The Finite Length Property of the Rado Graph and Friends

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Abstract

An infinite structure has the finite length property (over a given field) if, for each of its finite powers, strict chains of equivariant subspaces in the corresponding free vector space are bounded in length. Prior work showed that the countable pure set and the dense linear order without endpoints have this property. We generalise these results to (a) structures approximated by finite substructures with few orbits, provided the field is of characteristic zero, and (b) generically ordered expansions of Fraïssé limits with free amalgamation, in vocabularies consisting of unary and binary relations. As a special case, we deduce the finite length property of the Rado graph using both methods. We also describe some connections with function spaces, weighted register automata, and solving orbit-finite systems of linear equations.

CCS Concepts

• Theory of computation → Logic; **Automata over infinite objects**; **• Mathematics of computing → Random graphs**.

Keywords

Rado graph, orbit-finitely spanned vector spaces, finite length

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1 Introduction

This paper is part of a research programme focused on orbit-finite sets and structures. In this programme, one starts with an infinite relational structure $\mathbb{A}$ whose elements are called *atoms*. Based on these, one constructs sets which are called *orbit-finite*. Precise definitions will follow, but the understanding is that elements of an orbit-finite set are constructed using atoms, and there are only finitely many of them up to automorphisms of $\mathbb{A}$. For the theory to make sense, we must assume that $\mathbb{A}$ is *oligomorphic*, which means that $\mathbb{A}^d$ has finitely many orbits for every d. The simplest example of an oligomorphic atom structure is what we refer to as the *equality atoms*; this is the structure which has a countably infinite underlying set, and no relations except for equality. This structure, like all oligomorphic structures, arises by applying a model-theoretic construction (the Fraïssé limit) to a well-behaved class of finite structures. Figure 1 shows other examples of such structures.

	class of finite structures	its Fraïssé limit
1.	finite sets with equality only	$(\mathbb{N}, =)$
2.	finite orders	$(\mathbb{Q}, \leq)$
3.	finite graphs	(Rado graph, E)
4.	finite F_2-vector spaces	$\mathsf{F}_2 \oplus \mathsf{F}_2 \oplus \mathsf{F}_2 \oplus \cdots$

Figure 1: Examples of Fraïssé limits

When the underlying atom structure is oligomorphic, the orbit-finite sets have a robust theory, which resembles in some ways the theory of finite sets. This theory was originally developed to describe regular languages over infinite alphabets, by considering orbit-finite versions of various automata models [Boj13; BKL14], but it has since developed to cover other models, such as orbit-finite Turing machines [Boj+13] or orbit-finite constraint satisfaction problems [Kli+15]. Also, there are programming languages with data structures that can store orbit-finite sets [Boj+12; BT12] with working implementations [KT16; Szy22]. For a survey of the orbit-finite programme, we refer to the lecture notes [Boj25].

Some results do not depend on the choice of the atom structure, while others do. Here is an example of the latter case that arises for orbit-finite Turing machines [Boj+13]. If we choose the atoms to be the Fraïssé limit of linear orders, as in row 2 of Figure 1, then the orbit-finite version of P≠NP has the same answer as the classical version without atoms. On the other hand, if we choose any of the other three rows of the table, then one can prove unconditionally that P≠NP holds in the orbit-finite setting, by leveraging problems with choice. The dependency on the underlying atom structure will play a prominent role in this paper.

Vector spaces. One direction of the orbit-finite programme, motivated by the study of orbit-finite weighted automata, is focused on vector spaces [Boj+24]. In these spaces (taken over some fixed field), one can take linear combinations and apply automorphisms of the atom structure. We are interested in spaces which have an orbit-finite spanning set, which means that the entire space can be obtained from some finite subset by using atom automorphisms and linear combinations. A prototypical example is the space $\operatorname{Lin} X$, which consists of formal linear combinations of elements from some orbit-finite set X. The original application of these spaces was in automata theory, but they have also found applications in the study of orbit-finite linear programming [GHL23], function spaces for orbit-finite sets [BNS24], or the analysis of two-party communication protocols over infinite alphabets [Boj+26].

To be useful, the theory of orbit-finitely spanned vector spaces should have certain properties. For example, one would like to be able to represent these spaces in a finite way, or solve algorithmic problems such as solving systems of linear equations. One rather modest requirement is that the spaces be closed under taking *equivariant* subspaces (that is, it must be closed under both atom automorphisms and linear combinations): an equivariant subspace of an orbit-finitely spanned vector space V should itself be orbit-finitely spanned. We do not know if this closure property holds in general, and we think that it is an important open problem. This problem can be equivalently phrased in terms of ascending chains: is it true that every orbit-finitely spanned vector space is *Noetherian*, which means that one cannot find an infinite ascending chain of its equivariant subspaces? To the best of our knowledge, this question was first recognised by Camina and Evans [CE91, Q. 2] who identified a sufficient condition for this, namely the existence of an Ahlbrandt–Ziegler enumeration. Using this condition they showed certain vector spaces are Noetherian, notably Lin $\mathbb{A}$ over the ordered atoms (row 2 of Figure 1), Lin $\binom{\mathbb{A}}{d}$ over the equality atoms (row 1 of Figure 1), and a similar space over the bit-vector atoms (row 4 of Figure 1).

The above question is independently considered in [Boj+24, p. 21] where it is conjectured that every oligomorphic structure has the *ascending chain property*, meaning that all orbit-finitely spanned vector spaces over the structure are Noetherian. In such a vector space, if descending chains as well as ascending chains of equivariant subspaces are all finite, by the Jordan–Hölder Theorem, there is some finite upper bound on the length of chains of its equivariant subspaces. In that case, we say that the structure has the *finite length property*. This stronger property has been confirmed for:

- *The equality atoms.* This was proved in three different contexts independently: model theory [ER02, Thm. 3.9], representation theory [SS15, Prop. 6.1.6], and orbit-finite sets [Boj+24, Thm. 4.10]. The result from [SS15] assumes that the underlying field is the complex numbers, while the other two do not restrict the choice of a field.
- *The ordered atoms.* Here, the finite length property was proved in [Boj+24, Thm. 4.10]. The weaker ascending chain property for this structure was proved in [GL24, Thm. 27], using an alternative method based on Hilbert's Basis Theorem. That method can also be extended to the Fraïssé limit of trees. (We do not know if the limit of trees has the finite length property.)

The finite length property is strictly stronger than the ascending chain property. Examples of this can be exhibited using the bit-vector atoms, as found independently in [Gra97, Thm. 2.7] and [Boj+24, Thm. 4.16]. It is worth noting that the failure of the finite length property is connected to the Thomas Conjecture [Tho91, p. 177] in model theory. As observed by Evans [Eva25b, sld. 6], a counterexample to that conjecture would arise from a structure which: (a) is homogeneous over a finite relational vocabulary; and (b) fails the finite length property over some finite field. The known examples of failure of the finite length property are not good enough, since they use an infinite vocabulary (or functions).

Our contributions. Until now, the finite length property has been a theory of two examples: the equality and ordered atoms. We substantially improve this state of affairs, using two different techniques:

- In Theorem 5.4, we prove the finite length property for structures $\mathbb{A}$ which admit what we call oligomorphic approximation, a relaxed version of smooth approximation known from model theory — e.g. the equality atoms, the bit-vector atoms. This is under an additional assumption that the underlying field has characteristic 0.
- In Theorem 7.3, the finite length property is shown for $\mathbb{A}$ which arise as Fraïssé limits of generically ordered free amalgamation classes, over relational vocabularies of arity at most 2 — e.g. the ordered atoms. Here we do not restrict the underlying field.

In particular, a special case of either of these techniques is the Rado atoms (row 3 in Figure 1), where even the weaker ascending property was not known before.

Most of the paper is devoted to introducing the background necessary to understand these results (Sections 2-4), and to proving them (Sections 5, 7-8 and the technical Appendix). In Section 6 we discuss some connections with function spaces and weighted automata, and in Section 8.5 we briefly discuss applications to solving orbit-finite system of equations.

2 Structures

In this section, we briefly recall some basic notions from model theory and describe the main examples of structures that we will consider in this paper.

Let us begin by fixing some notation. A *vocabulary* is a set of relations, each with a specified arity (we do not use functions in this paper). For example, the vocabulary of graphs will contain one binary relation: the edge relation. (We do not include equality in the vocabulary, since it will be automatically present in all structures.) A *structure* $\mathbb{A}$ over a vocabulary consists of an underlying set, also denoted by $\mathbb{A}$, together with intepretations of relations from the vocabulary as actual relations on that set. An *isomorphism* between two structures over the same vocabulary is a bijection between their underlying sets that preserves and reflects all relations. An *automorphism* of a structure is an isomorphism from the structure to itself; these form a group.

Automorphisms of $\mathbb{A}$ act on tuples in $\mathbb{A}^d$ componentwise. When we speak of orbits in $\mathbb{A}^d$, we mean the orbits under this action of the automorphism group. For example, if $\mathbb{A}$ is a graph, then two pairs (a_1, a_2) and (b_1, b_2) are in the same orbit if and only if there is some automorphism of the graph that maps a_1 to b_1 and a_2 to b_2. In particular, the edge relation must be defined in the same way for both pairs.

All structures considered in this paper will be countable (or finite), and we will always want them to have finitely many orbits in every finite dimension, as in the following definition.

Definition 2.1 (Oligomorphic structure)**.** A structure $\mathbb{A}$ is *oligomorphic* if $\mathbb{A}^d$ has finitely many orbits for every $d \in \{1, 2, \ldots\}$.

A relation $R \subseteq \mathbb{A}^d$ on a structure is called *equivariant* if it is invariant under the action of the automorphism group. Equivalently, the relation is a union of orbits. If the structure is oligomorphic, then there are finitely many orbits to consider once the dimension d is fixed, and therefore only finitely many equivariant relations.

By the Ryll-Nardzewski Theorem [Hod93, Thm. 7.3.1], if the structure is oligomorphic and countable, then the equivariant relations are exactly those that can be defined in first-order logic — see for instance [Boj25, Lem. 5.9]. In fact, the infinite structures that we consider in this paper will satisfy a stronger property: the equivariant relations will be definable not only by first-order formulas, but even by quantifier-free ones. This will be ensured by the additional homogeneity condition defined below. In the condition, a *substructure* of $\mathbb{A}$ is any structure obtained by restricting $\mathbb{A}$ to some subset of its underlying set; we do not distinguish between substructures and subsets.

Definition 2.2 (Homogeneous structure)**.** A structure $\mathbb{A}$ is *homogeneous* if every isomorphism between finite substructures of $\mathbb{A}$ extends to an automorphism of $\mathbb{A}$.

In a homogeneous structure $\mathbb{A}$, an orbit in $\mathbb{A}^d$ consists of tuples that satisfy the same quantifier-free formulas — see for instance [Boj25, Thm. 6.3]. Every homogeneous structure arises via a construction called the Fraïssé limit [Hod93, Sec. 7.4], from classes of finite structures that satisfy certain closure properties (the so-called Fraïssé/amalgamation classes). For the Fraïssé limit to be not only homogeneous, but also oligomorphic, we need to assume that the underlying class — which, up to isomorphism, coincides with the finite substructures of the Fraïssé limit — has only finitely many non-isomorphic structures of each finite size.

The countable structures that we work with will be both oligomorphic and homogeneous, possibly over an infinite vocabulary. Here are some of the important examples that we will consider in this paper.

Example 2.3 (Equality atoms)**.** In this structure, the underlying set is countable and there are no relations other than equality. Automorphisms are arbitrary permutations, and two tuples are in the same orbit if and only if they have the same equality pattern. Since there are finitely many equality patterns for tuples of fixed length, this structure is oligomorphic. For example, in dimension $d = 2$ there are two orbits: $x_1 = x_2$ and $x_1 \neq x_2$. ▲

Example 2.4 (Ordered atoms)**.** In this structure, the underlying set is the set of rational numbers, equipped with the usual order — the vocabulary consists of this binary relation only. Automorphisms are order-preserving permutations, and two tuples are in the same orbit if and only if they have the same order pattern. Since there are finitely many order patterns for tuples of fixed length, this structure is oligomorphic. In dimension $d = 2$ there are three orbits: $x_1 < x_2$, $x_1 = x_2$, and $x_1 > x_2$. ▲

Example 2.5 (Vector atoms)**.** Fix some finite field k, and let $\mathbb{A}$ be the vector space of countable dimension over k. This vector space is seen as a structure over an infinite vocabulary, which contains a relation

$$\{ (a_1, \ldots, a_d) \in \mathbb{A}^d \mid \lambda_1 a_1 + \cdots + \lambda_d a_d = 0 \}$$

for every $d \in \{1, 2, \ldots\}$ and all coefficients $\lambda_1, \ldots, \lambda_d$ in k. The vocabulary is defined so that automorphisms are the same thing as permutations that are linear maps. Two tuples are in the same orbit if and only if they have the same linear dependencies. Over a finite field there are finitely many linear dependency patterns for tuples of fixed length, so this structure is oligomorphic. In particular, in dimension $d = 2$, there are three more orbits than there are elements in the field: $x_1 = 0 = x_2$; $x_1 = 0 \neq x_2$; $0 \neq x_1 = \lambda x_2$, where λ ranges over the field k; and x_1, x_2 being linearly independent.

This particular example can lead to some confusion, since here $\mathbb{A}$ is itself a vector space, and we will later on consider the vector space (possibly over a different field than k) with $\mathbb{A}$ as a basis. ▲

Example 2.6 (Rado atoms)**.** The Rado graph is the Fraïssé limit of the class of finite undirected graphs. Here, an undirected graph is viewed as a structure that has one binary relation that is symmetric and irreflexive. A famous characterisation of the Rado graph is that if one randomly selects a graph with a countable set of vertices by independently including each possible edge with probability $1/2$, then with probability 1 the resulting graph is isomorphic to the Rado graph. By homogeneity, two tuples in the Rado graph are in the same orbit if and only if they have the same equality and adjacency patterns, hence there are finitely many orbits in every dimension. For instance, in $d = 2$ there are three orbits: the two coordinates can be equal, adjacent hence distinct, or distinct but non-adjacent. ▲

3 Orbit-finite sets

We shall now briefly explain the concept of orbit-finiteness. We start with a countable oligomorphic structure $\mathbb{A}$, as described in Section 2, whose elements we call *atoms*. These can be used to construct sets that are finite up to the symmetries of $\mathbb{A}$, such as

$$\underbrace{\mathbb{A}^2}_{\text{pairs}} \qquad \underbrace{\{ (a, b) \in \mathbb{A}^2 \mid a \neq b \}}_{\text{non-repeating pairs}} \qquad \underbrace{\binom{\mathbb{A}}{2}}_{\text{unordered pairs}} .$$

There are several equivalent definitions of orbit-finiteness. Of these, we use one that is based on first-order interpretations [Hod93, Sec. 4.3]. To construct an orbit-finite set, we proceed in three steps: take a finite power of the atoms (as in the first example above), restrict this power to an equivariant subset (as in the second example above), and then take a quotient under an equivariant equivalence relation (as in the third example above, where the equivalence relation identifies two pairs if they differ only in their order). The formal definition is given below.

Definition 3.1 (Orbit-finite set)**.** An *orbit-finite set* over an oligomorphic structure $\mathbb{A}$ is any set that is obtained as follows:

(1) Start with a finite power $\mathbb{A}^d$ for some $d \in \{0, 1, \ldots\}$.
(2) Restrict it to an equivariant subset $X \subseteq \mathbb{A}^d$.
(3) Quotient X under an equivariant equivalence relation.

Let us justify the name "orbit-finite" in this definition. An orbit-finite set is equipped with an action of the automorphism group of the original structure $\mathbb{A}$, namely the action inherited from $\mathbb{A}^d$, suitably extended to the quotient. Under this action, the set has finitely many orbits: $\mathbb{A}^d$ has finitely many orbits by assumption on oligomorphism, and the number of orbits can only go down when restricting to an equivariant subset and quotienting under an equivariant equivalence relation.

There are other equivalent definitions of orbit-finite sets. One, see Definition 3.2 below, emphasises the role of the group action. In this definition, we use the following notion of support: if X is a set equipped with an action of the automorphism group of $\mathbb{A}$, then a *support* for an element $x \in X$ is any set $S \subseteq \mathbb{A}$ such that every automorphism π of $\mathbb{A}$ satisfies

$$\underbrace{\forall a \in S\ \pi(a) = a}_{\text{action of } \pi \text{ on } \mathbb{A}} \quad \Longrightarrow \quad \underbrace{\pi(x) = x}_{\text{action of } \pi \text{ on } X}.$$

Definition 3.2 (Orbit-finite sets, abstractly)**.** An *orbit-finite set* over an oligomorphic structure $\mathbb{A}$ is a set X equipped with an action of the automorphism group of $\mathbb{A}$ such that:

(1) there are finitely many orbits under the action;
(2) every element has some finite support.

