Cyclical Long Memory: Decoupling, Modulation, and Modeling

Stefanos Kechagias[1], Vladas Pipiras[2], and Pavlos Zoubouloglou*[2]

[1]TBD

[2]Department of Statistics and Operations Research, UNC Chapel Hill

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Abstract

A new model for general cyclical long memory is introduced, by means of random modulation of certain bivariate long memory time series. This construction essentially decouples the two key features of cyclical long memory: quasi-periodicity and long-term persistence. It further allows for a general cyclical phase in cyclical long memory time series. Several choices for suitable bivariate long memory series are discussed, including a parametric fractionally integrated vector ARMA model. The parametric models introduced in this work have explicit autocovariance functions that can be used readily in simulation, estimation, and other tasks.

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1 Introduction

The main goal of this work is to shed further light on the phenomenon of the so-called cyclical long memory concerning stationary time series. Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a second-order stationary time series with zero mean $\mathbb{E}X_n = 0$ for simplicity and autocovariance function (ACVF) $\gamma_X(h) = \mathbb{E}X_hX_0,\ h \in \mathbb{Z}$. If it exists, denote the spectral density of X by $f_X(\lambda), \lambda \in (-\pi, \pi)$. Under mild assumptions (and by the chosen convention), one expects that $f_X(\lambda) = (2\pi)^{-1}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_X(\lambda),\ \lambda \in (-\pi, \pi)$. By symmetry, it is enough to focus on $f_X(\lambda),\ \lambda \in (0, \pi)$.

The time series X exhibits *Cyclical Long Memory* (CLM, for short) if

$$\begin{aligned}\gamma_X(h) &\simeq c_\gamma \cos(\lambda_0 h + \phi)h^{2d-1} \\ &= c_{\gamma,1}\cos(\lambda_0 h)h^{2d-1} + c_{\gamma,2}\sin(\lambda_0 h)h^{2d-1}, \quad \text{as } h \to \infty,\end{aligned} \tag{1.1}$$

where $d \in (0, 1/2)$, $\lambda_0 \in (0, \pi), c_\gamma > 0$, and $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The notation $\gamma_1(h) \simeq \gamma_2(h)$ in (1.1) and throughout this work will stand for $\gamma_1(h) - \gamma_2(h) = o(h^{2d-1})$, as $h \to \infty$. A formal definition is given and investigated in the appendix. The parameter d is known as *long memory* parameter, and we will refer to λ_0 and ϕ as *cyclical frequency* and *cyclical phase* parameters. In fact, as shown in the paper (Appendix C.3), the cyclical phase

$$\phi \in \mathcal{I}_d \doteq \left[\left(d - \frac{1}{2}\right)\pi, \left(\frac{1}{2} - d\right)\pi\right], \tag{1.2}$$

*Corresponding author, e-mail: pavlos@ad.unc.edu.

which we refer to as the *set of admissible cyclical phase* parameters. The boundary points $\phi = \pm\left(\frac{1}{2}-d\right)\pi$ of the set play a special role as indicated below. The same also holds for $\phi = 0$.

In the spectral domain, CLM can be thought of as

$$f_X(\lambda) \sim \begin{cases} c_f^{+}(\lambda-\lambda_0)^{-2d}, & \text{as } \lambda \to \lambda_0^{+}, \\ c_f^{-}(\lambda_0-\lambda)^{-2d}, & \text{as } \lambda \to \lambda_0^{-}, \end{cases} \tag{1.3}$$

where $c_f^{+}, c_f^{-} \geq 0$ are constants so that $c_f^{+} + c_f^{-} > 0$. Namely, the spectral density diverges around the cyclical frequency λ_0. As usual, $\sim$ denotes asymptotic equivalence. Conditions and results for going from (1.3) to (1.1), and vice versa, are given in Appendices B and C below, along with relations connecting c_f^{+}, c_f^{-} and c_γ, ϕ. These results are of independent interest and, to the best of our knowledge, are new in this area. In particular, we have that

$$\phi = 0 \quad \text{if and only if} \quad c_f^{+} = c_f^{-} \doteq c_f, \tag{1.4}$$

so that (1.3) becomes $f_Y(\lambda) \sim c_f|\lambda-\lambda_0|^{-2d}$, as $\lambda \to \lambda_0$, that is, the divergence around $\lambda = \lambda_0$ is symmetric. In particular, the boundary points satisfy the relation:

$$\phi = \pm\left(\frac{1}{2}-d\right)\pi \quad \text{if and only if} \quad c_f^{\mp} = 0. \tag{1.5}$$

But we also caution the reader that the "boundary" case (1.5) needs to be treated more carefully, as explained in Appendices B.2 and C.2. We note in that regard that, by our convention, for example, $c_f^{+} = 0$ in (1.3) stands for $(\lambda-\lambda_0)^{2d} f_X(\lambda) \to 0$ as $\lambda \to \lambda_0^{+}$. It subsumes more refined asymptotics as $f_X(\lambda) \sim c(\lambda-\lambda_0)^{-2\delta}$ with $\delta < d$, which need to be taken into account as presented in, e.g., Appendix B.2. The models considered below will satisfy both (1.1) and (1.3). When $\lambda_0 = 0$, CLM becomes the usual well-studied Long Memory (LM) (e.g., [7, 17, 33, 36]), but the focus here is on $\lambda_0 > 0$. CLM is sometimes also called seasonal LM, though some authors use the latter term more specifically for the so-called seasonal FARIMA model.

The origins of CLM go back at least to the celebrated paper by Hosking [20], who considered fractional differencing in modeling LM, but in the last paragraph of the paper, also discussed briefly a CLM model exhibiting "both long-term peristance and quasiperiodic behavior." The model became known as the Gegenbauer series and has drawn most of the attention in this area, especially as likely being the most popular parametric model. Various developments and extensions for this model (ARMA counterparts, multiple divergence frequences, estimation, multivariate constructions, and so on) can be found in, e.g., [1, 11, 14, 16, 18, 19, 38, 39]. A review can be found in [10]. We note that the corresponding models usually exhibit CLM with cyclical phase $\phi = 0$ only and that their ACVFs are generally challenging to compute [5, 27]. Other aspects of CLM (limit theorems, semi-parametric estimations, and so on) were considered in [3, 5, 6, 15, 21, 28–30].

Our own interest in CLM stems from its relevence to naval applications [34]. Stationary spatio-temporal models are available in Naval Architecture, Oceanography and related fields for the wave height at given spatial locations and over time. The Longuet-Higgins model is a celebrated example [24] and involves a spectrum function, which can be thought of as the spectrum of the wave height process at a fixed location. The process has short memory. However, if the wave height is measured at a location traveling at a constant speed, the resulting process can be shown to exhibit CLM. This is relevant to ships traveling at constant speed, whose motions then inherit the CLM from that of the associated wave excitation. Aspects of this behavior have been known in Naval Architecture for years (e.g., [9, 23]), well before the paper by Hosking [20], but connections to CLM have been clarified only recently in [34]. As far as we know, this seems to be the first physical model of CLM in the sense of being constructed from first (physics) principles. We also note that the CLM phenomenon in the naval applications is associated with a non-zero cyclical phase $\phi \neq 0$.

A particularly interesting feature of the definition (1.1) is its multiplicative nature as the product of $\cos(\lambda_0 h + \phi)$ and h^{2d-1}. The term h^{2d-1} corresponds to the usual LM behavior, while the term $\cos(\lambda_0 h+\phi)$ is associated with modulation. This suggests that CLM series could be viewed as modulated LM series, where the LM and cyclical effects are decoupled. In fact, when $\phi = 0$, such modulated series exhibiting CLM are straightforward to construct. Take two independent copies $\{Y_{1,n}\}_{n\in\mathbb{Z}}, \{Y_{2,n}\}_{n\in\mathbb{Z}}$ of LM series Y satisfying $\gamma_Y(h) \sim c_\gamma h^{2d-1}$, as $h \to \infty$. Set

$$X_n \doteq \cos(\lambda_0 n) Y_{1,n} + \sin(\lambda_0 n) Y_{2,n}, \quad n \in \mathbb{Z}. \tag{1.6}$$

By construction,

$$\begin{aligned} \mathbb{E}X_{n+h}X_n &= \big(\cos(\lambda_0 h)\cos(\lambda_0(n+h)) + \sin(\lambda_0 h)\sin(\lambda_0(n+h))\big)\gamma_Y(h) \\ &= \cos(\lambda_0 h)\gamma_Y(h) \sim c_\gamma \cos(\lambda_0 h) h^{2d-1}, \quad \text{as } h \to \infty, \end{aligned} \tag{1.7}$$

satisfies (1.1) with $\phi = 0$. In the CLM literature, this construction was studied and exploited in applications in [26, 35], where the model (1.6) was termed the *fractional sinusoidal waveform process.*

The model (1.6) has in fact a long history in signal processing, where its construction is known as *random modulation* (RMod, for short). See, for example, a review paper [31]. A textbook on probability for engineers [32] also includes a section (Section 11.3) on this topic. In the context of random modulation, the series $\{Y_{1,n}\}_{n\in\mathbb{Z}}$ and $\{Y_{2,n}\}_{n\in\mathbb{Z}}$ need not to be independent for the Rmod model (1.6) to yield a stationary series. A sufficient condition is for the vector series

$$Y_n = \begin{pmatrix} Y_{1,n} \\ Y_{2,n} \end{pmatrix} \tag{1.8}$$

to be second-order stationary with the matrix ACVF

$$\gamma_Y(h) = \mathbb{E}Y_{n+h}Y_n^T = \begin{pmatrix} \mathbb{E}Y_{1,n+h}Y_{1,n} & \mathbb{E}Y_{1,n+h}Y_{2,n} \\ \mathbb{E}Y_{2,n+h}Y_{1,n} & \mathbb{E}Y_{2,n+h}Y_{2,n} \end{pmatrix} \doteq \begin{pmatrix} \gamma_{Y,11}(h) & \gamma_{Y,12}(h) \\ \gamma_{Y,21}(h) & \gamma_{Y,22}(h) \end{pmatrix} \tag{1.9}$$

satisfying

$$\gamma_{Y,11}(h) = \gamma_{Y,22}(h) \text{ and } \gamma_{Y,12}(h) = -\gamma_{Y,21}(h) \text{ for all } h \in \mathbb{Z}. \tag{P-T}$$

Under (P-T), the ACVF of the RMod model (1.6) can be checked to be (see the proof of Proposition 3.3 below)

$$\gamma_X(h) = \cos(\lambda_0 h)\gamma_{Y,11}(h) + \sin(\lambda_0 h)\gamma_{Y,12}(h). \tag{1.10}$$

Papoulis [31] reviews a number of questions addressed in the past regarding the RMod series (1.6) under the assumption (P-T). We will draw connections to this literature below, but one important difference is that the focus here will be on LM series. The property P-T is reformulated in the spectral domain in Proposition 2.3 below.

For uncorrelated LM series $\{Y_{1,n}\}_{n\in\mathbb{Z}}$ and $\{Y_{2,n}\}_{n\in\mathbb{Z}}$, the expression (1.10) reduced to (1.7) and led to CLM (1.1) with $\phi = 0$. The presence of both cosine and sine in (1.10) suggests that *correlated* LM series $\{Y_{1,n}\}_{n\in\mathbb{Z}}$ and $\{Y_{2,n}\}_{n\in\mathbb{Z}}$ might lead to the RMod series (1.6) exhibiting CLM with general cyclical phase ϕ. This leads to the following main questions addressed in this work:

Q1: Are there correlated LM series $\{Y_{1,n}\}_{n\in\mathbb{Z}}$ and $\{Y_{2,n}\}_{n\in\mathbb{Z}}$ (or bivariate LM series Y in (1.8)) having property (P-T) such that the RMod series (1.6) is CLM satisfying (1.1) with general ϕ?

Q2: Are there parametric bivariate LM series Y in Q1 which are suited for modeling CLM and related tasks, especially for resulting CLM models having explicit ACVF?

We show in this work that the answers to both questions are affirmative. More specifically, we characterize the bivariate LM series in Q1 and establish conditions for the resulting RMod model (1.6) to have CLM with a particular phase ϕ. We also propose a parametric model for such series Y with explicit ACVF, and hence the same for the RMod model (1.6) in view of the relation (1.10). This offers computational and modeling advantages over the Gegenbauer models discussed above. Various additional contributions regarding this construction and CLM will also be made, where we would draw the reader's attention to the extensions of our approach allowing for the exponents in (1.3) as $\lambda \to \lambda_0^+$ and $\lambda \to \lambda_0^-$ to be different, that is,

$$f_X(\lambda) \sim \begin{cases} c_{f,+}(\lambda - \lambda_0)^{-2d_+}, & \text{as } \lambda \to \lambda_0^+, \\ c_{f,-}(\lambda_0 - \lambda)^{-2d_-}, & \text{as } \lambda \to \lambda_0^-, \end{cases} \tag{1.11}$$

where $c_{f,+}, c_{f,-} > 0$ and $d_-, d_+ \in (0, 1/2)$. We naturally construct such series X as $X = X^+ + X^-$ by taking uncorrelated X^+, X^- satisfying (1.3) with

$$f_{X^+}(\lambda) \sim \begin{cases} c_{f,+}(\lambda - \lambda_0)^{-2d_+}, & \text{as } \lambda \to \lambda_0^+, \\ 0 \cdot (\lambda_0 - \lambda)^{-2d_+}, & \text{as } \lambda \to \lambda_0^-, \end{cases} \quad f_{X^-}(\lambda) \sim \begin{cases} 0 \cdot (\lambda - \lambda_0)^{-2d_-}, & \text{as } \lambda \to \lambda_0^+, \\ c_{f,-}(\lambda_0 - \lambda)^{-2d_-}, & \text{as } \lambda \to \lambda_0^-. \end{cases} \tag{1.12}$$

We, however, need to take into account the delicate issues around the "boundary" cases in (1.12) as noted following (1.5).

The rest of the paper is organized as follows. In Section 2, we recall some facts about bivariate LM series and provide some results on bivariate series satisfying (P-T). In Section 3, we establish connections between bivatiate LM and CLM series through the RMod construction (1.6) and provide a parametric model for the bivariate and CLM series. Section 4 contains extensions. Section 5 concludes. Appendices A–D gather results relating the definitions (1.1) and (1.3) of CLM.

2 Preliminaries

As discussed in Section 1, our CLM model will involve bivariate LM series and RMod. We recall basic facts about bivariate LM in Section 2.1 below. We also provide one general construction of bivariate series that satisfy property (P-T) and hence that can be used in the random modulation model (1.6); see Section 2.2. This construction will be adapted to bivariate LM series in Section 3 to construct a parametric model of CLM.

2.1 Bivariate Long Memory

Let $Y_n = (Y_{1,n}, Y_{2,n})^T, n \in \mathbb{Z}$, be a bivariate second-order stationary time series with matrix ACVF $\gamma_Y(h), h \in \mathbb{Z}$, in (1.9). Again, we assume $\mathbb{E}Y_n = 0$ for simplicity. The respective spectral density will be denoted $f_Y(\lambda), \lambda \in (-\pi, \pi)$. Under mild assumptions, one expects $f_Y(\lambda) = (2\pi)^{-1} \sum_{h=-\infty}^{\infty} e^{-i\lambda h}\gamma_Y(h)$.

The time series Y is called *bivariate LM* if one of the following two conditions is satisfied:

- Time domain: as $h \to \infty$,

$$\gamma_Y(h) = \begin{pmatrix} \gamma_{Y,11}(h) & \gamma_{Y,12}(h) \\ \gamma_{Y,21}(h) & \gamma_{Y,22}(h) \end{pmatrix} \sim \begin{pmatrix} R_{11}h^{2d_1-1} & R_{12}h^{d_1+d_2-1} \\ R_{21}h^{d_1+d_2-1} & R_{22}h^{2d_2-1} \end{pmatrix}, \tag{2.1}$$

  where $d_j \in (0, 1/2)$, $R_{jk} \in \mathbb{R}, j, k = 1, 2$, and $R_{11}, R_{22} \neq 0$.

- Spectral domain: as $\lambda \to 0^+$,

$$f_Y(\lambda) = \begin{pmatrix} f_{Y,11}(\lambda) & f_{Y,12}(\lambda) \\ f_{Y,21}(\lambda) & f_{Y,22}(\lambda) \end{pmatrix} \sim \begin{pmatrix} g_{11}\lambda^{-2d_1} & (g_{12}e^{-i\omega})\lambda^{-d_1-d_2} \\ (g_{12}e^{i\omega})\lambda^{-d_1-d_2} & g_{22}\lambda^{-2d_2} \end{pmatrix}, \tag{2.2}$$

  where $d_j \in (0, 1/2), j = 1, 2$, $g_{11}, g_{22} > 0, g_{12} \in \mathbb{R}$, and $\omega \in (-\pi, \pi)$.

