A Few More Vector Spaces with Atoms of Finite Lengths

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Abstract

An infinite structure has, given a field, the finite length property if the following holds for each of its finite powers: in the corresponding free vector space, strict chains of equivariant subspaces are bounded in length. It has been shown that the countable pure set and the dense linear ordering without endpoints have this property. In this paper, we generalise these two results to a) interpretations in smoothly approximable structures, provided the field has characteristic zero, and b) generaically ordered expansions of Fraïssé limits with free amalgamation over a graph language. As a special case, we prove the finite length property of the Rado graph using both methods.

CCS Concepts

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ACM Reference Format:
Jingjie Yang, Mikołaj Bojańczyk, and Bartek Klin. 2026. A Few More Vector Spaces with Atoms of Finite Lengths. In *Proceedings of Make sure to enter the correct conference title from your rights confirmation email (Conference acronym 'XX).* ACM, New York, NY, USA, 13 pages. https://doi.org/XXXX XXX.XXXXXXX

1 Introduction

2 Preliminaries

We briefly recall some basic notions from model theory, mainly in order to fix notation. A language is a set of relation and function symbols, each with a specified arity. A structure $\mathbb{A}$ over a language consists of an underlying set, together with interpretations of the symbols in the language as relations and functions on the underlying set. An automorphism of a structure $\mathbb{A}$ is a bijection from the underlying set of $\mathbb{A}$ to itself that preserves all relations and functions of $\mathbb{A}$. When we talk about orbits in $\mathbb{A}^d$, for some power $d \in \{1, 2, \ldots\}$, we mean the orbits under the component-wise action of the automorphism group. In other words, two tuples $(a_1, \ldots, a_d)$ and $(b_1, \ldots, b_d)$ are in the same orbit if there exists an automorphism π of $\mathbb{A}$ such that $\pi(a_i) = b_i$ for all $i \in \{1, \ldots, d\}$.

Definition 2.1 (Oligomorphic structure)**.** A structure $\mathbb{A}$ is *oligomorphic* if $\mathbb{A}^d$ has finitely many orbits for every $d \in \{1, 2, \ldots\}$.

All structures considered in this paper will be not only oligomorphic, but they will also arise as Fraïssé limits of suitable Fraïssé classes, and therefore they will also be homogeneous. Since homogeneity does not seem to play a role, we do not define it. We list below the main examples of structures that we consider in the paper, which are Fraïssé limits of the following classes of finite structures: (a) sets with equality only; (b) linear orders; (c) vector spaces over a finite field; and (d) graphs.

Example 2.2 (Equality only)**.** In this structure, the underlying set is countably infinite and there are no relations or functions. Automorphisms are arbitrary permutations, and two tuples are in the same orbit if and only if they have the same equality pattern. Since there are finitely many equality patterns for tuples of fixed length, this structure is oligomorphic. For example, in dimension $d = 2$ there are two orbits: $x_1 = x_2$ and $x_1 \neq x_2$.

Example 2.3 (Order)**.** In this structure, the underlying set is the set of rational numbers, equipped with the usual order. Automorphisms are order-preserving permutations, and two tuples are in the same orbit if and only if they have the same order pattern. Since there are finitely many order patterns for tuples of fixed length, this structure is oligomorphic.

Example 2.4 (Vector space)**.** Fix some finite field, and consider the vector space of countably infinite dimension over this field. This vector space can be seen as a structure, with functions for vector addition and scalar multiplication. Automorphisms are permutations that are linear maps, and two tuples are in the same orbit if and only if they have the same linear dependencies. Since there are finitely many linear dependency patterns for tuples of fixed length over a finite field, this structure is oligomorphic.

Example 2.5 (Rado graph)**.** The Rado graph is the Fraïssé limit of the class of finite undirected graphs. It is not so easy to describe it explicitly, but one of its descriptions is that if one randomly selects a graph with a countably infinite set of vertices, by independently including each possible edge with probability $1/2$, then with probability 1 the resulting graph is isomorphic to the Rado graph. Two tuples are in the same orbit if and only if they have the same equality and adjacency patterns, hence there are finitely many orbits in every dimension.

2.1 Vector spaces

We will use structures to construct vector spaces.

Conference acronym 'XX, Woodstock, NY

ACM ISBN 978-1-4503-XXXX-X/2018/06
https://doi.org/XXXXXXX.XXXXXXX

Definition 2.6 (Vector space with atoms)**.** For a structure $\mathbb{A}$, a field k, and $d \in \{1, 2, \ldots\}$, we write

$$\mathrm{Lin}_{\mathsf{k}}\, \mathbb{A}^d$$

for the free vector space which consists of finite formal linear combinations of d-tuples of elements in $\mathbb{A}$, using the field k. Any space of this form is called a *vector space with atoms*[1].

We use the name *atom dimension* for the parameter d in the above definition, so that we do not confuse it with the dimension of a vector space. The atom dimension tells us how many atoms can be stored in a basis vector. The atom dimension will be an important induction parameter in the proofs. The dimension, in the sense of vector spaces, will always be countably infinite, since we will always be interested in the case where the structure $\mathbb{A}$ is countably infinite.

Apart from the structure of a vector space, this space described above is also equipped with an action of the automorphism group of $\mathbb{A}$, and we will be interested in subsets which preserve both kinds of structure, i.e. they are closed under taking linear combinations, and applying automorphisms. Such subsets are called *equivariant subspaces.*

Example 2.7. Let $\mathbb{A}$ be the structure with equality only, let k be any field, and let the atom dimension be $d = 1$. As explained in [2, Example 4.2], this corresponding vector space with atoms

$$\mathrm{Lin}_{\mathsf{k}}\, \mathbb{A}$$

has only three equivariant subspaces: the zero subspace, the whole space, and the subspace which consists of vectors where all coefficients sum to zero.

The finite length property. The main topic of this paper is the study of chains of equivariant spaces

$$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n$$

which are contained in some vector space with atoms. The *length* of such a chain in the number n of strict inclusions.

Definition 2.8 (Finite length property)**.** The length of a vector space with atoms is the maximal length of a chain of its equivariant subspaces. A structure $\mathbb{A}$ has the *finite length property over a field* k if for every $d \in \{0, 1, \ldots\}$, the vector space with atoms $\mathrm{Lin}_{\mathsf{k}}\, \mathbb{A}^d$ has finite length.

The finite length property was studied in [2], where it was shown that the equality atoms (Example 2.2) and the order atoms (Example 2.3) have this property over any field. In this paper, we will establish the finite length property for more structures, including the Rado graph (Example 2.5) and the vector space atoms (Example 2.4). For the Rado graph, we will not make any assumptions on the field, but for the case of the vector space structure from Example 2.4, we will need to assume that the field k in $\mathrm{Lin}_{\mathsf{k}}\, \mathbb{A}^d$ has characteristic zero. (There are two fields involved here, namely the finite field used to define $\mathbb{A}$, and the field k used to define the vector space with atoms. The assumption is on the latter field.)

[1]There is a more general definition of vector spaces with atoms, see [1, Definition 8.1]. In particular, a vector space as in the above definition will necessarily have an equivariant basis, which is not the case in the more general definition. However, the results on finite length from this paper reduce to the special case described above. Therefore, in the interest of simplicity, we only work with this special case.

but for the albeit under the assumption that the field has characteristic zero. Definitions:

(1) oligomorphic
(2) homogeneous
(3) smooth approximation by homogeneous substructures [4] (N.B. not the 'smooth approximation' from [5, Definitino 4])
(4) *oligomorphic approximation* of a homogeneous structure by finite substructures with uniformly few orbits (i.e., types) that cover the age of $\mathbb{A}$

Definition 2.9. An *interpretation in* $\mathbb{A}$ is a structure $\mathbb{A}' = (D/E; R_1, R_2, \ldots)$, where

- D is an equivariant subset of $\mathbb{A}^n$ for some $n \geq 1$;
- E is an equivariant equivalence relation on D;
- D/E consists of equivalence classes $d/E \subseteq D$ for $d \in D$, which $\pi \in \mathrm{Aut}(\mathbb{A})$ acts on via $\pi \cdot d/E = (\pi \cdot d)/E$;
- every relation R_i of arity r_i is an equivariant subset of $(D/E)^{r_i}$.

We call $\mathbb{A}'$ a *reduct of* $\mathbb{A}$ if $D = \mathbb{A}$ and E is just equality.

Proposition 2.10. *If $\mathbb{A}$ has the finite length property over* F*, then so does any interpretation* $\mathbb{A}'$*.*

Proof. Let $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l$ be a chain of $\mathrm{Aut}(\mathbb{A}')$-equivariant subspaces in $\mathrm{Lin}_{\mathsf{F}}(\mathbb{A}')^k = \mathrm{Lin}_{\mathsf{F}}(D/E)^k$. Each V_j is then $\mathrm{Aut}(\mathbb{A})$-equivariant: given $\pi \in \mathrm{Aut}(\mathbb{A})$, notice that $d/E \mapsto (\pi \cdot d)/E$ is an automorphism of $\mathbb{A}'$. Now l is bounded above by the $\mathrm{Aut}(\mathbb{A})$-length of $\mathrm{Lin}_{\mathsf{F}}(D/E)^k$, which is finite — $\mathrm{Lin}_{\mathsf{F}}(D/E)^k$ is isomorphic to the quotient of $\mathrm{Lin}_{\mathsf{F}}\, D^k$ by the span of

$$\left\{ \begin{matrix} (d_1, \ldots, d_k) \\ -(d_1', \ldots, d_k') \end{matrix} \middle| d_1/E = d_1'/E, \ldots, d_k/E = d_k'/E \right\}.$$

□

3 Rado graph, sans cogs

In this section, we establish a method for proving the finite length property, assuming that the field has characterisatic zero. Using this method, we will establish the finite length property for the Rado graph, and for vector spaces over a finite field, under the assumption of characteristic zero. These are new results. Also, we think that the proof itself, even when applied to get already known results, is of independent interest and arguably simpler than previous proofs.

Oligomorphic approximation.

Definition 3.1 (Oligomorphic approximation)**.** We say that a structure $\mathbb{A}$ has *oligomorphic approximation* if for every $d \in \{1, 2, \ldots\}$ there exists a family $\mathscr{B}$ of finite substructures of $\mathbb{A}$ such that:

(1) every finite substructure of $\mathbb{A}$ embeds into some $\mathbb{B} \in \mathscr{B}$; and
(2) there is a common finite upper bound on the number of orbits in $\mathbb{B}^d$ for $\mathbb{B} \in \mathscr{B}$.

The above definition is a relaxation of a stronger notion from model theory that is called *smooth approximation.* In the stronger notion, the family $\mathscr{B}$ is independent of d, and there are other requirements as well [4, p. 440].

Theorem 3.2. *The following structures have oligomorphic approximation:*

(1) *the structure with equality only from Example 2.2;*

(2) *the vector space structure from Example 2.4, for any finite field;*
(3) *the Rado graph from Example 2.5.*

Before proving the above theorem, let us observe that the dense linear order does not have oligomorphic approximation.

Example 3.3 (Non-example: rational numbers with order)**.** A non-example is the rational number with the usual order. The finite substructures in this case are finite linear orders, and already for dimension $d = 1$, a finite linear order of size n will have n orbits. Hence, we cannot have a common finite upper bound on the number of orbits in $\mathbb{B}^d$ for $\mathbb{B} \in \mathscr{B}$.

Proof of Theorem 3.2.

