

Cogs span the projection kernel, version $n + 1$

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1 Notations

Let $\mathcal{O} \subseteq \mathbb{A}^I$ be an S -ordered orbit. Given a vector $v \in \text{Lin}_{\mathbb{E}} \mathcal{O}$, write \underline{v} for its set-theoretic support, i.e., the finite subset $v^{-1}(\mathbb{E}^*) \subseteq \mathcal{O}$. More generally, given any finite subset $\sigma \subseteq \mathcal{O}$, write

$$\bar{\sigma} = \{(i, a_i) \mid i \in I, a \in \sigma\}$$

and define two binary relations

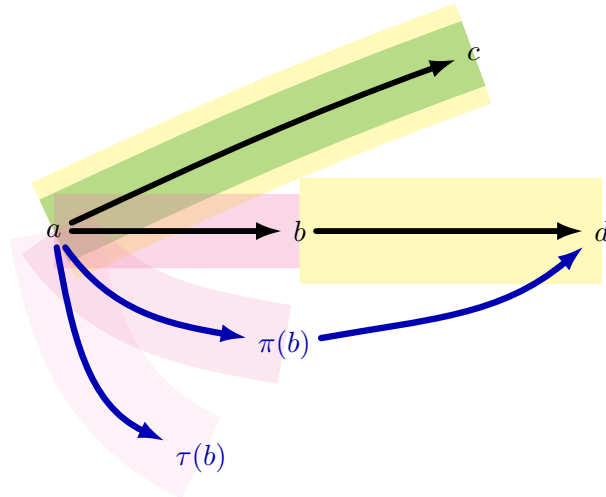
$$(i, a_i)?(j, b_j) \iff a_i = b_j \text{ but } i \neq j,$$

$$(i, a_i)!(j, b_j) \iff a_i, b_j \text{ are related but not in the same way as } a_i, a_j$$

called *ambiguities* and *obstructions*. Both relations are symmetric and $? \subseteq !$; denote their images by $?\bar{\sigma} \subseteq !\bar{\sigma} \subseteq \bar{\sigma}$. Lastly, I refer to the atoms that appear in σ by $\sqrt{\sigma} \subseteq \mathbb{A}$.

Remark 1 Assume $?\bar{\sigma} = \emptyset$. Given any $a_i \in \sqrt{\sigma}$, we have $(i, a_i) \in \bar{\sigma}$.

2 Lemmas



Lemma 2 *Let $\mathcal{V}, \mathcal{U}, \mathcal{W}$ be finite subsets of \mathcal{O} such that $\overline{?\mathcal{U} \cup \mathcal{V}} = \emptyset = \overline{?\mathcal{W}} \supseteq \overline{?\mathcal{U}}$. Then there exists $\pi \in \text{Aut}(\mathbb{A}/S \cup \sqrt{\mathcal{U}})$ that satisfies*

$$\overline{?\mathcal{W} \cup \pi(\mathcal{V})} = \emptyset.$$

PROOF. Fix \mathcal{U}, \mathcal{W} and induct on the size of $\overline{?\mathcal{W} \cup \mathcal{V}}$.

Let $(i, a_i)!(j, b_j)$; without loss of generality we may assume $(j, b_j) \in \overline{?\mathcal{W}}$ and $(i, a_i) \in \overline{\mathcal{V}} \setminus \overline{\mathcal{U}}$. Since $\overline{?\mathcal{U}} = \emptyset$, by Remark 1 we see that $a_i \notin \sqrt{\mathcal{U}}$; also $a_i \notin S$, as \mathcal{O} is S -ordered. With strong amalgamation, we may find some $\pi \in \text{Aut}(\mathbb{A}/S \cup \sqrt{\mathcal{U}})$ such that $\pi(a_i) \notin \sqrt{\overline{?\mathcal{W} \cup \mathcal{V}}}$. Now it is straightforward to check that

$$\overline{?\mathcal{W} \cup \pi(\mathcal{V})} \subseteq \overline{?\mathcal{W} \cup \mathcal{V}} \setminus \{(i, a_i)\}.$$

