BEM doc

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December 2, 2011

We solve the Helmholtz equation

$$\nabla^2 p(\mathbf{y}) + k^2 p(\mathbf{y}) = q(\mathbf{y}), \qquad \mathbf{y} \in V \tag{1}$$

where p denotes the acoustic pressure, and V is a closed volume, surrounded by the surface S. The surface is directed outward, as shown in Fig. **??**.

1 The Helmholtz equation

The solution method is formulated by testing the Helmholtz equation with the function $g(\mathbf{y})$ over the volume V

$$\int_V g(\mathbf{y}) \nabla^2 p(\mathbf{y}) \mathrm{d}V(\mathbf{y}) + k^2 \int_V g(\mathbf{y}) p(\mathbf{y}) \mathrm{d}V(\mathbf{y}) = 0 \tag{2}$$

We assume that the function $g(\mathbf{y})$ also satisfies the Helmholtz equation with a Dirac delta excitation located in $\mathbf{x}$

$$\nabla^2 g(\mathbf{y}) + k^2 g(\mathbf{y}) = -\delta(\mathbf{y} - \mathbf{x}) \tag{3}$$

testing the latter equation with the pressure field $p(\mathbf{y})$ results in

$$\int_V p(\mathbf{y}) \nabla^2 g(\mathbf{y}) \mathrm{d}V(\mathbf{y}) + k^2 \int_V p(\mathbf{y}) g(\mathbf{y}) \mathrm{d}V(\mathbf{y}) \\ = - \int_V p(\mathbf{y}) \delta(\mathbf{y} - \mathbf{x}) \mathrm{d}V(\mathbf{y}) \tag{4}$$

Subtracting the two volume integrals results in

$$\int_V \left(g(\mathbf{y}) \nabla^2 p(\mathbf{y}) - p(\mathbf{y}) \nabla^2 g(\mathbf{y}) \right) \mathrm{d}V(\mathbf{y}) = c(\mathbf{x}) p(\mathbf{x}) \tag{5}$$

Figure 1: The problem domain

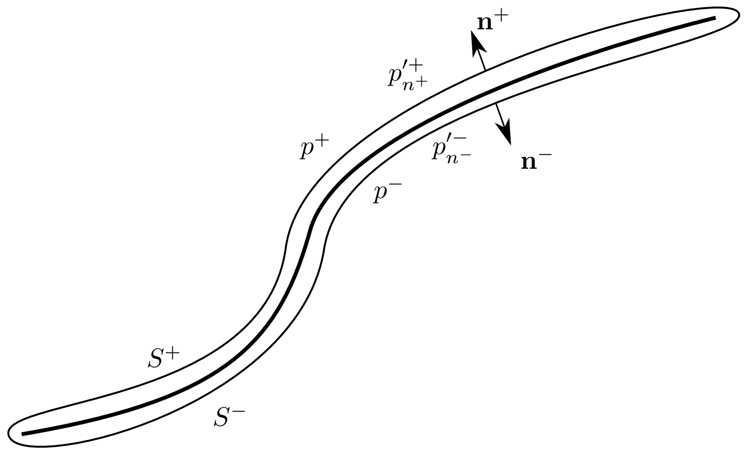

Figure 2: Sound field around a thin surface

where

$$c(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in V \\ \frac{1}{2} & \mathbf{x} \in S \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Green's theorem can be applied to the left hand side to transform the volume integral to surface integral as

$$\int_S \left(g(\mathbf{y}, \mathbf{x}) p'_n(\mathbf{y}) - p(\mathbf{y}) g'_n(\mathbf{y}, \mathbf{x}) \right) \mathrm{d}S(\mathbf{y}) = c(\mathbf{x}) p(\mathbf{x}) \tag{7}$$

By changing the direction of the surface normal so that n points inside V, the sign of the left hand side changes and we obtain

$$\int_S \left(p(\mathbf{y}) g'_n(\mathbf{y}, \mathbf{x}) - g(\mathbf{y}, \mathbf{x}) p'_n(\mathbf{y}) \right) \mathrm{d}S(\mathbf{y}) = c(\mathbf{x}) p(\mathbf{x}) \tag{8}$$

By applying Sommerfeld's radiation condition, the same integral equation is valid for outward problems, if the normal is pointing towards the external infinite volume V.

