More Vector Spaces with Atoms of Finite Lengths

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***Abstract*—*CRITICAL: Do Not Use Symbols, Special Characters, Footnotes, or Math in Paper Title or Abstract.**

I. Introduction

II. Rado graph, sans cogs

$\mathbb{A}$ is:

- oligomorphic if, for $d = 0, 1, 2, \ldots$, $\mathbb{A}^d$ only has finitely many orbits;
- $\mathfrak{F}$-oligomorphic if, for $d = 0, 1, 2, \ldots$, $\mathrm{Lin}_{\mathfrak{F}}\, \mathbb{A}^d$ only has finitely long chains.

Of note: with Stirling numbers of the second kind and Gaussian 2-binomial coefficients, the orbit counts are given by

$$\#\mathbb{N}^d = \sum_{k=0}^{d} \left\{ {d \atop k} \right\}$$

$$\#\mathbb{Q}^d = \sum_{k=0}^{d} \left\{ {d \atop k} \right\} k!$$

$$\#\mathbb{G}^d = \sum_{k=0}^{d} \left\{ {d \atop k} \right\} 2^{\binom{k}{2}}$$

$$\#\mathbb{V}_\infty^d = \sum_{k=0}^{d} \begin{bmatrix} d \\ k \end{bmatrix}_2$$

$$\#\mathbb{W}_\infty^d = \sum_{k=0}^{d} \begin{bmatrix} d \\ k \end{bmatrix}_2 2^{\binom{kW}{2}}$$

To introduce:

- *smooth approximation* by *homogeneous substructures* [3] (N.B. 'smooth approximation' from [4, Definition 4] seems to be entirely different)
- *rough approximation* of a homogeneous structure by finite substructures with few orbits (i.e., types) that cover the age of $\mathbb{A}$

A. Symplectic vector spaces

Throughout this subsection let $\mathfrak{f}$ denote a finite field.

Definition II.1. A *symplectic vector space* is an $\mathfrak{f}$-vector space $\mathbb{W}$ equipped with a bilinear form $\omega : \mathbb{W} \times \mathbb{W} \to \mathfrak{f}$ that is

- alternating: $\omega(v, v) = 0$ for all v; and
- non-degenerate: if $\omega(v, w) = 0$ for all w then $v = 0$.

Example II.2. Let $\mathbb{W}_n$ be the $\mathfrak{f}$-vector space with basis $e_1, \ldots, e_n, f_1, \ldots, f_n$. Define ω by bilinearly extending

$$\omega(e_i, f_i) = 1 = -\omega(f_i, e_i), \quad \omega(-, *) = 0 \text{ elsewhere}; \qquad (\S)$$

one may straightforwardly check that ω is alternating and non-degenerate. Moreover, noticing that $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$, we obtain a countable-dimensional symplectic vector space $\mathbb{W}_\infty = \bigcup_n \mathbb{W}_n$.

We will take a straight-line path to prove the following in a self-contained manner. Detailed expositions can be found in [1, §III.3].

Theorem II.3. *The symplectic vector space $\mathbb{W}_\infty$ is smoothly approximated by $\mathbb{W}_0 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \cdots$.*

To begin with, we will refer to vectors satisfying (§) as a *symplectic basis* — indeed, they must be linearly independent. Such bases behave very much like the usual bases.

Lemma II.4. *Assume that $\mathbb{W}$ is a symplectic vector space that is at most countable. Then any finite symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ can be extended to a symplectic basis that spans the whole $\mathbb{W}$.*

Proof. Suppose that $e_1, \ldots, e_n, f_1, \ldots, f_n$ does not already span $\mathbb{W}$; take v to be a witness (that is least according to some fixed enumeration of $\mathbb{W}$ in the case it is infinite). Put

$$e_{n+1} = v - \sum_{i=1}^{n} \omega(e_i, v) f_i + \sum_{i=1}^{n} \omega(f_i, v) e_i$$

so that $\omega(e_i, e_{n+1}) = 0 = \omega(f_i, e_{n+1})$. By the non-degeneracy of ω, there is — rescaling if necessary — some w such that $\omega(e_{n+1}, w) = 1$. Now define

$$f_{n+1} = w - \sum_{i=1^n} \omega(e_i, w) f_i + \sum_{i=1}^{n} \omega(f_i, w) e_i$$

in a similar manner, making $e_1, \ldots, e_n, e_{n+1}, f_1, \ldots, f_n, f_{n+1}$ a symplectic basis that spans v. We go through every element of $\mathbb{W}$ by continuing this way. $\square$

Given two symplectic vector spaces $\mathbb{W}$ and $\mathbb{W}'$, we call a function f between $X \subseteq \mathbb{W}$ and $X' \subseteq \mathbb{W}'$ *isometric* if $\omega(f(x_1), f(x_2)) = \omega(x_1, x_2)$ for all $x_1, x_2 \in X$.