Because $\overline{?\mathcal{U} \cup \pi(\mathcal{V})} = \overline{?\pi(\mathcal{U}) \cup \pi(\mathcal{V})} = \pi(\emptyset) = \emptyset$ still, the inductive hypothesis gives us some $\pi' \in \text{Aut}(\mathbb{A}/S \cup \sqrt{\mathcal{U}})$ such that $\overline{?\mathcal{W} \cup \pi'\pi(\mathcal{V})} = \emptyset$. ■

Lemma 3 *Let $\mathcal{V}, \mathcal{U}, \mathcal{W}$ be finite subsets of \mathcal{O} such that $\overline{!\mathcal{U} \cup \mathcal{V}} = \emptyset = \overline{!\mathcal{W}} \supseteq \overline{!\mathcal{U}}$. Then*

$$\overline{!\mathcal{W} \cup \pi(\mathcal{V})} = \emptyset$$

for some $\pi \in \text{Aut}(\mathbb{A}/S \cup \sqrt{\mathcal{U}})$.

PROOF. By Lemma 2 we may assume that $\overline{?\mathcal{W} \cup \mathcal{V}} = \emptyset$ already. As before, fix \mathcal{U}, \mathcal{W} and proceed by induction on the size of $\overline{!\mathcal{W} \cup \mathcal{V}}$.

Let $(i, a_i)!(j, b_j)$; without loss of generality we may assume $(j, b_j) \in \overline{!\mathcal{W}}$ and $(i, a_i) \in \overline{\mathcal{V}}$. Now $(i, a_i) \notin \overline{\mathcal{U}}$, so $a_i \notin \sqrt{\mathcal{U}}$ by Remark 1; further, whenever $(i, a_i)!(k, c_k)$ we observe that $c_k \notin S \cup \{a_i\} \cup \sqrt{\overline{\mathcal{V} \cup \mathcal{U}}}$. Let Y consist of all such c_k and put

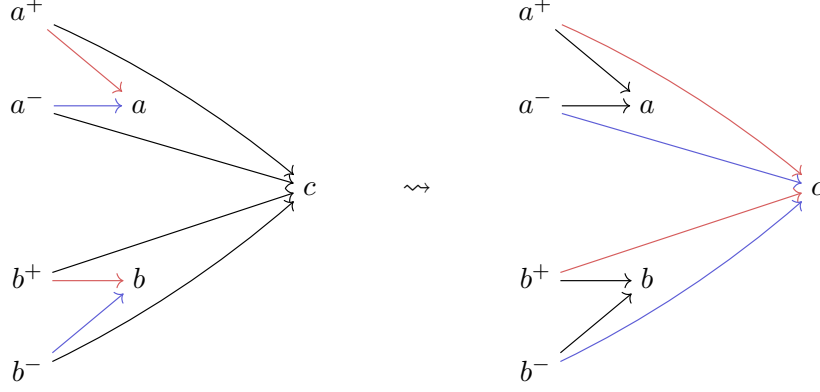
$$X = S \cup \sqrt{\overline{\mathcal{V} \cup \mathcal{W}}} \setminus (Y \cup \{a_i\}).$$

Then $X, Y, \{a_i\}$ are pairwise disjoint, and we see X contains $S \cup \sqrt{\mathcal{U}}$ as well as $\sqrt{\overline{\mathcal{V}}} \setminus \{a_i\}$. With free amalgamation (in \mathbb{A}_0) this time, we may find some $\tau \in \text{Aut}(\mathbb{A}/S \cup \sqrt{\mathcal{U}})$ such that $\tau(a_i) \notin X \cup Y \cup \{a_i\}$ is not related to any atom in Y . Again, we can check that

