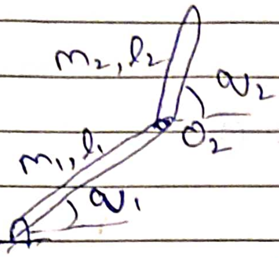
2R Manipulator :-



Planar elbow manipulator

2R → Revolute, Prismatic

Revolute ⇓ (hinge joint)

O_1 (Origin)

q_1, q_2 joint angles

→ E → end effector, (x, y) => end effector position

→ Note :- ~~to~~ Use q convention.

→ Let us assume we have a way to control either the torques τ_1 & τ_2 applied to the joints or control the angles q_1 & q_2 directly.

Let us consider 3 tasks ⇒

T_1 → Given arbitrary trajectory of end ~~effort~~ effector [x, y as a function of time], make the robot ~~to~~ follow its trajectory.

T_2 → Given location on a wall, make the robot to touch the wall and apply a prespecified (constant) force at that ~~loc~~ location.

T_3 → Make the robot behave like a virtual spring connected from the end effector (E) to the given point (x_0, y_0)

Now, $x = l_1 \cos q_1 + l_2 \cos q_2$
$y = l_1 \sin q_1 + l_2 \sin q_2$

Using simplified notation,

$$\left.\begin{aligned} x &= l_1 c q_1 + l_2 c q_2 \\ y &= l_1 s q_1 + l_2 s q_2 \end{aligned}\right\} \quad -(1)$$

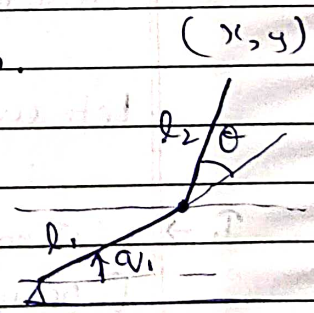
$$\left.\begin{aligned} \dot{x} &= -l_1 s q_1 \dot{q}_1 + - l_2 s q_2 \dot{q}_2 \\ \dot{y} &= l_1 \cos q_1 \dot{q}_1 + l_2 c q_2 \dot{q}_2 \end{aligned}\right\} \quad -(2)$$

→ End effector velocity $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 s q_1 & -l_2 s q_2 \\ l_1 c q_1 & l_2 c q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$

- We will need the inverse relationships. Given (x, y) we need to able to find q_1, q_2

Opt 1: Solve numerically

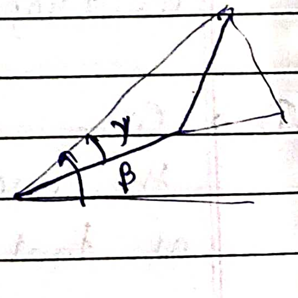
Opt 2: Derive a closed-form expression.
- Hard in general
- Multiple solⁿ



Cosine rule + switching to acute angles

$$x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta$$

→ $$\theta = \cos^{-1}\left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right)$$

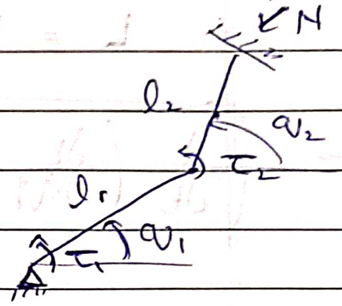


$$q_1 = \beta - \gamma$$
$$= \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2 \cos\theta}\right)$$

$$q_2 = q_1 + \theta$$

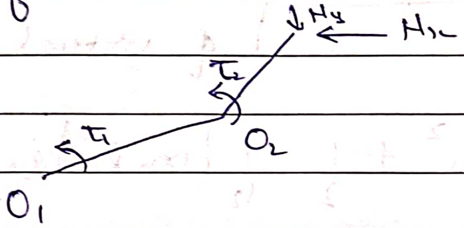
This is 1st level answer to τ_1.