(1) For the structure with equality only, we can choose $\mathscr{B}$ so that is indpendent of d, and this is simply all finite structures with equality only. For sufficiently large $\mathbb{B} \in \mathscr{B}$, the number of orbits in $\mathbb{B}^d$ is the same as the number of orbits in $\mathbb{A}^d$.
(2) Consider the vector space structure. As in the previous example, we can choose $\mathscr{B}$ independently of d, namely the family of vector spaces of finite dimension. If we changed the field to any finite field, the same argument would apply.
(3) The most interesting case is the Rado graph. For every $n \in \{1, 2, \ldots\}$ define a finite graph as follows. The set of vertices is 2^{2n}, which is seen as a vector space over the two-element field whose basis is
$$\{e_1, \ldots, e_n, f_1, \ldots, f_n\}.$$
Since the field has two elements, we can view vertices as subsets of the above basis. In this graph, there is an edge vw if and only if the sets
$$\{i \in \{1, \ldots, n\} \mid e_i \in v \text{ and } f_i \in w \}$$
$$\{i \in \{1, \ldots, n\} \mid f_i \in v \text{ and } e_i \in w \}$$
have different sizes modulo two. This graph might seem a bit articial, it is in fact a well-known construction, which is called the *symplecitic graph* over the two-element field; see e.g. [**<empty citation>**]. These graphs are placed in more context in the appendix, where we also prove that they satisfy the necessary properties from Definition 3.1. actually

□

The main result of this section is the following theorem.

Theorem 3.4. *If an oligomorphic structure $\mathbb{A}$ has oligomorphic approximation, then it has the finite length property over any field of characteristic* 0.

Combining Theorems 3.2 and 3.4, we can get the following results, both old and new, on the finite length property.

Corollary 3.5. *Over any field of characteristic* 0*, the following structures have the finite length property: (a) the equality atoms; (b) the bit-vector atoms; and (c) the Rado graph.*

As mentioned in the introduction, the finite length property was already known for the equality atoms, for arbitrary fields. The results for the bit-vector atoms and the Rado graph are new. The assumption on characteristic zero is important, at least in case of the bit-vector atoms, where the finite length property is known to fail over finite fields [2, Section 4.4]. Later on in this paper, we will prove the result for the Rado graph again using a different method that works for any field.

The rest of this section is devoted to proving Theorem 3.4.

Proof. Copy the 'bojań-trick' from §8.2 of Mikołaj's https://www.mimuw.edu.pl/~bojan/papers/notes-July3.pdf. □

Corollary 3.6. *Also for m cliques of n vertices — interpretable in the equality atoms.*

3.1 Symplectic vector spaces

Throughout this subsection let f denote a finite field.

Definition 3.7. A *symplectic vector space* is an f-vector space $\mathbb{W}$ equipped with a bilinear form $\omega : \mathbb{W} \times \mathbb{W} \to \mathsf{f}$ that is

(1) alternating: $\omega(v, v) = 0$ for all v; and
(2) non-degenerate: if $\omega(v, w) = 0$ for all w then $v = 0$.

Example 3.8. Let $\mathbb{W}_n$ be the f-vector space with basis $e_1, \ldots, e_n, f_1, \ldots, f_n$. Define ω by bilinearly extending
$$\omega(e_i, f_i) = 1 = -\omega(f_i, e_i), \quad \omega(-, *) = 0 \text{ elsewhere}; \qquad (\S)$$
one may straightforwardly check that ω is alternating and non-degenerate. Moreover, noticing that $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$, we obtain a countable-dimensional symplectic vector space $\mathbb{W}_\infty = \bigcup_n \mathbb{W}_n$.

We will refer to vectors satisfying (§) as a *symplectic basis.* Note such vectors must be linearly independent: if $v = \sum_i \lambda_i e_i + \mu_i f_i = 0$, then $\lambda_i = \omega(v, f_i) = 0$ and $\mu_i = \omega(e_i, v) = 0$ for each i. Such bases behave very much like the usual bases.

Proposition 3.9. *Assume that $\mathbb{W}$ is a symplectic vector space that is at most countable. Then any finite symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ can be extended to a symplectic basis that spans the whole $\mathbb{W}$.*

Proof. Suppose that $e_1, \ldots, e_n, f_1, \ldots, f_n$ does not already span $\mathbb{W}$; take v to be a witness (that is least according to some fixed enumeration of $\mathbb{W}$ in the case it is infinite). Put
$$e_{n+1} = v - \sum_{i=1}^{n} \omega(e_i, v) f_i + \sum_{i=1}^{n} \omega(f_i, v) e_i$$
so that $\omega(e_i, e_{n+1}) = 0 = \omega(f_i, e_{n+1})$. This cannot be the zero vector lest we contradict the choice of v. By the non-degeneracy of ω, there is — rescaling if necessary — some w such that $\omega(e_{n+1}, w) = 1$. Now define
$$f_{n+1} = w - \sum_{i=1^n} \omega(e_i, w) f_i + \sum_{i=1}^{n} \omega(f_i, w) e_i$$
in a similar manner, making $e_1, \ldots, e_n, e_{n+1}, f_1, \ldots, f_n, f_{n+1}$ a symplectic basis that spans v. We go through every element of $\mathbb{W}$ by continuing this way. □

In fact, we will also make use of the "symplectic basis and a half" variant below.

Proposition 3.10. *Now assume $\mathbb{W}$ is a finite-dimensional symplectic vector space. Let*

$$e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+k},$$
$$f_1, \ldots, f_n$$

be linearly independent vectors satisfying (§)*. Then we can find the missing* $f_{n+1}, \ldots, f_{n+k}$ *to complete the symplectic basis.*

Proof. Suppose we have found $f_{n+1}, \ldots, f_{n+i}$ already such that

$$e_1, \ldots, e_n, e_{n+1}, \ldots e_{n+i}, e_{n+i+1}, e_{n+i+2}, \ldots, e_{n+k},$$
$$f_1, \ldots, f_n, f_{n+1}, \ldots, f_{n+i}$$

satisfy (§). Notice these vectors are linearly independent: in a linear combination that sums to 0, the coefficients of $e_1, f_1, \ldots, e_{n+i}, f_{n+i}$ must be zero, and we assumed the linear independence of $e_{n+i+1}, \ldots, e_{n+k}$. By extending these to a basis B of $\mathbb{W}$, we may define a linear function

$$\psi : \mathbb{W} \to \mathrm{f}$$

which sends e_{n+i+1} to 1 but every other $b \in B$ to 0. Now apply Proposition 3.9 to obtain a symplectic basis $e'_1, f'_1, \ldots, e'_m, f'_m$ of $\mathbb{W}$, and put

$$f_{n+i+1} = \sum_{j=1}^{m} \psi(e'_j)f'_j - \psi(f'_j)e'_j;$$

then $\omega(-, f_{n+i+1})$ agrees with ψ on this symplectic basis, so by linearity they must be the same function. In particular

$$\omega(e_{n+i+1}, f_{n+i+1}) = \psi(e_{n+i+1}) = 1,$$

whereas $\psi(e_1), \ldots, \psi(e_{n+k}), \psi(f_1), \ldots, \psi(f_{n+i})$ are all 0 Thus we have (§) as required. □

Given two symplectic vector spaces $\mathbb{W}$ and $\mathbb{W}'$, we call a function α between $X \subseteq \mathbb{W}$ and $X' \subseteq \mathbb{W}'$ *isometric* if $\omega(\alpha(x_1), \alpha(x_2)) = \omega(x_1, x_2)$ for all $x_1, x_2 \in X$. We can make an easy observation:

Lemma 3.11. *Let* $\{e_i, f_i \mid i \in I\} \subseteq \mathbb{W}$, $\{e'_j, f'_j \mid j \in J\} \subseteq \mathbb{W}'$ *be two symplectic bases and let* $\alpha : I \to J$ *be a bijection. Then*

$$e_i \mapsto e'_{\alpha(i)}, f_i \mapsto f'_{\alpha(i)}$$

defines an isometric linear bijection $\langle e_i, f_i \rangle \to \langle e'_j, f'_j \rangle$.

It then follows from Proposition 3.9 that, up to isometric linear bijections, $\mathbb{W}_0, \mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_\infty$ are all the countable symplectic vector spaces. Whilst we may deduce that $\mathbb{W}_\infty$ is oligomorphic by appealing to Ryll-Nardzewski, we will opt for a more direct proof that also establishes smooth approximation.

Proposition 3.12 (Witt Extension). *Any isometric linear injection* $\alpha : \langle X \rangle \subseteq \mathbb{W}_n \to \mathbb{W}_n$ *can be extended to an automorphism of* $\mathbb{W}_n$ *(i.e., an isometric linear bijection) and in turn to one of* $\mathbb{W}_\infty$.

Proof. To begin with, find a basis $x_1, \ldots, x_k$ for the subspace $W = \{w \in \langle X \rangle \mid \forall x \in X : \omega(w, x) = 0\}$ and extend it to a basis $x_1, \ldots, x_k, x_{k+1}, \ldots, x_d$ for $\langle X \rangle$. Notice that

$$U = \langle x_{k+1}, \ldots, x_d \rangle$$

must be a symplectic subspace: as it intersects with W trivially, given any non-zero vector $u \in U$ we must have $0 \neq \omega(u, w + u') = \omega(u, u')$ for some $w \in W$ and $u' \in U$. Hence use Proposition 3.9 to find a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ for U. Observe that

$$e_1, \ldots, e_n, x_1, \ldots, x_k,$$
$$f_1, \ldots, f_n$$

form a basis for $\langle X \rangle$ and satisfy (§). On the other hand,

$$\alpha(e_1), \ldots, \alpha(e_n), \alpha(x_1), \ldots, \alpha(x_k),$$
$$\alpha(f_1), \ldots, \alpha(f_n)$$

form a basis for $\alpha(\langle X \rangle)$ and also satisfy (§). Therefore apply Proposition 3.10 twice to find the missing $y_1, \ldots, y_k$ and $y'_1, \ldots, y'_k$ to complete the two symplectic bases — call them $\mathcal{B}$ and $\mathcal{B}'$. They are of the same size.

Now, by using Proposition 3.9, extend $\mathcal{B}$ and $\mathcal{B}'$ to symplectic bases C and C' that span $\mathbb{W}_n$. These must both have size $2n$, so by Lemma 3.11 we obtain an isometric linear automorphism $\beta : \mathbb{W}_n \to \mathbb{W}_n$ extending α.

To finish, notice that $C, e_{n+1}, \ldots, f_{n+1}, \ldots$ as well as $C', e_{n+1}, \ldots, f_{n+1}, \ldots$ form a symplectic basis spanning $\mathbb{W}_\infty$. We obtain from Lemma 3.11 another time an isometric linear automorphism $\gamma : \mathbb{W}_\infty \to \mathbb{W}_\infty$ extending β that is the identity almost everywhere. □

Proposition 3.13. $\mathbb{W}^k_\infty$ *has precisely* $\sum_{d=0}^{k} \left[{k \atop d} \right]_q \cdot q^{\binom{d}{2}}$ *orbits under* $\mathrm{Aut}(\mathbb{W}_\infty)$*, where* $q = |\mathrm{f}|$ *and*

$$\begin{bmatrix} k \\ d \end{bmatrix}_q = \frac{(q^k - 1)(q^{k-1} - 1) \cdots (q^{k-d+1} - 1)}{(q^d - 1)(q^{d-1} - 1) \cdots (q^1 - 1)}$$

is the q-binomial coefficient.

Remark 3.14. To anticipate the next subsection, we note a similarity with the Rado graph: in $\mathbb{G}^k$ there are $\sum_{d=0}^{k} \left\{ {k \atop d} \right\} \cdot 2^{\binom{d}{2}}$ orbits — we may impose any edge relation on d vertices.

