University of Illinois Fall 2021

ECE 313: Midterm Exam I

Monday, October 04, 2021
8:45 p.m. — 10:00 p.m.

1. **[10+8+8 points]** The three parts of this problem are unrelated.

   (a) Let an experiment consist of rolling a fair die, and define the following events about the number showing: $A =$ "number is even," and $B =$ "number is greater than 3". Display the outcomes in a Karnaugh map and determine $P(A \cup B^C)$.
   Solution: The outcomes are displayed in a Karnaugh map, shown as follows:

	B^C	B
A^C	1, 3	5
A	2	4, 6

   $P(A \cup B^C) = 5/6$.

   (b) The outcome of an experiment can be any strictly positive integer k with probability $1/2^k$. What is the probability that the outcome is greater than or equal to 3?
   Solution: $P\{k \geq 3\} = 1 - P\{k \leq 2\} = 1 - P\{k = 1\} - P(\{k = 2\}) = 1 - 1/2 - 1/2^2 = 1/4$.

   (c) An engine of a test drone fails with probability p. The drone can still fulfill its mission if at least half of the engines are operational. Find the interval of values of p for which the design with 2 engines is not worse than the design with 4 engines.
   Solution: One can see that in the two engine design, the chances of failure are $p_2 = p^2$, while for the four engine design, the chances of failure are $p_4 = p^4 + 4p^3(1-p)$. Comparing these two polynomials, we see that

   $$p_2 \leq p_4 \Leftrightarrow p^2 \leq p^2(4p - 3p^2) \Leftrightarrow p \in [1/3, 1].$$

2. **[14+6 points]** Suppose two fair dice, one orange and one blue, are rolled. Define the following events:
   A: The number on the orange die is even.
   B: The number on the blue die is less than or equal to 3.
   C: The sum of the numbers are even.

   Answer questions (a) and (b). Hint: It is helpful to list all possible outcomes in a 6×6 matrix to compute probabilities.

   (a) Are events A, B, and C pairwise independent?
   Solution: $P(A) = 1/2, P(B) = 1/2, P(C) = 1/2, P(AB) = 1/4, P(BC) = 1/4, P(AC) = 1/4$
   $P(A)P(B) = P(AB), P(A)P(C) = P(AC), P(B)(PC) = P(BC) \Rightarrow A, B,$ and C are pairwise independent.

   (b) Are events A, B, and C three independent events?
   Solution: $P(ABC) = 1/12$.
   $P(A)P(B)P(C) = 1/8 \neq P(ABC) \Rightarrow A, B,$ and C are not independent.

3. **[6+6+8 points]** The three parts of this problem are unrelated.

(a) Let X be a Poisson random variable with unknown parameter λ, where $\lambda = \sqrt{\theta}$. Assume that you draw one sample (i.e., observe one outcome) from X, which has the value k. Find the ML estimator of θ.

Solution: We showed in class that the ML estimator for λ given one observation k equals $\hat{\lambda}_{ML} = k$. Therefore, $\theta = k^2$.

(b) Suppose the fraction p of students preferring in-person lectures over Zoom lectures at the University of Illinois system is estimated using the formula $\hat{p} = \frac{X}{n}$, where n stands for the number of students questioned about their lecture preferences, and X denotes the random variable counting the number of polled students in favor of in-person teaching. You would like to estimate p within 0.2 with at least 75% confidence. How many students should you question in your study?

Solution: Based on the confidence level, you can check that choosing $a = 2$ works. Now, we also have $\frac{a}{2\sqrt{n}} = 0.2$, which upon replacing $a = 2$ gives $n = 25$.

(c) Suppose X is a geometric random variable with parameter $p = \frac{1}{2}$. Using Markov's inequality, find an upper bound on the probability of X being greater than or equal to 2.

Solution: Markov's inequality asserts that $P\{X \geq k\} \leq \frac{E[X]}{k}$, for X a positive random variable. In our case, $E[X] = \frac{1}{p} = 2$ and $k = 2$, so Markov inequality does nothing more than to reassert that probabilities have to be ≤ 1. On the other hand, using the PMF of a geometric random variable, we see that the correct value of the probability obtained without using Markov's inequality equals $1 - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$.

