Breuer-Major Theorems for Hilbert Space-Valued Random Variables *†‡

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Abstract

Let $\{X_k\}_{k\in\mathbb{Z}}$ be a stationary Gaussian process with values in a separable Hilbert space $\mathcal{H}_1$, and let $G : \mathcal{H}_1 \to \mathcal{H}_2$ be an operator acting on X_k. Under suitable conditions on the operator G and the temporal and cross-sectional correlations of $\{X_k\}_{k\in\mathbb{Z}}$, we derive a central limit theorem (CLT) for the normalized partial sums of $\{G[X_k]\}_{k\in\mathbb{Z}}$. To prove a CLT for the Hilbert space-valued process $\{G[X_k]\}_{k\in\mathbb{Z}}$, we employ techniques from the recently developed infinite dimensional Malliavin-Stein framework. In addition, we provide quantitative and continuous time versions of the derived CLT. In a series of examples, we recover and strengthen limit theorems for a wide array of statistics relevant in functional data analysis, and present a novel limit theorem in the framework of neural operators as an application of our result.

1 Introduction

Consider a stationary Gaussian process $\{X_k\}_{k\in\mathbb{Z}}$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and taking values in a separable Hilbert space $\mathcal{H}_1$. Let $G : \mathcal{H}_1 \to \mathcal{H}_2$ be a measurable function mapping $\mathcal{H}_1$ into a (possibly different) separable Hilbert space $\mathcal{H}_2$, where $\mathcal{H}_1, \mathcal{H}_2$ are equipped with inner products $\langle\cdot,\cdot\rangle_{\mathcal{H}_i}$ and induced norms $\|\cdot\|_{\mathcal{H}_i}$, $i = 1, 2$. This work aims to find conditions on the operator G and the temporal and cross-sectional correlation structure of the underlying process $\{X_k\}_{k\in\mathbb{Z}}$ that ensure a Central Limit Theorem (CLT), i.e., the weak convergence of the normalized partial sums

$$S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(G[X_k] - \mathrm{E}\, G[X_k]\right), \quad n \in \mathbb{N}, \tag{1.1}$$

to a Gaussian random variable in $\mathcal{H}_2$, as $n \to \infty$. Here, $\{G[X_k]\}_{k\in\mathbb{Z}}$ is known as the class of *subordinated Gaussian processes*. Without loss of generality, we assume in Sections 1–5 that $\mathrm{E}\, G[X_1] = 0$.

For $\{X_k\}$ being independent and identically distributed (i.i.d.) random variables and taking values in a Hilbert space $\mathcal{H}$, Varadhan [44] was the first to prove a CLT for the normalized partial sums (1.1). Following [44], a new line of work developed, aiming to understand CLTs in infinite

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dimensions under more general dependence structures. The literature on CLTs for Hilbert space-valued processes typically assumes that the sequence of random variables $\{X_k\}$ either admits a linear representation (e.g., [12,20,27,28]), or meets suitable mixing conditions (e.g., [11,20,24,29]).

For $\mathscr{H}_1, \mathscr{H}_2$ being finite dimensional, the behavior of $\{S_n\}_{n\in\mathbb{N}}$ is by now well understood for Gaussian processes $\{X_k\}$ and under general assumptions on G, avoiding linearity or mixing conditions. The seminal work of Breuer and Major [7] ($\mathscr{H}_1 = \mathscr{H}_2 = \mathbb{R}$) showed the convergence of (1.1) to a Gaussian law whenever $\mathrm{E}\,|G(X_1)|^2 < \infty$ and $\{G(X_k)\}_{k\in\mathbb{Z}}$ exhibits short-range dependence, in the sense that its autocovariance functions are absolutely summable. Such theorems are now customarily referred to as Breuer-Major theorems. Later, [1] considered the multivariate setting $\mathscr{H}_1 = \mathbb{R}^d, \mathscr{H}_2 = \mathbb{R}$ with $\mathrm{E}\,|G(X_1)|^2 < \infty$ and $\mathrm{E}\,X_1 = 0$, $\mathrm{Cov}(X_1) = \mathrm{Id}_d$ (with Id_d the $d\times d$ identity matrix), assuming further that the autocorrelation functions satisfy

$$\sum_{v\in\mathbb{Z}} \sup_{r,s=1,\dots,d} |\tilde{\phi}_{rs}(v)|^q < \infty, \quad \text{with} \quad \tilde{\phi}_{rs}(v) \doteq \mathrm{E}\,X_1^{(r)} X_{1+v}^{(s)}, \tag{1.2}$$

where $X_k^{(r)}, r = 1,\dots,d$, denotes the $r-$th component of X_k and q the Hermite rank of the function G to be defined (in a more general setting) in Definition 4.1 below. Since then, several other cases for finite dimensional $\mathscr{H}_1, \mathscr{H}_2$ have been investigated, including $\mathscr{H}_2 = \mathbb{R}^m, m \geq 1$; see [3,16,17].

The proof techniques used in the seminal work [7] and its respective follow-up articles are based on moment and cumulant computations using diagram formulae. An alternative, more modern approach is based on a pairing of Malliavin calculus and Stein's lemma to derive quantitative CLTs under different distances implying weak convergence for subordinated Gaussian processes. We refer to [34] for a detailed outline of the tools that are used. This machinery has since then been leveraged to prove quantitative Breuer-Major theorems in numerous settings, including $\mathscr{H}_1 = \mathbb{R}^d, \mathscr{H}_2 = \mathbb{R}^m$ [22,32,35,36,37,38,39]; random fields, i.e., $n \in \mathbb{R}^p$ in (1.1) (see [33]), and functional settings [8,31]. Note that, for the functional setting, the former work refers to continuous time processes, and the latter work [31] to the functional convergence of processes $\frac{1}{\sqrt{n}}\sum_{k=1}^{\lfloor n\cdot\rfloor} G(X_k)$, with $G: \mathbb{R}\to\mathbb{R}$. In particular, they are both different from our setting (1.1). For future reference we use the term *continuous (time) CLT* to refer to the convergence of $\frac{1}{\sqrt{n}}\sum_{k=1}^{\lfloor n\cdot\rfloor} G[X_k]$, to avoid confusion with CLTs for functional data.

In a recent work, [6] introduced the technical tools to derive quantitative central limit theorems (in the so-called d_2 distance, see Section 3 below) in the context of random variables taking values in a separable Hilbert space. The authors in [6] state general conditions on possibly infinite chaos decompositions to converge weakly to a Gaussian measure on Hilbert spaces. Their motivation stems from the problem of proving continuous CLTs to a Gaussian random variable in a function space in the case $\mathscr{H}_1 = \mathscr{H}_2 = \mathbb{R}$, thus bypassing the usual procedure of proving convergence of the finite dimensional distributions and tightness of the family of random variables $\{S_n\}_{n\in\mathbb{N}}$ in the respective space.

In the present work, we leverage the tools developed by [6], to prove a Breuer-Major theorem for (1.1) allowing for general Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$. There are two main technical difficulties associated with our proofs. The first difficulty is in finding a chaotic decomposition for the process $\{S_n\}_{n\in\mathbb{N}}$. To be more precise, we decompose the Hilbert space $L^2(\mathscr{H}_1, \gamma : \mathscr{H}_2)$, where γ is a suitable Gaussian measure on $\mathscr{H}_1$ and L^2 is the usual space of square integrable functions from $\mathscr{H}_1$ to $\mathscr{H}_2$, into its chaotic components; see Section 4. Although the arguments in Section 4 are standard, we obtain tractable forms for the Hermite coefficients, which can then be used for applications; see Section 6.

The second difficulty is to impose suitable conditions on G and the temporal and cross-sectional correlations structure of the process $\{X_k\}$. Let

$$Q : \mathscr{H}_1 \to \mathscr{H}_1, \quad Q[\cdot] = \mathrm{E}\left(\langle X_1, \cdot\rangle_{\mathscr{H}_1} X_1\right) \tag{1.3}$$

be the covariance operator of X_1. For $\{u_r\}_{r\in\mathbb{N}}$ an orthonormal basis in $\mathscr{H}_1$, we define the autocorrelation function of the scores of X_k by

$$\rho_{rs}(v) \doteq \mathrm{E}\left(\langle X_1, Q^{-1/2}u_r\rangle_{\mathscr{H}_1}\langle X_{1+v}, Q^{-1/2}u_s\rangle_{\mathscr{H}_1}\right), \quad v \in \mathbb{Z}; \tag{1.4}$$

see Section 3 for more details on $Q^{-1/2}$. The proofs in finite dimension, e.g., [1] and [36], crucially exploit the fact that $\dim(\mathscr{H}_1) < \infty$ which results in condition (1.2). Attempting to employ the same tools and based on condition (1.2) and the autocorrelations (1.4), one may conjecture that

$$\sum_{v\in\mathbb{Z}} \sup_{r,s\in\mathbb{N}} |\rho_{rs}(v)|^q < \infty \tag{1.5}$$

can substitute assumption (1.2) as sufficient condition to prove a CLT in infinite dimensions. Interestingly, our study suggests that (1.5) is not sufficient when $\dim(\mathscr{H}_1) = \infty$. In general, our condition (see (2.1) below) is stronger than (1.5), but equivalent when $\dim(\mathscr{H}_1) < \infty$; see Remark 2.1 below.

Besides developing a CLT for operators of Gaussian Hilbert space-valued random variables $\{S_n\}_{n\in\mathbb{N}}$ in (1.1), we provide a continuous time as well as a quantitative version of the Breuer-Major theorem for such random variables. To the best of our knowledge, our results are the first to generalize the notion of subordination to Hilbert space-valued random processes, allowing for a general class of transformations G.

Our investigation is closely related to the modeling and analysis of functional data (e.g., data in $L^2([0,1])$), where CLTs can be used for downstream statistical tasks, such as the design of hypothesis tests, or the construction of confidence bands (for some expositions, see [5, 19, 43]). Standard models are Hilbert space-valued generalizations of autoregressive models [2,23]. Recently, the study of functional data has resurged, due to Machine Learning applications. Examples include data taking values in reproducing kernel Hilbert spaces [30, 45], as well as neural networks for learning maps between function spaces; see [21] who introduced the framework of neural operators for this task.

In Section 6, we illustrate the generality of our main result by applying it to statistics of functional data such as the sample covariance operator (also studied in [13, 25, 26]) and estimators for eigenvalues of the covariance operator (e.g., [5]). Such results already exist for linear processes, but we extend them to a different dependence structure determined through the process's autocorrelation structure. The neural network literature has been interested in proving CLTs for networks with randomly initialized weights as the width of the layers diverges; see [14, 21, 30]. We provide a novel result along these lines for neural operators giving a quantitative CLT in terms of the network's increasing width.

The rest of the paper is organized as follows. In Section 2, we present our main results. Section 3 focuses on notation, terminology, and some standard facts, and is followed by a chaotic decomposition in Section 4. The proofs are presented in Section 5 and supplemented by results in Appendices A and B. Section 6 concludes with a variety of applications of our results.

2 Main Results

We state in Section 2.1 our main results, providing a central limit theorem for the quantity in (1.1) and a continuous-time version of the same result. The statements are followed by a discussion on our assumptions and some examples in Section 2.2.

2.1 Statements

Prior to stating our main results, we collect here a minimal amount of notation necessary to formulate the statements and refer the reader to Section 3 for more details. Let $(\mathcal{X}, \mathcal{F}, \mu)$ be a measure space and $\mathcal{K}$ a separable Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ and norm $\|\cdot\|_{\mathcal{K}}$. We denote by $L^2(\mathcal{X}, \mathcal{F}, \mu : \mathcal{K})$ the space of functions from $\mathcal{X}$ to $\mathcal{K}$ that admit two moments with regard to μ. Whenever it is clear from the context, we omit the measure μ and write $L^2(\mathcal{X} : \mathcal{K})$. Moreover, if $\mathcal{K} = \mathbb{R}$, we may write $L^2(\mathcal{X})$.

Recall from (1.3) the covariance operator Q of X_1, and the induced autocorrelation function ρ_{rs} defined in (1.4). Central to our statements is the condition $\mathrm{E}\,\|G[X_1]\|^2_{\mathcal{H}_2} < \infty$ which can be recast as $G \in L^2(\mathcal{H}_1, \gamma_Q : \mathcal{H}_2)$, where γ_Q is the (unique) Gaussian measure on $\mathcal{H}_1$ associated with the covariance operator Q and mean-zero. As anticipated, one part of our assumptions can be written in terms of ρ_{rs} in (1.4) and the Hermite rank of G, formally defined in Definition 4.1 below. For clarity, we state our theorems for the important case $\dim(\mathcal{H}_1) = \infty$, but the results hold even with $\dim(\mathcal{H}_1) < \infty$. When $\dim(\mathcal{H}_1) < \infty$, the results here recover many existing theorems, e.g., Theorem 7.2.4 of [34] and Theorem 5.1 of [6].

Covariance operators are elements of the space of Hilbert-Schmidt operators $HS(\mathcal{H})$, for some Hilbert space $\mathcal{H}$; we refer to Section 3 for more details on $HS(\mathcal{H})$ and note here that $HS(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}$. This isomorphism is used throughout this work and gets also reflected in the representation of the limiting covariance operators of our statements.

Theorem 2.1 (Breuer-Major for Hilbert space-valued random variables)**.** *Let $\{X_k\}_{k\in\mathbb{Z}}$ be a zero-mean, stationary Gaussian process with values in $\mathcal{H}_1$ and covariance operator Q. Suppose $G \in L^2(\mathcal{H}_1, \gamma_Q : \mathcal{H}_2)$ with Hermite rank $q \geq 1$ and that*

$$\sum_{v\in\mathbb{Z}} \left(\sup_{r\geq 1} \sum_{s=1}^{\infty} |\rho_{rs}(v)| \right)^q < \infty, \tag{2.1}$$

where ρ_{rs} is defined in (1.4). Then, $S_n \xrightarrow{d} Z$, as $n \to \infty$, where Z is a centered Gaussian random variable with values in $\mathcal{H}_2$ and covariance operator

$$\mathcal{T}_Z \doteq \sum_{v=1}^{\infty} \Big(\mathrm{E}\, G[X_1] \otimes G[X_{v+1}] + \mathrm{E}\, G[X_{v+1}] \otimes G[X_1] \Big) + \mathrm{E}\, G[X_1] \otimes G[X_1] \in \mathcal{H}_2 \otimes \mathcal{H}_2. \tag{2.2}$$

In Theorem A.1 in Appendix A, we also give a quantitative version of Theorem 2.1 and its proof.

Theorem 2.1 and the used proof techniques allow us also to generalize the result to a continuous-time version. Define

$$V_n(t) \doteq \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} G[X_k], \quad t \in [0,1], \tag{2.3}$$

where $\{X_k\}_{k\in\mathbb{Z}}$, $X_k \in \mathcal{H}_1$, and $G : \mathcal{H}_1 \to \mathcal{H}_2$. Here, V_n is an element of $L^2([0,1] : \mathcal{H}_2) \cong L^2([0,1]) \otimes \mathcal{H}_2$, which is again a Hilbert space. For $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$, Theorem 5.1 of [6] established a CLT for V_n proving convergence to a Brownian motion in $L^2([0,1])$. Recall that the covariance operator $\mathcal{T}_B$ of a Brownian motion B in $L^2([0,1])$ is given by the integral operator

$$\mathcal{T}_B : L^2([0,1]) \to L^2([0,1]), \;\; \mathcal{T}_B[f] = \int_0^1 \int_0^1 f(t)\kappa(s,t)dt \quad \text{with} \quad \kappa(s,t) \doteq s \wedge t. \tag{2.4}$$

The following theorem states an analogue of Theorem 5.1 in [6] but allows for general Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$.

Theorem 2.2. *Let $\{V_n(t)\}_{t\in[0,1]}$ be as in (2.3) with $\{X_k\}_{k\in\mathbb{Z}}$ a zero-mean, stationary Gaussian process with values in $\mathcal{H}_1$ and covariance operator Q. Suppose $G \in L^2(\mathcal{H}_1, \gamma_Q : \mathcal{H}_2)$ with Hermite rank $q \geq 1$ and that (2.1) is true. Then,*

$$V_n \xrightarrow{d} W, \quad \text{as } n \to \infty,$$

in $L^2([0,1]) \otimes \mathcal{H}_2$, where $W \doteq B \otimes Z$ is a centered Gaussian element in $L^2([0,1]) \otimes \mathcal{H}_2$ with $B = \{B_t \mid t \in [0,1]\}$ denoting a standard Brownian motion in $L^2([0,1])$ with covariance operator $\mathcal{T}_B$, and Z being the Gaussian limit in Theorem 2.1 with covariance operator $\mathcal{T}_Z$ given in (2.2). In particular, the covariance operator of W is given by $\mathcal{T}_W : L^2([0,1]) \otimes \mathcal{H}_2 \to L^2([0,1]) \otimes \mathcal{H}_2$ with

$$\mathcal{T}_W = \mathcal{T}_B \otimes \mathcal{T}_Z.$$

2.2 Discussion on assumptions and examples

We first provide a few remarks comparing assumption (2.1) in Theorem 2.1 to the conjectured condition (1.5) followed by some examples for the process $\{X_k\}_{k\in\mathbb{Z}}$.

Remark 2.1. Note that, for all $v \in \mathbb{Z}$,

$$\sup_{r,s=1,\ldots,\dim(\mathcal{H}_1)} |\rho_{rs}(v)| \leq \sup_{r=1,\ldots,\dim(\mathcal{H}_1)} \sum_{s=1,\ldots,\dim(\mathcal{H}_1)} |\rho_{rs}(v)| \leq \dim(\mathcal{H}_1) \sup_{r,s=1,\ldots,\dim(\mathcal{H}_1)} |\rho_{rs}(v)|,$$

and so by the first inequality, if Assumption (2.1) is satisfied, then so is the relation (1.5). In addition, if $\dim(\mathcal{H}_1) < \infty$, then (1.5) is true (with $r, s = 1, \ldots, \dim(\mathcal{H}_1)$) if and only if (2.1) is true.

