Tutorial VIII - Exercise Sheet 7

Exercise 1: Show that if $Y_n \to Y$ in L^p, $1 \le p \le \infty$, then $\mathbb{E}(Y_n | \mathcal{G}) \to \mathbb{E}(Y|\mathcal{G})$ in L^p.

Recall that $|\cdot|^p$ is convex for $1 \le p \le \infty$ (with $p = \infty$ denoting the ess sup norm.)

Therefore,

$$\|\mathbb{E}(Y_n|\mathcal{G}) - \mathbb{E}(Y|\mathcal{G})\|_{L^p} = \mathbb{E}\,|\mathbb{E}(Y_n|\mathcal{G}) - \mathbb{E}(Y|\mathcal{G})|^p$$

$$= \mathbb{E}\,|\mathbb{E}(Y_n - Y\,|\mathcal{G})|^p \underset{\text{Jensen}}{\le} \mathbb{E}\left(\mathbb{E}(|Y_n - Y|^p|\mathcal{G})\right)$$

$$= \mathbb{E}\,|Y_n - Y|^p = \|Y_n - Y\|_{L^p} \to 0, \text{ as } n \to \infty.$$

Exercise 2 Let $T_a = \inf\{t : B_t = a\}$, $a \geq 0$, then prove that

$$\mathbb{E}^0 \exp(-\lambda T_a) = \exp\{-a\sqrt{2\lambda}\}$$

Recall that $M_t = e^{\sigma X_t - \frac{1}{2}\sigma^2 t}$ is a $\mathcal{F}_t$-mg/c. for σ

Moreover, T_a is an $\mathcal{F}_t$-stopping time $\Rightarrow T_a \wedge t$ is a bounded stopp. time $\forall t \geq 0$. Assume that $a \geq 0$

Hence $\mathbb{E}^0 M_{T_a \wedge t} = \mathbb{E}^0 M_0 = 1$,

where $\mathbb{E}^0 M_{T_a \wedge t} = \mathbb{E}^0 \exp\{\sigma X_{T_a \wedge t} - \frac{1}{2}\sigma^2 (T_a \wedge t)\} = 1$

Taking $t \to \infty$ by BCT $(a \geq 0)$, we have

$$\text{LHS} \to \mathbb{E}^0 \exp\{\sigma X_{T_a} - \tfrac{1}{2}\sigma^2 T_a\} = \mathbb{E} \exp\{\sigma a - \tfrac{1}{2}\sigma^2 T_a\} = 1$$

Setting $\frac{\sigma^2}{2} = \lambda \geq 0$, or $\sigma = \sqrt{2\lambda}$, we get $\mathbb{E}^0 \exp\{-\lambda T_a\} = \exp\{-a\sqrt{2\lambda}\}$

Exercise 3 : $T = \inf\{t \geq 0 : X_t \notin (-a,a)\}$

Find a Mgle of the form $X_t^6 - c_1 t X_t^4 + c_2 t^2 X_t^2 - c_3 t^3$ and use it to calculate $E_0 T^3$.

Recall the following lemma: (Thm 7.5.5 in Durrett) $u(t,x)$ is a polynomial in t and x. Solving $\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0$. Then $u(t, X_t)$ is a Mgle.

Let $u(t,x) = x^6 - c_1 t x^4 + c_2 t^2 x^2 - c_3 t^3$.

Then
$$\frac{\partial u}{\partial t} = -c_1 x^4 + 2c_2 t x^2 - 3c_3 t^2$$
$$\frac{\partial u}{\partial x} = 6x^5 - 4c_1 t x^3 + 2c_2 t^2 x$$
$$\frac{\partial^2 u}{\partial x^2} = 30x^4 - 12c_1 t x^2 + 2c_2 t^2$$

Then,
$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = -c_1 x^4 + 2c_2 t x^2 - 3c_3 t^2 + 15x^4 - 6c_1 t x^2 + c_2 t^2$$
$$= (15 - c_1)x^4 + (2c_2 t - 6c_1 t)x^2 + (c_2 - 3c_3)t^2$$

and by setting this $= 0$, we get

$$\begin{cases} c_1 = 15 \\ 2c_2 - 6c_1 = 0 \Rightarrow c_2 = 3c_1 = 45 \\ c_2 - 3c_3 = 0 \Rightarrow c_3 = \frac{c_2}{3} = 15 \end{cases}$$

Then, $X_t^6 - 15tX_t^4 + 45t^2X_t^2 - 15t^3$ is an $\mathcal{F}_t$-mgle.

