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Green's Theorem

Semester-II

Multivariable Calculus (18EMAB102)

Learning Outcomes

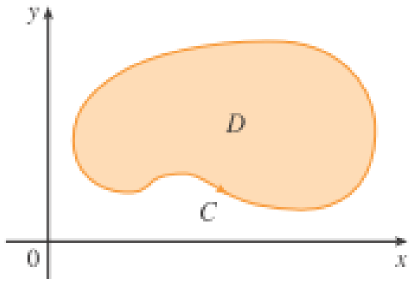
At the end of the session you will be able to

- Apply Green's theorem to convert line integral to double integral.

- Apply Green's theorem to compute area of plane region as a line integral over its boundary.

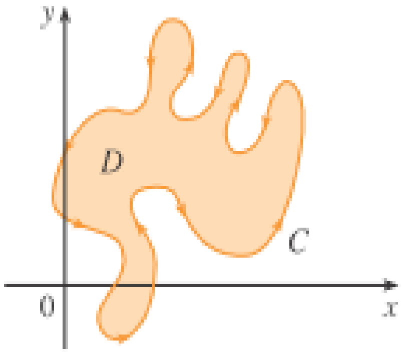
Introduction

- Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane D bounded by C.

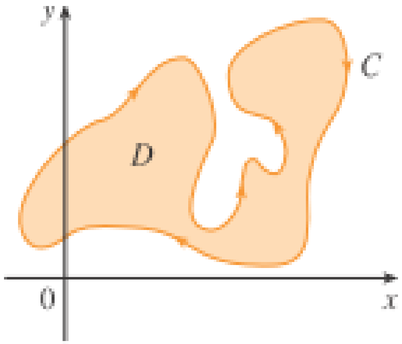


- Orientation

Positive orientation



Negative orientation



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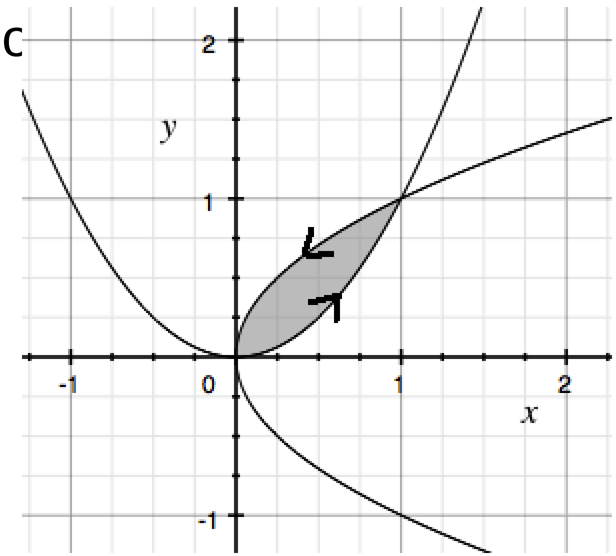
Green’s theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane $R^2$and let D be the region bounded by C. If $P(x, y)$ and $Q(x, y)$ be defined and continuous partial derivatives ,then

$$\oint_{C} Pdx + Qdy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Example1. Use Greens theorem to evaluate the line integral along the given positively oriented curve $\int_c (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy$; C is the boundary of the region enclosed by the parabo
$x = y^2$



Solution:

According to Green's Theorem

$\oint_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$dxdy ---------(1)

Where $P = y + e^{\sqrt{x}} Q = 2x + cosy^2$

$\frac{\partial P}{\partial y}$=1 $\frac{\partial Q}{\partial x}$=2

Substituting in (1),we get

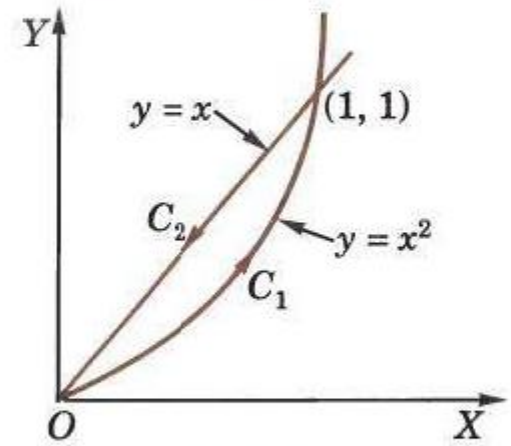
$\oint_c (y + e^{\sqrt{x}})dx + (2x + cosy^2)dy = \iint_R$ 1dxdy

x varies from 0 to1 , y varies from $x^2 to \sqrt{x}$

$=\int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} dydx = \int_0^1 (\sqrt{x} - x^2)$dx $= \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right) = \frac{1}{3}$.

