More Vector Spaces with Atoms of Finite Lengths

Jingjie Yang
University of Oxford

Mikołaj Bojańczyk
University of Warsaw

Bartek Klin
University of Oxford

***Abstract*—We say an infinite structure is oligomorphic over a field if the following holds for each of its finite powers: in the corresponding free vector space, strict chains of equivariant subspaces are bounded in length. It has been shown that the countable pure set and the dense linear ordering without endpoints have this property. In this paper, we generalise these two results to a) reducts of smoothly approximable structures, provided the field has characteristic zero, and b) generaically ordered expansions of Fraïssé limits with free amalgamation, in languages with at most binary relations. As a special case, we prove the Rado graph is oligomorphic over any field using both methods.**

I. Introduction

II. Rado graph, sans cogs

1) $\mathbb{A}$ is oligomorphic if, for $k = 0, 1, 2, \cdots$, $\mathbb{A}^k$ only has finitely many orbits;
2) $\mathbb{A}$ has the finite length property over $\mathfrak{F}$ if, for $k = 0, 1, 2, \cdots$, $\mathrm{Lin}_{\mathfrak{F}}\, \mathbb{A}^k$ only has finitely long chains.

Of note: with Stirling numbers of the second kind and Gaussian 2-binomial coefficients, the orbit counts are given by

$$\#\mathbb{N}^k = \sum_{d=0}^{k} \left\{ {k \atop d} \right\}$$

$$\#\mathbb{Q}^k = \sum_{d=0}^{k} \left\{ {k \atop d} \right\} d!$$

$$\#\mathbb{G}^k = \sum_{d=0}^{k} \left\{ {k \atop d} \right\} 2^{\binom{d}{2}}$$

$$\#\mathbb{V}_\infty^k = \sum_{d=0}^{k} \begin{bmatrix} k \\ d \end{bmatrix}_2$$

$$\#\mathbb{W}_\infty^k = \sum_{d=0}^{k} \begin{bmatrix} k \\ d \end{bmatrix}_2 2^{\binom{d}{2}}$$

To introduce:

1) *smooth approximation* by *homogeneous substructures* [3] (N.B. 'smooth approximation' from [4, Definition 4] seems to be entirely different)
2) *oligomorphic approximation* of a homogeneous structure by finite substructures with uniformly few orbits (i.e., types) that cover the age of $\mathbb{A}$
3) For the equality and ordered atoms, being supportively $\mathfrak{F}$-oligomorphic follows from being $\mathfrak{F}$-oligomorphic [1, Theorem 4.10]

A. Symplectic vector spaces

Throughout this subsection let $\mathfrak{f}$ denote a finite field.

Definition II.1. A *symplectic vector space* is an $\mathfrak{f}$-vector space $\mathbb{W}$ equipped with a bilinear form $\omega : \mathbb{W} \times \mathbb{W} \to \mathfrak{f}$ that is

1) alternating: $\omega(v, v) = 0$ for all v; and
2) non-degenerate: if $\omega(v, w) = 0$ for all w then $v = 0$.

Example II.2. Let $\mathbb{W}_n$ be the $\mathfrak{f}$-vector space with basis $e_1, \dots, e_n, f_1, \dots, f_n$. Define ω by bilinearly extending

$$\omega(e_i, f_i) = 1 = -\omega(f_i, e_i), \quad \omega(-, *) = 0 \text{ elsewhere;} \quad (\S)$$

one may straightforwardly check that ω is alternating and non-degenerate. Moreover, noticing that $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$, we obtain a countable-dimensional symplectic vector space $\mathbb{W}_\infty = \bigcup_n \mathbb{W}_n$.

We will refer to vectors satisfying (§) as a *symplectic basis*. Note such vectors must be linearly independent: if $v = \sum_i \lambda_i e_i + \mu_i f_i = 0$, then $\lambda_i = \omega(v, f_i) = 0$ and $\mu_i = \omega(e_i, v) = 0$ for each i. Such bases behave very much like the usual bases.

Proposition II.3. *Assume that $\mathbb{W}$ is a symplectic vector space that is at most countable. Then any finite symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$ can be extended to a symplectic basis that spans the whole $\mathbb{W}$.*

Proof. Suppose that $e_1, \dots, e_n, f_1, \dots, f_n$ does not already span $\mathbb{W}$; take v to be a witness (that is least according to some fixed enumeration of $\mathbb{W}$ in the case it is infinite). Put

$$e_{n+1} = v - \sum_{i=1}^{n} \omega(e_i, v) f_i + \sum_{i=1}^{n} \omega(f_i, v) e_i$$

so that $\omega(e_i, e_{n+1}) = 0 = \omega(f_i, e_{n+1})$. This cannot be the zero vector lest we contradict the choice of v. By the non-degeneracy of ω, there is — rescaling if necessary — some w such that $\omega(e_{n+1}, w) = 1$. Now define

$$f_{n+1} = w - \sum_{i=1^n} \omega(e_i, w) f_i + \sum_{i=1}^{n} \omega(f_i, w) e_i$$

in a similar manner, making $e_1, \dots, e_n, e_{n+1}, f_1, \dots, f_n, f_{n+1}$ a symplectic basis that spans v. We go through every element of $\mathbb{W}$ by continuing this way. $\square$

In fact, we will also make use of the "symplectic basis and a half" variant below.

Proposition II.4. *Now assume $\mathbb{W}$ is a finite-dimensional symplectic vector space. Let*

$$\begin{aligned}&e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+k},\\ &f_1, \ldots, f_n\end{aligned}$$

be linearly independent vectors satisfying (§)*. Then we can find the missing $f_{n+1}, \ldots, f_{n+k}$ to complete the symplectic basis.*

Proof. We first need the following notion. Given a subspace $V \subseteq \mathbb{W}$, consider its orthogonal complement

$$V^\perp = \{w \in \mathbb{W} \mid \forall v \in V : \omega(v, w) = 0\}.$$

It is the kernel of the composite linear map

$$\begin{aligned}\mathbb{W} &\to (\mathbb{W} \xrightarrow{\text{lin.}} \mathfrak{f}) \to (V \xrightarrow{\text{lin.}} \mathfrak{f})\\ w &\mapsto \omega(-, w) \mapsto \omega(-, w)|_V.\end{aligned}$$

Note this map is surjective: the first part is injective by non-degeneracy and hence surjective for dimension reasons, and the second part is surjective since we can extend a basis of V to one of $\mathbb{W}$. Therefore

$$\dim V^\perp = \dim \mathbb{W} - \dim V,$$

and in particular $V^{\perp\perp}$ is precisely equal to V.

Now suppose we have found $f_{n+1}, \ldots, f_{n+i}$ already. If e_{n+i+1} were to be spanned by

$$\begin{aligned}&e_1, \ldots, e_n, e_{n+1}, \ldots e_{n+i}, \cancel{e_{n+i+1}}, e_{n+i+2}, \ldots, e_{n+k},\\ &f_1, \ldots, f_n, f_{n+1}, \ldots, f_{n+i},\end{aligned}$$

it would be spanned by $e_{n+i+2}, \ldots, e_{n+k}$ alone because of (§); but this is impossible as we assumed linear independency. So

$$\begin{aligned}e_{n+i+1} &\notin \langle e_1, \ldots, e_{n+i}, e_{n+i+2}, f_1, \ldots, f_{n+i}\rangle\\ &= \langle e_1, \ldots, e_{n+i}, e_{n+i+2}, f_1, \ldots, f_{n+i}\rangle^{\perp\perp},\end{aligned}$$

i.e., some $f_{n+i+1} \in \langle e_1, \ldots, e_{n+i}, e_{n+i+2}, f_1, \ldots, f_{n+i}\rangle^\perp$ satisfies $\omega(e_{n+i+1}, f_{n+i+1}) = 1$. □

Given two symplectic vector spaces $\mathbb{W}$ and $\mathbb{W}'$, we call a function α between $X \subseteq \mathbb{W}$ and $X' \subseteq \mathbb{W}'$ *isometric* if $\omega(\alpha(x_1), \alpha(x_2)) = \omega(x_1, x_2)$ for all $x_1, x_2 \in X$. We can make an easy observation:

Lemma II.5. *Let $\{e_i, f_i \mid i \in I\} \subseteq \mathbb{W}$, $\{e'_j, f'_j \mid j \in J\} \subseteq \mathbb{W}'$ be two symplectic bases and let $\alpha : I \to J$ be a bijection. Then*

$$e_i \mapsto e'_{\alpha(i)}, f_i \mapsto f'_{\alpha(i)}$$

defines an isometric linear isomorphism $\langle e_i, f_i\rangle \to \langle e'_j, f'_j\rangle$.

