

# Cogs span the projection kernel, version $n + 1$

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## 1 Notations

Let  $\mathcal{O} \subseteq \mathbb{A}^I$  be an  $S$ -ordered orbit. Given a vector  $v \in \text{Lin}_{\mathbb{E}} \mathcal{O}$ , write  $\underline{v}$  for its set-theoretic support, i.e., the finite subset  $v^{-1}(\mathbb{E}^*) \subseteq \mathcal{O}$ . More generally, given any finite subset  $\sigma \subseteq \mathcal{O}$ , write

$$\bar{\sigma} = \{(i, a_i) \mid i \in I, a \in \sigma\}$$

and define two binary relations

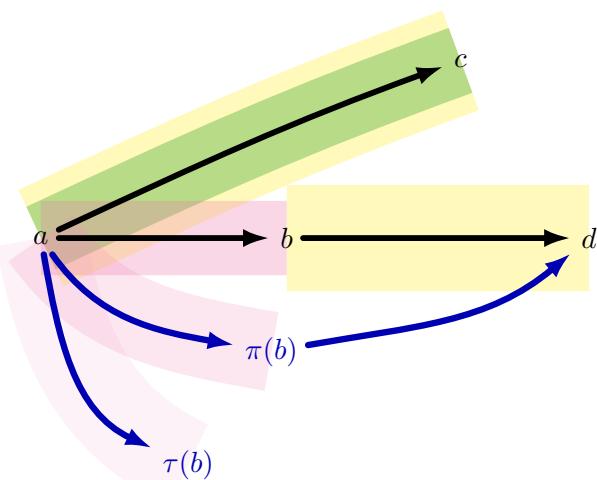
$$(i, a_i)?(j, b_j) \iff a_i = b_j \text{ but } i \neq j,$$

$$(i, a_i)!(j, b_j) \iff a_i, b_j \text{ are related but not in the same way as } a_i, a_j$$

called *ambiguities* and *obstructions*. Both relations are symmetric and  $? \subseteq !$ ; denote their images by  $? \bar{\sigma} \subseteq ! \bar{\sigma} \subseteq \bar{\sigma}$ . Lastly, I refer to the atoms that appear in  $\sigma$  by  $\sqrt{\sigma} \subseteq \mathbb{A}$ .

**Remark 1** Assume  $? \bar{\sigma} = \emptyset$ . Given any  $a_i \in \sqrt{\sigma}$ , we have  $(i, a_i) \in \bar{\sigma}$ .

## 2 Lemmas



**Lemma 2** Let  $\Phi, \Pi, \Theta$  be finite subsets of  $\mathcal{O}$  such that  $?(\overline{\Pi} \cup \overline{\Phi}) = \emptyset = ?(\overline{\Theta}) \supseteq ?(\overline{\Pi})$ . Then there exists  $\pi \in \text{Aut}(\mathbb{A}/S \cup \sqrt{\Pi})$  that satisfies

$$?(\overline{\Theta \cup \pi(\Phi)}) = \emptyset.$$

PROOF. Fix  $\Pi, \Theta$  and induct on the size of  $?(\overline{\Theta} \cup \overline{\Phi})$ .

Let  $(i, a_i)?(j, b_j)$ ; without loss of generality we may assume  $(j, b_j) \in \overline{\Theta}$  and  $(i, a_i) \in \overline{\Phi} \setminus \overline{\Pi}$ . Since  $?(\overline{\Pi}) = \emptyset$ , by Remark 1 we see that  $a_i \notin \sqrt{\Pi}$ ; also  $a_i \notin S$ , as  $\mathcal{O}$  is  $S$ -ordered. With strong amalgamation, we may find some  $\pi \in \text{Aut}(\mathbb{A}/S \cup \sqrt{\Pi})$  such that  $\pi(a_i) \notin \sqrt{\Theta \cup \Phi}$ . Now it is straightforward to check that

$$?(\overline{\Theta \cup \pi(\Phi)}) \subseteq ?(\overline{\Theta \cup \Phi}) \setminus \{(i, a_i)\}.$$