$$\overline{!\mathcal{W} \cup \tau(\mathcal{V})} \subseteq \overline{!\mathcal{W} \cup \mathcal{V}} \setminus (i, a_i)$$

so the conclusion follows straightforwardly from the inductive hypothesis. ■

3 Unobstructed vector



Proposition 4 *Let $v \in \text{Ker}_{\mathbb{E}} \mathcal{O}$ and suppose that $!\underline{v} = \emptyset$. Then we can write*

$$v = \sum_{a^{\pm} \in A^{\pm}} \lambda_{a^{\pm}} \cdot a^{+} \frown a^{-}$$

with $!\overline{v \cup A^{\pm}} = \emptyset$ and $\lambda_{a^{\pm}} \in v(\mathcal{O})$.

This is the same statement as in §IV.D; here I use a slightly different proof.

We proceed by induction on the dimension $|I|$, noting that when $I = \emptyset$ we just have $v = v() \cdot () = v() \cdot (\frown)$ without any possible obstructions.

So suppose I is non-empty; let $d \in I$ be the greatest. Group the terms in v by their greatest atom so that $v = v^1 + v^2 + \dots + v^k$. We now induct on k . If $k < 2$, we are done: as $v_{-d} = 0$ we must have $v = 0$, so the empty sum will do. Otherwise

$$v = v^{d:a_d} + v^{d:b_d} + v'.$$

By the outer inductive hypothesis, we get

$$v^{d:a_d} = v_{-d}^{d:a_d} a_d = \sum_{A^{\pm}} (\lambda_{a^{\pm}} \cdot a^{+} \frown a^{-}) a_d$$

where we only know $!\overline{v_{-d}^{d:a_d} \cup A^{\pm}} = \emptyset$ so that $!\overline{v^{d:a_d} \cup A^{\pm} a_d} = \emptyset$. But any $\pi \in \text{Aut}(\mathbb{A}/S \cup \sqrt{v^{d:a_d}})$ satisfies

$$v^{d:a_d} = \pi(v^{d:a_d}) = \sum_{a^{\pm} \in A^{\pm}} \lambda_{a^{\pm}} \cdot \pi a^{+} \frown \pi a^{-},$$

so by Lemma 3 we may assume without loss of generality that

$$!\overline{v \cup A^{\pm} a_d} = \emptyset.$$

Similarly, we can write

$$v^{d:a_d} = \sum_{B^\pm} (\lambda_{b^\pm} \cdot b^+ \wp b^-) b_d$$

where, in turn, we may upgrade the assumption that $\overline{!v^{d:b_d} \cup B^\pm b_d} = \emptyset$ to

$$\overline{!v \cup A^\pm a_d \cup B^\pm b_d} = \emptyset.$$

The key is that we may now invent a new element z , on which we impose the following relations with $S \cup \sqrt{A^\pm a_d \cup B^\pm b_d} \subseteq \mathbb{A}$:

1. $a_d, b_d < z$, and $z < s$ if $a_d, b_d < s$ for some $s \in S$ (enough to let s be the least such);
2. for any unary relation $P \in \mathcal{L}_0$:

$$P(z) :\iff P(a_d) \iff P(b_d)$$

— recall that $a, b \in \mathcal{O}$;

3. for any binary relation $R \in \mathcal{L}_0$ and $s \in S$, $a^\pm \in A^\pm$, $b^\pm \in B^\pm$, $i \in I \setminus \{d\}$:

$$(a) \ R(z, s) :\iff R(a_d, s) \iff R(b_d, s),$$

$$(b) \ R(z, a_i^\pm) :\iff R(a_d, a_i^\pm),$$

$$(c) \ R(z, a_d) :\iff \perp,$$

$$(d) \ R(z, b_i^\pm) :\iff R(b_d, b_i^\pm);$$

$$(e) \ R(z, b_d) :\iff \perp,$$

$$(f) \text{ and symmetrically for } R(-, z).$$

These are well-defined because $a^\pm a_d, b^\pm b_d \in \mathcal{O}$ and $i = j$ whenever $a_i^\pm = b_j^\pm$.