It then follows from Proposition II.3 that, up to isometric linear isomorphisms, $\mathbb{W}_0, \mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_\infty$ are all the countable symplectic vector spaces. Whilst we may deduce that $\mathbb{W}_\infty$ is oligomorphic by appealing to Ryll-Nardzewski, we will opt for a more direct proof that also establishes smooth approximation.

Proposition II.6 (Witt Extension)**.** *Any isometric linear injection $\alpha : \langle X\rangle \subseteq \mathbb{W}_n \to \mathbb{W}_n$ can be extended to an isometric linear automorphism of $\mathbb{W}_n$ and in turn to one of $\mathbb{W}_\infty$.*

Proof. To begin with, find a basis $x_1, \ldots, x_k$ for the subspace $W = \{w \in \langle X\rangle \mid \forall x \in X : \omega(w, x) = 0\}$ and extend it to a basis $x_1, \ldots, x_k, x_{k+1}, \ldots, x_d$ for $\langle X\rangle$. Notice that

$$U = \langle x_{k+1}, \ldots, x_d\rangle$$

must be a symplectic subspace: as it intersects with W trivially, given any non-zero vector $u \in U$ we must have $0 \neq \omega(u, w + u') = \omega(u, u')$ for some $w \in W$ and $u' \in U$. Hence use Proposition II.3 to find a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ for U. Observe that

$$\begin{aligned}&e_1, \ldots, e_n, x_1, \ldots, x_k,\\ &f_1, \ldots, f_n\end{aligned}$$

form a basis for $\langle X\rangle$ and satisfy (§). On the other hand,

$$\begin{aligned}&\alpha(e_1), \ldots, \alpha(e_n), \alpha(x_1), \ldots, \alpha(x_k),\\ &\alpha(f_1), \ldots, \alpha(f_n)\end{aligned}$$

form a basis for $\alpha(\langle X\rangle)$ and also satisfy (§). Therefore apply Proposition II.4 twice to find the missing $y_1, \ldots, y_k$ and $y'_1, \ldots, y'_k$ to complete the two symplectic bases — call them $\mathcal{B}$ and $\mathcal{B}'$. They are of the same size.

Now, by using Proposition II.3, extend $\mathcal{B}$ and $\mathcal{B}'$ to symplectic bases $\mathcal{C}$ and $\mathcal{C}'$ that span $\mathbb{W}_n$. These must both have size $2n$, so by Lemma II.5 we obtain an isometric linear automorphism $\beta : \mathbb{W}_n \to \mathbb{W}_n$ extending α.

To finish, notice that $\mathcal{C}, e_{n+1}, \ldots, f_{n+1}, \ldots$ as well as $\mathcal{C}', e_{n+1}, \ldots, f_{n+1}, \ldots$ form a symplectic basis spanning $\mathbb{W}_\infty$. We obtain from Lemma II.5 another time an isometric linear automorphism $\gamma : \mathbb{W}_\infty \to \mathbb{W}_\infty$ extending β that is the identity almost everywhere. □

Proposition II.7. *$\mathbb{W}^k_\infty$ has precisely $\sum_{d=0}^k \left[\begin{smallmatrix}k\\d\end{smallmatrix}\right]_q \cdot q^{\binom{d}{2}}$ orbits under isometric linear automorphisms, where $q = |\mathfrak{f}|$ and*

$$\begin{bmatrix}k\\d\end{bmatrix}_q = \frac{(q^k - 1)(q^{k-1} - 1)\cdots(q^{k-d+1} - 1)}{(q^d - 1)(q^{d-1} - 1)\cdots(q^1 - 1)}$$

is the q-binomial coefficient.

Remark. To anticipate the next subsection, we note a similarity with the Rado graph: in $\mathbb{G}^k$ there are $\sum_{d=0}^k \binom{k}{d} \cdot 2^{\binom{d}{2}}$ orbits — we may impose any edge relation on d vertices.

Proof. To each $v_\bullet \in \mathbb{W}^k_\infty$ we associate a *type*, which comprises the following data:

1) pivot indices $I \subseteq \{1, \ldots, k\}$ containing every i such that v_i is not spanned by $v_1, \ldots, v_{i-1}$ — so we inductively ensure that
$$\{v_{i'} \mid i' \in I, i' \leq i\}$$
is a basis for $\langle v_1, \ldots, v_i\rangle$;
2) for each $j \notin I$, an assignment $\Lambda_j : \{i \in I \mid i < j\} \to \mathfrak{f}$ such that $v_j = \sum_{i \in I, i<j} \Lambda_j(i) v_i$;

3) a map $\Omega : \binom{I}{2} \to \mathfrak{f}$ defined by $\Omega(\{i' < i\}) = \omega(v_{i'}, v_i)$. If $\pi : \mathbb{W}_\infty \to \mathbb{W}_\infty$ is an isometric linear automorphism, then $v_\bullet = (v_1, \dots, v_k)$ and $\pi \cdot v_\bullet = (\pi(v_1), \dots, \pi(v_k))$ evidently share the same type. Conversely, if $w_\bullet$ has the type of $v_\bullet$, then

$$\begin{aligned} \alpha : \langle v_i \mid i \in I\rangle &\to \langle w_i \mid i \in I\rangle \subseteq \mathbb{W}_n \\ v_i &\mapsto w_i \end{aligned}$$

gives an isometric linear injection for some large enough n. Observe that α must send $v_j \mapsto w_j$ for $j \notin I$ too, and that it may be extended to an isometric linear automorphism π of $\mathbb{W}_\infty$ by Propsoition II.6. Furthermore we can find some $v_\bullet$ that realises any given type $(I, \{\Lambda_j\}_j, \Omega)$: it suffices to put

$$v_i = e_i + \sum_{i' \in I, i' < i} \Omega(i', i) f_{i'}$$

for $i \in I$ and $v_j = \sum_{i \in I, i<j} \in \Lambda_j(i) v_i$ for $j \notin I$. Therefore the number of types is precisely the number of orbits in $\mathbb{W}_\infty^k$.

Finally, we do some combinatorics. Fix $0 \leq d \leq k$ and count the number of types with $|I| = d$. There are $q^{\binom{d}{2}}$ choices for Ω and say $\#_{k,d}$ choices for the Λ_j's; the two can be chosen independently. In total, this gives

$$\sum_{d=0}^{k} q^{\binom{d}{2}} \cdot \#_{k,d}$$

types for vectors in $\mathbb{W}_\infty^d$. So focus on $\#_{k,d}$, the number of *linear types* — i.e., $(I, \{\Lambda_j\}_j)$, ignoring Ω — in $\mathbb{W}_\infty^k$. (Incidentally $\sum_{d=0}^{k} \#_{k,d}$ is the number of orbits in $\mathbb{W}_\infty^k$ or, more generally, any countable-dimensional $\mathfrak{f}$-vector space under linear automorphisms.) On the small values we easily check that

$$\begin{aligned} &\#_{0,0} = 1 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_q, \\ &\#_{1,0} = 1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]_q, \qquad \#_{1,1} = 1 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]_q. \end{aligned}$$

Given a linear type in $\mathbb{W}_\infty^k$ with $|I| = d$, we either have $1 \in I$ or $I \subseteq \{2, \dots, k\}$. In the first case, the linear type is specified by one of the $\#_{k-1,d-1}$ linear types in $\mathbb{W}_\infty^{k-1}$ together with how v_1 is involved in the span of the $(k-1)-(d-1)$ non-pivot vectors. In the second case, the linear type is simply one of the $\#_{k-1,d}$ linear types in $\mathbb{W}_\infty^{k-1}$. Thus

$$\begin{aligned} \#_{k,d} &= q^{k-d} \cdot \#_{k-1,d-1} + \#_{k-1,d} \\ &= q^{k-d} \cdot \begin{bmatrix} k-1 \\ d-1 \end{bmatrix}_q + \begin{bmatrix} k-1 \\ d \end{bmatrix}_q = \begin{bmatrix} k \\ d \end{bmatrix}_q . \end{aligned}$$

□

Theorem II.8. *The symplectic vector space $\mathbb{W}_\infty$ is smoothly approximated by $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$.*

Corollary II.9. *The symplectic $\mathfrak{f}$-vector space $\mathbb{W}_\infty$ has the finite length property over any field of characteristic 0.*

B. Symplectic graphs

For this subsection let $\mathfrak{f}$ be the two-element field $\mathfrak{f}_2$.