Example 2: Use Green's theorem to evaluate

$\int_c (xy + y^2)dx + x^2 dy$ where C is the closed curve made up of the line $y = x$ and the parabola $y = x^2$.



Solution: According to Green's Theorem

$$\oint_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\text{dxdy} \quad \text{---------(1)}$$

Where $P = xy + y^2; \quad Q = x^2$

$$\frac{\partial P}{\partial y} = x + 2y, \qquad \frac{\partial Q}{\partial x} = 2x$$

Substituting in (1),we get

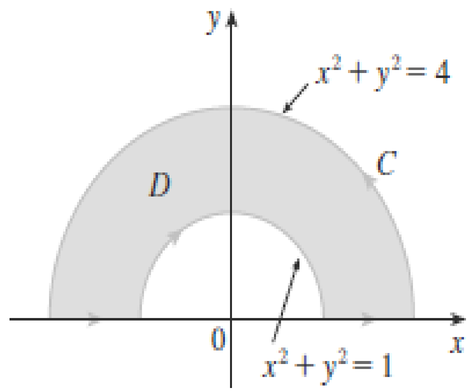
$$\oint_c (xy + y^2)dx + x^2 dy = \iint_R (x - 2y)dxdy$$

x varies from 0 to1, y varies from $x^2 to\ x$

$$=\int_{x=0}^{1}\int_{y=x^2}^{x}(x-2y)dydx=\int_0^1(x^4-x^3)dx = -\frac{1}{20}.$$

Example3.Evaluate $\oint_c \; y^2 \; dx + 3xy\, dy$, where C is the boundary of the semi annular region D in the upper half–plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

Solution: Notice that although D is not simple, the y-axis divides it into two simple regions .



In polar coordinates we can write

$D = \{(r, \theta) \;\; such\ that \;\; 1 \leq r \leq 2, \qquad 0 \leq \theta \leq \pi\}$

$$\oint_c y^2 dx + 3xydy = \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right] \mathrm{dA}$$

$$= \iint_D y.\, dA = \int_0^{\pi} \int_1^2 (r sin\theta) \mathrm{rdrd}\theta$$

$$= \int_0^{\pi} sin\theta \; d\theta \int_1^2 r^2 \mathrm{dr}$$

$$= \frac{14}{3}$$

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Application of Green's theorem

Use Green's theorem to prove that the area of a simple closed curve is $\frac{1}{2}\int_c \; xdy - ydx$.

Proof:

Consider $\int_c \; xdy - ydx = \int_c \; -ydx + xdy$

Here $\quad P = -y \;,\; Q = x \;,\; \frac{\partial P}{\partial y} = -1 \, \frac{\partial Q}{\partial x} = 1$

By Green's theorem

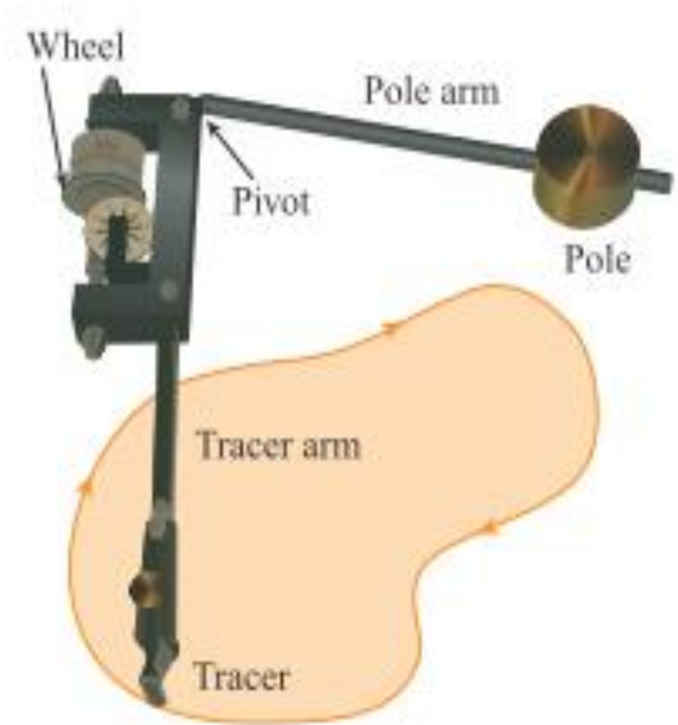
$$\int_C \; xdy - ydx = \iint_R \; [1-(-1)]dxdy$$

