IB Mathematics (Analysis & Approaches) HL Exploration:

Extension of Euler's method of superposition to magic cubes of odd orders

Length: 30 pages excluding cover, contents and bibliography.

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Contents

1 Introduction

While my mathematical curiosity was developed from a young age, a significant portion of my fascination was with squares and perpendicular grids of cells. This included anything from memorising square numbers to finding the number of different optimal paths through a perpendicular lattice. Magic squares were part of that mix, but because my first contact with this mathematical construct was before learning to access the internet, I was only taught one method of constructing magic squares of odd orders. However, the opportunity of the Exploration reminded me to resume my past interest and look further, into different construction methods of magic squares as well as its extensions into higher dimensions. The 3-dimensional extension of the magic square is named the magic cube.

Moreover, I recently started learning the skill of Python programming, and although my ideas and hypotheses in this Exploration are purely mathematical, implementing them on a larger scale for experimentation has never been easier. This Exploration will document my attempt at applying Euler's method of constructing magic squares in the third dimension, which was not originally done by Euler.

2 Mathematical concepts and methods used

2.1: Modulo operation

The *modulo operation* is often confused with *congruence* in modular arithmetic, which takes the notation $a \equiv b \pmod{d}$, where the "mod" is written in parentheses *separate from the equation*. Congruence establishes that two integers a and b gives the same remainder when divided by d, while the modulo operation directly evaluates the remainder itself and the mod is written *inside the expression*. The context of this Exploration requires the latter to be used.

I believe this explicit disambiguation is necessary as any mathematically knowledgeable reader may assume the inclusion of modular arithmetic upon seeing "mod", but in fact numerical manipulation through modular arithmetic is not at all a significant part of this Exploration.

The modulo operation mod will be used in this Exploration to find the least positive residue[1]. This means for dividend p and divisor d undergoing division with remainder, the remainder is denoted as $p \ mod \ d$. Note that p can be any integer and d can be any *positive* integer, and $p \ mod \ d$ is simply *the integer between* 0 *and* $d - 1$ *(inclusive) that results from* p *added or subtracted by an integer multiple of* d. For example:

$20 \ mod \ 3 = 2,$	since $d = 3, 20 = 6d + 2$	where $0 \leq 2 \leq d - 1$
$(-19) mod \ 5 = 1,$	since $d = 5, -19 = -4d + 1$	where $0 \leq 1 \leq d - 1$

Derived from its definition the modulo operation has the following three properties,

If $0 \leq p \leq d - 1, p \ mod \ d = p.$ …**(2.1.1)**

$p \ mod \ d \equiv (p + dz) mod \ d$ where z can be any integer. …**(2.1.2)**

If $R = p \ mod \ d, p = R + dz$ where z can be any integer. **…(2.1.3)**

Another property of the operation which will be stated axiomatically is that given *any* d *consecutive* integers $C, C + 1, C + 2, \ldots, C + d - 1$, the *set of*[2] least positive residues when they are divided by d must be the *set of* integers from 0 to $d - 1$, the sum of the elements of which is $\frac{1}{2}d(d - 1)$. Since addition is commutative and associative, we can deduce the following identity:

$$\sum_{m=0}^{d-1} [(C + m) mod \ d] \equiv \frac{1}{2}d(d - 1) \quad \textbf{…(2.1.4)}$$

I am aware that a lengthy segment was spent explaining this operation without yet establishing its clear relevance, and this is justified by its frequent and crucial usage throughout the Exploration as one reads on.

[1] Least positive residue is largely synonymous with "remainder", the only difference being that the term "remainder" is often used with the assumption of positive dividend.

[2] "Set" emphasising an *unordered* group.

2.2: Property of number bases

This essay will neither be changing to different bases nor using subscript notation to denote a number other than base 10. However, it will make use of the fact that:

> Given $n, r \in \mathbf{Z}^+$, and an ordered group of integers $(C_1, C_2 ..., C_r)$ where $0 \leq C_p \leq n - 1$ for all p, there will be a *one-to-one relationship* between the group and the sum $C_1 n^{r-1} + C_2 n^{r-2} + C_3 n^{r-3} + ... + C_{r-1} n + C_r$. …**(2.2.1)**

This means for each ordered group described above there will be a *single and unique* sum specified above. This is stated as axiomatic because it is intuitive that without the use of operators or excessive zeroes, there is only one way to write each number in terms of ordered place values. The property will also be used later in the Exploration as a crucial part of proof.

3 The magic square and magic cube

3.1: Square and cubic arrangements

This essay will mainly deal with square and cubic arrangements of positions, which can be shown as a square grid with a position in each cell, or a stack of grids as high as their side lengths in units to represent a cube. The order of a square or cube[3] is its side length. Each position is assigned a numerical value. Descriptions are defined as such:

row:	A horizontal line of positions from the left side/face to the right side/face of a square/cube.
column:	A vertical line of positions down the whole square, or a straight line of positions from the back face to the front face of a cube.
pillar	A vertical line of positions from the top face to the bottom face of a cube.
main diagonal:	A straight line of positions with ends at opposing corners of a square.

[3] In the Exploration, "square" will be used interchangeably with "square arrangement (of positions)", and "cube" will be used interchangeably with "cubic arrangement (of positions)"

main space diagonal:	A straight line of positions connecting one vertex of a cube to the vertex furthest away from it, and passes through the spatial centre of the cube.

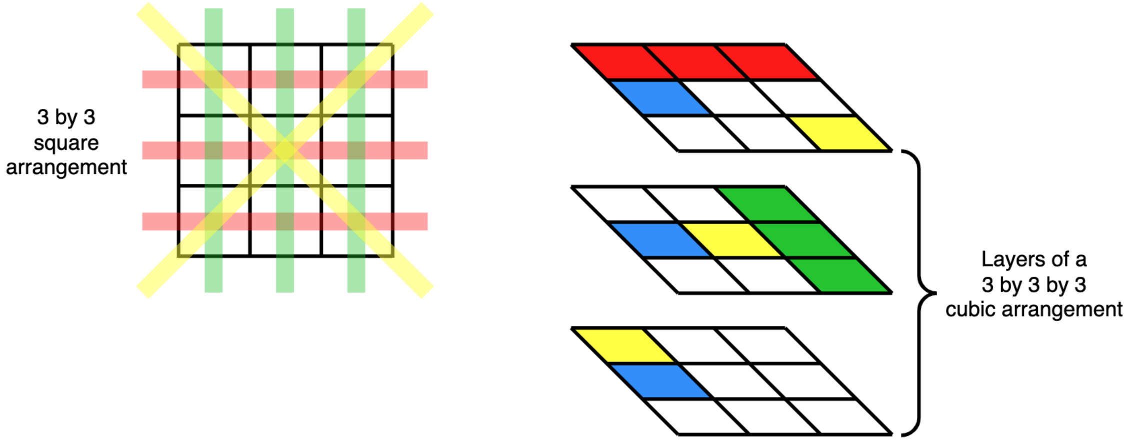


Fig. 1. *Visual depiction of a square arrangement (left) and cubic arrangement (right) of order 3. Rows are shown in red, columns in green, pillar in blue and main (space) diagonals in yellow. In the cube, only one of each is shown for neatness.*

3.2: The magic property

A square (arrangement) has the magic property when the sum of the values in each and every row, column and main diagonal are the same number (it is a common sum). A cube has the magic property when the sum of the values in each and every row, column, pillar and main space diagonal are the same number. This Exploration will define a "conventional magic square" of order n to be a square with the magic property, filled with integer values from 1 to n^2 without repetition. Similarly a "conventional magic cube" of order n will be a cube with the magic property, filled with integer values from 1 to n^3 without repetition.