Thanks to oligomorphism of the underlying atom structure, the two definitions above are equivalent in the sense that they describe the same sets, up to equivariant bijections — see [Boj25, Thm. 5.13]. Both definitions have their uses. Definition 3.1 is more concrete, and it comes with a finite representation, which can be used for algorithms that process orbit-finite sets. On the other hand, some constructions — e.g. disjoint unions, products — will result in sets that are consistent with the second, more relaxed, definition.

4 Orbit-finitely spanned vector spaces

We now introduce the main topic of this paper, which is orbit-finitely spanned vector spaces. We begin with the special case of spaces that have an orbit-finite basis; this special case has an elementary definition, and yet it will be the relevant case for almost all results of this paper.

Definition 4.1 (Orbit-finite basis)**.** A vector space with an orbit-finite basis is any space of the form $\mathrm{Lin}_{\mathsf{F}} X$ (where X is an orbit-finite set and F a field), i.e. the space of finite formal linear combinations of elements in X.

This definition has two important parameters: the field F and the oligomorphic structure $\mathbb{A}$ over which X is an orbit-finite set. The spaces defined here have two kinds of structure: that of a vector space, and the action of the automorphism group of $\mathbb{A}$. We will be interested in subsets that preserve both kinds of structure, i.e. they are closed under taking linear combinations, and applying automorphisms of $\mathbb{A}$. Such subsets are called *equivariant subspaces.*

Example 4.2. Let $\mathbb{A}$ be the equality atoms and let F be any field. As explained in [Boj+24, Ex. 4.2], this corresponding vector space with atoms $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}$ has only three equivariant subspaces: the zero subspace, the full space, and subspace which consists of those vectors where the coefficients add up to zero. ▲

Unfortunately, there is a price to pay for the elementary character of Definition 4.1, which is the failure of certain closure properties. In particular, the spaces are not closed under taking equivariant subspaces, or quotients under such spaces. The problem with subspaces is apparent already in Example 4.2, since the unique nontrivial subspace of $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}$ does not have any equivariant basis, regardless of the underlying field [Boj+24, Ex. 6.1]. For this reason, we use a more general notion of vector space, which is presented below, in a style that emphasises the group action, as was done in Definition 3.2.

Definition 4.3. [Orbit-finitely spanned vector space, abstractly] An *orbit-finitely spanned vector space* over an oligomorphic structure $\mathbb{A}$ is a vector space equipped with an action of automorphisms of $\mathbb{A}$ such that:

(1) vector addition and scalar multiplication are equivariant;[1]
(2) every vector is supported by a finite set of atoms;
(3) the space is spanned by some subset that is orbit-finite.

In (1), by equivariance of scalar multiplication we mean that for every field element λ, the operation $v \mapsto \lambda v$ is equivariant. The above definition is easily seen to be closed under taking quotients by equivariant subspaces. Closure under taking equivariant subspaces is less obvious: one could imagine that condition (3) above is violated by moving to an equivariant subspace. It turns out that closure under equivariant subspaces is intimately related to the ascending chain property discussed in the introduction:

Theorem 4.4. *For any field F and oligomorphic structure $\mathbb{A}$, the following conditions are equivalent:*

(1) orbit-finitely spanned vector spaces are closed under taking equivariant subspaces;
(2) for every $d \in \{0, 1, \ldots\}$, the vector space $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^d$ does not have any infinite ascending chain of equivariant subspaces;
(3) for every orbit-finitely spanned vector space V, there are no infinite ascending chains of equivariant subspaces in V.

Furthermore, if these conditions hold, then every orbit-finitely spanned vector space is isomorphic to one of the form V/U, where $V \subseteq U$ are equivariant subspaces of $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^d$ for some $d \in \{0, 1, \ldots\}$.

We say that an atom structure $\mathbb{A}$ has the *ascending chain property* over a field F if any of the equivalent conditions in Theorem 4.4 are satisfied. One interpretation of the theorem is that the ascending chain property is necessary for the theory of vector spaces to be well-behaved. In particular, thanks to the "furthermore" part, we get a similar result to the equivalence of Definitions 3.1 and 3.2 in the previous section, i.e. a concrete characterisation of the orbit-finitely spanned vector spaces that can be used in algorithms (provided we know how to represent equivariant subspaces — see Section 8.5).

We are therefore interested in atom structures that have the ascending chain property. As it turns out, the techniques used in this paper will yield a stronger property, namely a finite bound on the length of chains:

Definition 4.5 (Finite length property)**.** An oligomorphic structure $\mathbb{A}$ has the *finite length property over a field* F if for every orbit-finite set X over $\mathbb{A}$, there is a finite upper bound on the length of chains of equivariant subspaces of $\mathrm{Lin}_{\mathsf{F}} X$. (The supremum of the chain lengths, finite or not, is called the *length* of $\mathrm{Lin}_{\mathsf{F}} X$.)

In light of Theorem 4.4, we could have used $\mathbb{A}^d$ instead of X in the above theorem. As mentioned in the introduction, the finite length property can be strictly stronger than the ascending chain property, as witnessed by the vector atoms from Example 2.5. (Note that when we talk about the vector atoms, there two fields involved, namely the finite field used to define $\mathbb{A}$, and the field F used to define the vector space with atoms. In the counterexample, both fields are the two-element field.) The finite length property was studied in [Boj+24], where it was shown that the equality atoms (Example 2.3) and the ordered atoms (Example 2.4) have this property over any field.

[1] Equivalently, the group action $v \mapsto \pi(v)$ is linear.

The main contribution of this paper is to establish the finite length property for more structures. We will use two different techniques for this purpose.

5 Finite length in characteristic zero

In this section, we present the first of our two main results, which is a method for proving the finite length property, assuming that the field has characteristic zero. Under this assumption, we will establish the finite length property for the Rado atoms (Example 2.6) and for the vector atoms (Example 2.5). These are new results. We also think that the proof itself, even when applied to get already known results, is of independent interest and arguably simpler than previously known proofs.

The method that we use will work for structures that satisfy the following condition.

Definition 5.1 (Oligomorphic approximation)**.** We say that a structure $\mathbb{A}$ has *oligomorphic approximation* if it is homogeneous and, for every $d \in \{1, 2, \ldots\}$, there exists a family $\mathscr{B}$ of finite substructures of $\mathbb{A}$ such that:

(1) every finite substructure of $\mathbb{A}$ is contained in some $\mathbb{B} \in \mathscr{B}$; and
(2) there is a common finite upper bound on the number of orbits in $\mathbb{B}^d$ for $\mathbb{B} \in \mathscr{B}$.

This is a relaxation of a stronger notion of *smooth approximation* known in model theory. There, the family $\mathscr{B}$ is independent of d; also, in the place of (2) we require $\mathbb{B} \in \mathscr{B}$ to be homogeneous itself and $\mathbb{A}$ to be oligomorphic [KLM89, p. 440]. In that case, two tuples in $\mathbb{B}^d$ are in the same orbit if and only if they are in the same orbit of $\mathbb{A}^d$; hence the number of orbits in $\mathbb{B}^d$ is at most that of $\mathbb{A}^d$, which is finite by oligomorphism.

Theorem 5.2. *The following structures have oligomorphic approximation:*

(1) the equality atoms from Example 2.3;
(2) the vector atoms from Example 2.5, for any finite field k*;*
(3) the Rado atoms from Example 2.6.

Before proving the above theorem, let us observe that the ordered atoms (Example 2.4) do not have oligomorphic approximation.

Non-example 5.3. Consider the rational numbers with the usual order. The finite substructures in this case are finite linear orders, and already for dimension $d = 1$, a finite linear order of size n will have n orbits. So no family $\mathbb{B}$ satisfies both condition (1) and (2). ▼

Proof of Theorem 5.2. Note that, by relying on the homogeneity of $\mathbb{A}$, we can afford to be sloppy and only identify $\mathbb{B}$ up to isomorphism.

(1) For the equality atoms, we can simply choose $\mathscr{B}$ to be all finite sets. These are all homogeneous: we can extend a bijection to a permutation.
(2) For the vector atoms, we can also choose $\mathscr{B}$ independently of d to be the family of all vector spaces of finite dimension. These are all homogeneous, since we can extend linear bijections by extending a linearly independent set to a basis. This argument applies to any finite field k.
(3) The most interesting case is the Rado atoms. The witness for oligomorphic approximation[2] will be a family of *symplectic graphs*: see [GR01, Sec. 8.11].[3] For every $n \in \{1, 2, \ldots\}$ define a finite graph as follows. The set of vertices is the vector space over the two-element field with basis

$$\{e_1, \ldots, e_n, f_1, \ldots, f_n\}.$$

Since the field has two elements, we can view vertices as subsets of this basis. In this graph, there is an edge between vertices v and w if and only if the sets

$$\{i \in \{1, \ldots, n\} \mid e_i \in v \text{ and } f_i \in w\}$$
$$\{i \in \{1, \ldots, n\} \mid f_i \in v \text{ and } e_i \in w\}$$

have different sizes modulo two. These graphs satisfy condition (1) from Definition 5.1, i.e. every finite graph embeds in some symplectic graph [GR01, Thm. 8.11.2]. In Appendix B, we prove condition (2), i.e. that the number of orbits of d-tuples in symplectic graphs is uniformly bounded by a function of d only. □

The main result of this section is the following theorem.

Theorem 5.4. *If an oligomorphic structure $\mathbb{A}$ has oligomorphic approximation, then it has the finite length property over any field of characteristic* 0.

Combining Theorems 5.2 and 5.4, we can get the following results, both old and new, on the finite length property.

Corollary 5.5. *Over any field of characteristic* 0*, the following structures have the finite length property: (a) the equality atoms; (b) the vector atoms; and (c) the Rado atoms.*

As mentioned in the introduction, the finite length property was already known for the equality atoms, for arbitrary fields. The results for the vector atoms and the Rado atoms are new. The assumption on characteristic zero is important, at least in the case of the vector atoms, where the finite length property is known to fail over finite fields — see, most recently, [Boj+24, Sec. 4.4]. Later on in this paper, we will prove the result for the Rado atoms again using a different method that works for any field.

The rest of this section is devoted to proving Theorem 5.4.

Proof of Theorem 5.4. Fix a structure $\mathbb{A}$, which has oligomorphic approximation, and a field of characteristic zero. Since the field is fixed, we omit the field subscript and write $\operatorname{Lin} X$ for linear combinations of elements in X that use coefficients from that field. Fix some power $d \in \{1, 2, \ldots\}$. Our goal is to show that $\operatorname{Lin} \mathbb{A}^d$ has the finite length property. For technical reasons, we apply the assumption on oligomorphic approximation not to d, but to $2d$, yielding some class $\mathscr{B}$ of finite structures that satisfies Definition 5.1.

Lemma 5.6. *For every* $d \in \{1, 2, \ldots\}$ *we have*

$$\textit{length of } \operatorname{Lin} \mathbb{A}^d \quad \leq \quad \sup_{\mathbb{B} \in \mathscr{B}} \textit{ length of } \operatorname{Lin} \mathbb{B}^d.$$

[2] But there is no smooth approximation — see [CH03, Rem. 2.1.2]. Essentially, the classification of finite homogeneous graphs [Gar76, p. 100] means (1) cannot hold.
[3] We are grateful to Ehud Hrushovski for drawing our attention to this construction.

Proof. Consider some chain of equivariant subspaces

$$V_0 \subset V_1 \subset \ldots \subset V_n = \operatorname{Lin} \mathbb{A}^d,$$

where equivariance is with respect to automorphisms of $\mathbb{A}$. For each $i \in \{1, \ldots, n\}$, choose some vector that is in V_i but not in V_{i-1}. Let S be the (finite) set of atoms that appear in these chosen vectors, and choose some $\mathbb{B} \in \mathscr{B}$ which contains all atoms from S. Define

$$W_i = V_i \cap \operatorname{Lin} \mathbb{B}^d.$$

By homogeneity, every automorphism of $\mathbb{B}$ extends to an automorphism of $\mathbb{A}$, and therefore the space W_i is equivariant with respect to automorphisms of $\mathbb{B}$. The chain of W_i's continues to be strictly growing, since it contains vectors that witness the growth of the original chain. Hence, the new chain witnesses that the length of $\operatorname{Lin} \mathbb{B}^d$ is at least n. □

Thanks to the above lemma, it remains to show that the length of $\operatorname{Lin} \mathbb{B}^d$ is bounded by some number that depends only on d. In our proof, this bound will be the number of orbits in $\mathbb{A}^{2d}$. What we have gained by moving from $\mathbb{A}$ to $\mathscr{B}$ is that our vector spaces now have finite (albeit unbounded) linear dimension, and the group actions use finite (albeit unbounded) groups. This will let us leverage a central result from representation theory, called Maschke's Theorem (see e.g. [JL01, Chap. 8]), which decomposes spaces into irreducible parts. (In the theorem below, $\oplus$ stands for the direct sum of vector spaces, i.e. $V \oplus W$ is the set of pairs (v, w) where $v \in V$ and $w \in W$, with the operations defined coordinate-wise.)

Fact 5.7 (Maschke's Theorem). *Let V be a finite dimensional vector space over a field of characteristic zero, equipped with an action of a finite group G. Then V can be decomposed as*

$$V = V_1 \oplus \cdots \oplus V_n,$$

where each V_i is an equivariant subspace (with respect to the action of G) and is irreducible, i.e. the only equivariant subspaces of V_i are the zero space and the full space V_i.

We will use Maschke's Theorem to bound the length of the vector spaces $\operatorname{Lin} \mathbb{B}^d$. For two vector spaces V and W equipped with an action of the same group G, let us write

$$V \underset{\text{lineq}}{\longrightarrow} W$$

for the set of all those linear maps from V to W which are equivariant with respect to the action of G. (The group and its action are implicit in this notation.) Elements of this set are closed under taking linear combinations, and therefore the set can be seen as a vector space. (We do not care to equip this space with an action of G.) In particular, it is meaningful to talk about the dimension of this vector space.

Lemma 5.8. *Let V be a finite dimensional vector space over a field of characteristic zero, equipped with an action of a finite group G. Then*

$$\text{length of } V \quad \leq \quad \text{dimension of } V \underset{\text{lineq}}{\longrightarrow} V.$$

Proof. Apply Maschke's Theorem, yielding a decomposition

$$V = V_1 \oplus \cdots \oplus V_n,$$

where the subspaces $V_1, \ldots, V_n$ are equivariant and irreducible, with respect to the action of the group G. Irreducible spaces have length one by definition, and the length is additive with respect to direct sums, i.e.

$$\text{length of } V = \text{length of } V_1 + \cdots + \text{length of } V_n,$$

so the length of V is equal to n. We will now show that the dimension is at least n on the right-hand side of the inequality in the statement of the lemma. For every $i \in \{1, \ldots, n\}$ we can define an equivariant linear map from V to itself which is the identity on V_i and maps vectors from the other components to zero. This gives us at least n equivariant linear maps from V to itself. None of these maps is spanned by the other, and hence the dimension is at least n. □

Thanks to Lemmas 5.6 and 5.8, the length of the vector space with atoms $\operatorname{Lin} \mathbb{A}^d$ is bounded by the dimensions of the vector spaces

$$\operatorname{Lin} \mathbb{B}^d \underset{\text{lineq}}{\longrightarrow} \operatorname{Lin} \mathbb{B}^d, \tag{$*$}$$

where $\mathbb{B}$ ranges over the family $\mathscr{B}$. To complete the proof of the theorem, it remains to show that these dimensions are bounded by some constant that depends only on d. This is done in the following lemma, which completes the proof of Theorem 5.4.

Lemma 5.9. *For every $\mathbb{B} \in \mathscr{B}$ the dimension of the vector space in $(*)$ is at most the number of orbits in $\mathbb{B}^{2d}$.*

Proof. A linear map in the F-vector space $(*)$ is the same thing as a square matrix indexed by $\mathbb{B}^d$, i.e. a function of type

$$\mathbb{B}^d \times \mathbb{B}^d \to \mathsf{F}.$$

This function must be equivariant with respect to automorphisms of $\mathbb{B}$. This means that inputs in the same orbit must be mapped to the same field element. Therefore, to define such a function, we need to choose one element of F for each orbit of the input. Therefore, the dimension of the space in $(*)$ is equal to the number of orbits of $\mathbb{B}^{2d}$, under the action of the group of automorphisms of $\mathbb{B}$. This dimension is bounded by some constant that depends only on d, by the assumption of oligomorphic approximation. □

This completes the proof of Theorem 5.4. For convenience, we summarise the steps in Figure 2. □

The inequalities shown in Figure 2 give us upper bounds on the length. In the case of the equality atoms, this bound is exponential in d. In the case of the vector atoms from Example 2.5, the bound is the number of linear dependency patterns for $2d$-tuples over a finite field. Such a pattern is described by: (a) indicating a subset of the coordinates which is a basis for the tuple; and (b) indicating the basis decompositions for the remaining coordinates. This can be done in at most exponentially many ways in d, and therefore the overall bound is exponential in d. In Appendix B we give an upper bound in the case of the Rado atoms, also exponential in d.