Definition II.10. For $n = 0, 1, 2, \dots$, the *symplectic graph* $\widetilde{\mathbb{W}}_n$ has vertices $\mathbb{W}_n$ and edges

$$v_1 \sim v_2 \iff \omega(v_1, v_2) = 1.$$

This is indeed an undirected graph: as ω is alternating, we have $\omega(v_1, v_2) = -\omega(v_2, v_1) = \omega(v_2, v_1)$ over $\mathfrak{f}_2$.

Proposition II.11. $\mathrm{Aut}(\widetilde{\mathbb{W}}_n) = \mathrm{Aut}(\mathbb{W}_n)$.

Proof. Clearly any isometric linear automorphism of $\mathbb{W}_n$ is a graph automorphism of $\widetilde{\mathbb{W}}_n$. Conversely, any $f \in \widetilde{\mathbb{W}}_n$ is evidently isometric. To show that f is linear, take $\lambda_1, \lambda_2 \in \mathfrak{f}$ and $v_1, v_2 \in \mathbb{W}$. We calculate:

$$\begin{aligned} &\omega\Big(f(\sum_i \lambda_i v_i) - \sum_i \lambda_i f(v_i), f(w)\Big) \\ &= \omega\Big(f(\sum_i \lambda_i v_i), f(w)\Big) - \sum_i \lambda_i \omega\big(f(v_i), f(w)\big) \\ &= \omega\Big(\sum_i \lambda_i v_i, w\Big) - \sum_i \lambda_i \omega(v_i, w) \\ &= \omega(0, w) = 0 \end{aligned}$$

for all $f(w) \in f(\mathbb{W}_n) = \mathbb{W}_n$; since ω is non-degenerate, we conclude that $f(\sum_i \lambda_i v_i) = \sum_i \lambda_i f(v_i)$. □

So the number of orbits in $\widetilde{\mathbb{W}}_n^k$ is precisely equal to the number of orbits in $\mathbb{W}_n^k$ — in particular, it is bounded above by $\sum_{d=0}^{k} \left[\begin{smallmatrix} k \\ d \end{smallmatrix}\right]_2 \cdot 2^{\binom{d}{2}}$ independently of n by Proposition II.7.[1] It remains to show $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$ embeds all finite graphs:

Proposition II.12 ([2, Theorem 8.11.2])**.** *Every graph on at most $2n$ vertices embeds into $\widetilde{\mathbb{W}}_n$.*

Proof. Let G be a graph on at most $2n$ vertices. The conclusion is trivial when $n = 0$. Also, if G contains no edges, we can choose any $2n$ of the 2^n vectors in $\langle e_1, \dots, e_n\rangle \subseteq \widetilde{\mathbb{W}}_n$.

So suppose $n \geq 1$ and G has an edge $s \sim t$. Let $G_{s,t}$ be the graph on vertices $G \setminus \{s, t\}$ with edges which we will specify later. By induction, some embedding $f : G_{s,t} \to \widetilde{\mathbb{W}}_{n-1}$ exists. Define $f' : G \to \widetilde{\mathbb{W}}_n$ by

$$\begin{aligned} x \in G_{s,t} &\mapsto f(x) - [\![x \sim s]\!] f_n + [\![x \sim t]\!] e_n \\ s &\mapsto e_n \\ t &\mapsto f_n \end{aligned}$$

where $[\![\phi]\!]$ is 1 if ϕ holds and 0 otherwise. Then we have $\omega(f'(x), f'(s)) = [\![x \sim s]\!]$ and $\omega(f'(x), f'(t)) = [\![x \sim t]\!]$ as desired, on one hand. On the other,

$$\begin{aligned} \omega(f'(x_1), f'(x_2)) = [\![x_1 \sim x_2]\!] &+ [\![x_1 \sim s]\!][\![x_2 \sim t]\!] \\ &+ [\![x_1 \sim t]\!][\![x_2 \sim s]\!] \end{aligned}$$

tells us how we should define the edge relation in $G_{s,t}$ for f' to be an embedding of graphs. □

Theorem II.13. *The Rado graph is oligomorphically approximated by $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$.*

Corollary II.14. *The Rado graph has the finite length property over any field of characteristic 0.*

[1] This is the kth term in the OEIS sequence A028631.

This proof of finite length also applies to *oriented graphs* (i.e., $x \to y \implies y \not\to x$ but unlike in a tournament, it may occur that $x \not\to y \wedge y \not\to x$) — use the three-element field instead of $\mathfrak{f}_2$.

III. Rado graph, with cogs

In this section we work with the following setting:

- $\mathcal{L}_0$ is a (possibly infinite) relational language containing a binary symbol $=$;
- $\mathcal{C}_0$ is a free amalgamation class of $\mathcal{L}_0$-structures where $=$ is interpreted as true equality, but every other $R \in \mathcal{L}_0$ is interpreted irreflexively.[2]
- $\mathcal{L}$ consists of $\mathcal{L}_0$ together with a new binary symbol $<$;
- $\mathcal{C}$ consists of $\mathcal{L}$-structures obtained from $\mathcal{C}_0$ by expanding with all possible linear orderings;
- $\mathbb{A}_0$ and $\mathbb{A}$ are the respective Fraïssé limits of $\mathcal{C}_0$ and $\mathcal{C}$, where without loss of generality we assume $\mathbb{A}_0$ and $\mathbb{A}$ share the same domain so that $\mathrm{Aut}(\mathbb{A}_0) \supseteq \mathrm{Aut}(\mathbb{A})$.

Example III.1. Take $\mathcal{L}_0$ to consist of $=$ only and $\mathcal{C}_0$ to be all finite sets. Then $\mathbb{A}_0$ is isomorphic to the pure set $\mathbb{N}$, whereas $\mathbb{A}$ is isomorphic to $\mathbb{Q}$ with the usual order.

Example III.2. Let $\mathcal{L}_0$ consist of $=$ together with a single binary symbol $\sim$ and let $\mathcal{C}_0$ consist of all finite undirected graphs not embedding the complete graph K_n, where $3 \le n$ $(\le \infty)$. Then $\mathbb{A}_0$ is the K_n-free Henson graph (or the Rado graph when $n = \infty$), and $\mathbb{A}$ is its generically ordered counterpart. (Allowing $n = 2$ makes these degenerate to $\mathbb{N}$ and $\mathbb{Q}$ above).

Free amalgamation in $\mathcal{C}_0$ allows us to free atoms in $\mathbb{A}$ from undesired relations. Let us make this more precise. Given atoms $z_i, z_j \in Z \subseteq \mathbb{A}$, we say z_i is *related* to z_j in Z if z_i, z_j appear together in some tuple $z_\bullet \in Z^*$ such that $\mathbb{A} \models R(a_\bullet)$ for some $R \in \mathcal{L}_0$ (since z_i and z_j are certainly related by $<$). Otherwise we say z_i is *unrelated* to z_j in Z — notice it is still possible that z_i becomes related to z_j in some bigger Z unless every relation in $\mathcal{L}_0$ is at most binary.

