Jingjie Yang
HT'26

Sheet 1

Logic & Proof

Propositional logic

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A formula in n variables amounts to a function $\{0,1\}^n \to \{0,1\}$.

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Example: graph homomorphism

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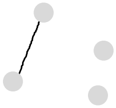

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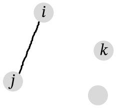

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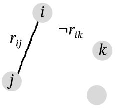

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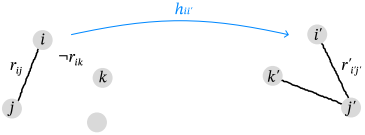

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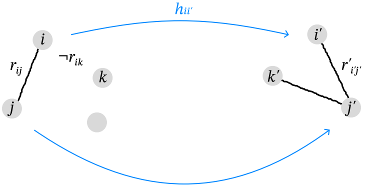

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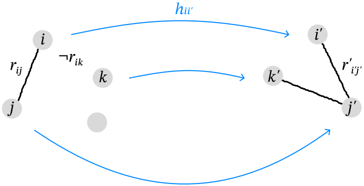

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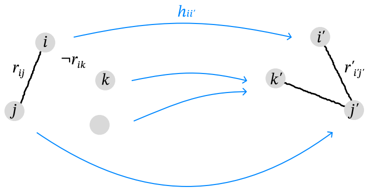

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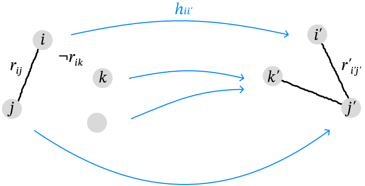

$$\bigwedge_{i} \bigvee_{i'} h_{ii'} \quad \wedge \quad \bigwedge_{i} \bigwedge_{i' \neq j'} \neg h_{ii'} \vee \neg h_{ij'} \quad \wedge \quad \bigwedge_{i,j,i',j'} \neg h_{ii'} \vee \neg h_{jj'} \vee \neg r_{ij} \vee r'_{i'j'}$$

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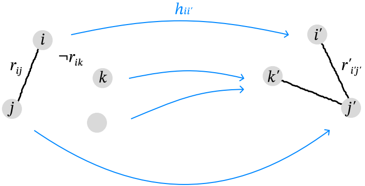

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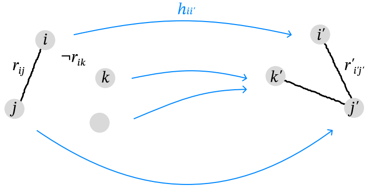

every vertex maps to ≥ 1 vertex′ every vertex maps to ≤ 1 vertex′

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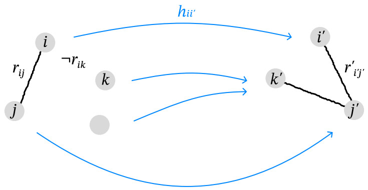

every vertex maps to ≥ 1 vertex′ — every vertex maps to ≤ 1 vertex′ — edges map to edges′

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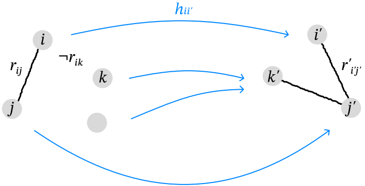

$$\underbrace{\bigwedge_i \bigvee_{i'} h_{ii'}}_{\text{every vertex maps to} \geq 1 \text{ vertex}'} \quad \wedge \quad \underbrace{\bigwedge_i \bigwedge_{i' \neq j'} \neg h_{ii'} \vee \neg h_{ij'}}_{\text{every vertex maps to} \leq 1 \text{ vertex}'} \quad \wedge \quad \underbrace{\bigwedge_{i,j,i',j'} \neg h_{ii'} \vee \neg h_{jj'} \vee \neg r_{ij} \vee r'_{i'j'}}_{\text{edges map to edges}'}$$

This formula in **h**, **r**, **r′** is satisfiable iff there is a graph homomorphism **h**.

Exercise 1

Write K_m for the complete graph on m vertices.