8	1	6
3	5	7
4	9	2

Order 3

Common sum: 15

18	22	1	10	14
24	3	7	11	20
5	9	13	17	21
6	15	19	23	2
12	16	25	4	8

Order 5

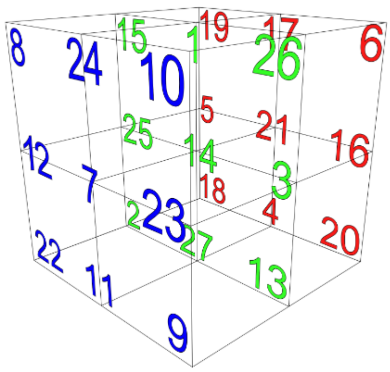
Common sum: 65

32	38	44	1	14	20	26
40	46	3	9	15	28	34
48	5	11	17	23	29	42
7	13	19	25	31	37	43
8	21	27	33	39	45	2
16	22	35	41	47	4	10
24	30	36	49	6	12	18

Order 7

Common sum: 175

Fig. 2. *Examples of possible conventional magic squares.*



8	24	10
12	7	23
22	11	9

15	1	26
25	14	3
2	27	13

19	17	6
5	21	16
18	4	20

Order 3. Common sum: 42

Fig. 3. *Depiction of a magic cube of order 3 created by Jaksmata (2008), In Wikipedia.*

Fig. 2 and Fig. 3 provide examples of conventional magic squares and a conventional magic cube. It is important to note that in the magic cube, the common sum does not apply to the main diagonals of its square cross-sections, only the main space diagonals. For example $8 + 14 + 20 = 42$.

3.3: Assumptions of the magic property

The following are intuitive facts about the magic property that derive from the associative and commutative properties of addition and summation. I will not prove these due to their simplicity:

1. The reflection of a square about its middle row or middle column retains the magic property.
2. The reflection of a cube about a middle plane parallel to two of its faces retains the magic property.
3. Multiplying an entire square or cube by a scale factor retains the magic property.
4. The superposition of two squares or cubes of the same order, which means summing the values in each of their corresponding positions to create a new square/cube of the same order, retains the magic property (associativity).

4 Euler's method of superposition

Note:

Out of all the methods of constructing conventional magic *squares*, I found Euler's method of superposition (Euler, 1849) to be the most intuitive and self-explanatory. It involves constructing two squares[4] with the magic property, except with repeating values, before superposing them to make a *conventional* magic square. This section will explain how Euler's method works visually and then create a mapping function for the magic square which inputs the position and outputs the value of the position. All mapping functions defined *from this point onward* only input and output integer values, and for brevity the $\in \mathbf{Z}$ will be omitted in the stated domain and range, allowing interval notation to be used.

[4] In Euler's paper they are named Greek and Latin squares.

This section will make many intuitive claims without rigorous proof, because I find it unnecessary given that the method is already published and recognised. I will instead utilise the limited space in the Exploration to rigorously prove my new work: the extension of this method to constructing ***magic cubes.***

4.1: Coordinate system

In order to input a position into a function, a method must be devised to refer to positions in a square. The simplest way to do this is through a discrete (integer) coordinate system.

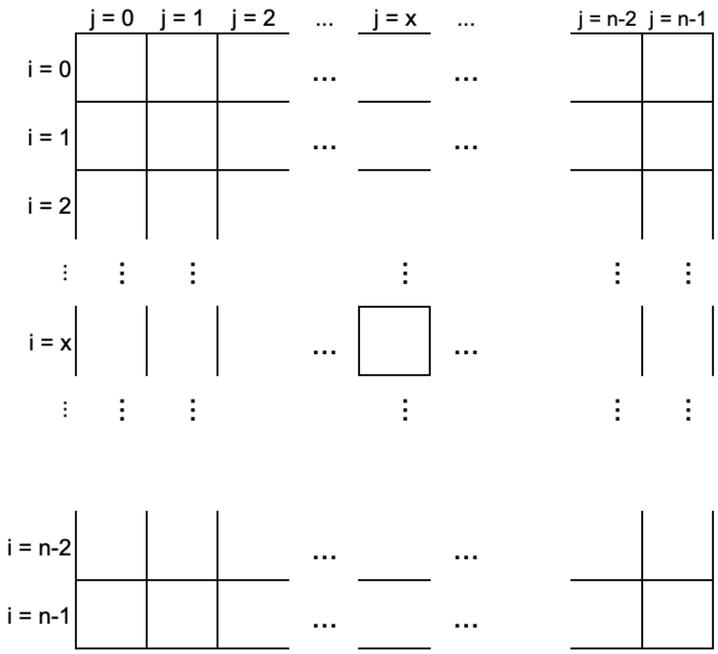


Fig. 4. *Coordinate system.*

As depicted in Fig. 4, the positions in a square of odd order n can be referenced through a vertical axis coordinate of $0 \leq i \leq n - 1$ to specify the row, and a horizontal axis coordinate of $0 \leq j \leq n - 1$ to specify the column. The coordinates are named and set according to Python conventions[5], so that implementing my mathematically-devised functions on all the positions by programming is made easier. The halfway point on the axes

[5] In Python, square arrangements are represented by nested lists. In a list, the first element has index 0, and the second has index 1, etc., hence beginning with 0. The lists can be made with for loops, which use letters *i* and *j* to keep track of the number of repetitions.

are significant because they refer to the centre of the square, hence we will specifically define it to be $x = \frac{n-1}{2}$ (the centre will be $i = x, j = x$, or position (x, x)), and due to the odd order n, x will be an integer. The variable x is completely tied to n in definition, and fixing n will mean fixing x. From here onwards, we can map a value to each position with any function $f(i, j)$ of the coordinates i and j.