6 Function spaces and weighted automata

The original motivation to introduce orbit-finitely spanned vector spaces in [Boj+24] was the study of orbit-finite weighted automata. In this section, we recall this motivation, and discuss how it is related to our new results. This discussion also involves the issue of function spaces, arguably more important, so we begin with that.

$$\begin{array}{c} \text{length of } \mathrm{Lin}\, \mathbb{A}^d \\ \wedge\!\mid \quad \text{(Lemma 5.6)} \\ \sup_{\mathbb{B}\in\mathscr{B}} \text{length of } \mathrm{Lin}\, \mathbb{B}^d \\ \wedge\!\mid \quad \text{(Lemma 5.8)} \\ \sup_{\mathbb{B}\in\mathscr{B}} \text{dimension of } \mathrm{Lin}\, \mathbb{B}^d \xrightarrow[\text{lineq}]{} \mathrm{Lin}\, \mathbb{B}^d \\ \wedge\!\mid \quad \text{(Lemma 5.9)} \\ \sup_{\mathbb{B}\in\mathscr{B}} \text{number of orbits in } \mathbb{B}^{2d} \\ \wedge \quad \text{(by definition)} \\ \infty. \end{array}$$

Figure 2: Summary of the proof of Theorem 5.4.

6.1 Function spaces

If we have two orbit-finitely spanned vector spaces V and W over the same atoms, then there are two natural ideas for a function space: the space of all linear maps from $V \to W$, and the subspace which consists of equivariant linear maps. As it turns out, the most relevant function space lies between them, as formalised in the following definition.

Definition 6.1 (Finitely supported function space)**.** For two orbit-finitely spanned vector spaces V and W, we define their *function space*, denoted by

$$V \xrightarrow[\text{linfs}]{} W,$$

to be the space of linear maps f which satisfy the following finite support condition: there is some finite set of atoms $S \subseteq \mathbb{A}$ such that

$$\pi(f(v)) = f(\pi(v)) \quad \text{for all } v \in V$$

holds for every atom automorphism π that fixes all atoms in S.

The notion of finite supports in the above definition is the same the one as used in Section 3, except that it is applied to the space of linear maps from V to W. It is also the standard restriction used in the study of orbit-finite sets. As argued in [Boj25, Sec. 8.3] using categorical arguments, the finitely supported function space is the most relevant kind of function space; a similar phenomenon occurs in nominal sets [Pit13, Thm. 3.13]. For this reason, we when talking about function spaces, we mean the finitely-supported function spaces from the above definition. This definition motivates the following property.

Definition 6.2 (Function space property)**.** We say that an atom structure has the *function space property* if orbit-finitely spanned vector spaces are closed under taking function spaces.

As it turns out, some oligomorphic atom structures have this property, and some do not. The equality and ordered atoms have it, which was shown in [Boj+24, Cor. 6.8] for a special case of function spaces, namely duals, with the general case of function spaces being treated in [Boj25, Sec. 8.3]. On the other hand, the Rado atoms fail this property, as explained in the following example. (In particular, the finite length property does not imply the function space property.)

Example 6.3. Assume that the $\mathbb{A}$ is the Rado atoms, and the field is arbitrary.[4] Consider the space

$$\mathbb{A} \xrightarrow[\text{fs}]{} \mathsf{F} \tag{1}$$

which consists of functions (not linear maps) from atoms to the field, which are finitely supported in the sense of Definition 6.1. This space is isomorphic to the function space

$$\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A} \xrightarrow[\text{linfs}]{} \mathsf{F}. \tag{2}$$

We will show that (1) is not orbit-finitely spanned, and hence the same is true for the isomorphic function space (2). For a finite set $S \subseteq \mathbb{A}$ of atoms, define a function $f_S : \mathbb{A} \to \mathsf{F}$ by

$$a \mapsto \begin{cases} 1 & \text{if } a \text{ is a neighbour of all atoms in } S \\ 0 & \text{otherwise.} \end{cases}$$

Define V to be the subspace of (1) that is spanned to be the functions f_S, where S ranges over finite sets of atoms.[5] The spanning set $\{f_S\}_S$ is not orbit-finite, and as we will show in Appendix C, no orbit-finite spanning set can be found. Being orbit-finitely spanned is closed under taking subspaces, which follows from Theorem 4.4 and the fact that the Rado atoms has the ascending chain property (which was already proved in Theorem 5.4 for characteristic zero and will be proved later for any field). In the appendix, we show that V is not orbit-finitely spanned, and thus the same is true for (1). ▲

6.2 Weighted automata

We now describe weighted automata, and explain how the issues with function spaces have an impact on the theory of these automata. There are several ways of defining weighted automata; we choose one that views them as deterministic automata, in which the states are endowed with a vector space structure.

Definition 6.4. An *orbit-finite weighted automaton* is given by:

- Σ an orbit-finite set;
- Q an orbit-finitely spanned vector space;
- q_0 an equivariant element of Q;
- δ an equivariant function of type $Q \times \Sigma \to Q$, which becomes a linear map from Q to itself after fixing any input letter;
- F an equivariant linear map of type $Q \to \mathsf{F}$.

An automaton as in the above definition computes a function of type $\Sigma^* \to \mathsf{F}$, which is defined in the same way as for deterministic automata.

From the finite length property, we can conclude decidability results about orbit-finite weighted automata. For example, a zeroness algorithm is derived in [Boj+24, Sec. 5]. In light of the results

[4] A variant of this example for the two-element field was shown in [Boj+24, Ex. 6.9], but here we can use any field, in particular a field of characteristic zero as treated in the previous section.

[5] The example would also work for the Henson graphs that will be defined later, but for that S will only range over anti-cliques.

from this paper, such an algorithm exists if we use the Rado graph as the atoms, or other structures for which we have established the finite length property. Also, weighted automata can be minimized [Boj+24, Sec. 7], effectively, using the bound for chains.

However, certain other results on weighted automata will depend on the function space property. Let us give one such example. Our definition of weighted automata is deterministic in the left-to-right direction. One could imagine a symmetric right-to-left model. Are these models equivalent? If the atoms have the function space property, then one can introduce a symmetric model based on monoids to show that the left-to-right and right-to-left variants of weighted automata are equivalent: see [Boj+24, Thm. 7.4]. However, as we show in the following example, the equivalence can fail without the function space property.

Example 6.5. Consider the Rado atoms and any field. In this example, we prove that the left-to-right and right-to-left variants of Definition 6.4 are not equivalent. The counterexample is the characteristic function of the language "the first letter is adjacent to all later ones", i.e. the function f defined by

$$a_1 \cdots a_n \in \mathbb{A}^* \quad \mapsto \quad \begin{cases} 1 & \text{if } a_1 \text{ is adjacent to all of } a_2, \ldots, a_n \\ 0 & \text{otherwise.} \end{cases}$$

We will show that this function is computed by a left-to-right orbit-finite weighted automaton, but not by a right-to-left one. To prove this, we use a Myhill-Nerode style argument. For an input word $w \in \mathbb{A}^*$, we define two functions of type $\mathbb{A}^* \to \mathsf{F}$ as follows:

$$\underbrace{v \mapsto f(wv)}_{\text{left derivative}} \quad \text{and} \quad \underbrace{v \mapsto f(vw)}_{\text{right derivative}}.$$

Using the usual Myhill-Nerode construction, one can show that a function is computed by a left-to-right orbit-finite weighted automaton if and only if its left derivatives are orbit-finitely spanned, and similarly for right-to-left derivatives. To finish the counterexample, we will now show that the left derivatives are orbit-finitely spanned, but the right ones are not.

The set of left derivatives is not only orbit-finitely spanned, but it is even orbit-finite: there is one left derivative for each $a \in \mathbb{A}$, plus one extra derivative for the always zero function. On the other hand, the set of right derviatives is the same as the spanning set of the vector space V from Example 6.3, and therefore it cannot be orbit-finitely spanned. ▲

7 Finite length from free amalgamation with a generic order

As we saw in Non-example 5.3, the ordered atoms do not have oligomorphic approximation. Nonetheless they do have the finite length property over any field [Boj+24, Thm. 4.8], not just over those of characteristic zero. We shall now generalise that result to a wide class of structures. In particular, we will deduce the finite length property for the Rado atoms and its variants, even with finitely many constants fixed.

We shall now state our assumptions and results, with proofs in Section 8.

Graph vocabulary. Consider a finite relational vocabulary σ_0 consisting of unary and binary relations only. (This allows us to talk about graphs with coloured vertices and edges, but in the running example we will consider a single edge relation.)

Free amalgamation class. Let $\mathscr{C}_0$ be a *free* amalgamation class of finite σ_0-structures. This is a standard notion in model theory — see [Mac11, Sec. 2.1]. Informally, it means that when we perform amalgamation, we do not need to glue together any new elements, or introduce new relations.

There is a useful characterisation [SS20, Lem. 4.5] of the amalgamation class $\mathscr{C}_0$ being free. Since $\mathscr{C}_0$ is closed under substructures and isomorphisms, we know that it consists of all the finite σ_0-structures which do not embed any structure from $\mathscr{F}$, where $\mathscr{F}$ consists of all the minimal finite σ_0-structures that do not belong to $\mathscr{C}_0$; we write

$$\mathscr{C}_0 = \mathrm{Forb}(\mathscr{F}).$$

That $\mathscr{C}_0$ is a free amalgamation class means that in each $F \in \mathscr{F}$, any two elements x, y are either equal or satisfy at least one of $R(x, y)$ and $R(y, x)$ for some binary relation $R \in \sigma_0$ — from here on, we will just say x, y are *related*. Conversely, given a family $\mathscr{F}$ of finite σ_0-structures where every two elements are related, the class $\mathrm{Forb}(\mathscr{F})$ of finite σ_0-structures form a free amalgamation class.

Example 7.1. Let σ_0 be empty. Then $\mathscr{C}_0 = \mathrm{Forb}(\{\})$ is a free amalgamation class consisting of all finite pure sets. ▲

Example 7.2. Let σ_0 consist of a single binary relation E. Consider σ_0-structures $\circlearrowright = \{x\}$ and $\rightarrow = \{y, z\}$, where the relation E is interpreted as $\{(x, x)\}$ and $\{(y, z)\}$. Then $\mathscr{C}_0 = \mathrm{Forb}(\{\circlearrowright, \rightarrow\})$ is a free amalgamation class consisting of all finite undirected graphs.

In addition, let K_n be the σ_0-structure representing a complete graph on n vertices. Then

$$\mathrm{Forb}(\{\circlearrowright, \rightarrow, K_3\}) \subseteq \mathrm{Forb}(\{\circlearrowright, \rightarrow, K_4\}) \subseteq \cdots \subseteq \mathrm{Forb}(\{\circlearrowright, \rightarrow\})$$

are all free amalgamation classes. ▲

Irreflexivity. For technical reasons, we restrict attention to classes $\mathscr{C}_0$ where all structures are *irreflexive*, i.e. such that if $R(x, y)$ then $x \neq y$ for each relation R. This does not lose generality: for any free amalgamation class $\mathscr{C}_0$, we may replace every binary relation R in the vocabulary with a new unary predicate $R_=$, interpreted in every structure so that $R_=(x)$ iff $R(x, x)$, and a binary relation interpreted as the irreflexive part of R. Then $\mathscr{C}_0$ interpreted over this modified language is still a free amalgamation class, and all our results transport back to the original $\mathscr{C}_0$. This way of encoding arbitrary structures as irreflexive ones is standard; see e.g. [Her98, Sec. 2.4] and [SS20, p. 121]. All examples considered here are already irreflexive.

Generically ordered expansion. Now, let σ consist of σ_0 together with a new binary relation $<$. Consider the class $\mathscr{C}$ of σ-structures obtained from $\mathscr{C}_0$, by interpreting $<$ in any σ_0-structure there as any total order.

Observe that $\mathscr{C}$ is an amalgamation class. Indeed, let X, Y_1, Y_2 be σ-structures in $\mathscr{C}$ with $X \subseteq Y_1 \cap Y_2$. Then we can amalgamate Y_1, Y_2 over X as σ_0-structures and as $\{<\}$-structures, both using the disjoint union of Y_1, Y_2 over X as the underlying set. Superposing these relations will give an σ-structure in $\mathscr{C}$, which is the desired amalgamation. (Because of the total order, this amalgamation in $\mathscr{C}$ is not free unless it is trivial.) Denote the Fraïssé limit of $\mathscr{C}$ by $\mathbb{A}$.

Theorem 7.3. *The ordered structure $\mathbb{A}$, even with finitely many constants fixed, has the finite length property over any field.*

Before we prove this theorem in Section 8, let us state an easy consequence of it and give some examples.

The Fraïssé limit $\mathbb{A}_0$ of $\mathscr{C}$ is a reduct of $\mathbb{A}$. To see this, notice that $\mathbb{A}$, when viewed as an σ_0-structure, shares the same age as $\mathbb{A}_0$ — namely, $\mathscr{C}_0$. Moreover, it follows from a back-and-forth argument [Hod93, Lem. 7.1.4] that the σ_0-structure $\mathbb{A}$ is also homogeneous and, therefore, that it is isomorphic to $\mathbb{A}_0$. So we may assume that $\mathbb{A}$ and $\mathbb{A}_0$ have the same underlying set; we then have $\mathrm{Aut}(\mathbb{A}) \subseteq \mathrm{Aut}(\mathbb{A}_0)$, where $\mathrm{Aut}(X)$ denotes the group of automorphisms of X.

Corollary 7.4. *The reduct $\mathbb{A}_0$ of $\mathbb{A}$, even with finitely many constants fixed, has the finite length property over any field.*

Proof. A chain of subspaces in $\mathrm{Lin}_\mathsf{F}\,\mathbb{A}_0^d = \mathrm{Lin}_\mathsf{F}\,\mathbb{A}^d$ each supported by $S \subseteq \mathbb{A}_0$ is also a chain of subspaces supported by $S \subseteq \mathbb{A}$; the latter has a bounded length by the theorem above. ◻

Conversely, we call $\mathbb{A}$ the *generically ordered expansion* of $\mathbb{A}_0$ or simply the "ordered $\mathbb{A}_0$".

Example 7.5. Continuing from Example 7.1, the ordered atoms (Example 2.4) is the generically ordered expansion of the equality atoms (Example 2.3). ▲

Example 7.6. Continuing from Example 7.2, we obtain generically ordered expansions of the Rado graph and of the K_n-free versions of it, which are called *Henson graphs*. The ordered Rado graph was studied in [BPP15] along with its first-order reducts (e.g., the universal homogeneous tournament). The triangle-free Henson graph will be studied in an extended example in Section 8.4. ▲

Using a classification result of Lachlan and Woodrow [LW80], we can get the following general result.

Corollary 7.7. *Every countable homogeneous undirected graph has the finite length property over any field.*

8 How the cogs turn: proof of Theorem 7.3

We will proceed in several steps, with the general idea similar to [Boj+24, Sec. 4.1], but with significant new complications arising from the presence of nontrivial relations in $\mathbb{A}_0$.

8.1 Orbits and projections

To start with, let us view $\mathbb{A}^d$ as $\mathbb{A}^{\{1,\dots,d\}}$. More generally, it will be convenient to consider $\mathbb{A}^I$ for a finite totally ordered indexing set $I \subseteq \mathbb{Q}$. Fix a finite support $S \subseteq \mathbb{A}$. We shall say that a tuple $a \in \mathbb{A}^I$ is *(S-)ordered* if $a_i \notin S$ for all i, and $a_i < a_j$ whenever $i < j$. Then the orbit $O = \mathrm{Aut}(\mathbb{A}/S) \cdot a$ only contains S-ordered tuples, and we will call the orbit S-ordered as well.[6] If a is not S-ordered, by removing the entries that repeat or come from S and reordering the rest, we can always find an $\mathrm{Aut}(\mathbb{A}/S)$-equivariant bijection from O to an S-ordered orbit.

