$ is S-ordered. Given $z \in S$,*

- *write $S' = S \setminus \{z\}$;*
- *let $j \notin I$ be such that $\mathcal{O}' = \mathrm{Aut}(\mathbb{A})_{(S')} \cdot (a_\bullet; z) \subseteq \mathbb{A}^{I\cup\{j\}}$ is ordered,*

- *let $X \subseteq \mathbb{A}$ be a finite set containing $\{a_i, b_i \mid i \in I\} \cup S'$ but not z;*
- *let $Y \subseteq \mathbb{A}$ be any finite set disjoint from $X \cup \{z\}$;*

then the $\tau \in \mathrm{Aut}(\mathbb{A})_{(X)}$ afforded by Lemma IV.3 gives us an $\mathcal{O}'$-duo $(a_\bullet; z) \parallel (b_\bullet; \tau(z))$.

Proof. First, notice that $b_\bullet; \tau(z) \in \mathcal{O}'$ and that we have the required order relations with z and $\tau(z)$. The remaining condition of Def. IV.8, for any R in $\mathcal{L}_0$, splits into the following cases (and their symmetric versions):

- $R(a_i, b_j) \iff R(a_i, a_j)$ since $a_\bullet \parallel b_\bullet$ is an $\mathcal{O}$-duo;
- $R(a_i, \tau(z)) \iff R(a_i, z)$ since τ is an automorphism that fixes all a_i;
- $R(a_i, z) \iff R(b_i, z)$ since $a_\bullet, b_\bullet \in \mathcal{O}$ and $z \in S$;
- $R(z, \tau(z))$ and $R(z, z)$ are both false: $\tau(z)$ is unrelated to z by Lemma IV.3, and R is irreflexive.

□

Starting from an empty duo, we may apply the previous lemma inductively.

Proposition IV.13. *Let $\mathcal{O} \subseteq \mathbb{A}^I$ be an S-ordered orbit. Then any $a_\bullet \in \mathcal{O}$ can be extended to an $\mathcal{O}$-duo $a_\bullet \parallel b_\bullet$.*

Proof. Enumerate the indices of I as $i_1, \ldots, i_d$. Suppose that we have found $b_{i_1}, \ldots, b_{i_k}$ such that

$$a|_\bullet^{\{i_1,\ldots,i_k\}} \parallel (i_1 \mapsto b_{i_1}, \ldots, i_k \mapsto b_{i_k})$$

is a duo for $\mathcal{O}_k = \mathrm{Aut}(\mathbb{A})_{(S \cup \{a_{i_{k+1}},\ldots,a_{i_d}\})} \cdot a|_\bullet^{\{i_1,\ldots,i_k\}}$ — note that $() \parallel ()$ is certainly a duo for $\mathcal{O}_0$ at the start. If $k < d$, with $z = a_{i_{k+1}}$, $X = \{a_{i_1}, b_{i_1}, \ldots, a_{i_k}, b_{i_k}\} \cup S \cup \{a_{i_{k+2}}, \ldots, a_{i_d}\}$, and $Y = \emptyset$, a straightforward application of Lemma IV.12 yields an atom $b_{i_{k+1}}$ that makes

$$a|_\bullet^{\{i_1,\ldots,i_k,i_{k+1}\}} \parallel (i_1 \mapsto b_{i_1}, \ldots, i_k \mapsto b_{i_k}, i_{k+1} \mapsto b_{i_{k+1}})$$

a duo for $\mathcal{O}_{k+1}$. For $k = d$ we thus obtain the desired duo for $\mathcal{O}_d = \mathcal{O}$. □

The result below substantiates the slogan that cogs are found everywhere.

Theorem IV.14. *Any $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace V of $\mathrm{Lin}_{\mathsf{E}}\,\mathcal{O}$ contains $\mathrm{Cog}_{\mathsf{E}(V)}\,\mathcal{O}$, where $\mathcal{O} \subseteq \mathbb{A}^I$ is S-ordered and $\mathsf{E}(V)$ is the subspace spanned by $\{v(a_\bullet) \mid v \in V, a_\bullet \in \mathcal{O}\}$ of E.*

Proof. Pick any $v \in V$ and $a_\bullet \in \mathcal{O}$; it is enough to show that V contains $v(a_\bullet) \cdot a_\bullet \between b_\bullet$ for some $\mathcal{O}$-duo $a_\bullet \parallel b_\bullet$. Actually, write

$$S' = S \cup \{c_i \mid v(c_\bullet) \neq 0, i \in I\} \setminus \{a_i \mid i \in I\} \supseteq S$$

and put $\mathcal{O}' = \mathrm{Aut}(\mathbb{A})_{(S')} \cdot a_\bullet \subseteq \mathcal{O}$ — then $\mathcal{O}'$ is S'-ordered. By Proposition IV.13, we can find $b_\bullet \in \mathcal{O}'$ such that $a_\bullet \parallel b_\bullet$ is an $\mathcal{O}'$-duo and *a fortiori* an $\mathcal{O}$-duo. Take the automorphisms $\pi_{i_1}, \ldots, \pi_{i_d}$ from Remark IV.9, where $i_1, \ldots, i_d$ enumerate I. Now define $v^{(0)} = v$ and

$$v^{(k)} = v^{(k-1)} - \pi_{i_k} \cdot v^{(k-1)}.$$

We can check inductively that for $k = 0, 1, \ldots, d$, with $\mathcal{O}^{(k)} = \{c_\bullet \mid v(c_\bullet) \neq 0, \{c_{i_1}, \ldots, c_{i_k}, \ldots, c_{i_d}\} \supseteq \{a_{i_1}, \ldots, a_{i_k}\}\}$ we have

$$v^{(k)} = \sum_{c_\bullet \in \mathcal{O}^{(k)}} \sum_{J \subseteq \{i_1,\ldots,i_k\}} (-1)^{|J|} v(c_\bullet) \prod_{j \in J} \pi_j \cdot c_\bullet.$$

But $\{c_{i_1}, \ldots, c_{i_d}\} \supseteq \{a_{i_1}, \ldots, a_{i_d}\}$ means that $c_\bullet = a_\bullet$, so at the end $v^{(d)}$ is the desired $\mathcal{O}$-cog. □

Corollary IV.15. $\mathrm{Cog}_{\mathsf{F}}\,\mathcal{O}$ *has length* 1.