Remark 2.2. Recall the covariance operator Q from (1.3). Suppose that the time and spatial decay of the autocovariance function decouple, i.e.,

$$\rho_{rs}(v) = \mathrm{E}\left(\langle X_1, u_r\rangle_{\mathcal{H}_1} \langle X_{1+v}, u_s\rangle_{\mathcal{H}_1}\right) = \langle Qu_r, u_s\rangle_{\mathcal{H}_1} \beta(v), \quad r, s \in \mathbb{N}, v \in \mathbb{Z}, \tag{2.5}$$

for a suitable function β with $\beta(0) = 1$. Since $\rho_{rs}(0) = \mathrm{E}\left(\langle X_1, Q^{-1/2}u_r\rangle_{\mathcal{H}_1} \langle X_1, Q^{-1/2}u_s\rangle_{\mathcal{H}_1}\right) = \delta_{rs}$ with δ_{rs} denoting the Kronecker delta, and given (2.5), we can infer that

$$\rho_{rs}(v) = \mathrm{E}\left(\langle X_1, Q^{-1/2}u_r\rangle_{\mathcal{H}_1} \langle X_{1+v}, Q^{-1/2}u_s\rangle_{\mathcal{H}_1}\right) = \beta(v)\langle QQ^{-\frac{1}{2}}u_r, Q^{-\frac{1}{2}}u_s\rangle_{\mathcal{H}_1} = \beta(v)\delta_{rs}.$$

Hence, assuming (2.5), Conditions (1.5) and (2.1) are equivalent since

$$\sup_{r\in\mathbb{N}} \sum_{s=1}^{\infty} |\rho_{rs}(v)| = \sup_{r\in\mathbb{N}} \sum_{s=1}^{\infty} |\beta(v)\delta_{rs}| = \sup_{r\in\mathbb{N}} |\rho_{rr}(v)| = \beta(v).$$

As a first example, we show that our results trivially extend (in a quantitative way) the standard CLT for $\mathcal{H}_1$-valued subordinated Gaussian processes.

Example 2.1 (i.i.d. case)**.** Let $\{X_k\}_{k\in\mathbb{Z}}$ be a sequence of zero-mean, Gaussian, i.i.d. random variables with covariance operator Q, and let $G \in L^2(\mathcal{H}_1, \gamma_Q : \mathcal{H}_2)$. Then,

$$\rho_{rs}(v) = \mathrm{E}\left(\langle X_1, Q^{-1/2}u_r\rangle_{\mathcal{H}_1} \langle X_{1+v}, Q^{-1/2}u_s\rangle_{\mathcal{H}_1}\right) = \delta_{v0}\langle QQ^{-\frac{1}{2}}u_r, Q^{-\frac{1}{2}}u_s\rangle_{\mathcal{H}_1} = \delta_{v0}\delta_{rs},$$

and so Assumption (2.1) trivially holds. By Theorem 2.1, it follows that, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} G[X_k] \xrightarrow{d} \mathcal{N}, \quad \mathcal{T}_{\mathcal{N}} \doteq \mathrm{E}\, G[X_1] \otimes G[X_1].$$

The following example identifies an equivalent condition to (2.1) for *m-dependent processes*. While m-dependence is a simple relaxation of independence, most time series models are not actually m-dependent, but can be approximated by an m-dependent sequence; see Chapter 16.1 in [18] for some examples. In particular, m-dependence does not require the underlying model to admit a linear representation.

Example 2.2 (m-dependent case)**.** Let $\{X_k\}_{k\in\mathbb{Z}}$ be a $\mathscr{H}_1$-valued, stationary, Gaussian, and m-dependent family, that is, for all $\ell \in \mathbb{N}$, the distribution of $\{X_k\}_{k\leq \ell}$ is independent of the distribution of $\{X_k\}_{k\geq \ell+m+1}$. Then, (2.1) can be rewritten as

$$\sum_{|v|\leq m} \left(\sup_{r\in\mathbb{N}} \sum_{s=1}^{\infty} |\rho_{rs}(v)| \right)^{q} < \infty. \tag{2.6}$$

In particular, (2.6) is true if and only if

$$\sup_{|v|\leq m} \sup_{r\in\mathbb{N}} \sum_{s=1}^{\infty} |\rho_{rs}(v)| < \infty. \tag{2.7}$$

Assuming (2.7) is true, the limiting autocovariance operator of S_n is

$$\mathscr{T}_Z \doteq \sum_{v=1}^{m} \left(\mathrm{E}\, G[X_1] \otimes G[X_{v+1}] + \mathrm{E}\, G[X_{v+1}] \otimes G[X_1] \right) + \mathrm{E}\, G[X_1] \otimes G[X_1] \in \mathscr{H}_2 \otimes \mathscr{H}_2.$$

Remark 2.3. A close look into the proof of Theorem 2.1 reveals that condition (2.7) is not necessary in the special case of $G = \mathrm{Id}_{\mathscr{H}_1}$ in (1.1). We refer to Remark 5.1 below for more details on this observation. Given that (2.7) is not necessary, we can recover classical CLTs for m-dependent random variables as in Theorem 16.3 in [18]. However, our results require Gaussianity of $\{X_k\}$. On the other hand, imposing condition (2.7), we can allow for general transformations G, strengthening existing results for m-dependent processes.

We now present a class of $L^2([0,1])$-valued random variables and prove that they satisfy assumption (2.1). The focus here is on linear processes with values in $L^2([0,1])$ (for an excellent exposition, see [5]), because the calculations of the autocorrelations are generally tractable, but we emphasize that the Gaussian processes considered in this work do not need to admit a linear representation.

A stationary sequence $\{X_k\}_{k\in\mathbb{Z}}$ of $\mathscr{H}_1-$valued random variables is called an AutoRegressive Hilbertian process of order one (or functional autoregressive process) (ARH(1)) associated with (μ, ε, ψ) if

$$X_k - \mu = \psi(X_{k-1} - \mu) + \varepsilon_k, \quad k \in \mathbb{Z}, \tag{2.8}$$

where $\varepsilon = \{\varepsilon_k : k \in \mathbb{Z}\}$ is an $\mathscr{H}_1-$valued white noise, $\mu \in \mathscr{H}_1$, and $\psi \in HS(\mathscr{H}_1)$ (the space $HS(\mathscr{H}_1)$ will be recalled in Section 3). Without loss of generality, we set $\mu = 0_{\mathscr{H}_1}$. Let, for simplicity, ψ be a compact symmetric operator, that is,

$$\psi = \sum_{j=0}^{\infty} \alpha_j u_j \otimes u_j, \quad \text{or} \quad \psi(u_j) = \alpha_j u_j, \quad \psi^v = \sum_{j=0}^{\infty} \alpha_j^{|v|} u_j \otimes u_j, \quad v \in \mathbb{Z}, \tag{2.9}$$

where α_j is a decreasing, positive sequence with $1 > \alpha_1$ and $\lim_{j\to\infty} \alpha_j = 0$. Note that $\alpha_j > 0$ is not a restriction since, if $\alpha_j < 0$, we can replace u_j with $-u_j$, leading to $\alpha_j > 0$.

Several options for the noise ε are possible. Here, we let $\{W_t : t \in \mathbb{R}\}$ be a Brownian motion and

$$\varepsilon_k(\cdot) \doteq W_{k+\cdot} - W_k \in L^2([0,1]), \quad k \in \mathbb{Z}. \tag{2.10}$$

In particular, $\{\varepsilon_k\}$ is a Gaussian sequence.

Example 2.3 (ARH(1) process)**.** Suppose $\{X_k\}_{k\in\mathbb{Z}}$ is an ARH(1) process associated with $(0_{\mathcal{H}_1}, \varepsilon, \psi)$ defined in (2.8)–(2.10). Denote the autocorrelation function

$$\rho_{rs}(v) \doteq \langle \psi^v(u_r), u_s\rangle_{\mathcal{H}_1} = \alpha_r^{|v|}\delta_{rs}, \quad v \in \mathbb{Z},$$

where the first equality holds from Theorem 3.2 in [5] and the last equality holds by using the compactness of ψ. For $q \geq 1$,

$$\sum_{v\in\mathbb{Z}}\left(\sup_{r\in\mathbb{N}}\sum_{s\in\mathbb{N}}\rho_{rs}(v)\right)^q = \sum_{v\in\mathbb{Z}}\left(\sup_{r\in\mathbb{N}}\sum_{s\in\mathbb{N}}\alpha_r^{|v|}\delta_{rs}\right)^q \leq 2\sum_{v=0}^{\infty}(\alpha_1)^{qv} \leq \frac{2}{1-\alpha_1^q} < \infty.$$

This says that Theorem 2.1 holds for any operator G with Hermite rank $q \geq 1$, whenever the underlying $\{X_k\}_{k\in\mathbb{Z}}$ is an ARH(1) process defined as above.

3 Preliminaries

Operators and norms on Hilbert spaces: Let $\{u_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. We denote by $HS(\mathcal{H})$ the Hilbert space of Hilbert-Schmidt operators equipped with the following inner product and its induced norm

$$\langle T, S\rangle_{HS(\mathcal{H})} \doteq \sum_{i=1}^{\infty}\langle T(u_i), S(u_i)\rangle_{\mathcal{H}}, \ \ \|T\|^2_{HS(\mathcal{H})} \doteq \sum_{i=1}^{\infty}\|T(u_i)\|^2_{\mathcal{H}}. \tag{3.1}$$

Closely related is the Banach space of trace class operators, denoted by $\mathcal{S}(\mathcal{H})$ and equipped with the norm

$$\|T\|_{\mathcal{S}(\mathcal{H})} = \mathrm{tr}(|T|) \doteq \sum_{i=1}^{\infty}\langle |T|u_i, u_i\rangle_{\mathcal{H}}, \tag{3.2}$$

with $|T| = \sqrt{T^*T}$. If T is a non-negative, self-adjoint operator (for instance a covariance operator), then $\|T\|_{\mathcal{S}(\mathcal{H})} = \mathrm{tr}(T) = \sum_{i=1}^{\infty}\langle Tu_i, u_i\rangle_{\mathcal{H}}$. The two norms (3.1) and (3.2) satisfy

$$\|\cdot\|_{HS(\mathcal{H})} \leq \|\cdot\|_{\mathcal{S}(\mathcal{H})}. \tag{3.3}$$

Recall the covariance operator Q from (1.3). Covariance operators are non-negative, self-adjoint trace-class operators. Let $\{u_j\}_{j\in\mathbb{N}}$ be the *eigenvectors* of Q and $\{\lambda_j\}_{j\in\mathbb{N}}$ its corresponding sequence of positive *eigenvalues*, such that $Qu_j = \lambda_j u_j$. Without loss of generality, we can re-enumerate $\{u_j\}_{j\in\mathbb{N}}$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots$. Then, the covariance operator Q satisfies

$$\mathrm{tr}(Q) = \sum_{j=1}^{\infty}\lambda_j = \mathrm{E}\,\|X_1\|^2, \quad \lambda_j = \mathrm{E}\langle X_1, u_j\rangle^2, \tag{3.4}$$

by Parseval's identity.

Crucial to our analysis are the isomorphisms $L^2(\Omega : \mathcal{H}) \cong L^2(\Omega : \mathbb{R}) \otimes \mathcal{H}$ and $HS(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}$. Furthermore, for $\mathcal{H}_i, i = 1, 2,$ Hilbert spaces, we can define an inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$, such that

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} = \langle x_1, x_2\rangle_{\mathcal{H}_1}\langle y_1, y_2\rangle_{\mathcal{H}_2}, \quad x_i \in \mathcal{H}_1, \ y_i \in \mathcal{H}_2, \ i = 1, 2. \tag{3.5}$$

Gaussian measures on Hilbert spaces: We provide here some elementary facts about measures on Hilbert spaces; for more details we refer to [10]. Fix a Hilbert space $\mathscr{H}$ and denote the corresponding Borel sets by $\mathscr{B}(\mathscr{H})$. A Gaussian measure $\gamma_{a,Q}$ (with mean a and covariance operator Q) on $(\mathscr{H}, \mathscr{B}(\mathscr{H}))$ is a measure whose Fourier transform is given by

$$N_{a,Q}(h) = \exp\left\{ i\langle a, h\rangle_{\mathscr{H}} + \frac{1}{2}\langle Q[h], h\rangle_{\mathscr{H}} \right\}.$$

Throughout the paper we use $\gamma_Q \doteq \gamma_{0_{\mathscr{H}},Q}$. The space $Q^{1/2}(\mathscr{H}) = \{y \in \mathscr{H} : y = Q^{1/2}x \text{ for some } x \in \mathscr{H}\}$ is called *Cameron-Martin space.* Introduce the *white noise mapping*

$$W : Q^{1/2}(\mathscr{H}) \to L^2(\mathscr{H}, \gamma_Q), \quad v \mapsto W_v, \quad W_v(x) \doteq \langle x, Q^{-1/2}v\rangle_{\mathscr{H}}, \quad x \in \mathscr{H}, \tag{3.6}$$

and from Section 1.7 in [10], since $Q^{1/2}$ is dense in $\mathscr{H}$ and W is an isometry, we can extend it uniquely to a map $W : \mathscr{H} \to L^2(\mathscr{H}, \gamma_Q)$, and to simplify notation, denote this map again by W. Then,

$$\int_{\mathscr{H}} W_{u_1}(x)W_{u_2}(x)\gamma_Q(dx) = \langle QQ^{-1/2}u_1, Q^{-1/2}u_2\rangle_{\mathscr{H}} = \langle u_1, u_2\rangle_{\mathscr{H}}. \tag{3.7}$$

Throughout this work, we use both the white-noise representation $W_u(x)$ and the notation $\langle x, Q^{-1/2}u\rangle_{\mathscr{H}}$ to either prioritize notational simplicity or to emphasize the statistical operation.

Weak convergence in infinite dimensions: Recall that, for an (everywhere) differentiable operator $h : \mathscr{H} \to \mathbb{R}$, the $k-$th (Fréchet) derivative $\nabla^k h$ is a linear map from $\mathscr{H}$ to $\mathscr{L}(\mathscr{H}^{\otimes k} : \mathbb{R})$, i.e. the space of linear operators from $\mathscr{H}^{\otimes k}$ to $\mathbb{R}$; see [9]. Further note that, for $\mathscr{L}(\mathscr{H} : \mathbb{R})$, equipped with the usual operator norm $\|T\|_{\mathscr{L}(\mathscr{H}:\mathbb{R})} = \sup_{\|x\|_{\mathscr{H}} \le 1} |T[x]|$, is a Banach space. We denote by $C_b^k(\mathscr{H})$ the space of bounded, $\mathbb{R}$-valued operators on $\mathscr{H}$, admitting k Fréchet derivatives, i.e., $h \in C_b^k(\mathscr{H})$ if

$$\|h\|_{C_b^k(\mathscr{H})} = \sup_{j=1,\dots,k} \sup_{x \in \mathscr{H}} \|\nabla^j h(x)\|_{\mathscr{L}(\mathscr{H}^{\otimes j}:\mathbb{R})} < \infty.$$

Then, the d_j metric on $\mathscr{H}$ is defined as

$$d_j(f, g) \doteq \sup_{h \in C_b^j(\mathscr{H}), \|h\|_{C_b^j(\mathscr{H})} \le 1} |h[f] - h[g]| \quad \text{for} \quad j \ge 1. \tag{3.8}$$

By Theorem 2.4 in [15], for $j \ge 1$, d_j (and in particular, d_2) metrizes weak convergence in $\mathscr{H}$.

Isonormal Gaussian processes and contractions: Denote by $\mathfrak{H}$ an underlying separable Hilbert space, and define an isonormal Gaussian process, i.e., a centered family of Gaussian random variables $\{W(h) \colon h \in \mathfrak{H}\}$, defined on a complete probability space $(\Omega, \mathscr{F}, P)$, such that

$$\mathrm{E}\, W(h_1)W(h_2) = \langle h_1, h_2\rangle_{\mathfrak{H}}, \qquad h_1, h_2 \in \mathfrak{H},$$

where the σ-algebra $\mathscr{F}$ is generated by W. We show in Lemma B.3 in Appendix B that it is possible to construct an isonormal Gaussian process on $\mathfrak{H}$ with the same autocorrelation function as an $\mathscr{H}-$valued Gaussian process. Denote by $\mathfrak{H}^{\otimes n}$ the the $n-$th tensor power of $\mathfrak{H}$, and by $\mathfrak{H}^{\odot n}$ the n-fold symmetrized tensor product of $\mathfrak{H}$, equipped with the norm $\sqrt{n!}\|\cdot\|_{\mathfrak{H}^{\otimes n}}$. For $n \ge 0$ and a kernel $f \in \mathfrak{H}^{\odot n}$, we write $I_n(f)$ to denote the multiple Wiener-Itô integral of order n of f; see [40].

Let $\{e_k \colon k \ge 0\}$ be an orthonormal basis of $\mathfrak{H}$. Fix $f \in \mathfrak{H}^{\odot n}$ and $g \in \mathfrak{H}^{\odot m}$, then for every $l = 0, \dots, n \wedge m$, the $l-$th contraction of f and g is defined as

$$f \otimes_l g = \sum_{i_1,\dots,i_l=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_l}\rangle_{\mathfrak{H}^{\otimes l}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_l}\rangle_{\mathfrak{H}^{\otimes l}} \in \mathfrak{H}^{\otimes(n+m-2l)}.$$

See also Appendix B in [34] for more details on contractions.