4.2: U_n, a square of repeated values with the magic property

Consider the function $U_n(i, j) = (i + j - x) mod\, n$ with constrained domain $i \in [0, n - 1]$ and $j \in [0, n - 1]$. The $mod\, n$ gives U_n a range of $U_n(i, j) \in [0, n - 1]$. Note that when i or j is increased by 1, $U_n(i, j)$ increases by 1 unless the result lies outside of the range, in which case the modulo operation subtracts n from the result and the value becomes 0. It is this "cycling back" property that necessitates the modulo operation.

The properties of this function can be seen visually (Fig. 5) in that moving one position to the right *or* one position down would result in the value increasing by 1 or cycling to 0 due to the modulo operation. In addition to that, moving one position right *and* one position up (or one left and one down) will not change the value. Hence the repetition of values "slopes upward" as shown:

	j = 0	j = 1	j = 2
i = 0	2	0	1
i = 1	0	1	2
i = 2	1	2	0

n = 3, x = 1

	j = 0	j = 1	j = 2	j = 3	j = 4
i = 0	3	4	0	1	2
i = 1	4	0	1	2	3
i = 2	0	1	2	3	4
i = 3	1	2	3	4	0
i = 4	2	3	4	0	1

n = 5, x = 2

	j = 0	j = 1	j = 2	j = 3	j = 4	j = 5	j = 6
i = 0	4	5	6	0	1	2	3
i = 1	5	6	0	1	2	3	4
i = 2	6	0	1	2	3	4	5
i = 3	0	1	2	3	4	5	6
i = 4	1	2	3	4	5	6	0
i = 5	2	3	4	5	6	0	1
i = 6	3	4	5	6	0	1	2

n = 7, x = 3

Fig. 5. *$U_n(i, j)$ at $n = 3$, $n = 5$ and $n = 7$ respectively.*

Moreover, to find the value at the central position, we substitute $i = x$ and $j = x$ so we have

$U_n(x, x) = (x + x - x) mod\, n$

$= x\, mod\, n$

$U_n(x, x) = x$ [by **(2.1.1)** on page 4]

Note that x is the average of the set $\{0, 1, 2, 3, ..., n - 1\}$, and the central position has the same value as the whole upward-sloping main diagonal due to repetitions.

Visually I find it intuitive and trivial that the arrangement created by U_n has the magic property, because every row, every column, and the downward-sloping main diagonal contain the same set of values, while each position in the upward-sloping main diagonal contains the average of the set.

4.3: V_n, the reflection of U_n

Consider the function $V_n(i, j) = (i + n - 1 - j - x) mod\, n$ as shown in Fig. 6, with the same domain and range as U_n.

	j = 0	j = 1	j = 2
i = 0	1	0	2
i = 1	2	1	0
i = 2	0	2	1

n = 3, x = 1

	j = 0	j = 1	j = 2	j = 3	j = 4
i = 0	2	1	0	4	3
i = 1	3	2	1	0	4
i = 2	4	3	2	1	0
i = 3	0	4	3	2	1
i = 4	1	0	4	3	2

n = 5, x = 2

	j = 0	j = 1	j = 2	j = 3	j = 4	j = 5	j = 6
i = 0	3	2	1	0	6	5	4
i = 1	4	3	2	1	0	6	5
i = 2	5	4	3	2	1	0	6
i = 3	6	5	4	3	2	1	0
i = 4	0	6	5	4	3	2	1
i = 5	1	0	6	5	4	3	2
i = 6	2	1	0	6	5	4	3

n = 7, x = 3

Fig. 6. *$V_n(i, j)$ at $n = 3$, $n = 5$ and $n = 7$ respectively.*

Since $V_n(i, x + j) \equiv (i + n - 1 - (x + j) - x) mod\, n$

$\equiv (i + 2x - x - j - x) mod\, n$

$\equiv (i + (x - j) - x) mod\, n$

$\equiv U_n(i, x - j)$,

V_n is simply U_n reflected about the middle column $j = x$. This means the mapping function $V_n(i, j)$ also creates a square with the magic property. This time the repetition of values may be described as “slopes downward”.

4.4: Superposition of nU_n and V_n

First of all, scaling $U_n(i, j)$ by a factor of n we know that the mapping expression $n[U_n(i, j)]$ has the magic property.

Now superposing[6] $n \times U_n$ and V_n entails addition of the mapping functions, resulting in the mapping expression $n[U_n(i, j)] + V_n(i, j)$ with the magic property.

Note from Fig. 5 on page 10 and Fig. 6 on page 11 that since the repetitions of $U_n(i, j)$ and $V_n(i, j)$ slope perpendicularly to each other, each position (i, j) would have a unique combination of $(U_n(i, j), V_n(i, j))$ as it is the intersection of one upward-sloping line of numbers and one downward-sloping line of numbers. This means there is a one-to-one relationship between (i, j) and $(U_n(i, j), V_n(i, j))$. In my extension to magic cubes this concept is proven in detail through the solving of a system of equations. Since there is also a one-to-one relationship between $(U_n(i, j), V_n(i, j))$ and $n[U_n(i, j)] + V_n(i, j)$ by **(2.2.1)** on page 5, there is a one-to-one relationship between (i, j) and $n[U_n(i, j)] + V_n(i, j)$.

[6] See 3.3: Assumptions of the magic property (page 8) for definition of superposition.

To find the range of $n[U_n(i, j)] + V_n(i, j)$ we use the ranges of U_n and V_n, the minimum possible value is $0n + 0 = 0$ and the maximum possible value is $n(n - 1) + (n - 1) = n^2 - 1$. Hence $n[U_n(i, j)] + V_n(i, j) \in [0, n^2 - 1]$. Since there are exactly n^2 integers within the range, each integer value should be in a position inside the square arrangement without repetition. We now add 1 to the entire expression to achieve the range $[1, n^2]$ and the convention magic square of odd order n is made with the function:

$M_n(i, j) = 1 + n[U_n(i, j)] + V_n(i, j)$

$$M_n(i, j) = 1 + n[(i + j - x) mod\, n] + (i + n - 1 - j - x) mod\, n$$

	1	1	1		6	0	3		1	0	2		8	1	6
Order 3	1	1	1	+	0	3	6	+	2	1	0	=	3	5	7
	1	1	1		3	6	0		0	2	1		4	9	2

	1	1	1	1	1		15	20	0	5	10		2	1	0	4	3		18	22	1	10	14
	1	1	1	1	1		20	0	5	10	15		3	2	1	0	4		24	3	7	11	20
Order 5	1	1	1	1	1	+	0	5	10	15	20	+	4	3	2	1	0	=	5	9	13	17	21
	1	1	1	1	1		5	10	15	20	0		0	4	3	2	1		6	15	19	23	2
	1	1	1	1	1		10	15	20	0	5		1	0	4	3	2		12	16	25	4	8

	1	1	1	1	1	1	1		28	35	42	0	7	14	21		3	2	1	0	6	5	4		32	38	44	1	14	20	26
	1	1	1	1	1	1	1		35	42	0	7	14	21	28		4	3	2	1	0	6	5		40	46	3	9	15	28	34
	1	1	1	1	1	1	1		42	0	7	14	21	28	35		5	4	3	2	1	0	6		48	5	11	17	23	29	42
Order 7	1	1	1	1	1	1	1	+	0	7	14	21	28	35	42	+	6	5	4	3	2	1	0	=	7	13	19	25	31	37	43
	1	1	1	1	1	1	1		7	14	21	28	35	42	0		0	6	5	4	3	2	1		8	21	27	33	39	45	2
	1	1	1	1	1	1	1		14	21	28	35	42	0	7		1	0	6	5	4	3	2		16	22	35	41	47	4	10
	1	1	1	1	1	1	1		21	28	35	42	0	7	14		2	1	0	6	5	4	3		24	30	36	49	6	12	18

Fig. 7. *The construction of magic squares of orders* 3, 5 *and* 7 *using superposition as described by* M_n.