Therefore $\mathbb{E}_0 u(t\wedge T, X_{t\wedge T}) = \mathbb{E}_0 u(0, X_0)$, since $t\wedge T$ is a bold stopping time.

Then $0 = \mathbb{E} X^6_{T\wedge t} - 15\, \mathbb{E}(T\wedge t)\, X^4_{T\wedge t} + 45\, \mathbb{E}\, (T\wedge t)^2 X^2_{T\wedge t} - 15\, \mathbb{E}(T\wedge t)^3$

$\overset{|X_T| = |a|}{=} a^6 - 15 a^4\, \mathbb{E}(T\wedge t) + 45 a^2\, \mathbb{E}(T\wedge t)^2 - 15\, \mathbb{E}(T\wedge t)^3$

$$\Rightarrow \mathbb{E}_0 (T\wedge t)^3 = \frac{1}{15}\left(a^6 - 15a^4\, \mathbb{E}(T\wedge t) + 45 a^2\, \mathbb{E}(T\wedge t)^2\right) \quad (*)$$

we must calculate $\mathbb{E}_0(T\wedge t)$, $\mathbb{E}_0(T\wedge t)^2$.

- Note that $\mathbb{E}_0(X_t^2 \mid \mathcal{F}_s) = \mathbb{E}_0((X_t - X_s + X_s)^2 \mid \mathcal{F}_s)$

$\Rightarrow \mathbb{E}_0((X_t - X_s)^2 \mid \mathcal{F}_t) + \mathbb{E}_0(X_s^2 \mid \mathcal{F}_s) + 2\,\mathbb{E}_0(\cancel{(X_t - X_s) X_s \mid \mathcal{F}_s})$

$= (t-s) + X_s^2 \Rightarrow \mathbb{E}_0(X_t^2 - t \mid \mathcal{F}_s) = X_s^2 - s$

This suggests that $X_t^2 - t$ is an $\mathcal{F}_t$-mgle.

Therefore $\mathbb{E}_0(X^2_{T\wedge t} - T\wedge t) = 0 \Rightarrow \mathbb{E}_0(T\wedge t) = \mathbb{E}_0(X^2_{T\wedge t})$

$$\mathbb{E}_0(T\wedge t) \xrightarrow{\text{MCT}} \mathbb{E}_0 T, \qquad \mathbb{E}_0(X^2_{T\wedge t}) \xrightarrow{\text{BCT}} a^2$$

Therefore $\mathbb{E}_0 T = a^2 \quad (**)$

- For $\mathbb{E}(T\wedge t)^2$, we must find a different martingale.

Claim: $X^4_{T\wedge t} - 6(T\wedge t) X^2_{T\wedge t} + 3(T\wedge t)^2$ is an $\mathcal{F}_t$-mgle. We check this using the Lemma.

we have $u(t,x) = x^4 - 6t x^2 + 3t^2$. Then

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = -6x^2 + 6t + \frac{1}{2}\left[12x^2 - 12t\right] = 0$$

$\Rightarrow u(T\wedge t, X_{T\wedge t})$ is a Mgle

Then, we get $\mathbb{E}_0 u(0, X_0) = 0 = \mathbb{E}_0 u(T\wedge t, X_{T\wedge t})$

$\Rightarrow -3\,\mathbb{E}_0(T\wedge t)^2 = \mathbb{E}_0(X^4_{T\wedge t}) - 6\,\mathbb{E}_0((T\wedge t) X^2_{T\wedge t})$

We know that $\mathbb{E}_0 T^2 = a^2 < \infty$

Then, as $t \to \infty$, $\mathbb{E}_0 (T \wedge t)^2 \to \mathbb{E}_0 T^2$ by MCT

$\mathbb{E}_0 (X_{T \wedge t})^4 \to \mathbb{E}_0 X_T^4 = a^4$ by BCT

$\mathbb{E}_0 ((T \wedge t) X^2_{T \wedge t}) \to \mathbb{E}_0 T \cdot X_T^2$ by DCT

and moreover $\mathbb{E}_0 T X_T^2 = a^2 \mathbb{E}_0 T = a^4$.

Combining these we get $\mathbb{E}_0 T^2 = -\frac{1}{3}\left[a^4 - 6a^4\right] = \frac{5}{3} a^4$ (***)

Now combining (*), (**), (***) and sending $t \to \infty$ with DCT,

$$\mathbb{E}_0 T^3 = \frac{1}{15}\left[a^6 - 15 a^4 \cdot a^2 + 45 \cdot a^2 \cdot \frac{5}{3} \cdot a^4\right]$$

$$= \frac{1}{15}\left[a^6 - 15 a^6 + 75 a^6\right] = \frac{1}{15}\left[61 a^6\right] = \frac{61}{15} a^6$$

Exercise 4: Show that $t \mapsto X_t(\omega)$ are a.s. $\omega \in \mathcal{C}$ non differentiable.