Under mild assumptions, the time-domain and spectral-domain definitions above can be shown to be equivalent (Proposition 2.1 in [22]). The assumptions will be satisfied for the models considered below, so we will use them interchangeably. There are also explicit formulas relating $R_{11}, R_{12}, R_{21}, R_{22}$ and $g_{11}, g_{12}, g_{21}, \omega$ in [22].

The parameter ω in (2.2) is called the *bivariate phase parameter*. It controls the (a)symmetry of the series at large lags, that is, one has $\omega = 0$ if and only if $R_{12} = R_{21}$ in (2.1). The parameter ω should not be confused with the cyclical phase parameter ϕ in (1.1), even if both are related to symmetry (see (1.4) for ϕ).

A way to construct bivariate LM series is through two-sided linear representations of the form

$$Y_n = \sum_{l=-\infty}^{\infty} A_l \varepsilon_{n-l}, \quad n \in \mathbb{Z}, \tag{2.3}$$

where $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ is a white noise series (i.e., $\mathbb{E}\varepsilon_n = 0$, $\mathbb{E}\varepsilon_n\varepsilon_m^T = 0, n \neq m$, and $\mathbb{E}\varepsilon_n\varepsilon_n^T = I_2$) and $A_l \in \mathbb{R}^{2\times 2}$ are such that

$$A_l \sim \begin{cases} \begin{pmatrix} A_{11}^{+} l^{d_1-1} & A_{12}^{+} l^{d_1-1} \\ A_{21}^{+} l^{d_2-1} & A_{22}^{+} l^{d_2-1} \end{pmatrix}, & l \to +\infty, \\ \begin{pmatrix} A_{11}^{-} (-l)^{d_1-1} & A_{12}^{-} (-l)^{d_1-1} \\ A_{21}^{-} (-l)^{d_2-1} & A_{22}^{-} (-l)^{d_2-1} \end{pmatrix}, & l \to -\infty. \end{cases} \tag{2.4}$$

There are similar formulas relating $A_{jk}^{\pm}$ in (2.4) to $R_{j,k}$ in (2.1) (Proposition 3.1 in [22]).

2.2 Bivariate Series for Random Modulation

Consider the RMod model in 1.6. Recall from Section 1 that the bivariate stationary series $Y = \{Y_{j,n}\}_{j=1,2,n\in\mathbb{Z}}$ should satisfy property (P-T) for the RMod model to be stationary. In this section, we find one convenient representation of the series Y that will satisfy (P-T). More specifically, consider the two-sided linear representation (2.3), where $A_l \in \mathbb{R}^{2\times 2}$ are such that $\sum_{l=-\infty}^{\infty} \|A_l\|_F^2 < \infty$, where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix.

Proposition 2.1. *Let a bivariate second-order stationary series $Y = \{Y_n\}_{n\in\mathbb{Z}}$ be given by* (2.3)*. If*

$$A_l = \begin{pmatrix} a_{0,l} & a_{1,l} \\ -a_{1,l} & a_{0,l} \end{pmatrix}, \quad l \in \mathbb{Z}, \tag{2.5}$$

then Y satisfies property (P-T)*.*

Proof. Recall that $\mathbb{E}\varepsilon_n\varepsilon_n^T = I_2$. Then,

$$\gamma_Y(h) = \mathbb{E}Y_h Y_0^T = \sum_{l=-\infty}^{\infty} A_{l+h} A_l^T = \sum_{l=-\infty}^{\infty} \begin{pmatrix} a_{0,l+h} & a_{1,l+h} \\ -a_{1,l+h} & a_{0,l+h} \end{pmatrix} \begin{pmatrix} a_{0,l} & -a_{1,l} \\ a_{1,l} & a_{0,l} \end{pmatrix}$$

$$= \sum_{l=-\infty}^{\infty} \begin{pmatrix} a_{0,l+h}a_{0,l} + a_{1,l+h}a_{1,l} & -a_{0,l+h}a_{1,l} + a_{1,l+h}a_{0,l} \\ -a_{1,l+h}a_{0,l} + a_{0,l+h}a_{1,l} & a_{0,l+h}a_{0,l} + a_{1,l+h}a_{1,l} \end{pmatrix}.$$

This says that γ_Y satisfies the required property. □

Remark 2.2. *We conjecture that any second-order stationary series Y satisfying property* (P-T) *has a linear representation* (2.3) *with coefficient matrices given by* (2.5)*.*

Property (P-T) and the resulting ACVF in (1.10) were expressed in the time domain. In the following result, we express them in the spectral domain.

Proposition 2.3. *Suppose the matrix ACVF γ_Y of a bivariate series Y satisfies property* (P-T). *Assume that Y has the spectral density $f_Y(\lambda) = (f_{Y,jk}(\lambda))_{j,k=1,2}$ connected to γ_Y through the usual relations: $f_Y(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-i\lambda h}\gamma_Y(h)$, $\gamma_Y(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda$. Then, property* (P-T) *is equivalent to:*

$$f_{Y,11}(\lambda) = f_{Y,22}(\lambda) \quad \text{and} \quad f_{Y,12}(\lambda) = -f_{Y,21}(\lambda), \quad \text{for all } \lambda \in (-\pi,\pi). \tag{P-S}$$

Furthermore, under (P-T), *the spectral density of the RMod series* (1.6) *can be expressed as*

$$f_X(\lambda) = \frac{1}{2}\left[f_{Y,11}(\lambda-\lambda_0) + f_{Y,11}(\lambda+\lambda_0)\right] + \frac{1}{2i}\left[f_{Y,12}(\lambda-\lambda_0) - f_{Y,12}(\lambda+\lambda_0)\right]. \tag{2.6}$$

Proof. The claim in (P-S) follows upon writing, by (P-T),

$$f_Y(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-i\lambda h}\gamma_Y(h) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-i\lambda h}\begin{pmatrix}\gamma_{Y,11}(h) & \gamma_{Y,12}(h)\\ -\gamma_{Y,12}(h) & \gamma_{Y,11}(h)\end{pmatrix}.$$

For the claim in (2.6), note that

$$\begin{aligned}
f_X(\lambda) &= \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_X(h)\\
&= \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\left(\cos(\lambda_0 h)\gamma_{Y,11}(h) + \sin(\lambda_0 h)\gamma_{Y,12}(h)\right)\\
&= \frac{1}{2}\frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\left[e^{-ih(\lambda-\lambda_0)}\gamma_{Y,11}(h) + e^{-ih(\lambda+\lambda_0)}\gamma_{Y,11}(h)\right]\\
&\quad + \frac{1}{2i}\frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\left[e^{-ih(\lambda-\lambda_0)}\gamma_{Y,12}(h) - e^{-ih(\lambda+\lambda_0)}\gamma_{Y,12}(h)\right]\\
&= \frac{1}{2}\left[f_{Y,11}(\lambda-\lambda_0) + f_{Y,11}(\lambda+\lambda_0)\right] + \frac{1}{2i}\left[f_{Y,12}(\lambda-\lambda_0) - f_{Y,12}(\lambda+\lambda_0)\right],
\end{aligned}$$

where we have used the formula for γ_X from (1.10) in the second line and the representations of cos and sin as complex exponentials in the third. □

We next provide a linear representation for the RMod series $\{X_n\}_{n\in\mathbb{Z}}$ in (1.6) when the bivariate series $\{Y_n\}_{n\in\mathbb{Z}}$ assumes the form in (2.3)–(2.5). Although a special case of the following result was proved in [35], we have not encountered an analogous result in the RMod literature. Recall the definition of the sign function

$$\operatorname{sign}(c) = \begin{cases} 1 & c > 0,\\ 0 & c = 0,\\ -1 & c < 0.\end{cases} \tag{2.7}$$

Theorem 2.4. *Let $\{X_n\}_{n\in\mathbb{Z}}$ and $\{Y_n\}_{n\in\mathbb{Z}}$ be given by* (1.6) *and* (2.3)–(2.5) *respectively. Then, $\{X_n\}_{n\in\mathbb{Z}}$ has the linear representation*

$$X_n = \sum_{j=-\infty}^{\infty}\begin{pmatrix}\cos(\lambda_0 j)a_{0,j} - \sin(\lambda_0 j)a_{1,j}\\ \cos(\lambda_0 j)a_{1,j} + \sin(\lambda_0 j)a_{0,j}\end{pmatrix}^T \tilde{\varepsilon}_{n-j}, \quad n\in\mathbb{Z},$$

where $\{\tilde{\varepsilon}_n\}_{n\in\mathbb{Z}}$ is a WN sequence defined as, for $n\in\mathbb{Z}$,

$$\tilde{\varepsilon}_n \doteq \begin{pmatrix}\cos(\lambda_0 n) & \operatorname{sign}(n)\sin(\lambda_0 n)\\ -\operatorname{sign}(n)\sin(\lambda_0 n) & \cos(\lambda_0 n)\end{pmatrix}\varepsilon_n.$$

In particular, $\mathbb{E}(\tilde{\varepsilon}_k\tilde{\varepsilon}_l^T) = I_2\delta_{kl}$, where δ_{kl} denotes the Kronecker delta.

Proof. Define the matrix

$$M \doteq \begin{pmatrix} \cos(\lambda_0) & \sin(\lambda_0) \\ -\sin(\lambda_0) & \cos(\lambda_0) \end{pmatrix}, \quad v_n \doteq \begin{pmatrix} \cos(\lambda_0 n) & \sin(\lambda_0 n) \end{pmatrix}^T, \quad v_0 \doteq \begin{pmatrix} 1 & 0 \end{pmatrix}^T,$$

and observe that the following identities hold

$$M^j = \begin{cases} \begin{pmatrix} \cos(\lambda_0 j) & \sin(\lambda_0 j) \\ -\sin(\lambda_0 j) & \cos(\lambda_0 j) \end{pmatrix} & \text{if } j \geq 1, \\ \begin{pmatrix} \cos(\lambda_0 j) & -\sin(\lambda_0 j) \\ \sin(\lambda_0 j) & \cos(\lambda_0 j) \end{pmatrix} & \text{if } j \leq 0. \end{cases} \qquad v_j^T = v_0^T M^j, \quad j \in \mathbb{Z},$$

Recall the linear representation of Y_n in (2.5), and write

$$X_n = v_n^T Y_n = v_0^T M^n \sum_{j=-\infty}^{\infty} A_j \varepsilon_{n-j} = \sum_{j=-\infty}^{\infty} v_0^T M^j M^{n-j} A_j \varepsilon_{n-j}.$$

Since for all $k \in \mathbb{Z}$, $M^k = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ for some α, β, the following commutativity property holds:

$$M^k A_l = A_l M^k,$$

for all $l, k \in \mathbb{Z}$. Therefore,

$$X_n = \sum_{j=-\infty}^{\infty} v_0^T M^j M^{n-j} A_j \varepsilon_{n-j} = \sum_{j=-\infty}^{\infty} v_0^T M^j A_j M^{n-j} \varepsilon_{n-j} = \sum_{j=-\infty}^{\infty} v_j^T A_j \tilde{\varepsilon}_{n-j} = \sum_{j=-\infty}^{\infty} \tilde{A}_j \tilde{\varepsilon}_{n-j},$$

where, for $k \in \mathbb{Z}$,

$$\tilde{A}_k \doteq v_k^T A_k = \begin{pmatrix} \cos(\lambda_0 k) a_{0,k} - \sin(\lambda_0 k) a_{1,k} \\ \cos(\lambda_0 k) a_{1,k} + \sin(\lambda_0 k) a_{0,k} \end{pmatrix}^T, \quad \tilde{\varepsilon}_k \doteq M^k \varepsilon_k.$$

In particular, upon noting that, for all $k \in \mathbb{Z}$, $M^k (M^k)^T = I_2$, we see that

$$\mathbb{E}\left(\tilde{\varepsilon}_k \tilde{\varepsilon}_k^T\right) = \mathbb{E}\left(M^k \varepsilon_k \varepsilon_k^T (M^k)^T\right) = M^k (M^K)^T = I_2.$$

This concludes the proof. $\square$

3 CLM Model From Random Modulation

3.1 CLM Model Construction

In view of Proposition 2.1, its condition (2.5), and the representation (2.3)–(2.4), it is now straightforward to construct bivariate LM series Y satisfying property (P-T). From property (P-T), note that for this Y, the limiting form of the ACVF in (2.1) necessarily has

$$d_1 = d_2 \doteq d, \tag{3.1}$$

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \doteq \begin{pmatrix} r_0 & -r_1 \\ r_1 & r_0 \end{pmatrix} \tag{3.2}$$

for some $r_0, r_1 \in \mathbb{R}, r_0 \neq 0$. In view of (3.2) and (P-S), the limiting form of the spectral density in (2.2) necessarily has

$$\begin{pmatrix} g_{11} & g_{12}e^{-i\omega} \\ g_{12}e^{i\omega} & g_{22} \end{pmatrix} = \begin{pmatrix} g_0 & ig_1 \\ -ig_1 & g_0 \end{pmatrix}, \tag{3.3}$$

for some $g_0, g_1 \in \mathbb{R}, g_0 \neq 0$, that is, a very special bivariate phase parameter $\omega = -\frac{\pi}{2}$. (Equivalently, we can set $\omega = \frac{\pi}{2}$, which would change the sign of g_1.)

Consider now a bivariatw LM series Y with the representation (2.3)–(2.4). For this Y to satisfy property (P-T), Proposition 2.1 requires A_l to have the form (2.5). This implies taking (3.1) and also $A_{11}^{\pm} = A_{22}^{\pm}$, $A_{12}^{\pm} = -A_{21}^{\pm}$, in (2.4). The memory parameter d aside, this formulation has 4 parameters (2 for + and 2 for −), compared to 2 parameters in (3.2) or (3.3). To reduce the number of parameters to 2, we suggest imposing

$$A^{+} \doteq \begin{pmatrix} A_{11}^{+} & A_{12}^{+} \\ A_{21}^{+} & A_{22}^{+} \end{pmatrix} = \begin{pmatrix} a_0 & a_1 \\ -a_1 & a_0 \end{pmatrix} = \begin{pmatrix} A_{11}^{-} & A_{12}^{-} \\ A_{21}^{-} & A_{22}^{-} \end{pmatrix}^T \doteq (A^{-})^T \tag{3.4}$$

for some $a_0, a_1 \in \mathbb{R}$. The transpose in (3.4) is important, e.g., without the transpose, the resulting model necessarily leads to $R_{12} = R_{21} = 0$.

The following two results relate the parameters in the various forms (3.1)–(3.4) above, and describe the CLM structure of the RMod series (1.6) resulting from the bivariate LM series Y chosen above. The fact that the RMod X in (1.6) will have CLM follows immediately from the construction above, since by using (1.10), (2.1), and (3.2),

$$\begin{aligned} \gamma_X(h) &= \gamma_{Y,11}(h)\cos(\lambda_0 h) + \gamma_{Y,12}(h)\sin(\lambda_0 h) \\ &\simeq R_{11}\cos(\lambda_0 h)h^{2d-1} + R_{12}\sin(\lambda_0 h)h^{2d-1} \\ &= [\cos(\lambda_0 h)r_0 - \sin(\lambda_0 h)r_1]\, h^{2d-1} \\ &= c_\gamma \cos(\lambda_0 h + \phi)h^{2d-1}, \quad \text{as } h \to \infty, \end{aligned} \tag{3.5}$$

for c_γ, ϕ specified in the second result below.