Proof. Let $V \subseteq \mathrm{Cog}_{\mathsf{F}}\,\mathcal{O}$ be a non-zero $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace. Then $\{0\} \subsetneq \mathsf{E}(V) \subseteq \mathsf{E} = \mathsf{F}$ so E must be the entire field F, and by above V must be $\mathrm{Cog}_{\mathsf{F}}\,\mathcal{O}$ itself. □

In light of Propositions IV.6 and IV.7, we will be able to prove the finite length property for an oligomorphic structure with free amalgamation over any field and support if we know $\mathrm{Ker}_{\mathsf{F}}\,\mathcal{O} = \mathrm{Cog}_{\mathsf{F}}\,\mathcal{O}$. Let us now attempt to show that.

C. Subvectors

This is a good time to recall a view we have tacitly taken: with $\mathcal{O}$ as a standard basis, a vector $v \in \mathrm{Lin}_{\mathsf{E}}\,\mathcal{O}$ is just a finite set of pairs in $\mathsf{E} \times \mathcal{O}$. A *subvector* of v is a subset of these pairs.

Now suppose as usual that $\mathcal{O} \subseteq \mathbb{A}^I$ is S-ordered. Given $i \in I$ and $a \in \mathcal{O}$, we write

$$\mathcal{O}^{i:a_i} = \{b \in \mathcal{O} \mid b_i = a_i\};$$

this is an $\mathrm{Aut}(\mathbb{A})_{Sa_i}$-orbit, and its projection $\mathcal{O}^{i:a_i}|^{-i} = \mathrm{Aut}(\mathbb{A})_{Sa_i} \cdot a|^{-i}$ is ordered. For a vector $v \in \mathrm{Lin}_{\mathsf{E}}\,\mathcal{O}$, by

$$v^{i:a_i} \in \mathrm{Lin}_{\mathsf{E}}\,\mathcal{O}^{i:a_i}$$

we mean the subvector consisting of all pairs in $\mathsf{E} \times \mathcal{O}^{i:a_i}$.

Lemma IV.16. *Let $v \in \mathrm{Lin}_{\mathsf{E}}\,\mathcal{O}$ be balanced. Then any projected subvector $v^{i:a_i}|^{-i} \in \mathrm{Lin}_{\mathsf{E}}\,\mathcal{O}^{i:a_i}|^{-i}$ is also balanced.*

Proof. Let $j \in I \setminus \{i\}$. By assumption we have

$$0 = v|^{-j} = \sum_a v^{i:a_i}|^{-j}$$

in $\mathrm{Lin}_{\mathsf{E}}\,\mathbb{A}^{I \setminus \{j\}}$, so by looking at ith entries we see that each $v^{i:a_i}|^{-j}$ must be the zero vector. Hence so is $v^{i:a_i}|^{-j}|^{-i} = v^{i:a_i}|^{-i}|^{-j}$, which shows that $v^{i:a_i}|^{-i}$ is in the projection kernel. □

So we can try to prove $\mathrm{Ker}_{\mathsf{E}}\,\mathcal{O} \subseteq \mathrm{Cog}_{\mathsf{E}}\,\mathcal{O}$ for any ordered $\mathcal{O} \subseteq \mathbb{A}^I$ by inducting on $|I|$.

D. Special case: unobstructed vectors

We will begin by showing any $v \in \mathrm{Ker}_{\mathsf{E}}\, \mathcal{O}$ lies in $\mathrm{Cog}_{\mathsf{E}}\, \mathcal{O}$ provided that v satisfies an additional condition which, as we will explain in the subsection, may be assumed without loss of generality. We motivate and introduce this condition now.

For a vector $v \in \mathrm{Lin}_{\mathsf{E}}\, \mathcal{O}$ where $\mathcal{O}$ is S-ordered, define:

$$[v] = \{a \in \mathcal{O} \mid v(a) \neq 0\}.$$

Take any atom a_i for $a \in [v]$, and let $P, R \in \mathcal{L}_0$ be unary and binary respectively. Then whether $P(a_i)$, $R(a_i, s)$, $R(s, a_i)$ hold in $\mathbb{A}$ for some $s \in S$ is determined by $\mathcal{O}$ and i, and so it does not depend on the choice of a.

Now take another atom b_j for some $b \in [v]$. What can be said about $R(a_i, b_j)$? Unless $a = b$, not much. The index j may not even be unique: we may well have $a_i = b_j$ even if $i \neq j$. We want to avoid such confusions:

Definition IV.17. Let $\mathcal{O}$ be an S-ordered orbit in $\mathbb{A}^I$, and fix some representative $o \in \mathcal{O}$. Call a finite family $V \subseteq \mathcal{O}$

1) *unambiguous* if the assignment $\sqrt{-} : a_i \mapsto o_i$ is a well-defined function on the atoms present in V, i.e., if for any $a, b \in V$, if $a_i = b_j$ then $i = j$;
2) *unobstructed* if it is unambiguous and for any $a, b \in V$ and $i, j \in I$, either a_i and b_j are unrelated or

$$R(a_i, b_j) \iff R(o_i, o_j)$$

for every binary relation $R \in \mathcal{L}_0$.

(Obviously these properties do not depend on the choice of $o \in \mathcal{O}$.) Furthermore, we call a vector $v \in \mathrm{Lin}_{\mathsf{E}}\, \mathcal{O}$ unambiguous or unobstructed if the set $[v]$ is so.

For example, it is easy to see that cogs are unobstructed.

Theorem IV.18. *Let $v \in \mathrm{Ker}_{\mathsf{E}}\, \mathcal{O}$ be unobstructed. Then we have*

$$v = \sum_{k \in K} v(b^{(k)}) \cdot x^{(k)} \between y^{(k)}$$

for some $b^{(k)}, x^{(k)}, y^{(k)} \in \mathcal{O}$, where moreover

$$[v] \cup \{x^{(k)}, y^{(k)} \mid k \in K\}$$

is unobstructed.