5 Extension to constructing magic cubes

5.1: Coordinate system

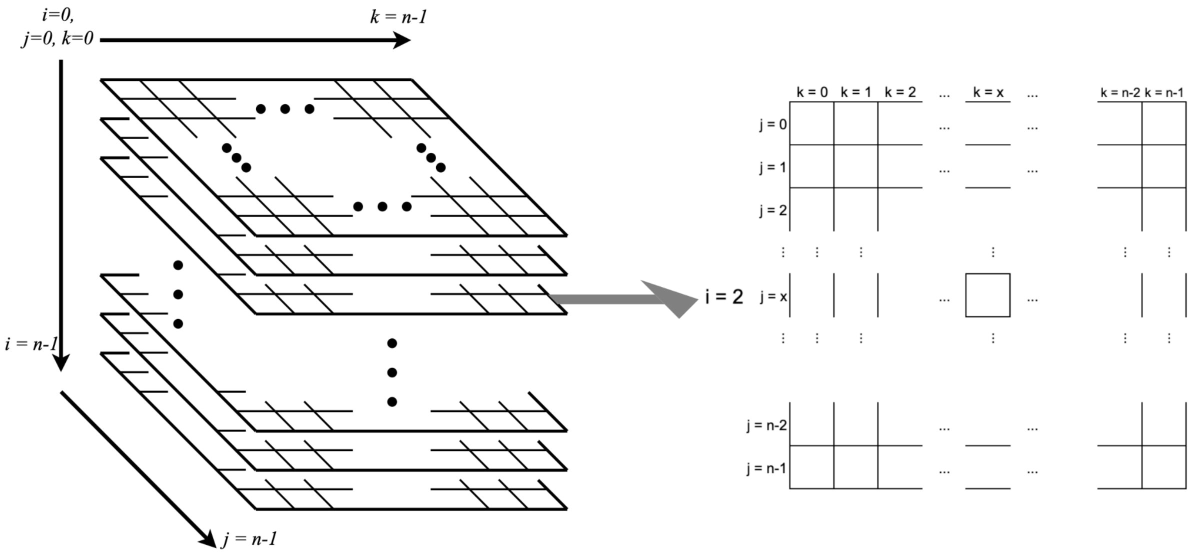


Fig. 8. *Depiction of coordinate system for magic cubes, showing example of a cross-section.*

The coordinate system used by a mapping function $f(i, j, k)$ to construct a cubic arrangement, is similar to that of a square. As mentioned before it is helpful to think of the cubic arrangement as n cross-sectional layers of squares of order n, where each layer has a different i-value as shown in Fig. 8. The axes i, j and k are defined as shown.

5.2: Proposed mapping function

The end product of this Exploration is the mapping function that inputs the position (i, j, k) of a cubic arrangement of order n and outputs the following:

$$M_n(i, j, k) = 1 + n^2[(i + j + k + 1) mod\, n]$$
$$+ n[(i + j - k) mod\, n]$$
$$+ (i - j + k) mod\, n, \text{ where } i, j, k \in [0, n - 1]$$

which I will rigorously prove to be a conventional magic cube. This is inspired by Euler's method as it is based on the superposition of three component cubes instead of two component squares.

5.3: Proof - Part I

Consider the mapping function $T_n(i, j, k) = (i + j + k + 1) mod\, n$ with domain constrained to $i \in [0, n - 1], j \in [0, n - 1], k \in [0, n - 1]$ and with range $T_n(i, j, k) \in [0, n - 1]$ due to the modulo operation. In this subsection I will prove that the arrangement mapped by T_n has the magic property.

Fig. 8 (page 14) specifies that a row of positions from left to right has varying k -values, hence any row can be located given fixed i and j values. The sum of any given row in terms of i and j in the arrangement mapped by T_n can hence be given by the expression $\sum_{k=0}^{n-1} T_n(i, j, k)$. We have:

$$\sum_{k=0}^{n-1} T_n(i, j, k) \equiv \sum_{k=0}^{n-1} [(i + j + k + 1) mod\, n] \quad \text{[by definition]}$$

$$\equiv \sum_{k=0}^{n-1} [(i + j + 1 + k) mod\, n]$$

Since only k is varied within the summation,

$C = i + j + 1, m = k, d = n$ can be substituted into **(2.1.4)** on page 4, which states $\sum_{m=0}^{d-1} [(C + m) mod\, d] \equiv \frac{1}{2}d(d - 1)$, hence

$$\sum_{k=0}^{n-1} [(i + j + 1 + k) mod\, n] \equiv \frac{1}{2}n(n - 1)$$

Since RHS is independent of i and j, it is now proven that:

The sum of *any* row of values from left to right can be evaluated by

$$\sum_{k=0}^{n-1} T_n(i, j, k) \equiv \frac{1}{2}n(n - 1). \quad \ldots\textbf{(5.3.1)}$$

Similarly, Fig. 8 (page 14) specifies that a column of positions from back to front has varying j-values, hence any column can be located given fixed i and k values. The sum of any given column in terms of i and k in the arrangement mapped by T_n can hence be given by the expression:

$$\sum_{j=0}^{n-1} T_n(i,j,k) \equiv \sum_{j=0}^{n-1} [(i + j + k + 1) mod\, n] \quad \text{[by definition]}$$

$$\equiv \sum_{j=0}^{n-1} [(i + k + 1 + j) mod\, n]$$

Since only j is varied within the summation,

$C = i + k + 1, m = j, d = n$ can be substituted into **(2.1.4)** on page 4, which states

$\sum_{m=0}^{d-1} [(C + m) mod\, d] \equiv \frac{1}{2}d(d - 1)$, hence

$$\sum_{j=0}^{n-1} [(i + k + 1 + j) mod\, n] \equiv \frac{1}{2}n(n - 1)$$

Since RHS is independent of i and k, it is now proven that:

The sum of *any* column of values from back to front can be evaluated by

$$\sum_{j=0}^{n-1} T_n(i,j,k) \equiv \frac{1}{2}n(n - 1). \quad \ldots\textbf{(5.3.2)}$$

Lastly, Fig. 8 (page 14) specifies that a pillar of positions from top to bottom has varying i-values, hence any pillar can be located given fixed j and k values. The sum of any given pillar in terms of j and k in the arrangement mapped by T_n can hence be given by the expression:

$$\sum_{i=0}^{n-1} T_n(i,j,k) \equiv \sum_{i=0}^{n-1} [(i + j + k + 1) mod\, n] \quad \text{[by definition]}$$

$$\equiv \sum_{i=0}^{n-1} [(j + k + 1 + i) mod\, n]$$

Since only i is varied within the summation,

$C = j + k + 1$, $m = i$, $d = n$ can be substituted into **(2.1.4)** on page 4, which states

$\sum_{m=0}^{d-1} [(C + m) mod\, d] \equiv \frac{1}{2} d(d - 1)$, hence

$\sum_{i=0}^{n-1} [(j + k + 1 + i) mod\, n] \equiv \frac{1}{2} n(n - 1)$

Since RHS is independent of j and k, it is now proven that:

> The sum of *any* pillar of values from back to front can be evaluated by
>
> $\sum_{i=0}^{n-1} T_n(i, j, k) \equiv \frac{1}{2} n(n - 1)$. …**(5.3.3)**

Having established the above, proving that each main space diagonal of T_n sums to

$\frac{1}{2} n(n - 1)$ would be sufficient for the magic property.

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A cubic arrangement has four main space diagonals, and given the corner positions they connect, it is relatively easy to find their equations by finding the "gradients" $\frac{\Delta j}{\Delta i}$, $\frac{\Delta k}{\Delta i}$ and $\frac{\Delta k}{\Delta j}$ then substituting points.

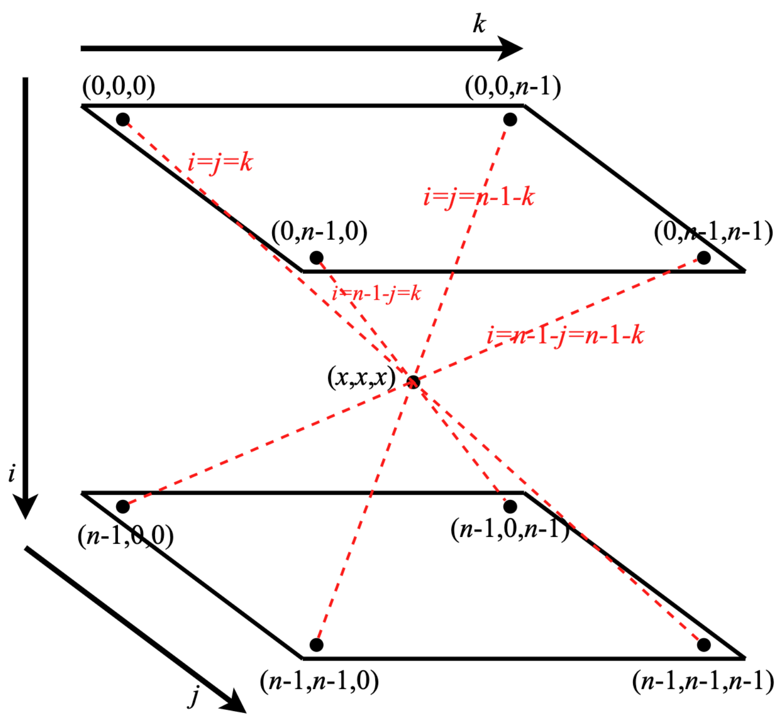

Fig. 9. *Illustration of main space diagonals, the positions they connect, and their equations.*

After doing so (which is not shown because the process is simplistic and only involve "gradients" of 1 and -1), we have as illustrated in Fig. 9:

$i = j = k$ connecting positions $(0, 0, 0)$ and $(n-1, n-1, n-1)$,

$i = j = n - 1 - k$ connecting $(0, 0, n-1)$ and $(n-1, n-1, 0)$,

$i = n - 1 - j = k$ connecting $(0, n-1, 0)$ and $(n-1, 0, n-1)$,

$i = n - 1 - j = n - 1 - k$ connecting $(0, n-1, n-1)$ and $(n-1, 0, 0)$.

Since $n - 1 = 2x$, the equations of the diagonals are rewritten as such:

$i = j = k, \quad i = j = 2x - k, \quad i = 2x - j = k, \quad i = 2x - j = 2x - k.$

Regarding the diagonal $i = j = 2x - k$:

since $i = 2x - k \Leftrightarrow k = 2x - i$, and $i = j$, the sum of values on this diagonal in the arrangement mapped by T_n will be $\sum_{i=0}^{n-1} T_n(i, i, 2x - i)$. Hence we have:

$$\sum_{i=0}^{n-1} T_n(i, i, 2x - i) \equiv \sum_{i=0}^{n-1} [(i + i + 2x - i + 1) mod\, n]$$

$$\equiv \sum_{i=0}^{n-1} [(i + 2x + 1) mod\, n]$$

$$\equiv \sum_{i=0}^{n-1} [(i + n) mod\, n]$$

$$\equiv \sum_{i=0}^{n-1} (i\, mod\, n) \quad \text{[by } \mathbf{(2.1.2)} \text{ on page 4]}$$

$$\equiv \sum_{i=0}^{n-1} i \quad \text{[by } \mathbf{(2.1.1)} \text{ on page 4, since } i \text{ is always less than } n\text{]}$$

$$\equiv \frac{1}{2}n(n - 1) \text{ as required.}$$

Regarding the diagonal $i = 2x - j = k$:

since $i = 2x - j \Leftrightarrow j = 2x - i$, and $i = k$, the sum of values on this diagonal in the arrangement mapped by T_n will be $\sum_{i=0}^{n-1} T_n(i, 2x - i, i)$. Hence we have:

$$\sum_{i=0}^{n-1} T_n(i, 2x - i, i) \equiv \sum_{i=0}^{n-1} [(i + 2x - i + i + 1) mod\, n]$$

$$\equiv \sum_{i=0}^{n-1} [(i + 2x + 1) mod\, n]$$

$$\equiv \frac{1}{2}n(n - 1) \quad \text{[skipped repeated steps above on the same page]}$$

as required.