Because  $?(\overline{\Pi} \cup \pi(\Phi)) = ?(\overline{\Pi} \cup \pi(\Phi)) = \pi(\emptyset) = \emptyset$  still, the inductive hypothesis gives us some  $\pi' \in \text{Aut}(\mathbb{A}/S \cup \sqrt{\Pi})$  such that  $?(\overline{\Theta \cup \pi' \pi(\Phi)}) = \emptyset$ . ■

**Lemma 3** Let  $\Phi, \Pi, \Theta$  be finite subsets of  $\mathcal{O}$  such that  $!(\overline{\Pi} \cup \overline{\Phi}) = \emptyset = !(\overline{\Theta}) \supseteq !(\overline{\Pi})$ . Then

$$!(\overline{\Theta \cup \pi(\Phi)}) = \emptyset$$

for some  $\pi \in \text{Aut}(\mathbb{A}/S \cup \sqrt{\Pi})$ .

PROOF. By Lemma 2 we may assume that  $?(\overline{\Theta} \cup \overline{\Phi}) = \emptyset$  already. As before, fix  $\Pi, \Theta$  and proceed by induction on the size of  $!(\overline{\Theta} \cup \overline{\Phi})$ .

Let  $(i, a_i)!?(j, b_j)$ ; without loss of generality we may assume  $(j, b_j) \in \overline{\Theta}$  and  $(i, a_i) \in \overline{\Phi}$ . Now  $(i, a_i) \notin \overline{\Pi}$ , so  $a_i \notin \sqrt{\Pi}$  by Remark 1; further, whenever  $(i, a_i)!?(k, c_k)$  we observe that  $c_k \notin S \cup \{a_i\} \cup \sqrt{\Phi \cup \Pi}$ . Let  $Y$  consist of all such  $c_k$  and put

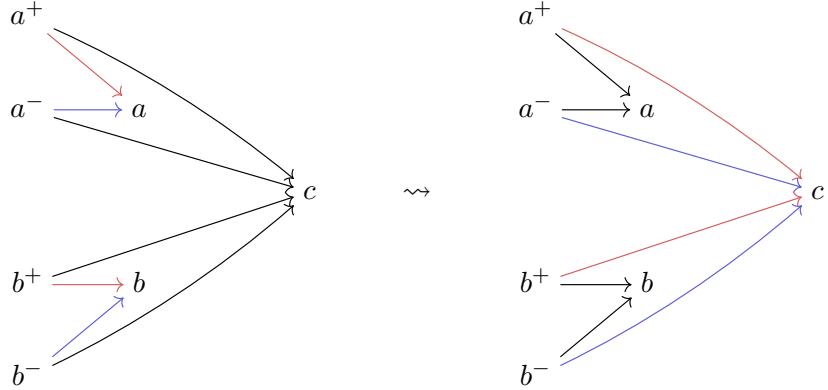
$$X = S \cup \sqrt{\Phi \cup \Pi} \setminus (Y \cup \{a_i\}).$$

Then  $X, Y, \{a_i\}$  are pairwise disjoint, and we see  $X$  contains  $S \cup \sqrt{\Pi}$  as well as  $\sqrt{\Phi} \setminus \{a_i\}$ . With free amalgamation (in  $\mathbb{A}_0$ ) this time, we may find some  $\tau \in \text{Aut}(\mathbb{A}/S \cup \sqrt{\Pi})$  such that  $\tau(a_i) \notin X \cup Y \cup \{a_i\}$  is not related to any atom in  $Y$ . Again, we can check that

$$!(\overline{\Theta \cup \tau(\Phi)}) \subseteq !(\overline{\Theta \cup \Phi}) \setminus (i, a_i)$$

so the conclusion follows straightforwardly from the inductive hypothesis. ■

### 3 Unobstructed vector



**Proposition 4** Let  $v \in \text{Ker}_{\mathbb{E}} \mathcal{O}$  and suppose that  $!\underline{v} = \emptyset$ . Then we can write

$$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ \between a^-$$

with  $!\underline{v} \cup A^\pm = \emptyset$  and  $\lambda_{a^\pm} \in v(\mathcal{O})$ .