Note that the last part of the theorem does not trivially follow from the first: $[x^{(k)} \between y^{(k)}]$ may contain some tuples which are absent from $[v]$ due to cancellations with other cogs.

We spend the rest of this subsection giving the proof, which proceeds by induction on $|I|$ for S-ordered orbits $\mathcal{O} \subseteq \mathbb{A}^I$ for every S at once.

The base case $I = \emptyset$ is immediate: as $\mathcal{O}$ is the singleton $\{()\}$, any vector $v \in \mathrm{Ker}_{\mathsf{E}}\, \mathcal{O} = \mathrm{Lin}_{\mathsf{E}}\, \mathcal{O} = \mathrm{Cog}_{\mathsf{E}}\, \mathcal{O}$ is a cog already; so the theorem says nothing more than $v = v$.

Now suppose that some $i^* \in I$ exists — in fact, let i^* be maximal. Consider an unobstructed vector $v \in \mathrm{Ker}_{\mathsf{E}}\, \mathcal{O}$. We can decompose v into finitely many subvectors to write

$$v = v^{i^*:a_1} + \cdots + v^{i^*:a_m}.$$

Then each projection $v^{i^*:a_j}|^{-i^*}$ lies in $\mathrm{Ker}_{\mathsf{E}}\, \mathcal{O}^{i^*:a_j}|^{-i^*}$ by Lemma IV.16, and we may straightforwardly check that it is unambiguous and unobstructed. It follows from the inductive hypothesis of Theorem IV.18 that

$$v^{i^*:a_j}|^{-i^*} = \sum_{k \in K_j} v^{i^*:a_j}|^{-i^*}(b^{(a_j,k)}) \cdot x^{(a_j,k)} \between y^{(a_j,k)}$$

for some tuples $b^{(a_j,k)}, x^{(a_j,k)}, y^{(a_j,k)} \in \mathcal{O}^{i^*:a_j}|^{-i^*}$ such that $[v^{i^*:a_j}|^{-i^*}] \cup \{x^{(a_j,k)}, y^{(a_j,k)} \mid k \in K_j\}$ is unobstructed. We can return to $\mathcal{O}$ by adding a_j back as the i^*th term to every tuple: we get

$$v^{i^*:a_j} = \sum_{k \in K_j} \Big(v(b^{(a_j,k)}; a_j) \cdot x^{(a_j,k)} \between y^{(a_j,k)}\Big); a_j,$$

where the family

$$U_j = [v^{i^*:a_j}] \cup \{(x^{(a_j,k)}; a_j), (y^{(a_j,k)}; a_j) \mid k \in K_j\}$$

is easily seen to be unobstructed. We can ask for more:

Claim IV.19. *We may choose $x^{(a_j,k)}$'s, and $y^{(a_j,k)}$'s so that*

$$V_0 = [v], \qquad V_j = V_{j-1} \cup U_j$$

is unobstructed for $j = 1, 2, \ldots, m$.

Proof. Let A_j denote the atoms that appear in $[v^{i^*:a_j}]$. Given any automorphism $\pi \in \mathrm{Aut}(\mathbb{A})$ that fixes S as well as A_j, using the tuples $\pi \cdot x^{(a_j,k)}$ and $\pi \cdot y^{(a_j,k)}$ gives us another decomposition of $v^{i^*:a_j}|^{-i^*}$ that is unobstructed. We will show that composing such automorphisms — which will be provided by Lemma IV.3 — suffices to make the claim hold.

Assume that V_{j-1} is unobstructed. Suppose that V_j is not even unambiguous. Then for some $b \in V_j \setminus V_{j-1} \subseteq U_j$ and $c \in V_{j-1}$ have $b_i = c_{i'}$ with $i \neq i'$. Now if we were to have $b_i \in A_j$ (e.g., if $b_i = a_j$), then $b_i = d_{i''}$ for some $d \in [v^{i^*:a_j}] \subseteq [v]$. Since U_j is unambiguous, $i = i''$; and since V_{j-1} is unambigous, $i'' = i'$. This contradicts the assumption that $i \neq i'$.

So the ambiguous atom b_i does not belong to A_j. In other words, the set X of all atoms that appear in V_j except b_i contains A_j (and a_j). Apply Lemma IV.3 to get an automorphism τ which fixes $S \cup X$ but sends b_i to a fresh atom. Then, in the new family

$$V_{j-1} \cup \{(\tau \cdot x^{(a_j,k)}; a_j), (\tau \cdot y^{(a_j,k)}; a_j) \mid k \in K_j\},$$

the ambiguous atoms are precisely the ones in the old family minus b_i — these are all fixed by τ. We continue this way until all ambiguous atoms are freshened.

Hence assume V_j is unambiguous. Suppose b_i and $c_{i'}$, for some $b, c \in V_j$, are the reason why V_j fails to be unobstructed. To have b_i in A_j is impossible: otherwise $b_i = d_{i''}$ for some $d \in [v^{i^*:a_j}] \subseteq [v]$ and since V_j is unambiguous, we get $i = i''$. But d_i (hence b_i) cannot be part of an obstruction in V_j because both U_j and V_{j-1} are unobstructed. Symmetrically, we see that $c_{i'} \notin A_j$.

Since V_{j-1} is unobstructed, b (or, symmetrically, c) must be one of the $x^{(a_j,k)}; a_j$ or $y^{(a_j,k)}; a_j$. This time, split all atoms that appear in V_j except b_i into two sets X and Y: put any atom $c_{i'}$ which makes b_i an obstruction in Y, and put everything else in X. Then X contains A_j as discussed above; it also contains $\{x_i^{(a_j,k)}, y_i^{(a_j,k)} \mid k \in K_j, i \in I\}$. Use Lemma IV.3 to get $\tau \in \mathrm{Aut}(\mathbb{A})_{(S\cup X)}$ which sends b_i to a fresh atom disjoint from $S \cup X \cup Y \cup \{b_i\}$ that is unrelated to everything in Y. Observe that the new family

$$V_{j-1} \cup \{(\tau \cdot x^{(a_j,k)}; a_j), (\tau \cdot y^{(a_j,k)}; a_j) \mid k \in K_j\},$$

remains unambiguous. Moreover, the obstructions here are precisely the ones from V_j except the ones that involve b_i, which are all fixed by τ. So we may repeat this process until all obstructions are removed. $\square$