To see that the \mathcal{L} -structure $S \cup \sqrt{A^\pm a_d \cup B^\pm b_d} \cup \{z\}$ still embeds into \mathbb{A} , suppose towards a contradiction that it contains a forbidden \mathcal{L}_0 -substructure F . Then F must contain z . Since any two elements in F are necessarily related, we must have $a_d, b_d \notin F$. Similarly, whenever F contains x_i where $x \in A^\pm \cup B^\pm$, $i \in I \setminus \{d\}$ it does not contain a distinct atom of the form x'_i . It follows that

$$s \mapsto s, \quad x_i \mapsto a_i, \quad z \mapsto a_d$$

defines an injective function $\phi : F \rightarrow \mathbb{A}_0$, which is furthermore an embedding because $\overline{!A^\pm \cup B^\pm} = \emptyset$ and any x_i, x'_i for $i \neq i'$ are related. This is impossible — therefore assume $z \in \mathbb{A}$.

It is now routine to check that $a^+ a_d \parallel a^- z$ and $b^+ a_d \parallel b^- z$ are \mathcal{O} -dipoles for $a^\pm \in A^\pm, b^\pm \in B^\pm$ and that $\overline{!A^+ a_d \cup A^- z \cup B^+ b_d \cup B^- z} = \emptyset$. By Lemma 3 we may assume that $\overline{!v \cup A^+ a_d \cup A^- z \cup B^+ b_d \cup B^- z} = \emptyset$. (Alternatively, we could have explicitly ensured this when defining z). Then

$$\begin{aligned} v'' &= v - \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_d \wp a^- z - \sum_{B^\pm} \lambda_{b^\pm} \cdot b^+ b_d \wp b^- z \\ &= v_{-d}^{d:a_d} z + v_{-d}^{d:b_d} z + v', \end{aligned}$$

when grouped into subvectors by the largest atom in each term, has at least one fewer component than v . By the inner inductive hypothesis, we may write we may write

$$v'' = \sum_{C^\pm} \lambda_{c^\pm} \cdot c^+ \wr c^-$$

where $!\overline{v''} \cup C^\pm = \emptyset$. But $\overline{v''} \subseteq \overline{v \cup A^+ a_d \cup A^- z \cup B^+ b_d \cup B^- z}$, so one last application of Lemma 3 allows us to assume that

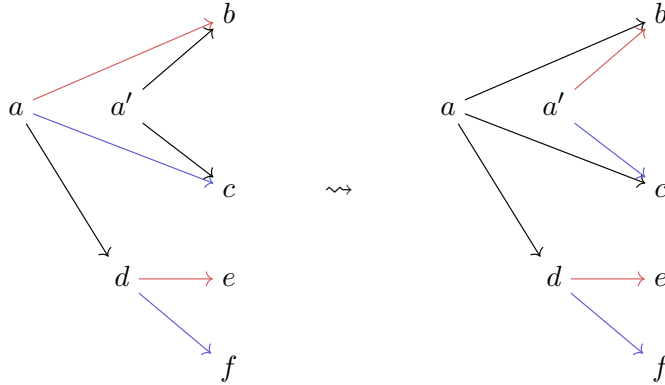
$$\overline{!v \cup A^+ a_d \cup A^- z \cup B^+ b_d \cup B^- z \cup C^\pm} = \emptyset.$$

We conclude that

$$v = \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_d \wr a^- z + \sum_{B^\pm} \lambda_{b^\pm} \cdot b^+ b_d \wr b^- z + \sum_{C^\pm} \lambda_{c^\pm} \cdot c^+ \wr c^-;$$

in other words, we have decomposed an unobstructed vector into an unobstructed family of cogs. (The notation $\lambda_{a^\pm}, \lambda_{b^\pm}$ is a bit sloppy ...)