Lemma III.3. *Let $X, Y, \{z\} \subseteq \mathbb{A}$ be disjoint and finite. Then there is some automorphism $\tau \in \mathrm{Aut}(\mathbb{A})$ such that*

1) *τ fixes every $x \in X$;*
2) *$\tau(z)$ is unrelated to any $y \in Y$ and to z in $X \cup Y \cup \{z, \tau(z)\}$;*
3) *$\tau(z) > z$.*

Proof. In $\mathbb{A}_0$, form the free amalgam

$$\begin{tikzcd}
 & X \cup Y \cup \{z\} \arrow[dr, dashed, hook, "\subseteq"] & \\
X \arrow[ur, hook] \arrow[dr, hook] & & X \cup Y \cup \{z, z'\} \\
 & X \cup \{z\} \arrow[ur, dashed, hook, "x \in X \mapsto x, z \mapsto z'"'] &
\end{tikzcd}$$

so that no element of $Y \cup \{z\}$ is related to z' in $X \cup Y \cup \{z, z'\}$. Now we make $X \cup Y \cup \{z, z'\}$ an $\mathcal{L}$-structure: inherit the order on $X \cup Y \cup \{z\}$ from $\mathbb{A}$, and declare that $z < z'$ as well as $z' < a$ if a, the next element of $X \cup Y$ larger than z, exists at all. Observe that

$$x \in X \mapsto x, z \mapsto z'$$

is still an embedding in presence of the order. By homogeneity, we may embed $X \cup Y \cup \{z, z'\}$ into $\mathbb{A}$ via some f which is the identity on $X \cup Y \cup \{z\}$; again by homogeneity, we may extend the embedding

$$f(x) = x \in X \mapsto f(x), f(z) \mapsto f(z')$$

to some automorphism τ which makes 1), 2), and 3) true. □

On the other hand, an $\mathcal{L}$-structure fails to embed into $\mathbb{A}$ precisely when it embeds some forbidden structure, in which every two distinct elements are related:

Lemma III.4. *Let $\mathcal{F}_0$ consist of minimal (with respect to $\subseteq$) $\mathcal{L}_0$-structures which do not appear in $\mathcal{C}_0$. Then*

1) *$\mathcal{C}_0$ consists of every $\mathcal{L}_0$-structure that does not embed any $F \in \mathcal{F}_0$.*
2) *$\mathcal{C}$ consists of every $\mathcal{L}$-structure whose $\mathcal{L}_0$-reduct does not embed any $F \in \mathcal{F}_0$.*
3) *In any $F \in \mathcal{F}_0$, every two distinct elements $x, y \in F$ are related by some $R \in \mathcal{L}_0$.*

Proof. As $\mathcal{C}_0$ is closed under substructures, its complement is closed under superstructures and thus is — since there are no infinite strictly descending chain of embedded substructures — determined by its minimal structures. 2) follows because an $\mathcal{L}$-structure is in $\mathcal{C}$ precisely when its $\mathcal{L}_0$-reduct is in $\mathcal{C}_0$. For 3), notice that $F \setminus \{x\}$, $F \setminus \{y\}$ are in $\mathcal{C}_0$ by minimality; therefore so is their free amalgam over $F \setminus \{x, y\}$, which then cannot agree with F. □

In what follows, we will juggle with Lemma III.3 just enough so that we avoid the forbidden structures described in Lemma III.4. A main result will be:

Theorem III.5. *Assume each $R \in \mathcal{L}_0$ has arity at most two. Then $\mathbb{A}$ is $\mathfrak{F}$-oligomorphic for any field $\mathfrak{F}$ even with finitely many constants fixed, provided that $\mathbb{A}$ is oligomorphic (for instance if $\mathcal{L}_0$ is finite).*

And a corollary will be that $\mathbb{A}$ from Examples III.1 and III.2 is $\mathfrak{F}$-oligomorphic; as is its reduct $\mathbb{A}_0$.

A. Two reductions: orbits and projections

To start with, let us view $\mathbb{A}^d$ as $\mathbb{A}^{\{1,\dots,d\}}$ and more generally consider $\mathbb{A}^I$ for a finite indexing set $I \subseteq \mathbb{N}$. Fix a finite support $S \subseteq \mathbb{A}$. If $\mathbb{A}$ is oligomorphic, the tuples in $\mathbb{A}^I$ split into finitely many $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant orbits. Let $\mathcal{O} = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot o_\bullet$ be one such orbit. We shall call $\mathcal{O}$ *(S-)orderly* if $o_i \notin S$ and if $o_i < o_j$ whenever $i < j$. By removing the entries in $o_\bullet$ that repeat or come from S and reordering the rest, we can always find an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant bijection to an S-orderly orbit.

[2] We may assume irreflexivity with no loss of generality: see [5, beginning of §2.4].

An easy observation is that we may focus on a single orderly orbit at a time:

Proposition III.6. *The following are equivalent:*

1) *For $d = 0, 1, 2, \dots$ and any finite $S \subseteq \mathbb{A}$, chains of $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces in $\mathrm{Lin}_{\mathfrak{F}} \mathbb{A}^d$ are bounded in length;*
2) *$\mathbb{A}$ is oligomorphic, and $\mathrm{Lin}_{\mathfrak{F}} \mathcal{O}$ has fintie length for any orderly orbit $\mathcal{O}$.*

Proof. We have $\mathrm{len}(\mathrm{Lin}_{\mathfrak{F}}(\biguplus_i \mathcal{O}_i)) = \mathrm{len}(\bigoplus_i \mathrm{Lin}_{\mathfrak{F}} \mathcal{O}_i) = \sum_i \mathrm{len}(\mathrm{Lin}_{\mathfrak{F}} \mathcal{O}_i)$. $\square$

So fix an orderly orbit $\mathcal{O} = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot o_\bullet \subseteq \mathbb{A}^I$. From here we take an inductive approach. By $o|^J_\bullet$ we mean the restriction of $o_\bullet : I \to \mathbb{A}$ to $J \subseteq I$; we will often write $o|^{-i}_\bullet$ instead of $o|^{I \setminus \{i\}}_\bullet$. Note the image $\mathcal{O}|^I$ of $\mathcal{O}$ under this projection agrees with $\mathrm{Aut}(\mathbb{A})_{(S)} \cdot o|^J_\bullet$ and is still orderly.

To anticipate more general statements later, Let $\mathfrak{E}$ be a finite-dimensional $\mathfrak{F}$-vector space — for instance, $\mathfrak{F}$ itself. Things become more interesting when we lift $(-)|^J$ to a linear $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant map

$$\begin{aligned}(-)|^J : \mathrm{Lin}_{\mathfrak{E}} \mathcal{O} &\to \mathrm{Lin}_{\mathfrak{E}} \mathcal{O}|^J \\ v &\mapsto v|^J.\end{aligned}$$

Many cancellations can occur under $(-)|^J$; the *projection kernel* is the $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace

$$\mathrm{Ker}_{\mathfrak{E}} \mathcal{O} = \bigcap_{i \in I} \ker{(-)|^{-i}}$$

of $\mathrm{Lin}_{\mathfrak{E}} \mathcal{O}$.

Proposition III.7. *The following are equivalent:*

1) *$\mathrm{Lin}_{\mathfrak{F}} \mathcal{O}$ has finite length for every orderly orbit $\mathcal{O}$;*
2) *$\mathrm{Ker}_{\mathfrak{F}} \mathcal{O}$ has finite length for every orderly orbit $\mathcal{O}$.*

Proof. That 1) implies 2) is clear as $\mathrm{Ker}_{\mathfrak{F}} \mathcal{O} \subseteq \mathrm{Lin}_{\mathfrak{F}} \mathcal{O}$.

To prove the other implication, assume 2) and let $\mathcal{O} \subseteq \mathbb{A}^I$. We proceed by induction on $|I|$. If $I = \emptyset$, then $\mathcal{O}$ must be the entire singleton $\mathbb{A}^\emptyset = \{()\}$; as $\mathrm{Lin}_{\mathfrak{F}} \mathcal{O}$ has no nontrivial subspaces (let alone finitely supported ones), it has length 1. Now if $|I| \geq 1$, assemble all $|I|$ projection maps into a single map

$$\begin{aligned}\mathrm{Lin}_{\mathfrak{F}} \mathcal{O} &\to \bigoplus_{i \in I} \mathcal{O}|^{-i} \\ v &\mapsto (v|^{-i})_{i \in I}\end{aligned}$$

whose kernel is precisely $\mathrm{Ker}_{\mathfrak{F}} \mathcal{O}$. We have

$$\mathrm{len}(\mathrm{Lin}_{\mathfrak{F}} \mathcal{O}) - \mathrm{len}(\mathrm{Ker}_{\mathfrak{F}} \mathcal{O}) \leq \sum_{i \in I} \mathrm{len}(\mathrm{Lin}_{\mathfrak{F}} \mathcal{O}|^{-i})$$

which shows that $\mathrm{len}(\mathrm{Lin}_{\mathfrak{F}} \mathcal{O})$ is finite from the assumptions. $\square$

We call a vector from the projection kernel *balanced*. As we will see in the next subsection, cogs are a prominent example.