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This is the same statement as in §IV.D; here I use a slightly different proof.

We proceed by induction on the dimension  $|I|$ , noting that when  $I = \emptyset$  we just have  $v = v() \cdot () = v() \cdot (\between)$  without any possible obstructions.

So suppose  $I$  is non-empty; let  $d \in I$  be the greatest. Group the terms in  $v$  by their greatest atom so that  $v = v^1 + v^2 + \dots + v^k$ . We now induct on  $k$ . If  $k < 2$ , we are done: as  $v_{-d} = 0$  we must have  $v = 0$ , so the empty sum will do. Otherwise

$$v = v^{d:a_d} + v^{d:b_d} + v'.$$

By the outer inductive hypothesis, we get

$$v^{d:a_d} = v_{-d}^{d:a_d} a_d = \sum_{A^\pm} (\lambda_{a^\pm} \cdot a^+ \between a^-) a_d$$

where we only know  $!\underline{v_{-d}^{d:a_d} \cup A^\pm} = \emptyset$  so that  $!\underline{v^{d:a_d} \cup A^\pm a_d} = \emptyset$ . But any  $\pi \in \text{Aut}(\mathbb{A}/S \cup \sqrt{v^{d:a_d}})$  satisfies

$$v^{d:a_d} = \pi(v^{d:a_d}) = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot \pi a^+ \between \pi a^-,$$

so by Lemma 3 we may assume without loss of generality that

$$!\underline{v \cup A^\pm a_d} = \emptyset.$$

Similarly, we can write

$$v^{d:a_d} = \sum_{B^\pm} (\lambda_{b^\pm} \cdot b^+ \between b^-) b_d$$

where, in turn, we may upgrade the assumption that  $\overline{!v^{d:b_d} \cup B^\pm b_d} = \emptyset$  to

$$\overline{!v \cup A^\pm a_d \cup B^\pm b_d} = \emptyset.$$

The key is that we may now invent a new element  $z$ , on which we impose the following relations with  $S \cup \sqrt{A^\pm a_d \cup B^\pm b_d} \subseteq \mathbb{A}$ :

1.  $a_d, b_d < z$ , and  $z < s$  if  $a_d, b_d < s$  for some  $s \in S$  (enough to let  $s$  be the least such);
2. for any unary relation  $P \in \mathcal{L}_0$ :

$$P(z) : \iff P(a_d) \iff P(b_d)$$

— recall that  $a, b \in \mathcal{O}$ ;

3. for any binary relation  $R \in \mathcal{L}_0$  and  $s \in S$ ,  $a^\pm \in A^\pm$ ,  $b^\pm \in B^\pm$ ,  $i \in I \setminus \{d\}$ :
  - (a)  $R(z, s) : \iff R(a_d, s) \iff R(b_d, s)$ ,
  - (b)  $R(z, a_i^\pm) : \iff R(a_d, a_i^\pm)$ ,
  - (c)  $R(z, a_d) : \iff \perp$ ,
  - (d)  $R(z, b_i^\pm) : \iff R(b_d, b_i^\pm)$ ;
  - (e)  $R(z, b_d) : \iff \perp$ ,
  - (f) and symmetrically for  $R(-, z)$ .

These are well-defined because  $a^\pm a_d, b^\pm b_d \in \mathcal{O}$  and  $i = j$  whenever  $a_i^\pm = b_j^\pm$ .