Proof. To each $v_\bullet \in \mathbb{W}^k_\infty$ we associate a *type*, which comprises the following data:

1. pivot indices $I \subseteq \{1, \ldots, k\}$ containing every i such that v_i is not spanned by $v_1, \ldots, v_{i-1}$ — so we inductively ensure that
$$\{v_{i'} \mid i' \in I, i' \leq i\}$$
is a basis for $\langle v_1, \ldots, v_i \rangle$;
2. for each $j \notin I$, an assignment $\Lambda_j : \{i \in I \mid i < j\} \to \mathrm{f}$ such that $v_j = \sum_{i \in I, i<j} \Lambda_j(i)v_i$;
3. a map $\Omega : \binom{I}{2} \to \mathrm{f}$ defined by $\Omega(\{i' < i\}) = \omega(v_{i'}, v_i)$.

If $\pi : \mathbb{W}_\infty \to \mathbb{W}_\infty$ is an isometric linear bijection, then $v_\bullet = (v_1, \ldots, v_k)$ and $\pi \cdot v_\bullet = (\pi(v_1), \ldots, \pi(v_k))$ evidently share the same type. Conversely, if $w_\bullet$ has the type of $v_\bullet$, then

$$\alpha : \langle v_i \mid i \in I \rangle \to \langle w_i \mid i \in I \rangle \subseteq \mathbb{W}_n$$
$$v_i \mapsto w_i$$

gives an isometric linear injection for some large enough n. Observe that α must send $v_j \mapsto w_j$ for $j \notin I$ too, and that it may be extended to an automorphism π of $\mathbb{W}_\infty$ by Propsoition 3.12. Furthermore we can find some $v_\bullet$ that realises any given type $(I, \{\Lambda_j\}_j, \Omega)$: it suffices to put

$$v_i = e_i + \sum_{i' \in I, i' < i} \Omega(i', i) f_{i'}$$

for $i \in I$ and $v_j = \sum_{i \in I, i<j} \in \Lambda_j(i)v_i$ for $j \notin I$. Therefore the number of types is precisely the number of orbits in $\mathbb{W}^k_\infty$.

Finally, we do some combinatorics. Fix $0 \leq d \leq k$ and count the number of types with $|I| = d$. There are $q^{\binom{d}{2}}$ choices for Ω and say

$\#_{k,d}$ choices for the Λ_j's; the two can be chosen independently. In total, this gives

$$\sum_{d=0}^{k} q^{\binom{d}{2}} \cdot \#_{k,d}$$

types for vectors in $\mathbb{W}_\infty^d$. So focus on $\#_{k,d}$, the number of *linear types* — i.e., $(I, \{\Lambda_j\}_j)$, ignoring Ω — in $\mathbb{W}_\infty^k$. (Incidentally $\sum_{d=0}^k \#_{k,d}$ is the number of orbits in $\mathbb{W}_\infty^k$ or, more generally, any countable-dimensional $\mathfrak{f}$-vector space under linear automorphisms.) On the small values we easily check that

$$\#_{0,0} = 1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q,$$
$$\#_{1,0} = 1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q, \qquad \#_{1,1} = 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q.$$

Given a linear type in $\mathbb{W}_\infty^k$ with $|I| = d$, we either have $1 \in I$ or $I \subseteq \{2, \ldots, k\}$. In the first case, the linear type is specified by one of the $\#_{k-1,d-1}$ linear types in $\mathbb{W}_\infty^{k-1}$ together with how v_1 is involved in the span of the $(k-1)-(d-1)$ non-pivot vectors. In the second case, the linear type is simply one of the $\#_{k-1,d}$ linear types in $\mathbb{W}_\infty^{k-1}$. Thus

$$\begin{aligned}\#_{k,d} &= q^{k-d} \cdot \#_{k-1,d-1} + \#_{k-1,d} \\ &= q^{k-d} \cdot \begin{bmatrix} k-1 \\ d-1 \end{bmatrix}_q + \begin{bmatrix} k-1 \\ d \end{bmatrix}_q = \begin{bmatrix} k \\ d \end{bmatrix}_q. \qquad \square\end{aligned}$$

Theorem 3.15. *The symplectic vector space $\mathbb{W}_\infty$ is smoothly approximated by $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$.*

Corollary 3.16. *The symplectic $\mathfrak{f}$-vector space $\mathbb{W}_\infty$ has the finite length property over any field of characteristic 0.*

3.2 Symplectic graphs

For this subsection let $\mathfrak{f}$ be the two-element field $\mathfrak{f}_2$.

Definition 3.17. For $n = 0, 1, 2, \ldots$, the *symplectic graph* $\widetilde{\mathbb{W}}_n$ has vertices $\mathbb{W}_n$ and edges

$$v_1 \sim v_2 \iff \omega(v_1, v_2) = 1.$$

This is indeed an undirected graph: as ω is alternating, we have $\omega(v_1, v_2) = -\omega(v_2, v_1) = \omega(v_2, v_1)$ over $\mathfrak{f}_2$.

Proposition 3.18. $\mathrm{Aut}(\widetilde{\mathbb{W}}_n) = \mathrm{Aut}(\mathbb{W}_n)$.

Proof. Clearly any isometric linear automorphism of $\mathbb{W}_n$ is a graph automorphism of $\widetilde{\mathbb{W}}_n$. Conversely, any $f \in \widetilde{\mathbb{W}}_n$ is evidently isometric. To show that f is linear, take $\lambda_1, \lambda_2 \in \mathfrak{f}$ and $v_1, v_2 \in \mathbb{W}$. We calculate:

$$\begin{aligned}&\omega\Big(f\big(\sum_i \lambda_i v_i\big) - \sum_i \lambda_i f(v_i), f(w)\Big) \\ &= \omega\Big(f\big(\sum_i \lambda_i v_i\big), f(w)\Big) - \sum_i \lambda_i \omega\big(f(v_i), f(w)\big) \\ &= \omega\Big(\sum_i \lambda_i v_i, w\Big) - \sum_i \lambda_i \omega(v_i, w) \\ &= \omega(0, w) = 0\end{aligned}$$

for all $f(w) \in f(\mathbb{W}_n) = \mathbb{W}_n$; since ω is non-degenerate, we conclude that $f(\sum_i \lambda_i v_i) = \sum_i \lambda_i f(v_i)$. $\square$

So the number of orbits in $\widetilde{\mathbb{W}}_n^k$ is precisely equal to the number of orbits in $\mathbb{W}_n^k$ — in particular, it is bounded above by $\sum_{d=0}^k \begin{bmatrix} k \\ d \end{bmatrix}_2 \cdot 2^{\binom{d}{2}}$ independently of n by Proposition 3.13.[2] It remains to show $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$ embeds all finite graphs:

Proposition 3.19 ([3, Theorem 8.11.2]). *Every graph on at most $2n$ vertices embeds into $\widetilde{\mathbb{W}}_n$.*

Proof. Let G be a graph on at most $2n$ vertices. The conclusion is trivial when $n = 0$. Also, if G contains no edges, we can choose any $2n$ of the 2^n vectors in $\langle e_1, \ldots, e_n\rangle \subseteq \widetilde{\mathbb{W}}_n$.

So suppose $n \geq 1$ and G has an edge $s \sim t$. Let $G_{s,t}$ be the graph on vertices $G \setminus \{s, t\}$ with edges which we will specify later. By induction, some embedding $f : G_{s,t} \to \widetilde{\mathbb{W}}_{n-1}$ exists. Define $f' : G \to \widetilde{\mathbb{W}}_n$ by

$$\begin{aligned} x \in G_{s,t} &\mapsto f(x) - [\![x \sim s]\!] f_n + [\![x \sim t]\!] e_n \\ s &\mapsto e_n \\ t &\mapsto f_n \end{aligned}$$

where $[\![\phi]\!]$ is 1 if ϕ holds and 0 otherwise. Then we have $\omega(f'(x), f'(s)) = [\![x \sim s]\!]$ and $\omega(f'(x), f'(t)) = [\![x \sim t]\!]$ as desired, on one hand. On the other,

$$\begin{aligned}\omega(f'(x_1), f'(x_2)) = [\![x_1 \sim x_2]\!] &+ [\![x_1 \sim s]\!][\![x_2 \sim t]\!] \\ &+ [\![x_1 \sim t]\!][\![x_2 \sim s]\!]\end{aligned}$$

tells us how we should define the edge relation in $G_{s,t}$ for f' to be an embedding of graphs. $\square$

Theorem 3.20. *The Rado graph is oligomorphically approximated by $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$.*

Corollary 3.21. *The Rado graph has the finite length property over any field of characteristic 0.*

This proof of finite length also applies to *oriented graphs* (i.e., $x \to y \implies y \not\to x$ but unlike in a tournament, it may occur that $x \not\to y \wedge y \not\to x$) — use the three-element field instead of $\mathfrak{f}_2$.

4 Rado graph, with cogs

In this section we work with the following setting:

- $\mathcal{L}_0$ is a (should we just assume finite?) relational language consisting of unary and binary symbols;
- C_0 is a free amalgamation class of finite $\mathcal{L}_0$-structures, where every $R \in \mathcal{L}_0$ is interpreted irreflexively.[3]
- $\mathcal{L}$ consists of $\mathcal{L}_0$ together with a new binary symbol $<$;
- C consists of $\mathcal{L}$-structures obtained from C_0 by expanding with all possible linear orderings — this is still an amalgamation class;
- $\mathbb{A}_0$ and $\mathbb{A}$ are the respective Fraïssé limits of C_0 and C, where without loss of generality (because of the extension property) we assume $\mathbb{A}_0$ and $\mathbb{A}$ share the same domain so that $\mathrm{Aut}(\mathbb{A}_0) \supseteq \mathrm{Aut}(\mathbb{A})$.

Example 4.1. Take $\mathcal{L}_0$ to consist of $=$ only and C_0 to be all finite sets. Then $\mathbb{A}_0$ is isomorphic to the pure set $\mathbb{N}$, whereas $\mathbb{A}$ is isomorphic to $\mathbb{Q}$ with the usual order.

[2]This is the kth term in the OEIS sequence A028631.

[3]We may assume irreflexivity with no loss of generality: see [6, beginning of §2.4].

Example 4.2. Let $\mathcal{L}_0$ consist of $=$ together with a single binary symbol $\sim$ and let $\mathcal{C}_0$ consist of all finite undirected graphs not embedding the complete graph K_n, where $3 \leq n\ (\leq \infty)$. Then $\mathbb{A}_0$ is the K_n-free Henson graph (or the Rado graph when $n = \infty$), and $\mathbb{A}$ is its generically ordered counterpart. (Allowing $n = 2$ makes these degenerate to $\mathbb{N}$ and $\mathbb{Q}$ above).

Free amalgamation in $\mathcal{C}_0$ allows us to free atoms in $\mathbb{A}$ from undesired relations. Let us make this more precise. Say that atoms $a, b \in \mathbb{A}$ are *related* if $a = b$ or $\mathbb{A} \models R(a,b) \vee R(b,a)$ for some $R \in \mathcal{L}_0$ (they are certainly related by $<$, but we disregard it here).