B. Cogs

Definition III.8. Let $\mathcal{O} = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot o_\bullet \subseteq \mathbb{A}^I$ be an S-orderly orbit. An *$\mathcal{O}$-cog duo* $a_\bullet \parallel b_\bullet$ consists of $2 \cdot |I|$ atoms in $\mathbb{A}$ with the following $\mathcal{L}$-structure on $\{a_i, b_i \mid i \in I\} \cup S$:

1) $a_{i_1} < b_{i_1} < a_{i_2} < b_{i_2} < \cdots < a_{i_d} < b_{i_d}$ where I consists of the indices $i_1 < i_2 < \cdots < i_d$;
2) $a_i, b_i < s$ if and only if $o_i < s$;
3) any relation in $\mathcal{L}_0$ (in particular $=$) holds for a tuple $c_\bullet$ with entries in $a_I \cup b_I \cup S$ if and only if it holds for $c_\bullet$ with each entry equal to a_i, b_i replaced by o_i.

Three remarks are in order. First, each a_i is unrelated to its counterpart b_i in $a_I \cup b_I \cup S$. Second, given any $J \subseteq I$, the combined tuple $a|^J_\bullet; b|^{I \setminus J}_\bullet$ lies in $\mathcal{O}$ by homogeneity: observe

$$\begin{aligned} a_j &\mapsto o_j, j \in J; \\ b_i &\mapsto o_i, i \in I \setminus J; \\ s &\mapsto s, s \in S \end{aligned}$$

defines an embedding. Third, by homogeneity still, any two cog duos $a_\bullet \parallel b_\bullet$ and $a'_\bullet \parallel b'_\bullet$ belong to the same $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant orbit.

Definition III.9. Given $\lambda \in \mathfrak{E}$ and an $\mathcal{O}$-cog duo $a_\bullet \parallel b_\bullet$, the corresponding *$\mathcal{O}$-cog* with coefficient λ is the vector

$$\lambda \cdot a_\bullet \between b_\bullet = \sum_{J \subseteq I} (-1)^{|J|} \lambda \cdot a|^J_\bullet; b^{I \setminus J}_\bullet$$

in $\mathrm{Lin}_{\mathfrak{E}} \mathcal{O}$. The linear span of all $\mathcal{O}$-cogs with coefficients from $\mathfrak{E}$ is denoted by $\mathrm{Cog}_{\mathfrak{E}} \mathcal{O}$.

As remarked above, given any two $\mathcal{O}$-cog duos there is some $\pi \in \mathrm{Aut}(\mathbb{A})_{(S)}$ such that $\pi \cdot (a_\bullet \parallel b_\bullet) = a'_\bullet \parallel b'_\bullet$ and thus $\pi \cdot (\lambda \cdot a_\bullet \between b_\bullet) = \lambda \cdot a'_\bullet \between b'_\bullet$. Hence $\mathrm{Cog}_{\mathfrak{E}} \mathcal{O}$ is an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace of $\mathrm{Lin}_{\mathfrak{E}} \mathcal{O}$ and it is generated by cogs based on a single duo.

Proposition III.10. $\mathrm{Cog}_{\mathfrak{E}} \mathcal{O}$ *is contained in* $\mathrm{Ker}_{\mathfrak{E}} \mathcal{O}$.

Proof. Let $\mathcal{O} \subseteq \mathbb{A}^I$, let $a_\bullet \parallel b_\bullet$ be an $\mathcal{O}$-cog duo, and let $i \in I$. The subsets of I come in pairs of J and $J \cup \{i\}$, where J is a subset of $I \setminus \{i\}$. The two tuples $a|^J_\bullet; b|^{I \setminus J}_\bullet$ and $a|^{J \cup \{i\}}_\bullet; b|^{I \setminus (J \cup \{i\})}_\bullet$ differ only on the ith entry. But this difference gets erased under $(-)|^{-i}$, so the two corresponding terms in $\lambda \cdot a_\bullet \between b_\bullet$ will cancel out and hence $(\lambda \cdot a_\bullet \between b_\bullet)|^{-i} = 0$ overall. $\square$

In fact, cogs arise anywhere.

Lemma III.11. *Suppose $a_\bullet \parallel b_\bullet$ is an $\mathcal{O}$-cog duo, where $\mathcal{O} \subseteq \mathbb{A}^I$ is S-orderly. Given $z \in S$,*

- *write $S' = S \setminus \{z\}$;*
- *let $j \notin I$ be such that $\mathcal{O}' = \mathrm{Aut}(\mathbb{A})_{(S')} \cdot (a_\bullet; z) \subseteq \mathbb{A}^{I \cup \{j\}}$ is orderly,*
- *let $X \subseteq \mathbb{A}$ be a finite set containing $\{a_i, b_i \mid i \in I\} \cup S'$ but not z;*
- *let $Y \subseteq \mathbb{A}$ be any finite set disjoint from $X \cup \{z\}$;*

then the $\tau \in \mathrm{Aut}(\mathbb{A})_{(X)}$ afforded by Lemma III.3 gives us an $\mathcal{O}'$-cog duo $(a_\bullet; z) \parallel (b_\bullet; \tau \cdot z)$.

Proof. To start with, notice that we have the required order relations with z and $\tau(z)$. Now let $R \in \mathcal{L}_0$ and let $c_\bullet$ be a tuple with entries in $a_{I\cup\{j\}} \cup b_{I\cup\{j\}} \cup S' = a_I \cup b_I \cup S \cup \{\tau \cdot z\}$; we want to show that

$$\mathbb{A} \models R(c_\bullet) \text{ if and only if } \mathbb{A} \models R(d_\bullet), \qquad (\P)$$

where $d_\bullet$ is obtained from $c_\bullet$ by replacing every b_i with a_i and $\tau \cdot z$ with z. We split into three cases.

1) Suppose z and $\tau \cdot z$ both appear in $c_\bullet$. Then $\mathbb{A} \not\models R(c_\bullet)$ because $\tau \cdot z$ is unrelated to z by design, and $\mathbb{A} \not\models R(d_\bullet)$ because R is irreflexive and z appears twice.
2) So if z appears in $c_\bullet$, we may assume that $\tau \cdot z$ does not — that is, $c_\bullet$ has entries in $a_I \cup b_I \cup S$. In this case (¶) follows from the assumption that $a_\bullet \parallel b_\bullet$ is a cog duo in $\mathcal{O} = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot a_\bullet$.
3) Finally, suppose that $\tau \cdot z$ appears in $c_\bullet$ but not z. Then only z appears in $\tau^{-1} \cdot c_\bullet$, and $d_\bullet$ is obtained by replacing every b_i with a_i in $\tau^{-1} \cdot c_\bullet$. But $\mathbb{A} \models R(c_\bullet)$ if and only if $\mathbb{A} \models R(\tau^{-1} \cdot c_\bullet)$, and now (¶) holds as we discussed for the case above. $\square$

Starting from an empty cog duo, we may apply the previous lemma inductively.

Proposition III.12. *Let $\mathcal{O} \subseteq \mathbb{A}^I$ be an S-orderly orbit. Then for any $a_\bullet \in \mathcal{O}$, there is:*

1) *some $b_\bullet \in \mathcal{O}$ such that $a_\bullet \parallel b_\bullet$ is an $\mathcal{O}$-cog duo; and*
2) *for each $i \in I$, some $\pi_i \in \mathrm{Aut}(\mathbb{A})_{(S\cup\{a_j, b_j | j \in I, j \neq i\})}$ sending $a_i \mapsto b_i$.*

Proof. Enumerate the indices of I as $i_1, \ldots, i_d$. Suppose that we have found $b_{i_1}, \ldots, b_{i_k}$ such that

$$a|_\bullet^{\{i_1,\ldots,i_k\}} \parallel (i_1 \mapsto b_{i_1}, \ldots, i_k \mapsto b_{i_k})$$

is a cog duo in $\mathcal{O}_k = \mathrm{Aut}(\mathbb{A})_{(S\cup\{a_{i_{k+1}},\ldots,a_{i_d}\})} \cdot a|_\bullet^{\{i_1,\ldots,i_k\}}$ — note that $() \parallel ()$ is certainly a cog duo in $\mathcal{O}_0$ at the start. If $k < d$, with $z = a_{i_{k+1}}$, $X = \{a_{i_1}, b_{i_1}, \ldots, a_{i_k}, b_{i_k}\} \cup S \cup \{a_{i_{k+2}}, \ldots, a_{i_d}\}$, and $Y = \emptyset$, a straightforward application of Lemma III.11 yields an atom $b_{i_{k+1}}$ that makes

$$a|_\bullet^{\{i_1,\ldots,i_k,i_{k+1}\}} \parallel (i_1 \mapsto b_{i_1}, \ldots, i_k \mapsto b_{i_k}, i_{k+1} \mapsto b_{i_{k+1}})$$

a cog duo in $\mathcal{O}_{k+1}$. We thus obtain the desired cog duo in $\mathcal{O}_d = \mathcal{O}$ as we reach $k = d$. The automorphisms π_{i_k} now come directly from homogeneity and the definition of an $\mathcal{O}$-cog duo: the map

$$\begin{array}{l} a_{i_1} \mapsto a_{i_1}, \ldots, a_{i_k} \mapsto b_{i_k}, \ldots, a_{i_d} \mapsto a_{i_d} \\ b_{i_1} \mapsto b_{i_1}, \ldots, \qquad\qquad\quad \ldots, b_{i_d} \mapsto b_{i_d}, s \in S \mapsto s \end{array}$$

is an embedding. $\square$

The result below substantiates the slogan that cogs are found everywhere.