To see that the  $\mathcal{L}$ -structure  $S \cup \sqrt{A^\pm a_d \cup B^\pm b_d} \cup \{z\}$  still embeds into  $\mathbb{A}$ , suppose towards a contradiction that it contains a forbidden  $\mathcal{L}_0$ -substructure  $F$ . Then  $F$  must contain  $z$ . Since any two elements in  $F$  are necessarily related, we must have  $a_d, b_d \notin F$ . Similarly, whenever  $F$  contains  $x_i$  where  $x \in A^\pm \cup B^\pm$ ,  $i \in I \setminus \{d\}$  it does not contain a distinct atom of the form  $x'_i$ . It follows that

$$s \mapsto s, \quad x_i \mapsto a_i, \quad z \mapsto a_d$$

defines an injective function  $\phi : F \rightarrow \mathbb{A}_0$ , which is furthermore an embedding because  $\overline{!A^\pm \cup B^\pm} = \emptyset$  and any  $x_i, x'_{i'}$  for  $i \neq i'$  are related. This is impossible — therefore assume  $z \in \mathbb{A}$ .

It is now routine to check that  $a^+ a_d \parallel a^- z$  and  $b^+ a_d \parallel b^- z$  are  $\mathcal{O}$ -dipoles for  $a^\pm \in A^\pm$ ,  $b^\pm \in B^\pm$  and that  $\overline{!A^+ a_d \cup A^- z \cup B^+ b_d \cup B^- z} = \emptyset$ . By Lemma 3 we may assume that  $\overline{!v \cup A^+ a_d \cup A^- z \cup B^+ b_d \cup B^- z} = \emptyset$ . (Alternatively, we could have explicitly ensured this when defining  $z$ ). Then

$$\begin{aligned} v'' &= v - \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_d \between a^- z - \sum_{B^\pm} \lambda_{b^\pm} \cdot b^+ b_d \between b^- z \\ &= v_{-d}^{d:a_d} z + v_{-d}^{d:b_d} z + v', \end{aligned}$$

when grouped into subvectors by the largest atom in each term, has at least one fewer component than  $v$ . By the inner inductive hypothesis, we may write we may write

$$v'' = \sum_{C^\pm} \lambda_{c^\pm} \cdot c^+ \between c^-$$

where  $!\underline{v''} \cup C^\pm = \emptyset$ . But  $\overline{\underline{v''}} \subseteq \overline{\underline{v}} \cup A^+ a_d \cup A^- z \cup B^+ b_d \cup B^- z$ , so one last application of Lemma 3 allows us to assume that

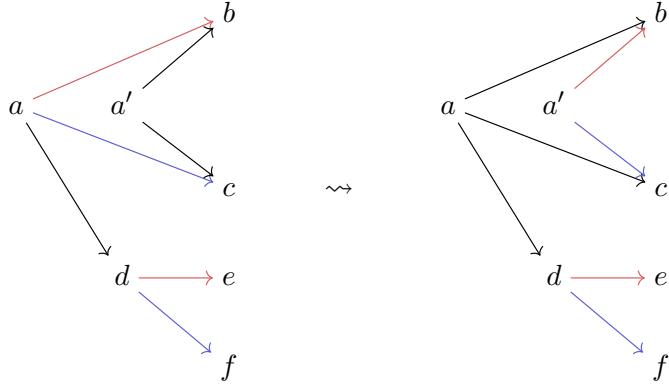
$$!\overline{\underline{v}} \cup A^+ a_d \cup A^- z \cup B^+ b_d \cup B^- z \cup C^\pm = \emptyset.$$

We conclude that

$$v = \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_d \between a^- z + \sum_{B^\pm} \lambda_{b^\pm} \cdot b^+ b_d \between b^- z + \sum_{C^\pm} \lambda_{c^\pm} \cdot c^+ \between c^-;$$

in other words, we have decomposed an unobstructed vector into an unobstructed family of cogs. (The notation  $\lambda_{a^\pm}, \lambda_{b^\pm}$  is a bit sloppy ...)