To study the lengths of orbit-finitely spanned spaces, we may focus on a single ordered orbit at a time:

Claim 8.1. *The following are equivalent:*

1. *For any d and finite $S \subseteq \mathbb{A}$, chains of $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspaces in $\mathrm{Lin}_\mathsf{F}\,\mathbb{A}^d$ are bounded in length;*
2. *$\mathrm{Lin}_\mathsf{F}\,O$ has finite length for any ordered orbit O.*

Proof. Since $\mathbb{A}$ is oligomorphic, given any d and S, we know that $\mathbb{A}^d$ is in an $\mathrm{Aut}(\mathbb{A}/S)$-equivariant bijection with a finite disjoint union $\biguplus_i O_i$ of S-ordered orbits. Hence the length of $\mathrm{Lin}_\mathsf{F}\,\mathbb{A}^d$, under the action of $\mathrm{Aut}(\mathbb{A}/S)$, equals

$$\mathit{length}(\mathrm{Lin}_\mathsf{F}(\biguplus_i O_i)) = \mathit{length}(\bigoplus_i \mathrm{Lin}_\mathsf{F}\,O_i) = \sum_i \mathit{length}(\mathrm{Lin}_\mathsf{F}\,O_i),$$

which is finite if and only if each summands is finite. ◻

So fix an ordered orbit $O = \mathrm{Aut}(\mathbb{A}/S)\cdot o \subseteq \mathbb{A}^I$. From here we take an inductive approach. By $o|^J$ we mean the restriction of $o : I \to \mathbb{A}$ to $J \subseteq I$; particularly, we will often write $o|^{-i}$ instead of $o|^{I\setminus\{i\}}$. The image $O|^J$ of O under this projection agrees with $\mathrm{Aut}(\mathbb{A}/S)\cdot o|^J$ and is still ordered.

The function $(-)|^J$ lifts to a linear $\mathrm{Aut}(\mathbb{A}/S)$-equivariant map

$$(-)|^J : \mathrm{Lin}_\mathsf{F}\,O \to \mathrm{Lin}_\mathsf{F}\,O|^J.$$

Many cancellations can occur under $(-)|^J$; the *projection kernel* is the $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspace of $\mathrm{Lin}_\mathsf{F}\,O$ below:

$$\mathrm{Ker}_\mathsf{F}\,O = \bigcap_{i\in I} \ker\,(-)|^{-i}.$$

Claim 8.2. *The following are equivalent:*

1. *$\mathrm{Lin}_\mathsf{F}\,O$ has finite length for every ordered orbit O;*
2. *$\mathrm{Ker}_\mathsf{F}\,O$ has finite length for every ordered orbit O.*

Proof. That (1) implies (2) is clear as $\mathrm{Ker}_\mathsf{F}\,O \subseteq \mathrm{Lin}_\mathsf{F}\,O$.

To prove the other implication, assume (2) and let $O \subseteq \mathbb{A}^I$. We proceed by induction on $|I|$. If $I = \emptyset$, then O must be the entire singleton $\mathbb{A}^\emptyset = \{()\}$; as $\mathrm{Lin}_\mathsf{F}\,O$ has no nontrivial subspaces (let alone finitely supported ones), it has length 1. Now if $|I| \geq 1$, assemble all $|I|$ projection maps into a single map

$$\mathrm{Lin}_\mathsf{F}\,O \to \bigoplus_{i\in I} O|^{-i}$$
$$v \mapsto (v|^{-i})_{i\in I}$$

whose kernel is precisely $\mathrm{Ker}_\mathsf{F}\,O$. We have

$$\mathit{length}(\mathrm{Lin}_\mathsf{F}\,O) - \mathit{length}(\mathrm{Ker}_\mathsf{F}\,O) \leq \sum_{i\in I} \mathit{length}(\mathrm{Lin}_\mathsf{F}\,O|^{-i})$$

which shows that $\mathit{length}(\mathrm{Lin}_\mathsf{F}\,O)$ is finite from the assumptions. ◻

Following the terminology used in [Boj+24, Eq. (4)], we shall call a vector from the projection kernel *balanced*. As we will see shortly, there exist balanced vectors other than 0.

8.2 Cogs

From now on we will use a lightweight notation for combining tuples of atoms: for disjoint indexing sets I and J, if $a \in \mathbb{A}^I$ and $b \in \mathbb{A}^J$ are both ordered, then $ab \in \mathbb{A}^{I\cup J}$ will denote their obvious combination. We will only use this notation if this ab is ordered. For an obvious example, for any S-ordered $a \in \mathbb{A}^I$ and $J \subseteq I$, we have $a|^{I\setminus J} a|^J = a$.

Definition 8.3. Let $O \subseteq \mathbb{A}^I$ be an S-ordered orbit. An *O-duo* $a \parallel b$ consists of tuples $a, b \in O$ such that:

[6] Here and in the following, $\mathrm{Aut}(\mathbb{A}/S)$ is the group of those automorphisms of $\mathbb{A}$ that fix every element of S. So "$\mathrm{Aut}(\mathbb{A}/S)$-equivariant" is synonymous with "S-supported".

(1) $a_i < b_i$ for all $i \in I$;
(2) $b_i < a_j$ for all $i < j \in I$;
(3) for any binary R in σ_0 (which we assumed to be irreflexive) and $i, j \in I$:

$$\begin{aligned} R(a_i, b_j) &\iff R(a_i, a_j) \\ &\qquad \Updownarrow \text{ as } a, b \in O \\ R(b_i, a_j) &\iff R(b_i, b_j). \end{aligned}$$

Remark 8.4. Conditions (1) and (2) specify a total order on the $2|I|$ atoms in a duo. Moreover, thanks to irreflexivity, each a_i is unrelated to its counterpart b_i. Further, given any $J \subseteq I$, the combined tuple $a|^{I \setminus J} b|^J$ satisfies the same relations as a (and b), so it lies in O. In particular, taking $J = \{i\}$, there is an automorphism π_i that sends a_i to b_i and fixes all the other elements of a, b and S.

For the special case of the ordered atoms (Ex. 2.4 & 7.5) the following construction was studied in [Boj+24, p. 11], and it already appeared earlier in [Gra97, p. 125] under the name of "polytabloids".

Definition 8.5. Given a O-duo $a \parallel b$, the corresponding *O-cog* is the vector

$$a \between b = \sum_{J \subseteq I} (-1)^{|J|} (a|^{I \setminus J} b|^J)$$

in $\mathrm{Lin}_{\mathsf{F}}\, O$. The linear span of all O-cogs is denoted by $\mathrm{Cog}_{\mathsf{F}}\, O$.

Note that, for a fixed S-ordered orbit O, all O-duos (hence all O-cogs) are in the same $\mathrm{Aut}(\mathbb{A}/S)$-equivariant orbit. As a result, $\mathrm{Cog}_{\mathsf{F}}\, O$ is an $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspace of $\mathrm{Lin}_{\mathsf{F}}\, O$ and it is generated by any single cog.

Claim 8.6. $\mathrm{Cog}_{\mathsf{F}}\, O \subseteq \mathrm{Ker}_{\mathsf{F}}\, O$.

Proof. Let $O \subseteq \mathbb{A}^I$, let $a \parallel b$ be an O-duo, and take any $i \in I$. The subsets of I come in pairs of J and $J \cup \{i\}$, where $J \subseteq I \setminus \{i\}$. The two tuples $a|^{I \setminus J} b|^J$ and $a|^{I \setminus (J \cup \{i\})} b|^{J \cup \{i\}}$ are present in $a \between b$ with the opposite coefficients, and they differ only on the i-th entry. Therefore they cancel out under $(-)|^{-i}$; hence $(a \between b)|^{-i} = 0$. □

So far, we have not shown that O-duos and O-cogs even exist in general. Let us rectify this by showing that they can be found in all but one $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspaces of $\mathrm{Lin}_{\mathsf{F}}\, O$.

8.3 Finding cogs

We begin with a technical lemma, which combines the free amalgamation in $\mathscr{C}_0$ and the generic order of $\mathbb{A}$.

Lemma 8.7. *Let $X, Y, \{z\} \subseteq \mathbb{A}$ be disjoint and finite. Then there is an automorphism $\tau \in \mathrm{Aut}(\mathbb{A})$ such that*

(1) τ fixes every $x \in X$;
(2) $\tau(z)$ is unrelated to all $y \in Y$ and to z;
(3) $\tau(z) > z$.

Proof. Form the free amalgam of structures in $\mathscr{C}_0$:

$$\begin{tikzcd} & X \cup Y \cup \{z\} \arrow[dr, dotted, "\subseteq"] & \\ X \arrow[ur] \arrow[dr] & & X \cup Y \cup \{z, z'\} \\ & X \cup \{z\} \arrow[ur, dotted, "x \in X \mapsto x,\ z \mapsto z'"'] & \end{tikzcd}$$

so that no element of $Y \cup \{z\}$ is related to z'. To make $X \cup Y \cup \{z, z'\}$ an σ-structure, inherit the order on $X \cup Y \cup \{z\}$ from $\mathbb{A}$, and declare that $z < z'$, as well as $z' < a$ whenever $z < a$ for $a \in X \cup Y$. This makes the above a diagram of embeddings in the presence of the order. By homogeneity, $X \cup Y \cup \{z, z'\}$ embeds into $\mathbb{A}$ via some f which is the identity on $X \cup Y \cup \{z\}$; again by homogeneity, we may extend the embedding

$$x \in X \mapsto x, \quad z \mapsto f(z')$$

to some automorphism τ that satisfies (1), (2), and (3). □

Lemma 8.8. *Suppose $a \parallel b$ is an O-duo, where $O \subseteq \mathbb{A}^I$ is S-ordered. Given $z \in S$,*

- *write $S' = S \setminus \{z\}$;*
- *let $j \notin I$ be such that $az \in \mathbb{A}^{I \cup \{j\}}$ — thus $O' = \mathrm{Aut}(\mathbb{A})_{(S')} \cdot az \subseteq \mathbb{A}^{I \cup \{j\}}$ — is S'-ordered;*
- *let $X \subseteq \mathbb{A}$ be any finite set containing $\{a_i, b_i \mid i \in I\} \cup S'$ but not z.*

Denote $z' = \tau(z)$, where $\tau \in \mathrm{Aut}(\mathbb{A}/X)$ is afforded by Lemma 8.7 (with an arbitrary Y). Then $az \parallel bz'$ is an O'-duo.

Proof. First, notice that $bz' \in O'$ and that we have the required order relations with z and z'. The remaining condition (3) of Definition 8.3, for any binary R in σ_0, splits into the following cases (and their symmetric counterparts):

- $R(a_i, b_{i'}) \iff R(a_i, a_{i'})$ since $a \parallel b$ is an O-duo;
- $R(a_i, z') \iff R(a_i, z)$ since τ is an automorphism that fixes all a_i;
- $R(a_i, z) \iff R(b_i, z)$ since $a, b \in O$ and $z \in S$;
- $R(z, z')$ and $R(z, z)$ are both false: z' is unrelated to z by Lemma 8.7, and R is irreflexive. □

Starting from an empty duo, we may apply Lemma 8.8 inductively and obtain:

Lemma 8.9. *Let $O \subseteq \mathbb{A}^I$ be an S-ordered orbit. Then any $a \in O$ can be extended to an O-duo $a \parallel b$.*

As some $a \in O$ always exists, it follows that Claim 8.6 was not vacuous: we now know $\mathrm{Cog}_{\mathsf{F}}\, O$, and hence $\mathrm{Ker}_{\mathsf{F}}\, O$, is nontrivial — but barely so. Indeed, as the result below shows, $\mathrm{Cog}_{\mathsf{F}}\, O$ admits no nontrivial $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspaces.

Theorem 8.10. *Any nontrivial $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspace $V \subseteq \mathrm{Lin}_{\mathsf{F}}\, O$ contains $\mathrm{Cog}_{\mathsf{F}}\, O$.*

Proof. Pick any $v \in V$ and $a \in O$ with $v(a) \neq 0$; it is enough to show that V contains $a \between b$ for some $b \in O$. Define:

$$S' = S \cup \{c_i \mid v(c) \neq 0, i \in I\} \setminus \{a_i \mid i \in I\} \supseteq S$$

and put $O' = \mathrm{Aut}(\mathbb{A})_{(S')} \cdot a \subseteq O$ — then O' is S'-ordered. By Lemma 8.9, we can find $b \in O'$ such that $a \parallel b$ is an O'-duo and *a fortiori* an O-duo. Take the automorphisms $\pi_{i_1}, \ldots, \pi_{i_d}$ from Remark 8.4, where $i_1, \ldots, i_d$ enumerate I. Now define $v^{(0)} = v$ and

$$v^{(k)} = v^{(k-1)} - \pi_{i_k} v^{(k-1)}.$$

Then each $v^{(k)}$ is in V. By induction on k, we have:

$$v^{(k)} = \sum_{c \in C_k} \sum_{J \subseteq \{i_1, \ldots, i_k\}} (-1)^{|J|} v(c) \left(\prod_{j \in J} \pi_j c \right),$$

where

$$C_k = \{c \mid v(c) \neq 0, \{c_{i_1}, \dots, c_{i_k}, \dots, c_{i_d}\} \supseteq \{a_{i_1}, \dots, a_{i_k}\}\}.$$

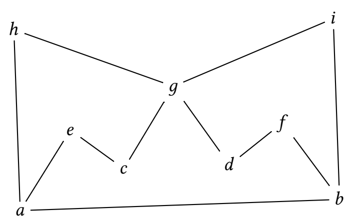
But $\{c_{i_1}, \dots, c_{i_d}\} \supseteq \{a_{i_1}, \dots, a_{i_d}\}$ means that $c = a$, so $C_d = \{a\}$ and $\frac{1}{v(a)} v^{(d)} = a \between b$. ◻

Corollary 8.11. $\mathrm{Cog}_\mathsf{F}\, O$ *has length* 1.

In light of Claims 8.1, 8.2 and 8.6, for the finite length property it is enough to prove that $\mathrm{Ker}_\mathsf{F}\, O \subseteq \mathrm{Cog}_\mathsf{F}\, O$. In words, we need to show that every balanced vector in $\mathrm{Lin}_\mathsf{F}\, O$ is a linear combination of O-cogs. Before we show a proof of this (stated as Theorem 8.12), let us illustrate its key ideas on an example.

8.4 Spanning by cogs: an extended example

Let $\mathbb{A}_0$ be the universal triangle-free (undirected) graph, and $\mathbb{A}$ its totally ordered version. Consider nine atoms $\{a, \dots, i\}$, ordered by $<$ alphabetically, with the edge relation as shown here:

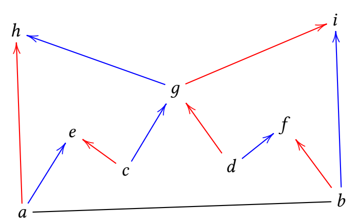


This graph is drawn so that the total order of the atoms corresponds to the vertical order.

Putting $S = \emptyset$ and $|I| = 2$, let O be the ordered orbit of pairs of atoms which are adjacent. Consider the following vector:

$$v = ah - ae + ce - cg + dg - df + bf - bi + gi - gh \in \mathrm{Lin}_\mathsf{F}\, O.$$

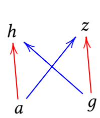
This can be graphically presented as the following graph:



where edges with coefficient $+1$ are marked as red, and with -1 as blue. The arrows on the chosen edges remind us that the pairs in O are ordered, but this is mere decoration: the definition of O means that all arrows must point upwards.

Note that v is balanced. Graphically, this means that every atom has as many outgoing red edges as outgoing blue edges, and as many incoming red edges as incoming blue edges.

It is easy to draw O-cogs in this way. Assuming some additional atom $z > h$ which is adjacent to a and g but not to h, the O-cog $ah \between gz$ can be drawn as:



We would like to present v as a sum of such O-cogs. Some additional atoms must be used for that, as no four atoms among the original nine form an O-duo. It would be very convenient to introduce a single new atom z to form all the O-duos that we will use. (If $\mathbb{A}$ is the ordered atoms, we can simply take z to be the largest atom present in v: cf. the proof of [Boj+24, Cl. 4.7].) We can naïvely require z to be:

- larger than every atom in v,
- adjacent to every atom that is a source of a directed edge in v (equivalently: that occurs as the first component of a pair in v), and
- not adjacent to any atom that is a target of a directed edge in v (equivalently: that occurs as the second component of a pair in v).