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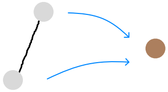

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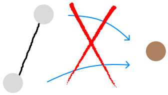

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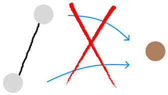

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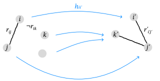

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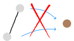

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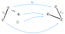

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Given F in CNF: $\overset{\vee \text{ clause in } F}{C_0}, \ \overset{\text{clause} \in F}{C_1}, \ \underset{C_0 = C_0' \cup \{p\}}{C_2}, \ \ldots, \ C_m$

Resolution proof

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$C_0 = C_0' \cup \{p\}$

$C_1 = C_1' \cup \{\neg p\}$

Resolution proof

Given F in CNF: $C_0, \; C_1, \; C_2, \; \ldots, \; C_m$

$\vee$ clause in F ; clause $\in F$

$C_0 = C_0' \cup \{p\}$

$C_1 = C_1' \cup \{\neg p\}$

$C_2 = C_1' \cup C_2'$

Claim: $F(p_1, \ldots, p_n)$ unsatisfiable $\Rightarrow$ $\exists$resolution proof of $\square$.

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Inductive step:

$$F[\mathbf{false}/p_n]$$

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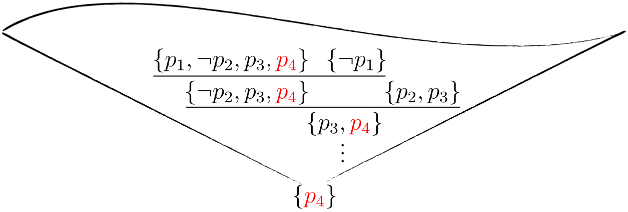

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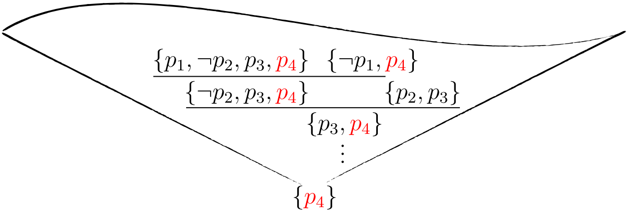

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Claim: $F(p_1, \ldots, p_n)$ unsatisfiable $\Rightarrow$ $\exists$resolution proof of $\square$.

Proof: by induction on n.

Base case: $n = 0$; only clause is $\square$; only CNF's are $\cancel{\{\}}$ or $\{\square\}$.

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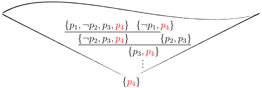

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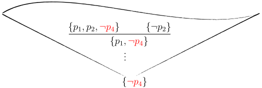

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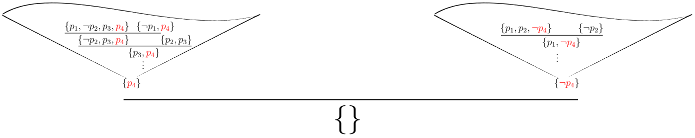

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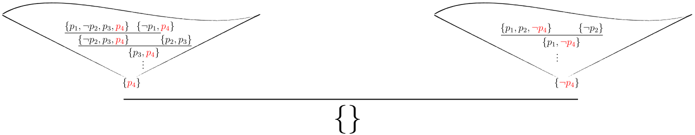

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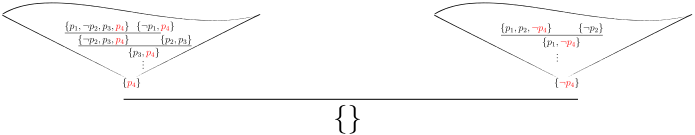

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This is a CNF, still unsatisfiable, in fewer variables. So:

Either p_n never occurs — done, or we have a proof of p_n.

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$$\cfrac{\cfrac{\cfrac{\{p_1, \neg p_2, p_3, p_4\} \quad \{\neg p_1, p_4\}}{\{\neg p_2, p_3, p_4\}} \quad \{p_2, p_3\}}{\{p_3, p_4\}}}{\vdots \atop \{p_4\}} \qquad \cfrac{\{p_1, p_2, \neg p_4\} \quad \{\neg p_2\}}{\{p_1, \neg p_4\}} \atop {\vdots \atop \{\neg p_4\}}$$

length $\leq 2^n - 1$

$$\frac{\{p_4\} \qquad \{\neg p_4\}}{\{\}}$$

of length $\leq 2^{n+1} - 1$

Claim: $F(p_1, \ldots, p_n)$ unsatisfiable $\Rightarrow$ $\exists$resolution proof of $\square$.