Regarding the diagonal $i = 2x - j = 2x - k$:

since $2x - j = 2x - k \Leftrightarrow j = k$, and $i = 2x - j$, the sum of values on this diagonal in the arrangement mapped by T_n will be $\sum_{j=0}^{n-1} T_n(2x - j, j, j)$. Hence we have:

$$\sum_{j=0}^{n-1} T_n(2x - j, j, j) \equiv \sum_{j=0}^{n-1} [(2x - j + j + j + 1) mod\, n]$$

$$\equiv \sum_{j=0}^{n-1} [(j + 2x + 1) mod\, n]$$

$$\equiv \sum_{j=0}^{n-1} [(j + n) mod\, n]$$

$$\equiv \sum_{j=0}^{n-1} (j\ mod\, n) \quad \text{[by } \mathbf{(2.1.2)} \text{ on page 4]}$$

$$\equiv \sum_{j=0}^{n-1} j \quad \text{[by } \mathbf{(2.1.1)} \text{ on page 4, since } j \text{ is always less than } n\text{]}$$

$$\equiv \frac{1}{2}n(n - 1) \text{ as required.}$$

It is significantly more difficult to prove that given the mapping function T_n, the last remaining diagonal $i = j = k$ has values on it that sum to $\frac{1}{2}n(n - 1)$. We know that $i = j$ and $i = k$, so it can be established that the sum is the expression $\sum_{i=0}^{n-1} T_n(i, i, i)$. After this however, I had to break the proof into three cases and prove each case, all of which exhaustively proves for integer values of n. The cases are:

1. n is a multiple of 3, i.e. $n\ mod\ 3 = 0$
2. n is 1 more than a multiple of 3, i.e. $n\ mod\ 3 = 1$
3. n is 2 more than a multiple of 3, i.e. $n\ mod\ 3 = 2$

<u>Case 1</u>: $n\, mod\, 3 = 0$. We define integer w in terms of n as follows,

Let $w = \frac{n}{3} - 1 = \frac{n-3}{3}$, so that $n = 3w + 3$. We have:

$$\sum_{i=0}^{n-1} T_n(i,i,i) \equiv \sum_{i=0}^{n-1} [(i + i + i + 1) mod\, n]$$

$$\equiv \sum_{i=0}^{3w+2} [(3i + 1) mod\, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod\, n] + \sum_{i=w+1}^{2w+1} [(3i + 1) mod\, n] + \sum_{i=2w+2}^{3w+2} [(3i + 1) mod\, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod\, n] + \sum_{i=0}^{w} [(3(i + w + 1) + 1) mod\, n]$$

$$+ \sum_{i=0}^{w} [(3(i + 2w + 2) + 1) mod\, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod\, n] + \sum_{i=0}^{w} [(3i + 3w + 3 + 1) mod\, n]$$

$$+ \sum_{i=0}^{w} [(3i + 2(3w + 3) + 1) mod\, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod\, n] + \sum_{i=0}^{w} [(3i + n + 1) mod\, n] + \sum_{i=0}^{w} [(3i + 2n + 1) mod\, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod\, n] + \sum_{i=0}^{w} [(3i + 1) mod\, n] + \sum_{i=0}^{w} [(3i + 1) mod\, n]$$

[by **(2.1.2)** on page 4]

$$\equiv 3 \sum_{i=0}^{w} [(3i + 1) mod\, n]$$

$$\equiv 3 \sum_{i=0}^{w} (3i + 1)$$ [by **(2.1.1)** on page 4]

$$\equiv 3[(3 \sum_{i=0}^{w} i) + \sum_{i=0}^{w} 1]$$

$$\equiv 3[\frac{3}{2}w(w + 1) + w + 1]$$

$$\equiv 3(\frac{3}{2}w + 1)(w + 1)$$

$$\equiv 3(\frac{n-3}{2} + 1)(\frac{n}{3})$$

$$\equiv \frac{1}{2}n(n - 1)$$

Case 2: $n \, mod \, 3 = 1$. We define integer w in terms of n as follows,

Let $w = \frac{n-1}{3}$, so that $n = 3w + 1$. We have:

$$\sum_{i=0}^{n-1} T_n(i, i, i) \equiv \sum_{i=0}^{n-1} [(i + i + i + 1) mod \, n]$$

$$\equiv \sum_{i=0}^{3w} [(3i + 1) mod \, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod \, n] + \sum_{i=w+1}^{2w} [(3i + 1) mod \, n] + \sum_{i=2w+1}^{3w} [(3i + 1) mod \, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod \, n] + \sum_{i=0}^{w-1} [(3(i + w + 1) + 1) mod \, n]$$

$$+ \sum_{i=0}^{w-1} [(3(i + 2w + 1) + 1) mod \, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod \, n] + \sum_{i=0}^{w-1} [(3i + (3w + 1) + 2 + 1) mod \, n]$$

$$+ \sum_{i=0}^{w-1} [(3i + 2(3w + 1) + 1 + 1) mod \, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod \, n] + \sum_{i=0}^{w-1} [(3i + n + 2 + 1) mod \, n]$$

$$+ \sum_{i=0}^{w-1} [(3i + 2n + 1 + 1) mod \, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod\, n] + \sum_{i=0}^{w-1} [(3i + 2 + 1) mod\, n]$$

$$+ \sum_{i=0}^{w-1} [(3i + 1 + 1) mod\, n]$$ [by **(2.1.2)** on page 4]

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod\, n] + \sum_{i=0}^{w-1} [(3i + 3) mod\, n] + \sum_{i=0}^{w-1} [(3i + 2) mod\, n]$$

$$\equiv \sum_{i=0}^{w-1} [(3i + 1) mod\, n] + [(3w + 1) mod\, n]$$ [separating out the first sigma]

$$+ \sum_{i=0}^{w-1} [(3i + 3) mod\, n] + \sum_{i=0}^{w-1} [(3i + 2) mod\, n]$$

$$\equiv \sum_{i=0}^{w-1} (3i + 1) + [(3w + 1) mod\, n] + \sum_{i=0}^{w-1} (3i + 3) + \sum_{i=0}^{w-1} (3i + 2)$$

[by **(2.1.1)** on page 4]

$$\equiv \sum_{i=0}^{w-1} (3i + 1) + 0 + \sum_{i=0}^{w-1} (3i + 3) + \sum_{i=0}^{w-1} (3i + 2)$$ [since $3w + 1 = n$]

$$\equiv \sum_{i=0}^{w-1} (9i + 6)$$

$$\equiv 9 \sum_{i=0}^{w-1} i + \sum_{i=0}^{w-1} 6$$

$$\equiv \frac{9}{2} w(w - 1) + 6w$$

$$\equiv \frac{9}{2} (\frac{n-1}{3})(\frac{n-1}{3} - 1) + 6(\frac{n-1}{3})$$

$$\equiv \frac{1}{2} (n - 1)(n - 4) + 2(n - 1)$$

$$\equiv \frac{1}{2} n(n - 1)$$

<u>Case 3</u>: $n \, mod \, 3 = 2$. We define integer w in terms of n as follows,