Proposition 3.1. *Let Y have the representation* (2.3)–(2.4) *with A_l in* (2.4) *satisfying* (2.5) *and* (3.4). *Let also the limiting form of the ACVF of Y (resp. spectral density) be given by* (3.2) *(resp.* (3.3)*). Then,*

$$r_0 = \frac{\Gamma^2(d)}{\Gamma(2d)}\left[(a_0^2 + a_1^2)\frac{1}{\cos(\pi d)} + (a_0^2 - a_1^2)\right], \quad r_1 = 2a_0 a_1 \frac{\Gamma^2(d)}{\Gamma(2d)} \tag{3.6}$$

and

$$g_0 = \frac{\Gamma(2d)}{\pi}\cos(\pi d) r_0, \quad g_1 = \frac{\Gamma(2d)}{\pi}\sin(\pi d) r_1. \tag{3.7}$$

Proof. The first part of (3.6) follows from Proposition 3.1 in [22]. To apply that proposition, we compute

$$\begin{aligned} C^1 &= A^{-}(A^{-})^T = \begin{pmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{pmatrix}\begin{pmatrix} a_0 & a_1 \\ -a_1 & a_0 \end{pmatrix} = \begin{pmatrix} a_0^2 + a_1^2 & 0 \\ 0 & a_0^2 + a_1^2 \end{pmatrix}, \\ C^2 &= A^{-}(A^{+})^T = \begin{pmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{pmatrix}\begin{pmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{pmatrix} = \begin{pmatrix} a_0^2 - a_1^2 & -2a_0a_1 \\ 2a_0a_1 & a_0^2 - a_1^2 \end{pmatrix}, \\ C^3 &= A^{+}(A^{+})^T = \begin{pmatrix} a_0 & a_1 \\ -a_1 & a_0 \end{pmatrix}\begin{pmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{pmatrix} = \begin{pmatrix} a_0^2 + a_1^2 & 0 \\ 0 & a_0^2 + a_1^2 \end{pmatrix}. \end{aligned}$$

This leads to

$$\begin{aligned} R_{11} = R_{22} &= \frac{\Gamma^2(d)}{\Gamma(2d)}\left[2(a_0^2 + a_1^2)\frac{\sin(\pi d)}{\sin(2\pi d)} + (a_0^2 - a_1^2)\right] = \frac{\Gamma^2(d)}{\Gamma(2d)}\left[(a_0^2 + a_1^2)\frac{1}{\cos(\pi d)} + (a_0^2 - a_1^2)\right], \\ R_{12} &= -2a_0a_1\frac{\Gamma^2(d)}{\Gamma(2d)}, \end{aligned}$$

confirming (3.6), since $R_{11} = r_0$ and $R_{12} = -r_1$.

The second part (3.7) follows from Proposition 2.1 in [22]. In particular, since $R_{11} = R_{22} = r_0$ and $R_{12} = -R_{21} = -r_1$, we obtain that

$$g_0 = g_{11} = g_{22} = \frac{\Gamma(2d)}{\pi} r_0 \cos(\pi d),$$

and

$$ig_1 = g_{12}e^{-i\omega} = \frac{\Gamma(2d)}{2\pi} \left(-i(-r_1 - r_1)\sin(\pi d)\right),$$

which yields

$$g_1 = r_1 \tfrac{\Gamma(2d)}{\pi} \sin(\pi d) \qquad \square$$

Remark 3.2. *The relations in* (3.6) *can be solved for* a_0, a_1. *By solving for* a_1 *in the second equation in* (3.6) *and plugging this into the first equation, we obtain a fourth order polynomial in* a_1 *which, by setting* $\nu = a_1^2$, *reduces to a quadratic of the form*

$$\alpha\nu^2 - r_0\nu + \gamma r_1^2 = 0,$$

where

$$\alpha \doteq \frac{\Gamma^2(d)}{\Gamma(2d)} \left(\frac{1}{\cos(\pi d)} - 1\right), \ \gamma \doteq \frac{\Gamma(2d)}{4\Gamma^2(d)} \left(\frac{1}{\cos(\pi d)} + 1\right). \tag{3.8}$$

This leads to the discriminant

$$\Delta \doteq r_0^2 - 4\alpha\gamma r_1^2 = r_0^2 - \left(\frac{1}{\cos^2(\pi d)} - 1\right) r_1^2 = r_0^2 - \tan^2(\pi d) r_1^2. \tag{3.9}$$

Then, for the values of r_0, r_1 *for which* $\Delta \geq 0$ *(i.e.,* $r_0 \in [-\tan(\pi d)r_1, \tan(\pi d)r_1]$*), we can solve the quadratic and obtain the solutions (possibly one)* $\nu_\pm = \frac{r_0 \pm \sqrt{\Delta}}{2\alpha}$. *Moreover,* $\nu_+ \geq \nu_-$ *and if* $\nu_+ \geq 0$, *then we can solve for* a_0, a_1 *in*

$$\begin{aligned} a_{1,+} &= \sqrt{\frac{r_0 + \sqrt{\Delta}}{2\alpha}} = \sqrt{\frac{r_0 + \sqrt{r_0^2 - \tan^2(\pi d) r_1^2}}{2\frac{\Gamma^2(d)}{\Gamma(2d)}\left(\frac{1}{\cos(\pi d)} - 1\right)}}, \\ a_{0,+} &= \frac{\sqrt{2\alpha} r_1 \Gamma(2d)}{2\sqrt{r_0 + \sqrt{\Delta}}\Gamma^2(d)} = \frac{\sqrt{2\Gamma(2d)\left(\frac{1}{\cos(\pi d)} - 1\right)} r_1}{2\sqrt{r_0 + \sqrt{r_0^2 - \tan^2(\pi d) r_1^2}}\Gamma(d)} \end{aligned} \tag{3.10}$$

Note that if $\nu_+ > \nu_- \geq 0$, *then these values of* $a_{0,+}, a_{1,+}$ *are not unique. From now on, we write* a_0 *(resp.,* a_1*) to mean* $a_{0,+}$ *(resp., ,*$a_{1,+}$*) for simplicity.*

Proposition 3.3. *Let* $\{Y_n\}_{n\in\mathbb{Z}}$ *be a stationary bivariate LM series satisfying* (P-T) *with limiting ACVF satisfying* (3.2). *Let also* $\{X_n\}_{n\in\mathbb{Z}}$ *be the RMod series* (1.6) *for some* $\lambda_0 \in (0, \pi)$. *Then,* $\{X_n\}_{n\in\mathbb{Z}}$ *is second-order stationary, its ACVF satisfies* (1.10), *and exhibits CLM with the same* $\lambda_0 \in (0, \pi)$ *and*

$$c_\gamma \doteq \sqrt{(r_0^2 + r_1^2)}, \quad \phi \doteq \arcsin\left(\frac{r_1}{\sqrt{(r_0^2 + r_1^2)}}\right). \tag{3.11}$$

Equivalently, $r_0 = c_\gamma \cos(\phi), r_1 = c_\gamma \sin(\phi)$.

Proof. For $h, n \in \mathbb{Z}$, we have

$$\begin{aligned}
\mathbb{E}X_{n+h}X_n^T &= \cos(\lambda_0 n)\cos(\lambda_0(n+h))\gamma_{Y,11}(h) + \sin(\lambda_0 n)\sin(\lambda_0(n+h))\gamma_{Y,22}(h) \\
&\quad + \cos(\lambda_0 n)\sin(\lambda_0(n+h))\gamma_{Y,12}(h) + \sin(\lambda_0 n)\cos(\lambda_0(n+h))\gamma_{Y,21}(h) \\
&= \gamma_{11}(h)\Big(\cos(\lambda_0 n)\cos(\lambda_0(n+h)) + \sin(\lambda_0 n)\sin(\lambda_0(n+h))\Big) \\
&\quad + \gamma_{12}(h)\Big(\cos(\lambda_0 n)\sin(\lambda_0(n+h)) - \sin(\lambda_0 n)\cos(\lambda_0(n+h))\Big),
\end{aligned}$$

by using (P-T). This implies that

$$\begin{aligned}
\gamma_X(h) &= \gamma_{11}(h)\cos(\lambda_0 h) + \gamma_{12}(h)\sin(\lambda_0 h) \\
&\simeq [r_0\cos(\lambda_0 h) - r_1\sin(\lambda_0 h)]\, h^{2d-1} \\
&= \sqrt{(r_0^2+r_1^2)}\left[\frac{r_0}{\sqrt{(r_0^2+r_1^2)}}\cos(\lambda_0 h) - \frac{r_1}{\sqrt{(r_0^2+r_1^2)}}\sin(\lambda_0 h)\right] h^{2d-1} \\
&= c_\gamma\left[\cos(\phi)\cos(\lambda_0 h) - \sin(\phi)\sin(\lambda_0 h)\right] h^{2d-1} \\
&= c_\gamma \cos(\lambda_0 h + \phi) h^{2d-1},
\end{aligned}$$

where c_γ, ϕ are defined in (3.11). It thus exhibits CLM with parameters c_γ, ϕ given in (3.11). □

Remark 3.4. *Note that the expression for ϕ in* (3.11) *both leads to the admissible set* (1.2) *and shows that any cyclical phase in the set can be achieved with a suitable choice of r_0, r_1. To see these points, substitute r_0, r_1 in* (3.11) *by g_0, g_1 using the relations in* (3.7)*. This leads to*

$$\phi = \arcsin\left(\frac{\pi g_1}{\Gamma(2d)\sin(\pi d)}\frac{1}{\sqrt{\frac{\pi^2 g_0^2}{\Gamma(2d)^2\cos^2(\pi d)} + \frac{\pi^2 g_1^2}{\Gamma^2(2d)\sin^2(\pi d)}}}\right) = \arcsin\left(\frac{\operatorname{sign}(g_1)}{\sqrt{\left(\frac{g_0}{g_1}\right)^2\tan^2(\pi d) + 1}}\right).$$

As $|g_1| \le g_0$ in the spectral domain, the cyclical phase takes its values in any point of the interval

$$\left[\arcsin\left(-\frac{1}{\sqrt{1+\tan^2(\pi d)}}\right), \arcsin\left(\frac{1}{\sqrt{1+\tan^2(\pi d)}}\right)\right] = [\arcsin(-\cos(\pi d)), \arcsin(\cos(\pi d))] = \mathcal{I}_d,$$

where $\mathcal{I}_d$ is the admissible set in (1.2)*. For the bounrary points, we have:*

$$\phi = \pm\left(\frac{1}{2} - d\right)\pi \Leftrightarrow g_1 = \pm g_0 \Leftrightarrow r_0 = \pm\tan(\pi d) r_1. \tag{3.12}$$

The relation (3.10) *shows that any admissible phase can be obtained with a suitable choice of a_0, a_1. For given $c_\gamma > 0$ and $\phi \in \mathcal{I}_d$, we have, from Proposition* 3.3 *and* (3.10) *that one choice of a_0, a_1 leading to c_γ and ϕ is*

$$a_1 = \sqrt{\frac{c_\gamma\cos(\phi) + c_\gamma\sqrt{\cos^2(\phi) - \tan^2(\pi d)\sin^2(\phi)}}{2\frac{\Gamma^2(d)}{\Gamma(2d)}\left(\frac{1}{\cos(\pi d)} - 1\right)}}, \quad a_0 = \frac{\sqrt{2c_\gamma\Gamma(2d)\left(\frac{1}{\cos(\pi d)} - 1\right)}\sin(\phi)}{2\Gamma(d)\sqrt{\cos(\phi) + \sqrt{\cos^2(\phi) - \tan^2(\pi d)\sin^2(\phi)}}}.$$

In particular, the boundary case (3.12) *corresponds to :*

$$\Delta = r_1^2\left(\frac{sin^2(\pi d) - 1}{\cos^2(\pi d)} + 1\right) = 0, \quad a_1 = \sqrt{\frac{r_0}{2\alpha}}, \quad a_0 = \frac{\sqrt{2\alpha} r_1 \Gamma(2d)}{2\sqrt{r_0}\Gamma^2(d)} = \frac{\sqrt{2\alpha r_0}\Gamma(2d)}{\tan(\pi d)\Gamma^2(d)},$$

where α was defined in (3.8) *and Δ in* (3.9).

3.2 Parametric Models

3.2.1 Capturing CLM effects

We shall now provide a parametric model for the bivariate LM series considered in Section 3.1, that can be used in the RMod construction of CLM series with general cyclical phase. Let B be the usual baskward shift operator acting as $B^k Z_n = Z_{n-k}, k \in \mathbb{Z}$, e.g., $B^{-1}Z_n = Z_{n+1}$. Set

$$D = \operatorname{diag}(d, d) = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}, \tag{3.13}$$

and define a bivariate fractional integration operator $(I - B^{\pm 1})^{-D} = \operatorname{diag}((I - B^{\pm 1})^{-d}, (I - B^{\pm 1})^{-d})$, where $(I - B^{\pm 1})^{-d} = \sum_{k=0}^{\infty} c_{d,k}(B_k^{\pm 1})$ and $c_{d,k}$ are the coefficients in the Taylor expansions

$$(1 - x)^{-d} = \sum_{k=0}^{\infty} c_{d,k} x^k.$$

The coefficients $c_{d,k}$ satisfy

$$c_{d,k} \doteq \frac{\Gamma(k + d)}{\Gamma(k + 1)\Gamma(d)} \sim \frac{k^{d-1}}{\Gamma(d)}, \quad \text{as } k \to +\infty \tag{3.14}$$

(e.g., (2.4.5) in [33]).

Take a bivariate WN series $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ with the covariance matrix $\mathbb{E}\varepsilon_n\varepsilon_n^T = I_2$. For the parametric bivariate LM series Y, set

$$Y_n = (I - B)^{-D}Q_+\varepsilon_n + (I - B^{-1})^{-D}Q_-\varepsilon_n, \tag{3.15}$$

where

$$Q_+ \doteq \begin{pmatrix} q_0 & q_1 \\ -q_1 & q_0 \end{pmatrix} \doteq Q_-^T. \tag{3.16}$$

The series $\{Y_n\}$ in (3.15) is an example of bivariate FARIMA$(0, D, 0)$ series considered in [22]. By construction, note that the series Y in (3.15)–(3.16) has the form of the series Y considered in Section 3.1 with

$$a_0 = \frac{q_0}{\Gamma(d)}, \quad a_1 = \frac{q_1}{\Gamma(d)}, \tag{3.17}$$

where we used (3.14). The series Y in (3.15)–(3.16) thus satisfies Property (P-T). Relations similar to those in Propositions 3.1 and 3.3 but in terms of q_0, q_1 follow by using (3.17). In the following results, we also derive the explicit expressions for the ACVF and spectral density of the parametric model (3.15)–(3.16).

Proposition 3.5. *Let Y be given by* (3.15)–(3.16)*. Then, $\{Y_n\}_{n\in\mathbb{Z}}$ satisfies* (P-T) *with, for $h \in \mathbb{Z}$,*

$$\begin{aligned} \gamma_{Y,11}(h) = \gamma_{Y,22}(h) &= q_0^2\left(2 + \frac{\Gamma(2d + h)}{\Gamma(2d)\Gamma(1 + h)}\right) + q_1^2\left(2 - \frac{\Gamma(2d + h)}{\Gamma(2d)\Gamma(1 + h)}\right), \\ \gamma_{Y,12}(h) = -\gamma_{Y,21}(h) &= \operatorname{sign}(h)\frac{q_0q_1}{2}\frac{2\pi}{\Gamma(2d)}\frac{\Gamma(h + 2d)}{\Gamma(1 + h)}, \end{aligned} \tag{3.18}$$

and has limiting ACVF as in (3.2) *with*

$$r_0 \doteq \frac{1}{\Gamma(2d)}\left[q_0^2\left(\frac{1}{\cos(\pi d)} + 1\right) + q_1^2\left(\frac{1}{\cos(\pi d)} - 1\right)\right], \quad r_1 \doteq \frac{2q_0q_1}{\Gamma(2d)}. \tag{3.19}$$

Proof. The series $\{Y_n\}_{n\in\mathbb{Z}}$ satisfies property (P-T) by Theorem 2.1. We still do calculations to obtain tractable formulas relating the ACVF of Y with the parameters q_0, q_1. Consider

$$b^1 \doteq Q_-Q_-^T = \begin{pmatrix} q_0^2+q_1^2 & 0 \\ 0 & q_0^2+q_1^2 \end{pmatrix}, \quad b^2 \doteq Q_-Q_+^T = \begin{pmatrix} q_0^2-q_1^2 & -2q_0q_1 \\ 2q_0q_1 & q_0^2-q_1^2 \end{pmatrix},$$
$$b^3 \doteq Q_+Q_+^T = \begin{pmatrix} q_0^2+q_1^2 & 0 \\ 0 & q_0^2+q_1^2 \end{pmatrix}, \quad b^4 \doteq Q_+Q_-^T = \begin{pmatrix} q_0^2-q_1^2 & 2q_0q_1 \\ -2q_0q_1 & q_0^2-q_1^2 \end{pmatrix}.$$