(Dvoretzky - Erdős - Kakutani, '61)

Set, $n \geq 1$, $A_n = \{\omega \in \mathcal{C} : X(\omega)$ is nowhere diff in $[0,n)\}$

we don't know if A_n is measurable, so we show that $\mathcal{C} \setminus A_n \subseteq N_n$, for some null set N_n.

Assume that X is differentiable at $t_0 \in [0,n)$. Then X is continuous at $t_0 \Rightarrow \exists \delta > 0$, $L > 0$ s.t. $\forall t \in B(t_0, \delta)$

$$|X(t) - X(t_0)| \leq L|t - t_0|$$

Consider the grid of $[0,n)$ $\{j/k,\ j = 1, -, nk\}$

and $\exists$ k large enough s.t.

$$t_0 \leq j/k \quad \text{and} \quad j/k,\ j+1/k,\ j+2/k,\ j+3/k \in B(t_0, \delta)$$

In particular, for $r = j+1, j+2, j+3$:

$$|X(r/k) - X(r-1/k)| \leq |X(r/k) - X(t_0)| + |X(t_0) - X(r-1/k)|$$
$$\leq L\left(|r/k - t_0| + |t_0 - r-1/k|\right)$$
$$\leq L\left(4/k + 3/k\right) = \frac{7L}{k}$$

Define $C_m^L = \bigcap_{k=m}^{\infty} \bigcup_{j=1}^{kn} \bigcap_{r=j+1}^{j+3} \left\{|X(r/k) - X(r-1/k)| \leq \frac{7L}{k}\right\}$

Then $\mathcal{C} \setminus A_n \subseteq \bigcup_{L=1}^{\infty} \bigcup_{m=1}^{\infty} C_m^L$

If $W^0(C_m^L) = 0$ for all $m, L \geq 1$ we are done.

for $k \geq m$,

$$\mathbb{W}^0(C^L_m) \leq \mathbb{W}^0\Big(\bigcup_{j=1}^{kn} \bigcap_{r=j+1}^{j+3} \{|X(r/k) - X(r-1/k)| \leq \tfrac{7L}{k}\}\Big)$$

$$\leq \sum_{j=1}^{kn} \mathbb{W}^0\Big(\bigcap_{r=j+1}^{j+3} \{|X(r/k) - X(r-1/k)| \leq \tfrac{7L}{k}\}\Big)$$

$$\overset{\text{indep. increm.}}{\leq} \sum_{j=1}^{kn} \prod_{r=j+1}^{j+3} \mathbb{W}^0\big(|X(r/k) - X(r-1/k)| \leq 7L/k\big)$$

$$\overset{\text{stat. increm.}}{=} \sum_{j=1}^{kn} \Big(\mathbb{W}^0\big(|X(1/k)| \leq 7L/k\big)\Big)^3$$

$$\leq kn \cdot \Big(\mathbb{W}^0\big(|X(1/k)| \leq 7L/k\big)\Big)^3$$

Now $\mathbb{W}^0\big(|X(1/k)| \leq 7L/k\big) \underset{\uparrow}{=} \mathbb{W}^0\big(|X(1)| \leq \frac{7L}{\sqrt{k}}\big) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{7L}{\sqrt{k}}}^{\frac{7L}{\sqrt{k}}} \underbrace{e^{-x^2/2}}_{\leq 1} dx$

$X(1/k) \overset{d}{=} \frac{X(1)}{\sqrt{k}}$

$$\leq \frac{2 \cdot 7L}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{k}}$$

Therefore $\mathbb{W}^0(C_m^L) \leq k \cdot n \cdot \Big(\frac{14L}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{k}}\Big)^3$

$$= n \cdot \Big(\frac{14L}{\sqrt{2\pi}}\Big)^3 \cdot \frac{1}{\sqrt{k}} \to 0, \text{ as } k \to \infty.$$

Since $\forall \varepsilon > 0$ we can find k large enough s.t.

$$\mathbb{W}^0(C_m^L) \leq n \cdot \Big(\frac{14L}{\sqrt{2\pi}}\Big)^3 \cdot \frac{1}{\sqrt{k}} < \varepsilon \Rightarrow \mathbb{W}^0(C_m^L) = 0$$

$\Rightarrow$ $e \setminus A_n$ is a null event

$\Rightarrow$ A_n is an a.s. event.