4 Unambiguous vector



Proposition 5 *Let $v \in \text{Ker}_E \mathcal{O}$ and suppose that $!\overline{v} = \emptyset$. Then we can write*

$$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ \wr a^-$$

with $!\overline{v} \cup A^\pm = \emptyset$ and $\lambda_{a^\pm} \in v(\mathcal{O})$.

This is an amended version of Proposition IV.25 (the one with \hat{v}). The proof there had a gap, which is filled by the additional conclusion about unambiguity here.

We proceed again by induction, first on the dimension $|I|$ then on the cardinality of $!\overline{v}$. The outer base case $I = \emptyset$ is trivial — we have $v = v() \cdot (\wr)$, and no ambiguities may arise — whilst the inner base case is just Proposition 4.

So suppose that $!\underline{v}$ contains some (i, a_i) . Since $\overline{v_{-i}^{i:a_i}} \subseteq \overline{v^{i:a_i}} \subseteq \underline{v}$, we know that $v_{-i}^{i:a_i}$ is unambiguous. Applying the outer inductive hypothesis, we may write

$$v^{i:a_i} = v_{-i}^{i:a_i} a_i = \sum_{A^\pm} (\lambda_{a^\pm} \cdot a^+ \frown a^-) a_i$$

where $\emptyset = \overline{?(v_{-i}^{i:a_i} \cup A^\pm) a_i} = \overline{?v^{i:a_i} \cup A^\pm a_i}$. Moreover, we may assume by Lemma 2 that

$$?\underline{v} \cup A^\pm a_i = \emptyset.$$

Now we can show that whenever $(j, b_j)!(i, a_i)$ in \underline{v} we have $b_j \notin \sqrt{A^\pm a_i}$. Already $b_j = a_i$ would imply $j = i$ as $?\underline{v} = 0$, which is impossible. So suppose to the contrary that $b_j = a_k^\pm$ for some $a^\pm \in A^\pm$ and $k \in I \setminus \{i\}$. By the assumption above, we must have $j = k$; this is a contradiction: note that $a^\pm a_i \in \mathcal{O}$.

Put $Y = \{b_j \mid (j, b_j) \in \underline{v}, (j, b_j)!(i, a_i)\}$. It follows that $X = S \cup \sqrt{\underline{v} \cup A^\pm} \setminus Y \setminus \{a_i\}$ contains $S \cup \sqrt{A^\pm}$ and that $X, Y, \{a_i\}$ are pairwise disjoint. Using free amalgamation in \mathbb{A}_0 , we may find $\tau \in \text{Aut}(\mathbb{A}/X)$ such that $\tau(a_i) \notin X \cup Y \cup \{a_i\}$, is greater than a_i , and is not related to any of $Y \cup \{a_i\}$. Then, given any $a^\pm \in A^\pm$, we can straightforwardly check that $a^+ a_i \parallel a^- \tau(a_i)$ is an \mathcal{O} -dipole, that $?\underline{v} \cup A^+ a_i \cup A^- \tau(a_i) = \emptyset$, and that

$$\begin{aligned} v - \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \frown a^- \tau(a_i) &= v - v^{i:a_i} + v^{i:\tau(a_i)} \\ &= v^{i:a_i \mapsto \tau(a_i)} \end{aligned}$$

satisfies $\overline{!v^{i:a_i \mapsto \tau(a_i)}} \subseteq \overline{!\underline{v} \setminus \{(i, a_i)\}}$. The inner inductive hypothesis tells us that $v^{i:a_i \mapsto \tau(a_i)} = \sum_{B^\pm} \lambda_{B^\pm} \cdot b^+ \frown b^-$ with

$$\overline{?v^{i:a_i \mapsto \tau(a_i)} \cup B^\pm} = \emptyset.$$

But $\overline{v^{i:a_i \mapsto \tau(a_i)}} \subseteq \underline{v} \subseteq \overline{A^+ a_i \cup A^- \tau(a_i)}$, so Lemma 2 allows us to assume that