Having chosen the fresh atoms carefully, let X consist of all the atoms appearing in the unobstructed family V_m. It will be convenient to fix a representative $o \in \mathcal{O}$ and recall the function

$$\sqrt{-} : X \to \{o_1, \ldots, o_d\}$$

introduced in Definition IV.17. We now add a new element b^* to the finite $\mathcal{L}$-structure $S \cup X \subseteq \mathbb{A}$ so that $(x^{(a_j,k)}; a_j) \parallel (y^{(a_j,k)}; b^*)$ becomes a well-formed $\mathcal{O}$-duo for each j. To this end we define b^* to be greater (wrt. $>$) than every atom in X, and the remaining relations are defined to make b^* mimic the greatest atom in o:

$$\begin{aligned}
b^* &> c \quad \text{for } c \in X \\
b^* > s &\iff \mathbb{A} \models o_{i^*} > s \\
P(b^*) &\iff \mathbb{A} \models P(o_{i^*}) \\
R(b^*, s) &\iff \mathbb{A} \models R(o_{i^*}, s) \\
R(s, b^*) &\iff \mathbb{A} \models R(s, o_{i^*}) \\
R(b^*, c) &\iff \mathbb{A} \models R(o_{i^*}, \sqrt{c}) \text{ for } c \in X \\
R(c, b^*) &\iff \mathbb{A} \models R(\sqrt{c}, o_{i^*}) \text{ for } c \in X
\end{aligned}$$

for every unary predicate P and binary relation R in $\mathcal{L}_0$. The last two clauses are well defined (i.e. they do not depend on the choice of $a \in V_m$ that contains the atom c) since V_m is unambiguous. Moreover, since V_m is unobstructed, our definition does not introduce any forbidden substructures:

Claim IV.20. *$S \cup X \cup \{b^*\}$ embeds into $\mathbb{A}$.*

Proof. If not, by Lemma IV.4 there is a forbidden $\mathcal{L}_0$-structure F of pairwise related elements which embeds into the $\mathcal{L}_0$-reduct of $S \cup X \cup \{b^*\}$, via ϕ say. Extend $\sqrt{-}$ from above to:

$$\sqrt{-} : X \cup S \cup \{b^*\} \to \{o_1, \ldots, o_d\} \cup S$$

by putting $\sqrt{b^*} = o_{i^*}$ and $\sqrt{s} = s$ for $s \in S$. We will check that $\sqrt{-} \circ \phi$ then embeds F into the $\mathcal{L}_0$-reduct of $\mathbb{A}$, namely $\mathbb{A}_0$, and reach a contradiction.

So let $P \in \mathcal{L}_0$ be unary and let $R \in \mathcal{L}_0$ be binary. It is immediate that $\sqrt{-}$ preserves and reflects P for all atoms in $X \cup S$ and for b^* — we do not even need to know they are in the image of ϕ. We can also see that $\sqrt{-}$ preserves and reflects R whenever at least one of the arguments is from $S \cup \{b^*\}$. Now consider the remaining case of $\phi(f), \phi(f') \in X$. As f and f' are related in F, their images under ϕ must be related in $\phi(F)$. But the family V_m that gave rise to X is unobstructed, which forces $\sqrt{-}$ to preserve and reflect R here as well. Since $\mathcal{L}_0$ only has unary and binary relations, this means that $\sqrt{\phi(F)} \subseteq \mathbb{A}_0$ is in $\mathcal{C}_0$ yet embeds F; this is a contradiction. $\square$

Using homogeneity we may assume that $S \cup X \cup \{b^*\} \subseteq \mathbb{A}$.

Claim IV.21. *$x^{(a_j,k)}; a_j \parallel y^{(a_j,k)}; b^*$ forms an $\mathcal{O}$-duo for $1 \le j \le m$ and $k \in K_j$. Furthermore, the family*

$$[v] \cup \{(x^{(a_j,k)}; a_j), (y^{(a_j,k)}; b^*) \mid k \in K_j, 1 \le j \le m\}$$

is unobstructed.

Proof. Recall that $x^{(a_j,k)} \parallel y^{(a_j,k)}$ is a duo for $\mathcal{O}^{i^*:a_j}|^{-i^*}$. Since $x^{(a_j,k)}, y^{(a_j,k)} \in X$, the remaining conditions for $x^{(a_j,k)}; a_j \parallel y^{(a_j,k)}; b^*$ being an $\mathcal{O}$-duo follow directly from our definition of relations on b^*.

The family in the claim is unambiguous since V_m is unambiguous and b^* is a fresh atom. Similarly, it is unobstructed since V_m is unobstructed, and the additional conditions that involve b^* follow from the relations that we imposed on b^*.

BK: This argument feels a bit shaky; I will think about it some more. $\square$

Claim IV.22. *We have $v = \sum_{j=1}^m \sum_{k\in K_j} v(b^{(a_j,k)}; a_j) \cdot (x^{(a_j,k)}; a_j \between y^{(a_j,k)}; b^*)$.*

Proof. Observe that, by definition of cogs, for any $\mathcal{O}$-duo $a; c \parallel b; d$ with $c, d \in \mathbb{A}$ we have

$$(a; c) \between (b; d) = (a \between b); c - (a \between b); d.$$

Using this, calculate:

$$\begin{aligned}
&\sum_{j=1}^{m} \sum_{k\in K_j} v(b^{(a_j,k)}) \cdot (x^{(a_j,k)}; a_j \between y^{(a_j,k)}; b^*) \\
&= \sum_{j=1}^{m} \sum_{k\in K_j} \left(v(b^{(a_j,k)}) \cdot x^{(a_j,k)} \between y^{(a_j,k)}\right); a_j \\
&\quad - \sum_{j=1}^{m} \sum_{k\in K_j} \left(v(b^{(a_j,k)}) \cdot x^{(a_j,k)} \between y^{(a_j,k)}\right); b^* \\
&= \sum_{j=1}^{m} v^{i^*:a_j} - \sum_{j=1}^{m} v^{i^*:a_j}|^{-i^*}; b^* \\
&= v - \sum_{j=1}^{m} v^{i^*:a_j}|^{-i^*}; b^*,
\end{aligned}$$

and it suffices to show the last sum vanishes. Recall from Proposition IV.11 that cogs are balanced. By projecting away the i^*th entry, we obtain

$$0 = 0 - \left(\sum_{j=1}^{m} v^{i^*:a_j}|^{-i^*}; b^* \right) \Bigg|^{-i^*}.$$

But we can simply add back b^* as the i^*th entry, yielding $0 = \sum_{j=1}^{m} v^{i^*:a_j}|^{-i^*}; b^*$ as needed — replacing the i^*th entry by a common atom achieves the same effect of projecting it away. $\square$

This completes the proof of Theorem IV.18: we have shown any balanced vector v is spanned by cogs as long as $[v]$ is unobstructed. Let us lift this restriction.