Theorem III.13. *Any $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace V of $\mathrm{Lin}_{\mathfrak{E}}\,\mathcal{O}$ contains $\mathrm{Cog}_{\mathfrak{E}'}\,\mathcal{O}$, where $\mathcal{O} \subseteq \mathbb{A}^I$ is S-orderly and $\mathfrak{E}'$ is the subspace of $\mathfrak{E}$ spanned by $\{v(a_\bullet) \mid v \in V, a_\bullet \in \mathcal{O}\}$.*

Proof. Pick any $v \in V$ and $a_\bullet \in \mathcal{O}$; it is enough to show that V contains $v(a_\bullet) \cdot a_\bullet \between b_\bullet$ for some $\mathcal{O}$-cog duo $a_\bullet \parallel b_\bullet$. Actually, write

$$S' = S \cup \{c_i \mid v(c_\bullet) \neq 0, i \in I\} \setminus \{a_i \mid i \in I\} \supseteq S$$

and put $\mathcal{O}' = \mathrm{Aut}(\mathbb{A})_{(S')} \cdot a_\bullet \subseteq \mathcal{O}$ — then $\mathcal{O}'$ is S'-orderly. By Proposition III.12, we can find $b_\bullet \in \mathcal{O}'$ such that $a_\bullet \parallel b_\bullet$ is a cog duo in $\mathcal{O}'$ and *a fortiori* a cog duo in $\mathcal{O}$. Also take the automorphisms $\pi_{i_1}, \ldots, \pi_{i_d}$ from there, where $i_1, \ldots, i_d$ enumerate I. Now define $v_0 = v$ and

$$v_k = v_{k-1} - \pi_{i_k} \cdot v_{k-1}.$$

We can check inductively that for $k = 0, 1, \ldots, d$, with $\mathcal{O}_k = \{c_\bullet \mid v(c_\bullet) \neq 0, \{c_{i_1}, \ldots, c_{i_k}, \ldots, c_{i_d}\} \supseteq \{a_{i_1}, \ldots, a_{i_k}\}\}$ we have

$$v_k = \sum_{c_\bullet \in \mathcal{O}_k} \sum_{J \subseteq \{i_1,\ldots,i_k\}} (-1)^{|J|} v(c_\bullet) \cdot c|_\bullet^J; b|_\bullet^{I\setminus J}.$$

But $\{c_{i_1}, \ldots, c_{i_d}\} \supseteq \{a_{i_1}, \ldots, a_{i_d}\}$ means that $c_\bullet = a_\bullet$, so at the end v_d is the desired $\mathcal{O}$-cog. $\square$

Corollary III.14. $\mathrm{Cog}_{\mathfrak{F}}\,\mathcal{O}$ *has length* 1.

Proof. Let $V \subseteq \mathrm{Cog}_{\mathfrak{F}}\,\mathcal{O}$ be a non-zero $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace. Then $\mathfrak{E}'$ is the entire field $\mathfrak{F}$, and by above V must be $\mathrm{Cog}_{\mathfrak{F}}\,\mathcal{O}$ itself. $\square$

In light of Propositions III.6 and III.7, we will be able to prove the finite length property for an oligomorphic structure with free amalgamation over any field and support if we know $\mathrm{Ker}_{\mathfrak{F}}\,\mathcal{O} = \mathrm{Cog}_{\mathfrak{F}}\,\mathcal{O}$. Let us now attempt to show that.

C. Subvectors

This is a good time to recall a view we have tacitly taken: with $\mathcal{O}$ as a standard basis, a vector $v \in \mathrm{Lin}_{\mathfrak{E}}\,\mathcal{O}$ is just a finite set of pairs in $\mathfrak{E} \times \mathcal{O}$. A *subvector* of v is a subset of these pairs.

Now suppose as usual that $\mathcal{O} \subseteq \mathbb{A}^I$ is S-orderly. Given $i \in I$ and $a = a_i \in \mathbb{A}$ for some $a_\bullet \in \mathcal{O}$, we write

$$\mathcal{O}^{i:a} = \{b_\bullet \in \mathcal{O} \mid b_i = a\};$$

this is an $\mathrm{Aut}(\mathbb{A})_{(S\cup\{a\})}$-orbit, and its projection $\mathcal{O}^{i:a}|^{-i} = \mathrm{Aut}(\mathbb{A})_{(S\cup\{a\})} \cdot a|_\bullet^{-i}$ is orderly. For a vector $v \in \mathrm{Lin}_{\mathfrak{E}}\,\mathcal{O}$, by

$$v^{i:a} \in \mathrm{Lin}_{\mathfrak{E}}\,\mathcal{O}^{i:a}$$

we mean the subvector consisting of all pairs in $\mathfrak{E} \times \mathcal{O}^{i:a}$.

Lemma III.15. *Let $v \in \mathrm{Lin}_{\mathfrak{E}}\,\mathcal{O}$ be balanced. Then any projected subvector $v^{i:a}|^{-i} \in \mathrm{Lin}_{\mathfrak{E}}\,\mathcal{O}^{i:a}|^{-i}$ is also balanced.*

Proof. Let $j \in I \setminus \{i\}$. By assumption we have

$$0 = v|^{-j} = \sum_a v^{i:a}|^{-j}$$

in $\mathrm{Lin}_{\mathfrak{E}}\,\mathbb{A}^{I\setminus\{j\}}$, so by looking at ith entries we see that each $v^{i:a}|^{-j}$ must be the zero vector. Hence so is $v^{i:a}|^{-j}|^{-i} =$

$v^{i:a}|^{-i}|^{-j}$, which shows that $v^{i:a}|^{-i}$ is in the projection kernel. □

So we can try to prove $\mathrm{Ker}_{\mathfrak{C}}\,\mathcal{O} \subseteq \mathrm{Cog}_{\mathfrak{C}}\,\mathcal{O}$ for any orderly $\mathcal{O} \subseteq \mathbb{A}^I$ by inducting on $|I|$; we just need to reassemble the various cogs in $\mathcal{O}^{i:a}|^{-i}$ back into $\mathcal{O}$-cogs. Unfortunately we were only able to do so under the hypothesis that $\mathcal{L}_0$ is an at most binary language, which we henceforth assume.

D. Special case: vectors free enough for binary relations

We will begin by showing any $v \in \mathrm{Ker}_{\mathfrak{C}}\,\mathcal{O}$ lies in $\mathrm{Cog}_{\mathfrak{C}}\,\mathcal{O}$ provided that v satisfies an additional condition which, as we will explain in the subsection, may be assumed without loss of generality. We motivate and introduce this condition now.

Consider the atoms in the tuples

$$\mathcal{O}[v] = \{b_\bullet \in \mathcal{O} \mid v(b_\bullet) \neq 0\}.$$

that appear in a vector $v \in \mathrm{Lin}_{\mathfrak{C}}\,\mathcal{O}$, with $\mathcal{O}$ being S-orderly. Take $a = b_i$ to be one such atom, and let $R_1, R_2 \in \mathcal{L}_0$ be unary and binary respectively. Then whether $R_1(a)$, $R_2(a,s)$, $R_2(s,a)$ hold in $\mathbb{A}$ given $s \in S$ is determined by $\mathcal{O}$ — more preciesly, by whether $R_1(o_i)$, $R_2(o_i,s)$, $R_2(s,o_i)$ hold in $\mathbb{A}$ for any $o_\bullet \in \mathcal{O}$.