Observe that, in view of (3.13) and the relation (72) of [22], we have that $\gamma_{1,kj}(h) = \gamma_{3,kj}(h), k,j = 1,2$ and $\gamma_{4,kj}(h) = \gamma_{2,kj}(-h), k,j = 1,2$ for all $h \in \mathbb{Z}$, where $\gamma_{l,kj}, l = 1,\dots,4$ and $k,j = 1,2$ are defined in Proposition 5.1 of [22]. Then, from the same proposition,

$$\begin{aligned}
\gamma_{Y,11}(h) &= \frac{1}{2\pi}\left[b^1_{11}\gamma_{1,11}(h) + b^2_{11}\gamma_{2,11}(h) + b^3_{11}\gamma_{3,11}(h) + b^4_{11}\gamma_{4,11}(h)\right] \\
&= \frac{1}{2\pi}\left[2(q_0^2+q_1^2)\gamma_{1,11}(h) + (q_0^2-q_1^2)\left(\gamma_{2,11}(h)\mathbf{1}_{\{h\le -1\}} + \gamma_{4,11}(h)\mathbf{1}_{\{h\ge 0\}}\right)\right] \\
&= q_0^2\left(2\frac{\Gamma(1-2d)\Gamma(h+d)\sin(\pi d)}{\Gamma(h+1-d)\pi} + \frac{\Gamma(2d+h)}{\Gamma(2d)\Gamma(1+h)}\right) \\
&\quad + q_1^2\left(2\frac{\Gamma(1-2d)\Gamma(h+d)\sin(\pi d)}{\Gamma(h+1-d)\pi} - \frac{\Gamma(2d+h)}{\Gamma(2d)\Gamma(1+h)}\right),
\end{aligned} \tag{3.20}$$

where for the second line we used $b^1_{11} = b^3_{11} = q_0^2+q_1^2$, $b^2_{11} = b^4_{11} = q_0^2-q_1^2$, and the properties for γ above, and for the third line we used the form of $\gamma_{1,11}, \gamma_{2,11}$. Similarly, on noticing that $b^1_{22} = b^3_{22} = q_0^2+q_1^2$, $b^2_{22} = b^4_{22} = q_0^2-q_1^2$, and that $\gamma_{i,11}(h) = \gamma_{i,22}(h)$ for all h and $i = 1,2,3,4$, we immediately obtain that

$$\gamma_{Y,22}(h) = \gamma_{Y,11}(h), \quad \text{for all } h \in \mathbb{Z}.$$

Next,

$$\begin{aligned}
\gamma_{Y,12}(h) &= \frac{1}{2\pi}\left[b^1_{12}\gamma_{1,12}(h) + b^2_{12}\gamma_{2,12}(h) + b^3_{12}\gamma_{3,12}(h) + b^4_{12}\gamma_{4,12}(h)\right] \\
&= \frac{q_0q_1}{2\pi}\left[-\gamma_{2,12}(h) + \gamma_{2,12}(-h)\right] \\
&= \operatorname{sign}(h)\frac{q_0q_1}{2}\frac{2\pi}{\Gamma(2d)}\frac{\Gamma(h+2d)}{\Gamma(1+h)},
\end{aligned} \tag{3.21}$$

and analogous calculations show that, for each $h \in \mathbb{Z}$,

$$\gamma_{Y,21}(h) = \frac{q_0q_1}{2\pi}\left[\gamma_{2,21}(h) - \gamma_{2,21}(-h)\right] = -\gamma_{Y,12}(h).$$

We proceed with calculating the asymptotics precisely. Recall that

$$\frac{\Gamma(x+a)}{\Gamma(x)} \sim x^a, \quad \text{as } x \to \infty.$$

By using the form of $\gamma_{1,jk}$ from [22], this says that, as $h \to \infty$, for all $j,k = 1,2$

$$\gamma_{1,jk}(h) = 2\Gamma(1-2d)\sin(d\pi)\frac{\Gamma(h+d)}{\Gamma(h+1-d)} \sim 2\Gamma(1-2d)\sin(d\pi)h^{2d-1}$$

and, as $h \to +\infty$, for all $j,k = 1,2$,

$$\gamma_{4,jk}(h) = \gamma_{2,12}(-h) = 2\pi\frac{1}{\Gamma(2d)}\frac{\Gamma(h+2d)}{\Gamma(h+1)} \sim 2\pi\frac{1}{\Gamma(2d)}h^{2d-1}.$$

The last two relations coupled with (3.20) and (3.21) imply that, as $h \to +\infty$ (and in particular for $h \geq 0$),

$$
\begin{aligned}
\gamma_{Y,11}(h) &\sim \frac{1}{2\pi}\left[4(q_0^2+q_1^2)\Gamma(1-2d)\sin(d\pi) + 2(q_0^2-q_1^2)\frac{\pi}{\Gamma(2d)}\right] h^{2d-1} \\
\gamma_{Y,12}(h) &\sim -\frac{2q_0q_1}{\Gamma(2d)} h^{2d-1}.
\end{aligned}
\tag{3.22}
$$

In particular, in view of the identity $\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}$ for $0 < x < 1$, $\{Y_n\}_{n\in\mathbb{Z}}$ has the limiting ACVF given by (3.2), with r_0, r_1 given by (3.19) □

Proposition 3.6. *Let Y be given by* (3.15)–(3.16)*. Then, $\{Y_n\}_{n\in\mathbb{Z}}$ satisfies* (P-S) *with*

$$
\begin{aligned}
f_{Y,11}(\lambda) &= \frac{2^{-2d}\sin^{-2d}\left(\frac{\lambda}{2}\right)}{\pi}\left[(q_0^2-q_1^2)\cos\left((\lambda-\pi)d\right) + (q_0^2+q_1^2)\right], \\
f_{Y,12}(\lambda) &= \frac{i}{\pi}\left[q_0q_1\left(2^{1-2d}\sin^{-2d}\left(\frac{\lambda}{2}\right)\sin\left((\lambda-\pi)d\right)\right)\right],
\end{aligned}
\tag{3.23}
$$

and has the limiting spectral decomposition as in (3.3) *with*

$$
g_0 = \frac{1}{\pi}\left[q_0^2(1+\cos(\pi d)) + q_1^2(1-\cos(\pi d))\right], \quad g_1 = 2\frac{\sin(\pi d)}{\pi}q_0q_1 \tag{3.24}
$$

Proof. The spectral density matrix is given by

$$
f(\lambda) = \frac{1}{2\pi}G(\lambda)G^*(\lambda),
$$

where $G(\lambda) = (1-e^{-i\lambda})^{-D}Q_+ + (1-e^{i\lambda})^{-D}Q_-$. Then, by denoting $g_j(\lambda)$ the jth row of $G(\lambda)$, we can write the (j,k) component as

$$
f_{jk}(\lambda) = \frac{1}{2\pi}g_j(\lambda)g_k^*(\lambda).
$$

In particular, we get that

$$
\begin{aligned}
f_{11}(\lambda) &= \frac{1}{2\pi}g_1(\lambda)g_1^*(\lambda) \\
&= \frac{1}{2\pi}\left[q_0^2\left((1-e^{-i\lambda})^{-d} + (1-e^{i\lambda})^{-d}\right)^2 - q_1^2\left((1-e^{i\lambda})^{-d} - (1-e^{-i\lambda})^{-d}\right)^2\right] \\
&= \frac{1}{2\pi}\left[\left(q_0^2-q_1^2\right)(1-e^{-i\lambda})^{-2d} + \left(q_0^2-q_1^2\right)(1-e^{i\lambda})^{-2d} + 2\left(q_0^2+q_1^2\right)(2-2\cos(\lambda))^{-d}\right] \\
&= f_{22}(\lambda).
\end{aligned}
$$

Upon recalling the identities $(1-e^{\pm i\lambda})^{-2d} = 2^{-2d}\sin^{-2d}\left(\frac{\lambda}{2}\right)e^{\mp id(\lambda-\pi)}$ and $(1-\cos(\lambda))^{-d} = 2^{-d}\sin^{-2d}\left(\frac{\lambda}{2}\right)$, we obtain

$$
\begin{aligned}
f_{11}(\lambda) &= \frac{1}{\pi}\left[\left(q_0^2-q_1^2\right)\left(2^{-2d}\sin^{-2d}\left(\frac{\lambda}{2}\right)\cos\left((\lambda-\pi)d\right)\right) + 2^{-2d}\left(q_0^2+q_1^2\right)\sin^{-2d}\left(\frac{\lambda}{2}\right)\right] \\
&= \frac{2^{-2d}\sin^{-2d}\left(\frac{\lambda}{2}\right)}{\pi}\left[(q_0^2-q_1^2)\cos\left((\lambda-\pi)d\right) + (q_0^2+q_1^2)\right].
\end{aligned}
$$

Moreover,

$$\begin{aligned}
f_{12}(\lambda) &= \frac{1}{2\pi} g_1(\lambda) g_2^*(\lambda) \\
&= \frac{1}{2\pi}\left[2q_0q_1\left((1-e^{-i\lambda})^{-2d} - (1-e^{i\lambda})^{-2d}\right)\right] \\
&= \frac{i}{\pi}\left[q_0q_1\left(2^{1-2d}\sin^{-2d}\left(\frac{\lambda}{2}\right)\sin\left((\lambda-\pi)d\right)\right)\right] \\
&= -f_{21}(\lambda).
\end{aligned}$$

The relation (3.24) is immediate upon substituting r_0, r_1 from Proposition 3.3 in the formula for g_0, g_1 of Proposition 3.1. $\square$

3.2.2 "Boundary" case

Note that in the "boundary" case $\phi = \pm\left(\frac{1}{2}-d\right)\pi$, since $r_0 = \pm\tan(\pi d) r_1$ and in view of (3.19), we obtain

$$\left(q_0\sqrt{\frac{1}{\cos(\pi d)}+1} \mp q_1\sqrt{\frac{1}{\cos(\pi d)}-1}\right)^2 = 0,$$

corresponding to

$$q_0 = \pm\sqrt{\frac{1-\cos(\pi d)}{1+\cos(\pi d)}}q_1 = \pm\frac{\sin(\pi d)}{1+\cos(\pi d)}q_1.$$

Then, due to (3.18), we have, for $h \in \mathbb{Z}$,

$$\begin{aligned}
\gamma_{Y,11}(h) &= q_1^2\left(2 - \frac{\Gamma(2d+h)}{\Gamma(2d)\Gamma(1+h)} + \frac{1-\cos(\pi d)}{1+\cos(\pi d)}\left(2+\frac{\Gamma(2d+h)}{\Gamma(2d)\Gamma(1+h)}\right)\right), \\
\gamma_{Y,12}(h) &= \pm\mathrm{sign}(h)\sqrt{\frac{1-\cos(\pi d)}{1+\cos(\pi d)}}q_1^2\frac{\pi}{\Gamma(2d)}\frac{\Gamma(h+2d)}{\Gamma(1+h)}.
\end{aligned} \tag{3.25}$$

On the other hand, we can easily see that,

$$q_0^2 - q_1^2 = q_1^2\left(\frac{1-\cos(\pi d)}{1+\cos(\pi d)} - 1\right) = -\frac{2\cos(\pi d)}{1+\cos(\pi d)}q_1^2, \quad q_0^2 + q_1^2 = q_1^2\frac{2}{1+\cos(\pi d)},$$

leading to the following spectral densities in (3.23) for $\lambda > 0$,

$$\begin{aligned}
f_{Y,11}(\lambda) &= \frac{2^{1-2d}\sin^{-2d}\left(\frac{\lambda}{2}\right)q_1^2}{\pi}\frac{1-\cos(\pi d)\cos\left((\lambda-\pi)d\right)}{1+\cos(\pi d)}, \\
f_{Y,12}(\lambda) &= \pm i\frac{2^{1-2d}\sin^{-2d}\left(\frac{\lambda}{2}\right)q_1^2}{\pi}\frac{\sin(\pi d)\sin\left((\lambda-\pi)d\right)}{1+\cos(\pi d)}.
\end{aligned} \tag{3.26}$$

We now investigate the spectral density of the parametric RMod series in the "boundary" case. First, let $r_0 = \tan(\pi d) r_1$, corresponding to the case

$$f_{Y,12}(\lambda) = i\frac{2^{1-2d}\sin^{-2d}\left(\frac{\lambda}{2}\right)q_1^2}{\pi}\frac{\sin(\pi d)\sin\left((\lambda-\pi)d\right)}{1+\cos(\pi d)}.$$

By combining (2.6) with (3.26), we have, as $\lambda \to \lambda_0^+$,

$$\begin{aligned}
&\frac{1}{2} f_{Y,11}(\lambda-\lambda_0) + \frac{1}{2i} f_{Y,12}(\lambda-\lambda_0) \\
&= \frac{2^{-2d}\sin^{-2d}\left(\frac{\lambda-\lambda_0}{2}\right) q_1^2}{\pi} \frac{1-\cos(\pi d)\cos\left((\lambda-\lambda_0-\pi)d\right)+\sin(\pi d)\sin\left((\lambda-\lambda_0-\pi)d\right)}{1+\cos(\pi d)} \\
&= \frac{2^{-2d}\sin^{-2d}\left(\frac{\lambda-\lambda_0}{2}\right) q_1^2}{\pi(1+\cos(\pi d))} \left(1-\cos\left((\lambda-\lambda_0)\pi\right)\right) \\
&\sim \frac{2^{-2d} q_1^2}{\pi(1+\cos(\pi d))} (\lambda-\lambda_0)^{-2d+2}.
\end{aligned} \tag{3.27}$$

In addition, as $\lambda \to \lambda_0^+$,

$$\begin{aligned}
\frac{1}{2} f_{Y,11}(\lambda+\lambda_0) + \frac{1}{2i} f_{Y,12}(\lambda+\lambda_0) &= \frac{2^{-2d}\sin^{-2d}\left(\frac{\lambda+\lambda_0}{2}\right) q_1^2}{\pi(1+\cos(\pi d))} \left(1-\cos\left((\lambda+\lambda_0)\pi\right)\right) \\
&\sim \frac{2^{-2d}\sin^{-2d}(\lambda_0)}{\pi(1+\cos(\pi d))} \left(1-\cos(2\lambda_0\pi)\right) q_1^2 = O(1),
\end{aligned}$$

which shows that, from (2.6), as $\lambda \to \lambda_0^+$,

$$\begin{aligned}
f_X(\lambda) &= \left[\frac{1}{2} f_{Y,11}(\lambda-\lambda_0) + \frac{1}{2i} f_{Y,12}(\lambda-\lambda_0) + \frac{1}{2} f_{Y,11}(\lambda+\lambda_0) - \frac{1}{2i} f_{Y,12}(\lambda+\lambda_0)\right] \\
&\sim \frac{2^{-2d}\sin^{-2d}(\lambda_0)}{\pi(1+\cos(\pi d))} \left(1-\cos(2\lambda_0\pi)\right) q_1^2, \quad \text{as } \lambda \to \lambda_0^+.
\end{aligned} \tag{3.28}$$

When $r_0 = \tan(\pi d) r_1$ and $\lambda < 0$,

$$f_{Y,12}(\lambda) = -i \frac{2^{1-2d}\sin^{-2d}\left(\frac{\lambda}{2}\right) q_1^2}{\pi} \frac{\sin(\pi d)\sin\left((\lambda-\pi)d\right)}{1+\cos(\pi d)},$$

which implies that, as $\lambda \to \lambda_0^-$,

$$\begin{aligned}
f_X(\lambda) &\sim \frac{2^{-2d}\sin^{-2d}\left(\frac{\lambda-\lambda_0}{2}\right) q_1^2}{\pi} \frac{1-\cos(\pi d)\cos\left((\lambda-\lambda_0-\pi)d\right)-\sin(\pi d)\sin\left((\lambda-\lambda_0-\pi)d\right)}{1+\cos(\pi d)} \\
&= \frac{2^{-2d}\sin^{-2d}\left(\frac{\lambda-\lambda_0}{2}\right) q_1^2}{\pi(1+\cos(\pi d))} \left(1-\cos\left(2\pi d-(\lambda-\lambda_0)\pi\right)\right) \\
&\sim \frac{2^{-2d} q_1^2 (1-\cos(2\pi d))}{\pi(1+\cos(\pi d))} (\lambda-\lambda_0)^{-2d}.
\end{aligned}$$

Finally, we note that in the case $r_0 = -\tan(\pi d) r_1$, we can reverse the calculations as $\lambda \to \lambda_0^+$, $\lambda \to \lambda_0^-$, yielding

$$\begin{aligned}
f_X(\lambda) &\sim \frac{2^{-2d} q_1^2 (1-\cos(2\pi d))}{\pi(1+\cos(\pi d))} (\lambda-\lambda_0)^{-2d}, \quad \text{as } \lambda \to \lambda_0^+, \\
f_X(\lambda) &\sim \frac{2^{-2d}\sin^{-2d}(\lambda_0)}{\pi(1+\cos(\pi d))} \left(1-\cos(2\lambda_0\pi)\right) q_1^2, \quad \text{as } \lambda \to \lambda_0^-.
\end{aligned} \tag{3.29}$$

3.2.3 Adding short memory effects

The parametric model for the bivariate LM series Y in (3.15)–(3.16) and the resulting RMod series X in (1.6) were constructed in Section 3.2.1 to capture the general form of CLM, which is the asymptotic notion in the time and spectral domains having to do with long memory. Short memory effects can be added to the model in a standard way. For this purpose, it is convenient to introduce the following terminology and models akin to fractionally integrated ARMA (AutoRegressive Moving Average) models. Let

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p,$$
$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q,$$

be the AR and MA polynomials of orders $p, q \in \mathbb{N}_0$. Assume their roots are outside the unit disc.