4 A General Wiener-Itô Chaotic Decomposition

4.1 Hermite expansion

We aim to introduce an orthonormal basis of the space $L^2(\mathscr{H}_1, \gamma_Q : \mathscr{H}_2)$. Simpler forms of this result have appeared elsewhere, e.g., for $L^2(\mathscr{H}_1, \gamma_1 : \mathbb{R})$ in [10]. For completeness, we start with rephrasing some basic properties of Hermite polynomials in the real valued case. The Hermite polynomials $\{H_n\}_{n\in\mathbb{N}_0}$ build an orthogonal basis of $L^2(\mathbb{R}, \phi(x)dx : \mathbb{R})$, where ϕ is the standard normal density; see Proposition 5.1.3 in [42]. As a result, every function $f \in L^2(\mathbb{R}, \phi(x)dx : \mathbb{R})$ has an expansion in Hermite polynomials. The Hermite polynomial of order n is defined as $H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$, for $x \in \mathbb{R}$, $n \geq 1$ and $H_0(x) = 1$ for all $x \in \mathbb{R}$. Moreover, recall their crucial orthogonality property

$$\mathrm{E}\left(H_n(X)H_m(Y)\right) = n! \left(\mathrm{E}\, XY\right)^n \delta_{nm}; \tag{4.1}$$

see Proposition 5.1.1 in [42].

We continue with some more notation. Let $\mathscr{L} = \ell^1(\mathbb{N}_0) = \{\boldsymbol{l} \in \mathbb{N}_0^\infty : \sum_{k=1}^\infty l_k < \infty\}$ denote the space of summable sequences with values in $\mathbb{N}_0$. Note that, for $\boldsymbol{l} \in \mathscr{L}$, there are only finitely many non-zero elements of $\boldsymbol{l}$. For fixed $\boldsymbol{l} \in \mathscr{L}$, we define

$$H_{\boldsymbol{l}}(x) \doteq \prod_{m=1}^{\infty} H_{l_m}(W_{u_m}(x)) = \prod_{m=1}^{M_{\boldsymbol{l}}} H_{l_m}(W_{u_m}(x)), \quad x \in \mathscr{H}_1, \tag{4.2}$$

where W_{u_m} is the white noise mapping defined in (3.6), $\{u_m\}_{m\in\mathbb{N}}$ is a basis for $\mathscr{H}_1$, and

$$M_{\boldsymbol{l}} \doteq \max\{m \in \mathbb{N} : l_m \geq 1\} \tag{4.3}$$

is the order of the highest non-zero element of the sequence $\boldsymbol{l}$. For $m \in \mathbb{N}_0$, we also write $H_m[W_u]$ to denote a map from $\mathscr{H}_1$ to $\mathbb{R}$ evaluated by $H_m[W_u](x) \doteq H_m(W_u(x))$. We have the following result.

Lemma 4.1. *Consider the operator $f \in L^2(\mathscr{H}_1, \gamma_Q : \mathscr{H}_2) \cong L^2(\mathscr{H}_1, \gamma_Q) \otimes \mathscr{H}_2$, where γ_Q is a Gaussian measure with covariance operator Q. Then, f admits the generalized Hermite expansion*

$$f[x] = \sum_{i=1}^{\infty} \sum_{\boldsymbol{r}\in\mathscr{L}} c_{i,\boldsymbol{r}} H_{\boldsymbol{r}}(x) \otimes v_i, \quad x \in \mathscr{H}_1, \tag{4.4}$$

with $H_{\boldsymbol{r}}$ as in (4.2) and $\{c_{i,\boldsymbol{r}}\}_{i\in\mathbb{N},\boldsymbol{r}\in\mathscr{L}}$ is given by (with some abuse of notation)

$$c_{i,\boldsymbol{r}} \doteq \frac{1}{\prod_{m=1}^{\infty} r_m!} \left\langle f, H_{\boldsymbol{r}} \otimes v_i \right\rangle_{L^2(\mathscr{H}_1,\gamma_Q)\otimes\mathscr{H}_2}. \tag{4.5}$$

Proof: For shortness' sake we write

$$\{\Gamma_{\boldsymbol{r}i}(\cdot)\}_{i\in\mathbb{N},\boldsymbol{r}\in\mathscr{L}}, \quad \text{with} \quad \Gamma_{\boldsymbol{r}i}(x) \doteq H_{\boldsymbol{r}}(x) \otimes v_i, \quad x \in \mathscr{H}_1.$$

We prove that the family $\{\Gamma_{\boldsymbol{r}i}\}_{i\in\mathbb{N},\boldsymbol{r}\in\mathscr{L}}$ is orthogonal and complete in $L^2(\mathscr{H}_1, \gamma_Q : \mathbb{R}) \otimes \mathscr{H}_2$.

Orthogonality: Fix $i, j \in \mathbb{N}$ and $\boldsymbol{r}, \boldsymbol{s} \in \mathscr{L}$, and let $M \doteq \max\{M_{\boldsymbol{r}}, M_{\boldsymbol{s}}\} < \infty$ (see (4.3)). Denote $\delta_{\boldsymbol{rs}} = \prod_{k=1}^{d} \delta_{r_k s_k}$ whenever $\boldsymbol{r}, \boldsymbol{s} \in \mathbb{N}_0^d$. Then, with explanations given below,

$$\langle \Gamma_{\boldsymbol{r}i}, \Gamma_{\boldsymbol{s}j} \rangle_{L^2(\mathscr{H}_1,\gamma)\otimes\mathscr{H}_2}$$

$$
\begin{aligned}
&= \left\langle \prod_{m=1}^{M} H_{r_m}[W_{u_m}] \otimes v_i, \prod_{m=1}^{M} H_{s_m}[W_{u_m}] \otimes v_j \right\rangle_{L^2(\mathcal{H}_1,\gamma_Q)\otimes\mathcal{H}_2} \\
&= \left\langle \prod_{m=1}^{M} H_{r_m}[W_{u_m}] \prod_{m=1}^{M} H_{s_m}[W_{u_m}] \right\rangle_{L^2(\mathcal{H}_1,\gamma_Q)} \langle v_i, v_j\rangle_{\mathcal{H}_2} && (4.6) \\
&= \int_{\mathcal{H}_1} \prod_{m=1}^{M} H_{r_m}(W_{u_m}(x)) H_{s_m}(W_{u_m}(x))\gamma_Q(dx)\delta_{ij} \\
&= \prod_{m=1}^{M} \int_{\mathcal{H}_1} H_{r_m}(W_{u_m}(x)) H_{s_m}(W_{u_m}(x))\gamma_Q(dx)\delta_{ij} && (4.7) \\
&= \prod_{m=1}^{M} \langle Q^{-1/2}u_m, QQ^{-1/2}u_m\rangle^{r_m} r_m!\delta_{ij}\delta_{\boldsymbol{rs}} = \prod_{m=1}^{M} l_m!\delta_{r_m s_m}\delta_{ij}, && (4.8)
\end{aligned}
$$

where (4.6) follows from (3.5), (4.7) follows since the variables W_u, W_v are uncorrelated for $u \neq v$, and (4.8) follows from (4.1) and (3.7).

Completeness: Let $\widetilde{\psi} \in L^2(\mathcal{H}_1, \gamma_Q : \mathcal{H}_2)$, and identify $\widetilde{\psi}$ with its isomorphic element $\psi \otimes \kappa \in L^2(\mathcal{H}_1, \gamma_Q) \otimes \mathcal{H}_2$. We need to show that, if for all $\boldsymbol{l} \in \mathcal{L}, i \in \mathbb{N}$,

$$
\langle \widetilde{\psi}, \Gamma_{\boldsymbol{l}i}\rangle_{L^2(\mathcal{H}_1,\gamma_Q)\otimes\mathcal{H}_2} = 0, \tag{4.9}
$$

then $\widetilde{\psi}(x) = 0_{\mathcal{H}_2}, \gamma_Q(dx)$ a.s., i.e., $\kappa = 0_{\mathcal{H}_2}$ or $\psi(x) = 0, \gamma_Q(dx)$ a.s.. Suppose (4.9) is true. Then, (4.9) implies, for all $\boldsymbol{l} \in \mathcal{L}, i \in \mathbb{N}$,

$$
\langle \widetilde{\psi}, \Gamma_{\boldsymbol{l}i}\rangle_{L^2(\mathcal{H}_1,\gamma_Q)\otimes\mathcal{H}_2} = \langle \kappa, v_i\rangle_{\mathcal{H}_2} \int_{\mathcal{H}_1} \psi(x) H_{\boldsymbol{l}}(x)\gamma_Q(dx) = 0.
$$

Note that, if $\langle \kappa, v_i\rangle_{\mathcal{H}_2} = 0$ for all $i \in \mathbb{N}$, then $\kappa = 0$ since $\{v_i\}_{i\in\mathbb{N}}$ is a basis of $\mathcal{H}_2$. Assume this is not the case, i.e., there is some $i \in \mathbb{N}$ such that $\langle \kappa, v_i\rangle_{\mathcal{H}_2} > 0$. Therefore, it is left to prove that

$$
\int_{\mathcal{H}_1} \psi(x) H_{\boldsymbol{l}}(x)\gamma_Q(dx) = 0 \quad \text{for all } \boldsymbol{l} \in \mathcal{L}, \tag{4.10}
$$

implies $\psi = 0$ $\gamma_Q(dx)$-almost surely. That this implication is true follows from, e.g., Theorem 1.1.1 of [40] or Theorem 9.7 of [10]. $\square$

Definition 4.1. Let $f \in L^2(\mathcal{H}_1, \gamma_Q : \mathcal{H}_2)$ and let $c_{i,\boldsymbol{l}}$, $i \in \mathbb{N}, \boldsymbol{l} \in \mathcal{L}$ be the coefficients given by (4.5) in the representation (4.4). Define the *Hermite rank* of the operator f to be

$$
\operatorname{rank}(f) = \min\left\{ q \in \mathbb{N} \mid \text{there is a } \boldsymbol{l} \in \mathcal{L}, \text{with } \sum_{k=1}^{\infty} l_k = q \text{ and } c_{i,\boldsymbol{l}} \neq 0 \text{ for some } i \in \mathbb{N} \right\}.
$$

4.2 Wiener chaos expansion

The following lemma formalizes the chaos decomposition for the partial sums, by using the Hermite expansion developed in Section 4.1.

Lemma 4.2. *The partial sums S_n in (1.1) admit the Wiener-Itô chaos decomposition*

$$
S_n = \sum_{p=q}^{\infty} (I_p \otimes \mathrm{Id}_{\mathcal{H}_2})[h_{p,n}] \quad \textit{with} \quad h_{p,n} = \sum_{i=1}^{\infty} \left(\widetilde{h}_{p,n,i} \otimes v_i \right), \tag{4.11}
$$

where I_p is the $p-$th Wiener chaos, $q \doteq \operatorname{rank}(G)$, and

$$\widetilde{h}_{p,n,i} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \sum_{\boldsymbol{j} \in \mathbb{N}^p} b_{i,\boldsymbol{j}} (\varepsilon_{j_1 k} \otimes \cdots \otimes \varepsilon_{j_p k}) \in \mathfrak{H}^{\odot p}, \tag{4.12}$$

where, for each $p \geq q$, $\{b_{i,\boldsymbol{j}} \doteq b_i(j_1, \ldots, j_p) \mid j_1, \ldots, j_p \geq 1\}$ with $\sum_{\boldsymbol{j} \in \mathbb{N}^p} (b_{i,\boldsymbol{j}})^2 < \infty$ is a symmetric array of real numbers.

Proof: Using the Hermite expansion from Lemma 4.1, we can rewrite, with further explanations given below, the series of partial sums as

$$\begin{aligned} S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} G[X_k] &= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \sum_{i=1}^{\infty} \sum_{\boldsymbol{l} \in \mathcal{L}} c_{i,\boldsymbol{l}} \prod_{j=1}^{M_{\boldsymbol{l}}} H_{l_j}(\langle Q^{-\frac{1}{2}} u_j, X_k \rangle_{\mathcal{H}_1}) \otimes v_i \\ &= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \sum_{i=1}^{\infty} \sum_{\boldsymbol{l} \in \mathcal{L}} c_{i,\boldsymbol{l}} \prod_{j=1}^{M_{\boldsymbol{l}}} H_{l_j}(X(\varepsilon_{jk})) \otimes v_i \qquad (4.13) \\ &= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \sum_{i=1}^{\infty} \sum_{p=q}^{\infty} \sum_{\boldsymbol{l} \in \mathcal{L}_p} c_{i,\boldsymbol{l}} \prod_{j=1}^{M_{\boldsymbol{l}}} H_{l_j}(X(\varepsilon_{jk})) \otimes v_i \qquad (4.14) \\ &= \sum_{i=1}^{\infty} \sum_{p=q}^{\infty} I_p(\widetilde{h}_{p,n,i}) \otimes v_i, \qquad (4.15) \end{aligned}$$

where (4.13) follows from Proposition 7.2.3 of [34] by setting $W_{u_j}(X_k) = X(\varepsilon_{jk})$, where $\{\varepsilon_{jk}\}_{j \in \mathbb{N}, k \in \mathbb{Z}}$ is an underlying isonormal process. Line (4.14) follows by defining $\mathcal{L}_m \doteq \{\boldsymbol{l} \in \mathbb{N}_0^\infty : \sum_{k=1}^\infty l_k = m\}$ and recalling that $\operatorname{rank}(G) = q$. Line (4.15) follows from the discussion in Chapter 8 of [41] and the linearity of I_p. Finally, (4.15) gives

$$S_n = \sum_{p=q}^{\infty} (I_p \otimes \mathrm{Id}_{\mathcal{H}_2}) \left[\sum_{i=1}^{\infty} \left(\widetilde{h}_{p,n,i} \otimes v_i \right) \right] = \sum_{p=q}^{\infty} (I_p \otimes \mathrm{Id}_{\mathcal{H}_2})[h_{p,n}], \tag{4.16}$$

by recalling the identities in (4.11)–(4.12) and the operation $(\mathcal{A} \otimes \mathcal{B})[a \otimes b] = (\mathcal{A}[a] \otimes \mathcal{B}[b])$. The relation (4.16) concludes the proof. $\square$

5 Proofs of Main Results

5.1 Proof of Theorem 2.1

The proof of the Breuer-Major theorem of [36] for $\mathcal{H}_1 = \mathbb{R}^d$ crucially exploits the finite dimensionality of the underlying Gaussian process $\{X_k\}_{k \in \mathbb{Z}}$ and is no longer available. Yet many of the arguments of this section resemble these of the finite dimensional case, presented in [36], and the proof follows similar lines as the proof of Lemma 4.1 in [36].

For the proof of Theorem 2.1, we use the results in [6]. From Theorems 3.14 and 4.2 in [6], we obtain the following implication: Given the chaos decomposition (4.11)–(4.12) of S_n, suppose

(i) for every $p \in \mathbb{N}$, there exists $h_p \in \mathfrak{H}^{\odot p} \otimes \mathcal{H}_2$ such that $\|h_{p,n} - h_p\|_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2} \to 0$, as $n \to \infty$,

$$\sum_{p=1}^{\infty} p! \|h_p\|^2_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2} < \infty \quad \text{and} \quad \lim_{N \to \infty} \sup_{n \geq 1} \sum_{p=N+1}^{\infty} p! \|h_{p,n}\|^2_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2} = 0. \tag{5.1}$$

(ii) a) for $p \geq q$, and $l = 1, \ldots, p-1$, it holds that $\|h_{p,n} \otimes_l h_{p,n}\|_{\mathfrak{H}^{\otimes 2(p-l)} \otimes \mathcal{H}_2^{\otimes 2}} \to 0$, as $n \to \infty$.

b) for $p \wedge r \geq q, p \neq r$, and $l = 1, \ldots, p \wedge r$, it holds that $\|h_{p,n} \otimes_l h_{r,n}\|_{\mathfrak{H}^{\otimes (p+r-2l)} \otimes \mathcal{H}_2^{\otimes 2}} \to 0$, as $n \to \infty$.

Then, $S_n \xrightarrow{d} Z$ with Z a Gaussian element of $\mathcal{H}_2$ with covariance operator $\mathcal{T}_Z = \sum_{p=q}^{\infty} p! \|h_p\|^2_{\mathfrak{H}^{\otimes p}}$.

The implication above becomes clear in the proof of Theorem A.1 in the appendix (see (A.8) and (A.13)–(A.15)), which states a quantitative version of our main result. Note also that Condition (ii) b) is crucial in the infinite dimensional case, but can be omitted in the finite dimensional case since it can be inferred from a); see (3.38) and the bound below in [36].

We introduce the following identities based on the chaos decomposition (4.11)–(4.12)

$$p! \sum_{i=1}^{\infty} \sum_{\boldsymbol{l} \in \mathbb{N}^p} b^2_{i,\boldsymbol{l}} = \mathrm{E}\, \|G^p[X_1]\|^2_{\mathcal{H}_2} < \infty, \quad \sum_{p=q}^{\infty} p! \sum_{i=1}^{\infty} \sum_{\boldsymbol{l} \in \mathbb{N}^p} b^2_{i,\boldsymbol{l}} = \mathrm{E}\, \|G[X_1]\|^2_{\mathcal{H}_2} < \infty, \tag{5.2}$$

where $G^p, q \leq p < \infty$ is the term of order p appearing in the chaotic expansion (4.11)–(4.12) of $S_1 = G[X_1]$.

We now show that the derived chaos decomposition (4.11) for S_n satisfies Conditions (i) and (ii).

Condition (i): With $h_{p,n}$ as in (4.11)–(4.12), we have

$$\begin{aligned}
p! \|h_{p,n}\|^2_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2} &= p! \left\langle \sum_{i=1}^{\infty} \left(\widetilde{h}_{p,n,i} \otimes v_i \right), \sum_{j=1}^{\infty} \left(\widetilde{h}_{p,n,j} \otimes v_j \right) \right\rangle_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2} \\
&= p! \sum_{i,j=1}^{\infty} \left\langle \widetilde{h}_{p,n,i}, \widetilde{h}_{p,n,j} \right\rangle_{\mathfrak{H}^{\otimes p}} \langle v_i, v_j \rangle_{\mathcal{H}_2} \\
&= \frac{p!}{n} \sum_{i=1}^{\infty} \sum_{k_1,k_2=1}^{n} \sum_{\boldsymbol{r},\boldsymbol{s} \in \mathbb{N}^p} b_{i,\boldsymbol{r}} b_{i,\boldsymbol{s}} \prod_{j=1}^{p} \rho_{r_j s_j}(k_1 - k_2) \\
&= p! \sum_{i=1}^{\infty} \sum_{|v|<n} \left(1 - \frac{|v|}{n} \right) \sum_{\boldsymbol{r},\boldsymbol{s} \in \mathbb{N}^p} b_{i,\boldsymbol{r}} b_{i,\boldsymbol{s}} \prod_{j=1}^{p} \rho_{r_j s_j}(v).
\end{aligned} \tag{5.3}$$

Now denote

$$\theta \doteq \sum_{v \in \mathbb{Z}} \theta^q(v) < \infty \quad \text{with} \quad \theta(v) \doteq \sup_{r \geq 1} \sum_{s=1}^{\infty} |\rho_{rs}(v)|, \quad K \doteq \inf_{k \in \mathbb{N}} \{\theta(v) \leq 1 \text{ for all } |v| \geq k\} < \infty, \tag{5.4}$$

where the finiteness of θ follows from assumption (2.1). Note that $\rho_{rs}(v) = \rho_{sr}(-v)$ given (1.4).