2 Application to a thin surface

Let's introduce $S = S^+ \cup S^-$ surrounding a thin surface, as shown in Fig. **??**. S^+ denotes the positive, S^- denotes the negative side of the closed surface. The pressure in an external point $\mathbf{x}$ can be expressed by writing the Helmholtz integral to the closed surface S as

$$p(\mathbf{x})$$

$$= \int_{S^+ \cup S^-} \left(p(\mathbf{y})g'_n(\mathbf{y},\mathbf{x}) - g(\mathbf{y},\mathbf{x})p'_n(\mathbf{y})\right) \mathrm{d}S(\mathbf{y}) \quad (9)$$

Taking into account that S^+ and S^- are the same surface but with opposite normal directions ($\mathbf{n}^+ = -\mathbf{n}^-$), we can split the integral into two sub integrals over the middle surface S_0. If the pressure on the positive and negative sides is denoted by p^+ and p^-, respectively, and we take into account that the normal derivatives with respect to $\mathbf{n}^+$ and $\mathbf{n}^-$ are of opposite signs, the integrals can be written as

$$\begin{aligned} p(\mathbf{x}) &= \int_{S_0} \left[\left(p^+ g'_{n^+} + p^- g'_{n^-}\right) - g\left(p'^+_{n^+} + g p'^-_{n^-}\right)\right] \mathrm{d}S \\ &= \int_{S_0} \left[\left(p^+ - p^-\right) g'_{n^+} - g\left(p'^+_{n^+} - p'^-_{n^+}\right)\right] \mathrm{d}S \\ &= \int_{S_0} \left(p^{\mathrm{d}}(\mathbf{y})g'_n(\mathbf{y},\mathbf{x}) - g(\mathbf{y},\mathbf{x})p'^{\mathrm{d}}_n(\mathbf{y})\right) \mathrm{d}S(\mathbf{y}) \end{aligned} \quad (10)$$

where

$$p^{\mathrm{d}}(\mathbf{y}) = p^+(\mathbf{y}) - p^-(\mathbf{y}) \quad (11)$$

is the pressure difference (or pressure jump) and

$$p'^{\mathrm{d}}_n(\mathbf{y}) = p'^+_n(\mathbf{y}) - p'^-_n(\mathbf{y}) \quad (12)$$

is the normal derivative difference (or velocity jump) between the negative and postive sides of S_0. It is important to notice that p'^+_n and p'^-_n are defined using the same (positive) normal direction.

If $\mathbf{x} \in S$ is located on the smooth surface, then the result is the mean of the positive and negative side pressure's

$$p^{\mathrm{m}}(\mathbf{x}) = \int_{S_0} \left(p^{\mathrm{d}}(\mathbf{y})g'_{n_y}(\mathbf{y},\mathbf{x}) - g(\mathbf{y},\mathbf{x})p'^{\mathrm{d}}_{n_y}(\mathbf{y})\right) \mathrm{d}S(\mathbf{y}) \quad (13)$$

where

$$p^{\mathrm{m}}(\mathbf{x}) = \frac{1}{2}\left(p^+(\mathbf{x}) + p^-(\mathbf{x})\right) \quad (14)$$

differentiation with respect to n_x yields

$$p'^{\mathrm{m}}_{n_x}(\mathbf{x}) =$$

$$= \int_{S_0} \left(p^{\mathrm{d}}(\mathbf{y})g''_{n_y n_x}(\mathbf{y},\mathbf{x}) - g'_{n_x}(\mathbf{y},\mathbf{x})p'^{\mathrm{d}}_{n_y}(\mathbf{y})\right) \mathrm{d}S(\mathbf{y}) \quad (15)$$

where

$$p'^{\mathrm{m}}_{n_x}(\mathbf{x}) = \frac{1}{2}\left(p'^+_{n_x}(\mathbf{x}) + p'^-_{n_x}(\mathbf{x})\right) \quad (16)$$

3 Discretisation

All surface quantities are represented by nodal values and interpolating functions. For the case of the pressure

$$p(\mathbf{y}) = \sum_j p_j N_j(\mathbf{y}) \tag{17}$$

where p_j is the j-th nodal value and $N_j(\mathbf{x})$ is the corresponding interpolation function.

The surface integral equation is solved by means of collocation method, where a szstem of equations is assembled by locating $\mathbf{x}$ in each i-th node:

$$\sum_j H_{ij} p_j - \sum_j G_{ij} p'_j = c(\mathbf{x}_i) p_i \tag{18}$$

where

$$H_{ij} = \int_S N_j(\mathbf{y}) g'_{n_y}(\mathbf{y}, \mathbf{x}_i) \mathrm{d}S(\mathbf{y}) \tag{19}$$

$$G_{ij} = \int_S N_j(\mathbf{y}) g(\mathbf{y}, \mathbf{x}_i) \mathrm{d}S(\mathbf{y}) \tag{20}$$

It is important to notice that the integrals where $\mathbf{x}_i$ is located within the support of $N_j(\mathbf{x})$ are singular, but the $\mathcal{O}(1/r)$ singularity of g and the $\mathcal{O}(1/r^2)$ singularity of g' is cancelled by $\mathrm{d}S$, so the integrals have a finite limit and can be evaluated numerically.