Proposition II.5. *$\mathbb{W}_0, \mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_\infty$ are all the countable symplectic vector spaces up to linear isometric isomorphisms.*

Proof. Let $\mathbb{W}$ be a countable symplectic vector space. By Lemma II.4, we can extend the empty symplectic basis to one that spans $\mathbb{W}$ — call the basis vectors e_i', f_i'. Then $e_i' \mapsto e_i, f_i' \mapsto f_i$ exhibits a linear isomorphism from $\mathbb{W}$ to $\mathbb{W}_n$ or $\mathbb{W}_\infty$, which one easily checks to be isometric. $\square$

In particular $\mathbb{W}_\infty$ is the essentially unique countable symplectic vector space; by Ryll-Nardzewski it is oligomorphic. We need one last fact about its finite counterparts.

Proposition II.6 (Witt Extension)**.** *Any isometric injective linear map $f : X \subseteq \mathbb{W}_n \to \mathbb{W}_n$ can be extended to an isometric linear isomorphism $\mathbb{W}_n \to \mathbb{W}_n$.*

Proof. We proceed by induction on the dimension of $X^{\perp X}$, where we write

$$Y^{\perp Z} = \{y \in Y \mid \forall z \in Z : \omega(y, z) = 0\}$$

for subspaces $Y, Z \subseteq \mathbb{W}_n$.

Suppose first that $X^{\perp X} = X \cap \mathbb{W}_n^{\perp X}$ is the zero space. Notice that $\dim \mathbb{W}_n^{\perp X} = 2n - \dim X$:

- the map $\mathbb{W}_n \to (\mathbb{W}_n \xrightarrow{\text{lin.}} \mathfrak{f}), v \mapsto \omega(v, -)$ is linear and injective since ω is bilinear and non-degenrate, so for dimension reasons it is also surjective;
- the restriction map $(\mathbb{W}_n \xrightarrow{\text{lin.}} \mathfrak{f}) \to (X \xrightarrow{\text{lin.}} \mathfrak{f})$ is linear and surjective, since any basis for $X \subseteq \mathbb{W}_n$ can be extended to one for $\mathbb{W}_n$;
- their composition $v \mapsto \omega(v, -)|_X$ is therefore linear, surjective, and has kernel $\mathbb{W}_n^{\perp X}$.

It follows that $\mathbb{W}_n^{\perp X}$ is the orthogonal complement of X in $\mathbb{W}_n$: by assumption and the above we have $X \cap \mathbb{W}_n^{\perp X} = \{0\}$, $X + \mathbb{W}_n^{\perp X} = \mathbb{W}_n$, and ω restricted to $X \times \mathbb{W}_n^{\perp X}$ is the zero function; we will use the notation

$$\mathbb{W}_n = X \ominus \mathbb{W}_n^{\perp X}.$$

On the other hand, as f is isometric, $f(X)^{\perp f(X)} = f(X^{\perp X})$ must also be the zero space, meaning $\mathbb{W}_n = f(X) \ominus \mathbb{W}_n^{\perp f(X)}$ by the same analysis. But $\dim \mathbb{W}_n^{\perp f(X)} = 2n - \dim f(X) = 2n - \dim X = \dim \mathbb{W}_n^{\perp X}$, so by Proposition II.5 there is a isometric linear isomorphism $g : \mathbb{W}_n^{\perp X} \to \mathbb{W}_n^{\perp f(X)}$. Combining f and g yields a linear isomorphism

$$\begin{gathered}\mathbb{W}_n = X \ominus \mathbb{W}_n^{\perp X} \to f(X) \ominus \mathbb{W}_n^{\perp f(X)} = \mathbb{W}_n \\ x + y \mapsto f(x) + g(y)\end{gathered}$$

which is isometric.