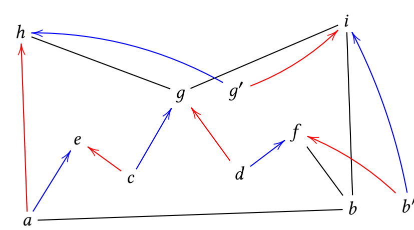
However, such a z does not exist in the triangle-free graph $\mathbb{A}$. There are two problems:

- The atom g occurs both as the first and as the second component in pairs present in v. Our specification of whether z is adjacent to g is therefore inconsistent.
- Atoms a and b both occur as first components in v, and they are adjacent in $\mathbb{A}$. As a result, an atom z as prescribed would create a triangle abz in $\mathbb{A}$, which is forbidden.

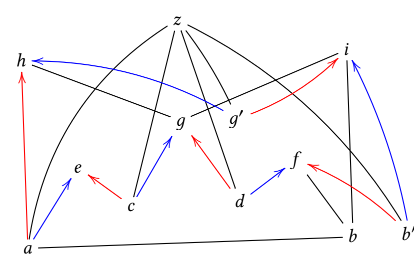
We resolve such *conflicts* by considering auxiliary atoms $g' > g$ and $b' > b$, with just enough edges to make $gh \parallel g'i$ and $bf \parallel b'i$ valid O-duos. Specifically, we postulate edges $E(g', h)$, $E(g', i)$, $E(b', f)$, $E(b', i)$ and no more. Such atoms exist by Lemma 8.7. We then define:

$$v' = v + (gh \between g'i) - (bf \between b'i)$$

which can be drawn as:



Now an atom z as postulated above does not create any triangles:



and it is easy to calculate:

$$v' = (ah \between g'z) - (ae \between cz) - (cg \between dz) + (b'f \between dz) - (b'i \between g'z)$$

which presents $v = v' - (gh \between g'i) + (bf \between b'i)$ as a linear combination of O-cogs.

8.5 All those equivariant subspaces

Inspired by the example, we shall now assert:

Theorem 8.12. *Every $v \in \mathrm{Ker}_{\mathsf{E}}\, O$ can be written as*

$$v = \sum_{a \| b} \lambda_{a\|b} \cdot a \between b$$

with $\lambda_{a\|b} \in \mathsf{E}$, where E is a subspace of some F^n.

Remark 8.13. We must tread carefully here, as E is not a field. (E.g. E is $\{(\kappa, -\kappa) \mid \kappa \in \mathsf{F}\}$ in F^2. Then $\mathrm{Lin}_{\mathsf{E}}\, O$ is an equivariant subspace of $\mathrm{Lin}_{\mathsf{F}^2}\, O$; the latter is more traditionally viewed as $\mathrm{Lin}_{\mathsf{F}}\, O \oplus \mathrm{Lin}_{\mathsf{F}}\, O$.) Given $\lambda \in \mathsf{E}$ and $a \in O$, we need to understand

$$\lambda \cdot a$$

as a formal expression for an element of $\mathrm{Lin}_{\mathsf{E}}\, O$, instead of the result of a scalar multiplication. Accordingly, we need to redefine $\mathrm{Cog}_{\mathsf{E}}\, O$ to be spanned by formal expressions $\lambda \cdot (a \between b)$ for $\lambda \in \mathsf{E}$. We still have $\mathrm{Lin}_{\mathsf{F}^n}\, O \supseteq \mathrm{Lin}_{\mathsf{E}}\, O \supseteq \mathrm{Ker}_{\mathsf{E}}\, O = \mathrm{Cog}_{\mathsf{E}}\, O$ as equivariant spaces over F.

Leaving the proof to Appendix E, we have — as discussed at the end of Section 8.3 — established Theorem 7.3, by taking $\mathsf{E} = \mathsf{F}$. But by allowing E to be finite-dimensional vector spaces, we can now describe all equivariant subspaces.

Theorem 8.14. *Fix $d \in \{1, 2, \dots\}$. Then there exists a finite family of equivariant linear maps of the form*

$$\restriction_i \colon \mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^d \to \mathrm{Lin}_{\mathsf{F}^{n_i}}\, O_i,$$

where O_i is an ordered orbit of $\mathbb{A}^{d_i}$ with $d > d_i$, such that every equivariant subspace $W \subseteq \mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^d$ is equal to

$$\{v \in \mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^d \mid \forall i, \forall a \in O_i : v\restriction_i(a) \in \mathsf{E}_i\} \tag{3}$$

with the finite-dimensional subspaces $\mathsf{E}_i \subseteq \mathsf{F}^{n_i}$ given by

$$\mathsf{E}_i = \{w\restriction_i(b) \mid w \in W, b \in O_i\}.$$

Example 8.15. Let $\mathbb{A}$ be the ordered atoms. For $d = 1$ there are two maps, $\restriction_1 \colon \mathrm{Lin}_{\mathsf{F}}\, \mathbb{A} \to \mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}$ and $\restriction_2 \colon \mathrm{Lin}_{\mathsf{F}}\, \mathbb{A} \to \mathsf{F}$, given by

$$v\restriction_1(a) = v(a), \qquad v\restriction_2() = \sum_{-} v(-).$$

For an equivariant vector space $V \subseteq \mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}$, the subspace $V\restriction_1(\mathbb{A}) \subseteq \mathsf{F}$ is either $\{0\}$ or the entire F.

- In the first case, V must be the zero space.
- Otherwise, we similarly distinguish two cases for $V\restriction_2() \subseteq \mathsf{F}$:
  – if it is the entire F, then V must be the full space $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}$ by Theorem 8.14;
  – if it is $\{0\}$, then V must be the zero-sum space spanned by $\{a - b \mid a, b \in \mathbb{A}\}$ again using Theorem 8.14.

So $\mathrm{Lin}\, \mathbb{A}$ has the same structure as the space over the equality atoms in Example 4.2.

For $d = 2$, there will be a map $\restriction_1 \colon \mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^2 \to \mathrm{Lin}_{\mathsf{F}^5}\, \mathbb{A}$. By considering subspaces of F^5, when the field is infinite we can exhibit infinitely many equivariant subspaces of $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^2$. Note that $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^2$ has finite length (equal to $2^1 + 2^2 + 2^2$, by below) nonetheless. ▲

Corollary 8.16. *The length of $\mathrm{Lin}_{\mathsf{F}} O$, where $O \subseteq \mathbb{A}^d$ is an ordered orbit, is precisely 2^d.*

Variants of Theorem 8.14 have been shown over the equality and the ordered atoms: see [Gra97, Cor. 3.17], [HLT17, Thm. 15], [HR22, Thm. 3.4], and [Gho22, Thm. 19], [GHL22, Sec. 6]. As described in [GL24, Sec. 9], such results allow us to decide the solvability of orbit-finite systems of linear equations. We briefly repeat the explanation. Checking whether the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ admits a solution amounts to checking whether $\mathbf{b}$ is spanned by the columns of $\mathbf{A}$. In the orbit-finite setting, we assume that the rows and columns are indexed by $\mathbb{A}^n$, that every column has finitely many non-zero entries, and that $j \mapsto \mathbf{A}_{-,j}$ is equivariant. That is, we ask whether $\mathbf{b} \in \mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^d$ is in the span of $\mathbf{A}_{-,j}$. By Theorem 8.14, it suffices to check whether $\mathbf{b}\restriction_i(O_i) \subseteq \mathbf{A}_{-,j_1}\restriction_i(O_i) + \cdots + \mathbf{A}_{-,j_n}\restriction_i(O_i)$, in the finite dimensional space F^{n_i}, having chosen orbit representatives $j_1, \dots, j_n \in \mathbb{A}^d$.

9 Conclusions

With Theorems 5.4 and 7.3 and their corollaries, we have extended the finite length property far beyond equality and ordered atoms, the two examples considered in [Boj+24]. We have used two approaches, one based on cogs, and one without them.

In the cog-based approach of Sections 7–8.5, we cover the generically ordered Fraïssé limit of any free amalgamation class over a finite relational vocabulary of relational symbols of arity at most 2. We do not know how to drop the arity restriction from our proofs, so it is interesting that recently, using a different approach, Evans proved [Eva25a, Prop. 3.10] that $\mathrm{Lin}\, \mathbb{A}^2$ has finite length for similar structures with vocabularies of arbitrarily high arity. We do not know how to combine these results. Our main cog-based result is Theorem 8.12, which allows us to describe all equivariant subspaces in any orbit-finite-dimensional vector space, and to deduce the finite length property with a tight bound. However, as we saw in Section 6, these structures can give rise to ill-behaved models of computation: over the Rado and Henson graphs, orbit-finitely spanned sets do not admit orbit-finitely spanned function spaces, and weighted register automata are not closed under reversal.

The cog-less approach to oligomorphically approximated structures in Section 5 is almost complementary. The finite length property is proved there only over fields of characteristic 0; indeed some structures covered in that setting, notably vector atoms (Ex. 2.5), do not have the finite length property over finite fields [Boj+24, Sec. 4.4]. But the proof here is quick and elegant, and covers two important subclasses of oligomorphic structures. The first is ω-stable structures, which admit orbit-finitely spanned dual spaces according to [Prz24, Thm. 3.7]. The second is weakly Lie coordinatisable structures, which are thoroughly studied in [CH03] and provide a rich variety of examples.

We finish the paper by reiterating some open questions:

- [CE91, Q. 2] Does every oligomorphic structure have the ascending chain property?
- [Eva25a, Q. 1.4] Does every structure homogeneous over a finite relational vocabulary have the finite length property?

We also ask a new one: Does every oligomorphic structure have the finite length property over fields of characteristic 0?

We do not even know the answer to these questions for some concrete and well-studied Fraïssé limits, notably the universal partial order or the countable atomless Boolean algebra.

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A Proof of Theorem 4.4

Proof of Theorem 4.4. The equivalence of (3) and (1) is a classical result in module theory.[7] Let us recall the proof. For (3) $\Rightarrow$ (1), we build an orbit-finite spanning set for the subspace greedily, by adding orbit after orbit of vectors. Thanks to (3), this must stabilise. For (1) $\Rightarrow$ (3), we use (1) to get an orbit-finite spanning set for the union of the chain from (3), and then we observe that orbits from this set can only be added finitely often in the chain.

Let us now prove the "furthermore" part. Take some orbit-finitely spanned vector space V. By Definition 3.1, the spanning set can be obtained from some equivariant subset of $\mathbb{A}^d$, by quotienting under an equivariant equivalence relation. This gives us a surjective equivariant linear map to V from an equivariant subspace of $\mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}^d$, thus establishing the "furthermore" part.

From the "furthermore" part we also get equivalence of (2) and (3). The implication (3) $\Rightarrow$ (2) is immediate, and the converse implication follows from the "furthermore" part, since the lack of infinite ascending chains is preserved by taking equivariant subspaces and images under equivariant linear maps. □

B Proof of Theorem 5.2(3)

The symplectic vector spaces are used as a basic building block in [KLM89]. Described as a classical geometry, it can indeed be found in classical textbooks such as [Art57, pp. 114–121] and [Asc00, pp. 75–83], though it is often not clear whether we assume finite dimension or characteristic zero for the results we require. As symplectic vector spaces are an important and interesting structure to understand, we provide a self-contained exposition here, taking a straight-line path to the oligomorphic approximation of the Rado graph in Theorem B.14.

B.1 Symplectic vector spaces

Let k denote a finite field.

Definition B.1. A *symplectic vector space* is an k-vector space $\mathbb{W}$ equipped with a bilinear form $\omega : \mathbb{W} \times \mathbb{W} \to \mathsf{k}$ that is

(1) alternating: $\omega(v, v) = 0$ for all v; and
(2) non-degenerate: if $\omega(v, w) = 0$ for all w then $v = 0$.

Example B.2. Let $\mathbb{W}_n$ be the k-vector space with basis

$$e_1, \ldots, e_n, f_1, \ldots, f_n.$$

Define ω by bilinearly extending

$$\omega(e_i, f_i) = 1 = -\omega(f_i, e_i), \quad \omega(-, *) = 0 \text{ elsewhere}; \tag{‡}$$

one may straightforwardly check that ω is alternating and non-degenerate. Moreover, noticing that $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$, we obtain a countable-dimensional symplectic vector space $\mathbb{W}_\infty = \bigcup_n \mathbb{W}_n$. ▲

We will refer to vectors satisfying (‡) as a *symplectic subbasis*. As the name suggests, such vectors must be linearly independent: if $v = \sum_i \lambda_i e_i + \mu_i f_i = 0$, then $\lambda_i = \omega(v, f_i) = 0$ and $\mu_i = \omega(e_i, v) = 0$ for each i. Such bases behave very much like the usual bases.

[7] An orbit-finitely spanned vector space over an oligomorphic structure $\mathbb{A}$ as in Definition 4.1 amounts to a finitely generated module over the group ring $\mathrm{Lin}_{\mathsf{F}}\,\mathrm{Aut}(\mathbb{A})$, with the additional continuity condition — see [Hod93, Lemma 4.1.5] — that every element is finitely supported. This condition is inherited by subspaces. An equivariant subspace is the same as a submodule.

Proposition B.3. *Assume that $\mathbb{W}$ is a symplectic vector space that is at most countable. Then any finite symplectic subbasis*

$$e_1, \ldots, e_n, f_1, \ldots, f_n$$

can be extended to a symplectic subbasis that spans the whole $\mathbb{W}$ (i.e. a symplectic basis).

Proof. Suppose that $e_1, \ldots, e_n, f_1, \ldots, f_n$ does not already span $\mathbb{W}$; take v to be a witness (that is least according to some fixed enumeration of $\mathbb{W}$ in the case it is infinite). Put

$$e_{n+1} = v - \sum_{i=1}^{n} \omega(e_i, v) f_i + \sum_{i=1}^{n} \omega(f_i, v) e_i$$

so that $\omega(e_i, e_{n+1}) = 0 = \omega(f_i, e_{n+1})$. This cannot be the zero vector lest we contradict the choice of v. By the non-degeneracy of ω, there is — rescaling if necessary — some w such that $\omega(e_{n+1}, w) = 1$. Now define

$$f_{n+1} = w - \sum_{i=1}^{n} \omega(e_i, w) f_i + \sum_{i=1}^{n} \omega(f_i, w) e_i$$

in a similar manner, making $e_1, \ldots, e_n, e_{n+1}, f_1, \ldots, f_n, f_{n+1}$ a symplectic subbasis that spans v. We go through every element of $\mathbb{W}$ by continuing this way. □

We may even forget some pairs.

Proposition B.4. *Now assume $\mathbb{W}$ is a finite-dimensional symplectic vector space. Let*

$$e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+k},$$
$$f_1, \ldots, f_n$$

be linearly independent vectors satisfying (‡). Then we can find the missing $f_{n+1}, \ldots, f_{n+k}$ to form a symplectic subbasis.

Proof. Suppose we have found $f_{n+1}, \ldots, f_{n+i}$ already such that

$$e_1, \ldots, e_n, e_{n+1}, \ldots e_{n+i}, e_{n+i+1}, e_{n+i+2}, \ldots, e_{n+k},$$
$$f_1, \ldots, f_n, f_{n+1}, \ldots, f_{n+i}$$

satisfy (‡). Notice these vectors are linearly independent: in a linear combination that sums to 0, the coefficients of $e_1, f_1, \ldots, e_{n+i}, f_{n+i}$ must be zero, and we assumed that $e_{n+i+1}, \ldots, e_{n+k}$ are linearly independent. By extending these to a basis B of $\mathbb{W}$, we may define a linear function

$$\psi : \mathbb{W} \to \mathsf{k}$$

which sends e_{n+i+1} to 1 but every other $b \in B$ to 0. Now apply Proposition B.3 to obtain a symplectic basis $e'_1, f'_1, \ldots, e'_m, f'_m$ of $\mathbb{W}$, and put

$$f_{n+i+1} = \sum_{j=1}^{m} \psi(e'_j) f'_j - \psi(f'_j) e'_j;$$

then $\omega(-, f_{n+i+1})$ agrees with ψ on this symplectic basis, so by linearity they must be the same function. In particular

$$\omega(e_{n+i+1}, f_{n+i+1}) = \psi(e_{n+i+1}) = 1,$$

whereas $\psi(e_1), \ldots, \psi(e_{n+k}), \psi(f_1), \ldots, \psi(f_{n+i})$ are all 0 Thus we have (‡) as required. □

Given two symplectic vector spaces $\mathbb{W}$ and $\mathbb{W}'$, we call a function α between $X \subseteq \mathbb{W}$ and $X' \subseteq \mathbb{W}'$ *isometric* if $\omega(\alpha(x_1), \alpha(x_2)) = \omega(x_1, x_2)$ for all $x_1, x_2 \in X$. We can make an easy observation:

Lemma B.5. *Let $\{e_i, f_i \mid i \in I\} \subseteq \mathbb{W}$, $\{e'_j, f'_j \mid j \in J\} \subseteq \mathbb{W}'$ be two symplectic bases and let $\alpha : I \to J$ be a bijection. Then*

$$e_i \mapsto e'_{\alpha(i)}, f_i \mapsto f'_{\alpha(i)}$$

defines an isometric linear bijection $\langle e_i, f_i \rangle \to \langle e'_j, f'_j \rangle$.