By equation (5.3) and using Cauchy-Schwarz and Young's inequality in (5.5) below, we get

$$\begin{aligned}
p! \|h_{p,n}\|^2_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2} &\leq p! \sum_{i=1}^{\infty} \sum_{|v| \leq K-1} \left| \sum_{\boldsymbol{r},\boldsymbol{s} \in \mathbb{N}^p} b_{i,\boldsymbol{r}} b_{i,\boldsymbol{s}} \prod_{j=1}^{p} \rho_{r_j s_j}(v) \right| \\
&\quad + p! \sum_{i=1}^{\infty} \sum_{|v| \geq K} \sum_{\boldsymbol{r},\boldsymbol{s} \in \mathbb{N}^p} |b_{i,\boldsymbol{r}} b_{i,\boldsymbol{s}}| \prod_{j=1}^{p} |\rho_{r_j s_j}(v)| \\
&\leq 2K\, \mathrm{E}\, \|G^p[X_1]\|^2_{\mathcal{H}_2} + p! \sum_{i=1}^{\infty} \sum_{|v| \geq K} \sum_{\boldsymbol{r},\boldsymbol{s} \in \mathbb{N}^p} \frac{1}{2} \left(b^2_{i,\boldsymbol{r}} + b^2_{i,\boldsymbol{s}} \right) \prod_{j=1}^{p} |\rho_{r_j s_j}(v)|.
\end{aligned} \tag{5.5}$$

Moreover, for the second summand in (5.5),

$$
\begin{aligned}
& p! \sum_{i=1}^{\infty} \sum_{|v| \geq K} \sum_{\boldsymbol{r}, \boldsymbol{s} \in \mathbb{N}^p} \frac{1}{2} \left(b_{i,\boldsymbol{r}}^2 + b_{i,\boldsymbol{s}}^2 \right) \prod_{j=1}^{p} |\rho_{r_j s_j}(v)| \\
& \leq p! \sum_{i=1}^{\infty} \sum_{|v| \geq K} \sum_{\boldsymbol{r} \in \mathbb{N}^p} b_{i,\boldsymbol{r}}^2 \max \left\{ \sup_{\boldsymbol{r} \in \mathbb{N}^p} \sum_{\boldsymbol{s} \in \mathbb{N}^p} \prod_{j=1}^{p} |\rho_{r_j s_j}(v)|, \sup_{\boldsymbol{s} \in \mathbb{N}^p} \sum_{\boldsymbol{r} \in \mathbb{N}^p} \prod_{j=1}^{p} |\rho_{r_j s_j}(v)| \right\} \\
& \leq p! \sum_{|v| \geq K} \left(\max\{\theta(v), \theta(-v)\} \right)^q \sum_{i=1}^{\infty} \sum_{\boldsymbol{r} \in \mathbb{N}^p} b_{i,\boldsymbol{r}}^2 \\
& \leq 2\theta \, \mathrm{E} \, \|G^p[X_1]\|_{\mathcal{H}_2}^2 .
\end{aligned} \tag{5.6}
$$

By combining (5.5) and (5.6), we obtain that

$$
p! \|h_{p,n}\|_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2}^2 \leq \mathrm{E} \, \|G^p[X_1]\|_{\mathcal{H}_2}^2 \, (2K + 2\theta) \, , \tag{5.7}
$$

where the constants θ and K were defined in (5.4). Since the RHS of (5.7) does not depend on n, by DCT, take h_p such that, as $n \to \infty$,

$$
\|h_{p,n} - h_p\|_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2} \to 0, \quad \|h_p\|_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2}^2 = \sum_{i=1}^{\infty} \sum_{v \in \mathbb{Z}} \sum_{\boldsymbol{r}, \boldsymbol{s} \in \mathbb{N}^p} b_{i,\boldsymbol{r}} b_{i,\boldsymbol{s}} \prod_{j=1}^{p} \rho_{r_j s_j}(v)
$$

and this also implies that

$$
\sum_{p=q}^{\infty} p! \|h_p\|_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2}^2 = \sum_{p=q}^{\infty} p! \lim_{n \to \infty} \|h_{p,n}\|_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2}^2 \leq \mathrm{E} \, \|G[X_1]\|_{\mathcal{H}_2}^2 (2K + 2\theta). \tag{5.8}
$$

Since the bound in (5.8) does not depend on p and n, it follows that

$$
\lim_{N \to \infty} \sup_{n \geq 1} \sum_{p=N+1}^{\infty} p! \|h_{p,n}\|_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2}^2 = 0.
$$

Condition (ii): We focus here on proving b), the second part of the condition, since it is more involved. Condition a) can be proved using analogous arguments.

Recall from (4.11) $h_{p,n}$ written in terms of $\widetilde{h}_{p,n,i}$ in (4.12). Then, for $l = 1, \ldots, p \wedge r$,

$$
\begin{aligned}
& \|h_{p,n} \otimes_l h_{r,n}\|_{\mathfrak{H}^{\otimes (p+r-2l)} \otimes \mathcal{H}_2^{\otimes 2}}^2 \\
& = \left\| \sum_{i=1}^{\infty} (\widetilde{h}_{p,n,i} \otimes v_i) \otimes_l \sum_{j=1}^{\infty} (\widetilde{h}_{r,n,j} \otimes v_j) \right\|_{\mathfrak{H}^{\otimes (p+r-2l)} \otimes \mathcal{H}_2^{\otimes 2}}^2 \\
& = \sum_{i,j=1}^{\infty} \|\widetilde{h}_{p,n,i} \otimes_l \widetilde{h}_{r,n,j}\|_{\mathfrak{H}^{\otimes (p+r-2l)}}^2 \|v_i \otimes v_j\|_{\mathcal{H}_2^{\otimes 2}}^2 = \sum_{i,j=1}^{\infty} \|\widetilde{h}_{p,n,i} \otimes_l \widetilde{h}_{r,n,j}\|_{\mathfrak{H}^{\otimes (p+r-2l)}}^2 .
\end{aligned} \tag{5.9}
$$

Then, for $l = 1, \ldots, p \wedge r, p \neq r, p, r \geq q$, we have the following identity for the contractions

$$
\widetilde{h}_{p,n,i} \otimes_l \widetilde{h}_{r,n,j}
$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{k_1,k_2=1}^{n}\sum_{\boldsymbol{r}\in\mathbb{N}^p,\boldsymbol{s}\in\mathbb{N}^r} b_{i,\boldsymbol{r}}b_{j,\boldsymbol{s}}(\varepsilon_{r_1k_1}\otimes\cdots\otimes\varepsilon_{r_pk_1})\otimes_l(\varepsilon_{s_1k_2}\otimes\cdots\otimes\varepsilon_{s_rk_2})\\
&= \frac{1}{n}\sum_{k_1,k_2=1}^{n}\sum_{\boldsymbol{r}\in\mathbb{N}^p,\boldsymbol{s}\in\mathbb{N}^r} b_{i,\boldsymbol{r}}b_{j,\boldsymbol{s}}\prod_{m=1}^{l}\langle\varepsilon_{r_mk_1},\varepsilon_{s_mk_2}\rangle_{\mathfrak{H}}(\varepsilon_{r_{l+1}k_1}\otimes\cdots\otimes\varepsilon_{r_pk_1}\otimes\varepsilon_{s_{l+1}k_2}\otimes\cdots\otimes\varepsilon_{s_rk_2})\\
&= \frac{1}{n}\sum_{k_1,k_2=1}^{n}\sum_{\boldsymbol{r}\in\mathbb{N}^p,\boldsymbol{s}\in\mathbb{N}^r} b_{i,\boldsymbol{r}}b_{j,\boldsymbol{s}}\prod_{m=1}^{l}\rho_{r_ms_m}(k_1-k_2)(\varepsilon_{r_{l+1}k_1}\otimes\cdots\otimes\varepsilon_{r_pk_1}\otimes\varepsilon_{s_{l+1}k_2}\otimes\cdots\otimes\varepsilon_{s_rk_2}).
\end{aligned}
\tag{5.10}
$$

It follows that, by taking the norm $\|\cdot\|^2_{\mathfrak{H}^{\otimes(p+r-2l)}}$ and denoting by $r_{k,m}, s_{k,m}$ respectively the $m-$th elements of the sequences $\boldsymbol{r}_k, \boldsymbol{s}_k$, $k=1,2$, we get

$$
\begin{aligned}
&\sum_{i,j=1}^{\infty}\|\widetilde{h}_{p,n,i}\otimes_l\widetilde{h}_{r,n,j}\|^2_{\mathfrak{H}^{\otimes(p+r-2l)}}\\
&= \sum_{i,j=1}^{\infty}\frac{1}{n^2}\sum_{k_1,k_2,k_3,k_4=1}^{n}\sum_{\substack{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathbb{N}^p\\ \boldsymbol{s}_1,\boldsymbol{s}_2\in\mathbb{N}^r}} b_{i,\boldsymbol{r}_1}b_{j,\boldsymbol{s}_1}b_{i,\boldsymbol{r}_2}b_{j,\boldsymbol{s}_2}\prod_{m=1}^{l}\rho_{r_{1,m}s_{1,m}}(k_1-k_2)\prod_{m=1}^{l}\rho_{r_{2,m}s_{2,m}}(k_3-k_4)\\
&\qquad\qquad\prod_{m=l+1}^{p}\rho_{r_{1,m}r_{2,m}}(k_1-k_3)\prod_{m=l+1}^{r}\rho_{s_{1,m}s_{2,m}}(k_2-k_4)\\
&\le \sum_{i,j=1}^{\infty}\frac{1}{n^2}\sum_{k_1,k_2,k_3,k_4=1}^{n}\sum_{\substack{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathbb{N}^p\\ \boldsymbol{s}_1,\boldsymbol{s}_2\in\mathbb{N}^r}} \frac{1}{2}\left((b_{i,\boldsymbol{r}_1}b_{j,\boldsymbol{s}_2})^2+(b_{i,\boldsymbol{r}_2}b_{j,\boldsymbol{s}_1})^2\right)\prod_{m=1}^{l}|\rho_{r_{1,m}s_{1,m}}(k_1-k_2)|\\
&\qquad\times\prod_{m=1}^{l}|\rho_{r_{2,m}s_{2,m}}(k_3-k_4)|\prod_{m=l+1}^{p}|\rho_{r_{1,m}r_{2,m}}(k_1-k_3)|\prod_{m=l+1}^{r}|\rho_{s_{1,m}s_{2,m}}(k_2-k_4)|\\
&= \sum_{i,j=1}^{\infty}\frac{1}{n^2}\sum_{k_1,k_2,k_3,k_4=1}^{n}\sum_{\substack{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathbb{N}^p\\ \boldsymbol{s}_1,\boldsymbol{s}_2\in\mathbb{N}^r}} \frac{1}{2}(b_{i,\boldsymbol{r}_1}b_{j,\boldsymbol{s}_2})^2\prod_{m=1}^{l}|\rho_{r_{1,m}s_{1,m}}(k_1-k_2)|\prod_{m=1}^{l}|\rho_{r_{2,m}s_{2,m}}(k_3-k_4)|\\
&\qquad\times\prod_{m=l+1}^{p}|\rho_{r_{1,m}r_{2,m}}(k_1-k_3)|\prod_{m=l+1}^{r}|\rho_{s_{1,m}s_{2,m}}(k_2-k_4)|\\
&\quad+\sum_{i,j=1}^{\infty}\frac{1}{n^2}\sum_{k_1,k_2,k_3,k_4=1}^{n}\sum_{\substack{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathbb{N}^p\\ \boldsymbol{s}_1,\boldsymbol{s}_2\in\mathbb{N}^r}} \frac{1}{2}(b_{i,\boldsymbol{r}_2}b_{j,\boldsymbol{s}_1})^2\prod_{m=1}^{l}|\rho_{r_{1,m}s_{1,m}}(k_1-k_2)|\\
&\qquad\times\prod_{m=1}^{l}|\rho_{r_{2,m}s_{2,m}}(k_3-k_4)|\prod_{m=l+1}^{p}|\rho_{r_{1,m}r_{2,m}}(k_1-k_3)|\prod_{m=l+1}^{r}|\rho_{s_{1,m}s_{2,m}}(k_2-k_4)|.
\end{aligned}
\tag{5.11}
$$

We now consider the first summand in (5.11), and note that the second can be treated similarly. We have

$$
\sum_{\boldsymbol{r}_2\in\mathbb{N}^p,\boldsymbol{s}_1\in\mathbb{N}^r}\prod_{m=1}^{l}|\rho_{r_{1,m}s_{1,m}}(k_1-k_2)|\prod_{m=1}^{l}|\rho_{r_{2,m}s_{2,m}}(k_3-k_4)|
$$

$$
\times \prod_{m=l+1}^{p} |\rho_{r_{1,m}r_{2,m}}(k_1-k_3)| \prod_{m=l+1}^{r} |\rho_{s_{1,m}s_{2,m}}(k_2-k_4)|
$$

$$
= \left(\sum_{s_{1,1},s_{1,2},\dots,s_{1,l}=1}^{\infty} \prod_{m=1}^{l} |\rho_{r_{1,m}s_{1,m}}(k_1-k_2)| \right) \left(\sum_{s_{1,l+1},\dots,s_{1,r}=1}^{\infty} \prod_{m=l+1}^{r} |\rho_{s_{1,m}s_{2,m}}(k_2-k_4)| \right) \tag{5.12}
$$

$$
\times \left(\sum_{r_{2,1},r_{2,2},\dots,r_{2,l}=1}^{\infty} \prod_{m=1}^{l} |\rho_{r_{2,m}s_{2,m}}(k_3-k_4)| \right) \left(\sum_{r_{2,l+1},\dots,r_{2,p}=1}^{\infty} \prod_{m=l+1}^{p} |\rho_{r_{1,m}r_{2,m}}(k_1-k_3)| \right)
$$

$$
\le (\theta(k_1-k_2))^l (\theta(k_4-k_3))^l (\theta(k_1-k_3))^{p-l} (\theta(k_4-k_2))^{r-l}, \tag{5.13}
$$

with $\theta(\cdot)$ as in (5.4). Therefore, still focusing on the first summand in (5.11), combined with (5.13), for $l = 1, \dots, p \wedge r$,

$$
\sum_{i,j=1}^{\infty} \sum_{\substack{\boldsymbol{r}_1 \in \mathbb{N}^p \\ \boldsymbol{s}_2 \in \mathbb{N}^r}} (b_{i,\boldsymbol{r}_1} b_{j,\boldsymbol{s}_2})^2 \frac{1}{n^2} \sum_{k_1,k_2,k_3,k_4=1}^{n} (\theta(k_1-k_2))^l (\theta(k_4-k_3))^l (\theta(k_1-k_3))^{p-l} (\theta(k_4-k_2))^{r-l}
$$

$$
\le \sum_{i=1}^{\infty} \sum_{\boldsymbol{r}_1 \in \mathbb{N}^p} (b_{i,\boldsymbol{r}_1})^2 \sum_{j=1}^{\infty} \sum_{\boldsymbol{s}_2 \in \mathbb{N}^r} (b_{j,\boldsymbol{s}_2})^2 \frac{1}{n^2} \sum_{k_1,k_2,k_3,k_4=1}^{n} (\theta(k_1-k_2))^l \Big((\theta(k_4-k_3))^p
$$

$$
+ (\theta(k_1-k_3))^p \Big) (\theta(k_4-k_2))^{r-l}
$$

$$
\le \mathrm{E} \, \|G^p[X_1]\|^2_{\mathscr{H}_2} \, \mathrm{E} \, \|G^q[X_1]\|^2_{\mathscr{H}_2} \frac{2}{n} \sum_{v \in \mathbb{Z}} \theta^p(v) \sum_{|v| \le n} \theta^l(v) \sum_{|v| \le n} \theta^{r-l}(v), \tag{5.14}
$$

where (5.14) follows by

$$
\frac{1}{n^2} \sum_{k_1,k_2,k_3,k_4=1}^{n} (\theta(k_1-k_2))^l (\theta(k_4-k_3))^p (\theta(k_2-k_4))^{r-l} \le \frac{1}{n} \sum_{v \in \mathbb{Z}} \theta^p(v) \sum_{|v| \le n} \theta^l(v) \sum_{|v| \le n} \theta^{r-l}(v).
$$

Finally, combining (5.11) and (5.14), we get

$$
\sum_{i,j=1}^{\infty} \|\widetilde{h}_{p,n,i} \otimes_l \widetilde{h}_{r,n,j}\|^2_{\mathfrak{H}^{\otimes(p+r-2l)}}
$$

$$
\le 4 \, \mathrm{E} \, \|G^p[X_1]\|^2_{\mathscr{H}_2} \, \mathrm{E} \, \|G^r[X_1]\|^2_{\mathscr{H}_2} \sum_{v \in \mathbb{Z}} \theta^p(v) \left(n^{-1+\frac{l}{r}} \sum_{|v| \le n} \theta^l(v) \right) \left(n^{-1+\frac{r-l}{r}} \sum_{|v| \le n} \theta^{r-l}(v) \right) \to 0, \tag{5.15}
$$

as $n \to \infty$. Indeed, the convergence in (5.15) holds due to (5.2) and because $n^{-1+\frac{l}{p}} \sum_{|v| \le n} \theta^l(v) \to 0$, as $n \to \infty$, by calculations identical to the ones at the end of p. 132 of [34]. By combining (5.9) with (5.15) we see that, for $p, r \ge q, p \ne r$, and $l = 1, \dots, p \wedge r$,

$$
\|h_{p,n} \otimes_l h_{r,n}\|^2_{\mathfrak{H}^{\otimes(p+r-2l)} \otimes \mathscr{H}_2^{\otimes 2}} \to 0.
$$