Now take another atom $a' = b'_j$ occurring in $\mathcal{O}[v]$. What can be said about $R_2(a,a')$? Except in those happy dispositions where $b'_\bullet = b_\bullet$, not much — whether $\mathbb{A} \models R_2(o_i,o_j)$ need not be a constraint for a and a'. The index j may not even be unique: we may well have $b'_j = b''_k$ for $j \neq k$. We want to avoid such confusions:

Definition III.16. Let $\mathcal{O}$ be S-orderly in $\mathbb{A}^I$, and take any $o_\bullet \in \mathcal{O}$. We say a finite family $\{b^{(k)}_\bullet \in \mathcal{O} \mid k \in K\}$ is

1) *rooted* if $\sqrt{-} : b^{(k)}_i \mapsto o_i$ for $k \in K$ is a function — i.e., if $i = i'$ whenever $b^{(k)}_i = b^{(k')}_{i'}$;
2) *free enough* if it is well-indexed and for any $b^{(k)}_i, b^{(k')}_{i'}$, either they are unrelated (in $\mathbb{A}$) or

$$\mathbb{A} \models R(b^{(k)}_i, b^{(k')}_{i'}) \leftrightarrow R(\sqrt{b^{(k)}_i}, \sqrt{b^{(k')}_{i'}}) \\ \wedge R(b^{(k')}_{i'}, b^{(k)}_i) \leftrightarrow R(\sqrt{b^{(k')}_{i'}}, \sqrt{b^{(k)}_i})$$

for every binary relation $R \in \mathcal{L}_0$.[3]

We can now state the precise result.

Theorem III.17. *Let $v \in \mathrm{Ker}_{\mathfrak{C}}\,\mathcal{O}$ be such that $\mathcal{O}[v]$ is free enough. Then (assuming that $\mathcal{L}_0$ is at most binary) we have*

$$v = \sum_{k \in K} v(b^{(k)}_\bullet) \cdot x^{(k)}_\bullet \between y^{(k)}_\bullet$$

for some $b^{(k)}_\bullet$'s, $x^{(k)}_\bullet$'s, and $y^{(k)}_\bullet$'s, where moreover

$$\mathcal{O}[v] \cup \{x^{(k)}_\bullet, y^{(k)}_\bullet \mid k \in K\}$$

is also free enough.

[3]Of course, whether the family is rooted or free enough does not depend on the choice of the specific representative $o_\bullet \in \mathcal{O}$!

We spend the rest of this subsection giving the proof, which proceeds by induction on $|I|$ for S-orderly orbits $\mathcal{O} \subseteq \mathbb{A}^I$ for every S at once.

The base case $I = \emptyset$ is immediate as before: as $\mathcal{O}$ is the singleton $\{()\}$, any vector $v \in \mathrm{Ker}_{\mathfrak{C}}\,\mathcal{O} = \mathrm{Lin}_{\mathfrak{C}}\,\mathcal{O} = \mathrm{Cog}_{\mathfrak{C}}\,\mathcal{O}$ is a cog already, where $\mathcal{O}[v]$ is vacuously free enough; so the theorem says nothing more than $v = v$.

Now suppose that some $i^* \in I$ exists — in fact, let i^* be maximal. Consider a vector $v \in \mathrm{Ker}_{\mathfrak{C}}\,\mathcal{O}$ with $\mathcal{O}[v]$ being free enough. We can decompose v into finitely many subvectors to write

$$v = v^{i^*:a_1} + \cdots + v^{i^*:a_m}.$$

Then each projection $v^{i^*:a_j}|^{-i^*}$ lies in $\mathrm{Ker}_{\mathfrak{C}}\,\mathcal{O}^{i^*:a_j}|^{-i^*}$ by Lemma III.15, and we may straightforwardly check that *a fortiori* $\mathcal{O}^{i^*:a_j}|^{-i^*}[v^{i^*:a_j}|^{-i^*}]$ is rooted and free enough. It follows from the inductive hypothesis of Theorem III.17 that

$$v^{i^*:a_j}|^{-i^*} = \sum_{k \in K_j} v^{i^*:a_j}|^{-i^*}(b^{(a_j,k)}_\bullet) \cdot x^{(a_j,k)}_\bullet \between y^{(a_j,k)}_\bullet$$

for some tuples $b^{(a_j,k)}_\bullet$'s, $x^{(a_j,k)}_\bullet$'s, and $y^{(a_j,k)}_\bullet$'s such that $\mathcal{O}^{i^*:a_j}|^{-i^*}[v^{i^*:a_j}|^{-i^*}] \cup \{x^{(a_j,k)}_\bullet, y^{(a_j,k)}_\bullet \mid k \in K_j\}$ is free enough. We can return to $\mathcal{O}$ by adding a_j back as the i^*th term to every tuple: we get

$$v^{i^*:a_j} = \sum_{k \in K_j} \left(v(b^{(a_j,k)}_\bullet; a_j) \cdot x^{(a_j,k)}_\bullet \between y^{(a_j,k)}_\bullet \right); a_j,$$

where the family

$$\mathcal{O}[v^{i^*:a_j}] \cup \{(x^{(a_j,k)}_\bullet; a_j), (y^{(a_j,k)}_\bullet; a_j) \mid k \in K_j\}$$

is easily seen to be free enough. We can ask for more:

Claim III.18. *We may choose $x^{(a_j,k)}_\bullet$'s, and $y^{(a_j,k)}_\bullet$'s so that*

$$\mathcal{O}[v] \cup \{(x^{(a_{j'},k)}_\bullet; a_{j'}), (y^{(a_{j'},k)}_\bullet; a_{j'}) \mid k \in K_{j'}, 1 \leq j' \leq j\}$$

is free enough for $j = 1, 2, \ldots, m$.

Proof. Let A_j denote the atoms that appear in $\mathcal{O}[v^{i^*:a_j}]$. Given any automorphism $\pi \in \mathrm{Aut}(\mathbb{A})$ that fixes S as well as A_j, using the tuples $\pi \cdot x^{(a_j,k)}_\bullet$ and $\pi \cdot y^{(a_j,k)}_\bullet$ gives us another decomposition of $v^{i^*:a_j}|^{-i^*}$ that is free enough. We will show that composing such automorphisms — which will be provided by Lemma III.3 — suffices to make the claim hold for j.

Assume the claim for $j-1$. Suppose that the family in the claim for j is not even rooted. We must have $b^{(k)}_i = b^{(k')}_{i'}$ with $i \neq i'$ where $b^{(k)}_\bullet$, say, is some $x^{(a_j,k)}_\bullet; a_j$ or $y^{(a_j,k)}_\bullet; a_j$. Now if we were to have $b^{(k)}_i \in A_j$ — e.g., if $b^{(k)}_i = a_j$ — then $b^{(k)}_i = b^{(k'')}_{i''}$ for some $b^{(k'')}_\bullet \in \mathcal{O}[v^{i^*:a_j}] \subseteq \mathcal{O}[v]$. This is impossible: we know two families that are free enough which imply $i = i''$ and $i'' = i'$.

So the problematic atom $b^{(k)}_i$ does not belong to A_j. In other words, the set X of all atoms except $b^{(k)}_i$ that appear in $\mathcal{O}[v] \cup \{(x^{(a_{j'},k)}_\bullet; a_{j'}), (y^{(a_{j'},k)}_\bullet; a_{j'}) \mid k \in K_{j'}, 1 \leq j' \leq j\}$ contains A_j and a_j. Apply Lemma III.3 — particularly, use the fact that $\mathbb{A}$ has no algebraicity — to get an automorphism τ which fixes $S \cup X$ but sends $b_i^{(k)}$ to a fresh atom. Then, in the new family

$$\begin{aligned}\mathcal{O}[v] &\cup \{(x_\bullet^{(a_{j'},k)}; a_{j'}), (y_\bullet^{(a_{j'},k)}; a_{j'}) \mid k \in K_{j'}, 1 \le j' < j\} \\ &\cup \{(\tau \cdot x_\bullet^{(a_j,k)}; a_j), (\tau \cdot y_\bullet^{(a_j,k)}; a_j) \mid k \in K_j\},\end{aligned}$$

the problematic atoms are precisely the ones in the old family minus $b_i^{(k)}$ — they are all fixed by τ. We thus continue this way until all problematic atoms are freshened.