E. Removing ambiguities and obstructions

Theorem IV.23. *Let $v \in \mathrm{Ker}_{\mathsf{E}}\,\mathcal{O}$ be arbitrary. Then (assuming as before that $\mathcal{L}_0$ is at most binary) we have*

$$v = \sum_{k\in K} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

for some $b_\bullet^{(k)}, x_\bullet^{(k)}, y_\bullet^{(k)} \in \mathcal{O}$.

As a reminder, $\mathcal{O} \subseteq \mathbb{A}^I$ is an S-ordered orbit and E is a finite-dimensional F-vector space. We induct on $|I|$, noting that when $I = \emptyset$ we are just saying $v = v() \cdot (\between)$. Hereafter assume $I \neq \emptyset$.

Proposition IV.24. *Let $v \in \mathrm{Ker}_{\mathsf{E}}\,\mathcal{O}$. Then we can find $b_\bullet^{(k)}, x_\bullet^{(k)}, y_\bullet^{(k)} \in \mathcal{O}$ so that*

$$\ddot{v} = v - \sum_{k\in K_1} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

satisfies $\ddot{v}(\mathcal{O}) = v(\mathcal{O}) \subseteq \mathsf{E}$ and $\mathcal{O}(\ddot{v})$ is unambiguous.

Proof. Assume there is an ambiguity in $\mathcal{O}(v)$, i.e., there are $a_\bullet, a'_\bullet \in \mathcal{O}(v)$ such that $a_i = a'_{i'}$ but $i \neq i'$ (so $|I| \geq 2$). Let us prevent the atom-index pair (a_i, i) from causing ambiguities. By Lemma IV.16, the projected subvector $v^{i:a_i}|^{-i}$ belongs to $\mathrm{Ker}_{\mathsf{E}}\,\mathcal{O}^{i:a_i}|^{-i}$; by the inductive hypothesis of Theorem IV.23 at hand, we have

$$v^{i:a_i} = v^{i:a_i}|^{-i}; a_i = \sum_{k\in K} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); a_i.$$

We can invoke Lemma IV.3 to find an automorphism τ which fixes all of the atoms except a_i that appear in $x_\bullet^{(k)}, y_\bullet^{(k)}$, $k \in K$ and in $b_\bullet \in \mathcal{O}(v)$. It follows from the appropriate modification of Lemma IV.12 that we get $\mathcal{O}$-duos $x_\bullet^{(k)}; a_i \parallel y_\bullet^{(k)}; \tau \cdot a_i$ for $k \in K$. Observe that

$$\begin{aligned} &v^{i:a_i} - \sum_{k\in K} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)}; a_i \between y_\bullet^{(k)}; \tau \cdot a_i) \\ &= v^{i:a_i} - \sum_{k\in K} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); a_i \\ &\quad + \sum_{k\in K_1} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); \tau \cdot a_i \end{aligned}$$

is equal to $v^{i:a_i}|^{-i}; \tau \cdot a_i$, so

$$\begin{aligned} v' &= v - \sum_{k\in K} v(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)}; \tau \cdot a_i \between y_\bullet^{(k)}; a_i) \\ &= v - v^{i:a_i} + v^{i:a_i}|^{-i}; \tau \cdot a_i \end{aligned}$$

is the vector obtained by changing the ith entry in every tuple of v from a_i to $\tau \cdot a_i$, a fresh atom disjoint from all atoms present. Then $v'(\mathcal{O}) = v(\mathcal{O})$. Moreover, we can directly check that an ambiguous atom-index pair (b_j, j) in $\mathcal{O}(v')$ cannot be $(\tau \cdot a_i, i)$, and hence it must already be one in $\mathcal{O}(v)$ except it cannot be (a_i, i). Therefore we may remove all ambiguities by iterating this process. $\square$

Next we tackle the obstructions.

Proposition IV.25. *Let $\ddot{v} \in \mathrm{Ker}_{\mathsf{E}}\,\mathcal{O}$ be such that $\mathcal{O}(\ddot{v})$ is unambiguous. Then we can find $b_\bullet^{(k)}, x_\bullet^{(k)}, y_\bullet^{(k)} \in \mathcal{O}$ so that*

$$\acute{\ddot{v}} = \ddot{v} - \sum_{k\in K_2} \ddot{v}(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

satisfies $\acute{\ddot{v}}(\mathcal{O}) = \ddot{v}(\mathcal{O})$ and $\mathcal{O}(\acute{\ddot{v}})$ is unobstructed.

Proof. We follow the same strategy: let the atom a_i be an obstruction with some other $a'_{i'}$ in $\mathcal{O}(\ddot{v})$. Consider the projected subvector $\ddot{v}^{i:a_i}|^{-i}$. By Lemma IV.16 and the inductive hypothesis of Theorem IV.23, we can write

$$\ddot{v}^{i:a_i} = \sum_{k\in K} \ddot{v}(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)} \between y_\bullet^{(k)}); a_i.$$

Leveraging no algebraicity like we did at the beginning of the proof of Claim IV.19, we may assume that $\mathcal{O}(\ddot{v}) \cup \{(x_\bullet^{(k)}; a_i), (y_\bullet^{(k)}; a_i) \mid k \in K\}$ is unambiguous.