Since Conditions (i) and (ii) are satisfied, we can infer that S_n converges in distribution to a centered Gaussian random variable Z with covariance operator $\mathscr{T}_Z$ given by $\mathscr{T}_Z = \sum_{p=q}^{\infty} p! \|h_p\|^2_{\mathfrak{H}^{\otimes p}}$. We conclude now that this coincides with the representation of $\mathscr{T}_Z$ in (2.2). By Corollary 4.2,

$$
\sum_{p=q}^{\infty} p! \|h_p\|^2_{\mathfrak{H}^{\otimes p}} = \sum_{p=q}^{\infty} p! \lim_{n \to \infty} \|h_{p,n}\|^2_{\mathfrak{H}^{\otimes p}}
$$

$$= \lim_{n\to\infty} \sum_{p=q}^{\infty} \mathrm{E}\left(\sum_{i=1}^{\infty} (I_p \otimes \mathrm{Id}_{\mathcal{H}_2}) \left[\widetilde{h}_{p,n,i} \otimes v_i\right] \otimes \sum_{i=1}^{\infty} (I_p \otimes \mathrm{Id}_{\mathcal{H}_2}) \left[\widetilde{h}_{p,n,i} \otimes v_i\right] \right)$$

$$= \lim_{n\to\infty} \mathrm{E}\left(S_n \otimes S_n\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k_1,k_2=1}^{n} \mathrm{E}\, G[X_{k_1}] \otimes G[X_{k_2}]$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) \left(\mathrm{E}\, G[X_1] \otimes G[X_{k+1}] + \mathrm{E}\, G[X_{k+1}] \otimes G[X_1]\right) + \mathrm{E}\, G[X_1] \otimes G[X_1] = \mathcal{T}_Z.$$

Remark 5.1. For the case of G being the identity operator in (1.1), the chaos expansion (4.15) simplifies to

$$\sum_{i=1}^{\infty} \sum_{p=q}^{\infty} I_p(\widetilde{h}_{p,k,i}) \otimes v_i = \sum_{i=1}^{\infty} I_1(\widetilde{h}_{1,k,i}) \otimes v_i, \quad h_{p,n} = 0 \quad \text{for } p \geq 2,$$

since $G[X_k] = X_k = \sum_{i=1}^{\infty} \sum_{\boldsymbol{l} \in \mathcal{L}_1} c_{i,\boldsymbol{l}} H_{\boldsymbol{l}}(X_k) \otimes v_i$ with $\mathcal{L}_1 = \{\boldsymbol{l} \in \mathbb{N}_0^{\infty} : \sum_{k=1}^{\infty} l_k = 1\}$. In this case, Condition (ii) is not necessary since it assumes $p \neq 1$. When Condition (ii) is not needed, it is worth pointing out that Condition (i) is satisfied whenever there is an M such that

$$\sum_{|v|>M}^{\infty} \left(\sup_{r\geq 1} \sum_{s=1}^{\infty} |\rho_{rs}(v)| \right)^q < \infty,$$

which weakens assumption (2.1). In particular, when $\{X_k\}$ is m-dependent as in Example 2.2, (5.1) is satisfied with $M = m$.

5.2 Proof of Theorem 2.2

Our proof borrows ideas from the proof of Theorem 5.1 in [6]. First, note that, in law,

$$V_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} G[X_k] = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \mathbf{1}_{\left[\frac{k}{n},1\right]}(t) G[X_k].$$

Remark that now the quantity inside the sum depends on k and n, and so Theorem 2.1 is not readily available. Using Lemma 4.2, we can infer that, in law,

$$V_n(t) = \sum_{p=q}^{\infty} (I_p \otimes \mathrm{Id}_{\mathcal{H}_2}) \left[h_{p,n,t}\right] \quad \text{with} \quad h_{p,n,t} \doteq \sum_{i=1}^{\infty} \left(\widehat{h}_{p,n,i,t} \otimes v_i\right), \tag{5.16}$$

where

$$\widehat{h}_{p,n,i,t} \doteq \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \sum_{\boldsymbol{l} \in \mathbb{N}^p} \mathbf{1}_{\left[\frac{k}{n},1\right]}(t) b_{i,\boldsymbol{l}} \left(\varepsilon_{l_1 k} \otimes \cdots \otimes \varepsilon_{l_p k}\right) \tag{5.17}$$

for coefficients $\{b_{i,\boldsymbol{l}}\}_{i\in\mathbb{N},\boldsymbol{l}\in\mathbb{N}^p}$ as in Lemma 4.2. Moreover, recall $\widetilde{h}_{p,n,i}$ defined in (4.12). For $k = 1, \ldots, n$, we view $\mathbf{1}_{\left[\frac{k}{n},1\right]}(\cdot)$ as an element of $L^2([0,1])$, and let $\mathcal{T}_{V_n}$ denote the covariance operator of V_n in $L^2([0,1])^{\otimes 2} \otimes \mathcal{H}_2^{\otimes 2}$ given by

$$\mathcal{T}_{V_n} = \sum_{p=q}^{\infty} \sum_{i,j=1}^{\infty} \sum_{k_1,k_2=1}^{n} \frac{1}{n} p! \left(\mathbf{1}_{\left[\frac{k_1}{n},1\right]} \otimes \mathbf{1}_{\left[\frac{k_2}{n},1\right]} \right) \sum_{\boldsymbol{l},\boldsymbol{r} \in \mathbb{N}^p} b_{i,\boldsymbol{l}} b_{j,\boldsymbol{r}} \prod_{m=1}^{p} \rho_{l_m r_m}(k_1 - k_2)(v_i \otimes v_j). \tag{5.18}$$

Recall the covariance operator $\mathcal{T}_Z$ of Z from (2.2) and write it in terms of Z's chaos decomposition as

$$\mathcal{T}_Z \doteq \sum_{p=q}^{\infty} \sum_{i,j=1}^{\infty} \sum_{k\in\mathbb{Z}} p! \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} b_{i,\boldsymbol{l}} b_{j,\boldsymbol{r}} \prod_{m=1}^{p} \rho_{l_m r_m}(k)(v_i \otimes v_j).$$

We first introduce a truncated version of the chaos expansion (5.16) of V_n and its associated covariance operator $\mathcal{T}_{V_{n,M}}$, that is,

$$\begin{aligned} V_{n,M}(t) &= \sum_{p=q}^{M} (I_p \otimes \mathrm{Id}_{\mathcal{H}_2})\,[h_{p,n,t}]\,, \\ \mathcal{T}_{V_{n,M}} &= \sum_{p=q}^{M} \sum_{i,j=1}^{\infty} \sum_{k_1,k_2=1}^{n} \frac{1}{n} p! \left(\mathbf{1}_{\left[\frac{k_1}{n},1\right]} \otimes \mathbf{1}_{\left[\frac{k_2}{n},1\right]} \right) \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} b_{i,\boldsymbol{l}} b_{j,\boldsymbol{r}} \prod_{m=1}^{p} \rho_{l_m r_m}(k_1-k_2)(v_i \otimes v_j) \end{aligned} \tag{5.19}$$

for some $M \geq q$ and with $h_{p,n,t}$ as in (5.16)–(5.17), and we define further a truncated Gaussian random sequence $W_{n,M}$ with covariance operator

$$\mathcal{T}_{W_{n,M}} \doteq \mathcal{T}_B \otimes \mathcal{T}_{S_{n,M}}, \quad \mathcal{T}_{S_{n,M}} \doteq \sum_{p=q}^{M} \sum_{i,j=1}^{\infty} \sum_{k_1,k_2=1}^{n} \frac{1}{n} p! \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} b_{i,\boldsymbol{l}} b_{j,\boldsymbol{r}} \prod_{m=1}^{p} \rho_{l_m r_m}(k_1-k_2)(v_i \otimes v_j), \tag{5.20}$$

where $S_{n,M}$ denotes the truncated version of S_n with chaos decomposition as in (A.6) in Appendix A, with $\widetilde{h}_{p,n,i}$ defined in (4.12) and $\mathcal{T}_B$ defined in (2.4). Then, for all $M \geq q$,

$$d_2(V_n, W) \leq d_2(V_n, V_{n,M}) + d_2(V_{n,M}, W_{n,M}) + d_2(W_{n,M}, W), \tag{5.21}$$

and we consider the three distances in (5.21) separately. We start with some preliminary estimates. First, recalling the operator $\mathcal{T}_B$ from (2.4), we have

$$\|\mathcal{T}_B\|_{HS(L^2([0,1]))} = \left(\int_{[0,1]^2} \kappa^2(s,t)\,ds\,dt \right)^{1/2} \leq 1. \tag{5.22}$$

Moreover, if $\mathcal{H}_1, \mathcal{H}_2$ are two Hilbert spaces with respective bases $\{e_j\}_{j\in\mathbb{N}}, \{u_j\}_{j\in\mathbb{N}}$, an operator $\mathcal{Q} \in HS(\mathcal{H}_1 \otimes \mathcal{H}_2)$ can be written as $\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{Q}_2$, with $\mathcal{Q}_i \in HS(\mathcal{H}_i)$, $i = 1,2$. Then, by (3.5),

$$\|\mathcal{Q}\|_{HS(\mathcal{H}_1\otimes\mathcal{H}_2)} = \sum_{j_1,j_2,r_1,r_2=1}^{\infty} \langle \mathcal{Q}_1[e_{j_1}], e_{j_2} \rangle_{\mathcal{H}_1} \langle \mathcal{Q}_2[u_{r_1}], u_{r_2} \rangle_{\mathcal{H}_2} = \|\mathcal{Q}_1\|_{HS(\mathcal{H}_1)} \|\mathcal{Q}_2\|_{HS(\mathcal{H}_2)}. \tag{5.23}$$

Now for the third summand in (5.21), we get

$$d_2(W_{n,M}, W) \leq \frac{1}{2} \|\mathcal{T}_{W_{n,M}} - \mathcal{T}_W\|_{HS(L^2([0,1])\otimes\mathcal{H}_2)} \tag{5.24}$$

$$= \|\mathcal{T}_B\|_{HS(L^2([0,1]))} \|\mathcal{T}_{S_{n,M}} - \mathcal{T}_Z\|_{HS(\mathcal{H}_2)} \leq \|\mathcal{T}_{S_{n,M}} - \mathcal{T}_Z\|_{HS(\mathcal{H}_2)}, \tag{5.25}$$

where (5.24) follows by Corollary 3.3 in [6] since $W_{n,M}, W$ are Gaussian elements, the equality in (5.25) is due to (5.23), and the inequality follows by (5.22). Furthermore, by defining $\mathcal{T}_{Z_M}$ as in (A.7),

$$\|\mathcal{T}_{S_{n,M}} - \mathcal{T}_Z\|_{HS(\mathcal{H}_2)} \leq \|\mathcal{T}_{S_{n,M}} - \mathcal{T}_{Z_M}\|_{HS(\mathcal{H}_2)} + \|\mathcal{T}_{Z_M} - \mathcal{T}_Z\|_{HS(\mathcal{H}_2)}$$

$$\leq \|\mathcal{T}_{S_{n,M}} - \mathcal{T}_{Z_M}\|_{HS(\mathcal{H}_2)} + \|\mathcal{T}_{Z_M} - \mathcal{T}_Z\|_{\mathcal{S}(\mathcal{H}_2)} \tag{5.26}$$
$$\leq \mathcal{R}_{4,n,M} + \mathcal{R}_{1,M} \to 0, \tag{5.27}$$

where (5.26) is due to (3.3), (5.27) is inferred from (A.16) and (A.18) in Appendix A, with $\mathcal{R}_{1,M}$, $\mathcal{R}_{4,n,M}$ defined in (A.4). Finally, $\mathcal{R}_{1,M} \to 0$ as $M \to \infty$ and $\mathcal{R}_{4,n,M} \to 0$ as $n \to \infty$ for all $M \geq q$.

For the second term on the RHS of (5.21), we apply Theorem 4.3 in [6] (note that there is a minor typo in the statement of the Theorem in [6], omitting the square root of $\sqrt{\Lambda_1(n) + \Lambda_2(n)}$) to get

$$d_2(V_{n,M}, W_{n,M}) \leq \frac{1}{2}\left(\sqrt{\Lambda_1(n) + \Lambda_2(n)} + \|\mathcal{T}_{V_{n,M}} - \mathcal{T}_{W_{n,M}}\|_{HS(L^2([0,1])\otimes\mathcal{H}_2)}\right) \tag{5.28}$$

with $\Lambda_1(n), \Lambda_2(n)$ as defined in (A.14) and (A.15) respectively, with $\mathcal{H}_2^{\otimes 2}$ replaced by $L^2([0,1])^{\otimes 2} \otimes \mathcal{H}_2^{\otimes 2}$, $h_{p,n}$ replaced by $h_{p,n,t}$ defined in (5.16). To show that the second summand $\|\mathcal{T}_{V_{n,M}} - \mathcal{T}_{W_{n,M}}\|_{HS(L^2([0,1])\otimes\mathcal{H}_2)}$ in (5.28) vanishes, we apply Lemma B.1 below. For the first summand in (5.28), in view of (A.14)–(A.15), it suffices to verify that (a) for $s = 1, \ldots, p-1$,

$$\|h_{p,n,\cdot} \otimes_s h_{p,n,\cdot}\|^2_{\mathfrak{H}^{\otimes 2(p-s)}\otimes L^2([0,1])^{\otimes 2}\otimes\mathcal{H}_2^{\otimes 2}} \to 0,$$

and that (b) for $s = 1, \ldots, p \wedge r, p \neq r$,

$$\|h_{p,n,\cdot} \otimes_s h_{r,n,\cdot}\|^2_{\mathfrak{H}^{\otimes (p+r-2s)}\otimes L^2([0,1])^{\otimes 2}\otimes\mathcal{H}_2^{\otimes 2}} \to 0.$$

Indeed, for (a), upon recalling $\widetilde{h}_{p,n,i}$ from (4.12) and $h_{p,n,t}, \widehat{h}_{p,n,i,t}$ from (5.16)–(5.17), we get

$$\begin{aligned}\|h_{p,n,\cdot} \otimes_s h_{p,n,\cdot}\|^2_{\mathfrak{H}^{\otimes 2(p-s)}\otimes L^2([0,1])^{\otimes 2}\otimes\mathcal{H}_2^{\otimes 2}} &\leq \sum_{i,j=1}^{\infty} \|\widehat{h}_{p,n,i,\cdot} \otimes_s \widehat{h}_{p,n,j,\cdot}\|^2_{\mathfrak{H}^{\otimes 2(p-s)}\otimes L^2([0,1])^{\otimes 2}} \\ &\leq \sum_{i,j=1}^{\infty} \|\widetilde{h}_{p,n,i} \otimes_s \widetilde{h}_{p,n,j}\|^2_{\mathfrak{H}^{\otimes 2(p-s)}} \to 0,\end{aligned} \tag{5.29}$$

where the inequality in (5.29) follows since $\left\langle \mathbf{1}_{\left[\frac{k_1}{n},1\right]}, \mathbf{1}_{\left[\frac{k_2}{n},1\right]} \right\rangle_{L^2([0,1])} \leq 1$, and the convergence is due to the same calculations as in (5.14) for $p = r$. The same arguments show that, for $s = 1, \ldots, p \wedge r, p \neq r$, and as $n \to \infty$,

$$\|h_{p,n,\cdot} \otimes_s h_{r,n,\cdot}\|^2_{\mathfrak{H}^{\otimes (p+r-2s)}\otimes L^2([0,1])^{\otimes 2}\otimes\mathcal{H}_2^{\otimes 2}} \leq \sum_{i,j=1}^{\infty} \|\widetilde{h}_{p,n,i} \otimes_s \widetilde{h}_{r,n,j}\|^2_{\mathfrak{H}^{\otimes (p+r-2s)}} \to 0. \tag{5.30}$$

For the first term on the RHS of (5.21), with further explanations given below,

$$d_2(V_{n,M}, V_n) \leq \sqrt{\mathrm{E}\left(\|V_n - V_{n,M}\|^2_{L^2([0,1])\otimes\mathcal{H}_2}\right)} = \sqrt{\mathrm{tr}(\mathcal{T}_{V_n - V_{n,M}})} \tag{5.31}$$
$$= \sqrt{\sum_{p=M+1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{n} \sum_{k_1,k_2=1}^{n} \mathrm{tr}\left(\mathbf{1}_{\left[\frac{k_1}{n},1\right]} \otimes \mathbf{1}_{\left[\frac{k_2}{n},1\right]}\right) p! \sum_{\boldsymbol{r},\boldsymbol{s}\in\mathbb{N}^p} b_{i,\boldsymbol{r}} b_{i,\boldsymbol{s}} \prod_{m=1}^{p} \rho_{r_m s_m}(k_1 - k_2)}$$
$$\leq \sqrt{\sum_{p=M+1}^{\infty} p! \sum_{i=1}^{\infty} \sum_{|v|\leq n-1} \left| \sum_{\boldsymbol{r},\boldsymbol{s}\in\mathbb{N}^p} b_{i,\boldsymbol{r}} b_{i,\boldsymbol{s}} \prod_{j=1}^{p} \rho_{r_j s_j}(v) \right|} \to 0, \quad \text{as} \quad M \to \infty, \tag{5.32}$$

where the inequality in (5.31) follows from the Lipschitz continuity of the functions considered for the d_2 distance and Jensen's inequality, and the equality in (5.31) is due to (5.18) and the definition of the trace of the covariance operator in (3.4). Finally, (5.32) follows from (5.5) and since $\mathrm{tr}\left(\mathbf{1}_{\left[\frac{k_1}{n},1\right]}\otimes\mathbf{1}_{\left[\frac{k_2}{n},1\right]}\right)=\left\langle\mathbf{1}_{\left[\frac{k_1}{n},1\right]},\mathbf{1}_{\left[\frac{k_2}{n},1\right]}\right\rangle_{L^2([0,1])}\leq 1$.