It then follows from Proposition B.3 that, up to isometric linear bijections, $\mathbb{W}_0, \mathbb{W}_1, \mathbb{W}_2, \dots, \mathbb{W}_\infty$ are all the countable symplectic vector spaces. Whilst we may deduce that $\mathbb{W}_\infty$ is oligomorphic by appealing to Ryll-Nardzewski, we will opt for another approach that establishes smooth approximation and gives an explicit orbit count.

Proposition B.6 (Witt Extension)**.** *Any isometric linear injection $\alpha : \langle X \rangle \subseteq \mathbb{W}_n \to \mathbb{W}_n$ can be extended to an isometric linear bijection of $\mathbb{W}_n$ to itself (i.e. automorphism), and in turn to one of $\mathbb{W}_\infty$ to itself.*

Proof. To begin with, find a basis $x_1, \dots, x_k$ for the subspace $W = \{w \in \langle X \rangle \mid \forall x \in X : \omega(w, x) = 0\}$ and extend it to a basis $x_1, \dots, x_k, x_{k+1}, \dots, x_d$ for $\langle X \rangle$. Notice that

$$U = \langle x_{k+1}, \dots, x_d \rangle$$

must be a symplectic subspace: as it intersects with W trivially, given any non-zero vector $u \in U$ we must have $0 \neq \omega(u, w + u') = \omega(u, u')$ for some $w \in W$ and $u' \in U$. Hence use Proposition B.3 to find a symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$ for U. Observe that

$$\begin{gathered} e_1, \dots, e_n, x_1, \dots, x_k, \\ f_1, \dots, f_n \end{gathered}$$

form a basis for $\langle X \rangle$ and satisfy (‡). On the other hand,

$$\begin{gathered} \alpha(e_1), \dots, \alpha(e_n), \alpha(x_1), \dots, \alpha(x_k), \\ \alpha(f_1), \dots, \alpha(f_n) \end{gathered}$$

form a basis for $\alpha(\langle X \rangle)$ and also satisfy (‡). Therefore apply Proposition B.4 twice to find the missing $y_1, \dots, y_k$ and $y'_1, \dots, y'_k$ and form two symplectic subbases — call them $\mathcal{B}$ and $\mathcal{B}'$. They are of the same size.

Now, by using Proposition B.3, extend $\mathcal{B}$ and $\mathcal{B}'$ to symplectic subbases C and C' that span $\mathbb{W}_n$. These must both have size $2n$, so by Lemma B.5 we obtain an isometric linear automorphism $\beta : \mathbb{W}_n \to \mathbb{W}_n$ extending α.

To finish, notice that $C, e_{n+1}, \dots, f_{n+1}, \dots$ form a symplectic subbasis spanning $\mathbb{W}_\infty$, as do $C', e_{n+1}, \dots, f_{n+1}, \dots$. We obtain from Lemma B.5 another time an isometric linear automorphism $\gamma : \mathbb{W}_\infty \to \mathbb{W}_\infty$ extending β that is the identity almost everywhere. ◻

Proposition B.7. *$\mathbb{W}^k_\infty$ has precisely $\sum_{d=0}^{k} \left[{k \atop d} \right]_q \cdot q^{\binom{d}{2}}$ orbits under* $\mathrm{Aut}(\mathbb{W}_\infty)$*, where $q = |\mathrm{k}|$ and*

$$\begin{bmatrix} k \\ d \end{bmatrix}_q = \frac{(q^k - 1)(q^{k-1} - 1) \cdots (q^{k-d+1} - 1)}{(q^d - 1)(q^{d-1} - 1) \cdots (q^1 - 1)}$$

is the q-binomial coefficient.

Remark B.8. To anticipate the next subsection, we note a similarity with the Rado graph: in $\mathbb{G}^k$ there are

$$\sum_{d=0}^{k} \left\{ {k \atop d} \right\} \cdot 2^{\binom{d}{2}} \geq \sum_{d=0}^{k} \begin{bmatrix} k \\ d \end{bmatrix}_2 \cdot 2^{\binom{d}{2}}$$

orbits, where the curly brackets count the number equivalence relations with d classes on a k-element set.

Proof. To each $v_\bullet \in \mathbb{W}^k_\infty$ we associate a *type*, which comprises the following data:

(1) pivot indices $I \subseteq \{1, \dots, k\}$ containing every i such that v_i is not spanned by $v_1, \dots, v_{i-1}$ — so we inductively ensure that
$$\{v_{i'} \mid i' \in I, i' \leq i\}$$
is a basis for $\langle v_1, \dots, v_i \rangle$;
(2) for each $j \notin I$, an assignment $\Lambda_j : \{i \in I \mid i < j\} \to \mathrm{k}$ such that $v_j = \sum_{i \in I, i < j} \Lambda_j(i) v_i$;
(3) a map $\Omega : \binom{I}{2} \to \mathrm{k}$ defined by $\Omega(\{i' < i\}) = \omega(v_{i'}, v_i)$.

If $\pi : \mathbb{W}_\infty \to \mathbb{W}_\infty$ is an isometric linear bijection, then $v_\bullet = (v_1, \dots, v_k)$ and $\pi \cdot v_\bullet = (\pi(v_1), \dots, \pi(v_k))$ evidently share the same type. Conversely, if $w_\bullet$ has the type of $v_\bullet$, then

$$\begin{gathered} \alpha : \langle v_i \mid i \in I \rangle \to \langle w_i \mid i \in I \rangle \subseteq \mathbb{W}_n \\ v_i \mapsto w_i \end{gathered}$$

gives an isometric linear injection for some large enough n. Observe that α must send $v_j \mapsto w_j$ for $j \notin I$ too, and that it may be extended to an isometric linear automorphism π of $\mathbb{W}_\infty$ by Proposition B.6. Furthermore we can find some $v_\bullet$ that realises any given type $(I, \{\Lambda_j\}_j, \Omega)$: it suffices to put

$$v_i = e_i + \sum_{i' \in I, i' < i} \Omega(i', i) f_{i'}$$

for $i \in I$ and $v_j = \sum_{i \in I, i < j} \Lambda_j(i) v_i$ for $j \notin I$. Therefore the number of types is precisely the number of orbits in $\mathbb{W}^k_\infty$.

Finally, we do some combinatorics. Fix $0 \leq d \leq k$ and count the number of types with $|I| = d$. There are $q^{\binom{d}{2}}$ choices for Ω and say $\#_{k,d}$ choices for the Λ_j's; the two can be chosen independently. In total, this gives

$$\sum_{d=0}^{k} q^{\binom{d}{2}} \cdot \#_{k,d}$$

types for vectors in $\mathbb{W}^d_\infty$. So focus on $\#_{k,d}$, the number of *linear types* — i.e. $(I, \{\Lambda_j\}_j)$, ignoring Ω — in $\mathbb{W}^k_\infty$. (Incidentally $\sum_{d=0}^{k} \#_{k,d}$ is the number of orbits in $\mathbb{W}^k_\infty$ or, more generally, any countable-dimensional k-vector space under linear automorphisms.) On the small values we easily check that

$$\begin{gathered} \#_{0,0} = 1 = \left[{0 \atop 0} \right]_q, \\ \#_{1,0} = 1 = \left[{1 \atop 0} \right]_q, \qquad \#_{1,1} = 1 = \left[{1 \atop 1} \right]_q. \end{gathered}$$

Given a linear type in $\mathbb{W}^k_\infty$ with $|I| = d$, we either have $1 \in I$ or $I \subseteq \{2, \dots, k\}$. In the first case, the linear type is specified by one of the $\#_{k-1,d-1}$ linear types in $\mathbb{W}^{k-1}_\infty$ together with how v_1 is involved in the span of the $(k-1)-(d-1)$ non-pivot vectors. In the second case, the linear type is simply one of the $\#_{k-1,d}$ linear types in $\mathbb{W}^{k-1}_\infty$. Thus

$$\begin{aligned} \#_{k,d} &= q^{k-d} \cdot \#_{k-1,d-1} + \#_{k-1,d} \\ &= q^{k-d} \cdot \begin{bmatrix} k-1 \\ d-1 \end{bmatrix}_q + \begin{bmatrix} k-1 \\ d \end{bmatrix}_q = \begin{bmatrix} k \\ d \end{bmatrix}_q. \end{aligned}$$ ◻

Theorem B.9. *The symplectic vector space $\mathbb{W}_\infty$ is smoothly approximated by $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$.*

As smooth approximation is a special case of oligomorphic approximation, by Theorem 5.4 we establish the finite length property for yet another class of structures:

Corollary B.10. *The symplectic $\Bbbk$-vector space $\mathbb{W}_\infty$ has the finite length property over any field of characteristic 0.*

B.2 Symplectic graphs

Let us finish the proof of Theorem 5.2(3). Here, let $\Bbbk$ be the two-element field.

Definition B.11. For $n = 0, 1, 2, \ldots$, the *symplectic graph* $\widetilde{\mathbb{W}}_n$ has vertices $\mathbb{W}_n$ and edges

$$E(v_1, v_2) \iff \omega(v_1, v_2) = 1.$$

This is indeed an undirected graph: as ω is alternating, we have $\omega(v_1, v_2) = -\omega(v_2, v_1) = \omega(v_2, v_1)$ over $\Bbbk_2$.

Proposition B.12. $\mathrm{Aut}(\widetilde{\mathbb{W}}_n) = \mathrm{Aut}(\mathbb{W}_n)$.

Proof. Clearly any isometric linear bijection of $\mathbb{W}_n$ is a graph automorphism of $\widetilde{\mathbb{W}}_n$. Conversely, any $f \in \widetilde{\mathbb{W}}_n$ is evidently isometric. To show that f is linear, take $\lambda_1, \lambda_2 \in \Bbbk$ and $v_1, v_2 \in \mathbb{W}$. We calculate:

$$\begin{aligned}
&\omega\Big(f(\sum_i \lambda_i v_i) - \sum_i \lambda_i f(v_i), f(w)\Big) \\
&= \omega\Big(f(\sum_i \lambda_i v_i), f(w)\Big) - \sum_i \lambda_i \omega\big(f(v_i), f(w)\big) \\
&= \omega\Big(\sum_i \lambda_i v_i, w\Big) - \sum_i \lambda_i \omega(v_i, w) \\
&= \omega(0, w) = 0
\end{aligned}$$

for all $f(w) \in f(\mathbb{W}_n) = \mathbb{W}_n$; since ω is non-degenerate, we conclude that $f(\sum_i \lambda_i v_i) = \sum_i \lambda_i f(v_i)$. ◻

So the number of orbits in $\widetilde{\mathbb{W}}_n^k$ is precisely equal to the number of orbits in $\mathbb{W}_n^k$ — in particular, it is bounded above by $\sum_{d=0}^{k} \left[\begin{smallmatrix} k \\ d \end{smallmatrix} \right]_2 \cdot 2^{\binom{d}{2}}$ independently of n by Proposition B.7.[8] It remains to show $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$ embeds all finite graphs:

Proposition B.13 ([GR01, Thm. 8.11.2])**.** *Every graph on at most $2n$ vertices embeds into $\widetilde{\mathbb{W}}_n$.*

Proof (without graph-theoretic jargon). Let G be a graph on at most $2n$ vertices. The conclusion is trivial when $n = 0$. Also, if G contains no edges, we can choose any $2n$ of the 2^n vectors in $\langle e_1, \ldots, e_n \rangle \subseteq \widetilde{\mathbb{W}}_n$.

So suppose $n \geq 1$ and G has an edge $E(s, t)$. Let $G_{s,t}$ be the graph on vertices $G \setminus \{s, t\}$ with edges which we will specify later. By induction, some embedding $f : G_{s,t} \to \widetilde{\mathbb{W}}_{n-1}$ exists. Define $f' : G \to \widetilde{\mathbb{W}}_n$ by

$$\begin{aligned}
x \in G_{s,t} &\mapsto f(x) - [\![E(x, s)]\!] f_n + [\![E(x, t)]\!] e_n \\
s &\mapsto e_n \\
t &\mapsto f_n
\end{aligned}$$

where $[\![\phi]\!]$ is 1 if ϕ holds and 0 otherwise. On one hand, we have

$$\begin{aligned}
\omega(f'(x), f'(s)) &= [\![E(x, s)]\!], \\
\omega(f'(x), f'(t)) &= [\![E(x, t)]\!]
\end{aligned}$$

as desired. On the other,

$$\begin{aligned}
\omega(f'(x_1), f'(x_2)) = [\![E(x_1, x_2)]\!] &+ [\![E(x_1, s)]\!][\![E(x_2, t)]\!] \\
&+ [\![E(x_1, t)]\!][\![E(x_2, s)]\!]
\end{aligned}$$

tells us how we should define the edge relation in $G_{s,t}$ for f' to be an embedding of graphs. ◻

As the Rado graph is homogeneous, we may consider the symplectic graphs $\widetilde{\mathbb{W}}_i$ as its induced subgraphs.

Theorem B.14. *The Rado graph is oligomorphically approximated by $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$.*

Corollary B.15. *The Rado graph has the finite length property over any field of characteristic 0.*

A different finite field $\Bbbk$ can encode different structures. For example, the 3-element field allows us to prove the finite length property of the Fraïssé limit of *oriented graphs*, where the edge relation is irreflexive and anti-symmetric.[9]

C Example 6.3 continued

In this part of the appendix, we prove that the space V from Example 6.3 is not orbit-finitely spanned.

Suppose towards a contradiction that V has an orbit-finite spanning set. Without loss of generality we can assume that this spanning set uses only vectors of the form f_S, i.e. the spanning set is $\{f_S \mid S \in \mathcal{S}\}$ for some orbit-finite family $\mathcal{S}$ of finite subsets of $\mathbb{A}$. Take any finite set $T \subseteq \mathbb{A}$ that is strictly bigger in terms of size than any set in $\mathcal{S}$. Then we can write

$$f_T = \lambda_1 \cdot f_{S_1} + \cdots + \lambda_n \cdot f_{S_n}$$

for $S_i \in \mathcal{S}$. We will prove that all coefficients λ_i are zero, and hence f_T is zero. This cannot be true, since one can find an atom that is a common neighbour of all atoms in T.

To prove that $\lambda_i = 0$, we use induction on n. In the induction proof we assume without loss of generality that the sets $S_1, \ldots, S_n$ are sorted by size in a non-decreasing way. Suppose that we have proved that all coefficients λ_j are zero for $j < i$. By the assumption on non-decreasing sizes, each of the sets $S_{i+1}, \ldots, S_n$ and T contains at least one atom that is not in S_i. Therefore, we can find some $a \in \mathbb{A}$ that is adjacent to all atoms in S_i, but which is non-adjacent to at least one atom in each of the sets $S_{i+1}, \ldots, S_n$ and T. As $\lambda_1, \ldots, \lambda_{i-1} = 0$, we see that

$$0 = f_T(a) = \sum_j \lambda_j \cdot f_{S_j}(a) = 0 + \lambda_i + 0$$

and hence $\lambda_i = 0$.

[8] This is the kth term in the OEIS sequence A028361.

[9] We are again grateful to Ehud Hrushovski for pointing this out.

D Proof of Corollary 7.7

We rely on the intricate classification result of Lachlan and Woodrow [LW80] which says that any countable homogeneous graph, or its complement, is isomorphic to one of:

(a) the Rado graph;
(b) the K_n-free Henson graph, where $n \geq 3$;
(c) m disjoint copies of the complete graph K_n, where at least one of m and n is infinite.

Since a graph $\mathbb{A}$ and its complement share the same automorphism group, when studying the length of $\mathrm{Lin}_{\mathsf{F}}\,\mathbb{A}^d$ we only need to examine the structures of kinds (a), (b) and (c). For the first two kinds, the finite length property follows directly from Corollary 7.4. The structures of kind (c) admit a first-order interpretation in the equality atoms, and hence they inherit the finite length property from them — we can only have fewer orbit-finite sets and orbit-finitely spanned vector spaces.