4. **[8+6+6 points]** The three parts of this problem are unrelated.

(a) Let X be a Binomial random variable with parameters $n = 40$ and $p = \frac{1}{2}$. Let $Y = 3X + 2$. Find $E[Y]$ and the variance of Y, var(Y).

Solution: Clearly, $E[Y] = 3 \times E[X] + 2 = 3 \times 40 \times \frac{1}{2} + 2 = 62$. Similarly, var($Y$)= 9×var($X$)= 90.

(b) Suppose you have an unusual die with eight faces, and each face is equally likely to show up when you roll the die. You roll the die over and over again until each of the eight faces shows up at least one time. Find the expected number of rolls until the event of interest (i.e., each of eight faces has shown up at least once) occurs. You do not need to find the value numerically, it is enough to write down the complete expression for the expectation.

Solution: This is a straightforward modification of Problem 2.5.1 from the text. Note that the probability of success for the ith block of roles (leading up to the first observation of the ith new face) is now $\frac{8-i+1}{8}$ rather than $\frac{6-i+1}{6}$. The solution therefore equals

$$8 \times \left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{8} \right).$$

(c) Suppose that you are suffering from insomnia and need to count sheep until you fall asleep. The count until success is governed by a geometric random variable with probability of success (i.e., you falling asleep) equal to $p = \frac{1}{10}$. What is the probability of you having to count a total of more than 10 sheep given that you did not fall asleep up until you counted 5 sheep? You do not need to compute the result numerically as long as you write down the expression for the probability.

Solution: Here, all you need to use is the memoryless property of geometric random variables. Based on Chapter 2.5, you have $k = 5$, $n = 5$ so that

$$P\{X > k + n | X > n\} = (1 - p)^k = \left(\frac{9}{10} \right)^5.$$

5. **[6+8 points]** The two parts of this problem are unrelated.

(a) A test for an infectious disease has sensitivity 95% (this is the probability that the test detects the infection correctly, conditioned on the patient being infected) and specificity 90% (this is the probability that the the test detects the absence of infection correctly). If the prevalence of the infection (i.e., the fraction of infected people) in a population is 10%, what are the chances, conditioned on the test outcome being positive, that the person is sick?

Solution: If we denote by E_1 the state of being infected, by E_2 the complement, and A the event of the test outcome being positive, then, by Bayes formula,

$$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{P(A|E_1)P(E_1) + P(A|E_2)P(E_2)} = \frac{p_{\mathtt{sens}}p}{p_{\mathtt{sens}}p + (1-p_{\mathtt{spec}})(1-p)},$$

where we denote by p the prevalence, by $p_{\mathtt{sens}}$ the test sensitivity, and by $p_{\mathtt{spec}}$ the test specificity. Hence the answer is

$$\frac{.95 \times .1}{.95 \times .1 + .1 \times .9} = \frac{19}{37}.$$

(b) A network consists of 3 consecutive legs (i.e., connected in series), each of which is comprised of 3 parallel links. Each link has a capacity of 1, with the probability p of failure. Link failures are independent. Find the probability that the capacity of the network is exactly 2, if $p = 1/2$.

Solution: The probability that a leg in the network has capacity of 3 is $P_3 := (1-p)^3$, and the probability that a leg in the network has capacity at least 2 is $P_2 := (1-p)^3 + 3p(1-p)^2 = (1+2p)(1-p)^2$. The chances that the whole network has capacity at least 2 are the chances that each leg has the capacity at least 2, i.e. P_2^3. The chances that the whole network has capacity at least 3 are the chances that each leg has the capacity at least 3, i.e. P_3^3. The chances that the capacity is exactly 2 are therefore

$$P_2^3 - P_3^3 = 9p(1-p)^6(1+p+p^2) = \frac{63}{512}.$$