The conclusion that $d_2(V_n, W)\to 0$ follows by combining (5.25), (5.27) (5.28), (5.29), (5.30), Lemma B.1 in Appendix B and (5.32).

6 Examples and Applications

We provide in this section some examples for possible operators G. Suppose that X_k is a Gaussian and stationary process taking values in a Hilbert space $\mathscr{H}$, with covariance (and hence nuclear) operator $Q[\cdot]\doteq \mathrm{E}\langle X_1,\cdot\rangle_{\mathscr{H}}X_1$. Denote by $\{v_j\}_{j\in\mathbb{N}}$ the basis of $\mathscr{H}$.

6.1 Sample covariance operator

A natural estimator for the covariance operator Q, which has been studied in, e.g., Section 4.1 of [5] (see also [26]) is given by

$$\Gamma_n\in\mathscr{L}(\mathscr{H}:\mathscr{H}),\quad \Gamma_n[\cdot]\doteq\frac{1}{n}\sum_{k=1}^{n}\langle X_k,\cdot\rangle_{\mathscr{H}}X_k.$$

We suppose that $\Gamma_n[\cdot]$ is a random element in $\mathrm{HS}(\mathscr{H})$, namely that $\|\Gamma_n\|^2_{HS(\mathscr{H})}=\sum_{k=1}^{\infty}\|\Gamma_n[v_k]\|^2_{\mathscr{H}}<\infty$. Then, S_n defined in (1.1) can be rewritten as $S_n=\sqrt{n}\Gamma_n$, with $G[x]=\langle x,\cdot\rangle_{\mathscr{H}}x$, $\mathscr{H}_1=\mathscr{H}$ and $\mathscr{H}_2=HS(\mathscr{H})$. For the condition $\mathrm{E}\,\|G[X_1]\|^2_{HS(\mathscr{H})}<\infty$ in Theorem 2.1 to be satisfied, we can impose an assumption on the eigenvalues of the covariance operator Q. Recall that $\{\langle v_r,\cdot\rangle_{\mathscr{H}}v_s\}_{r,s\in\mathbb{N}}$ is a basis of $HS(\mathscr{H})$. Then,

$$\mathrm{E}\,\|G[X_1]\|^2_{HS(\mathscr{H})}=\sum_{r,s=1}^{\infty}\mathrm{E}\left(\langle X_k,v_r\rangle_{\mathscr{H}}\langle X_k,v_s\rangle_{\mathscr{H}}\right)\leq\left(\sum_{r=1}^{\infty}\lambda_r^{\frac{1}{2}}\right)^2<\infty \tag{6.1}$$

by Cauchy-Schwarz inequality, and with $\lambda_r=\mathrm{E}\langle X_1,v_r\rangle^2_{\mathscr{H}}$ denoting the eigenvalues of Q. The eigenvalue assumption in (6.1) coincides with the assumption made in the i.i.d. case; see Proposition 5 in [26].

We argue now that the Hermite rank in Definition 4.1 of the map $G[x]\doteq\langle x,\cdot\rangle_{\mathscr{H}}x$ is equal to two. We aim to write the sample covariance operator using the Hermite expansion (4.4). Recall that $G[X_k]$ is an unbiased estimator, i.e., $\mathrm{E}\,G[X_k]=Q$. Then, we can calculate explicitly the Hermite coefficients for $r,l\in\mathbb{N},\boldsymbol{l}\in\mathscr{L}$ as follows:

$$\begin{aligned}
c_{r,s,\boldsymbol{l}}&\doteq\frac{1}{\prod_{j=1}^{\infty}l_j!}\langle G[X_1]-\mathrm{E}\,G[X_1],H_{\boldsymbol{l}}(X_1)\otimes(\langle v_r,\cdot\rangle_{\mathscr{H}}v_s)\rangle_{L_2(\Omega:\mathbb{R})\otimes HS(\mathscr{H})}\\
&=\frac{1}{\prod_{j=1}^{\infty}l_j!}\left\langle\langle X_1,v_r\rangle_{\mathscr{H}}\langle X_1,v_s\rangle_{\mathscr{H}}-\langle Q[v_r],v_s\rangle_{\mathscr{H}},\prod_{j=1}^{\infty}H_{l_j}(W_{v_j}(X_1))\right\rangle_{L^2(\Omega:\mathbb{R})} &(6.2)\\
&=\frac{1}{\prod_{j=1}^{\infty}l_j!}\left[\mathrm{E}\left(\langle X_1,v_r\rangle_{\mathscr{H}}\langle X_1,v_s\rangle_{\mathscr{H}}\prod_{j=1}^{\infty}H_{l_j}(W_{v_j}(X_1))\right)-\langle Q[v_r],v_s\rangle_{\mathscr{H}}\,\mathrm{E}\left(\prod_{j=1}^{\infty}H_{l_j}(W_{v_j}(X_1))\right)\right] &(6.3)
\end{aligned}$$

$$= \frac{\lambda_r^{\frac{1}{2}} \lambda_s^{\frac{1}{2}}}{\prod_{j=1}^{\infty} l_j!} \, \mathrm{E} \left[\langle X_1, Q^{-\frac{1}{2}} v_r \rangle_{\mathscr{H}} \langle X_1, Q^{-\frac{1}{2}} v_s \rangle_{\mathscr{H}} \prod_{j=1}^{\infty} H_{l_j}(W_{v_j}(X_1)) \right] \tag{6.4}$$

$$= \frac{\lambda_r^{\frac{1}{2}} \lambda_s^{\frac{1}{2}}}{\prod_{j=1}^{\infty} l_j!} \, \mathrm{E} \left[H_1(W_{v_r}(X_1)) H_1(W_{v_s}(X_1)) \prod_{j=1}^{\infty} H_{l_j}(W_{v_j}(X_1)) \right],$$

where (6.2) follows from the definition of G and by taking the inner product with respect to $HS(\mathscr{H})$, (6.4) follows upon noticing that the second summand in (6.3) is zero and that $v_r = \lambda_r^{1/2} Q^{-1/2} v_r$. Now note that, for $r \neq s$,

$$c_{r,s,\boldsymbol{l}} = \begin{cases} \frac{\lambda_r^{\frac{1}{2}} \lambda_s^{\frac{1}{2}}}{\prod_{j=1}^{\infty} l_j!} \, \mathrm{E} \left[(H_1(W_{v_r}(X_1)))^2 (H_1(W_{v_s}(X_1)))^2 \right] & \text{if } l_r = l_s = 1, \boldsymbol{l} \in \mathscr{L}_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \lambda_r^{\frac{1}{2}} \lambda_s^{\frac{1}{2}} & \text{if } l_r = l_s = 1, \boldsymbol{l} \in \mathscr{L}_2, \\ 0 & \text{otherwise,} \end{cases}$$

with $\mathscr{L}_2 = \{\boldsymbol{l} \in \mathbb{N}_0^{\infty} : \sum_{k=1}^{\infty} l_k = 2\}$, while for $r = s$

$$c_{r,r,\boldsymbol{l}} = \begin{cases} \frac{\lambda_r}{\prod_{j=1}^{\infty} l_j!} \, \mathrm{E} \left[(H_2(W_{v_r}(X_1)) + 1) H_2(W_{v_r}(X_1)) \right] & \text{if } l_r = 2, \boldsymbol{l} \in \mathscr{L}_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \lambda_r & \text{if } l_r = 2, \boldsymbol{l} \in \mathscr{L}_2, \\ 0 & \text{otherwise.} \end{cases}$$

Altogether, this says that

$$c_{r,s,\boldsymbol{l}} = \begin{cases} \lambda_r^{\frac{1}{2}} \lambda_s^{\frac{1}{2}} & \text{if } l_r = l_s = 1, \boldsymbol{l} \in \mathscr{L}_2, \\ \lambda_r & \text{if } l_r = 2, \boldsymbol{l} \in \mathscr{L}_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we can infer that for every Gaussian process $\{X_k\}_{k \in \mathbb{Z}}$ satisfying assumption (2.1) for $q = 2$, its sample covariance operator satisfies

$$\sqrt{n} \, (\Gamma_n - Q) \xrightarrow{d} Z, \tag{6.5}$$

where Z is a $HS(\mathscr{H})$-valued centered Gaussian element with covariance operator given by (2.2), with $\mathscr{H}_2 = HS(\mathscr{H})$ and $G[x] = \langle x, \cdot \rangle_{\mathscr{H}} x - Q[\cdot]$.

Analogous (but simpler) calculations can be done for the sample mean $\frac{1}{n} \sum_{k=1}^{n} X_k$ (here $G = \mathrm{Id}_{\mathscr{H}}$ and $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$), giving rise to the generalized Hermite coefficients $c_{i,\boldsymbol{l}} = \lambda_i^{1/2} \delta_{l_i=1} \delta_{\boldsymbol{l} \in \mathscr{L}_1}$, for $i \in \mathbb{N}$. This says that the generalized Hermite rank is $q = 1$, and so one can apply Theorem 2.1 whenever $\{X_k\}_{k \in \mathbb{Z}}$ satisfies (2.1) for $q = 1$.

6.2 Eigenvalue estimation

Suppose that the eigenvectors $\{v_j\}_{j \in \mathbb{N}}$ of the covariance operator Q are known and consider the problem of estimating their corresponding positive eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$. The following consistent estimators $\hat{\lambda}_{jn}$ for λ_j were considered in Section 4.2 of [5],

$$\hat{\lambda}_{jn} = \frac{1}{n} \sum_{k=1}^{n} \langle X_k, v_j \rangle_{\mathscr{H}}^2, \quad \mathrm{E} \, \hat{\lambda}_{jn} = \lambda_j, \quad j, n \in \mathbb{N},$$

giving rise to the setting of Theorem 2.1 with $\mathscr{H}_1 = \mathscr{H}, \mathscr{H}_2 = \mathbb{R}, S_n = \sqrt{n}(\hat{\lambda}_{jn} - \lambda_j)$ and $G_j[x] = \langle x, v_j \rangle^2_{\mathscr{H}}$. Fix some $j \in \mathbb{N}$ and assume that $\mathrm{E}\,|G_j[X_1]|^2 = \langle X_1, v_j \rangle^4_{\mathscr{H}} < \infty$. Moreover, by arguing as in the previous section, we have that the coefficients of G_j are $c_{\boldsymbol{l}} = \lambda_j \delta_{l_j 2}$, and so G_j has Hermite rank $q = 2$. Theorem 2.1 implies that if assumption (2.1) is met with $q = 2$, then for all $j \in \mathbb{N}$,

$$\sqrt{n}(\hat{\lambda}_{jn} - \lambda_j) \xrightarrow{d} N(0, \sigma_j^2),$$

where $\sigma_j^2 \doteq \sum_{k=1}^{\infty} EG_j^2[X_1] + 2\sum_{k=1}^{\infty} \mathrm{E}\,G_j[X_1]G_j[X_{k+1}]$. When $\{X_k\}$ is a Gaussian ARH(1) process, this result recovers Theorem 4.10 in [5].

In fact, we can strengthen this result by considering the simultaneous estimation of all eigenvalues. Let $\lambda \doteq (\lambda_1, \lambda_2, \dots) \in \ell^2, \hat{\lambda}_n = (\hat{\lambda}_{1n}, \hat{\lambda}_{2n}, \dots) \in \ell^2$. Then, $G[x] \in \ell^2(\mathbb{N})$, where $G_j[x] = \langle x, v_j \rangle^2$. Assume that $\mathrm{E}\,\|G[X_1]\|^2_{\ell^2} = \sum_{j=1}^{\infty} \langle X_1, v_j \rangle^4 < \infty$. Here, G has Hermite coefficients $c_{j,\boldsymbol{l}} = \lambda_j \delta_{l_j 2}$ and Hermite rank $q = 2$. Then, by Theorem 2.1,

$$\sqrt{n}(\hat{\lambda}_n - \lambda) \to S,$$

where S is a Gaussian element of $\ell^2(\mathbb{N})$ with covariance operator $\mathscr{T}_S$, such that $\langle \mathscr{T}_S(v_i), v_j \rangle = \delta_{ij}\sigma_j^2$.

6.3 Shallow neural operators with Gaussian initializations

In a recent work, [21] introduced a framework for learning operators, termed *neural operators*. Numerous quantitative CLTs have been investigated in the context of neural networks, e.g., [4, 14, 21]. In this section, we study the limiting distribution of a single layer neural operator with random initializations of the parameters, as the width of the network goes to infinity. The single layer update for these approximation schemes takes the form of some output $u \in L^2(D : \mathbb{R}^m)$, where D is a bounded domain $D \subset \mathbb{R}^l$; more precisely, in a simplified form,

$$u(x) = \sigma\left(\int_D \kappa(x, y)v(y)d\mu(y)\right), \quad x \in D,$$

where $\kappa : D \times D \to \mathbb{R}^{m \times m}$ is a suitable kernel, $v \in L^2(D : \mathbb{R}^m)$ is the input function, μ is a suitable measure, and σ is an *activation function* such that $\sigma \in HS(\mathscr{H})$. Here, we take μ to be the Lebesgue measure. There are several options for kernels, but here we select the so-called *low-rank neural operators*; see Section 4.2 in [21]. Low-rank neural operators are defined through

$$\kappa(x, y) = \sum_{j=1}^{r} \Phi^{(j)}(x)(\Psi^{(j)}(y))', \quad \text{implying} \quad \int_D \kappa(\cdot, y)v(y)d\mu(y) = \sum_{j=1}^{r} \langle \Psi^{(j)}, v \rangle_{L^2(D:\mathbb{R}^m)} \Phi^{(j)}(\cdot),$$

where $r \in \mathbb{N}$ is a constant termed the *rank* of the kernel and, for $i = 1, \dots, r$, $\Phi^{(i)}, \Psi^{(i)}$ are $L^2(D : \mathbb{R}^m)$-valued Gaussian centered random variables. Recall the Cartesian inner product $\langle x, y \rangle_{\mathscr{H}^{2r}} = \sum_{i=1}^{2r} \langle x_i, y_i \rangle_{\mathscr{H}}$, where x_i denotes the $i-$th marginal of $x \in \mathscr{H}^{2r}$. Now let $\{\Phi_k^{(i)}, \Psi_k^{(i)}\}_{i=1,\dots,r,k\in\mathbb{Z}}$ be a sequence such that $\{X_k\}_{k\in\mathbb{Z}}$ is stationary, where, for $k \in \mathbb{N}$, $X_k \doteq \Phi_k^{(1)} \times \cdots \times \Phi_k^{(r)} \times \Psi_k^{(1)} \times \cdots \times \Psi_k^{(r)} \in \left(L^2(D : \mathbb{R}^m)\right)^{2r} \doteq \mathscr{H}_1$. For such an X_k, define

$$G_\sigma[X_k][\cdot] \doteq \sigma\left[\sum_{j=1}^{r} \langle \Psi_k^{(j)}, \cdot \rangle_{L^2(D:\mathbb{R}^m)} \Phi_k^{(j)}\right],$$

and so here $\mathscr{H}_2 \doteq HS\left(\left(L^2(D:\mathbb{R}^m)\right)^r\right)$. Suppose that $\operatorname{rank}(G_\sigma) = q_\sigma$ and that (2.1) is satisfied for such a q_σ and a sequence $\{X_k\}$ with values in $\mathscr{H}_1$. Then, for the one layer, $n-$width neural operator

$$S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} G_\sigma[X_k] \xrightarrow{d} Z_\sigma,$$

where Z_σ is a $\mathscr{H}_2$ Gaussian random variable with covariance operator given in (2.2).

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A Quantitative Version of Theorem 2.1

We state and prove here a quantitative version of Theorem 2.1. For that, we derive an explicit upper bound for the d_2 distance (3.8) between the partial sums (1.1) and its limiting variable Z derived in Theorem 2.1. To be more precise, we aim to derive an inequality of the form

$$|\operatorname{E}(h[S_n]) - \operatorname{E}(h[Z])| \leq \mathscr{R}_n, \quad n \in \mathbb{N}, \tag{A.1}$$

where $h \in C_b^2(\mathscr{H}_2)$ with $\|h\|_{C_b^2(\mathscr{H}_2)} \leq 1$ and $\mathscr{R}_n \to 0$ as $n \to \infty$. An upper bound (A.1) quantifies the error that one makes when replacing the partial sum S_n by its limiting variable Z.