Split the atoms except a_i appearing in that family into two parts X and Y, where Y consists of all the atoms equal to some $a'_{i'}$ that make a_i an obstruction. Then $x_{i''}^{(k)}$, $y_{i''}^{(k)}$ cannot belong to Y for any $i'' \in I$. Indeed, if say $x_{i''}^{(k)}$ is equal to an obstructive atom $a'_{i'}$, we must have $i'' = i'$ by unambiguity; we now have a contradiction: $a'_{i'} = x_{i''}^{(k)}$ and a_i necessarily satisfy the right binary relations, since $x_\bullet^{(k)}; a_i \in \mathcal{O}$.

Next, we invoke the Lemma IV.3 to find an automorphism $\tau \cdot \mathrm{Aut}(\mathbb{A})$ making $\tau \cdot a_i$ greater than a_i, distinct from any $x \in X$, and unrelated to any $y \in Y$. By Lemma IV.12, we have $\mathcal{O}$-duos $x_\bullet^{(k)}; a_i \parallel y_\bullet^{(k)}; \tau \cdot a_i$ for $k \in K$. The refined vector

$$\begin{aligned} \ddot{v}' &= \ddot{v} - \sum_{k\in K} \ddot{v}(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)}; \tau \cdot a_i \between y_\bullet^{(k)}; a_i) \\ &= \ddot{v} - \ddot{v}^{i:a_i} + \ddot{v}^{i:\tau \cdot a_i} \end{aligned}$$

satisfies $\ddot{v}'(\mathcal{O}) = \ddot{v}(\mathcal{O})$. As we are just changing a_i to the fresh atom $\tau \cdot a_i$ in the ith entry of every tuple appearing in $\ddot{v}$, the family $\mathcal{O}(\ddot{v}')$ remains unambiguous; note also that a_i can only appear in the ith entry as $\mathcal{O}(\ddot{v})$ is unambiguous, so we have completely removed a_i from $\ddot{v}'$. But $\tau \cdot a_i$ by design cannot cause an obstruction in $\mathcal{O}(\ddot{v}')$, and consequently any atom causing obstructions in $\mathcal{O}(\ddot{v}')$ must already do so in $\mathcal{O}(\ddot{v})$ except that it cannot be a_i. Hence we can repeat this procedure until all obstructions are excised. $\square$

We are at last in a position to prove the inductive step of Theorem IV.23. Given $v \in \mathrm{Ker}_{\mathsf{E}}\,\mathcal{O}$, we apply Propositions IV.24 and IV.25 to obtain

$$\acute{\ddot{v}} = v - \sum_{k\in K_1\cup K_2} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

where $\acute{\ddot{v}}(\mathcal{O}) = \ddot{v}(\mathcal{O}) = v(\mathcal{O})$ and $\mathcal{O}(\acute{\ddot{v}})$ is unobstructed. Recalling Proposition IV.11, we see that $\acute{\ddot{v}} \in \mathrm{Ker}_{\mathsf{E}}\,\mathcal{O}$. The special Theorem IV.18 tells us that $\acute{\ddot{v}}$ is a sum of $\mathcal{O}$-cogs with coefficients from $\acute{\ddot{v}}(\mathcal{O}) = v(\mathcal{O})$. It follows that so is

$$v = \acute{\ddot{v}} + \sum_{k \in K_1 \cup K_2} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)},$$

which establishes the general Theorem IV.23.

F. All those equivariant subspaces

We finish this section with an important corollary of Theorem IV.23. Let $\mathcal{O}_1 \subseteq \mathbb{A}^{I_1}, \dots, \mathcal{O}_n \subseteq \mathbb{A}^{I_n}$ all be S-ordered orbits. Then $\mathrm{len}(\mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)) = 2^{|I_1|} + \cdots + 2^{|I_n|}$; in fact, we know and can characterise all the $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces of $\mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n) \simeq \mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1) \oplus \cdots \oplus \mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_n)$.

a) Local coefficients: First we set up some notations. Consider the $\sum_k 2^{|I_k|}$ projected S-ordered orbits $\mathcal{O}_k|^J$ for $1 \le k \le n$, $J \subseteq I_k$. Suppose

$$f : \mathcal{O}_k|^J \to \mathcal{O}_{k'}|^{J'}$$

is an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant bijection. Take any $o_\bullet \in \mathcal{O}_k|^J$, and enumerate its entries as $o_1 < \cdots < o_{|J|}$. Similarly, enumerate the entries of $f(o_\bullet)$ as $o'_1 < \cdots < o'_{|J'|}$. Then $\{o_1, \dots, o_{|J|}\} = \{o'_1, \dots, o'_{|J'|}\}$ because $\mathbb{A}$ has no algebraicity; since the orbits are ordered, we must have $|J| = |J'|$ and $o_1 = o'_1, \dots, o_{|J|} = o'_{|J'|}$. That is, f must be the obvious function that reindexes a J-tuple to a J'-tuple — hence we will write $o_\bullet^{/J'}$ instead of $f(o_\bullet)$, leaving f implicit.

Now, let $\mathcal{Q}_1 = \mathcal{O}_{k_1}|^{J_1}, \dots, \mathcal{Q}_t = \mathcal{O}_{k_t}|^{J_t}$ be the distinct S-ordered orbits up to $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant bijections, which we enumerate in such a way that $|J_1| \ge |J_2| \ge \cdots \ge |J_t| = 0$.

Definition IV.26. For $i = 1, \dots, t$, let P_i consist of pairs (k, J) such that $\mathcal{O}_k|^J$ is $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariantly isomorphic to $\mathcal{Q}_i$. Assemble all $|P_i|$ projections into a single map

$$(-)\restriction_i : \mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n) \to \mathrm{Lin}_{\mathsf{F}^{P_i}}\,\mathcal{Q}_i.$$

More precisely $(v_1, \dots, v_n)\restriction_i(a_\bullet)$ is the P_i-tuple whose entry at (k, J) is $v_k|^J(a_\bullet^{/J}) \in \mathsf{F}$. It is straightforward to check that $(-)\restriction_i$ is $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant and linear.