(In more detail, a *first-order interpretation* $\mathbb{B}$ in $\mathbb{A}$ consists of an orbit-finite set over $\mathbb{A}$ equipped with equivariant relations. Observe that, given $\pi \in \mathrm{Aut}(\mathbb{A})$, the induced action $b \mapsto \pi(b)$ on $\mathbb{B}$ is an automorphism of $\mathbb{B}$. So an $\mathrm{Aut}(\mathbb{B})$-orbit of tuples in $\mathbb{B}^d$ is equivariant under $\mathrm{Aut}(\mathbb{A})$, i.e., a union of $\mathrm{Aut}(\mathbb{A})$-orbits; since $\mathbb{A}$ is oligomorphic, so is $\mathbb{B}$. Similarly, consider a chain of $\mathrm{Aut}(\mathbb{B})$-equivariant subspaces in $\mathrm{Lin}_{\mathsf{F}}\,\mathbb{B}^d$. Then these spaces are necessarily $\mathrm{Aut}(\mathbb{A})$-equivariant, so the chain has bounded length because $\mathbb{B}^d$ is an orbit-finite set over $\mathbb{A}$.)

E Proof of Theorem 8.12

Recall the setting of Section 7 and the notations of Section 8. Consider a subspace $\mathsf{E} \subseteq \mathsf{F}^n$, and heed the warning of 8.13: remember that a cog is of the form $\lambda \cdot a \between b$ now. Finally, fix an S-ordered orbit $O \subseteq \mathbb{A}^I$.

E.1 Subvectors, locations, conflicts

We begin by introducing some additional terminology and notation. First, let us make explicit a view we have tacitly taken: with O as a standard basis, a vector $v \in \mathrm{Lin}_{\mathsf{E}}\,O$ is just a finite set of pairs in $\mathsf{E} \times O$. A *subvector* of v is a subset of these pairs. We write $[v] \subseteq O$ for the set of tuples which are present in v. For a finite subset $T \subseteq O$, we write $\sqrt{T} \subseteq \mathbb{A}$ for the set of atoms present anywhere in T.

For any $i \in I$ and $a \in \mathbb{A}$ (which is equal to b_i for some $b \in O$), we write

$$O^{i:a} = \{c \in O \mid c_i = a\};$$

this is an $\mathrm{Aut}(\mathbb{A})_{(Sa)}$-orbit (containing b), and its projection $O^{i:a}|^{-i}$ is Sa-ordered. For a vector $v \in \mathrm{Lin}_{\mathsf{E}}\,O$, by

$$v^{i:a} \in \mathrm{Lin}_{\mathsf{E}}\,O^{i:a}$$

we mean the subvector of v consisting of all pairs in $\mathsf{E} \times O^{i:a}$.

Lemma E.1. *Let $v \in \mathrm{Lin}_{\mathsf{E}}\,O$ be balanced. Then any projected subvector $v^{i:a}|^{-i} \in \mathrm{Lin}_{\mathsf{E}}\,O^{i:a}|^{-i}$ is also balanced.*

Proof. Let $j \in I \setminus \{i\}$. By assumption we have

$$0 = v|^{-j} = \sum_a v^{i:a}|^{-j}.$$

This sum is finite: it runs over those atoms a that occur as the i-th entries in $[v]$. By looking at i-th entries, we see that each $v^{i:a}|^{-j}$ must be the zero vector. Hence so is $v^{i:a}|^{-j}|^{-i} = v^{i:a}|^{-i}|^{-j}$, which shows that $v^{i:a}|^{-i}$ is balanced. ◻

For a finite subset $T \subseteq O$, a *location* in T is a pair $(i, a) \in I \times \mathbb{A}$ such that $a = c_i$ for some $c \in T$. Note that for any fixed $i, j \in I$, for all $c \in O$ the atoms c_i and c_j are related the same way in $\mathbb{A}_0$ (i.e. with respect to equality and binary relations in σ_0). We say that two locations (i, a) and (j, b) in T are in:

- an *equational conflict*, if $i \neq j$ but $a = b$, and
- a *relational conflict*, if a and b are related in $\mathbb{A}_0$ but not in the same way as c_i and c_j for $c \in O$.

(A situation where $a \neq b$ are not related by any relation in σ_0 at all does not constitute a conflict, even if c_i and c_j are related.) Recalling that a and b are related if they are equal, an equational conflict is a special case of a relational one. A location in a vector v means a location in the set $[v]$.

The prototypical examples of vectors which are free from any conflicts are cogs (or any subvectors of cogs). Note that the locations in a cog $a^+ \between a^-$ are exactly those in $\{a^+, a^-\}$, and these have no conflicts if $a^+ \parallel a^-$ is a duo.

In the following proof we will often manipulate many duos and cogs at once, so we will benefit from a concise notation for them. An O-duo will be denoted by a single letter as $a^\pm$; its constituent parts will then be denoted by a^+ and a^-, so that $a^\pm = a^+ \parallel a^-$. Sets of duos will be denoted with capital letters such as $A^\pm$, and sometimes we will slightly abuse this notation and write $A^\pm$ to mean $\bigcup_{a^\pm \in A^\pm} \{a^+, a^-\}$, A^+ to mean $\bigcup_{a^\pm \in A^\pm} \{a^+\}$, and A^+a to mean $\bigcup_{a^\pm \in A^\pm} \{a^+a\}$.

E.2 Conflict resolution lemmas

The following sister lemmas, relying on free amalgamation as distilled in Lemma 8.7, show how to merge conflict-free subsets of O in a way that avoids introducing new conflicts. This will be useful in Sections E.3 and E.4.

Lemma E.2. *Let $K, V_0 \subseteq V$ be finite subsets of O such that both $V_0 \cup K$ and V are free from equational conflicts. Then there exists a $\pi \in \mathrm{Aut}(\mathbb{A})$ that, while fixing all atoms in S and in $\sqrt{V_0}$, makes $V \cup \pi(K)$ free from equational conflicts.*

Proof. Fix V_0, V and induct on the number of equationally conflicting locations in $V \cup K$. Take any such locations (i, a) and (j, a), where $i \neq j$; without loss of generality (i, a) is a location in K and (j, a) a location in V.

Suppose that (i', a) were a location in V_0. Since there are no equational conflicts in $V_0 \cup K$ or in V, we see that $i = i'$ and $i' = j$, which is impossible. So $a \notin \sqrt{V_0}$. Also $a \notin S$, as O is S-ordered. Put:

$$X = S \cup \sqrt{K \cup V} \setminus \{a\},$$

and note that X contains $S \cup \sqrt{V_0}$. Use Lemma 8.7 (putting $z = a$ and $Y = \emptyset$) to obtain an automorphism $\pi \in \mathrm{Aut}(\mathbb{A}/X)$ such that $\pi(a) \notin X \cup \{a\}$.

In the set $V \cup \pi(K)$, the conflicting location (i, a) disappears and no new equational conflicts are created, so the number of equationally conflicting locations drops compared to $V \cup K$. Because $V_0 \cup \pi(K)$ is still conflict-free, the inductive hypothesis gives us some $\pi' \in \mathrm{Aut}(\mathbb{A})_{(S \cup \sqrt{V_0})}$ such that $V \cup \pi'\pi(K)$ is free from equational conflicts. ◻

Lemma E.3. *Let $K, V_0 \subseteq V$ be finite subsets of O such that both $V_0 \cup K$ and V are free from relational conflicts. Then there exists a $\pi \in \mathrm{Aut}(\mathbb{A})$ that, while fixing all atoms in S and in $\sqrt{V_0}$, makes $V \cup \pi(K)$ free from relational conflicts.*

Proof. By Theorem E.2 we may assume that $V \cup K$ is free from *equational* conflicts. As before, fix V_0, V and proceed by induction on the number of relationally conflicting locations in $V \cup K$.

Let (i, a) and (j, b) be in a relational conflict; without loss of generality (i, a) is a location in K and (j, b) in V. This is not an equational conflict, so $a \neq b$ (but possibly $i = j$).

Since there are no conflicts in $V_0 \cup K$ or in V, we see that (i, a) is not a location in V and (j, b) is not a location in $V_0 \cup K$ — i.e. since there are no equational conflicts, that $a \notin \sqrt{V}$ and $b \notin \sqrt{V_0 \cup K}$. Also, $a, b \notin S$. Let Y consist of all the atoms b that are in a relational conflict with (i, a) in $V \cup K$; we have just shown that Y does not contain a and is disjoint with $S \cup \sqrt{V_0 \cup K}$. Put:

$$X = S \cup \sqrt{K \cup V} \setminus (Y \cup \{a\}).$$

Then $X, Y, \{a\}$ are pairwise disjoint, and X contains $S \cup \sqrt{V_0}$. It also contains all atoms in $\sqrt{K}$ except a. Using Lemma 8.7, find some $\pi \in \mathrm{Aut}(\mathbb{A}/X)$ such that $\pi(a) \notin X \cup Y \cup \{a\}$ and $\pi(a)$ is not related to any atom in Y. In $V \cup \pi(K)$ the conflicting location (i, a) disappears and no new conflicts are created, so the conclusion follows from the inductive hypothesis. □

E.3 Conflict-free vectors

Claim E.4. *If $v \in \mathrm{Ker}_{\mathsf{E}}\, O$ is free from conflicts, then it can be written as a sum of O-cogs:*

$$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ \between a^-$$

with $\lambda_{a^\pm} \in \mathsf{E}$, where moreover $[v] \cup A^\pm$ is free from conflicts.

Proof. We proceed by induction on the dimension $|I|$, noting that when $I = \emptyset$ we just have $v = \lambda \cdot () = \lambda \cdot (\between)$.

So suppose I is non-empty; let $j \in I$ be the greatest element. Group the terms in v by their greatest atom so that $v = v^1 + v^2 + \cdots + v^k$. We now induct on k. If $k \leq 1$, we are done: as $v|^{-j} = 0$ we must have $v = 0$ (and $k = 0$), so the empty sum will do. Otherwise

$$v = v^{j:a} + v^{j:b} + v'$$

for some $a \neq b \in \mathbb{A}$. By Lemma E.1, $v^{j:a}|^{-j}$ is balanced, and it is conflict-free, as every location in it is also a location in v. By the outer inductive hypothesis, we get

$$v^{j:a} = (v^{j:a}|^{-j})a = \sum_{a^\pm \in A^\pm} (\lambda_{a^\pm} \cdot a^+ \between a^-)a$$

where $[v^{j:a}|^{-j}] \cup A^\pm$ is free from conflicts, which immediately implies that $[v^{j:a}] \cup A^\pm a$ is free from conflicts as well. Note that if a $\pi \in \mathrm{Aut}(\mathbb{A}/S)$ fixes every atom in $v^{j:a}$ — in particular, a — then

$$v^{j:a} = \pi(v^{j:a}) = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot \pi a^+ \between \pi a^-,$$

so by Lemma E.3 (putting $K = A^\pm a$, $V_0 = [v^{j:a}]$, and $V = [v]$) we may assume without loss of generality that $[v] \cup A^\pm a$ is free from conflicts.

Similarly, we can write

$$v^{j:b} = \sum_{b^\pm \in B^\pm} (\mu_{b^\pm} \cdot b^+ \between b^-)b$$

and apply Lemma E.3 again (putting $K = B^\pm b$, $V_0 = [v^{j:b}]$, and $V = [v] \cup A^\pm a$) to conclude that

$$[v] \cup A^\pm a \cup B^\pm b$$

is free from conflicts.

We now invent a new element z, on which we impose the following relations with $S \cup \sqrt{A^\pm a \cup B^\pm b} \subseteq \mathbb{A}$:

(1) $a, b < z$, and $z < s$ iff $a, b < s$ for any $s \in S$;
(2) for any unary relation $P \in \sigma_0$:

$$P(z) \;:\Longleftrightarrow\; P(a) \;\Longleftrightarrow\; P(b);$$

(3) for any binary relation $R \in \sigma_0$ and $s \in S$, $a^\pm \in A^\pm$, $b^\pm \in B^\pm$, $i \in I \setminus \{d\}$:
   - $R(z, s) \;:\Longleftrightarrow\; R(a, s) \;\Longleftrightarrow\; R(b, s)$,
   - $R(z, a_i^+) \;:\Longleftrightarrow\; R(a, a_i^+)$,
   - $R(z, b_i^+) \;:\Longleftrightarrow\; R(b, b_i^+)$,
   - $R(z, a_i^-) \;:\Longleftrightarrow\; R(a, a_i^-)$,
   - $R(z, b_i^-) \;:\Longleftrightarrow\; R(b, b_i^-)$,
   - $R(z, a)$ and $R(z, b)$ are both false,
   - and symmetrically for $R(-, z)$.

   These are consistent as there are no equational conflicts. (For instance, if $a_i^+ = b_{i'}^-$ then $i = i'$, and $R(a, a_i^+) \iff R(b, b_i^-)$ holds since a^+a and b^-b are both in O.)

To see that the σ-structure $S \cup \sqrt{A^\pm a \cup B^\pm b} \cup \{z\}$ still embeds into $\mathbb{A}$, suppose towards a contradiction that it contains a forbidden σ_0-substructure F. Then F must contain z. Since any two elements in F are necessarily related, we must have $a, b \notin F$. Similarly, whenever F contains an atom x_i for any $x \in A^\pm \cup B^\pm$, it does not contain y_i for any other $y \in A^\pm \cup B^\pm$. It follows that, fixing any $a^\pm \in A^\pm$,

$$s \mapsto s, \quad x_i \mapsto a_i^+, \quad z \mapsto a$$

defines an injective function $\phi : F \longrightarrow \mathbb{A}_0$, which is furthermore an embedding (we only need to check this for pairs!) because $A^\pm \cup B^\pm$ is conflict-free and any $x_i, y_{i'}$ for $i \neq i'$ are related. This is a contradiction. We may therefore assume that $z \in \mathbb{A}$.

It is now routine to check that $a^+a \parallel a^-z$ and $b^+b \parallel b^-z$ are O-duos for all $a^\pm \in A^\pm, b^\pm \in B^\pm$, and that

$$A^+a \cup A^-z \cup B^+b \cup B^-z$$

is free from conflicts. From Lemma E.3 we may assume that

$$[v] \cup A^+a \cup A^-z \cup B^+b \cup B^-z$$

is also free from conflicts. (Alternatively, we could have explicitly ensured this when defining z.) Then the vector:

$$\begin{aligned} v'' &= v - \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+a \between a^-z - \sum_{b^\pm \in B^\pm} \mu_{b^\pm} \cdot b^+b \between b^-z \\ &= v^{j:a}|^{-j}z + v^{j:b}|^{-j}z + v', \end{aligned}$$

when grouped into subvectors by the largest atom in each term, has at least one fewer component than v. By the inner inductive hypothesis, we may write

$$v'' = \sum_{c^\pm \in C^\pm} \kappa_{c^\pm} \cdot c^+ \between c^-$$

with $[v''] \cup C^\pm$ conflict-free, and one last application of Lemma E.3 allows us to conclude that

$$[v] \cup A^+a \cup A^-z \cup B^+b \cup B^-z \cup C^\pm$$

is conflict-free as well. We conclude that

$$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+a \between a^-z + \sum_{b^\pm \in B^\pm} \mu_{b^\pm} \cdot b^+b \between b^-z + \sum_{c^\pm \in C^\pm} \kappa_{c^\pm} \cdot c^+ \between c^-$$

is a decomposition of v into a sum of O-cogs as required. □

E.4 Vectors without equational conflicts

Claim E.5. *If $v \in \mathrm{Ker}_{\mathsf{E}}\, O$ has no equational conflicts, then it can be written as a sum of O-cogs:*

$$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ \between a^-$$

with $\lambda_{a^\pm} \in \mathsf{E}$, where moreover $[v] \cup A^\pm$ has no equational conflicts.

Proof. We proceed again by induction, first on $|I|$ then on the number of relational conflicts in v. The outer base case $I = \emptyset$ is trivial — we have $v = \lambda \cdot (\between)$, and no conflicts arise — and the inner base case is just Claim E.4.

Suppose that a location (i, a) is part of a relational conflict in v. Since every location in $v^{i:a}|^{-i}$ is also a location in v, we know that $v^{i:a}|^{-i}$ has no equational conflicts, and by Lemma E.1 it is balanced. By the outer inductive hypothesis, we get:

$$v^{i:a} = (v^{i:a}|^{-i})a = \sum_{a^\pm \in A^\pm} (\lambda_{a^\pm} \cdot a^+ \between a^-)a$$

where $[v^{i:a}|^{-i}] \cup A^\pm$ has no equational conflicts, which immediately implies that $[v^{i:a}] \cup A^\pm a$ is free from equational conflicts as well. By Lemma E.2 (putting $K = A^\pm a$, $V_0 = [v^{i:a}]$, and $V = [v]$) we may assume that $[v] \cup A^\pm a$ has no equational conflicts.

