$$g(\mathbf{x},\mathbf{y}) = \frac{\mathrm{e}^{-\mathrm{i}kr}}{4\pi r}, \qquad r = |\mathbf{x}-\mathbf{y}| \tag{1}$$

$$\frac{\partial^2 g(\mathbf{x},\mathbf{y})}{\partial n_{\mathbf{y}} \partial n_{\mathbf{x}}} = \nabla_{\mathbf{y}} \left(\nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y}) \mathbf{n}_{\mathbf{x}} \right) \mathbf{n}_{\mathbf{y}} \tag{2}$$

from Fischer and Matlab we know that:

$$\nabla_{\mathbf{y}} \left(\nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y}) \mathbf{n}_{\mathbf{x}} \right) \mathbf{n}_{\mathbf{y}} = (\mathbf{n}_x \mathbf{n}_y)(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{y}} g(\mathbf{x},\mathbf{y})) - \mathbf{n}_y \cdot \left[\nabla_{\mathbf{y}} \times (\mathbf{n}_x \times \nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y})) \right] \tag{3}$$

Since $\nabla_{\mathbf{x}} g = -\nabla_{\mathbf{y}} g$, the first term simplifies to $-\nabla^2_{\mathbf{x}} g = k^2 g$:

$$\nabla_{\mathbf{y}} \left(\nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y}) \mathbf{n}_{\mathbf{x}} \right) \mathbf{n}_{\mathbf{y}} = k^2 (\mathbf{n}_x \mathbf{n}_y) g(\mathbf{x},\mathbf{y}) - \mathbf{n}_y \cdot \left[\nabla_{\mathbf{y}} \times (\mathbf{n}_x \times \nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y})) \right] \tag{4}$$

$$\begin{aligned} &\int_{S_x} p(\mathbf{x}) \int_{S_y} q(\mathbf{y}) \frac{\partial^2 g(\mathbf{x},\mathbf{y})}{\partial n_{\mathbf{y}} \partial n_{\mathbf{x}}} \mathrm{d}S_y \mathrm{d}S_x \\ &\qquad = k^2 \int_{S_x} p(\mathbf{x}) \int_{S_y} q(\mathbf{y}) (\mathbf{n}_x \mathbf{n}_y) g(\mathbf{x},\mathbf{y}) \mathrm{d}S_y \mathrm{d}S_x \\ &\qquad - \int_{S_x} p(\mathbf{x}) \int_{S_y} q(\mathbf{y}) \mathbf{n}_y \cdot \left[\nabla_{\mathbf{y}} \times (\mathbf{n}_x \times \nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y})) \right] \mathrm{d}S_y \mathrm{d}S_x \end{aligned} \tag{5}$$

The first term on the rhs is weakly singular and can be evaluated easily. The internal integral over S_y in the second term can be transformed using the identity $\nabla \times (\alpha \mathbf{v}) = \nabla \alpha \times \mathbf{v} + \alpha \nabla \times \mathbf{v}$ and Stoke's theorem as

$$\begin{aligned} &\int_{S_y} q(\mathbf{y}) \mathbf{n}_y \cdot \left[\nabla_{\mathbf{x}} \times (\mathbf{n}_x \times \nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y})) \right] \mathrm{d}S_y \\ &\qquad = \int_{S_y} q(\mathbf{y}) \left[\nabla_{\mathbf{y}} \times (\mathbf{n}_x \times \nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y})) \right] \mathrm{d}\mathbf{S}_y \\ &= \int_{S_y} \nabla_{\mathbf{y}} \times \left[q(\mathbf{y}) \left(\mathbf{n}_x \times \nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y}) \right) \right] \mathrm{d}\mathbf{S}_y - \int_{S_y} \nabla_{\mathbf{y}} q(\mathbf{y}) \times \left(\mathbf{n}_x \times \nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y}) \right) \mathrm{d}\mathbf{S}_y \\ &= \int_{\partial S_y} \mathbf{n}_x \times \nabla_{\mathbf{x}} \left(g(\mathbf{x},\mathbf{y}) q(\mathbf{y}) \right) \mathrm{d}\mathbf{y} - \int_{S_y} \nabla_{\mathbf{y}} q(\mathbf{y}) \times \left(\mathbf{n}_x \times \nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y}) \right) \mathbf{n}_y \mathrm{d}S_y \\ &= \int_{\partial S_y} \nabla_{\mathbf{x}} \left(g(\mathbf{x},\mathbf{y}) q(\mathbf{y}) \right) \times \mathrm{d}\mathbf{y} \cdot \mathbf{n}_x - \int_{S_y} \left(\mathbf{n}_y \times \nabla_{\mathbf{y}} q(\mathbf{y}) \right) \cdot \left(\mathbf{n}_x \times \nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y}) \right) \mathrm{d}S_y \\ &= \nabla_{\mathbf{x}} \times \int_{\partial S_y} g(\mathbf{x},\mathbf{y}) q(\mathbf{y}) \mathrm{d}\mathbf{y} \cdot \mathbf{n}_x - \int_{S_y} \nabla_{\mathbf{x}} g(\mathbf{x},\mathbf{y}) \times \left(\mathbf{n}_y \times \nabla_{\mathbf{y}} q(\mathbf{y}) \right) \cdot \mathbf{n}_x \mathrm{d}S_y \\ &\qquad = \nabla_{\mathbf{x}} \times \left(\int_{\partial S_y} g(\mathbf{x},\mathbf{y}) q(\mathbf{y}) \mathrm{d}\mathbf{y} - \int_{S_y} g(\mathbf{x},\mathbf{y}) \left(\mathbf{n}_y \times \nabla_{\mathbf{y}} q(\mathbf{y}) \right) \mathrm{d}S_y \right) \cdot \mathbf{n}_x \end{aligned} \tag{6}$$