Let $W \subseteq \mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)$ be an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace. Using the t finite-dimensional vector spaces $W\restriction_1(\mathcal{Q}_1) \subseteq \mathsf{F}^{P_1}, \dots, W\restriction_t(\mathcal{Q}_t) \subseteq \mathsf{F}^{P_t}$ we define $\widetilde{W}$, which consists of all vectors $v \in \mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)$ such that

$$v\restriction_1(\mathcal{Q}_1) \subseteq W\restriction_1(\mathcal{Q}_1), \dots, v\restriction_t(\mathcal{Q}_t) \subseteq W\restriction_t(\mathcal{Q}_t).$$

Then $\widetilde{W}$ is an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace that contains W. It turns out these two are equal:

Lemma IV.27.

$$\begin{aligned} &\widetilde{W} \cap \ker(\restriction_{i+1}) \cap \cdots \cap \ker(\restriction_t) \\ &\subseteq W \cap \ker(\restriction_{i+1}) \cap \cdots \cap \ker(\restriction_t). \end{aligned}$$

In particular $\widetilde{W} \subseteq W$ when $i = t$.

b) Proof of the lemma: by induction on i. When $i = 0$, this containment is trivial:

Claim IV.28. $\ker(\restriction_1) \cap \ker(\restriction_2) \cdots \cap \ker(\restriction_t) = \{0\}$.

Proof. Let $v = (v_1, \dots, v_n) \in \ker(\restriction_1) \cap \ker(\restriction_2) \cdots \cap \ker(\restriction_t)$. Each (k, I_k) belongs to some P_i; that $v\restriction_i = 0$ implies $0 = v_k|^{I_k} = v_k$. □

To prove the containment for $i + 1$, we allow $v\restriction_{i+1}$ to be non-zero. But $v\restriction_{i+1}$ satisfies the next best property:

Claim IV.29. *The image of*

$$\widetilde{W} \cap \ker(\restriction_{i+2}) \cap \cdots \cap \ker(\restriction_t)$$

under $\restriction_{i+1}$ is contained in $\mathrm{Ker}_{W\restriction_{i+1}(\mathcal{Q}_{i+1})}\,\mathcal{Q}_{i+1}$.

Proof. Take any $v = (v_1, \dots, v_n)$ satisfying

$$v\restriction_{i+2} = 0, \dots, v\restriction_t = 0$$

from $\widetilde{W}$. That $v\restriction_{i+1} \in \mathrm{Lin}_{W\restriction_{i+1}(\mathcal{Q}_{i+1})}\,\mathcal{Q}_{i+1}$ is clear from the definition of $\widetilde{W}$. Recall that $\mathcal{Q}_{i+1} = \mathcal{O}_{k_{i+1}}|^{J_{i+1}}$. Given $j \in J_{i+1}$, we need to prove that $v\restriction_{i+1}|^{-j} = 0$.

To do so, take $(k, J) \in P_{i+1}$; the unique monotone bijection between J_{i+1} and J restricts to one between $J_{i+1} \setminus \{j\}$ and $J \setminus \{j'\}$. Now $(k, J \setminus \{j'\})$ belongs to some $P_{i'}$ with $i' > i+1$, so $v\restriction_{i'} = 0$. We calculate that $v\restriction_{i+1}|^{-j}(a_\bullet)_{k,J}$ is equal to

$$\sum_{b_\bullet \in \mathcal{O}_k, b|_\bullet^{J\setminus\{j'\}} = a_\bullet^{/J\setminus\{j'\}}} v_k(b_\bullet),$$

and that so is $0 = v\restriction_{i'}(a_\bullet^{/J\setminus\{j'\}})_{k,J\setminus\{j'\}}$. □

Now Theorem IV.23 tells us that $\mathrm{Ker}_{\widetilde{W}\restriction_{i+1}(\mathcal{Q}_{i+1})}\,\mathcal{Q}_{i+1} \subseteq \mathrm{Cog}_{\widetilde{W}\restriction_{i+1}(\mathcal{Q}_{i+1})}\,\mathcal{Q}_{i+1}$, which is good news:

Claim IV.30. $\mathrm{Cog}_{W\restriction_{i+1}(\mathcal{Q}_{i+1})}\,\mathcal{Q}_{i+1}$ *is contained in the image of*

$$W \cap \ker(\restriction_{i+2}) \cap \cdots \cap \ker(\restriction_t)$$

under $\restriction_{i+1}$.

Proof. Let $w\restriction_{i+1}(a_\bullet) \in W\restriction_{i+1}(\mathcal{Q}_{i+1})$. Let S' consist of S together with every atom appearing in w but not in $a_\bullet$. We generalise the proof of Theorem IV.14.

Start by applying Proposition IV.13 and Remark IV.9 to get automorphisms π_j for $j \in J_{i+1}$ such that $a_\bullet \parallel \prod_{j \in J_{i+1}} \pi_j \cdot a_\bullet$ is an $\mathcal{Q}_{i+1}$-duo, where π_j fixes S' and $a_{j'}, \pi_{j'} \cdot a_{j'}$ for $j' \in J \setminus \{j\}$. Put

$$w' = \prod_{j \in J_{i+1}} (1 - \pi_j) \cdot w \in W.$$

Given $1 \le i' \le t$, observe that as no more atoms can appear in $w\restriction_{i'}$ than in w, we have

$$\begin{aligned} w'\restriction_{i'} &= \prod_{j \in J_{i+1}} (1 - \pi_j) \cdot w\restriction_{i'} \\ &= \sum_{b_\bullet \in \mathcal{Q}_{i'}, \{b_j|j\} \supseteq \{a_j|j\}} \sum_{J' \subseteq J} (-1)^{|J'|} w\restriction_{i'}(b_\bullet) \prod_{j \in J'} \pi_j \cdot b_\bullet. \end{aligned}$$

Suppose $\{b_j \mid j \in J_{i'}\} \supseteq \{a_j \mid j \in J_{i+1}\}$. Then $i' \leq i+1$; if $i' = i+1$, we must have $b_\bullet = a_\bullet$. We therefore have

$$w'\restriction_{i+1} = w\restriction_{i+1}(a_\bullet) \cdot a_\bullet \between \prod_{j \in J_{i+1}} \pi_j \cdot a_\bullet$$

and $w'\restriction_{i+2} = 0, \ldots, w'\restriction_t = 0$. This proves that $(W \cap \ker \restriction_{i+2} \cap \cdots \cap \ker \restriction_t)\restriction_{i+1}$ contains $\mathrm{Cog}_{W\restriction_{i+1}(\mathcal{Q}_{i+1})}\, \mathcal{Q}_{i+1}$. □

