

# Chapter 1

## Linear Equations

# Section 1.1

## Systems of Linear Equations

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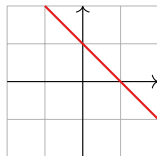
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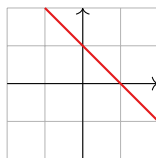


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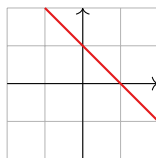
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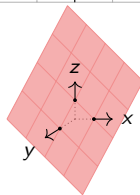
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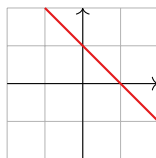


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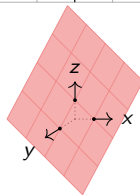
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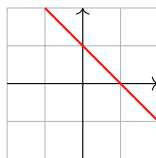


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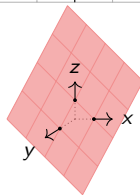
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~~~~~> a "3-plane" in "4-space"...

[not pictured here]

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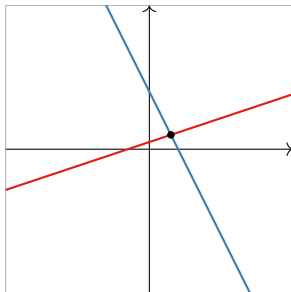
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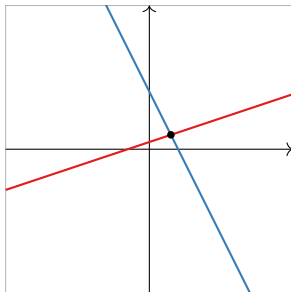


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In general it's an intersection of lines, planes, etc.

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In what other ways can two lines intersect?

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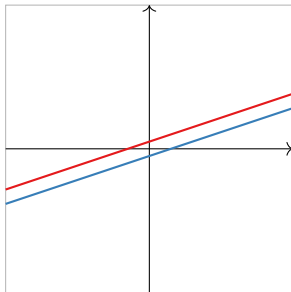
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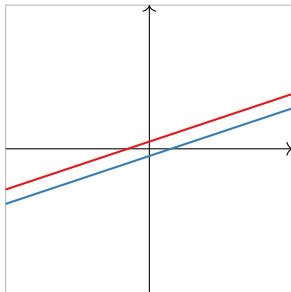


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A system of equations with no solutions is called **inconsistent**.

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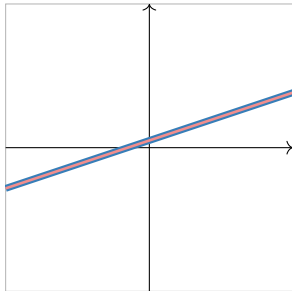
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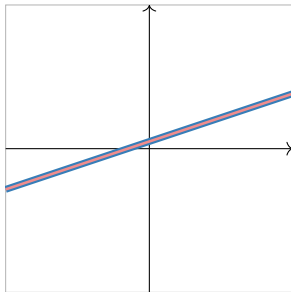
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$$\begin{aligned}x - 3y &= -3 \\ 2x - 6y &= -6\end{aligned}$$



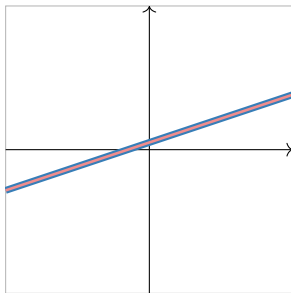
Note that multiplying an equation by a nonzero number gives the *same solution set*.

## Kinds of Solution Sets

In what other ways can two lines intersect?

$$x - 3y = -3$$

$$2x - 6y = -6$$



Note that multiplying an equation by a nonzero number gives the *same solution set*. In other words, they are *equivalent* (systems of) equations.

## Poll

What about in three variables?

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Poll

In how many different ways can three planes intersect in space?

- A. One
- B. Two
- C. Three
- D. Four
- E. Five
- F. Six
- G. Seven



# Solving Systems of Equations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

This is the kind of problem we'll talk about for the first half of the course.

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What is a *systematic* way to solve a system of equations?

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What strategies do you know?

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**Elimination method:** in what ways can you manipulate the equations?



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Now I've eliminated  $x$  from the last equation!

...but there's a long way to go still. Can we make our lives easier?

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- ▶ Multiply all entries in a row by a nonzero number. (scale)
- ▶ Add a multiple of each entry of one row to the corresponding entry in another. (row replacement)
- ▶ Swap two rows. (swap)

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**Strategy:** fiddle with it so we only have ones and zeros.

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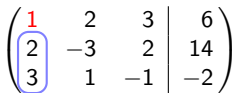
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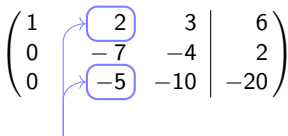
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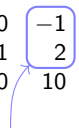
Continued

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$



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Success!

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Success!

Check:

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substitute solution  
~~~~~>

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So the linear equations of row-equivalent matrices have the *same solution set*.

## A Bad Example

### Example

Solve the system of equations

$$x + y = 2$$

$$3x + 4y = 5$$

$$4x + 5y = 9$$



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Now clear this by  
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$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array} \right)$$

# A Bad Example

Continued

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right) \begin{array}{l} \text{translates into} \\ \text{~~~~~}\rightarrow \end{array}$$

## Continued

translates into

In other words, the original equations

$$x + y = 2$$

$$3x + 4y = 5$$

$$4x + 5y = 9$$

have the same solutions as

$$x + y = 2$$

$$y = -1$$

$$0 = 2$$

## Continued

translates into

$$0 = 2$$

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.



## Continued

translates into

$$\begin{array}{rcl} x + y = 2 & & x + y = 2 \\ 3x + 4y = 5 & \text{have the same solutions as} & y = -1 \\ 4x + 5y = 9 & & 0 = 2 \end{array}$$

## Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.