Now suppose $X^{\perp X}$ contains some non-zero vector x. By extending x to a basis of X we can find a complement Y for $\langle x \rangle$ in X; then $X = \langle x \rangle \ominus Y$ by the assumption on x. Writing $Z = \mathbb{W}_n^{\perp Y}$, notice $Y \subseteq \mathbb{W}_n^{\perp Z}$ and $\dim Y = 2n - (2n - \dim Y) = \dim \mathbb{W}_n^{\perp Z}$. It follows that $\mathbb{W}_n^{\perp Z} = Y$ does not contain x, i.e., that some $z \in Z$ satisfies $\omega(x, z) = 1$. Consider $X' = \langle x, z \rangle + Y$; we must have

$$X' = \langle x, z \rangle \ominus Y,$$

because if $\lambda x + \mu z \in \langle x, z \rangle$ lies also in $Y \subseteq X$, then $0 = \omega(x, \lambda x + \mu z) = \mu$ and so $\lambda = 0$ too. Similarly, as $f(x)$ is a non-zero vector in $f(X)^{\perp f(X)}$ and $f(X) = \langle f(x) \rangle \ominus f(Y)$, we can find a vector z' orthogonal to $\mathbb{W}_n^{\perp f(Y)}$ and satisfying $\omega(f(x), z') = 1$. Hence

$$x \mapsto f(x), z \mapsto z', y \mapsto f(y)$$

defines an isometric linear embedding

$$f' : X' = \langle x, z \rangle \ominus Y \to \langle f(x), z' \rangle \ominus f(Y) \subseteq \mathbb{W}_n$$

extending f. Finally, we can apply the inductive hypothesis to extend f'. Indeed, if $v = \lambda x + \mu z + y$ is in $X'^{\perp X'}$ then $\lambda = \omega(v, z) = 0 = \omega(v, x) = \mu$, so $v = y$ belongs to $X^{\perp X}$; as $x \in X^{\perp X} \setminus X'^{\perp X'}$, we have $\dim X'^{\perp X'} \leq \dim X^{\perp X} - 1$. $\square$

Smooth approximation is now immediate.

Proof of Theorem II.3. Firstly, observe that the restriction $\mathrm{Aut}(\mathbb{W}_\infty)_{\{\mathbb{W}_n\}} \to \mathrm{Aut}(\mathbb{W}_n)$ is surjective: any isometric linear automorphism of $\mathbb{W}_n$ maps the standard symplectic basis to another symplectic basis, both of which can be extended to a symplectic basis of $\mathbb{W}_\infty$ by Lemma II.4.

Now suppose $\pi \in \mathrm{Aut}(\mathbb{W}_\infty)$ maps $(x_1, \ldots, x_d) \in \mathbb{W}_n^d$ to $(y_1, \ldots, y_d) \in \mathbb{W}_n^d$. By Proposition II.6, we may extend $\pi|_{\langle x_1, \ldots, x_d \rangle} : \langle x_1, \ldots, x_d \rangle \to \mathbb{W}_n$ to some $f \in \mathrm{Aut}(\mathbb{W}_n)$ which still maps $(x_1, \ldots, x_d)$ to $(y_1, \ldots, y_d)$. $\square$

Corollary II.7. *Provided $\mathfrak{F}$ is of characteristic 0, the symplectic $\mathfrak{f}$-vector space $\mathbb{W}_\infty$ is $\mathfrak{F}$-oligomorphic.*

B. Symplectic graphs

For this subsection let $\mathfrak{f}$ be the two-element field.

Definition II.8. The *symplectic graph* $\widetilde{\mathbb{W}}_n$ has vertices $\mathbb{W}_n$ and edges

$$v_1 \sim v_2 \iff \omega(v_1, v_2) = 1.$$

This is indeed an undirected graph: as ω is alternating, we have $\omega(v_1, v_2) = -\omega(v_2, v_1) = \omega(v_2, v_1)$.

Proposition II.9. $\mathrm{Aut}(\widetilde{\mathbb{W}}_n) = \mathrm{Aut}(\mathbb{W}_n)$.