Definition 3.7. *The RMod series in* (1.6) *constructed from the bivariate LM series in* (3.15)–(3.16) *will be called fractional RMod series and denoted as FRMod*$(0, d, 0)$*. The second-order stationary series* X *will be denoted FRMod*(p, d, q) *and called (cyclical) fractional RMod series of orders* p, d, q *if the series*

$$\Theta^{-1}(B)\Phi(B)X_n$$

is FRMod$(0, d, 0)$*.*

When $p = 0$, the FRMod$(0, d, q)$ series can be written as

$$X_n = \Theta(B)\tilde{X}_n = \tilde{X}_n + \Theta_1 \tilde{X}_{n-1} + \cdots + \Theta_q \tilde{X}_{n-q}, \tag{3.30}$$

where $\tilde{X}$ is FRMod$(0, d, 0)$. Since the ACVF of $\tilde{X}$ can be represented explicitly, the same holds for the series (3.30). The polynomial $\Theta(B)$ allows modeling the ACVF for smaller lags and the spectral density away from the divergence around the cyclical phase λ_0. When $q = 0$, the FRMod$(p, d, 0)$ model can be fitted in practice since

$$\Phi(B)X_n = X_n - \phi_1 X_{n-1} - \cdots - \phi_p X_{n-p}$$

is FRMod$(0, d, 0)$ with a known ACVF.

3.3 Insights From Random Modulation

In Sections 3.1 and 3.2, we constructed CLM models by random modulation of a bivariate LM series satisfying certain ACVF properties. This raises the question: Is the converse procedure also possible, i.e., given a CLM series with cyclical frequency λ_0, can it be represented as a random modulation of a LM bivariate series $(Y_{1,n}, Y_{2,n})$? The answer to this question will follow from general developments for RMod series as, for example, in [31].

Thus, let $\{X_{1,n}\}$ be a CLM series. Assume $\{X_{2,n}\}$ is another series such that the bivariate series $\{(X_{1,n}, X_{2,n})\}$ satisfies property (P-T). Examples of the series $\{X_{2,n}\}$ are given below. Set

$$\begin{aligned} Y_{1,n} &= \cos(\lambda_0 n)X_{1,n} - \sin(\lambda_0 n)X_{2,n}, \\ Y_{2,n} &= \sin(\lambda_0 n)X_{1,n} + \cos(\lambda_0 n)X_{2,n}, \end{aligned} \tag{3.31}$$

which is equivalent to

$$\begin{aligned} X_{1,n} &= \cos(\lambda_0 n)Y_{1,n} + \sin(\lambda_0 n)Y_{2,n}, \\ X_{2,n} &= -\sin(\lambda_0 n)Y_{1,n} + \cos(\lambda_0 n)Y_{2,n}. \end{aligned} \tag{3.32}$$

If $\{(X_{1,n}, X_{2,n})\}$ satisfies property (P-T), one can check that the bivariate series $\{Y_n\} = \{(Y_{1,n}, Y_{2,n})\}$ is second-order stationary (e.g., [31]). The proposition below will show that the bivariate series $\{Y_n\}$ is LM when $\{X_{1,n}\}$, $\{X_{2,n}\}$ are CLM, that is, in view of the first equation in (3.31), answering the question above in the affirmative.

Different choices of the series $\{X_{2,n}\}$ above are available in the random modulation literature and lead to different bivariate series. Popular choices are:

- Uncorrelated case: Take $\{X_{2,n}\}$ uncorrelated with $\{X_{1,n}\}$. In this casee, the ACVF of $\{(X_{1,n}, X_{2,n})\}$ is: for all $h \in \mathbb{Z}$,

$$\gamma_{X,22}(h) = \gamma_{X,11}(h), \quad \gamma_{X,12}(h) = 0. \tag{3.33}$$

  Similarly, the spectral density satisfies $f_{X,22}(\lambda) = f_{X,11}(\lambda)$ and $f_{X,12}(\lambda) = 0$.

- Hilbert transform case: Take $\{X_{2,n}\}$ to be the Hilbert transform of $\{X_{1,n}\}$. In this case, the spectral density of $\{(X_{1,n}, X_{2,n})\}$ satisfies

$$f_{X,22}(\lambda) = f_{X,11}(\lambda),\ f_{X,12}(\lambda) = -f_{X,21}(\lambda) = i f_{X,11}(\lambda)\text{sign}(\lambda). \tag{3.34}$$

  We also note that in the random modulation literature, λ_0 in (3.31) and (3.32) is a free parameter. Here, we fix it to be equal to the cyclical frequency. The representation of $\{X_{1,n}\}_{n\in\mathbb{Z}}$ in (3.32) with $\{X_{2,n}\}_{n\in\mathbb{Z}}$ as the Hilbert transform of $\{X_{1,n}\}_{n\in\mathbb{Z}}$ is referred to as *Rice's representation.*

Proposition 3.8. *Let $\{X_{1,n}\}$ be a CLM series satisfying* (1.3) *and let $\{X_{2,n}\}$ be the series constructed in the uncorrelated or Hilbert transform case above. Let also $\{Y_n\}_{n\in\mathbb{Z}} = \{(Y_{1,n}, Y_{2,n})\}_{n\in\mathbb{Z}}$ be given by* (3.31)*. Then, $\{Y_n\}_{n\in\mathbb{Z}}$ is a bivariate LM series satisfying* (2.2) *with* (3.3)*. Furthermore, we have:*

- *Uncorrelated case:*

$$g_0 = \frac{1}{2}\left(c_f^+ + c_f^-\right),\ g_1 = \frac{1}{2}\left(c_f^+ - c_f^-\right) \tag{3.35}$$

- *Hilbert transform case:*

$$g_0 = \left(c_f^+ + c_f^-\right),\ g_1 = \left(c_f^+ - c_f^-\right) \tag{3.36}$$

Proof. We start by showing that $\{Y_n\}_{n\in\mathbb{Z}}$ is a bivariate LM series satisfying (2.2) with (3.3). Indeed, we see that

$$\begin{aligned} f_Y(\lambda) &= \sum_{h\in\mathbb{Z}} e^{-ih\lambda}\gamma_Y(h) \\ &= \sum_{h\in\mathbb{Z}} e^{-i\lambda h}\begin{pmatrix} \gamma_{X,11}(h)\cos(\lambda_0 h) - \gamma_{X,12}(h)\sin(\lambda_0 h) & \gamma_{X,12}(h)\cos(\lambda_0 h) + \gamma_{X,11}(h)\sin(\lambda_0 h) \\ -\gamma_{X,12}(h)\cos(\lambda_0 h) - \gamma_{X,11}(h)\sin(\lambda_0 h) & \gamma_{X,11}(h)\cos(\lambda_0 h) - \gamma_{X,12}(h)\sin(\lambda_0 h) \end{pmatrix} \\ &= \begin{pmatrix} f_{Y,11}(\lambda) & f_{Y,12}(\lambda) \\ f_{Y,21}(\lambda) & f_{Y,22}(\lambda) \end{pmatrix}, \end{aligned} \tag{3.37}$$

where the second line follows from standard calculations that were shown in Proposition 3.3. Moreover

$$\begin{aligned} f_{Y,11}(\lambda) &= \sum_{h\in\mathbb{Z}}\left(\frac{1}{2}\left[\gamma_{X,11}(h)\left(e^{-i(\lambda-\lambda_0)h} + e^{-i(\lambda+\lambda_0)h}\right)\right] - \frac{1}{2i}\left[]\gamma_{X,12}(h)\left(e^{-i(\lambda-\lambda_0)h} - e^{-i(\lambda+\lambda_0)h}\right)\right]\right) \\ &= \frac{1}{2}\left[f_{X,11}(\lambda-\lambda_0) + f_{X,11}(\lambda+\lambda_0)\right] - \frac{1}{2i}\left[f_{X,12}(\lambda-\lambda_0) - f_{X,12}(\lambda+\lambda_0)\right], \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} f_{Y,12}(\lambda) &= \sum_{h\in\mathbb{Z}}\left(\frac{1}{2}\left[\gamma_{X,12}(h)\left(e^{-i(\lambda-\lambda_0)h} + e^{-i(\lambda+\lambda_0)h}\right)\right] + \frac{1}{2i}\left[\gamma_{X,11}(h)\left(e^{-i(\lambda-\lambda_0)h} - e^{-i(\lambda+\lambda_0)h}\right)\right]\right) \\ &= \frac{1}{2}\left[f_{X,12}(\lambda-\lambda_0) + f_{X,12}(\lambda+\lambda_0)\right] + \frac{1}{2i}\left[f_{X,11}(\lambda-\lambda_0) - f_{X,11}(\lambda+\lambda_0)\right]. \end{aligned} \tag{3.39}$$

From (3.37), (3.38), and (3.39), we see that, in the uncorrelated case (3.33), as $\lambda \to 0$,

$$f_{Y,11}(\lambda) \sim \frac{1}{2}\left(c_f^+ + c_f^-\right)|\lambda|^{-2d}, \quad f_{Y,12}(\lambda) \sim \frac{i}{2}\left(c_f^+ - c_f^-\right)|\lambda|^{-2d},$$

which leads to g_0, g_1 given in (3.35).

On the other hand, considering the Hilbert transform case (3.34), we see that, as $\lambda \to 0$,

$$f_{Y,11}(\lambda) \sim \left(c_f^+ + c_f^-\right)|\lambda|^{-2d}, \quad f_{Y,12}(\lambda) \sim i\left(c_f^+ - c_f^-\right)|\lambda|^{-2d},$$

leading to the respective constants g_0, g_1 presented in (3.36). □

Remark 3.9. *From the perspective of signal processing, there are physical reasons to select the Hilbert transform; see the discussion in [31]. It is not immediately clear whether any of these reasons carries on to the RMod series constructed here.*

3.4 Linear Representations

We identify here the linear representation for the parametric RMod series $\{X_n\}_{n\in\mathbb{Z}}$ when the series Y is constructed in (3.14)–(3.16). First note that the construction in (3.14)–(3.16) is a special case of the general form in (2.3)–(2.5) with (upon noticing that $c_{d,0} = 1$)

$$a_{0,k} = \begin{cases} c_{d,-k}q_0 & k \le -1, \\ 2q_0 & k = 0, \\ c_{d,k}q_0 & k \ge 1, \end{cases} \qquad a_{1,k} = \operatorname{sign}(k)c_{d,|k|}q_1 = \begin{cases} -c_{d,-k}q_1 & k \le -1, \\ 0 & k = 0, \\ c_{d,k}q_1 & k \ge 1. \end{cases} \tag{3.40}$$

The following result provides a linear representation for the RMod Series $\{X_n\}_{n\in\mathbb{Z}}$ and is an immediate corollary of Theorem 2.4.

Corollary 3.10. *Let $\{X_n\}_{n\in\mathbb{Z}}$ and $\{Y_n\}_{n\in\mathbb{Z}}$ be given by* (1.6) *and* (3.14)–(3.16) *respectively. Then, $\{X_n\}_{n\in\mathbb{Z}}$ has the linear representation obtained in Theorem* 2.4 *with $a_{0,k}, a_{1,k}, k \in \mathbb{Z}$ given in* (3.40).

We finally note that this general representation retrieves Proposition 1 of [35] for the two-sided representation.

4 Extensions

4.1 Assymetry in Memory Parameter

Random modulation and our parametric approach can also be used to construct CLM series satisfying (1.11), that is

$$f_X(\lambda) \sim \begin{cases} c_{f,+}(\lambda - \lambda_0)^{-2d_+}, & \text{as } \lambda \to \lambda_0^+, \\ c_{f,-}(\lambda_0 - \lambda)^{-2d_-}, & \text{as } \lambda \to \lambda_0^-, \end{cases} \tag{4.1}$$

where $c_{f,+}, c_{f,-} > 0$ and $d_-, d_+ \in \left(0, \frac{1}{2}\right)$, with $d_- \neq d_+$. As noted in Section 1, it is natural to set

$$X_n = X_n^+ + X_n^-, \tag{4.2}$$

where $\{X_n^+\}_{n\in\mathbb{Z}}$ and $\{X_n^-\}_{n\in\mathbb{Z}}$ are uncorrelated and satisfy (1.12). Recall that the asymptotics (1.12) are associated with the "boundary" cases (1.5). The asymptotic behavior $0 \cdot |\lambda - \lambda_0|^{-2d_\pm}$ in (1.12), however, is not specific enough to guarantee (4.1). Conditions implying (4.1) are

$$f_{X^+}(\lambda) \sim \begin{cases} c_{f,+}(\lambda - \lambda_0)^{-2d_+}, & \text{as } \lambda \to \lambda_0^+, \\ b_{f,+} \cdot (\lambda_0 - \lambda)^0, & \text{as } \lambda \to \lambda_0^-, \end{cases} \quad f_{X^-}(\lambda) \sim \begin{cases} b_{f,-} \cdot (\lambda - \lambda_0)^0, & \text{as } \lambda \to \lambda_0^+, \\ c_{f,-}(\lambda_0 - \lambda)^{-2d_-}, & \text{as } \lambda \to \lambda_0^-, \end{cases} \tag{4.3}$$

where $b_{f,+}, b_{f,-} \geq 0$. We argue here that the parametric RMod model constructed in Section 3.2.2 in the "boundary" cases (1.5) satisfy (4.3), and thus yields the model (4.2) satisfying (4.1).

Following the development of Section 3.2.2, we take $\{X_n^+\}$ in (4.2) as a parametric model FRMod$(0,d,0)$ constructed in Section 3.2.1 with

$$d = d_+, \quad q_0 = -\frac{\sin(\pi d_+)}{1+\cos(\pi d_+)} q_1. \tag{4.4}$$

By (3.29), the spectral density of $\{X_n^+\}_{n\in\mathbb{Z}}$ satisfies (4.3) with

$$c_{f,+} = \frac{2^{-2d_+} q_1^2 (1-\cos(2\pi d_+))}{\pi(1+\cos(\pi d_+))}, \quad b_{f,+} = \frac{2^{-2d_+} \sin^{-2d_+}(\lambda_0)}{\pi(1+\cos(\pi d_+))} \left(1-\cos\left(2\lambda_0\pi\right)\right) q_1^2. \tag{4.5}$$

Similarly, we take $\{X_n^-\}$ in (4.2) as the same parametric model with

$$d = d_-, \quad q_0 = \frac{\sin(\pi d_-)}{1+\cos(\pi d_-)} q_1. \tag{4.6}$$

By (3.27)-(3.28), the spectral density of $\{X_n^-\}$ satisfies (4.3) with

$$c_{f,-} = \frac{2^{-2d_-} q_1^2 (1-\cos(2\pi d_-))}{\pi(1+\cos(\pi d_-))}, \quad b_{f,-} = \frac{2^{-2d_-} \sin^{-2d_-}(\lambda_0)}{\pi(1+\cos(\pi d_-))} \left(1-\cos\left(2\lambda_0\pi\right)\right) q_1^2. \tag{4.7}$$

We can thus construct the series (4.2) satisfying (4.1) with $c_{f,+}$ and $c_{f,-}$ given in (4.5) and (4.7). The construction above could be extended to FRMod(p,d,q) models in the "boundary" case.