Proof: by induction on n.

length $1 \leq 2^1 - 1$

Base case: $n = 0$; only clause is $\square$; only CNF's are $\cancel{\{\}}$ or $\{\square\}$.

Inductive step:

$$F[\mathbf{false}/p_n] = \bigcup_{C \in F : \neg p_n \notin C} C \setminus \{p_n\}$$

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$$\dfrac{\{p_1, p_2, \neg p_4\} \quad \{\neg p_2\}}{\{p_1, \neg p_4\}}$$
$$\vdots$$

$$\dfrac{\{p_4\} \quad \text{length} \leq 2^n - 1 \qquad \text{length} \leq 2^n - 1 \quad \{\neg p_4\}}{\{\}}$$

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$$\{p_3, p_4\}$$
$$\vdots$$
$$\{p_4\}$$

$$\dfrac{\{p_1, p_2, \neg p_4\} \quad \{\neg p_2\}}{\{p_1, \neg p_4\}}$$
$$\vdots$$
$$\{\neg p_4\}$$

length $\leq 2^n - 1$ length $\leq 2^n - 1$

$$\dfrac{\{p_4\} \qquad \{\neg p_4\}}{\{\}}$$

length $\leq 2 \cdot (2^n - 1) + 1$

of length $\leq 2^{n+1} - 1$

Claim: $F(p_1, \ldots, p_n)$ unsatisfiable $\Rightarrow$ $\exists$resolution proof of $\square$.

Proof: by induction on n.

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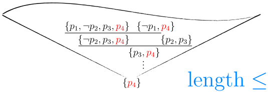

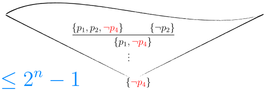

length $\leq 2^n - 1$ length $\leq 2^n - 1$

$$\frac{\{p_4\} \qquad \{\neg p_4\}}{\{\}}$$

length $\leq 2 \cdot (2^n - 1) + 1 = 2^{n+1} - 1$

Resolution proof

$$\text{Given } F \text{ in CNF:} \quad \overset{\vee \text{ clause in } F}{C_0}, \ \overset{\text{clause} \in F}{C_1}, \ C_2, \ \ldots, \ C_m$$

$C_0 = C_0' \cup \{p\}$

$C_1 = C_1' \cup \{\neg p\}$

$C_2 = C_1' \cup C_2'$

Claim: $F(p_1, \ldots, p_n)$ unsatisfiable $\Rightarrow$ $\exists$resolution proof of $\square$. of length $\leq 2^{n+1} - 1$

Proof: by induction on n.

Base case: $n = 0$; only clause is $\square$; only CNF's are ~~{}~~ or $\{\square\}$. length $1 \leq 2^1 - 1$

Inductive step:

$$F[\mathbf{false}/p_n] = \bigcup_{C \in F : \neg p_n \notin C} C \setminus \{p_n\}$$

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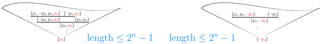


length $\leq 2^n - 1$ length $\leq 2^n - 1$

$$\frac{\{p_4\} \qquad \{\neg p_4\}}{\{\}}$$

length $\leq 2 \cdot (2^n - 1) + 1 = 2^{n+1} - 1$

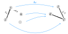
Propositional logic

A formula in n variables amounts to a function $\{0,1\}^n \to \{0,1\}$.

(CNF)

/ straight-line program

Example: graph homomorphism



every vertex maps to ≥ 1 vertex' $\quad$ every vertex maps to ≤ 1 vertex' $\quad$ edges map to edges'

$$\bigwedge_i \bigvee_{i'} h_{ii'} \quad \wedge \quad \bigwedge_i \bigwedge_{i' \neq j'} \neg h_{ii'} \vee \neg h_{ij'} \quad \wedge \quad \bigwedge_{i,j,i',j'} \neg h_{ii'} \vee \neg h_{jj'} \vee \neg r_{ij} \vee r'_{i'j'}$$

This formula in **h**, **r**, **r'** is satisfiable iff there is a graph homomorphism **h**.