Lemma 4.3. *Let $X, Y, \{z\} \subseteq \mathbb{A}$ be disjoint and finite. Then there is some automorphism $\tau \in \mathrm{Aut}(\mathbb{A})$ such that*

(1) *τ fixes every $x \in X$;*
(2) *$\tau(z)$ is unrelated to all $y \in Y$ and to z;*
(3) *$\tau(z) > z$.*

Proof. In $\mathbb{A}_0$, form the free amalgam

$$\begin{tikzcd}
 & X \cup Y \cup \{z\} \arrow[dr, hook, dashed, "\subseteq"] & \\
X \arrow[ur, hook] \arrow[dr, hook] & & X \cup Y \cup \{z, z'\} \\
 & X \cup \{z\} \arrow[ur, hook, dashed, "x \in X \mapsto x, z \mapsto z'"'] &
\end{tikzcd}$$

so that no element of $Y \cup \{z\}$ is related to z'. Now we make $X \cup Y \cup \{z, z'\}$ an $\mathcal{L}$-structure: inherit the order on $X \cup Y \cup \{z\}$ from $\mathbb{A}$, and declare that $z < z'$ as well as $z' < a$ if a, the next element of $X \cup Y$ larger than z, exists at all. Observe that

$$x \in X \mapsto x, z \mapsto z'$$

is still an embedding in presence of the order. By homogeneity, we may embed $X \cup Y \cup \{z, z'\}$ into $\mathbb{A}$ via some f which is the identity on $X \cup Y \cup \{z\}$; again by homogeneity, we may extend the embedding

$$f(x) = x \in X \mapsto f(x), f(z) \mapsto f(z')$$

to some automorphism τ which makes 1), 2), and 3) true. □

On the other hand, an $\mathcal{L}$-structure fails to embed into $\mathbb{A}$ precisely when it embeds some forbidden structure, in which every two distinct elements are related:

Lemma 4.4. *Let $\mathcal{F}_0$ consist of minimal (with respect to $\subseteq$) $\mathcal{L}_0$-structures which do not appear in $\mathcal{C}_0$. Then*

(1) *$\mathcal{C}_0$ consists of every $\mathcal{L}_0$-structure that does not embed any $F \in \mathcal{F}_0$.*
(2) *$\mathcal{C}$ consists of every $\mathcal{L}$-structure whose $\mathcal{L}_0$-reduct does not embed any $F \in \mathcal{F}_0$.*
(3) *In any $F \in \mathcal{F}_0$, every two distinct elements $x, y \in F$ are related by some $R \in \mathcal{L}_0$.*

Proof. As $\mathcal{C}_0$ is closed under substructures, its complement is closed under superstructures and thus is — since there are no infinite strictly descending chain of embedded substructures — determined by its minimal structures. 2) follows because an $\mathcal{L}$-structure is in $\mathcal{C}$ precisely when its $\mathcal{L}_0$-reduct is in $\mathcal{C}_0$. For 3), notice that $F \setminus \{x\}$, $F \setminus \{y\}$ are in $\mathcal{C}_0$ by minimality; therefore so is their free amalgam over $F \setminus \{x, y\}$, which then cannot agree with F. □

In what follows, we will juggle with Lemma 4.3 just enough so that we avoid the forbidden structures described in Lemma 4.4. A main result will be:

Theorem 4.5. *$\mathbb{A}$ has the finite length property even with finitely many constants fixed, provided that $\mathbb{A}$ is oligomorphic (for instance if $\mathcal{L}_0$ is finite).*

And a corollary will be that $\mathbb{A}$ from Examples 4.1 and 4.2 has the finite length property; as is its reduct $\mathbb{A}_0$.

JY: I changed $o_\bullet$'s to o in subsections A and C

4.1 Two reductions: orbits and projections

To start with, let us view $\mathbb{A}^d$ as $\mathbb{A}^{\{1,\ldots,d\}}$ and more generally consider $\mathbb{A}^I$ for a finite totally ordered indexing set $I \subseteq \mathbb{Q}$. Fix a finite support $S \subseteq \mathbb{A}$. If $\mathbb{A}$ is oligomorphic, the tuples in $\mathbb{A}^I$ split into finitely many $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant orbits. Let $O = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot o$ be one such orbit. We shall call O *(S-)ordered* if $o_i \notin S$ and if $o_i < o_j$ whenever $i < j$. By removing the entries in o that repeat or come from S and reordering the rest, we can always find an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant bijection to an S-ordered orbit. An easy observation is that we may focus on a single ordered orbit at a time:

Proposition 4.6. *The following are equivalent:*

(1) *For $d = 0, 1, 2, \ldots$ and any finite $S \subseteq \mathbb{A}$, chains of $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces in $\mathrm{Lin}_{\mathsf{F}}\, \mathbb{A}^d$ are bounded in length;*
(2) *$\mathbb{A}$ is oligomorphic, and $\mathrm{Lin}_{\mathsf{F}}\, O$ has finite length for any ordered orbit O.*

Proof. We have $\mathrm{len}(\mathrm{Lin}_{\mathsf{F}}(\biguplus_i O_i)) = \mathrm{len}(\bigoplus_i \mathrm{Lin}_{\mathsf{F}}\, O_i) = \sum_i \mathrm{len}(\mathrm{Lin}_{\mathsf{F}}\, O_i)$. □

So fix an ordered orbit $O = \mathrm{Aut}(\mathbb{A})_{(S)} \cdot o \subseteq \mathbb{A}^I$. From here we take an inductive approach. By $o|^J$ we mean the restriction of $o : I \to \mathbb{A}$ to $J \subseteq I$; we will often write $o|^{-i}$ instead of $o|^{I \setminus \{i\}}$. Note the image $O|^I$ of O under this projection agrees with $\mathrm{Aut}(\mathbb{A})_{(S)} \cdot o|^J$ and is still ordered.

To anticipate more general statements later, Let E be a finite-dimensional F-vector space — for instance, F itself. Things become more interesting when we lift $(-)|^J$ to a linear $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant map

$$(-)|^J : \mathrm{Lin}_{\mathsf{E}}\, O \to \mathrm{Lin}_{\mathsf{E}}\, O|^J$$
$$v \mapsto v|^J.$$

Many cancellations can occur under $(-)|^J$; the *projection kernel* is the $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace

$$\mathrm{Ker}_{\mathsf{E}}\, O = \bigcap_{i \in I} \ker (-)|^{-i}$$

of $\mathrm{Lin}_{\mathsf{E}}\, O$.

Proposition 4.7. *The following are equivalent:*

(1) *$\mathrm{Lin}_{\mathsf{F}}\, O$ has finite length for every ordered orbit O;*
(2) *$\mathrm{Ker}_{\mathsf{F}}\, O$ has finite length for every ordered orbit O.*

Proof. That 1) implies 2) is clear as $\mathrm{Ker}_{\mathsf{F}}\, O \subseteq \mathrm{Lin}_{\mathsf{F}}\, O$.

To prove the other implication, assume 2) and let $O \subseteq \mathbb{A}^I$. We proceed by induction on $|I|$. If $I = \emptyset$, then O must be the entire singleton $\mathbb{A}^\emptyset = \{()\}$; as $\mathrm{Lin}_{\mathsf{F}}\, O$ has no nontrivial subspaces (let alone finitely supported ones), it has length 1. Now if $|I| \geq 1$, assemble all $|I|$ projection maps into a single map

$$\operatorname{Lin}_{\mathsf{F}} O \to \bigoplus_{i \in I} O|^{-i}$$
$$v \mapsto (v|^{-i})_{i \in I}$$

whose kernel is precisely $\operatorname{Ker}_{\mathsf{F}} O$. We have

$$\operatorname{len}(\operatorname{Lin}_{\mathsf{F}} O) - \operatorname{len}(\operatorname{Ker}_{\mathsf{F}} O) \leq \sum_{i \in I} \operatorname{len}(\operatorname{Lin}_{\mathsf{F}} O|^{-i})$$

which shows that $\operatorname{len}(\operatorname{Lin}_{\mathsf{F}} O)$ is finite from the assumptions. □

We call a vector from the projection kernel *balanced.* As we will see in the next subsection, cogs are a prominent example.

4.2 Cogs

Definition 4.8. Let $O \subseteq \mathbb{A}^I$ be an S-ordered orbit. An *O-duo* $a_\bullet \parallel b_\bullet$ consists of tuples $a_\bullet, b_\bullet \in O$ such that:

1. $a_i < b_i$ for all $i \in I$;
2. $b_i < a_j$ for all $i < j \in I$;
3. for any binary R in $\mathcal{L}_0$ (except for $=$) and $i, j \in I$:

$$R(a_i, b_j) \iff R(b_i, a_j) \iff R(a_i, a_j).$$

Remark 4.9. Conditions (1) and (2) specify a total order on the $2|I|$ atoms in a duo. Moreover, thanks to irreflexivity, each a_i is unrelated to its counterpart b_i. Further, given any $J \subseteq I$, the combined tuple $a|_\bullet^{I \setminus J}; b|_\bullet^J$ satisfies the same relations as $a_\bullet; b_\bullet$, so it lies in O. In particular, taking $J = \{i\}$, there is an automorphism π_i that sends a_i to b_i and fixes all the other elements of $a_\bullet$, $b_\bullet$ and S. Finally, all O-duos are in the same $\operatorname{Aut}(\mathbb{A})_{(S)}$-equivariant orbit.

Definition 4.10. Given $\lambda \in \mathsf{E}$ and an O-duo $a_\bullet \parallel b_\bullet$, the corresponding *$O$-cog* with coefficient λ is the vector

$$\lambda \cdot a_\bullet \between b_\bullet = \sum_{J \subseteq I} (-1)^{|J|} \lambda \cdot a|_\bullet^{I \setminus J}; b_\bullet^J$$

in $\operatorname{Lin}_{\mathsf{E}} O$. The linear span of all O-cogs with coefficients from E is denoted by $\operatorname{Cog}_{\mathsf{E}} O$.

As remarked above, given any two O-duos there is some $\pi \in \operatorname{Aut}(\mathbb{A})_{(S)}$ such that $\pi \cdot (a_\bullet \parallel b_\bullet) = a'_\bullet \parallel b'_\bullet$ and thus $\pi \cdot (\lambda \cdot a_\bullet \between b_\bullet) = \lambda \cdot a'_\bullet \between b'_\bullet$. Hence $\operatorname{Cog}_{\mathsf{E}} O$ is an $\operatorname{Aut}(\mathbb{A})_{(S)}$-equivariant subspace of $\operatorname{Lin}_{\mathsf{E}} O$ and it is generated by cogs based on a single duo.

Proposition 4.11. *$\operatorname{Cog}_{\mathsf{E}} O$ is contained in $\operatorname{Ker}_{\mathsf{E}} O$.*

Proof. Let $O \subseteq \mathbb{A}^I$, let $a_\bullet \parallel b_\bullet$ be an O-duo, and let $i \in I$. The subsets of I come in pairs of J and $J \cup \{i\}$, where J is a subset of $I \setminus \{i\}$. The two tuples $a|_\bullet^J; b|_\bullet^{I \setminus J}$ and $a|_\bullet^{J \cup \{i\}}; b|_\bullet^{I \setminus (J \cup \{i\})}$ differ only on the ith entry. But this difference gets erased under $(-)|^{-i}$, so the two corresponding terms in $\lambda \cdot a_\bullet \between b_\bullet$ will cancel out and hence $(\lambda \cdot a_\bullet \between b_\bullet)|^{-i} = 0$ overall. □

In fact, cogs arise anywhere.

Lemma 4.12. *Suppose $a_\bullet \parallel b_\bullet$ is an O-duo, where $O \subseteq \mathbb{A}^I$ is S-ordered. Given $z \in S$,*

- *write $S' = S \setminus \{z\}$;*
- *let $j \notin I$ be such that $O' = \operatorname{Aut}(\mathbb{A})_{(S')} \cdot (a_\bullet; z) \subseteq \mathbb{A}^{I \cup \{j\}}$ is ordered,*
- *let $X \subseteq \mathbb{A}$ be a finite set containing $\{a_i, b_i \mid i \in I\} \cup S'$ but not z;*
- *let $Y \subseteq \mathbb{A}$ be any finite set disjoint from $X \cup \{z\}$;*

then the $\tau \in \operatorname{Aut}(\mathbb{A})_{(X)}$ afforded by Lemma 4.3 gives us an O'-duo $(a_\bullet; z) \parallel (b_\bullet; \tau(z))$.