One can also compute the ACVF of X. Indeed, let $\{Y_n^+\}_{n\in\mathbb{Z}}$ and $\{Y_n^-\}_{n\in\mathbb{Z}}$ be given by (3.15)–(3.16), with parameters specified by (4.4) and (4.6) respectively. Then, we write

$$\gamma_X(h) = \gamma_{X^+}(h) + \gamma_{X^-}(h) = \cos(\lambda_0 h)\left(\gamma_{Y^+,11}(h) + \gamma_{Y^-,11}(h)\right) + \sin(\lambda_0 h)\left(\gamma_{Y^+,12}(h) + \gamma_{Y^-,12}(h)\right), \tag{4.8}$$

where the second equality follows from the first line of (3.5) for $\gamma_{X^+}(h)$ and $\gamma_{X^-}(h)$ separately. In view of (3.25), this leads to

$$\begin{aligned}
\gamma_X(h) = \cos(\lambda_0 h) q_1^2 \Bigg[4 &- \frac{\Gamma(2d_+ + h)}{\Gamma(2d_+)\Gamma(1+h)} - \frac{\Gamma(2d_- + h)}{\Gamma(2d_-)\Gamma(1+h)} + \frac{1-\cos(\pi d_+)}{1+\cos(\pi d_+)} \left(2 + \frac{\Gamma(2d_+ + h)}{\Gamma(2d_+)\Gamma(1+h)}\right) \\
&+ \frac{1-\cos(\pi d_-)}{1+\cos(\pi d_-)} \left(2 + \frac{\Gamma(2d_- + h)}{\Gamma(2d_-)\Gamma(1+h)}\right)\Bigg] \\
&+ \sin(\lambda_0 h)\mathrm{sign}(h) q_1^2 \frac{\pi}{\Gamma(1+h)} \left(\sqrt{\frac{1-\cos(\pi d_-)}{1+\cos(\pi d_-)}} \frac{\Gamma(h+2d_-)}{\Gamma(2d_-)} - \sqrt{\frac{1-\cos(\pi d_+)}{1+\cos(\pi d_+)}} \frac{\Gamma(h+2d_+)}{\Gamma(2d_+)} \right).
\end{aligned}$$

To identify the asymptotics of $\gamma_X(h)$, we need more refined approximations than the one provided in (3.22). From Equation (1) of [12] and (3.22), it follows that

$$\begin{aligned}
\gamma_{Y^+,11}(h) &= \frac{1}{2\pi}\left[4(q_0^2+q_1^2)\Gamma(1-2d_+)\sin(d_+\pi) + 2(q_0^2-q_1^2)\frac{\pi}{\Gamma(2d_+)}\right] h^{2d_+-1} + O(h^{2d_+-2}) \\
\gamma_{Y^+,12}(h) &= -\frac{2q_0q_1}{\Gamma(2d_+)} h^{2d_+-1} + O(h^{2d_+-2}),
\end{aligned} \tag{4.9}$$

and similarly

$$\begin{aligned}
\gamma_{Y^-,11}(h) &= \frac{1}{2\pi}\left[4(q_0^2+q_1^2)\Gamma(1-2d_-)\sin(d_-\pi) + 2(q_0^2-q_1^2)\frac{\pi}{\Gamma(2d_-)}\right] h^{2d_--1} + O(h^{2d_--2}) \\
\gamma_{Y^-,12}(h) &= -\frac{2q_0q_1}{\Gamma(2d_-)} h^{2d_--1} + O(h^{2d_--2}),
\end{aligned} \tag{4.10}$$

Let $d^* \doteq \max\{d_+, d_-\}, d_* \doteq \min\{d_+, d_-\}$. Then, since $2d^* - 2 < 2d_* - 1 < 2d^* - 1$ and from (4.8), (4.9), and (4.10), we have

$$\begin{aligned}
\gamma_X(h) &= h^{2d^*-1}\left[\cos(\lambda_0 h)\frac{1}{2\pi}\left[4(q_0^2+q_1^2)\Gamma(1-2d^*)\sin(d^*\pi) + 2(q_0^2-q_1^2)\frac{\pi}{\Gamma(2d^*)}\right] - \sin(\lambda_0 h)\frac{2q_0q_1}{\Gamma(2d^*)}\right] \\
&\quad + h^{2d_*-1}\left[\cos(\lambda_0 h)\frac{1}{2\pi}\left[4(q_0^2+q_1^2)\Gamma(1-2d_*)\sin(d_*\pi) + 2(q_0^2-q_1^2)\frac{\pi}{\Gamma(2d_*)}\right] - \sin(\lambda_0 h)\frac{2q_0q_1}{\Gamma(2d_*)}\right] \\
&\quad + \cos(\lambda_0 h)O(h^{2d^*-2}) + \sin(\lambda_0 h)O(h^{2d^*-2})
\end{aligned}$$

The only CLM model we are aware of having the form (4.1) is the Seasonal Cyclical Asymetric Long Memory (CSALM) model of [6]. The CSALM model, unlike the one considered here, does not have explicit ACVF.

4.2 CLM with Multiple Singularities

Our approach extends easily to the case of CLM with multiple singularities, expressed in the spectral domain as: for $\lambda_{0,m} \in (0, \pi)$, $m = 1, \dots, M$,

$$f_X(\lambda - \lambda_{0,m}) \sim \begin{cases} c_{f,m}^+ \lambda^{-2d_m}, & \text{as } \lambda \to 0^+, \\ c_{f,m}^- (-\lambda)^{-2d_m}, & \text{as } \lambda \to 0^-, \end{cases} \tag{4.11}$$

where $d_m \in \left(0, \frac{1}{2}\right)$ and $c_{f,m}^+, c_{f,m}^- \geq 0$, $c_{f,m}^+ + c_{f,m}^- > 0$. We refer to this as CLM with M factors (M-CLM, for short). As with (1.1) and (1.3), it is expected that under suitable conditions, the condition (4.11) is equivalent to the time-domain condition: as $h \to \infty$,

$$\gamma_X(h) = \sum_{m=1}^{M} \left(c_{\gamma,m}\cos(\lambda_{0,m} h + \phi_m) h^{2d_m - 1} + o(h^{2d_m-1})\right),$$

where $\phi_m \in \mathcal{I}_{d_m}$ and $c_{\gamma,m} > 0$. M-CLM was considered in, e.g., [6, 16, 25, 35].

The series $\{X_n\}_{n\in\mathbb{Z}}$ satisfying (4.11) can be constructed as

$$X_n = \sum_{m=1}^{M} X_{m,n},$$

where $\{X_{m,n}\}$, $m = 1, \dots, M$, are independent series and the spectral density $f_{X_m}(\lambda)$ of each $\{X_{m,n}\}$ satisfies (4.11). One or more of the parametric RMod models introduced in this work can be taken for $\{X_{m,n}\}$, including in the "boundary" case. For the resulting parametric model, when $\{X_{m,n}\}$ are constructed in (3.15)–(3.16), the ACVF and spectral densities can be computed explicitly by

$$\gamma_X(h) = \sum_{m=1}^{M} \gamma_{X_m}(h), \quad f_X(\lambda) = \sum_{m=1}^{M} f_{X_m}(\lambda),$$

where γ_{X_m} and f_{X_m} are given explicitly in Propositions 3.5 and 3.6, respectively. It is evident that one can consider one or more of the $\{X_{m,n}\}$ to be FRMod(p, d, q). Again, explicit formulas for the ACVF of the $M-$factor Gegenbauer processes are not available (see, e.g., the discussion at the end of Section 3 in [5] or [10])).

5 Conclusions

We have introduced a way to construct time series exhibiting CLM based on random modulation (RMod). We have also provided a class of parametric models capturing general CLM, admitting explicit autocovariance functions, spectral densities, and linear representations with regard to a white noise sequence. A crucial element of these constructions is the decoupling of the quasi-periodicity and long memory (LM) that appear in the autocovariance functions of the CLM series.

The constructed parametric RMod series, on the one hand, significantly extend the basic model that was considered in [35] and, on the other hand, rely on delicate constructions of bivariate parametric LM series considered in [22]. We mention here two advantages compared to other parametric CLM models: first, our model allows for more flexibility with regard to modeling, due to the presence of a cyclical phase ϕ and a careful analysis of the model when ϕ is on the "boundary" of its admissible set; second, the construction of the RMod series allows for explicit calculations of ACVF and spectra that are often not available for other CLM models, such as the Gegenbauer series. These explicit quantities can be used readily for downstream tasks, such as simulation.

The flexibility in the modeling of CLM with RMod series is also evidenced by the extensions that were undertaken in Section 4, including the case of multiple singularities in the spectrum and the "boundary" case. We conjecture that it is also possible to consider RMod series in several different settings, e.g. multivariate RMod series (for the multivariate Gegenbauer processes see, e.g., [39]) or RMod random fields (again, for the Gegenbauer counterpart, see, e.g., [13]). It is also interesting to consider statistical tasks for RMod series, e.g., estimation for the location of the singularity λ_0 and the memory parameter.

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A Definitions of CLM in Time and Spectral Domains

In this appendix, we formalize the definitions of CLM in the spectral and time domains, incorporating both the "non-boundary" and "boundary" cases. In Appendix B (resp. C) below, we provide some conditions for a series exhibiting CLM in the spectral domain (resp. time domain) to also exhibit CLM in the time domain (resp. spectral domain). Finally, we prove a technical lemma that will be useful for the "boundary" case (of Appendix C.2) in Appendix D below.

For clarity, Definition A.1 below is given for series exhibiting CLM with one singularity, but analogues can be developed in the case of multiple singularities.

Definition A.1. *(Spectral Domain) A second-order stationary time series $\{X_n\}_{n\in\mathbb{Z}}$ is said to exhibit CLM if its spectral density satisfies*

$$f_X(\lambda) = \begin{cases} \left(c_f^- + R_f^-(\lambda_0-\lambda)\right)(\lambda_0-\lambda)^{-2d}, & 0<\lambda<\lambda_0, \\ \left((c_f^+ + R_f^+(\lambda-\lambda_0)\right)(\lambda-\lambda_0)^{-2d}, & \lambda_0<\lambda<\pi, \end{cases} \tag{A.1}$$

where $\lambda_0\in(0,\pi), d\in\left(0,\frac{1}{2}\right)$, $c_f^\pm\geq 0$ with $c_f^+ + c_f^- > 0$, and $R_f^\pm:(0,\infty)\to[-c_f^\pm,\infty)$ are functions with $R_f^\pm(x)\to 0$, as $x\to 0^+$. If, in addition, $c_f^+=0$ or $c_f^-=0$, then we say that $\{X_n\}_{n\in\mathbb{Z}}$ exhibits CLM in the "boundary" case. The "non-boundary" case corresponds to $c_f^+>0$ and $c_f^->0$.

The next definition provides a definition for CLM in the time domain.

Definition A.2. *(Time domain) A second-order stationary time series $\{X_n\}_{n\in\mathbb{Z}}$ is said to exhibit CLM if its autocovariance function satisfies*

$$\gamma_X(h) = \left(c_\gamma\cos(\lambda_0 h+\phi) + R_\gamma(h)\right)h^{2d-1}, \quad h\in\mathbb{N}_0, \tag{A.2}$$

where $\lambda_0\in(0,\pi), d\in(0,1/2), c_\gamma\in(0,\infty), \phi\in\left[-\left(\frac{1}{2}-d\right)\pi, \left(\frac{1}{2}-d\right)\pi\right]$, and $R_\gamma:[0,\infty)\to\mathbb{R}$ is a function with $R_\gamma(x)\to 0, \quad \text{as } x\to+\infty$. If, in addition, $\phi=-\left(\frac{1}{2}-d\right)\pi$ or $\phi=\left(\frac{1}{2}-d\right)\pi$, then we say that $\{X_n\}_{n\in\mathbb{Z}}$ exhibits CLM in the "boundary" case. The "non-boundary" case is $\phi\in\left(-\left(\frac{1}{2}-d\right)\pi, \left(\frac{1}{2}-d\right)\pi\right)$.

The next remark provides an alternative useful formulation for CLM in the time domain.

Remark A.3. *The relation* (A.2) *can be recast as*

$$\gamma_X(h) = c_{1,\gamma}\cos(\lambda_0 h)h^{2d-1} + c_{2,\gamma}\sin(\lambda_0 h)h^{2d-1} + R_\gamma(h)h^{2d-1}, \tag{A.3}$$

where R_γ is as in Definition A.2, *and*

$$c_{1,\gamma} \doteq c_\gamma\cos(\phi)\in(0,\infty), \quad c_{2,\gamma}\doteq -c_\gamma\sin(\phi)\in\mathbb{R}. \tag{A.4}$$

Remark A.4. *Definitions* A.1 *and* A.2 *suggest the relationship $c_f^\mp=0 \Leftrightarrow \phi=\pm\left(\frac{1}{2}-d\right)\pi$ in the "boundary" case. This will be discussed in Appendix* C.3 *below.*

Remark A.5. *More general definitions of CLM could be based on using slowly varying functions instead of constants $c_f^\pm$ in* (A.1) *and $c_{i,\gamma}$ in* (A.3). *We restrict our analysis to the case of constants, which is more relevant to modeling in practice. The role of functions $R_f^\pm$ in* (A.1) *is to allow for deviations from the constants $c_f^\pm$, and Definition* A.1 *naturally captures the divergence of the spectral density around frequency λ_0. By having the function $R_\gamma(h)$ in* (A.2) *or* (A.3), *we aimed to have a general definition of CLM in the time domain. There may, however, be ways to make it more inclusive. For example, take two uncorrelated series X and Y with X being CLM in the sense of* A.2 *and Y being LM with the memory parameter $\tilde{d}>d$. One may want to call $X+Y$ CLM, but $X+Y$ is not CLM in the sense of Definition* A.2, *since the LM decay $h^{2\tilde{d}-1}$ associated with Y cannot be incorporated in the remainder term $R_\gamma(h)h^{2d-1}$ in* (A.2) .

We note that series exhibiting *Seasonal Cyclical Assymetric Long Memory* (SCALM) presented in, e.g., [3, 4], also exhibit CLM in the boundary case (i.e., $c_f^+ = 0$ or $c_f^- = 0$) according to Definition A.1.

Definitions A.1 and A.2 are, in general, not equivalent. We are aware of only two works relating some versions of these two definitions. First, in Proposition 2 of [37], an argument is provided for passing from the spectral to the time domain definition for extended fractional ARMA processes with seasonal effects in the "non-boundary" case. However, even for this specific class of processes, the reader is referred to a different, but similar proof, and so the arguments are not complete. Second, Lemma 1 of Chapter 2 in [2] considers an argument for passing from the spectral to the time domain definition in the "boundary" case. While the proof strategy is valid (and, in fact, similar to the one used in Appendix B.2), it ignores second-order asymptotic expansions (see Appendix D), thus rendering the stated result imprecise. We are not aware of any results obtaining a spectral domain representation from the time domain.

In relating Definitions A.1 and A.2 we shall use slowly varying functions converging to a constant, but also having the following quasi-monotonicity property. A slowly varying function $L : [0, \infty) \to (0, \infty)$ at infinity is called *quasi-monotone* (see, e.g., Chapter 2.7 in [8]) if the following two conditions hold: (i) it is of bounded variation on any compact interval of $[0, \infty)$ and (ii) for some $\delta > 0$, $\int_0^x u^\delta |dL(u)| = O(x^\delta L(x))$, as $x \to \infty$.

B From Spectral to Time Domain

When going from the spectral to the time domain, the terms $c_f^-(\lambda_0 - \lambda)^{-2d}$ and $c_f^+(\lambda - \lambda_0)^{-2d}$ in Definition A.1 will lead to $c_\gamma \cos(\lambda_0 h + \phi) h^{2d-1}$ in Definition A.2, plus a remainder of order h^{-1}. The ultimate form of the remainder $R_\gamma(h) h^{2d-1}$ in A.2 will be determined by the exact forms of $R_f^-(\lambda_0 - \lambda)$ and $R_f^+(\lambda - \lambda_0)$. Various forms are possible. Results below provide a few examples and illustrations.

In Appendix B.1 we state and prove a result in the "non-boundary" case. We emphasize that special treatment is required for the "boundary" case, and state a related result in Appendix B.2. Finally, we derive the set of admissible cyclical phases and note some special cases for the cyclical phase ϕ in Appendix C.3.

B.1 "Non-boundary" case

The following proposition provides sufficient conditions for Definition A.1 to imply Definition A.2 in the "non-boundary" case.