Take any location (j, b) which is a in a relational conflict with (i, a) in v. Then $b \notin \sqrt{A^\pm a}$. To see this, note that $b = a$ would imply $j = i$ (since v has no equational conflicts), but that would mean no conflict. On the other hand, if $b = a^+_k$ for some $a^+ \in A^\pm$ and $k \in I \setminus \{i\}$ (the case of a^- is identical) then $j = k$, but this is not a conflict either, since $a^+a \in O$ and $a = (a^+a)_i$. Also, $b \notin S$.

Let Y consist of all the atoms b that are in a relational conflict with (i, a) in V. As $a \notin S$ and $a \notin \sqrt{A^\pm}$, we have shown that

$$X = S \cup \sqrt{[v] \cup A^\pm} \setminus (Y \cup \{a\})$$

contains $S \cup \sqrt{A^\pm}$ and that $X, Y, \{a\}$ are pairwise disjoint. Use Lemma 8.7 (putting $z = a$) to find $\tau \in \mathrm{Aut}(\mathbb{A}/X)$ such that $\tau(a) \notin X \cup Y \cup \{a\}$, is greater than a, and is not related to any of $Y \cup \{a\}$. Denote $a' = \tau(a)$. Then, for any $a^\pm \in A^\pm$, it follows by Lemma 8.8 that $a^+a \parallel a^-a'$ is an O-duo. Moreover, $[v] \cup A^+a \cup A^-a'$ has no equational conflicts, and the vector

$$v' = v - \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+a \between a^-a' = v - v^{i:a} + v^{i:a'}$$

has strictly fewer relationally conflicting locations than v, as the location (i, a) disappears from it. The inner inductive hypothesis tells us that we may write

$$v' = \sum_{b^\pm \in B^\pm} \mu_{B^\pm} \cdot b^+ \between b^-$$

with $[v'] \cup B^\pm$ free from equational conflicts.

Since $[v'] \subseteq [v] \cup A^+a \cup A^-a'$, Lemma E.2 allows us to assume that

$$[v] \cup A^+a \cup A^-a' \cup B^\pm$$

is also free from equational conflicts. We conclude that

$$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+a \between a^-a' + \sum_{b^\pm \in B^\pm} \mu_{b^\pm} \cdot b^+ \between b^-$$

as required. □

E.5 Arbitrary vectors

We restate and prove Theorem 8.12:

Theorem E.6. *Any $v \in \mathrm{Ker}_{\mathsf{E}}\, O$ can be written as*

$$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ \between a^-$$

with $\lambda_{a^\pm} \in \mathsf{E}$.

Proof. This is similar to the proof of Claim E.5, but simpler. We proceed again by induction, first on $|I|$ then on the number of equational conflicts in v. The outer base case $I = \emptyset$ is trivial as before, and and the inner base case is Claim E.5.

Suppose that a location (i, a) is part of an equational conflict in v. By the outer inductive hypothesis, we get:

$$v^{i:a} = (v^{i:a}|^{-i})a = \sum_{a^\pm \in A^\pm} (\lambda_{a^\pm} \cdot a^+ \between a^-)a.$$

Then neither S nor $\sqrt{A^\pm}$ contains a, so

$$X = S \cup \sqrt{[v] \cup A^\pm} \setminus \{a\}$$

contains $S \cup \sqrt{A^\pm}$. Using Lemma 8.7 (putting $z = a$ and $Y = \emptyset$), find $\pi \in \mathrm{Aut}(\mathbb{A}/X)$ such that $\pi(a)$ is not in X, is greater than a, and is otherwise unrelated to a. Denote $a' = \tau(a)$. Then, for any $a^\pm \in A^\pm$, it follows by Lemma 8.8 that $a^+a \parallel a^-a'$ is an O-duo. Moreover, the vector

$$v' = v - \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+a \between a^-a' = v - v^{i:a} + v^{i:a'}$$

has strictly fewer equationally conflicting locations than v, as the location (i, a) disappears from it. It follows from the inner inductive hypothesis that we can write

$$v' = \sum_{b^\pm \in B^\pm} \mu_{b^\pm} \cdot b^+ \between b^-,$$

which gives a decomposition of $v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+a \between a^-a'$ as required. □

F Proofs missing in Section 8.5

F.1 Proof of Theorem 8.14

For notational simplicity, we will prove the result for a single ordered orbit $O \subseteq \mathbb{A}^I$. The general multi-orbited case is very similar, because we will be projecting onto a single orbit anyway.

Consider the $2^{|I|}$ projected S-ordered orbits $O|^J$ for $J \subseteq I$. Suppose that

$$f : O|^J \to O|^{J'}$$

is an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant bijection. Take any $a \in O|^J$, and enumerate its entries as $a_1 < \cdots < a_{|J|}$. Similarly, enumerate the entries of $f(a)$ as $b_1 < \cdots < b_{|J'|}$. Then $\{a_1, \ldots, a_{|J|}\} = \{b_1, \ldots, b_{|J'|}\}$ because $\mathbb{A}$ has no algebraicity; since the orbits are ordered, we must have $|J| = |J'|$ and $a_1 = b_1, \ldots, a_{|J|} = b_{|J'|}$. That is, f must be the obvious function that reindexes a J-tuple to a J'-tuple – hence we will write $a^{/J'}$ instead of $f(a)$, leaving f implicit.

Now, let $Q_1 = O|^{J_1}, \ldots, Q_t = O|^{J_t}$ be distinct projected S-ordered orbits up to $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant bijections, enumerated in such a way that $|J_1| \geq |J_2| \geq \cdots \geq |J_t|$. (In particular, $J_1 = I$ and $J_t = \emptyset$.)

For $i = 1, \ldots, t$, let $\mathscr{J}_i$ consist of all sets J such that $O|^J$ is in $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant bijection with Q_i. Assemble all $|\mathscr{J}_i|$ projections into a single map

$$(-)\upharpoonright_i : \mathrm{Lin}_{\mathsf{F}}\, O \to \mathrm{Lin}_{\mathsf{F}^{\mathscr{J}_i}}\, Q_i.$$

To be more precise $v\upharpoonright_i(a)$ is, for $a \in Q_i$, the $\mathscr{J}_i$-tuple whose J-th entry is $v|^J(a^{/J}) \in \mathsf{F}$. It is straightforward to check that $(-)\upharpoonright_i$ is $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant and linear.

Lemma F.1. *For $i = 1, \ldots, t, t+1$,*

$$\begin{aligned} &W \cap \ker(\upharpoonright_i) \cap \ker(\upharpoonright_{i+1}) \cap \cdots \cap \ker(\upharpoonright_t) \\ &\subseteq \widehat{W} \cap \ker(\upharpoonright_i) \cap \ker(\upharpoonright_{i+1}) \cap \cdots \cap \ker(\upharpoonright_t). \end{aligned}$$

This gives Theorem 8.14 when $i = t+1$.

Proof. We proceed by induction on i. The base case $i = 0$ is trivial, since $\ker(\upharpoonright_1) = \{0\}$. Indeed, $\mathscr{J}_1 = \{I\}$, so $v\upharpoonright_1$ is the identity map.

To prove the containment for some $i > 0$, we allow $v\upharpoonright_i$ to be non-zero. But $v\upharpoonright_i$ satisfies the next best property:

Claim F.2. *The image of*

$$\widehat{W} \cap \ker(\upharpoonright_{i+1}) \cap \cdots \cap \ker(\upharpoonright_t)$$

under $\upharpoonright_i$ is contained in $\mathrm{Ker}_{W\upharpoonright_i(Q_i)}\, Q_i$.

Proof. Take any $v \in \widehat{W}$ such that $v\upharpoonright_{i'} = 0$ for all $i' > i$. That $v\upharpoonright_i \in \mathrm{Lin}_{W\upharpoonright_i(Q_i)}\, Q_i$ is clear from the definition of $\widehat{W}$. Recall that $Q_i = O|^{J_i}$. For every $j \in J_i$, we need to prove that $v\upharpoonright_i|^{-j} = 0$. In more elementary terms, given any $a \in O|^{J_i \setminus \{j\}}$, we need the J-th entry of $v\upharpoonright_i|^{-j}(a)$ to be 0, for every $J \in \mathscr{J}_i$.

So take any $J \in \mathscr{J}_i$. The unique ordered bijection between J_i and J restricts to one between $J_i \setminus \{j\}$ and $J \setminus \{j'\}$, for some $j' \in J$. Denote $J' = J \setminus \{j'\}$. Then J' belongs to some $\mathscr{J}_{i'}$ with $i' > i$, so $v\upharpoonright_{i'} = 0$. Now calculate (with $a \in O|^{J_i \setminus \{j\}}$, $b \in Q_i$, $c \in O|^J$ and $d \in O$):

$$\begin{aligned} (v\upharpoonright_i|^{-j}(a))_J &= \sum_{b|^{-j}=a} (v\upharpoonright_i(b))_J = \sum_{b|^{-j}=a} v|^J(b^{/J}) = \sum_{c|^{-j'}=a^{/J'}} v|^J(c) \\ &= \sum_{d|^{J'}=a^{/J'}} v(d) \ = v|^{J'}(a^{/J'}) = (v\upharpoonright_{i'}(a))_{J'} = 0. \end{aligned}$$ ◻

In light of Remark 8.13, from Theorem 8.12 we get:

$$\mathrm{Ker}_{W\upharpoonright_i(Q_i)}\, Q_i \subseteq \mathrm{Cog}_{W\upharpoonright_i(Q_i)}\, Q_i.$$

Furthermore:

Claim F.3. *$\mathrm{Cog}_{W\upharpoonright_i(Q_i)}\, Q_i$ is contained in the image of*

$$W \cap \ker(\upharpoonright_{i+1}) \cap \cdots \cap \ker(\upharpoonright_t)$$

under $\upharpoonright_i$.

Proof. Consider any Q_i-cog with a coefficient $\lambda \in W\upharpoonright_i(Q_i)$, and let $w \in W$ and $a \in Q_i$ be such that $\lambda = w\upharpoonright_i(a)$. Let S' consist of S together with every atom appearing in w but not in a. We generalise the construction used in the proof of Theorem 8.10.

Apply Lemma 8.9 and Remark 8.4 to get automorphisms π_j for $j \in J_i$ such that $a \parallel \prod_{j \in J_i} \pi_j a$ is an Q_i-duo, where each π_j fixes S' and all $a_{j'}$ and $\pi_{j'}(a_{j'})$ for $j' \neq j$. Since all Q_i-duos are in the same orbit, it is enough to show that the cog corresponding to this particular duo, with the coefficient λ, belongs to the $\upharpoonright_i$-image as in the statement of the claim.

Put:

$$w' = \prod_{j \in J_i} (\mathrm{id} - \pi_j) w \in W.$$

For any $1 \leq i' \leq t$, noting that as no more atoms can appear in $w\upharpoonright_{i'}$ than in w, we have

$$w'\upharpoonright_{i'} = \prod_{j \in J_i} (\mathrm{id} - \pi_j) w\upharpoonright_{i'} = \sum_{c \in C_i} \sum_{J' \subseteq J_i} (-1)^{|J'|} w\upharpoonright_{i'}(c) \cdot \left(\prod_{j \in J'} \pi_j c \right)$$

where

$$C_i = \{c \in Q_{i'} : \{c_j \mid j \in J_{i'}\} \supseteq \{a_j \mid j \in J_i\}\}.$$

(The formula used in the proof of Theorem 8.10 is a special case of this for $i' = 1$ so that $J_{i'} = I$ and $Q_{i'} = O$.) Now, if $i' > i$ then C_i is empty, and so $w'\upharpoonright_{i+1} = \cdots = w'\upharpoonright_t = 0$. Moreover, if $i' = i$ then $C_i = \{a\}$ and we obtain the cog from before:

$$w'\upharpoonright_i = \lambda \cdot \left(a \between \prod_{j \in J_i} \pi_j a \right).$$

So w' is a witness for the inclusion from the claim. ◻

This is enough to establish Lemma F.1 for i, assuming it for $i-1$. Indeed, given $v \in \widehat{W} \cap \ker(\upharpoonright_{i+1}) \cap \cdots \cap \ker(\upharpoonright_t)$, by the preceding claims we can find $w \in W \cap \ker(\upharpoonright_{i+1}) \cap \cdots \cap \ker(\upharpoonright_t) \subseteq \widehat{W}$ such that $v\upharpoonright_i = w\upharpoonright_i$. But then $(v - w)\upharpoonright_i = 0$, so $v - w$ lies in $\ker(\upharpoonright_i)$ as well as $\widehat{W} \cap \ker(\upharpoonright_{i+1}) \cap \cdots \cap \ker(\upharpoonright_t)$. It follows from the inductive hypothesis that

$$v - w \in W \cap \ker(\upharpoonright_i) \cap \ker(\upharpoonright_{i+1}) \cap \cdots \cap \ker(\upharpoonright_t),$$

so $v = (v - w) + w$ is in $W \cap \ker(\upharpoonright_{i+1}) \cap \cdots \cap \ker(\upharpoonright_t)$ as well.

This completes the proof of Theorem 8.14. ◻

F.2 Proof of Corollary 8.16

Now we specialise to the case where we have a single ordered orbit $O \subseteq \mathbb{A}^d$. The maps $\{\restriction_i \colon \mathrm{Lin}_{\mathsf{F}}\, O \to \mathrm{Lin}_{\mathsf{F}\mathscr{J}_i}\, Q_i\}_{1 \le i \le t}$ are as before.

For two equivariant subspaces $W, W' \subseteq \mathrm{Lin}_{\mathsf{F}}\, O$, if we have $W\restriction_i(Q_i) = W'\restriction_i(Q_i)$ for all $i = 1, \ldots, t$, then $W = \widehat{W} = \widehat{W'} = W'$ by Lemma F.1. As a consequence:

Proposition F.4. *Let $W_0 \subset W_1 \subset \cdots \subset W_l$ be a chain of $\mathrm{Aut}(\mathbb{A}/S)$-equivariant subspaces of $\mathrm{Lin}_{\mathsf{F}}(O)$. Then $l \le 2^{|I|}$.*

Proof. We obtain t chains:

$$\begin{aligned}
W_0\restriction_1(Q_1) \subseteq W_1\restriction_1(Q_1) \subseteq \cdots \subseteq W_l\restriction_1(Q_1) &\subseteq \mathsf{F}^{\mathscr{J}_1},\\
W_0\restriction_2(Q_2) \subseteq W_1\restriction_2(Q_2) \subseteq \cdots \subseteq W_l\restriction_2(Q_2) &\subseteq \mathsf{F}^{\mathscr{J}_2},\\
&\vdots\\
W_0\restriction_t(Q_t) \subseteq W_1\restriction_t(Q_t) \subseteq \cdots \subseteq W_l\restriction_t(Q_t) &\subseteq \mathsf{F}^{\mathscr{J}_t}.
\end{aligned}$$

At each of the l steps, one of the t containments must be strict. Hence $l \le |\mathscr{J}_1| + |\mathscr{J}_2| + \cdots + |\mathscr{J}_t| = 2^{|I|}$. □

To complement the upper bound from Proposition F.4, we now exhibit a chain of equivariant subspaces whose length is precisely $\sum_{i=1}^{t} |\mathscr{J}_i|$, using the same idea as [Boj+24, Cor. 4.12].

Pick some $a \in O$, and let π_j (for $j \in I$) be the automorphisms from Lemma 8.9. Given $J \in \mathscr{J}_i$, define a vector

$$v_J = \prod_{j \in J} (1 - \pi_j) a$$

in $\mathrm{Lin}_{\mathsf{F}}(O)$. Let $\langle v_J \rangle$ be the linear span of permutations of v_J; this is an equivariant subspace. Given $J' \in \mathscr{J}_{i'}$, we compute:

$$(\langle v_J \rangle \restriction_{i'}(Q_{i'}))_{J'} = \begin{cases} \mathsf{F} & \text{if } J \subseteq J', \\ \{0\} & \text{otherwise.} \end{cases}$$

Enumerating each $\mathscr{J}_i$ in any order as $J_i^1, J_i^2, \ldots, J_i^{|\mathscr{J}_i|}$, we obtain a chain

$$\begin{aligned}
\langle\rangle \subset \langle v_{J_t^1} \rangle \subset \langle v_{J_t^1}, v_{J_t^2} \rangle \subset \cdots \subset \langle v_{J_t^1}, v_{J_t^2}, \ldots, v_{J_t^{|\mathscr{J}_t|}} \rangle \\
\subset \langle v_{J_t^1}, v_{J_t^2}, \ldots, v_{J_t^{|\mathscr{J}_t|}}, v_{J_{t-1}^1} \rangle \subset \cdots
\end{aligned}$$

of length $|\mathscr{J}_t| + |\mathscr{J}_{t-1}| + \cdots + |\mathscr{J}_1| = 2^{|I|}$. We conclude that the upper bound from Proposition F.4 is tight.