$$= 2\iint_R \; dxdy$$

$$= 2A$$

$$A = \frac{1}{2}\int_c \; xdy - ydx$$

Planimeter



- A planimeter is a mechanical instrument used for measuring the area of a region by tracing its boundary curve.

  in biology for measuring the area of leaves or wings,

  in medicine for measuring the size of cross-sections of organs or tumors, in forestry for

  estimating the size of forested regions from photographs.

  Figure shows the operation of a polar planimeter: The pole is fixed and, as the tracer

  is moved along the boundary curve of the region, the wheel partly slides and partly rolls

  perpendicular to the tracer arm. The planimeter measures the distance that the wheel rolls and this is proportional to the area of the enclosed region.

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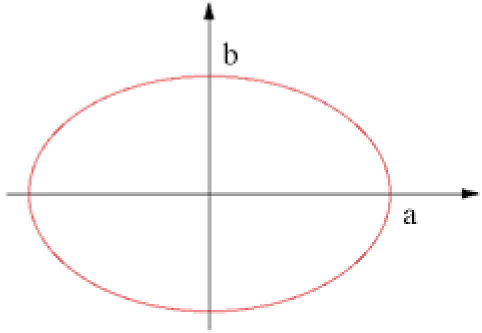
Example4. Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution:



The ellipse has parametric equations

$x = acost$ and $y = bsint$ where $0 \le t \le 2\pi$

Using the formula

$$A = \frac{1}{2}\int_c xdy - ydx$$

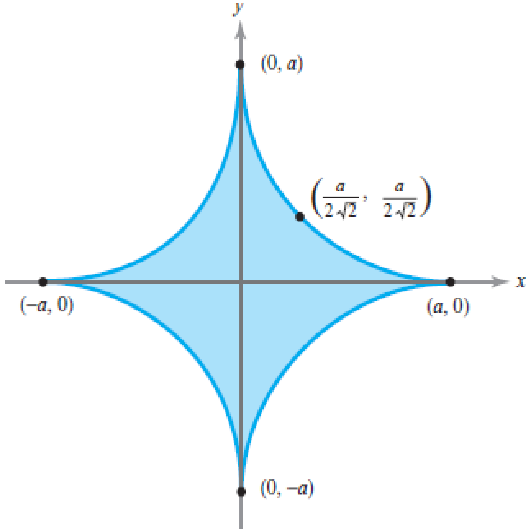
$$=\frac{1}{2}\int_0^{2\pi}(acost)(bcost)dt - (bsint)(-asint)dt$$

$$=\frac{ab}{2}\int_0^{2\pi} dt$$

$$=\pi ab$$

Example5.Find the area of asteroid

$x = a\cos^3 t\,,\ \ y = a\sin^3 t$



Solution: Using the formula , $A = \frac{1}{2}\int_c xdy - ydx$

$=\frac{1}{2}\int_0^{2\pi}(\mathrm{a}\cos^3 t)(3acostsin^2t)dt - (\mathrm{a}\sin^3 t)(-3asintcos^2t)dt$

$=\frac{3a^2}{2}\int_0^{2\pi}(\sin^2 tcos^4t + \cos^2 tsin^4t)\,dt$

$=\frac{3a^2}{2}\int_0^{2\pi}(sin^2 t + \cos^2 t)\cos^2 t\, sin^2t\,dt$

$=\frac{3a^2}{2}\int_0^{2\pi}\cos^2 t\, sin^2t\,dt = \frac{3a^2}{8}\int_0^{2\pi} sin^2 2t\,dt$

$=\frac{3a^2}{8}\int_0^{2\pi}\left(\frac{1-cos4t}{2}\right)dt = \frac{3\pi a^2}{8}$