For the case of the thin boundary, the $\mathcal{O}(1/r^3)$ singularity of the kernel g'' does not allow to apply a collocation solution method. Instead, the surface integral equation is discretised using a Galerkin approach, where the integrals are tested by each shape function over the surface S. The discretisation of (**??**) results in

$$\sum_j M_{ij} p_j^{\mathrm{m}} = \sum_j H_{ij} p_j^{\mathrm{d}} - \sum_j G_{ij} p_j'^{\mathrm{d}} \tag{21}$$

where

$$M_{ij} = \int_{S_0} N_i(\mathbf{x}) N_j(\mathbf{x}) \mathrm{d}S(\mathbf{x}) \tag{22}$$

$$H_{ij} = \int_{S_0} \int_{S_0} N_i(\mathbf{x}) N_j(\mathbf{y}) g'_{n_y}(\mathbf{y}, \mathbf{x}) \mathrm{d}S(\mathbf{y}) \mathrm{d}S(\mathbf{x}) \tag{23}$$

$$G_{ij} = \int_{S_0} \int_{S_0} N_i(\mathbf{x}) N_j(\mathbf{y}) g(\mathbf{y}, \mathbf{x}) \mathrm{d}S(\mathbf{y}) \mathrm{d}S(\mathbf{x}) \tag{24}$$

and similarly, the discretisation of (**??**) results in

$$\sum_j M_{ij} p_j'^{\mathrm{m}} = \sum_j E_{ij} p_j^{\mathrm{d}} - \sum_j H^*_{ij} p_j'^{\mathrm{d}} \tag{25}$$

where

$$E_{ij} = \int_{S_0} \int_{S_0} N_i(\mathbf{x}) N_j(\mathbf{y}) g''_{n_y n_x}(\mathbf{y}, \mathbf{x}) \mathrm{d}S(\mathbf{y}) \mathrm{d}S(\mathbf{x}) \tag{26}$$

$$H^*_{ij} = \int_{S_0} \int_{S_0} N_i(\mathbf{x}) N_j(\mathbf{y}) g'_{n_x}(\mathbf{y}, \mathbf{x}) \mathrm{d}S(\mathbf{y}) \mathrm{d}S(\mathbf{x}) \tag{27}$$

3.1 Simple subcases

3.1.1 Neumann radiation from rigid surface

Rigid surface: $p'^{\mathrm{m}}_{n_x}(\mathbf{x}) = p'_{n_x}(\mathbf{x})$ prescribed, $p'^{\mathrm{d}}_{n_x}(\mathbf{y}) = 0$

1. Solve

$$p'_{n_x}(\mathbf{x}) = \int_{S^+} p^{\mathrm{d}}(\mathbf{y}) g''_{n_y n_x}(\mathbf{y}, \mathbf{x}) \mathrm{d}S(\mathbf{y}) \tag{28}$$

   by forcing $p^{\mathrm{d}}(\mathbf{y}) = 0$ on the boundary of S^+ in order to obtain $p^{\mathrm{d}}(\mathbf{y})$

2. evaluate

$$p^{\mathrm{m}}(\mathbf{x}) = \int_{S^+} p^{\mathrm{d}}(\mathbf{y}) g'_{n_y}(\mathbf{y}, \mathbf{x}) \mathrm{d}S(\mathbf{y}) \tag{29}$$

   to obtain $p^{\mathrm{m}}(\mathbf{x})$ on the surface, and compute p^+ and p^- from p^{d} and p^{m}

3. evaluate

$$p(\mathbf{x}) = \int_{S^+} p^{\mathrm{d}}(\mathbf{y}) g'_{n}(\mathbf{y}, \mathbf{x}) \mathrm{d}S(\mathbf{y}) \tag{30}$$

   to obtain the radiated pressure

3.1.2 Dirichlet radiation from flexible surface

Flexible surface: $p^{\mathrm{m}}(\mathbf{y}) = p(\mathbf{y})$ prescribed on the surface, $p^{\mathrm{d}}_{n_x} = 0$

1. solve

$$p(\mathbf{x}) = -\int_{S^+} g(\mathbf{y}, \mathbf{x}) p'^{\mathrm{d}}_{n_y}(\mathbf{y}) \mathrm{d}S(\mathbf{y}) \tag{31}$$

   to obtain $p'^{\mathrm{d}}_{n_y}(\mathbf{y})$ on the surface.

2. evaluate

$$p'^{\mathrm{m}}_{n_x}(\mathbf{x}) = -\int_{S^+} g'_{n_x}(\mathbf{y}, \mathbf{x}) p'^{\mathrm{d}}_{n_y}(\mathbf{y}) \mathrm{d}S(\mathbf{y}) \tag{32}$$

   to obtain $p'^{\mathrm{m}}_{n_x}(\mathbf{x})$ and compute p'^+ and p'^- from p'^{d} and p'^{m}

3. evaluate

$$p(\mathbf{x}) = -\int_{S^+} g(\mathbf{y}, \mathbf{x}) p'^{\mathrm{d}}_{n}(\mathbf{y}) \mathrm{d}S(\mathbf{y}) \tag{33}$$

   to obtain the external pressure.