## 4 Unambiguous vector



**Proposition 5** Let  $v \in \text{Ker}_E \mathcal{O}$  and suppose that  $? \overline{v} = \emptyset$ . Then we can write

$$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ \between a^-$$

with  $? \overline{v} \cup A^\pm = \emptyset$  and  $\lambda_{a^\pm} \in v(\mathcal{O})$ .

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This is an amended version of Proposition IV.25 (the one with  $\hat{v}$ ). The proof there had a gap, which is filled by the additional conclusion about unambiguity here.

We proceed again by induction, first on the dimension  $|I|$  then on the cardinality of  $! \overline{v}$ . The outer base case  $I = \emptyset$  is trivial — we have  $v = v() \cdot (\between)$ , and no ambiguities may arise — whilst the inner base case is just Proposition 4.

So suppose that  $!\underline{v}$  contains some  $(i, a_i)$ . Since  $\overline{v_{-i}^{i:a_i}} \subseteq \overline{v^{i:a_i}} \subseteq \overline{\underline{v}}$ , we know that  $v_{-i}^{i:a_i}$  is unambiguous. Applying the outer inductive hypothesis, we may write

$$v^{i:a_i} = v_{-i}^{i:a_i} a_i = \sum_{A^\pm} (\lambda_{a^\pm} \cdot a^+ \between a^-) a_i$$

where  $\emptyset = ?(\overline{v_{-i}^{i:a_i} \cup A^\pm}) a_i = ?\overline{v^{i:a_i} \cup A^\pm} a_i$ . Moreover, we may assume by Lemma 2 that

$$?\overline{v \cup A^\pm} a_i = \emptyset.$$

Now we can show that whenever  $(j, b_j)!(i, a_i)$  in  $\overline{\underline{v}}$  we have  $b_j \notin \sqrt{A^\pm a_i}$ . Already  $b_j = a_i$  would imply  $j = i$  as  $?_v = 0$ , which is impossible. So suppose to the contrary that  $b_j = a_k^\pm$  for some  $a^\pm \in A^\pm$  and  $k \in I \setminus \{i\}$ . By the assumption above, we must have  $j = k$ ; this is a contradiction: note that  $a^\pm a_i \in \mathcal{O}$ .

Put  $Y = \{b_j \mid (j, b_j) \in \overline{\underline{v}}, (j, b_j)!(i, a_i)\}$ . It follows that  $X = S \cup \sqrt{v \cup A^\pm} \setminus Y \setminus \{a_i\}$  contains  $S \cup \sqrt{A^\pm}$  and that  $X, Y, \{a_i\}$  are pairwise disjoint. Using free amalgamation in  $\mathbb{A}_0$ , we may find  $\tau \in \text{Aut}(\mathbb{A}/X)$  such that  $\tau(a_i) \notin X \cup Y \cup \{a_i\}$ , is greater than  $a_i$ , and is not related to any of  $Y \cup \{a_i\}$ . Then, given any  $a^\pm \in A^\pm$ , we can straightforwardly check that  $a^+ a_i \parallel a^- \tau(a_i)$  is an  $\mathcal{O}$ -dipole, that  $?_v \cup A^+ a_i \cup A^- \tau(a_i) = \emptyset$ , and that

$$\begin{aligned} v - \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \between a^- \tau(a_i) &= v - v^{i:a_i} + v^{i:\tau(a_i)} \\ &= v^{i:a_i \mapsto \tau(a_i)} \end{aligned}$$

satisfies  $!_v \subseteq !\overline{v} \setminus \{(i, a_i)\}$ . The inner inductive hypothesis tells us that  $v^{i:a_i \mapsto \tau(a_i)} = \sum_{B^\pm} \lambda_{B^\pm} \cdot b^+ \between b^-$  with  
 $?_v \cup B^\pm = \emptyset$ .