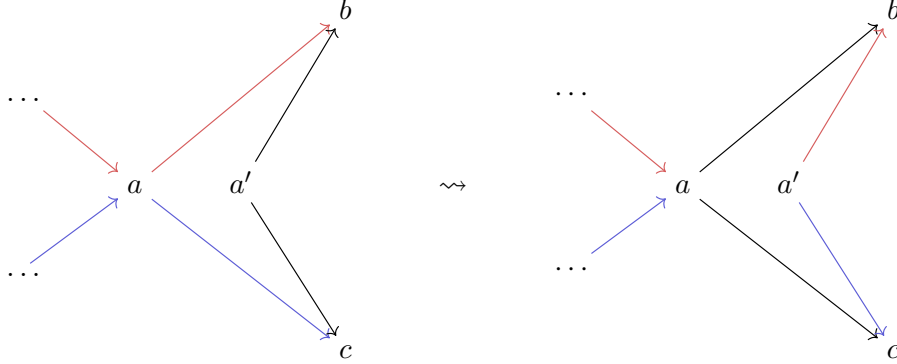
$$\overline{?\underline{v} \cup A^+ a_i \cup A^- \tau(a_i) \cup B^\pm} = \emptyset.$$

We conclude that

$$v = \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \frown a^- \tau(a_i) + \sum_{B^\pm} \lambda_{b^\pm} \cdot b^+ \frown b^-$$

— in other words, we have decomposed an unambiguous vector into an unambiguous family of cogs.

5 General vector



Proposition 6 *Let $v \in \text{Ker}_E \mathcal{O}$. Then we can write*

$$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ \mathbin{\mathbb{X}} a^-$$

with $\lambda_{a^\pm} \in v(\mathcal{O})$.

This is the same statement and proof as Proposition IV.24 (the one with \bar{v}).

This is an easier induction on $|I|$ then on $? \bar{v}$. If $I = \emptyset$, the decomposition is trivial; if v is unambiguous already, the decomposition comes from Proposition 5.

Now suppose $? \bar{v}$ contains some (i, a_i) , and use the outer inductive hypothesis to write

$$v^{i:a_i} = v_{-i}^{i:a_i} a_i = \sum_{A^\pm} (\lambda_{a^\pm} \cdot a^+ \mathbin{\mathbb{X}} a^-) a_i.$$

Then neither S nor $\sqrt{A^\pm}$ contains a_i , so $X = S \cup \sqrt{v \cup A^\pm} \setminus \{a_i\}$ contains $S \cup \sqrt{A^\pm}$. Using free amalgamation in \mathbb{A}_0 and the generic order in \mathbb{A} , we may find some $\pi \in \text{Aut}(\mathbb{A}/X)$ such that $\pi(a_i) \notin X$, $\pi(a_i) > a_i$, $\pi(a_i) \perp a_i$. We can check that

1. $a^+ a_i \parallel a^- \pi(a_i)$ is an \mathcal{O} -dipole, given any $a^\pm \in A^\pm$;
2. $v - \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \mathbin{\mathbb{X}} a^- \pi(a_i) = v - v^{i:a_i} + v^{i:\pi(a_i)} = v^{i:a_i \mapsto \pi(a_i)}$ satisfies

$$\overline{? v^{i:a_i \mapsto \pi(a_i)}} \subseteq ? \bar{v} \setminus \{(i, a_i)\}.$$

It follows from the inner inductive hypothesis that

$$\begin{aligned} v &= \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \mathbin{\mathbb{X}} a^- \pi(a_i) + v^{i:a_i \mapsto \pi(a_i)} \\ &= \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \mathbin{\mathbb{X}} a^- \pi(a_i) + \sum_{b^\pm \in B^\pm} \lambda_{b^\pm} \cdot b^+ \mathbin{\mathbb{X}} b^-. \end{aligned}$$