Prior to stating our theorem, we recall $\theta(v)$ and θ from (5.4), as well as the corresponding constant $K \doteq \inf_{k\in\mathbb{N}}\{\theta(v) \leq 1 \text{ for all } |v| \geq k\}$. Moreover, define the constants

$$c_{p,r}(l) \doteq p^2(l-1)!\binom{p-1}{l-1}\binom{r-1}{l-1}(p+r-2l)!, \quad p, r \in \mathbb{N}, l = 1, \dots, p \wedge r, \tag{A.2}$$

and, for $p, n \in \mathbb{N}, l = 1, \dots, p \wedge r$, set

$$\begin{aligned} A(p,r,n,s) \doteq\ & \operatorname{E}\|G^p[X_1]\|^2_{\mathscr{H}_2} \operatorname{E}\|G^r[X_1]\|^2_{\mathscr{H}_2} \\ & \times \sum_{v\in\mathbb{Z}} \theta^p(v) \left(n^{-1+\frac{s}{r}} \sum_{|v|\leq n} \theta^s(v) \right) \left(n^{-1+\frac{r-s}{r}} \sum_{|v|\leq n} \theta^{r-s}(v) \right). \end{aligned} \tag{A.3}$$

We further introduce the following quantities that are used in the theorem below:

$$\begin{aligned} \mathscr{R}_{1,M} &\doteq \sqrt{(2K+2\theta)\sum_{p=M+1}^{\infty} \operatorname{E}\|G^p[X_1]\|^2_{\mathscr{H}_2}} \\ \mathscr{R}_{2,n,M}, &\doteq \sum_{p=1}^{M} \sqrt{\sum_{s=1}^{p-1} c^2_{p,p}(s) A(p,p,n,s)} \\ \mathscr{R}_{3,n,M}, &\doteq \sum_{\substack{1\leq p,r\leq M \\ p\neq r}}^{\infty} \sqrt{\sum_{s=1}^{p\wedge r} c^2_{p,r}(s) A(p,r,n,s)} \\ \mathscr{R}_{4,n,M}, &= \sqrt{\sum_{p=q}^{M} \operatorname{E}\|G^p[X_1]\|^2_{\mathscr{H}_2} \left(\frac{2K(K+1)}{n} + \sum_{|v|\leq n} \theta^q(v)\frac{|v|}{n} + \sum_{|v|\geq n} \theta^q(v) \right)} \end{aligned} \tag{A.4}$$

with $c_{p,r}(l)$ as in (A.2).

One can easily infer that, under the assumptions of Theorem 2.1, $R_{1,M} \to 0$, as $M \to \infty$, and $R_{i,n,M} \to 0$ as $n \to \infty$, for all fixed $M \geq q$ and $i = 1, 2, 3$. Hence, Theorem A.1 below recovers Theorem 2.1.

Theorem A.1 (Quantitative Breuer-Major theorem for Hilbert space-valued random variables)**.** *Suppose the assumptions of Theorem 2.1 hold. Then, for all $n \in \mathbb{N}$,*

$$d_2(S_n, Z) \leq \inf_{M \geq q} \left\{ 2\mathcal{R}_{1,M} + \sqrt{\mathcal{R}_{2,n,M} + \mathcal{R}_{3,n,M}} + \mathcal{R}_{4,n,M} \right\}. \tag{A.5}$$

Proof: Analogous to the representation in Lemma 4.2, we set, for $M \geq q$,

$$S_{n,M} \doteq \sum_{p=q}^{M} (I_p \otimes \mathrm{Id}_{\mathcal{H}_2}) \left[\sum_{i=1}^{\infty} \widetilde{h}_{p,n,i} \otimes v_i \right], \tag{A.6}$$

where $\widetilde{h}_{p,n,i} \in \mathfrak{H}^{\odot p}$ are defined in (4.12).

We introduce further the limiting random variable Z_M that corresponds to the truncated chaos in (A.6). Let Z_M be an $\mathcal{H}_2$-valued Gaussian object with covariance operator

$$\mathcal{T}_{Z_M} \doteq \sum_{p=q}^{M} \sum_{i,j=1}^{\infty} \sum_{k \in \mathbb{Z}} p! \sum_{\boldsymbol{r},\boldsymbol{s} \in \mathbb{N}^p} b_{i,\boldsymbol{r}} b_{j,\boldsymbol{s}} \prod_{m=1}^{p} \rho_{r_m s_m}(k) (v_i \otimes v_j) \in \mathcal{H}_2 \otimes \mathcal{H}_2. \tag{A.7}$$

From the triangle inequality, we get

$$d_2(S_n, Z) \leq d_2(S_n, S_{n,M}) + d_2(S_{n,M}, Z_M) + d_2(Z_M, Z) \tag{A.8}$$

and it remains to bound the three quantities in (A.8) separately.

For the first term on the right hand side of (A.8), we have, by definition (3.8),

$$d_2(S_n, S_{n,M}) = \sup_{h \in C_b^2(\mathcal{H}_2), \|h\|_{C_b^2} \leq 1} |\,\mathrm{E}(h[S_n]) - \mathrm{E}(h[S_{n,M}])|.$$

Fix some $h \in C_b^2(\mathcal{H}_2)$ with $\|h\|_{C_b^2} \leq 1$, implying that $\sup_{x \in \mathcal{H}_2} \|Dh[x]\|_{\mathcal{L}(\mathcal{H}_2:\mathbb{R})} \leq 1$. From Theorem 3.3.2 of [9], since Hilbert spaces are convex, it follows that

$$\|h[y] - h[x]\|_{\mathcal{L}(\mathcal{H}_2:\mathbb{R})} \leq \|y - x\|_{\mathcal{H}_2}, \tag{A.9}$$

implying that

$$|\,\mathrm{E}(h[S_n]) - \mathrm{E}(h[S_{n,M}])| \leq \|S_n - S_{n,M}\|_{L^2(\Omega:\mathcal{H}_2)}. \tag{A.10}$$

Since $h \in C_b^2$, it is also possible to provide a bound based on a second-order expansion; see, e.g., [36] for the finite dimensional analogue. Since first- and second-order expansions result in comparable bounds, we use the first-order expansion (A.9). Note that, from (5.7),

$$\|S_n - S_{n,M}\|_{L^2(\Omega:\mathcal{H}_2)} = \sqrt{\sum_{p=M+1}^{\infty} p! \|h_{p,n}\|^2_{\mathfrak{H}^{\otimes p} \otimes \mathcal{H}_2^{\otimes 2}}} \leq \sqrt{(2K + 2\theta) \sum_{p=M+1}^{\infty} \mathrm{E}\,\|G^p[X_1]\|^2_{\mathcal{H}_2}}. \tag{A.11}$$

Combining (A.10) and (A.11), and with $\mathcal{R}_{1,M}$ defined in (A.4), we get

$$d_2(S_n, S_{n,M}) \leq \mathcal{R}_{1,M}. \tag{A.12}$$

We now turn to estimating the second distance on the RHS of (A.8). From Theorem 4.3 of [6] (after correcting a minor typo),

$$d_2(S_{n,M}, Z_M) \leq \frac{1}{2} \left(\sqrt{\Lambda_1(n) + \Lambda_2(n)} + \|\mathcal{T}_{S_{n,M}} - \mathcal{T}_{Z_M}\|_{HS(\mathcal{H}_2)} \right), \tag{A.13}$$

where $\Lambda_i(n), i = 1, 2$, are defined in (A.14)–(A.15) below. We argue further that $\Lambda_i(n), i = 1, 2$, can be bounded in terms of (A.4) as follows

$$\Lambda_1(n) \doteq \sum_{p=1}^{M} \sqrt{\sum_{s=1}^{p-1} c_{p,p}^2(s) \|h_{p,n} \otimes_s h_{p,n}\|^2_{\mathfrak{H}^{\otimes 2(p-s)} \otimes \mathcal{H}_2^{\otimes 2}}}$$

$$\le \sum_{p=1}^{M} \sqrt{\sum_{s=1}^{p-1} c_{p,p}^2(s) \sum_{i,j=1}^{\infty} \|\widetilde{h}_{p,n,i} \otimes_s \widetilde{h}_{p,n,j}\|^2_{\mathfrak{H}^{\otimes 2(p-s)}}} \le \mathcal{R}_{2,n,M}; \tag{A.14}$$

$$\Lambda_2(n) \doteq \sum_{\substack{1\le p,r\le M\\ p\ne r}} \sqrt{\sum_{s=1}^{p\wedge r} c_{p,r}^2(s) \|h_{p,n} \otimes_s h_{r,n}\|^2_{\mathfrak{H}^{\otimes(p+r-2s)}\otimes \mathcal{H}_2^{\otimes 2}}}$$

$$= \sum_{\substack{1\le p,r\le M\\ p\ne r}} \sqrt{\sum_{s=1}^{p\wedge r} c_{p,r}^2(s) \sum_{i,j=1}^{\infty} \|\widetilde{h}_{p,n,i} \otimes_s \widetilde{h}_{r,n,j}\|^2_{\mathfrak{H}^{\otimes(p+r-2s)}}} \le \mathcal{R}_{3,n,M}, \tag{A.15}$$

where the first inequality in (A.14) follows from (5.9) and the second from calculations analogous to (5.14). For (A.15), we used again (5.9) and (5.14). Recall $\mathcal{T}_{Z_M}$ in (A.7) and $\mathcal{T}_{S_{n,M}}$ in (5.20). Furthermore, the difference $\mathcal{T}_{S_{n,M}} - \mathcal{T}_{Z_M}$ is a self-adjoint operator such that $\mathrm{tr}(|\mathcal{T}_{S_{n,M}} - \mathcal{T}_{Z_M}|) = \mathrm{tr}(\mathcal{T}_{S_{n,M}} - \mathcal{T}_{Z_M})$. Then, using (3.3) and (3.4),

$$\begin{aligned}
\|\mathcal{T}_{S_{n,M}} - \mathcal{T}_{Z_M}\|_{HS(\mathcal{H}_2)} &\le \sqrt{\mathrm{tr}(\mathcal{T}_{S_{n,M}} - \mathcal{T}_{Z_M})} \le \sqrt{\sum_{p=q}^{M} p!\|h_{p,n} - h_p\|^2_{\mathfrak{H}^{\otimes p}\otimes\mathcal{H}_2}} \\
&\le \sqrt{\sum_{p=q}^{M}\sum_{i=1}^{\infty} p! \left| \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} b_{i,\boldsymbol{l}} b_{i,\boldsymbol{r}} \left[\left(\sum_{|v|\le K} \frac{|v|}{n} + \sum_{K<|v|<n} \frac{|v|}{n} + \sum_{|v|\ge n} \right) \prod_{j=1}^{p} \rho_{l_j r_j}(v) \right] \right|} \\
&\le \sqrt{\sum_{p=q}^{M} \mathrm{E}\,\|G^p[X_1]\|^2_{\mathcal{H}_2} \left(\frac{2K(K+1)}{n} + \sum_{|v|<n} \theta^q(v) \frac{|v|}{n} + \sum_{|v|\ge n} \theta^q(v) \right)} \qquad \text{(A.16)} \\
&\le \mathcal{R}_{4,n,M},
\end{aligned}$$

where (A.16) follows from calculations similar to (5.5).

Finally, for the third summand in (A.8), recall that Z_M, Z are both Gaussian elements in $\mathcal{H}_2$, with respective covariance operators given by $\mathcal{T}_{Z_M}$ in (A.7) and $\mathcal{T}_Z$ in (2.2). Since $h \in C_b^2(\mathcal{H}_2)$, we get by Lipschitz-continuity, and with further explanations given below,

$$\begin{aligned}
d_2(Z_M, Z) \le \mathrm{E}\,(\|Z_M - Z\|_{\mathcal{H}_2}) &\le \sqrt{\mathrm{tr}[\mathcal{T}_{Z-Z_M}]} \qquad \text{(A.17)} \\
&= \sqrt{\sum_{p=M+1}^{\infty} \sum_{k\in\mathbb{Z}} \sum_{i=1}^{\infty} p! \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} b_{i,\boldsymbol{l}} b_{i,\boldsymbol{r}} \prod_{m=1}^{p} \rho_{l_m r_m}(k)} \\
&= \sqrt{\sum_{p=M+1}^{\infty} p!\|h_p\|^2_{\mathfrak{H}^{\otimes p}\otimes\mathcal{H}_2}} \le \mathcal{R}_{1,M}, \qquad \text{(A.18)}
\end{aligned}$$

where the second inequality in (A.17) is due to Cauchy-Schwarz and (3.4) with $\mathcal{T}_{Z-Z_M}$ denoting the covariance operator of $Z - Z_M$. Using (5.3) and (5.8), we can infer (A.18). The result follows by combining (A.8), (A.12), (A.13)–(A.16), and (A.18). $\square$

B Auxiliary Results

The following lemma is analogous to Lemma 5.3 in [6].

Lemma B.1. *Suppose the assumptions of Theorem 2.2 hold. The covariance operators $\mathcal{T}_{W_{n,M}}$ and $\mathcal{T}_{V_{n,M}}$ in (5.20) and (5.19) corresponding to $W_{n,M}$ and $V_{n,M}$, satisfy, for all $M \ge q$,*

$$\|\mathcal{T}_{V_{n,M}} - \mathcal{T}_{W_{n,M}}\|_{HS(L^2([0,1])\otimes\mathcal{H}_2)} \to 0, \quad \text{as } n \to \infty.$$

Proof: Recall that $W_{n,M}$ and $V_{n,M}$ are viewed as random elements in $L^2([0,1]) \otimes \mathcal{H}_2$, with corresponding covariance operators $\mathcal{T}_{W_{n,M}}$ and $\mathcal{T}_{V_{n,M}}$ that are Hilbert-Schmidt operators in $HS(L^2([0,1])\otimes\mathcal{H}_2)$. Note first, that

$$
\begin{aligned}
&\mathcal{T}_{V_{n,M}} - \mathcal{T}_{W_{n,M}} \\
&= \sum_{p=q}^{M} \sum_{i,j=1}^{\infty} \sum_{k_1,k_2=1}^{n} \frac{1}{n} p! \left(\mathbf{1}_{[\frac{k_1}{n},1]} \otimes \mathbf{1}_{[\frac{k_2}{n},1]} - \mathrm{E}(B\otimes B) \right) \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} b_{i,\boldsymbol{l}} b_{j,\boldsymbol{r}} \prod_{m=1}^{p} \rho_{l_m r_m}(k_1-k_2)(v_i\otimes v_j) \\
&= \sum_{i,j=1}^{\infty} \sum_{k_1,k_2=1}^{n} \Upsilon_n^{k_1,k_2} A_{i,j,M}^{k_1,k_2}(v_i\otimes v_j),
\end{aligned} \tag{B.1}
$$

with

$$
\Upsilon_n^{k_1,k_2} \doteq \mathbf{1}_{[\frac{k_1}{n},1]} \otimes \mathbf{1}_{[\frac{k_2}{n},1]} - \mathrm{E}(B\otimes B), \quad A_{i,j,M}^{k_1,k_2} \doteq \sum_{p=q}^{M} \frac{1}{n} p! \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} b_{i,\boldsymbol{l}} b_{j,\boldsymbol{r}} \prod_{m=1}^{p} \rho_{l_m r_m}(k_1-k_2), \tag{B.2}
$$

where $\Upsilon_n^{k_1,k_2}$ are integral operators with corresponding kernels $\upsilon_n^{k_1,k_2}$. Define further the following kernels

$$
\boldsymbol{\kappa}_{n,M}^{i,j}(s,t) \doteq \sum_{k_1,k_2=1}^{n} \left(\mathbf{1}_{[\frac{k_1}{n},1]}(t) \mathbf{1}_{[\frac{k_2}{n},1]}(s) - t\wedge s \right) A_{i,j,M}^{k_1,k_2}, \tag{B.3}
$$

and denote the respective integral operators in $L^2([0,1])$ by $\mathcal{K}_{n,M}^{i,j}$. Let $\{e_r\otimes v_s\}_{r,s\geq 1}$ be an orthonormal basis in $L^2([0,1])\otimes\mathcal{H}_2$. Then, using the representation (B.1), we get

$$
\|\mathcal{T}_{V_{n,M}} - \mathcal{T}_{W_{n,M}}\|^2_{HS(L^2([0,1])\otimes\mathcal{H}_2)} \tag{B.4}
$$

$$
= \sum_{r_1,r_2,s_1,s_2=1}^{\infty} \left| \sum_{i,j=1}^{\infty} \sum_{k_1,k_2=1}^{n} \left\langle \Upsilon_n^{k_1,k_2}, e_{r_1}\otimes e_{r_2} \right\rangle_{L^2([0,1])^{\otimes 2}} \left\langle A_{i,j,M}^{k_1,k_2}(v_i\otimes v_j), v_{s_1}\otimes v_{s_2} \right\rangle_{\mathcal{H}_2^{\otimes 2}} \right|^2 \tag{B.5}
$$

$$
\begin{aligned}
&= \sum_{r_1,r_2=1}^{\infty} \sum_{i,j=1}^{\infty} \left| \sum_{k_1,k_2=1}^{n} A_{i,j,M}^{k_1,k_2} \int_{[0,1]\times[0,1]} \upsilon_n^{k_1,k_2}(s,t) e_{r_1}(s) e_{r_2}(t) ds dt \right|^2 \\
&= \sum_{i,j=1}^{\infty} \|\mathcal{K}_{n,M}^{i,j}\|^2_{HS(L^2([0,1]))} = \sum_{i,j=1}^{\infty} \|\boldsymbol{\kappa}_{n,M}^{i,j}\|^2_{L^2([0,1]^2)} \leq \sum_{i,j=1}^{\infty} \sup_{s,t\in[0,1]} |\boldsymbol{\kappa}_{n,M}^{i,j}(s,t)|^2,
\end{aligned} \tag{B.6}
$$

where (B.5) follows from the first equality in (5.23) and (B.6) follows from (B.3) and the last display on page 28 of [6], by bounding with the supremum over s,t.