Proof. First, notice that $b_\bullet; \tau(z) \in O'$ and that we have the required order relations with z and $\tau(z)$. The remaining condition of Def. 4.8, for any R in $\mathcal{L}_0$, splits into the following cases (and their symmetric versions):

- $R(a_i, b_j) \iff R(a_i, a_j)$ since $a_\bullet \parallel b_\bullet$ is an O-duo;
- $R(a_i, \tau(z)) \iff R(a_i, z)$ since τ is an automorphism that fixes all a_i;
- $R(a_i, z) \iff R(b_i, z)$ since $a_\bullet, b_\bullet \in O$ and $z \in S$;
- $R(z, \tau(z))$ and $R(z, z)$ are both false: $\tau(z)$ is unrelated to z by Lemma 4.3, and R is irreflexive.

□

Starting from an empty duo, we may apply the previous lemma inductively.

Proposition 4.13. *Let $O \subseteq \mathbb{A}^I$ be an S-ordered orbit. Then any $a_\bullet \in O$ can be extended to an O-duo $a_\bullet \parallel b_\bullet$.*

Proof. Enumerate the indices of I as $i_1, \ldots, i_d$. Suppose that we have found $b_{i_1}, \ldots, b_{i_k}$ such that

$$a|_\bullet^{\{i_1, \ldots, i_k\}} \parallel (i_1 \mapsto b_{i_1}, \ldots, i_k \mapsto b_{i_k})$$

is a duo for $O_k = \operatorname{Aut}(\mathbb{A})_{(S \cup \{a_{i_{k+1}}, \ldots, a_{i_d}\})} \cdot a|_\bullet^{\{i_1, \ldots, i_k\}}$ — note that $() \parallel ()$ is certainly a duo for O_0 at the start. If $k < d$, with $z = a_{i_{k+1}}$, $X = \{a_{i_1}, b_{i_1}, \ldots, a_{i_k}, b_{i_k}\} \cup S \cup \{a_{i_{k+2}}, \ldots, a_{i_d}\}$, and $Y = \emptyset$, a straightforward application of Lemma 4.12 yields an atom $b_{i_{k+1}}$ that makes

$$a|_\bullet^{\{i_1, \ldots, i_k, i_{k+1}\}} \parallel (i_1 \mapsto b_{i_1}, \ldots, i_k \mapsto b_{i_k}, i_{k+1} \mapsto b_{i_{k+1}})$$

a duo for O_{k+1}. For $k = d$ we thus obtain the desired duo for $O_d = O$. □

The result below substantiates the slogan that cogs are found everywhere.

Theorem 4.14. *Any $\operatorname{Aut}(\mathbb{A})_{(S)}$-equivariant subspace V of $\operatorname{Lin}_{\mathsf{E}} O$ contains $\operatorname{Cog}_{\mathsf{E}(V)} O$, where $O \subseteq \mathbb{A}^I$ is S-ordered and $\mathsf{E}(V)$ is the subspace spanned by $\{v(a_\bullet) \mid v \in V, a_\bullet \in O\}$ of E.*

Proof. Pick any $v \in V$ and $a_\bullet \in O$; it is enough to show that V contains $v(a_\bullet) \cdot a_\bullet \between b_\bullet$ for some O-duo $a_\bullet \parallel b_\bullet$. Actually, write

$$S' = S \cup \{c_i \mid v(c_\bullet) \neq 0, i \in I\} \setminus \{a_i \mid i \in I\} \supseteq S$$

and put $O' = \operatorname{Aut}(\mathbb{A})_{(S')} \cdot a_\bullet \subseteq O$ — then O' is S'-ordered. By Proposition 4.13, we can find $b_\bullet \in O'$ such that $a_\bullet \parallel b_\bullet$ is an O'-duo and *a fortiori* an O-duo. Take the automorphisms $\pi_{i_1}, \ldots, \pi_{i_d}$ from Remark 4.9, where $i_1, \ldots, i_d$ enumerate I. Now define $v^{(0)} = v$ and

$$v^{(k)} = v^{(k-1)} - \pi_{i_k} \cdot v^{(k-1)}.$$

We can check inductively that for $k = 0, 1, \ldots, d$, with $O^{(k)} = \{c_\bullet \mid v(c_\bullet) \neq 0, \{c_{i_1}, \ldots, c_{i_k}, \ldots, c_{i_d}\} \supseteq \{a_{i_1}, \ldots, a_{i_k}\}\}$ we have

$$v^{(k)} = \sum_{c_\bullet \in O^{(k)}} \sum_{J \subseteq \{i_1, \ldots, i_k\}} (-1)^{|J|} v(c_\bullet) \prod_{j \in J} \pi_j \cdot c_\bullet.$$

But $\{c_{i_1}, \ldots, c_{i_d}\} \supseteq \{a_{i_1}, \ldots, a_{i_d}\}$ means that $c_\bullet = a_\bullet$, so at the end $v^{(d)}$ is the desired O-cog. □

Corollary 4.15. *$\mathrm{Cog}_\mathsf{F}\, O$ has length* 1.

Proof. Let $V \subseteq \mathrm{Cog}_\mathsf{F}\, O$ be a non-zero $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace. Then $\{0\} \subsetneq \mathsf{E}(V) \subseteq \mathsf{E} = \mathsf{F}$ so E must be the entire field F, and by above V must be $\mathrm{Cog}_\mathsf{F}\, O$ itself. □

In light of Propositions 4.6 and 4.7, we will be able to prove the finite length property for an oligomorphic structure with free amalgamation over any field and support if we know $\mathrm{Ker}_\mathsf{F}\, O = \mathrm{Cog}_\mathsf{F}\, O$. Let us now attempt to show that.

4.3 Subvectors

This is a good time to recall a view we have tacitly taken: with O as a standard basis, a vector $v \in \mathrm{Lin}_\mathsf{E}\, O$ is just a finite set of pairs in $\mathsf{E} \times O$. A *subvector* of v is a subset of these pairs.

Now suppose as usual that $O \subseteq \mathbb{A}^I$ is S-ordered. Given $i \in I$ and $a \in O$, we write

$$O^{i:a_i} = \{b \in O \mid b_i = a_i\};$$

this is an $\mathrm{Aut}(\mathbb{A})_{Sa_i}$-orbit, and its projection $O^{i:a_i}|^{-i} = \mathrm{Aut}(\mathbb{A})_{Sa_i} \cdot a|^{-i}$ is ordered. For a vector $v \in \mathrm{Lin}_\mathsf{E}\, O$, by

$$v^{i:a_i} \in \mathrm{Lin}_\mathsf{E}\, O^{i:a_i}$$

we mean the subvector consisting of all pairs in $\mathsf{E} \times O^{i:a_i}$.

Lemma 4.16. *Let $v \in \mathrm{Lin}_\mathsf{E}\, O$ be balanced. Then any projected subvector $v^{i:a_i}|^{-i} \in \mathrm{Lin}_\mathsf{E}\, O^{i:a_i}|^{-i}$ is also balanced.*

Proof. Let $j \in I \setminus \{i\}$. By assumption we have

$$0 = v|^{-j} = \sum_a v^{i:a_i}|^{-j}$$

in $\mathrm{Lin}_\mathsf{E}\, \mathbb{A}^{I \setminus \{j\}}$, so by looking at ith entries we see that each $v^{i:a_i}|^{-j}$ must be the zero vector. Hence so is $v^{i:a_i}|^{-j}|^{-i} = v^{i:a_i}|^{-i}|^{-j}$, which shows that $v^{i:a_i}|^{-i}$ is in the projection kernel. □

So we can try to prove $\mathrm{Ker}_\mathsf{E}\, O \subseteq \mathrm{Cog}_\mathsf{E}\, O$ for any ordered $O \subseteq \mathbb{A}^I$ by inducting on $|I|$.

4.4 Special case: unobstructed vectors

We will begin by showing any $v \in \mathrm{Ker}_\mathsf{E}\, O$ lies in $\mathrm{Cog}_\mathsf{E}\, O$ provided that v satisfies an additional condition which, as we will explain in the subsection, may be assumed without loss of generality. We motivate and introduce this condition now.

For a vector $v \in \mathrm{Lin}_\mathsf{E}\, O$ where O is S-ordered, define:

$$[v] = \{a \in O \mid v(a) \neq 0\}.$$

Take any atom a_i for $a \in [v]$, and let $P, R \in \mathcal{L}_0$ be unary and binary respectively. Then whether $P(a_i)$, $R(a_i, s)$, $R(s, a_i)$ hold in $\mathbb{A}$ for some $s \in S$ is determined by O and i, and so it does not depend on the choice of a.

Now take another atom b_j for some $b \in [v]$. What can be said about $R(a_i, b_j)$? Unless $a = b$, not much. The index j may not even be unique: we may well have $a_i = b_j$ even if $i \neq j$. We want to avoid such confusions:

Definition 4.17. Let O be an S-ordered orbit in $\mathbb{A}^I$, and fix some representative $o \in O$. Call a finite family $V \subseteq O$

(1) *unambiguous* if the assignment $\sqrt{-} : a_i \mapsto o_i$ is a well-defined function on the atoms present in V, i.e., if for any $a, b \in V$, if $a_i = b_j$ then $i = j$;
(2) *unobstructed* if it is unambiguous and for any $a, b \in V$ and $i, j \in I$, either a_i and b_j are unrelated or

$$R(a_i, b_j) \iff R(o_i, o_j)$$

for every binary relation $R \in \mathcal{L}_0$.

(Obviously these properties do not depend on the choice of $o \in O$.) Furthermore, we call a vector $v \in \mathrm{Lin}_\mathsf{E}\, O$ unambiguous or unobstructed if the set $[v]$ is so.

For example, it is easy to see that cogs are unobstructed.

Theorem 4.18. *Let $v \in \mathrm{Ker}_\mathsf{E}\, O$ be unobstructed. Then we have*

$$v = \sum_{k \in K} v(b^{(k)}) \cdot x^{(k)} \between y^{(k)}$$

for some $b^{(k)}, x^{(k)}, y^{(k)} \in O$, where moreover

$$[v] \cup \{x^{(k)}, y^{(k)} \mid k \in K\}$$

is unobstructed.

Note that the last part of the theorem does not trivially follow from the first: $[x^{(k)} \between y^{(k)}]$ may contain some tuples which are absent from $[v]$ due to cancellations with other cogs.

We spend the rest of this subsection giving the proof, which proceeds by induction on $|I|$ for S-ordered orbits $O \subseteq \mathbb{A}^I$ for every S at once.

The base case $I = \emptyset$ is immediate: as O is the singleton $\{()\}$, any vector $v \in \mathrm{Ker}_\mathsf{E}\, O = \mathrm{Lin}_\mathsf{E}\, O = \mathrm{Cog}_\mathsf{E}\, O$ is a cog already; so the theorem says nothing more than $v = v$.

Now suppose that some $i^* \in I$ exists — in fact, let i^* be maximal. Consider an unobstructed vector $v \in \mathrm{Ker}_\mathsf{E}\, O$. We can decompose v into finitely many subvectors to write

$$v = v^{i^*:a_1} + \cdots + v^{i^*:a_m}.$$

Then each projection $v^{i^*:a_j}|^{-i^*}$ lies in $\mathrm{Ker}_\mathsf{E}\, O^{i^*:a_j}|^{-i^*}$ by Lemma 4.16, and we may straightforwardly check that it is unambiguous and unobstructed. It follows from the inductive hypothesis of Theorem 4.18 that

$$v^{i^*:a_j}|^{-i^*} = \sum_{k \in K_j} v^{i^*:a_j}|^{-i^*}(b^{(a_j,k)}) \cdot x^{(a_j,k)} \between y^{(a_j,k)}$$

for some tuples $b^{(a_j,k)}, x^{(a_j,k)}, y^{(a_j,k)} \in O^{i^*:a_j}|^{-i^*}$ such that $[v^{i^*:a_j}|^{-i^*}] \cup \{x^{(a_j,k)}, y^{(a_j,k)} \mid k \in K_j\}$ is unobstructed. We can return to O by adding a_j back as the i^*th term to every tuple: we get

$$v^{i^*:a_j} = \sum_{k \in K_j} \Big(v(b^{(a_j,k)}; a_j) \cdot x^{(a_j,k)} \between y^{(a_j,k)}\Big); a_j,$$

where the family

$$U_j = [v^{i^*:a_j}] \cup \{(x^{(a_j,k)}; a_j), (y^{(a_j,k)}; a_j) \mid k \in K_j\}$$

is easily seen to be unobstructed. We can ask for more:

Claim 4.19. *We may choose $x^{(a_j,k)}$'s, and $y^{(a_j,k)}$'s so that*

$$V_0 = [v], \qquad V_j = V_{j-1} \cup U_j$$

is unobstructed for $j = 1, 2, \ldots, m$.