This is enough to establish Lemma IV.27 for $i+1$ assuming the result for i. Indeed, given $v \in \widetilde{W} \cap \ker(\restriction_{i+2}) \cap \cdots \cap \ker(\restriction_t)$, we can find $w \in W \cap \ker(\restriction_{i+2}) \cap \cdots \cap \ker(\restriction_t) \subseteq \widetilde{W}$ such that

$$v\restriction_{i+1} = w\restriction_{i+1}$$

by the preceding claims. But $(v-w)\restriction_{i+1} = 0$ — that is, $v-w$ lies in in $\ker(\restriction_{i+1})$ as well as $\widetilde{W} \cap \ker(\restriction_{i+2}) \cap \cdots \cap \ker(\restriction_t)$. It follows from the inductive hypothesis that

$$v - w \in W \cap \ker(\restriction_{i+2}) \cap \cdots \cap \ker(\restriction_t),$$

so $v = (v-w)+w$ is a member of $W \cap \ker(\restriction_{i+2}) \cap \cdots \cap \ker(\restriction_t)$ as well.

c) Lengths: Let W, W' be two $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces of $\mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)$. If we have $W\restriction_1(\mathcal{Q}_1) = W'\restriction_1(\mathcal{Q}_1), \ldots, W\restriction_t(\mathcal{Q}_t) = W'\restriction_t(\mathcal{Q}_t)$, then $W = \widetilde{W} = \widetilde{W'} = W'$ by Lemma IV.27. An immediate consquence is:

Proposition IV.31. *Let $W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_l$ be a chain of $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces in $\mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)$. Then $l \leq 2^{|I_1|} + \cdots + 2^{|I_n|}$.*

Proof. We obtain t chains

$$\begin{gathered} W_0\restriction_1(\mathcal{Q}_1) \subseteq W_1\restriction_1(\mathcal{Q}_1) \subseteq \cdots \subseteq W_l\restriction_1(\mathcal{Q}_1) \subseteq \mathsf{F}^{P_1}, \\ W_0\restriction_2(\mathcal{Q}_2) \subseteq W_1\restriction_2(\mathcal{Q}_2) \subseteq \cdots \subseteq W_l\restriction_2(\mathcal{Q}_2) \subseteq \mathsf{F}^{P_2}, \\ \vdots \\ W_0\restriction_t(\mathcal{Q}_t) \subseteq W_1\restriction_t(\mathcal{Q}_t) \subseteq \cdots \subseteq W_l\restriction_t(\mathcal{Q}_t) \subseteq \mathsf{F}^{P_t}. \end{gathered}$$

At each of the l steps, one of the t containments must be strict. Hence $l \leq |P_1| + |P_2| + \cdots + |P_t| = 2^{|I_1|} + \cdots + 2^{|I_t|}$. □

It follows that any $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace of $\mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)$ is finitely generated. We can compute the local coefficients of such subspaces easily:

Remark IV.32. For $v \in \mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)$, let $\langle v \rangle$ denote the $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace it generates. Then:

1) $\langle v \rangle\restriction_i(\mathcal{Q}_i)$ is the subspace of F^{P_i} generated by vectors of the form $v\restriction_i(a_\bullet)$, which is zero unless every atom appearing in $a_\bullet$ appears in v — there are only finitely many such $a_\bullet$'s;
2) $\langle v, v' \rangle\restriction_i(\mathcal{Q}_i) = \langle v \rangle\restriction_i(\mathcal{Q}_i) + \langle v' \rangle\restriction_i(\mathcal{Q}_i)$.

We may now exhibit a chain of $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces whose length is precisely $\sum_{i=1}^t 2^{|P_i|}$, generalising [1, Corollary 4.12]. Take any $(k, J) \in P_i$. Pick some $a_\bullet \in \mathcal{O}_k$, and let $\pi_j, j \in I_k$ be the automorphisms from Proposition IV.13. Define a vector

$$\begin{aligned} v^i_{k,J} &\in \mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n) \\ &\simeq \mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1) \oplus \cdots \oplus \mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_n) \end{aligned}$$

with $\prod_{j \in J}(1 - \pi_j) \cdot a_\bullet$ as its kth component and zero everywhere else. Then

$$\langle v^i_{k,J} \rangle\restriction_{i'}(\mathcal{Q}_{i'})_{k',J'} = \begin{cases} \mathsf{F} & \text{if } k = k' \text{ and } J \subseteq J', \\ \{0\} & \text{otherwise.} \end{cases}$$

Enumerating each P_i as $(k^i_1, J^i_1), (k^i_2, J^i_2), \ldots, (k^i_{|P_i|}, J^i_{|P_i|})$, we obtain a chain

$$\begin{aligned} &\langle\rangle \\ \subsetneq\ &\langle v^t_{k^t_1, J^t_1} \rangle \\ \subsetneq\ &\langle v^t_{k^t_1, J^t_1}, v^t_{k^t_2, J^t_2} \rangle \\ \subsetneq\ &\cdots \\ \subsetneq\ &\langle v^t_{k^t_1, J^t_1}, v^t_{k^t_2, J^t_2}, \ldots, v^t_{k^t_{|P_t|}, J^t_{|P_t|}} \rangle \\ \subsetneq\ &\langle v^t_{k^t_1, J^t_1}, v^t_{k^t_2, J^t_2}, \ldots, v^t_{k^t_{|P_t|}, J^t_{|P_t|}}, v^{t-1}_{k^{t-1}_1, J^{t-1}_1} \rangle \\ \subsetneq\ &\cdots \end{aligned}$$

of length $|P_t| + |P_{t-1}| + \cdots + |P_1| = 2^{|I_1|} + \cdots + 2^{|I_n|}$. With the upper bound in Proposition IV.31, we conclude:

Theorem IV.33. $\mathrm{len}(\mathrm{Lin}_{\mathsf{F}}(\mathcal{O}_1 \uplus \cdots \uplus \mathcal{O}_n)) = 2^{|I_1|} + \cdots + 2^{|I_n|}$.

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