Proposition B.1. *Let*

$$R_f^\pm(x) = L_f^\pm\left(\frac{1}{x}\right) - c_f^\pm,$$

where $L_f^- : \left(\frac{1}{\lambda_0}, \infty\right) \to (0, \infty)$ and $L_f^+ : \left(\frac{1}{\pi - \lambda_0}, \infty\right) \to (0, \infty)$ are quasi-monotone slowly varying functions at $+\infty$, with $L_f^\pm(x) \sim c_f^\pm \in (0, \infty)$ as $x \to +\infty$. Then, Definition A.1 *implies Definition* A.2 *with*

$$\begin{aligned} c_\gamma &\doteq 2\Gamma(1-2d)\sqrt{(c_f^+)^2 + (c_f^-)^2 - 2c_f^+ c_f^- \left(\cos(2\pi d)\right)^2}, \\ \phi &\doteq \arcsin\left(\frac{(c_f^+ - c_f^-)\cos(\pi d)}{\sqrt{(c_f^+)^2 + (c_f^-)^2 - 2c_f^+ c_f^- \left(\cos(2\pi d)\right)^2}}\right). \end{aligned} \tag{B.1}$$

Proof. We have

$$\begin{aligned} \gamma_X(h) &= 2\int_0^\pi \cos(h\lambda) f_X(\lambda) d\lambda = 2\int_0^{\lambda_0} \cos(h\lambda) f_X(\lambda) d\lambda + 2\int_{\lambda_0}^\pi \cos(h\lambda) f_X(\lambda) d\lambda \\ &\doteq 2\left[\gamma_-(h) + \gamma_+(h)\right]. \end{aligned} \tag{B.2}$$

We consider these two quantities separately. First,

$$\gamma_-(h) = \int_0^{\lambda_0} \cos(h\lambda) L_f^- \left(\frac{1}{\lambda_0 - \lambda}\right) (\lambda_0 - \lambda)^{-2d} d\lambda = \int_0^{\lambda_0} \cos(h(\lambda_0 - \omega)) L_f^- \left(\frac{1}{\omega}\right) \omega^{-2d} d\omega$$
$$= \cos(h\lambda_0) \int_0^{\lambda_0} \cos(h\omega) L_f^- \left(\frac{1}{\omega}\right) \omega^{-2d} d\omega + \sin(h\lambda_0) \int_0^{\lambda_0} \sin(h\omega) L_f^- \left(\frac{1}{\omega}\right) \omega^{-2d} d\omega,$$

where the second equality follows from the change of variables $\lambda_0 - \lambda = \omega$. By Proposition A.2.2 of [33] and since L_f^- is quasi-monotone, we have that, for large h,

$$\int_0^{\lambda_0} \cos(h\omega) L_f^- \left(\frac{1}{\omega}\right) \omega^{-2d} d\omega = h^{2d-1} L_f^-(h) \Gamma(1-2d) \sin(\pi d) + o(h^{2d-1}),$$

and similarly

$$\int_0^{\lambda_0} \sin(h\omega) L_f^- \left(\frac{1}{\omega}\right) \omega^{-2d} d\omega = h^{2d-1} L_f^-(h) \Gamma(1-2d) \cos(\pi d) + o(h^{2d-1}).$$

Analogous calculations show that

$$\begin{aligned}\gamma_+(h) &= \int_{\lambda_0}^{\pi} \cos(h\lambda) L_f^+ \left(\frac{1}{\lambda - \lambda_0}\right) (\lambda - \lambda_0)^{-2d} d\lambda = \int_0^{\pi-\lambda_0} \cos(h(\lambda_0 + \omega)) L_f^+ \left(\frac{1}{\omega}\right) \omega^{-2d} d\omega \\ &= \cos(h\lambda_0) \int_0^{\pi-\lambda_0} \cos(h\omega) L_f^+ \left(\frac{1}{\omega}\right) \omega^{-2d} d\omega - \sin(h\lambda_0) \int_0^{\pi-\lambda_0} \sin(h\omega) L_f^+ \left(\frac{1}{\omega}\right) \omega^{-2d} d\omega,\end{aligned} \tag{B.3}$$

where

$$\int_0^{\pi-\lambda_0} \cos(h\omega) L_f^+ \left(\frac{1}{\omega}\right) \omega^{-2d} d\omega = h^{2d-1} L_f^+(h) \Gamma(1-2d) \sin(\pi d) + o(h^{2d-1}),$$

and

$$\int_0^{\pi-\lambda_0} \sin(h\omega) L_f^+ \left(\frac{1}{\omega}\right) \omega^{-2d} d\omega = h^{2d-1} L_f^+(h) \Gamma(1-2d) \cos(\pi d) + o(h^{2d-1}).$$

Since $L_f^{\pm}(h) \to c_f^{\pm}$ as $h \to \infty$, (B.2) can be written as

$$\begin{aligned}\gamma_X(h) = \cos(h\lambda_0) &\left[h^{2d-1} c_f^- 2\Gamma(1-2d) \sin(\pi d) + R_1^-(h))\right] \\ + \sin(h\lambda_0) &\left[h^{2d-1} c_f^- 2\Gamma(1-2d) \cos(\pi d) + R_2^-(h))\right] \\ + \cos(h\lambda_0) &\left[h^{2d-1} c_f^+ 2\Gamma(1-2d) \sin(\pi d) + R_1^+(h))\right] \\ - \sin(h\lambda_0) &\left[h^{2d-1} c_f^+ 2\Gamma(1-2d) \cos(\pi d) + R_2^+(h))\right],\end{aligned}$$

where $R_1^{\pm}(h) \doteq 2(L_f^{\pm}(h) - c_f^{\pm})\Gamma(1-2d)\sin(\pi d) h^{2d-1} + o(h^{2d-1})$ and $R_2^{\pm}(h) \doteq 2(L_f^{\pm}(h) - c_f^{\pm})\Gamma(1-2d)\cos(\pi d) h^{2d-1} + o(h^{2d-1})$. We thus have that

$$\begin{aligned}\gamma_X(h) &= 2\Gamma(1-2d) h^{2d-1} \left[(c_f^+ + c_f^-) \sin(\pi d) \cos(\lambda_0 h) + (c_f^- - c_f^+) \cos(\pi d) \sin(\lambda_0 h)\right] + R_\gamma(h) h^{2d-1} \\ &= c_\gamma \cos(\lambda_0 h + \phi) h^{2d-1} + R_\gamma(h) h^{2d-1},\end{aligned} \tag{B.4}$$

where c_γ, ϕ are given in (B.1), and

$$R_\gamma(h) h^{2d-1} \doteq \cos(h\lambda_0) R_1^-(h) + \sin(h\lambda_0) R_2^-(h) + \cos(h\lambda_0) R_1^+(h) - \sin(h\lambda_0) R_2^+(h) = o(1).$$

This concludes the proof. □

Remark B.2. *Note that* (B.4) *can be reformulated as in* (A.3) *with*

$$c_{1,\gamma} \doteq 2\Gamma(1-2d)(c_f^- + c_f^+) \sin(\pi d), \quad c_{2,\gamma} \doteq 2\Gamma(1-2d)(c_f^- - c_f^+) \cos(\pi d).$$

B.2 "Boundary" case

As stated in Definition A.1, the "boundary" case corresponds to $c_f^+ = 0$ or $c_f^- = 0$. For simplicity, we consider here only the case $c_f^- = 0$, leading to a phase $\phi = \left(\frac{1}{2} - d\right)\pi$. An analogous result can be stated for the case $c_f^+ = 0$, corresponding to $\phi = -\left(\frac{1}{2} - d\right)\pi$.

Proposition B.3. *Let*

$$c_f^- = 0, \quad R_f^-(x) = L_f^-\left(\frac{1}{x}\right)x^{-2\varepsilon}, \quad R_f^+(x) = L_f^+\left(\frac{1}{x}\right) - c_f^+,$$

where $L_f^- : \left(\frac{1}{\lambda_0}, \infty\right) \to (0,\infty)$, $L_f^+ : \left(\frac{1}{\pi-\lambda_0}, \infty\right) \to (0,\infty)$ *are two quasi-monotone slowly varying functions at* $+\infty$*, with* $L_f^+(x) \sim c_f^+$ *as* $x \to \infty$*, and* $\varepsilon \in (0, d)$*. Then, Definition* A.1 *implies Definition* A.2 *in the "boundary" case with*

$$c_\gamma \doteq 2\Gamma(1-2d)c_f^+, \quad \phi \doteq \left(\frac{1}{2} - d\right)\pi. \tag{B.5}$$

Proof. Recall the relation (B.2). In view of (B.3) and the form of R_f^+ in the assumption, we write

$$\begin{aligned}
2\gamma_+(h) &= 2\cos(h\lambda_0)\int_0^{\pi-\lambda_0}\cos(h\omega)L_f^+\left(\frac{1}{\omega}\right)\omega^{-2d}d\omega \\
&\quad - 2\sin(h\lambda_0)\int_0^{\pi-\lambda_0}\sin(h\omega)L_f^+\left(\frac{1}{\omega}\right)\omega^{-2d}d\omega \\
&= 2\Gamma(1-2d)\sin(d\pi)L_f^+(h)\cos(\lambda_0 h)h^{2d-1} - 2\Gamma(1-2d)\cos(d\pi)L_f^+(h)\sin(\lambda_0 h)h^{2d-1} \\
&\quad + o(h^{2d-1}) \\
&= 2\Gamma(1-2d)\sin(d\pi)c_f^+\cos(\lambda_0 h)h^{2d-1} - 2\Gamma(1-2d)\cos(d\pi)c_f^+\sin(\lambda_0 h)h^{2d-1} + o(h^{2d-1}) \\
&= c_\gamma\cos(\lambda_0 h + \phi)h^{2d-1} + o(h^{2d-1}),
\end{aligned} \tag{B.6}$$

where the first equality follows from Proposition A.2.2 of [33], and ϕ, c_γ were defined in (B.5).

Now we compute the contribution of γ_- in (B.3). From Proposition A.2.2 of [33] and the form of R_f^-, as $h \to \infty$,

$$\begin{aligned}
2\gamma_-(h) &= 2\cos(h\lambda_0)\int_0^{\lambda_0}\cos(h\lambda)R_f^-(\omega)\,\omega^{-2d}d\omega + 2\sin(h\lambda_0)\int_0^{\lambda_0}\sin(h\omega)R_f^-(\omega)\,\omega^{-2d}d\omega \\
&= 2\cos(h\lambda_0)\int_0^{\lambda_0}\cos(h\lambda)L_f^-\left(\frac{1}{\omega}\right)\omega^{-2d+2\varepsilon_-}d\omega + 2\sin(h\lambda_0)\int_0^{\lambda_0}\sin(h\omega)L_f^-\left(\frac{1}{\omega}\right)\omega^{-2d+2\varepsilon_-}d\omega \\
&= 2\Gamma(1-2d+2\varepsilon_-)L_f^-(h)\left(\cos(\pi(d-\varepsilon_-))\sin(h\lambda_0) + \sin(\pi(d-\varepsilon_-))\cos(h\lambda_0)\right)h^{2d-1-2\varepsilon_-} \\
&\quad + o(h^{2(d-\varepsilon)-1}) = o(h^{2d-1}).
\end{aligned} \tag{B.7}$$

Combining (B.2), (B.6), and (B.7), we write

$$\gamma_X(h) = 2\gamma_+(h) + 2\gamma_-(h) = c_\gamma\cos(\lambda_0 h + \phi)h^{2d-1} + o(h^{2d-1})$$

where c_γ, ϕ are defined in (B.5). $\square$

C From Time to Spectral Domain

When going from the time to the spectral domain, the term $c_\gamma \cos(\lambda_0 h + \phi) h^{2d-1}$ in Definition A.2 will lead to $c_f^-(\lambda_0 - \lambda)^{-2d}$, $c_f^+(\lambda - \lambda_0)^{-2d}$ in Definition A.2, plus a remainder. The ultimate remainders $R_f^{\pm}$ in Definition A.2 will be determined by the exact form of $R_\gamma(h)$.

In Appendix C.1 we state and prove a result in the "non-boundary" case $\phi \in \left(-\left(\frac{1}{2} - d\right)\pi, \left(\frac{1}{2} - d\right)\pi\right)$. A special treatment is required for the "boundary" case $\phi = \pm\left(\frac{1}{2} - d\right)\pi$, and a related result is stated in Appendix C.2. Recall the sign function in (2.7).

C.1 "Non-boundary" case

Proposition C.1. *Assume that $\phi \in \left(\left(d - \frac{1}{2}\right)\pi, \left(\frac{1}{2} - d\right)\pi\right)$, and that for $h \geq 1$,*

$$R_\gamma(h) \doteq (L_{1,\gamma}(h) - c_\gamma \cos(\phi)) \cos(\lambda_0 h) + \operatorname{sign}(-\sin(\phi))(L_{2,\gamma}(h) - |c_\gamma \sin(\phi)|) \sin(\lambda_0 h), \tag{C.1}$$

where $L_{1,\gamma}, L_{2,\gamma} : (0,\infty) \to (0,\infty)$ are quasi-monotone slowly varying functions at $+\infty$, with $L_{1,\gamma}(x) \sim c_\gamma \cos(\phi) \in (0,\infty)$ and for $\phi \neq 0$, $L_{2,\gamma}(x) \sim |c_\gamma \sin(\phi)| \in (0,\infty)$ as $x \to +\infty$. Then, Definition A.2 *implies Definition* A.1 *with*

$$c_f^{\pm} \doteq \frac{c_\gamma}{2\pi} \Gamma(2d) \cos(\pi d \mp \phi). \tag{C.2}$$

Proof. As in the proof of Proposition 2.2.14 in the Appendix of [33], for $\lambda \in [0,\pi) \setminus \{\lambda_0\}$,

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma_X(h).$$

Strictly speaking, the arguments below are first used to show that the spectral density can be written this way and then the asymptotics are established. In particular,

$$\begin{aligned} f_X(\lambda) &= \frac{1}{2\pi} \left[\gamma_X(0) + 2 \sum_{h=1}^{\infty} \cos(h\lambda) \left(L_{1,\gamma}(h) \cos(\lambda_0 h) + \operatorname{sign}(-\sin(\phi)) L_{2,\gamma}(h) \sin(\lambda_0 h) \right) h^{2d-1} \right] \\ &= \frac{1}{2\pi} \left[\gamma_X(0) + f_1(\lambda) - \operatorname{sign}(-\sin(\phi)) f_2(\lambda) + f_3(\lambda) + \operatorname{sign}(-\sin(\phi)) f_4(\lambda) \right], \end{aligned} \tag{C.3}$$

where

$$f_1(\lambda) \doteq \sum_{h=1}^{\infty} \cos\big((\lambda - \lambda_0)h\big) L_{1,\gamma}(h) h^{2d-1}, \quad f_2(\lambda) \doteq \sum_{h=1}^{\infty} \sin\big((\lambda - \lambda_0)h\big) L_{2,\gamma(h)} h^{2d-1},$$

$$f_3(\lambda) \doteq \sum_{h=1}^{\infty} \cos((\lambda + \lambda_0)h) L_{1,\gamma}(h) h^{2d-1}, \quad f_4(\lambda) \doteq \sum_{h=1}^{\infty} \sin((\lambda + \lambda_0)h) L_{2,\gamma}(h) h^{2d-1}.$$

By Proposition A.2.1 of [33], as $\lambda \to \lambda_0$,

$$\begin{aligned} f_1(\lambda) &= |\lambda - \lambda_0|^{-2d} L_{1,\gamma}\left(\frac{1}{|\lambda - \lambda_0|}\right) \Gamma(2d) \cos(\pi d) + o(|\lambda - \lambda_0|^{-2d}), \\ f_2(\lambda) &= \operatorname{sign}(\lambda - \lambda_0) |\lambda - \lambda_0|^{-2d} L_{2,\gamma}\left(\frac{1}{|\lambda - \lambda_0|}\right) \Gamma(2d) \sin(\pi d) + o(|\lambda - \lambda_0|^{-2d}). \end{aligned} \tag{C.4}$$