Proof. Let A_j denote the atoms that appear in $[v^{i^*:a_j}]$. Given any automorphism $\pi \in \mathrm{Aut}(\mathbb{A})$ that fixes S as well as A_j, using the tuples $\pi \cdot x^{(a_j,k)}$ and $\pi \cdot y^{(a_j,k)}$ gives us another decomposition of $v^{i^*:a_j}|^{-i^*}$ that is unobstructed. We will show that composing such automorphisms — which will be provided by Lemma 4.3 — suffices to make the claim hold.

Assume that V_{j-1} is unobstructed. Suppose that V_j is not even unambiguous. Then for some $b \in V_j \setminus V_{j-1} \subseteq U_j$ and $c \in V_{j-1}$ have $b_i = c_{i'}$ with $i \neq i'$. Now if we were to have $b_i \in A_j$ (e.g., if $b_i = a_j$), then $b_i = d_{i''}$ for some $d \in [v^{i^*:a_j}] \subseteq [v]$. Since U_j is unambiguous, $i = i''$; and since V_{j-1} is unambigous, $i'' = i'$. This contradicts the assumption that $i \neq i'$.

So the ambiguous atom b_i does not belong to A_j. In other words, the set X of all atoms that appear in V_j except b_i contains A_j (and a_j). Apply Lemma 4.3 to get an automorphism τ which fixes $S \cup X$ but sends b_i to a fresh atom. Then, in the new family

$$V_{j-1} \cup \{(\tau \cdot x^{(a_j,k)}; a_j), (\tau \cdot y^{(a_j,k)}; a_j) \mid k \in K_j\},$$

the ambiguous atoms are precisely the ones in the old family minus b_i — these are all fixed by τ. We continue this way until all ambiguous atoms are freshened.

Hence assume V_j is unambiguous. Suppose b_i and $c_{i'}$, for some $b, c \in V_j$, are the reason why V_j fails to be unobstructed. To have b_i in A_j is impossible: otherwise $b_i = d_{i''}$ for some $d \in [v^{i^*:a_j}] \subseteq [v]$ and since V_j is unambiguous, we get $i = i''$. But d_i (hence b_i) cannot be part of an obstruction in V_j because both U_j and V_{j-1} are unobstructed. Symmetrically, we see that $c_{i'} \notin A_j$.

Since V_{j-1} is unobstructed, b (or, symmetrically, c) must be one of the $x^{(a_j,k)}; a_j$ or $y^{(a_j,k)}; a_j$. This time, split all atoms that appear in V_j except b_i into two sets X and Y: put any atom $c_{i'}$ which makes b_i an obstruction in Y, and put everything else in X. Then X contains A_j as discussed above; it also contains $\{x_i^{(a_j,k)}, y_i^{(a_j,k)} \mid k \in K_j, i \in I\}$. Use Lemma 4.3 to get $\tau \in \mathrm{Aut}(\mathbb{A})_{(S \cup X)}$ which sends b_i to a fresh atom disjoint from $S \cup X \cup Y \cup \{b_i\}$ that is unrelated to everything in Y. Observe that the new family

$$V_{j-1} \cup \{(\tau \cdot x^{(a_j,k)}; a_j), (\tau \cdot y^{(a_j,k)}; a_j) \mid k \in K_j\},$$

remains unambiguous. Moreover, the obstructions here are precisely the ones from V_j except the ones that involve b_i, which are all fixed by τ. So we may repeat this process until all obstructions are removed. □

Having chosen the fresh atoms carefully, let X consist of all the atoms appearing in the unobstructed family V_m. It will be convenient to fix a representative $o \in O$ and recall the function

$$\sqrt{-} : X \to \{o_1, \ldots, o_d\}$$

introduced in Definition 4.17. We now add a new element b^* to the finite $\mathcal{L}$-structure $S \cup X \subseteq \mathbb{A}$ so that $(x^{(a_j,k)}; a_j) \parallel (y^{(a_j,k)}; b^*)$ becomes a well-formed O-duo for each j. To this end we define b^* to be greater (wrt. >) than every atom in X, and the remaining relations are defined to make b^* mimic the greatest atom in o:

$$\begin{aligned}
b^* &> c \quad \text{for } c \in X \\
b^* > s &\iff \mathbb{A} \models o_{i^*} > s \\
P(b^*) &\iff \mathbb{A} \models P(o_{i^*}) \\
R(b^*, s) &\iff \mathbb{A} \models R(o_{i^*}, s) \\
R(s, b^*) &\iff \mathbb{A} \models R(s, o_{i^*}) \\
R(b^*, c) &\iff \mathbb{A} \models R(o_{i^*}, \sqrt{c}) \text{ for } c \in X \\
R(c, b^*) &\iff \mathbb{A} \models R(\sqrt{c}, o_{i^*}) \text{ for } c \in X
\end{aligned}$$

for every unary predicate P and binary relation R in $\mathcal{L}_0$. The last two clauses are well defined (i.e. they do not depend on the choice of $a \in V_m$ that contains the atom c) since V_m is unambiguous. Moreover, since V_m is unobstructed, our definition does not introduce any forbidden substructures:

Claim 4.20. *$S \cup X \cup \{b^*\}$ embeds into $\mathbb{A}$.*

Proof. If not, by Lemma 4.4 there is a forbidden $\mathcal{L}_0$-structure F of pairwise related elements which embeds into the $\mathcal{L}_0$-reduct of $S \cup X \cup \{b^*\}$, via ϕ say. Extend $\sqrt{-}$ from above to:

$$\sqrt{-} : X \cup S \cup \{b^*\} \to \{o_1, \ldots, o_d\} \cup S$$

by putting $\sqrt{b^*} = o_{i^*}$ and $\sqrt{s} = s$ for $s \in S$. We will check that $\sqrt{-} \circ \phi$ then embeds F into the $\mathcal{L}_0$-reduct of $\mathbb{A}$, namely $\mathbb{A}_0$, and reach a contradiction.

So let $P \in \mathcal{L}_0$ be unary and let $R \in \mathcal{L}_0$ be binary. It is immediate that $\sqrt{-}$ preserves and reflects P for all atoms in $X \cup S$ and for b^* — we do not even need to know they are in the image of ϕ. We can also see that $\sqrt{-}$ preserves and reflects R whenever at least one of the arguments is from $S \cup \{b^*\}$. Now consider the remaining case of $\phi(f), \phi(f') \in X$. As f and f' are related in F, their images under ϕ must be related in $\phi(F)$. But the family V_m that gave rise to X is unobstructed, which forces $\sqrt{-}$ to preserve and reflect R here as well. Since $\mathcal{L}_0$ only has unary and binary relations, this means that $\sqrt{\phi(F)} \subseteq \mathbb{A}_0$ is in C_0 yet embeds F; this is a contradiction. □

Using homogeneity we may assume that $S \cup X \cup \{b^*\} \subseteq \mathbb{A}$.

Claim 4.21. *$x^{(a_j,k)}; a_j \parallel y^{(a_j,k)}; b^*$ forms an O-duo for $1 \leq j \leq m$ and $k \in K_j$. Furthermore, the family*

$$[v] \cup \{(x^{(a_j,k)}; a_j), (y^{(a_j,k)}; b^*) \mid k \in K_j, 1 \leq j \leq m\}$$

is unobstructed.

Proof. Recall that $x^{(a_j,k)} \parallel y^{(a_j,k)}$ is a duo for $O^{i^*:a_j}|^{-i^*}$. Since $x^{(a_j,k)}, y^{(a_j,k)} \in X$, the remaining conditions for $x^{(a_j,k)}; a_j \parallel y^{(a_j,k)}; b^*$ being an O-duo follow directly from our definition of relations on b^*.

The family in the claim is unambiguous since V_m is unambiguous and b^* is a fresh atom. Similarly, it is unobstructed since V_m is unobstructed, and the additional conditions that involve b^* follow from the relations that we imposed on b^*.

BK: This argument feels a bit shaky; I will think about it some more. □

Claim 4.22. *We have $v = \sum_{j=1}^{m} \sum_{k \in K_j} v(b^{(a_j,k)}; a_j) \cdot (x^{(a_j,k)}; a_j \between y^{(a_j,k)}; b^*)$.*

Proof. Observe that, by definition of cogs, for any O-duo $a;c \parallel b;d$ with $c,d \in \mathbb{A}$ we have

$$(a;c) \parr (b;d) = (a \parr b);c - (a \parr b);d.$$

Using this, calculate:

$$\begin{aligned}
&\sum_{j=1}^{m} \sum_{k\in K_j} v(b^{(a_j,k)}) \cdot (x^{(a_j,k)};a_j \parr y^{(a_j,k)};b^*) \\
&= \sum_{j=1}^{m} \sum_{k\in K_j} \left(v(b^{(a_j,k)}) \cdot x^{(a_j,k)} \parr y^{(a_j,k)}\right);a_j \\
&\quad - \sum_{j=1}^{m} \sum_{k\in K_j} \left(v(b^{(a_j,k)}) \cdot x^{(a_j,k)} \parr y^{(a_j,k)}\right);b^* \\
&= \sum_{j=1}^{m} v^{i^*:a_j} - \sum_{j=1}^{m} v^{i^*:a_j}|^{-i^*};b^* \\
&= v - \sum_{j=1}^{m} v^{i^*:a_j}|^{-i^*};b^*,
\end{aligned}$$

and it suffices to show the last sum vanishes. Recall from Proposition 4.11 that cogs are balanced. By projecting away the i^*th entry, we obtain

$$0 = 0 - \left(\sum_{j=1}^{m} v^{i^*:a_j}|^{-i^*};b^*\right)\Bigg|^{-i^*}.$$

But we can simply add back b^* as the i^*th entry, yielding $0 = \sum_{j=1}^{m} v^{i^*:a_j}|^{-i^*};b^*$ as needed — replacing the i^*th entry by a common atom achieves the same effect of projecting it away. □

This completes the proof of Theorem 4.18: we have shown any balanced vector v is spanned by cogs as long as $[v]$ is unobstructed. Let us lift this restriction.

4.5 Removing ambiguities and obstructions

Theorem 4.23. *Let $v \in \mathrm{Ker}_{\mathsf{E}}\, O$ be arbitrary. Then (assuming as before that $\mathcal{L}_0$ is at most binary) we have*

$$v = \sum_{k\in K} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \parr y_\bullet^{(k)}$$

for some $b_\bullet^{(k)}, x_\bullet^{(k)}, y_\bullet^{(k)} \in O$.

As a reminder, $O \subseteq \mathbb{A}^I$ is an S-ordered orbit and E is a finite-dimensional F-vector space. We induct on $|I|$, noting that when $I = \emptyset$ we are just saying $v = v() \cdot (\parr)$. Hereafter assume $I \neq \emptyset$.