We will later start using the notation x_d and y_d (q_{1d} and q_{2d}) for the desired values. (They are not ~~ne~~ necessary actual values)



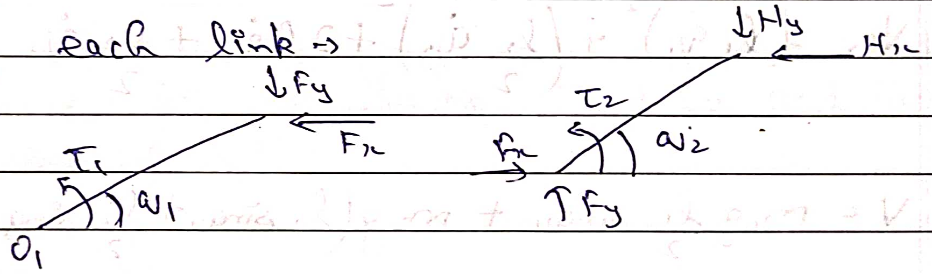
TASK 2:

FBD of entire robot



$F_y = -H_y$, $F_x = -H_x$ (Neglecting gravity)

FBD of each link →



$\Sigma M_{O_1} = 0$

$H_y l_1 \cos q_1 - H_x l_1 \sin q_1 = \tau_1$

$\Sigma M_{O_2} = 0$

$H_y l_2 \cos q_2 - H_x \sin q_2 \, l_2 = \tau_2$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 \sin q_1 & l_1 \cos q_1 \\ -l_2 \sin q_2 & l_2 \cos q_2 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \end{bmatrix} \quad (4)$$

For T_3 and next-level answer to T_1. Need to understand dynamics.

~~Ko~~ Langrange's Equation:

$L = K - V$ where K is kinetic energy, V - Potential Energy

(5) — $$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i'}$$ Q_i' are generalized forces derived using principle of virtual work.

$$K = \underbrace{\frac{1}{2}\left(\frac{1}{3}m_1 l_1^2\right)\dot{q}_1^2}_{\text{Pure rotation of } l_1} + \frac{1}{2}\frac{1}{12}\left(m_2 l_2^2\right)\dot{q}_2 + \frac{1}{2}m_2 \underbrace{V_{C_2}^2}_{\text{velocity of COM of } l_2}$$

~~$V_{C_2} = l_1\dot{q}_1$~~

Note: $$V_{C_2} = (l_1\dot{q}_1)^2 + \left(\frac{l_2}{2}\dot{q}_2\right)^2 + 2 l_1\dot{q}_1 + \frac{l_2}{2}\dot{q}_2 \cos(q_2 - q_1)$$

$$V = m_1 g \frac{l_1}{2}\sin q_1 + m_2 g\left(l_1 \sin q_1 + \frac{l_2}{2}\sin q_2\right) + \frac{1}{3}m_1 l_1^2 \ddot{q}_1$$

$$+ m_2 l_1^2 \ddot{q}_1 + m_2 \frac{l_1 l_2}{2}\ddot{q}_2 \cos(q_2 - q_1) - m_2 \frac{l_1 l_2}{2}\dot{q}_2(\dot{q}_2 - \dot{q}_1)\sin(q_2 - q_1)$$

$$+ m_1 g \frac{l_1}{2}\cos q_1 + m_2 g l_1 \cos q_1 = \tau_1$$

$$\frac{1}{3}m_2 l_2^2 \ddot{q}_2 + m_2 \frac{l_2^2}{24}\ddot{q}_2 + m_2 \frac{l_1 l_2}{2}\ddot{q}_1 \cos(q_1 - q_2) - m_2 \frac{l_1 l_2}{2}\dot{q}_1 \cancel{(\dot{q}_2 - \dot{q}_1)}$$

$$(\dot{q}_2 - \dot{q}_1)\sin(q_2 - q_1) + m_2 g \frac{l_2}{2}\sin q_2 = \tau_2$$

— (6)

$f_x = \cancel{k}\, K x \quad , \quad f_y = K y$

→ $f_x = K_x (x - x_0) \quad , \quad f_y = K_y (y - y_0)$ [General eq.]

From eq. 1

$f_x = \cancel{k}\, k (l_1 c\theta_1 + l_2 c\theta_2)$

$f_y = k (l_1 s\theta_1 + l_2 s\theta_2)$

From eq. 4

[Eq. 7]

$$\begin{cases} k(l_1 s\theta_1 + l_2 s\theta_2)\, l_2 c\theta_2 - k(l_1 c\theta_1 + l_2 c\theta_2)\, l_2 s\theta_2 = \tau_{2s} \\ k(l_1 s\theta_1 + l_2 s\theta_2)\, l_1 c\theta_1 - k(l_1 c\theta_1 + l_2 c\theta_2)\, l_1 s\theta_1 = \tau_{1s} \end{cases}$$