Hence assume that the family in the claim for j is rooted. Suppose $b_i^{(k)}$ and some $b_{i'}^{(k')}$ are the reason why the family fails to be free enough; call the atom $b_i^{(k)}$ an obstruction. To have $b_i^{(k)} \in A_j$ is impossible: otherwise $b_i^{(k)} = b_{i''}^{(k'')}$ for some $b_\bullet^{(k'')} \in \mathcal{O}[v^{i^*:a_j}] \subseteq \mathcal{O}[v]$; then $i = i''$ by the rooted assumption, yet $b_{i''}^{(k'')}$ — hence $b_i^{(k)}$ — cannot be an obstruction because we have two subfamilies that are free enough. Symmetrically, we see that $b_{i'}^{(k')} \notin A_j$.

Finally, notice that by the claim for $j-1$, an obstruction $b_i^{(k)}$ must again come from $b_\bullet^{(k)}$ being some $x_\bullet^{(a_j,k)}; a_j$ or $y_\bullet^{(a_j,k)}; a_j$. This time, split all atoms except $b_i^{(k)}$ which appear in $\mathcal{O}[v] \cup \{(x_\bullet^{(a_{j'},k)}; a_{j'}), (y_\bullet^{(a_{j'},k)}; a_{j'}) \mid k \in K_{j'}, 1 \le j' \le j\}$ into two sets X and Y: put any atom $b_{i'}^{(k')}$ which makes $b_i^{(k)}$ an obstruction in Y, and put everything else in X. Then X contains A_j as discussed above; in fact, X contains $\{x_i^{(a_j,k)}, y_i^{(a_j,k)} \mid k \in K_j, i \in I\}$. Use Lemma III.3 to get $\tau \in \mathrm{Aut}(\mathbb{A})_{(S\cup X)}$ which sends $b_i^{(k)}$ to a fresh atom disjoint from $S \cup X \cup Y \cup \{b_i^{(k)}\}$ that is unrelated to everything in Y. Observe that the new family

$$\begin{aligned}\mathcal{O}[v] &\cup \{(x_\bullet^{(a_{j'},k)}; a_{j'}), (y_\bullet^{(a_{j'},k)}; a_{j'}) \mid k \in K_{j'}, 1 \le j' < j\} \\ &\cup \{(\tau \cdot x_\bullet^{(a_j,k)}; a_j), (\tau \cdot y_\bullet^{(a_j,k)}; a_j) \mid k \in K_j\},\end{aligned}$$

remains rooted. Moreover, the obstructions here are precisely the ones from the old family minus $b_k^{(i)}$, which are all fixed by τ. So we may repeat this process until all obstructions are removed. □

Having chosen the fresh atoms carefully, let X consist of all the atoms appearing in $\mathcal{O}[v] \cup \{(x_\bullet^{(a_{j'},k)}; a_{j'}), (y^{(a_{j'},k)\bullet;a_{j'}}) \mid k \in K_{j'}, 1 \le j' \le m\}$. The trick now is to add a new element b^* to the finite $\mathcal{L}$-structure $S \cup Z \subseteq \mathbb{A}$. We define the new relations by

$$\begin{cases} R_1(b^*) \iff R_1(\sqrt{b^*}) \\ R_2(b^*, s) \iff R_2(\sqrt{b^*}, \sqrt{s}) \\ R_2(s, b^*) \iff R_2(\sqrt{s}, \sqrt{b^*}) \\ R_2(b^*, b_i^{(k)}) \iff R_2(\sqrt{b^*}, \sqrt{b_i^{(k)}}) \\ R_2(b_i^{(k)}, b^*) \iff R_2(\sqrt{b_i^{(k)}}, \sqrt{b^*}) \\ b^* > s \iff \sqrt{b^*} > \sqrt{s} \\ b^* > b_i^{(k)} \end{cases}$$

for every unary relation R_1 and binary relation R_2 in $\mathcal{L}_0$, where we extend $\sqrt{-} : X \to \{o_1, \ldots, o_d\}$ to

$$\sqrt{-} : S \cup X \cup \{b^*\} \to \{o_1, \ldots, o_d\} \cup S$$

by letting $\sqrt{s} = s$ and $\sqrt{b^*} = o_{i^*}$; this is indeed a function.

Claim III.19. *$S \cup X \cup \{b^*\}$ embeds into $\mathbb{A}$.*

Proof. TODO 1

If not, by Lemma III.4 there is a forbidden $\mathcal{L}_0$-structure F which embeds into $S \cup X \cup \{b^*\}$ via ϕ. We will show that $\sqrt{-} \circ \phi$ then embeds F into $\mathbb{A}$, which is clearly impossible. $\sqrt{s}$ □

Using homogeneity we may assume that $S \cup X \cup \{b^*\} \subseteq \mathbb{A}$.

Claim III.20. *$x_\bullet^{(a_j,k)}; a_j \parallel y_\bullet^{(a_j,k)}; b^*$ forms an $\mathcal{O}$-cog duo for $1 \le j \le m$ and $k \in K_j$.*

Proof. TODO 2 □

Claim III.21. *We have $v = \sum_{j=1}^m \sum_{k \in K_j} v(b_\bullet^{(a_j,k)}; a_j) \cdot (x_\bullet^{(a_j,k)}; a_j \between y_\bullet^{(a_j,k)}; b^*)$.*

Proof. Observe that

$$\begin{aligned} &\sum_{j=1}^m \sum_{k \in K_j} v(b_\bullet^{(a_j,k)}) \cdot (x_\bullet^{(a_j,k)}; a_j \between y_\bullet^{(a_j,k)}; b^*) \\ &= \sum_{j=1}^m \sum_{k \in K_j} \left(v(b_\bullet^{(a_j,k)}) \cdot x_\bullet^{(a_j,k)} \between y_\bullet^{(a_j,k)}\right); a_j \\ &\quad - \sum_{j=1}^m \sum_{k \in K_j} \left(v(b_\bullet^{(a_j,k)}) \cdot x_\bullet^{(a_j,k)} \between y_\bullet^{(a_j,k)}\right); b^* \\ &= \sum_{j=1}^m v^{i^*:a_j} - \sum_{j=1}^m v^{i^*:a_j}|^{-i^*}; b^* \\ &= v - \sum_{j=1}^m v^{i^*:a_j}|^{-i^*}; b^*, \end{aligned}$$

so it suffices to show the last sum vanishes. Recall from Proposition III.10 that cogs are balanced. By projecting away the i^*th entry, we obtain

$$0 = 0 - \left(\sum_{j=1}^m v^{i^*:a_j}|^{-i^*}; b^* \right)\Bigg|^{-i^*}.$$

But we can simply add back b^* as the i^*th entry, yielding $0 = \sum_{j=1}^m v^{i^*:a_j}|^{-i^*}; b^*$ as desired. □

Claim III.22. *The family $\mathcal{O}[v] \cup \{(x_\bullet^{(a_j,k)}; a_j), (y_\bullet^{(a_j,k)}; b^*) \mid k \in K_j, 1 \le j \le m\}$ is free enough.*

Proof. TODO 3 □

E. Reducing to the special case

Acknowledgements

Hrushovski
Evans

References

[1] Mikołaj Bojańczyk et al. "Orbit-Finite-Dimensional Vector Spaces and Weighted Register Automata". In: *TheoretiCS* Volume 3 (May 2024). ISSN: 2751-4838. DOI: 10.46298/theoretics.24.13. URL: http://dx.doi.org/10.46298/theoretics.24.13.

[2] Chris Godsil and Gordon Royle. *Algebraic graph theory*. 1st ed. Graduate Texts in Mathematics. Springer, 2001. ISBN: 978-0-387-95241-3.

[3] W. M. Kantor, Martin W. Liebeck, and H. D. Macpherson. "$\aleph_0$-Categorical Structures Smoothly Approximated by Finite Substructures". In: *Proceedings of the London Mathematical Society* s3-59.3 (1989), pp. 439–463. DOI: https://doi.org/10.1112/plms/s3-59.3.439.

[4] Antoine Mottet and Michael Pinsker. "Smooth approximations: An algebraic approach to CSPs over finitely bounded homogeneous structures". In: *J. ACM* 71.5 (Oct. 2024). DOI: 10.1145/3689207.

[5] Daoud Siniora and Sławomir Solecki. "Coherent extension of partial automorphisms, free amalgamation and automorphism groups". In: *The Journal of Symbolic Logic* 85.1 (2020), pp. 199–223. DOI: 10.1017/jsl.2019.32.