Calculus and Optimization for Machine Learning

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1 Week 1

1.1 Numerical sets and mappings

Set. A *set* (is commonly called non-definable and fundamental) is an entity, a *collection* of some objects:

- An object either belongs to the set or do not
- One object could be included in the set only one time
- There is *no order* (even if it is trivial) on objects of the set

Sets are usually denoted by capital letters: $X, Y, A, B, R, ...$ The fact of belonging to the set is denoted as: $a \in A$.
Another essential concept concerning sets is subsets. Basically, the subset is any set of (not necessarily all) elements of the given set. We consider primarily numerical sets:

- Natural numbers $\mathbb{N} = \{1, 2, 3, 4, ...\}$
- Integer numbers $\mathbb{Z} = \{0, 1, -1, 2, -2, ...\}$
- Rational numbers $\mathbb{Q} = \{\frac{m}{n} | m \in Z, n \in N\}$
- Real numbers $\mathbb{R} = \{a_0, a_1, ...\}$

Mapping. Assume that we have two sets X and Y. A mapping between them is, generally speaking, ordered realtion between elements of X and Y. Consider two logic notations: quantifiers "for all" $\forall$ and "exists" $\exists$.

Axiomatic definition of real numbers.

- $x + y = y + x, y \cdot x = x \cdot y$ - the commutative rule
- $(x + y) + z = x + (y + z), (x \cdot y) \cdot z = x \cdot (x \cdot z)$ - associativity
- $(x + y) \cdot z = x \cdot z + y \cdot z$ - distributivity
- Existence of two neutral elements 1 and 0: $a \cdot 1 = a, a + 0 = a$
- Existence of the inverse elements (exept 0): $a + (-a) = 0, a \cdot a^{-1} = 1$
- Non-triviality $0 \neq 1$
- For any two real numbers one is able to say $a > b, a < b$ or $a = b$
- This order is transitive: if $a < b, b < c$, then $a < c$
- Completeness: Assume the we consider some section of our set into two non-intersecting sets: $\mathbb{R} = A \cup B, A \cap B = \emptyset$, such as any element $a \in A$ is smaller than any element $b \in B$. Then there is a pivot real number $c \in \mathbb{R}$ that $a \leq c \leq b$ for any a and b elements from the corresponding sets.

Functions. Consider two sets X and Y and mapping $f : X \mapsto Y$. This mapping (relation) called *functional* (or a *function*) if and only it associates each element of the X set to exactly on element of the Y set. X set is called domain (set of arguments) and Y set is called codomain (set of values).
Function graph is a cetrain curve on the plane: the set of all points $(x, f(x))$ for all x belonging to the function's domain.
Domain - $D(f)$
Codomain - $E(f)$
Support - $supp(f) = \{x \in X : f(x) \neq 0\}$
Composite function or composition - $g(f(x))$
Vertical shift - $y = f(x) + c$
Horizontal shift - $y = f(x + C)$
Vertical contraction - $y = C \cdot f(x)$
Horizontal contraction - $y = f(C \cdot x)$
Absolute value - $y = |f(x)|$

1.2 Limits, sequences

Limit. Limit of the sequence - the real number that resembles our sequence the most as the element's number infinitely grows(approaches infinity). The notation:

$$\lim_{n\to\infty} a_n = C$$

$\forall\varepsilon > 0 : \exists N \in \mathbb{N}\ that\ \forall n \in \mathbb{N}, n \geq N \Rightarrow |a_n - C| < \varepsilon$
If a sequence has the limit equal to 0, it is called *infinitesimal.*