Introducing the result into the surface integral over S_x and aplpying Stokes' theorem results in

$$\begin{aligned}
&\int_{S_x} p(\mathbf{x})\nabla_{\mathbf{x}} \times \left(\int_{\partial S_y} g(\mathbf{x},\mathbf{y})q(\mathbf{y})\mathrm{d}\mathbf{y} - \int_{S_y} g(\mathbf{x},\mathbf{y}) \left(\mathbf{n}_y \times \nabla_{\mathbf{y}} q(\mathbf{y})\right) \mathrm{d}S_y \right) \cdot \mathbf{n}_x \mathrm{d}S_x \\
&= \int_{\partial S_x} p(\mathbf{x}) \left(\int_{\partial S_y} g(\mathbf{x},\mathbf{y})q(\mathbf{y})\mathrm{d}\mathbf{y} - \int_{S_y} g(\mathbf{x},\mathbf{y}) \left(\mathbf{n}_y \times \nabla_{\mathbf{y}} q(\mathbf{y})\right) \mathrm{d}S_y \right) \mathrm{d}\mathbf{x} \\
&- \int_{S_x} \nabla_{\mathbf{x}} p(\mathbf{x}) \times \left(\int_{\partial S_y} g(\mathbf{x},\mathbf{y})q(\mathbf{y})\mathrm{d}\mathbf{y} - \int_{S_y} g(\mathbf{x},\mathbf{y}) \left(\mathbf{n}_y \times \nabla_{\mathbf{y}} q(\mathbf{y})\right) \mathrm{d}S_y \right) \cdot \mathbf{n}_x \mathrm{d}S_x \\
&= \int_{\partial S_x} p(\mathbf{x}) \int_{\partial S_y} g(\mathbf{x},\mathbf{y})q(\mathbf{y})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \\
&- \int_{\partial S_x} p(\mathbf{x}) \int_{S_y} g(\mathbf{x},\mathbf{y}) \left(\mathbf{n}_y \times \nabla_{\mathbf{y}} q(\mathbf{y})\right) \mathrm{d}S_y \mathrm{d}\mathbf{x} \\
&- \int_{S_x} \left(\mathbf{n}_x \times \nabla_{\mathbf{x}} p(\mathbf{x})\right) \int_{\partial S_y} g(\mathbf{x},\mathbf{y})q(\mathbf{y})\mathrm{d}\mathbf{y}\mathrm{d}S_x \\
&+ \int_{S_x} \left(\mathbf{n}_x \times \nabla_{\mathbf{x}} p(\mathbf{x})\right) \int_{S_y} g(\mathbf{x},\mathbf{y}) \left(\mathbf{n}_y \times \nabla_{\mathbf{y}} q(\mathbf{y})\right) \mathrm{d}S_y \mathrm{d}S_x \qquad (7)
\end{aligned}$$

1 Computing $\nabla_{\mathbf{x}} N_i$

$$\begin{bmatrix} x'_\xi & y'_\xi & z'_\xi \\ x'_\eta & y'_\eta & z'_\eta \\ n_x & n_y & n_z \end{bmatrix} \begin{Bmatrix} N'_{i,x} \\ N'_{i,y} \\ N'_{i,z} \end{Bmatrix} = \begin{Bmatrix} N'_{i,\xi} \\ N'_{i,\eta} \\ 0 \end{Bmatrix} \tag{8}$$

or in compact form

$$\begin{bmatrix} \mathbf{r}'^{\mathrm{T}}_\xi \\ \mathbf{r}'^{\mathrm{T}}_\eta \\ \mathbf{n}^{\mathrm{T}} \end{bmatrix} \nabla_{\mathbf{x}} N_i = \begin{Bmatrix} \nabla_\xi N_i \\ 0 \end{Bmatrix} \tag{9}$$

where $\mathbf{n} = \mathbf{r}'_\xi \times \mathbf{r}'_\eta$ Inverting the matrix yields

$$\begin{aligned}
\nabla_{\mathbf{x}} N_i &= \frac{(\mathbf{r}'_\eta \times \mathbf{n})N'_{i,\xi} + (\mathbf{n} \times \mathbf{r}'_\xi)N'_{i,\eta}}{\|\mathbf{n}\|^2} && (10) \\
&= \frac{\mathbf{n} \times (N'_{i,\eta}\mathbf{r}'_\xi - N'_{i,\xi}\mathbf{r}'_\eta)}{\|\mathbf{n}\|^2} && (11) \\
&= \frac{\mathbf{n} \times \sum_j (N'_{i,\eta}L'_{j,\xi} - N'_{i,\xi}L'_{j,\eta})\mathbf{r}_j}{\|\mathbf{n}\|^2} && (12) \\
&= \frac{\mathbf{n} \times \sum_j Q_{ij}\mathbf{r}_j}{\|\mathbf{n}\|^2} && (13)
\end{aligned}$$

For the linear triangle element

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \tag{14}$$

For the linear quad element

$$\mathbf{Q} = \frac{1}{8}\begin{bmatrix} 0 & \eta - 1 & \xi - \eta & 1 - \xi \\ 1 - \eta & 0 & -\xi - 1 & \eta + \xi \\ \eta - \xi & \xi + 1 & 0 & -\eta - 1 \\ \xi - 1 & -\eta - \xi & \eta + 1 & 0 \end{bmatrix} \mathbf{Q}_0 = \frac{1}{8}\begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$
$$\mathbf{Q}_\xi = \frac{1}{8}\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \mathbf{Q}_\eta = \frac{1}{8}\begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \tag{15}$$