On the other hand, fix some $0 < \alpha < \lambda_0 < \beta < \pi$ so that $\alpha - \lambda_0 < 0$, $\beta - \lambda_0 > 0$, and $\sup_{\hat{\lambda} \in [\alpha,\beta]} \{\frac{2}{\sin(\hat{\lambda}/2)}, \frac{2}{\cos(\hat{\lambda}/2)}\} < \infty$. Since $L_{1,\gamma}, L_{2,\gamma}$ are quasi-monotone, the relation (A.2.6) in Proposition A.2.1 of [33] says that, for $\lambda \in [\alpha - \lambda_0, \beta - \lambda_0]$,

$$\left| \sum_{h=n}^{\infty} \cos(h(\lambda + \lambda_0)) \frac{L_{1,\gamma}(h)}{h^{1-2d}} \right| \leq \frac{2}{\sup_{\hat{\lambda} \in [\alpha,\beta]} \{\frac{2}{\sin(\hat{\lambda}/2)}\}} \frac{|L_{1,\gamma}(n)|}{n^{1-2d}} (1 + O(1)),$$

which implies that, for all $\epsilon > 0$, there exists some $M \geq 1$ such that

$$\sup_{\lambda \in [\alpha-\lambda_0, \beta-\lambda_0]} \left| \sum_{h=M}^{\infty} \cos(h(\lambda+\lambda_0)) \frac{L_{1,\gamma}(h)}{h^{1-2d}} \right| < \epsilon.$$

Likewise, we obtain

$$\sup_{\lambda \in [\alpha-\lambda_0, \beta-\lambda_0]} \left| \sum_{h=M}^{\infty} \sin(h(\lambda+\lambda_0)) \frac{L_{2,\gamma}(h)}{h^{1-2d}} \right| < \epsilon.$$

By truncating these series, this shows that, as $\lambda \to \lambda_0$,

$$f_3(\lambda) \to \sum_{h=1}^{\infty} \cos(2\lambda_0 h) \frac{L_{1,\gamma}(h)}{h^{1-2d}} = O(1), \quad f_4(\lambda) \to \sum_{h=1}^{\infty} \sin(2\lambda_0 h) \frac{L_{2,\gamma}(h)}{h^{1-2d}} = O(1). \tag{C.5}$$

Now define

$$\begin{aligned} R(\lambda) \doteq \frac{\Gamma(2d)}{2\pi} \Bigg[&\left(L_{1,\gamma}\left(\frac{1}{|\lambda-\lambda_0|}\right) - c_\gamma \cos(\phi) \right) \cos(d\pi) \\ &- \operatorname{sign}(\lambda-\lambda_0) \left(\operatorname{sign}(-\sin(\phi)) L_{2,\gamma}\left(\frac{1}{|\lambda-\lambda_0|}\right) + c_\gamma \sin(\phi) \right) \sin(d\pi) \Bigg]. \end{aligned}$$

Since $L_{1,\gamma}(x) \to c_\gamma \cos(\phi), \operatorname{sign}(-\sin(\phi)) L_{2,\gamma}(x) \to -c_\gamma \sin(\phi)$ as $x \to +\infty$, it follows that $R(\lambda) = o(1)$. By combining (C.3), (C.4) and (C.5), we write

$$\begin{aligned} f_X(\lambda) = &\frac{\Gamma(2d)}{2\pi} \Bigg[L_{1,\gamma}\left(\frac{1}{|\lambda-\lambda_0|}\right) \cos(d\pi) \\ &\quad - \operatorname{sign}(-\sin(\phi)) \operatorname{sign}(\lambda-\lambda_0) L_{2,\gamma}\left(\frac{1}{|\lambda-\lambda_0|}\right) \sin(d\pi) \Bigg] |\lambda-\lambda_0|^{-2d} \\ &+ \frac{\Gamma(2d)}{2\pi} \left(\gamma_X(0) + f_3(\lambda) + \operatorname{sign}(-\sin(\phi)) f_4(\lambda) \right) + o(|\lambda-\lambda_0|^{-2d}) \\ = &\frac{\Gamma(2d)}{2\pi} \left[c_\gamma \cos(\phi) \cos(d\pi) + \operatorname{sign}(\lambda-\lambda_0) c_\gamma \sin(\phi) \sin(d\pi) \right] |\lambda-\lambda_0|^{-2d} + R(\lambda) |\lambda-\lambda_0|^{-2d} \\ &+ O(1) + o(|\lambda-\lambda_0|^{-2d}) \\ = &\left(\frac{\Gamma(2d)}{2\pi} c_\gamma \cos(\pi d - \operatorname{sign}(\lambda-\lambda_0)\phi) + o(1) \right) |\lambda-\lambda_0|^{-2d}, \end{aligned} \tag{C.6}$$

where c_γ, ϕ were defined in (B.1). $\square$

Remark C.2. *In view of Remark* B.2*, we can recast* (C.1) *as*

$$R_\gamma(h) = (L_{1,\gamma}(h) - c_{1,\gamma}) \cos(\lambda_0 h) h^{2d-1} + \operatorname{sign}(c_{2,\gamma})(L_{2,\gamma}(h) - |c_{2,\gamma}|) \sin(\lambda_0 h) h^{2d-1},$$

where $c_{1,\gamma}, c_{2,\gamma}$ are given in (A.4)*. Then, from* (C.2)*, we have that*

$$c_f^{\pm} \doteq \frac{\Gamma(2d)}{2\pi} \left(c_{1,\gamma} \cos(\pi d) \mp c_{2,\gamma} \sin(\pi d) \right) \in (0, \infty).$$

C.2 "Boundary" case

For clarity, we shall focus on the case

$$\phi = \left(\frac{1}{2} - d\right) \pi,$$

which is expected to correspond to $c_f^- = 0$ (see Appendix C.3). An analogous statement can be obtained in the case $\phi = -\left(\frac{1}{2} - d\right)\pi$, corresponding to $c_f^+ = 0$.

Proposition C.3. *Assume that $\phi = \left(\frac{1}{2} - d\right)\pi$, and that for $h \geq 1$,*

$$R_\gamma(h) \doteq L_{1,\gamma}(h)\cos(\lambda_0 h)h^{-2\varepsilon} + \xi L_{2,\gamma}(h)\sin(\lambda_0 h)h^{-2\varepsilon},$$

where $\varepsilon \in (0,d), \xi = \pm 1$, $L_{1,\gamma}, L_{2,\gamma} : (0,\infty) \to (0,\infty)$ are quasi-monotone slowly varying functions at $+\infty$, with $L_{1,\gamma}(x) \sim b_1 \in (0,\infty)$ and $\xi L_{2,\gamma}(x) \sim b_2 \in \mathbb{R}$ as $x \to +\infty$, where

$$b_1 \in \mathbb{R} \text{ and } b_2 = 0, \quad \text{or} \quad \frac{b_1}{b_2} \in (-\tan(\pi(d-\varepsilon)), \tan(\pi(d-\varepsilon))). \tag{C.7}$$

Then, Definition A.2 *implies Definition* A.1 *with*

$$c_f^- = 0, \quad c_f^+ \doteq \frac{c_\gamma}{2\pi}\Gamma(2d)\sin(2\pi d). \tag{C.8}$$

Proof. We can write, for $h \in \mathbb{Z}$,

$$\gamma_X(h) = c_\gamma \cos(\lambda_0 h + \phi)h^{2d-1} + L_{1,\gamma}(h)\cos(\lambda_0 h)h^{2d-2\varepsilon-1} + \xi L_{2,\gamma}(h)\sin(\lambda_0 h)h^{2d-2\varepsilon-1} = \gamma_1(h) + \gamma_2(h),$$

with

$$\gamma_1(h) \doteq c_\gamma \cos(\lambda_0 h + \phi)h^{2d-1}, \quad \gamma_2(h) \doteq L_{1,\gamma}(h)\cos(\lambda_0 h)h^{2d-2\varepsilon-1} + \xi L_{2,\gamma}(h)\sin(\lambda_0 h)h^{2d-2\varepsilon-1}.$$

Calculations similar to (C.3)–(C.6) and the fact that $\phi = \left(\frac{1}{2} - d\right)\pi$, imply that, as $\lambda \to \lambda_0^+$,

$$\begin{aligned}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_1(h) &= [\cos(\phi)\cos(\pi d) - \operatorname{sign}(\lambda - \lambda_0)\sin(\phi)\sin(\pi d)]\, c_\gamma \frac{1}{2\pi}|\lambda - \lambda_0|^{-2d} + O(1) \\ &= \sin(2d\pi)\frac{c_\gamma}{2\pi}\Gamma(2d)|\lambda - \lambda_0|^{-2d} + O(1).\end{aligned} \tag{C.9}$$

This yields the formula for c_f^+ in (C.8), and the remainder $o(1)|\lambda - \lambda_0|^{2d}$ gets absorbed into R_f^+ in (A.1) since $\varepsilon < d$. On the other hand, as $\lambda \to \lambda_0^-$,

$$\begin{aligned}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_1(h) = \frac{1}{2\pi}\Bigg[&\gamma_1(0) + \sin(d\pi)c_\gamma \sum_{h=1}^{\infty}\cos((\lambda - \lambda_0)h)h^{2d-1} \\ &- \cos(d\pi)c_\gamma \sum_{h=1}^{\infty}\sin((\lambda_0 - \lambda)h)h^{2d-1} \\ &+ \sin(d\pi)c_\gamma \sum_{h=1}^{\infty}\cos((\lambda + \lambda_0)h)h^{2d-1} - \cos(d\pi)c_\gamma \sum_{h=1}^{\infty}\sin((\lambda + \lambda_0)h)h^{2d-1}\Bigg] \\ = \frac{1}{2\pi}\Bigg[&\sin(d\pi)c_\gamma \sum_{h=1}^{\infty}\cos((\lambda + \lambda_0)h)h^{2d-1} \\ &- \cos(d\pi)c_\gamma \sum_{h=1}^{\infty}\sin((\lambda + \lambda_0)h)h^{2d-1} + O(1)\Bigg] \\ = O(1),&\end{aligned} \tag{C.10}$$

where we have used the form of ϕ and Lemma D.1 to replace the two series (and $\gamma_1(0)$) by $O(1)$ in the second equality, and that the series in the fourth and fifth line are O(1) in the last equality, which follows from the same arguments as in (C.5). The relation (C.10) verifies the form of c_f^- in (C.8), and gets absorbed into $R_f^- = o(|\lambda - \lambda_0|^{-2d})$ in (A.1).

We now investigate the spectral density corresponding to $\gamma_2(h)$, which will also get absorbed in $R_f^{\pm}(\lambda)$ in (A.1). Since $0 < \varepsilon < d < \frac{1}{2}$ and since b_1, b_2 meet condition (C.7), this spectral density is non-negative. Calculations identical to the ones in Appendix C.1 show that, as $\lambda \to \lambda_0$,

$$\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_2(h) = o(|\lambda-\lambda_0|^{-2(d-\varepsilon)}) + \frac{\Gamma(2(d-\varepsilon))}{2\pi}\left[L_{1,\gamma}\left(\frac{1}{|\lambda-\lambda_0|}\right)\cos((d-\varepsilon)\pi)\right.$$
$$\left. - \operatorname{sign}(\lambda-\lambda_0)\xi L_{2,\gamma}\left(\frac{1}{|\lambda-\lambda_0|}\right)\sin((d-\varepsilon)\pi)\right]|\lambda-\lambda_0|^{-2(d-\varepsilon)}. \tag{C.11}$$

The relations (C.9)– (C.11) show that $R_f^{\pm} = o(1)$. □

C.3 Set of admissible cyclical phases and special cases

Since $c_f^+, c_f^- > 0$, note that (C.2) yields

$$-\frac{\pi}{2} \le d\pi - \phi \le \frac{\pi}{2}, -\frac{\pi}{2} \le \phi - d\pi \le \frac{\pi}{2}$$

so that, as stated in (1.2),

$$\phi \in \mathcal{I}_d \doteq \left[\left(d-\frac{1}{2}\right)\pi, \left(\frac{1}{2}-d\right)\pi\right].$$

When $c_f^+ = c_f^- = c_f$ (symmetric case), we have, from Proposition B.1,

$$c_\gamma = 2\sqrt{2}\Gamma(1-2d)c_f\sin(2\pi d), \quad \phi = 0,$$

or, in terms of $c_{1,\gamma}, c_{2,\gamma}$, from Remark B.2,

$$c_{1,\gamma} = 4\Gamma(1-2d)c_f\sin(\pi d), \quad c_{2,\gamma} = 0.$$

If $c_f^- = 0$, then Proposition B.3 says that

$$c_\gamma = 2\Gamma(1-2d)c_f^+, \quad \phi = \left(\frac{1}{2}-d\right)\pi,$$

or, through Remark B.2,

$$c_{1,\gamma} = 2\Gamma(1-2d)c_f^+\sin(\pi d), \quad c_{2,\gamma} = -2\Gamma(1-2d)c_f^+\cos(\pi d), \quad \frac{c_{1,\gamma}}{c_{2,\gamma}} = -\tan(\pi d).$$

Similarly, $c_f^+ = 0$ corresponds to

$$c_\gamma = 2\Gamma(1-2d)c_f^-, \quad \phi = \left(d-\frac{1}{2}\right)\pi,$$

or, equivalently,

$$c_{1,\gamma} = 2\Gamma(1-2d)c_f^-\sin(\pi d), \quad c_{2,\gamma} = 2\Gamma(1-2d)c_f^-\cos(\pi d), \quad \frac{c_{1,\gamma}}{c_{2,\gamma}} = \tan(\pi d).$$

D Auxiliary Lemma

We present here results on higher-order behavior of the Fourier series of power-law coefficients that was used in Appendix C.2.

Lemma D.1. *For $d \in (0, 1/2)$, as $\omega \to 0^+$,*

$$\sum_{k=1}^{\infty} \sin(k\omega) k^{2d-1} = \omega^{-2d}\Gamma(2d)\sin(\pi d) + R_1(\omega),$$
$$\sum_{k=1}^{\infty} \cos(k\omega) k^{2d-1} = \omega^{-2d}\Gamma(2d)\cos(\pi d) + R_2(\omega),$$

where $R_i(\omega) \to c_i \in \mathbb{R}$, $i = 1, 2$.

Proof. Write

$$\begin{aligned}\int_0^{\infty} e^{iu\omega} u^{2d-1} du - \sum_{k=1}^{\infty} e^{ik\omega} k^{2d-1} &= \sum_{k=1}^{\infty} \left(\int_{k-1}^{k} e^{iu\omega} u^{2d-1} du - e^{ik\omega} k^{2d-1} \right) \\ &= \sum_{k=1}^{\infty} \left(\int_{k-1}^{k} e^{iu\omega} (u^{2d-1} - k^{2d-1}) du + k^{2d-1} \left(\int_{k-1}^{k} e^{iu\omega} du - e^{ik\omega} \right) \right).\end{aligned} \tag{D.1}$$

Next, note that, by Taylor's expansion,

$$\int_{k-1}^{k} (u^{2d-1} - k^{2d-1}) du = k^{2d} \int_{1-1/k}^{1} (z^{2d-1} - 1) dz \sim -k^{2d-2} \frac{2d-1}{2},$$

which implies, from the bounded convergence theorem, as $\omega \to 0^+$,

$$\sum_{k=1}^{\infty} \int_{k-1}^{k} e^{iu\omega} (u^{2d-1} - k^{2d-1}) du \to -C. \tag{D.2}$$

Moreover,

$$\int_{k-1}^{k} e^{iu\omega} du - e^{ik\omega} = \frac{e^{i\omega k} - e^{i\omega(k-1)}}{i\omega} - e^{i\omega k} = e^{ik\omega} \frac{1 - e^{-i\omega} - i\omega}{i\omega}.$$

Thus,

$$\sum_{k=1}^{\infty} k^{2d-1} \left(\int_{k}^{k+1} e^{iu\omega} du - e^{ik\omega} \right) = \frac{1 - e^{-i\omega} - i\omega}{i\omega} \sum_{k=1}^{\infty} (k^{2d-1} e^{ik\omega}) \sim \omega^{-2d+1} \Gamma(2d) e^{i\pi(\frac{1}{2} - d)} = o(1), \tag{D.3}$$

where the last asymptotic relation holds from Proposition A.2.1 of [33]. By combining (D.1)–(D.3),

$$\begin{aligned}\sum_{k=1}^{\infty} e^{ik\omega} k^{2d-1} &= \int_0^{\infty} e^{iu\omega} u^{2d-1} du \\ &\quad - \sum_{k=1}^{\infty} \int_{k-1}^{k} e^{iu\omega} (u^{2d-1} - k^{2d-1}) du - \sum_{k=1}^{\infty} k^{2d-1} \left(\int_{k-1}^{k} e^{iu\omega} du - e^{ik\omega} \right) \\ &= \omega^{-2d} \Gamma(2d) e^{id\pi} + R(\omega),\end{aligned}$$

where, as $\omega \to 0^+$, $R(\omega) \to C - \frac{1}{2d}$ and C is given in (D.2). ∎