Let $w = \frac{n-2}{3}$, so that $n = 3w + 2$. We have:

$$\sum_{i=0}^{n-1} T_n(i, i, i) \equiv \sum_{i=0}^{n-1} [(i + i + i + 1) mod \, n]$$

$$\equiv \sum_{i=0}^{3w+1} [(3i + 1) mod \, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod \, n] + \sum_{i=w+1}^{2w} [(3i + 1) mod \, n] + \sum_{i=2w+1}^{3w+1} [(3i + 1) mod \, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod \, n] + \sum_{i=0}^{w-1} [(3(i + w + 1) + 1) mod \, n]$$

$$+ \sum_{i=0}^{w} [(3(i + 2w + 1) + 1) mod \, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod \, n] + \sum_{i=0}^{w-1} [(3i + (3w + 2) + 1 + 1) mod \, n]$$

$$+ \sum_{i=0}^{w} [(3i + 2(3w + 2) - 1 + 1) mod \, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod \, n] + \sum_{i=0}^{w-1} [(3i + n + 2) mod \, n]$$

$$+ \sum_{i=0}^{w} [(3i + 2n) mod \, n]$$

$$\equiv \sum_{i=0}^{w} [(3i + 1) mod \, n] + \sum_{i=0}^{w-1} [(3i + 2) mod \, n]$$

$$+ \sum_{i=0}^{w} [(3i) mod \, n] \quad \text{[by } \textbf{(2.1.2)} \text{ on page 4]}$$

$$\equiv \sum_{i=0}^{w} (3i + 1) + \sum_{i=0}^{w-1} (3i + 2) + \sum_{i=0}^{w} (3i) \quad \text{[by } \textbf{(2.1.1)} \text{ on page 4]}$$

$$\equiv \sum_{i=0}^{w} (3i + 1) + \sum_{i=0}^{w} (3i + 2) - (3w + 2) + \sum_{i=0}^{w} (3i)$$

$$\equiv \sum_{i=0}^{w}(9i + 3) - n \quad \text{[since } 3w + 2 = n\text{]}$$

$$\equiv 9\sum_{i=0}^{w} i + \sum_{i=0}^{w} 3 - n$$

$$\equiv \frac{9}{2}w(w + 1) + 3(w + 1) - n$$

$$\equiv \frac{9}{2}(\frac{n-2}{3})(\frac{n-2}{3} + 1) + 3(\frac{n-2}{3} + 1) - n$$

$$\equiv \frac{1}{2}(n - 2)(n + 1) + (n + 1) - n$$

$$\equiv \frac{1}{2}(n^2 - n - 2 + 2n + 2 - 2n)$$

$$\equiv \frac{1}{2}n(n - 1)$$

All three cases proved as required.

Therefore it is established that:

> The sum of values on any main space diagonal of the arrangement mapped by T_n evaluate to $\frac{1}{2}n(n - 1)$. …**(5.3.4)**

By **(5.3.1)**, **(5.3.2)**, **(5.3.3)** and **(5.3.4)** on pages 15, 16, 17 and 25 respectively, it has been proven that the cube mapped by $T_n(i, j, k)$ has the magic property.

5.4: Proof - Part II

Consider the mapping function $U_n(i, j, k) = (i + j - k) mod\, n$ with the same domain and range as T_n, we can see that:

$U_n(i, j, x + k) \equiv (i + j - (x + k)) mod\, n$

$\equiv (i + j - x - k) mod\, n$

$\equiv (i + j + (x - k) - 2x) mod\, n$

$\equiv (i + j + (x - k) + 1) mod\, n$ [by **(2.1.2)** on page 4]

$\equiv T_n(i, j, x - k)$

$U_n(i, j, x + k) \equiv T_n(i, j, x - k)$

This means that the cube mapped by $U_n(i, j, k)$ is simply the cube mapped by $T_n(i, j, k)$ reflected about the middle plane $k = x$, and hence retains the magic property.

Similarly, the mapping function $V_n(i, j, k) = (i - j + k) mod\, n$ with the same domain and range as T_n allows the fact that:

$V_n(i, x + j, k) \equiv (i - (x + j) + k) mod\, n$

$\equiv (i - x - j + k) mod\, n$

$\equiv (i + (x - j) + k - 2x) mod\, n$

$\equiv (i + (x - j) + k + 1) mod\, n$ [by **(2.1.2)** on page 4]

$\equiv T_n(i, x - j, k)$

$V_n(i, x + j, k) \equiv T_n(i, x - j, k)$

This means that the cube mapped by $V_n(i, j, k)$ is simply the cube mapped by $T_n(i, j, k)$ reflected about the plane $j = x$, and hence retains the magic property.

Since the magic property is also associative and retained after multiplication of the whole cubic arrangement by a scalar. We know that the mapping expression $n^2[T_n(i, j, k)] + n[U_n(i, j, k)] + V_n(i, j, k)$ has the magic property.

5.5: Proof - Part III

The final segment is required to prove that there is a one-to-one relationship between the position (i, j, k) and the expression $n^2[T_n(i, j, k)] + n[U_n(i, j, k)] + V_n(i, j, k)$. Note that since the ranges of T_n, U_n and V_n are all $[0, n - 1]$, **(2.2.1)** from page 5 will be applicable.

Hence:

> There is a one-to-one relationship between the expression $n^2[T_n(i, j, k)] + n[U_n(i, j, k)] + V_n(i, j, k)$ and the ordered triplet $(T_n(i, j, k), U_n(i, j, k), V_n(i, j, k))$. …**(5.5.1)**

Now it will suffice to prove that there is a one-to-one relationship between the position (i, j, k) and the ordered triplet $(T_n(i, j, k), U_n(i, j, k), V_n(i, j, k))$, which I will abbreviate by omitting the function parameters, as (T_n, U_n, V_n).

From the definitions of the mapping functions T_n, U_n and V_n we have the equations below, which already show that a single (i, j, k) will only evaluate a single (T_n, U_n, V_n). To also show the converse would prove the one-to-one relationship, and to do that it is sufficient to prove that solving these equations as a system would only lead to one single solution.

$$T_n = (i + j + k + 1) mod\, n \quad \ldots①$$

$$U_n = (i + j - k) mod\, n \quad \ldots②$$

$$V_n = (i - j + k) mod\, n \quad \ldots③$$

First we rearrange all the equations according to **(2.1.3)** on page 4.