In order to bound (B.6), we start with the kernels in (B.3). Note that, with further explanations given below,

$$
\begin{aligned}
|\boldsymbol{\kappa}_{n,M}^{i,j}(s,t)| &= \left| \sum_{p=q}^{M} p! \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} b_{i,\boldsymbol{l}} b_{j,\boldsymbol{r}} \frac{1}{n} \sum_{k_1,k_2=1}^{n} \left(\mathbf{1}_{[\frac{k_1}{n},1]}(t) \mathbf{1}_{[\frac{k_2}{n},1]}(s) - (t\wedge s) \right) \prod_{m=1}^{p} \rho_{l_m r_m}(k_1-k_2) \right| \\
&= \left| \sum_{p=q}^{M} p! \sum_{k=-(n-1)}^{n-1} f_{kn}(s,t) \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} b_{i,\boldsymbol{l}} b_{j,\boldsymbol{r}} \prod_{m=1}^{p} \rho_{l_m r_m}(k) \right| \\
&\leq \sum_{p=q}^{M} p! |f_{0,n}(s,t)| \left(\sum_{\boldsymbol{l}\in\mathbb{N}^p} b_{i,\boldsymbol{l}}^2 \right)^{\frac{1}{2}} \left(\sum_{\boldsymbol{r}\in\mathbb{N}^p} b_{j,\boldsymbol{r}}^2 \right)^{\frac{1}{2}} \\
&\qquad + \sum_{p=q}^{M} p! \sum_{|k|=1}^{K-1} |f_{kn}(s,t)| \left| \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} b_{i,\boldsymbol{l}} b_{j,\boldsymbol{r}} \prod_{j=1}^{p} \rho_{l_j r_j}(k) \right|
\end{aligned} \tag{B.7}
$$

$$+\sum_{p=q}^{M} p! \sum_{|k|=K}^{n-1} |f_{kn}(s,t)| \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} |b_{i,\boldsymbol{l}} b_{j,\boldsymbol{r}}| \prod_{j=1}^{p} |\rho_{l_j r_j}(k)| \tag{B.8}$$

$$\le \frac{1}{n}(1+3K(K+1)) \sum_{p=q}^{M} p! \left(\sum_{\boldsymbol{l}\in\mathbb{N}^p} b_{i,\boldsymbol{l}}^2\right)^{\frac{1}{2}} \left(\sum_{\boldsymbol{r}\in\mathbb{N}^p} b_{j,\boldsymbol{r}}^2\right)^{\frac{1}{2}}$$

$$+3\sum_{p=q}^{M} p! \sum_{|k|=K+1}^{n-1} \frac{|k|}{n} \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} |b_{i,\boldsymbol{l}} b_{j,\boldsymbol{r}}| \prod_{m=1}^{p} |\rho_{l_m r_m}(k)|, \tag{B.9}$$

where (B.7) is due to Lemma B.2 with f_{kn} defined in (B.14) below and (B.8) follows by Cauchy-Schwarz with K defined in (5.4). Finally, (B.9) is due to (B.15) in Lemma B.2. We continue providing estimates for (B.6) by using (B.9):

$$\sum_{i,j=1}^{\infty} \sup_{s,t\in[0,1]} |\boldsymbol{k}_{n,M}^{i,j}(s,t)|^2$$

$$\le \frac{2(1+3K(K+1))^2}{n^2} \sum_{i,j=1}^{\infty} \left(\sum_{p=q}^{M} p! \left(\sum_{\boldsymbol{l}\in\mathbb{N}^p} b_{i,\boldsymbol{l}}^2\right)^{\frac{1}{2}} \left(\sum_{\boldsymbol{r}\in\mathbb{N}^p} b_{j,\boldsymbol{r}}^2\right)^{\frac{1}{2}}\right)^2$$

$$+18\sum_{i,j=1}^{\infty} \left(\sum_{p=q}^{M} p! \sum_{|k|=K+1}^{n-1} \frac{|k|}{n} \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}^p} |b_{i,\boldsymbol{l}} b_{j,\boldsymbol{r}}| \prod_{m=1}^{p} |\rho_{l_m r_m}(k)|\right)^2 \tag{B.10}$$

$$\le \frac{2(1+3K(K+1))^2}{n^2} \left(\sum_{i=1}^{\infty}\sum_{p=q}^{M} p! \sum_{\boldsymbol{l}\in\mathbb{N}^p} b_{i,\boldsymbol{l}}^2\right)^2$$

$$+18\left(2\sum_{k=K+1}^{n-1} \frac{k}{n} \left(\max\{\theta(k),\theta(-k)\}\right)^q\right)^2 \left(\sum_{i=1}^{\infty}\sum_{p=q}^{M} p! \sum_{\boldsymbol{l}\in\mathbb{N}^p} b_{i,\boldsymbol{l}}^2\right)^2 \tag{B.11}$$

$$\le 2\left(\mathrm{E}\,\|G[X_1]\|_{\mathcal{H}_2}^2\right)^2 \left(\frac{(1+3K(K+1))^2}{n^2} + 72\left(\sum_{k=K+1}^{n-1} \frac{k}{n} \left(\max\{\theta(k),\theta(-k)\}\right)^q\right)^2\right), \tag{B.12}$$

where (B.10) follows from (B.9) and Young's inequality, and (B.11) is due to Cauchy-Schwarz and (5.2). The conclusion follows by combining (B.6), and (B.12), as $n\to\infty$. □

The following lemma is similar to the inequality derived in the second display on page 28 in [6]. For completeness, we derive an upper bound of our kernel function in (B.4).

Lemma B.2. *Using the notation of Lemma B.1, we have*

$$\frac{1}{n}\sum_{k_1,k_2=1}^{n} \left(\mathbf{1}_{[\frac{k_1}{n},1]}(t)\mathbf{1}_{[\frac{k_2}{n},1]}(s) - (t\wedge s)\right) \prod_{m=1}^{p} \rho_{l_m r_m}(k_1-k_2) = \sum_{k=-(n-1)}^{n-1} f_{kn}(s,t) \prod_{m=1}^{p} \rho_{l_m r_m}(k) \tag{B.13}$$

with, for $|k| = 1,\dots,n-1$,

$$f_{kn}(s,t) = \begin{cases} \frac{\lfloor ns\rfloor}{n} - (t\wedge s) + (t\wedge s)\frac{|k|}{n} + \frac{k}{n}\mathbf{1}_{\{k\le -1\}}, & \text{if } \ t-s > \frac{k}{n}, \\ \frac{\lfloor nt\rfloor}{n} - (t\wedge s) + (t\wedge s)\frac{|k|}{n} - \frac{k}{n}\mathbf{1}_{\{k\ge 1\}}, & \text{if } \ t-s \le \frac{k}{n}, \end{cases} \tag{B.14}$$

and $f_{0n}(s,t) = \frac{\lfloor n(s\wedge t)\rfloor}{n} - (s\wedge t)$. *In particular, for all* $n\in\mathbb{N}$,

$$\sup_{s,t\in[0,1]} |f_{kn}(s,t)| \le 3\frac{|k|}{n}, \quad |k| = 1,\dots,n-1, \quad \sup_{s,t\in[0,1]} |f_{0n}(s,t)| \le \frac{1}{n}. \tag{B.15}$$

Proof: By the same calculations as in (5.14) of [6] and the subsequent estimates, we have

$$\frac{1}{n}\sum_{k_1,k_2=1}^{n}\mathbf{1}_{[\frac{k_1}{n},1]}(s)\mathbf{1}_{[\frac{k_2}{n},1]}(t)\prod_{m=1}^{p}\rho_{l_m r_m}(k_1-k_2)=A_n+B_n+C_n, \tag{B.16}$$

with

$$A_n \doteq \sum_{k=-(n-1)}^{-1}\prod_{m=1}^{p}\rho_{l_m r_m}(k)\times\begin{cases}\frac{\lfloor ns\rfloor}{n}+\frac{k}{n} & \text{if } t-s>\frac{k}{n}\\ \frac{\lfloor nt\rfloor}{n} & \text{if } t-s\le\frac{k}{n}\end{cases}, \quad B_n \doteq \frac{\lfloor n(s\wedge t)\rfloor}{n}\delta_{\boldsymbol{lr}},$$

$$C_n \doteq \sum_{k=1}^{n-1}\prod_{m=1}^{p}\rho_{l_m r_m}(k)\times\begin{cases}\frac{\lfloor ns\rfloor}{n} & \text{if } t-s>\frac{k}{n}\\ \frac{\lfloor nt\rfloor}{n}-\frac{k}{n} & \text{if } t-s\le\frac{k}{n}\end{cases}.$$

Then,

$$\begin{aligned}&\frac{1}{n}\sum_{k_1,k_2=1}^{n}\Big(\mathbf{1}_{[\frac{k_1}{n},1]}(t)\mathbf{1}_{[\frac{k_2}{n},1]}(s)-(t\wedge s)\Big)\prod_{m=1}^{p}\rho_{l_m r_m}(k_1-k_2)\\ &=A_n+B_n+C_n-(t\wedge s)\sum_{k=-(n-1)}^{n-1}\left(1-\frac{|k|}{n}\right)\prod_{m=1}^{p}\rho_{l_m r_m}(k)\\ &=A_n+C_n-(\mathscr{A}_n+\mathscr{C}_n)+B_n-\mathscr{B}_n\end{aligned} \tag{B.17}$$

with $\mathscr{B}_n \doteq (t\wedge s)\delta_{\boldsymbol{lr}}$ and

$$\mathscr{A}_n \doteq (t\wedge s)\sum_{k=-(n-1)}^{-1}\left(1-\frac{|k|}{n}\right)\prod_{m=1}^{p}\rho_{l_m r_m}(k), \quad \mathscr{C}_n \doteq (t\wedge s)\sum_{k=1}^{n-1}\left(1-\frac{|k|}{n}\right)\prod_{m=1}^{p}\rho_{l_m r_m}(k). \tag{B.18}$$

Note first that

$$B_n-\mathscr{B}_n=\left(\frac{\lfloor n(s\wedge t)\rfloor}{n}-t\wedge s\right)\delta_{\boldsymbol{lr}}=f_{0n}(s,t)\delta_{\boldsymbol{lr}}. \tag{B.19}$$

For the remaining quantities in (B.17), we distinguish the cases $t-s>\frac{k}{n}$ and $t-s\le\frac{k}{n}$.
For $t-s>\frac{k}{n}$:

$$\begin{aligned}A_n-\mathscr{A}_n+C_n-\mathscr{C}_n&=\sum_{k=-(n-1)}^{-1}\left(\frac{\lfloor ns\rfloor}{n}+\frac{k}{n}-(t\wedge s)\Big(1-\frac{|k|}{n}\Big)\right)\prod_{m=1}^{p}\rho_{l_m r_m}(k)\\ &\quad+\sum_{k=1}^{n-1}\left(\frac{\lfloor ns\rfloor}{n}-(t\wedge s)\Big(1-\frac{|k|}{n}\Big)\right)\prod_{m=1}^{p}\rho_{l_m r_m}(k)\\ &=\sum_{|k|=1}^{n-1}f_{kn}(s,t)\prod_{m=1}^{p}\rho_{l_m r_m}(k).\end{aligned} \tag{B.20}$$

For $t-s\le\frac{k}{n}$:

$$\begin{aligned}A_n-\mathscr{A}_n+C_n-\mathscr{C}_n&=\sum_{k=-(n-1)}^{-1}\left(\frac{\lfloor nt\rfloor}{n}-(t\wedge s)\Big(1-\frac{|k|}{n}\Big)\right)\prod_{m=1}^{p}\rho_{l_m r_m}(k)\\ &\quad+\sum_{k=1}^{n-1}\left(\frac{\lfloor nt\rfloor}{n}-\frac{k}{n}-(t\wedge s)\Big(1-\frac{|k|}{n}\Big)\right)\prod_{m=1}^{p}\rho_{l_m r_m}(k)\\ &=\sum_{|k|=1}^{n-1}f_{kn}(s,t)\prod_{m=1}^{p}\rho_{l_m r_m}(k).\end{aligned} \tag{B.21}$$

This finishes the proof of (B.13). To verify (B.15), note that, since $s,t\in[0,1]$,

$$\left|(t\wedge s)\frac{|k|}{n}+\frac{k}{n}\mathbf{1}_{\{k\leq -1\}}\right|\leq 2\frac{|k|}{n},\quad \left|(t\wedge s)\frac{|k|}{n}-\frac{k}{n}\mathbf{1}_{\{k\geq 1\}}\right|\leq 2\frac{|k|}{n}. \tag{B.22}$$

Now we estimate $\left|\frac{\lfloor ns\rfloor}{n}-(t\wedge s)\right|$ whenever $t-s>\frac{k}{n}, |k|=1,\dots,n-1$. First, note that, $s\wedge t=t$ implies $t-s\geq\frac{k}{n}$ for $k\leq -1$, and so $s-t\leq\frac{|k|}{n}$. Then, we see that

$$\begin{aligned}-\frac{1}{n}\leq\frac{\lfloor ns\rfloor}{n}-\frac{\lceil ns\rceil}{n}\leq\frac{\lfloor ns\rfloor}{n}-s\leq\frac{\lfloor ns\rfloor}{n}-\frac{\lfloor ns\rfloor}{n}=0,&\quad\text{if } s\wedge t=s,\\ -\frac{1}{n}\leq\frac{\lfloor ns\rfloor}{n}-s\leq\frac{\lfloor ns\rfloor}{n}-s\wedge t\leq s-t\leq\frac{|k|}{n},&\quad\text{if } s\wedge t=t,\end{aligned} \tag{B.23}$$

which says that $\left|\frac{\lfloor ns\rfloor}{n}-(t\wedge s)\right|\leq\frac{|k|}{n}, |k|=1,\dots,n-1$ whenever $t-s>\frac{k}{n}$.

We now turn to estimating $\left|\frac{\lfloor nt\rfloor}{n}-(t\wedge s)\right|$ whenever $t-s\leq\frac{k}{n}, |k|=1,\dots,n-1$. Arguing as in (B.23) gives

$$\left|\frac{\lfloor nt\rfloor}{n}-(t\wedge s)\right|\leq\frac{|k|}{n}. \tag{B.24}$$

Combining (B.22)–(B.24) verifies (B.15). □

For the following lemma, let $\{X_k\}_{k\in\mathbb{Z}}$ be a $\mathscr{H}-$valued stationary stochastic process, with covariance operator Q and autocorrelation function given by

$$\rho_{rs}(k-l)=\mathrm{E}\left[\langle Q^{-1/2}u_r,X_k\rangle_{\mathscr{H}_1}\langle Q^{-1/2}u_s,X_l\rangle_{\mathscr{H}_1}\right].$$

The following result allows us to rewrite the normalized series with regard to an isonormal Gaussian process. The proof is very closely related to that of Proposition 7.2.3 of [34].

Lemma B.3. *There exists a real separable Hilbert space $\mathfrak{H}$, as well as an isonormal Gaussian process over $\mathfrak{H}$, written $\{X(h):h\in\mathfrak{H}\}$, with the property that there exists a set $E=\{\varepsilon_{ik}:k\in\mathbb{Z},i\in\mathbb{N}\}\subset\mathfrak{H}$ such that 1. E generates $\mathfrak{H}$; 2. $\langle\varepsilon_{ik},\varepsilon_{jl}\rangle_{\mathfrak{H}}=\rho_{ij}(k-l)$, 3. $\langle Q^{-1/2}u_i,X_k\rangle_{\mathscr{H}_1}=X(\varepsilon_{ik})$*

Proof: We start by defining $\mathfrak{H}$. First define a set $\mathscr{E}$, such that $h\in\mathscr{E}$ if and only if $h=\{h_{ik},k\in\mathbb{Z},i\in\mathbb{N}\mid h_{ik}=0 \text{ for all but finitely many } h_{ik}\}$. Equip the space $\mathscr{E}$ with the inner product

$$\langle f,g\rangle_{\mathfrak{H}}=\sum_{\substack{k,l\in\mathbb{Z}\\ i,j\in\mathbb{N}}}f_{ik}g_{jl}\rho_{ij}(k-l)=\sum_{\substack{k,l\in\mathbb{Z}\\ i,j\in\mathbb{N}}}f_{ik}g_{jl}\,\mathrm{E}\left[\langle Q^{-1/2}u_i,X_k\rangle_{\mathscr{H}_1}\langle Q^{-1/2}u_j,X_l\rangle_{\mathscr{H}_1}\right], \tag{B.25}$$

whenever $f,g\in\mathscr{E}$, and now define $\mathfrak{H}$ as the closure of $\mathscr{E}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{H}}$.

We proceed with defining the generating set E. First, we take, for $k\in\mathbb{Z},i\in\mathbb{N}$, sequences of the form

$$\varepsilon_{ik}=\{\delta_{kl}\delta_{ij},j\in\mathbb{N},l\in\mathbb{Z}\}=\{\delta_{kl}\langle u_i,u_j\rangle_{\mathscr{H}_1},j\in\mathbb{N},l\in\mathbb{Z}\}.$$

Now consider the set $E\doteq\{\varepsilon_{ik},i\in\mathbb{N},k\in\mathbb{Z}\}\subset\mathfrak{H}$. Clearly E is a generating set for $\mathfrak{H}$, and so Property 1 in the statement is satisfied. For $h\in\mathscr{E}$, define

$$X(h)\doteq\sum_{k\in\mathbb{Z},i\in\mathbb{N}}h_{ik}\langle Q^{-1/2}u_i,X_k\rangle_{\mathscr{H}_1}. \tag{B.26}$$

Then, for $h\in\mathfrak{H}$, select a sequence $h_n\in\mathscr{E}$ converging to h and take $X(h)=\lim_{n\to\infty}X(h_n)$, where the limit is understood in the $L^2(\Omega)$ and the a.s. sense. Now we prove that $\{X(h):h\in\mathfrak{H}\}$ is an isonormal process over $\mathfrak{H}$. Indeed, for $f,g\in\mathfrak{H}$, from (B.25) and (B.26),

$$\begin{aligned}\mathrm{E}\left[X(f)X(g)\right]&=\mathrm{E}\left[\left(\sum_{k\in\mathbb{Z},i\in\mathbb{N}}f_{ik}\langle Q^{-1/2}u_i,X_k\rangle_{\mathscr{H}_1}\right)\times\left(\sum_{l\in\mathbb{Z},j\in\mathbb{N}}g_{jl}\langle Q^{-1/2}u_j,X_l\rangle_{\mathscr{H}_1}\right)\right]\\&=\sum_{\substack{k,l\in\mathbb{Z}\\ i,j\in\mathbb{N}}}f_{ik}g_{jl}\rho_{ij}(k-l)=\langle f,g\rangle_{\mathfrak{H}}\end{aligned}$$

and so, in particular, Property 2 follows. Finally, by construction,

$$X(\varepsilon_{ik}) = \sum_{l\in\mathbb{Z},j\in\mathbb{N}} \delta_{ij}\delta_{kl}\langle Q^{-1/2}u_j, X_l\rangle_{\mathscr{H}_1} = \langle Q^{-1/2}u_i, X_k\rangle_{\mathscr{H}_1},$$

such that Property 3 is also satisfied. $\square$