Proposition 4.24. *Let $v \in \mathrm{Ker}_{\mathsf{E}}\, O$. Then we can find $b_\bullet^{(k)}, x_\bullet^{(k)}, y_\bullet^{(k)} \in O$ so that*

$$\ddot{v} = v - \sum_{k\in K_1} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \parr y_\bullet^{(k)}$$

satisfies $\ddot{v}(O) = v(O) \subseteq \mathsf{E}$ and $O(\ddot{v})$ is unambiguous.

Proof. Assume there is an ambiguity in $O(v)$, i.e., there are $a_\bullet, a'_\bullet \in O(v)$ such that $a_i = a'_{i'}$ but $i \neq i'$ (so $|I| \geq 2$). Let us prevent the atom-index pair (a_i, i) from causing ambiguities. By Lemma 4.16, the projected subvector $v^{i:a_i}|^{-i}$ belongs to $\mathrm{Ker}_{\mathsf{E}}\, O^{i:a_i}|^{-i}$; by the inductive hypothesis of Theorem 4.23 at hand, we have

$$v^{i:a_i} = v^{i:a_i}|^{-i};a_i = \sum_{k\in K} v(b_\bullet^{(k)};a_i) \cdot (x_\bullet^{(k)} \parr y_\bullet^{(k)});a_i.$$

We can invoke Lemma 4.3 to find an automorphism τ which fixes all of the atoms except a_i that appear in $x_\bullet^{(k)}, y_\bullet^{(k)}$, $k \in K$ and in $b_\bullet \in O(v)$. It follows from the appropriate modification of Lemma 4.12 that we get O-duos $x_\bullet^{(k)};a_i \parallel y_\bullet^{(k)};\tau\cdot a_i$ for $k \in K$. Observe that

$$\begin{aligned}
&v^{i:a_i} - \sum_{k\in K} v(b_\bullet^{(k)};a_i) \cdot (x_\bullet^{(k)};a_i \parr y_\bullet^{(k)};\tau\cdot a_i) \\
&= v^{i:a_i} - \sum_{k\in K} v(b_\bullet^{(k)};a_i) \cdot (x_\bullet^{(k)} \parr y_\bullet^{(k)});a_i \\
&\qquad + \sum_{k\in K_1} v(b_\bullet^{(k)};a_i) \cdot (x_\bullet^{(k)} \parr y_\bullet^{(k)});\tau\cdot a_i
\end{aligned}$$

is equal to $v^{i:a_i}|^{-i};\tau\cdot a_i$, so

$$\begin{aligned}
v' &= v - \sum_{k\in K} v(b_\bullet^{(k)};a_i) \cdot (x_\bullet^{(k)};\tau\cdot a_i \parr y_\bullet^{(k)};a_i) \\
&= v - v^{i:a_i} + v^{i:a_i}|^{-i};\tau\cdot a_i
\end{aligned}$$

is the vector obtained by changing the ith entry in every tuple of v from a_i to $\tau\cdot a_i$, a fresh atom disjoint from all atoms present. Then $v'(O) = v(O)$. Moreover, we can directly check that an ambiguous atom-index pair (b_j, j) in $O(v')$ cannot be $(\tau\cdot a_i, i)$, and hence it must already be one in $O(v)$ except it cannot be (a_i, i). Therefore we may remove all ambiguities by iterating this process. □

Next we tackle the obstructions.

Proposition 4.25. *Let $\ddot{v} \in \mathrm{Ker}_{\mathsf{E}}\, O$ be such that $O(\ddot{v})$ is unambiguous. Then we can find $b_\bullet^{(k)}, x_\bullet^{(k)}, y_\bullet^{(k)} \in O$ so that*

$$\acute{\ddot{v}} = \ddot{v} - \sum_{k\in K_2} \ddot{v}(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \parr y_\bullet^{(k)}$$

satisfies $\acute{\ddot{v}}(O) = \ddot{v}(O)$ and $O(\acute{\ddot{v}})$ is unobstructed.

Proof. We follow the same strategy: let the atom a_i be an obstruction with some other $a'_{i'}$ in $O(\ddot{v})$. Consider the projected subvector $\ddot{v}^{i:a_i}|^{-i}$. By Lemma 4.16 and the inductive hypothesis of Theorem 4.23, we can write

$$\ddot{v}^{i:a_i} = \sum_{k\in K} \ddot{v}(b_\bullet^{(k)};a_i) \cdot (x_\bullet^{(k)} \parr y_\bullet^{(k)});a_i.$$

Leveraging no algebraicity like we did at the beginning of the proof of Claim 4.19, we may assume that $O(\ddot{v}) \cup \{(x_\bullet^{(k)};a_i), (y_\bullet^{(k)};a_i) \mid k \in K\}$ is unambiguous.

Split the atoms except a_i appearing in that family into two parts X and Y, where Y consists of all the atoms equal to some $a'_{i'}$ that make a_i an obstruction. Then $x_{i''}^{(k)}, y_{i''}^{(k)}$ cannot belong to Y for any $i'' \in I$. Indeed, if say $x_{i''}^{(k)}$ is equal to an obstructive atom $a'_{i'}$, we must have $i'' = i'$ by unambiguity; we now have a contradiction: $a'_{i'} = x_{i''}^{(k)}$ and a_i necessarily satisfy the right binary relations, since $x_\bullet^{(k)};a_i \in O$.

Next, we invoke the Lemma 4.3 to find an automorphism $\tau \cdot \mathrm{Aut}(\mathbb{A})$ making $\tau\cdot a_i$ greater than a_i, distinct from any $x \in X$, and unrelated to any $y \in Y$. By Lemma 4.12, we have O-duos $x_\bullet^{(k)}; a_i \parallel y_\bullet^{(k)}; \tau \cdot a_i$ for $k \in K$. The refined vector

$$\begin{aligned} \ddot{v}' &= \ddot{v} - \sum_{k \in K} \ddot{v}(b_\bullet^{(k)}; a_i) \cdot (x_\bullet^{(k)}; \tau \cdot a_i \between y_\bullet^{(k)}; a_i) \\ &= \ddot{v} - \ddot{v}^{i:a_i} + \ddot{v}^{i:\tau \cdot a_i} \end{aligned}$$

satisfies $\ddot{v}'(O) = \ddot{v}(O)$. As we are just changing a_i to the fresh atom $\tau \cdot a_i$ in the ith entry of every tuple appearing in $\ddot{v}$, the family $O(\ddot{v}')$ remains unambiguous; note also that a_i can only appear in the ith entry as $O(\ddot{v})$ is unambiguous, so we have completely removed a_i from $\ddot{v}'$. But $\tau \cdot a_i$ by design cannot cause an obstruction in $O(\ddot{v}')$, and consequently any atom causing obstructions in $O(\ddot{v}')$ must already do so in $O(\ddot{v})$ except that it cannot be a_i. Hence we can repeat this procedure until all obstructions are excised. □

We are at last in a position to prove the inductive step of Theorem 4.23. Given $v \in \mathrm{Ker}_{\mathsf{E}}\, O$, we apply Propositions 4.24 and 4.25 to obtain

$$\acute{\ddot{v}} = v - \sum_{k \in K_1 \cup K_2} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)}$$

where $\acute{\ddot{v}}(O) = \ddot{v}(O) = v(O)$ and $O(\acute{\ddot{v}})$ is unobstructed. Recalling Proposition 4.11, we see that $\acute{\ddot{v}} \in \mathrm{Ker}_{\mathsf{E}}\, O$. The special Theorem 4.18 tells us that $\acute{\ddot{v}}$ is a sum of O-cogs with coefficients from $\acute{\ddot{v}}(O) = v(O)$. It follows that so is

$$v = \acute{\ddot{v}} + \sum_{k \in K_1 \cup K_2} v(b_\bullet^{(k)}) \cdot x_\bullet^{(k)} \between y_\bullet^{(k)},$$

which establishes the general Theorem 4.23.

4.6 All those equivariant subspaces

We finish this section with an important corollary of Theorem 4.23. Let $O_1 \subseteq \mathbb{A}^{I_1}, \ldots, O_n \subseteq \mathbb{A}^{I_n}$ all be S-ordered orbits. Then $\mathrm{len}(\mathrm{Lin}_{\mathsf{F}}(O_1 \uplus \cdots \uplus O_n)) = 2^{|I_1|} + \cdots + 2^{|I_n|}$; in fact, we know and can characterise all the $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces of $\mathrm{Lin}_{\mathsf{F}}(O_1 \uplus \cdots \uplus O_n) \simeq \mathrm{Lin}_{\mathsf{F}}(O_1) \oplus \cdots \oplus \mathrm{Lin}_{\mathsf{F}}(O_n)$.

Local coefficients. First we set up some notations. Consider the $\sum_k 2^{|I_k|}$ projected S-ordered orbits $O_k|^J$ for $1 \le k \le n, J \subseteq I_k$. Suppose

$$f : O_k|^J \to O_{k'}|^{J'}$$

is an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant bijection. Take any $o_\bullet \in O_k|^J$, and enumerate its entries as $o_1 < \cdots < o_{|J|}$. Similarly, enumerate the entries of $f(o_\bullet)$ as $o'_1 < \cdots < o'_{|J'|}$. Then $\{o_1, \ldots, o_{|J|}\} = \{o'_1, \ldots, o'_{|J'|}\}$ because $\mathbb{A}$ has no algebraicity; since the orbits are ordered, we must have $|J| = |J'|$ and $o_1 = o'_1, \ldots, o_{|J|} = o'_{|J'|}$. That is, f must be the obvious function that reindexes a J-tuple to a J'-tuple — hence we will write $o_\bullet^{/J'}$ instead of $f(o_\bullet)$, leaving f implicit.

Now, let $Q_1 = O_{k_1}|^{J_1}, \ldots, Q_t = O_{k_t}|^{J_t}$ be the distinct S-ordered orbits up to $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant bijections, which we enumerate in such a way that $|J_1| \ge |J_2| \ge \cdots \ge |J_t| = 0$.

Definition 4.26. For $i = 1, \ldots, t$, let P_i consist of pairs (k, J) such that $O_k|^J$ is $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariantly isomorphic to Q_i. Assemble all $|P_i|$ projections into a single map

$$(-){\restriction_i} : \mathrm{Lin}_{\mathsf{F}}(O_1 \uplus \cdots \uplus O_n) \to \mathrm{Lin}_{\mathsf{F}^{P_i}}\, Q_i.$$

More precisely $(v_1, \ldots, v_n){\restriction_i}(a_\bullet)$ is the P_i-tuple whose entry at (k, J) is $v_k|^J(a_\bullet^{/J}) \in \mathsf{F}$. It is straightforward to check that $(-){\restriction_i}$ is $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant and linear.

Let $W \subseteq \mathrm{Lin}_{\mathsf{F}}(O_1 \uplus \cdots \uplus O_n)$ be an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace. Using the t finite-dimensional vector spaces $W{\restriction_1}(Q_1) \subseteq \mathsf{F}^{P_1}, \ldots, W{\restriction_t}(Q_t) \subseteq \mathsf{F}^{P_t}$ we define $\widetilde{W}$, which consists of all vectors $v \in \mathrm{Lin}_{\mathsf{F}}(O_1 \uplus \cdots \uplus O_n)$ such that

$$v{\restriction_1}(Q_1) \subseteq W{\restriction_1}(Q_1), \ldots, v{\restriction_t}(Q_t) \subseteq W{\restriction_t}(Q_t).$$

Then $\widetilde{W}$ is an $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace that contains W. It turns out these two are equal:

Lemma 4.27.

$$\begin{aligned} &\widetilde{W} \cap \ker({\restriction_{i+1}}) \cap \cdots \cap \ker({\restriction_t}) \\ &\subseteq W \cap \ker({\restriction_{i+1}}) \cap \cdots \cap \ker({\restriction_t}). \end{aligned}$$

In particular $\widetilde{W} \subseteq W$ when $i = t$.