Rearranging ①: $i + j + k + 1 = T_n + nz_1 \quad \ldots④$

Rearranging ②: $i + j - k = U_n + nz_2 \quad \ldots⑤$

Rearranging ③: $i - j + k = V_n + nz_3, \quad \ldots⑥$ where $z_1, z_2, z_3 \in \mathbf{Z}$

④ - ⑤: $2k + 1 = T_n - U_n + n(z_1 - z_2)$

$k = \frac{1}{2}[T_n - U_n - 1 + n(z_1 - z_2)]$ …⑦a

$k = \frac{1}{2}(T_n - U_n - 1) + \frac{1}{2}n(z_1 - z_2)$ …⑦b note $0 \leq k \leq n - 1, k \in \mathbf{Z}$

④ - ⑥: $2j + 1 = T_n - V_n + n(z_1 - z_3)$

$j = \frac{1}{2}[T_n - V_n - 1 + n(z_1 - z_3)]$ …⑧a

$j = \frac{1}{2}(T_n - V_n - 1) + \frac{1}{2}n(z_1 - z_3)$ …⑧b note $0 \leq j \leq n - 1, j \in \mathbf{Z}$

⑤ + ⑥: $2i = U_n + V_n + n(z_2 + z_3)$

$i = \frac{1}{2}[U_n + V_n + n(z_2 + z_3)]$ …⑨a

$i = \frac{1}{2}(U_n + V_n) + \frac{1}{2}n(z_2 + z_3)$ …⑨b note $0 \leq i \leq n - 1, i \in \mathbf{Z}$

Consider equations ⑦, ⑧, ⑨ which contain the solution $(i, j. k)$ in terms of T_n, U_n, and V_n. To prove that the solution is singular, it is now sufficient to prove that given T_n, U_n, and V_n are already known, the terms $z_1 - z_2$, $z_1 - z_3$, and $z_2 + z_3$ can each only take one single value.

We will first make use of the constraint that $i, j, k \in \mathbf{Z}$. The $\frac{1}{2}$ in front of the square brackets in each of ⑦a, ⑧a, ⑨a limits the quantity inside the square brackets to be an even number. Since all terms are integers, this constrains the terms $n(z_1 - z_2)$, $n(z_1 - z_3)$ and $n(z_2 + z_3)$ each to a single parity (i.e. constrained to only odd numbers or only even numbers). Since n is odd, the terms $z_1 - z_2$, $z_1 - z_3$, and $z_2 + z_3$ are also each constrained to a single parity[7].

[7] Since (odd)(odd) = odd, (odd)(even) = (even).

A variable constrained to a single parity has the property that the minimum difference between two distinct possible values of that variable is 2. This means that

the minimum difference between two distinct possible values of $z_1 - z_2$ is 2, and likewise for $z_1 - z_3$ and $z_2 + z_3$,

which implies that

- the minimum difference between the values of $\frac{1}{2}n(z_1 - z_2)$ given distinct values of $z_1 - z_2$ is n,
- the minimum difference between the values of $\frac{1}{2}n(z_1 - z_3)$ given distinct values of $z_1 - z_3$ is n,
- the minimum difference between the values of $\frac{1}{2}n(z_2 + z_3)$ given distinct values of $z_2 + z_3$ is n.

By equations ⑦b, ⑧b, ⑨b respectively, the above imply that:

- the minimum difference between the values of k given distinct values of $z_1 - z_2$ is n,
- the minimum difference between the values of j given distinct values of $z_1 - z_3$ is n,
- the minimum difference between the values of i given distinct values of $z_2 + z_3$ is n.

Since $i, j, k \in [0, n - 1]$, and the difference between the upper and lower bounds is $n - 1$, the statements above imply that

- only one value of $z_1 - z_2$ allows a valid value of k,
- only one value of $z_1 - z_3$ allows a valid value of j,
- only one value of $z_2 + z_3$ allows a valid value of i.

Hence, the system of equations ①, ② and ③ can only have one single solution regardless of the values of T_n, U_n, and V_n. We have proven that (i, j, k) is one-to-one with (T_n, U_n, V_n)

which is one-to-one with $n^2[T_n(i,j,k)] + n[U_n(i,j,k)] + V_n(i,j,k)$ by **(5.5.1)** on page 27.

This means:

The mapping expression $n^2[T_n(i,j,k)] + n[U_n(i,j,k)] + V_n(i,j,k)$ would give each position (i,j,k) in the cube a unique value. …**(5.5.2)**

5.6: Proof - Part IV

In this subsection I first find the upper and lower bounds of the mapping expression $n^2[T_n(i,j,k)] + n[U_n(i,j,k)] + V_n(i,j,k)$ by using the ranges of the mapping functions T_n, U_n and V_n.

Upper bound: $n^2(n-1) + n(n-1) + (n-1) \equiv (n-1)(n^2+n+1)$

$\equiv n^3 - 1$ [by difference of cubes]

Lower bound: $n^2(0) + n(0) + 0 \equiv 0$

We have $n^2[T_n(i,j,k)] + n[U_n(i,j,k)] + V_n(i,j,k) \in [0, n^3-1]$, which includes n^3 distinct integers. Since there are also n^3 positions in the cube, the one-to-one relationship proven earlier [**(5.5.2)**] shows that every distinct value in $[0, n^3-1]$ is assigned to a position without repetition. Since $n^2[T_n(i,j,k)] + n[U_n(i,j,k)] + V_n(i,j,k)$ has the magic property, it maps almost a conventional magic cube, with the slight distinction of range that can be fixed by adding 1 to every position so that the range is $[1, n^3]$. This involves adding 1 to the mapping expression, becoming the final formula:

$$M_n(i, j, k) = 1 + n^2[T_n(i, j, k)] + n[U_n(i, j, k)] + V_n(i, j, k)$$

$$\equiv 1 + n^2[(i + j + k + 1) mod\, n]$$

$$+ n[(i + j - k) mod\, n]$$

$$+ (i - j + k) mod\, n$$

End of proof.

6 Conclusion

In summary, I believe I have successfully achieved the aim and focus of this Exploration with the establishment of my final mapping function to extend Euler's method of superposition to the third dimension. My mathematical work heavily utilises the topics "Number and algebra" and "Functions" in its algebraic manipulation, as well as including content outside of the course but at a similar level of complexity in my opinion. To the best of my knowledge my extension work is original.

Upon testing with Python programming, my final mapping function was able to construct magic cubes and be verified up to order 41 before encountering display limitations. This I see as supporting the validity of my proof to an extent.

A key strength in my Exploration is the numerical labelling of important equations and statements for reference, which provides a framework to enhance the organisation of my points and proof. This was the result of numerous large edits and restructuring, in which I felt challenged and endeavoured to communicate my idea to the best of my ability. I am also quite pleased at the fact that the majority of my proof did not require as much textual explanation as mathematical expressions.

One exception to this is **5.5**, which I consider to be a weaker segment of the proof due to necessitating the heavy use of words, slightly compromising rigour. In addition to this, I am concerned that the reader may not completely be able to follow my intuition behind my claims in section **4**, and to make up for this was my main motivation for the lengthy 18-page proof. Lastly, the inconsistency of pronumeral fonts in the figures I provided from the equations is due to the different tools I used for the figures that did not support “insert equation”, though I do not believe it severely impacts communication.

Bibliography

For sources the provide recommended citations, they are used. Otherwise APA 7th referencing is used. All figures except the magic cube example are created by myself using Google Sheets and Diagrams.net

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