Proof. Clearly any isometric linear automorphism of $\mathbb{W}_n$ is a graph automorphism of $\widetilde{\mathbb{W}}_n$. Conversely, any $f \in \widetilde{\mathbb{W}}_n$ is evidently isometric. To show that f is linear, take $\lambda_1, \lambda_2 \in \mathfrak{f}$ and $v_1, v_2 \in \mathbb{W}$. We calculate:

$$\begin{aligned}&\omega\Big(f(\sum_i \lambda_i v_i) - \sum_i \lambda_i f(v_i), f(w)\Big) \\ &= \omega\Big(f(\sum_i \lambda_i v_i), f(w)\Big) - \sum_i \lambda_i \omega\big(f(v_i), f(w)\big) \\ &= \omega\Big(\sum_i \lambda_i v_i, w\Big) - \sum_i \lambda_i \omega(v_i, w) \\ &= \omega(0, w) = 0\end{aligned}$$

for all $f(w) \in f(\mathbb{W}_n) = \mathbb{W}_n$; since ω is non-degenerate, we conclude that $f(\sum_i \lambda_i v_i) = \sum_i \lambda_i f(v_i)$. $\square$

Proposition II.10. *The number of orbits in $\widetilde{\mathbb{W}}_n^d$ is at most $\prod_{i=1}^d (2^{i-1} + 1) = O(2^{d(d-1)/2})$ for all n.*

Proof. By Proposition II.9 and Theorem II.3, the number of orbits in $\mathbb{W}_\infty^d$ is an upper bound; this number is the OEIS sequence A028361. $\square$

Proposition II.11 ([2, Theorem 8.11.2])**.** *Every graph on at most $2n$ vertices embeds into $\widetilde{\mathbb{W}}_n$.*

Proof. Let G be a graph on at most $2n$ vertices. The conclusion is trivial when $n = 0$. Also, if G contains no edges, we can choose any $2n$ of the 2^n vectors in $\langle e_1, \dots, e_n \rangle \subseteq \widetilde{\mathbb{W}}_n$.

So suppose $n \geq 1$ and G has an edge $s \sim t$. Let $G_{s,t}$ be the graph on vertices $G \setminus \{s, t\}$ with edges which we will specify later. By induction, some embedding $f : G_{s,t} \to \widetilde{\mathbb{W}}_{n-1}$ exists. Define $f' : G \to \widetilde{\mathbb{W}}_n$ by

$$\begin{aligned} x \in G_{s,t} &\mapsto f(x) - [\![x \sim s]\!] f_n + [\![x \sim t]\!] e_n \\ s &\mapsto e_n \\ t &\mapsto f_n \end{aligned}$$

where $[\![\phi]\!]$ is 1 if ϕ holds and 0 otherwise. Then we have $\omega(f'(x), f'(s)) = [\![x \sim s]\!]$ and $\omega(f'(x), f'(t)) = [\![x \sim t]\!]$ as desired, on one hand. On the other,

$$\begin{aligned} \omega(f'(x_1), f'(x_2)) = [\![x_1 \sim x_2]\!] &+ [\![x_1 \sim s]\!][\![x_2 \sim t]\!] \\ &+ [\![x_1 \sim t]\!][\![x_2 \sim s]\!] \end{aligned}$$

tells us how we should define the edge relation in $G_{s,t}$ for f' to be an embedding of graphs. $\square$

Theorem II.12. *The Rado graph is roughly approximated by* $\widetilde{\mathbb{W}}_0 \subseteq \widetilde{\mathbb{W}}_1 \subseteq \widetilde{\mathbb{W}}_2 \subseteq \cdots$.

Corollary II.13. *Provided* $\mathfrak{F}$ *is of characteristic* 0*, the Rado graph is* $\mathfrak{F}$*-oligomorphic.*

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References

[1] Emil Artin. *Geometric Algebra*. Tracts in Pure and Applied Mathematics. Interscience Publishers, 1957.

[2] Chris Godsil and Gordon Royle. *Algebraic graph theory*. 1st ed. Graduate Texts in Mathematics. Springer, 2001. ISBN: 978-0-387-95241-3.

[3] W. M. Kantor, Martin W. Liebeck, and H. D. Macpherson. "$\aleph_0$-Categorical Structures Smoothly Approximated by Finite Substructures". In: *Proceedings of the London Mathematical Society* s3-59.3 (1989), pp. 439–463. DOI: https://doi.org/10.1112/plms/s3-59.3.439. eprint: https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/plms/s3-59.3.439. URL: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/plms/s3-59.3.439.

[4] Antoine Mottet and Michael Pinsker. "Smooth approximations: An algebraic approach to CSPs over finitely bounded homogeneous structures". In: *J. ACM* 71.5 (Oct. 2024). ISSN: 0004-5411. DOI: 10.1145/3689207. URL: https://doi.org/10.1145/3689207.