Proof of the lemma. by induction on i. When $i = 0$, this containment is trivial:

Claim 4.28. $\ker({\restriction_1}) \cap \ker({\restriction_2}) \cdots \cap \ker({\restriction_t}) = \{0\}$.

Proof. Let $v = (v_1, \ldots, v_n) \in \ker({\restriction_1}) \cap \ker({\restriction_2}) \cdots \cap \ker({\restriction_t})$. Each (k, I_k) belongs to some P_i; that $v{\restriction_i} = 0$ implies $0 = v_k|^{I_k} = v_k$. □

To prove the containment for $i+1$, we allow $v{\restriction_{i+1}}$ to be non-zero. But $v{\restriction_{i+1}}$ satisfies the next best property:

Claim 4.29. *The image of*

$$\widetilde{W} \cap \ker({\restriction_{i+2}}) \cap \cdots \cap \ker({\restriction_t})$$

under ${\restriction_{i+1}}$ *is contained in* $\mathrm{Ker}_{W{\restriction_{i+1}}(Q_{i+1})}\, Q_{i+1}$.

Proof. Take any $v = (v_1, \ldots, v_n)$ satisfying

$$v{\restriction_{i+2}} = 0, \ldots, v{\restriction_t} = 0$$

from $\widetilde{W}$. That $v{\restriction_{i+1}} \in \mathrm{Lin}_{W{\restriction_{i+1}}(Q_{i+1})}\, Q_{i+1}$ is clear from the definition of $\widetilde{W}$. Recall that $Q_{i+1} = O_{k_{i+1}}|^{J_{i+1}}$. Given $j \in J_{i+1}$, we need to prove that $v{\restriction_{i+1}}|^{-j} = 0$.

To do so, take $(k, J) \in P_{i+1}$; the unique monotone bijection between J_{i+1} and J restricts to one between $J_{i+1} \setminus \{j\}$ and $J \setminus \{j'\}$. Now $(k, J \setminus \{j'\})$ belongs to some $P_{i'}$ with $i' > i+1$, so $v{\restriction_{i'}} = 0$. We calculate that $v{\restriction_{i+1}}|^{-j}(a_\bullet)_{k,J}$ is equal to

$$\sum_{b_\bullet \in O_k, b|_\bullet^{J \setminus \{j'\}} = a_\bullet^{/J \setminus \{j'\}}} v_k(b_\bullet),$$

and that so is $0 = v{\restriction_{i'}}(a_\bullet^{/J \setminus \{j'\}})_{k, J \setminus \{j'\}}$. □

Now Theorem 4.23 tells us that $\mathrm{Ker}_{\widetilde{W}{\restriction_{i+1}}(Q_{i+1})}\, Q_{i+1} \subseteq \mathrm{Cog}_{\widetilde{W}{\restriction_{i+1}}(Q_{i+1})}\, Q_{i+1}$, which is good news:

Claim 4.30. $\mathrm{Cog}_{W{\restriction_{i+1}}(Q_{i+1})}\, Q_{i+1}$ *is contained in the image of*

$$W \cap \ker({\restriction_{i+2}}) \cap \cdots \cap \ker({\restriction_t})$$

under ${\restriction_{i+1}}$.

Proof. Let $w{\restriction_{i+1}}(a_\bullet) \in W{\restriction_{i+1}}(Q_{i+1})$. Let S' consist of S together with every atom appearing in w but not in $a_\bullet$. We generalise the proof of Theorem 4.14.

Start by applying Proposition 4.13 and Remark 4.9 to get automorphisms π_j for $j \in J_{i+1}$ such that $a_\bullet \parallel \prod_{j\in J_{i+1}} \pi_j \cdot a_\bullet$ is an Q_{i+1}-duo, where π_j fixes S' and $a_{j'}, \pi_{j'} \cdot a_{j'}$ for $j' \in J \setminus \{j\}$. Put

$$w' = \prod_{j\in J_{i+1}} (1-\pi_j)\cdot w \in W.$$

Given $1 \le i' \le t$, observe that as no more atoms can appear in $w{\restriction_{i'}}$ than in w, we have

$$\begin{aligned} w'{\restriction_{i'}} &= \prod_{j\in J_{i+1}} (1-\pi_j)\cdot w{\restriction_{i'}} \\ &= \sum_{b_\bullet \in Q_{i'}, \{b_j|j\}\supseteq\{a_j|j\}} \sum_{J'\subseteq J} (-1)^{|J'|} w{\restriction_{i'}}(b_\bullet) \prod_{j\in J'} \pi_j \cdot b_\bullet. \end{aligned}$$

Suppose $\{b_j \mid j \in J_{i'}\} \supseteq \{a_j \mid j \in J_{i+1}\}$. Then $i' \le i+1$; if $i' = i+1$, we must have $b_\bullet = a_\bullet$. We therefore have

$$w'{\restriction_{i+1}} = w{\restriction_{i+1}}(a_\bullet)\cdot a_\bullet \between \prod_{j\in J_{i+1}} \pi_j \cdot a_\bullet$$

and $w'{\restriction_{i+2}} = 0, \ldots, w'{\restriction_t} = 0$. This proves that $(W \cap \ker {\restriction_{i+2}} \cap \cdots \cap \ker {\restriction_t}){\restriction_{i+1}}$ contains $\mathrm{Cog}_{W{\restriction_{i+1}}(Q_{i+1})}\, Q_{i+1}$. ◻

This is enough to establish Lemma 4.27 for $i+1$ assuming the result for i. Indeed, given $v \in \widetilde{W} \cap \ker({\restriction_{i+2}}) \cap \cdots \cap \ker({\restriction_t})$, we can find $w \in W \cap \ker({\restriction_{i+2}}) \cap \cdots \cap \ker({\restriction_t}) \subseteq \widetilde{W}$ such that

$$v{\restriction_{i+1}} = w{\restriction_{i+1}}$$

by the preceding claims. But $(v-w){\restriction_{i+1}} = 0$ — that is, $v - w$ lies in in $\ker({\restriction_{i+1}})$ as well as $\widetilde{W} \cap \ker({\restriction_{i+2}}) \cap \cdots \cap \ker({\restriction_t})$. It follows from the inductive hypothesis that

$$v - w \in W \cap \ker({\restriction_{i+2}}) \cap \cdots \cap \ker({\restriction_t}),$$

so $v = (v-w)+w$ is a member of $W \cap \ker({\restriction_{i+2}}) \cap \cdots \cap \ker({\restriction_t})$ as well.

Lengths. Let W, W' be two $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces of $\mathrm{Lin}_{\mathsf{F}}(O_1 \uplus \cdots \uplus O_n)$. If we have $W{\restriction_1}(Q_1) = W'{\restriction_1}(Q_1), \ldots, W{\restriction_t}(Q_t) = W'{\restriction_t}(Q_t)$, then $W = \widetilde{W} = \widetilde{W'} = W'$ by Lemma 4.27. An immediate consquence is:

Proposition 4.31. *Let $W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_l$ be a chain of $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces in $\mathrm{Lin}_{\mathsf{F}}(O_1 \uplus \cdots \uplus O_n)$. Then $l \le 2^{|I_1|} + \cdots + 2^{|I_n|}$.*

Proof. We obtain t chains

$$\begin{aligned} W_0{\restriction_1}(Q_1) &\subseteq W_1{\restriction_1}(Q_1) \subseteq \cdots \subseteq W_l{\restriction_1}(Q_1) \subseteq \mathsf{F}^{P_1}, \\ W_0{\restriction_2}(Q_2) &\subseteq W_1{\restriction_2}(Q_2) \subseteq \cdots \subseteq W_l{\restriction_2}(Q_2) \subseteq \mathsf{F}^{P_2}, \\ &\vdots \\ W_0{\restriction_t}(Q_t) &\subseteq W_1{\restriction_t}(Q_t) \subseteq \cdots \subseteq W_l{\restriction_t}(Q_t) \subseteq \mathsf{F}^{P_t}. \end{aligned}$$

At each of the l steps, one of the t containments must be strict. Hence $l \le |P_1| + |P_2| + \cdots + |P_t| = 2^{|I_1|} + \cdots + 2^{|I_t|}$. ◻

It follows that any $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace of $\mathrm{Lin}_{\mathsf{F}}(O_1 \uplus \cdots \uplus O_n)$ is finitely generated. We can compute the local coefficients of such subspaces easily:

Remark 4.32. For $v \in \mathrm{Lin}_{\mathsf{F}}(O_1 \uplus \cdots \uplus O_n)$, let $\langle v\rangle$ denote the $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspace it generates. Then:

(1) $\langle v\rangle{\restriction_i}(Q_i)$ is the subspace of F^{P_i} generated by vectors of the form $v{\restriction_i}(a_\bullet)$, which is zero unless every atom appearing in $a_\bullet$ appears in v — there are only finitely many such $a_\bullet$'s;
(2) $\langle v, v'\rangle{\restriction_i}(Q_i) = \langle v\rangle{\restriction_i}(Q_i) + \langle v'\rangle{\restriction_i}(Q_i)$.

We may now exhibit a chain of $\mathrm{Aut}(\mathbb{A})_{(S)}$-equivariant subspaces whose length is precisely $\sum_{i=1}^t 2^{|P_i|}$, generalising [2, Corollary 4.12]. Take any $(k, J) \in P_i$. Pick some $a_\bullet \in O_k$, and let $\pi_j, j \in I_k$ be the automorphisms from Proposition 4.13. Define a vector

$$\begin{aligned} v^i_{k,J} &\in \mathrm{Lin}_{\mathsf{F}}(O_1 \uplus \cdots \uplus O_n) \\ &\simeq \mathrm{Lin}_{\mathsf{F}}(O_1) \oplus \cdots \oplus \mathrm{Lin}_{\mathsf{F}}(O_n) \end{aligned}$$

with $\prod_{j\in J}(1-\pi_j)\cdot a_\bullet$ as its kth component and zero everywhere else. Then

$$\langle v^i_{k,J}\rangle{\restriction_{i'}}(Q_{i'})_{k',J'} = \begin{cases} \mathsf{F} & \text{if } k = k' \text{ and } J \subseteq J', \\ \{0\} & \text{otherwise.} \end{cases}$$

Enumerating each P_i as $(k^i_1, J^i_1), (k^i_2, J^i_2), \ldots, (k^i_{|P_i|}, J^i_{|P_i|})$, we obtain a chain

$$\begin{aligned} &\langle\rangle \\ &\subsetneq \langle v^t_{k^t_1,J^t_1}\rangle \\ &\subsetneq \langle v^t_{k^t_1,J^t_1}, v^t_{k^t_2,J^t_2}\rangle \\ &\subsetneq \cdots \\ &\subsetneq \langle v^t_{k^t_1,J^t_1}, v^t_{k^t_2,J^t_2}, \ldots, v^t_{k^t_{|P_t|},J^t_{|P_t|}}\rangle \\ &\subsetneq \langle v^t_{k^t_1,J^t_1}, v^t_{k^t_2,J^t_2}, \ldots, v^t_{k^t_{|P_t|},J^t_{|P_t|}}, v^{t-1}_{k^{t-1}_1,J^{t-1}_1}\rangle \\ &\subsetneq \cdots \end{aligned}$$

of length $|P_t| + |P_{t-1}| + \cdots + |P_1| = 2^{|I_1|} + \cdots + 2^{|I_n|}$. With the upper bound in Proposition 4.31, we conclude:

Theorem 4.33. $\mathrm{len}(\mathrm{Lin}_{\mathsf{F}}(O_1 \uplus \cdots \uplus O_n)) = 2^{|I_1|} + \cdots + 2^{|I_n|}$.

Acknowledgements

The first-named author is partially funded by the NCN grant 2022/45/N/ST6/03242.
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Received 20 February 2007; revised 12 March 2009; accepted 5 June 2009