But  $\overline{v^{i:a_i \mapsto \tau(a_i)}} \subseteq \overline{v \subseteq A^+ a_i \cup A^- \tau(a_i)}$ , so Lemma 2 allows us to assume that

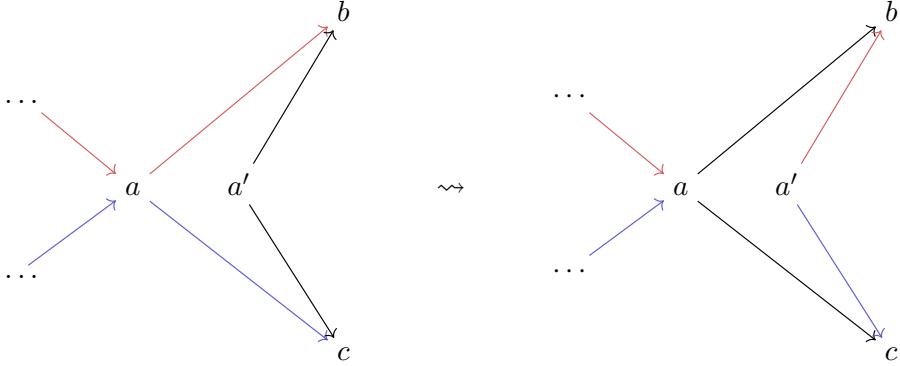
$$?\overline{v \cup A^+ a_i \cup A^- \tau(a_i) \cup B^\pm} = \emptyset.$$

We conclude that

$$v = \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \between a^- \tau(a_i) + \sum_{B^\pm} \lambda_{b^\pm} \cdot b^+ \between b^-$$

— in other words, we have decomposed an unambiguous vector into an unambiguous family of cogs.

## 5 General vector



**Proposition 6** *Let  $v \in \text{Ker}_{\mathbb{E}} \mathcal{O}$ . Then we can write*

$$v = \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ \between a^-$$

*with  $\lambda_{a^\pm} \in v(\mathcal{O})$ .*

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This is the same statement and proof as Proposition IV.24 (the one with  $\bar{v}$ ).

This is an easier induction on  $|I|$  than on  $?_{\bar{v}}$ . If  $I = \emptyset$ , the decomposition is trivial; if  $v$  is unambiguous already, the decomposition comes from Proposition 5.

Now suppose  $?_{\bar{v}}$  contains some  $(i, a_i)$ , and use the outer inductive hypothesis to write

$$v^{i:a_i} = v_{-i}^{i:a_i} a_i = \sum_{A^\pm} (\lambda_{a^\pm} \cdot a^+ \between a^-) a_i.$$

Then neither  $S$  nor  $\sqrt{A^\pm}$  contains  $a_i$ , so  $X = S \cup \sqrt{v \cup A^\pm \setminus \{a_i\}}$  contains  $S \cup \sqrt{A^\pm}$ . Using free amalgamation in  $\mathbb{A}_0$  and the generic order in  $\mathbb{A}$ , we may find some  $\pi \in \text{Aut}(\mathbb{A}/X)$  such that  $\pi(a_i) \notin X$ ,  $\pi(a_i) > a_i$ ,  $\pi(a_i) \perp a_i$ . We can check that

1.  $a^+ a_i \parallel a^- \pi(a_i)$  is an  $\mathcal{O}$ -dipole, given any  $a^\pm \in A^\pm$ ;
2.  $v - \sum_{A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \between a^- \pi(a_i) = v - v^{i:a_i} + v^{i:\pi(a_i)} = v^{i:a_i \mapsto \pi(a_i)}$  satisfies

$$\overline{?v^{i:a_i \mapsto \pi(a_i)}} \subseteq ?_{\bar{v}} \setminus \{(i, a_i)\}.$$

It follows from the inner inductive hypothesis that

$$\begin{aligned} v &= \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \between a^- \pi(a_i) + v^{i:a_i \mapsto \pi(a_i)} \\ &= \sum_{a^\pm \in A^\pm} \lambda_{a^\pm} \cdot a^+ a_i \between a^- \pi(a_i) + \sum_{b^\pm \in B^\pm} \lambda_{b^\pm} \cdot b^+ \between b^- \end{aligned}$$