Exercise 1

Write K_m for the complete graph on m vertices.

(a) homomorphism $K_m \to G$ $\quad = \quad$ m-clique in G

(b) homomorphism $G \to K_m$ $\quad = \quad$ m-colouring of G

Jingjie Yang

HT'26

Sheet 1

Logic & Proof

Resolution proof

Given F in CNF: $\quad C_0, \; C_1, \; C_2, \; \ldots, \; C_m$

$\vee$ clause in F $\quad$ clause $\in F$

$C_0 = C'_0 \cup \{p\}$

$C_1 = C'_1 \cup \{\neg p\}$

$C_2 = C'_1 \cup C'_2$

Claim: $F(p_1, \ldots, p_n)$ unsatisfiable $\Rightarrow$ $\exists$resolution proof of $\square$.

of length $\leq 2^{n+1} - 1$

Proof: by induction on n.

Base case: $n = 0$; only clause is $\square$; only CNF's are $\cancel{\emptyset}$ or $\{\square\}$.

length $1 \leq 2^1 - 1$

Inductive step:

$F[\mathbf{false}/p_n] = \bigcup_{C \in F: \neg p_n \notin C} C \setminus \{p_n\}$

This is a CNF, still unsatisfiable, in fewer variables. So:

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$F[\mathbf{true}/p_n] = \bigcup_{C \in F: p_n \notin C} C \setminus \{\neg p_n\}$

This is a CNF, still unsatisfiable, in fewer variables. So:

Either $\neg p_n$ never occurs — done, or we have a proof of $\neg p_n$.

length $\leq 2^n - 1$ $\quad$ length $\leq 2^n - 1$

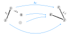
$\{\}$

length $\leq 2 \cdot (2^n - 1) + 1 = 2^{n+1} - 1$

Propositional logic

A formula (CNF) / straight-line program in n variables amounts to a function $\{0,1\}^n \to \{0,1\}$.

Example: graph homomorphism



$$\underbrace{\bigwedge_i \bigvee_{i'} h_{ii'}}_{\text{every vertex maps to} \geq 1 \text{ vertex}'} \wedge \underbrace{\bigwedge_i \bigwedge_{i' \neq j'} \neg h_{ii'} \vee \neg h_{ij'}}_{\text{every vertex maps to} \leq 1 \text{ vertex}'} \wedge \underbrace{\bigwedge_{i,j,i',j'} \neg h_{ii'} \vee \neg h_{jj'} \vee \neg r_{ij} \vee r'_{i'j'}}_{\text{edges map to edges}'}$$

This formula in **h**, **r**, **r'** is satisfiable iff there is a graph homomorphism **h**.

Exercise 1

Write K_m for the complete graph on m vertices.

(a) homomorphism $K_m \to G$ = m-clique in G

(b) homomorphism $G \to K_m$ = m-colouring of G

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Sheet 1

Logic & Proof

Resolution proof

Given F in CNF: $\underbrace{C_0}_{\vee \text{ clause in } F}, \underbrace{C_1}_{\text{clause} \in F}, C_2, \ldots, C_m$

$C_0 = C_0' \cup \{p\}$
$C_1 = C_1' \cup \{\neg p\}$
$C_2 = C_1' \cup C_2'$

Claim: $F(p_1, \ldots, p_n)$ unsatisfiable $\Rightarrow$ $\exists$resolution proof of $\square$ (of length $\leq 2^{n+1} - 1$).

Proof: by induction on n.

Base case: $n = 0$; only clause is $\square$; only CNF's are ~~{}~~ or $\{\square\}$. (length $1 \leq 2^1 - 1$)

Inductive step:

$F[\mathbf{false}/p_n] = \bigcup_{C \in F: \neg p_n \notin C} C \setminus \{p_n\}$

This is a CNF, still unsatisfiable, in fewer variables. So:
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length $\leq 2^n - 1$ length $\leq 2^n - 1$

$\{\}$

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