Section 1.8

Introduction to Linear Transformations

Motivation

Let A be an $m \times n$ matrix. For the matrix equation $Ax = b$ we have learned to describe

- the solution set: all x in $\mathbf{R}^n$ making the equation true.
- the column span: the set of all b in $\mathbf{R}^m$ making the equation consistent.

It turns out these two sets are very closely related to each other.

In order to understand this relationship, it helps to think of the matrix A as a *transformation* from $\mathbf{R}^n$ to $\mathbf{R}^m$.

It's a special kind of transformation called a *linear transformation*.

This is also a way to understand the *geometry* of *matrices*.

Transformations

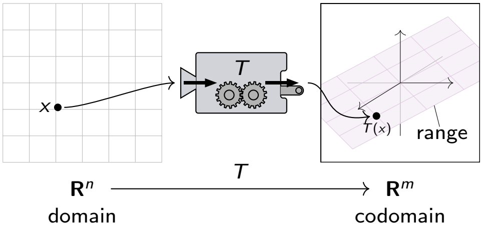
Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

- $\mathbf{R}^n$ is called the **domain** of T (the inputs).
- $\mathbf{R}^m$ is called the **codomain** of T (the outputs).
- For x in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the **image** of x under T.
  Notation: $x \mapsto T(x)$.
- The set of all images $\{T(x) \mid x \text{ in } \mathbf{R}^n\}$ is the **range** of T.

Notation:

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{means} \quad T \text{ is a transformation from } \mathbf{R}^n \text{ to } \mathbf{R}^m.$$



It may help to think of T as a "machine" that takes x as an input, and gives you $T(x)$ as the output.

Functions from Calculus

Many of the functions you know and love have domain and codomain $\mathbf{R}$.

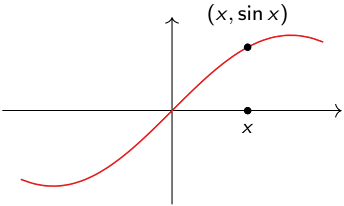
$$\sin\colon \mathbf{R} \longrightarrow \mathbf{R} \qquad \sin(x) = \begin{pmatrix} \text{the length of the opposite edge over the} \\ \text{hypotenuse of a right triangle with angle} \\ x \text{ in radians} \end{pmatrix}$$

Note how I've written down the *rule* that defines the function sin.

$$f\colon \mathbf{R} \longrightarrow \mathbf{R} \qquad f(x) = x^2$$

Note that "x^2" is sloppy (but common) notation for a function: it doesn't have a name!

You may be used to thinking of a function in terms of its graph.



The horizontal axis is the domain, and the vertical axis is the codomain.

This is fine when the domain and codomain are $\mathbf{R}$, but it's hard to do when they're $\mathbf{R}^2$ and $\mathbf{R}^3$! You need five dimensions to draw that graph.

Matrix Transformations

Most of the transformations we encounter in this class will come from (surprise) matrices!

Definition

Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \qquad \text{defined by} \qquad T(x) = Ax.$$

In other words, T takes the vector x in $\mathbf{R}^n$ to the vector Ax in $\mathbf{R}^m$.

Your life will be much easier if you just remember these.

- The *domain* of T is $\mathbf{R}^n$, which is the number of *columns* of A.
- The *codomain* of T is $\mathbf{R}^m$, which is the number of *rows* of A.
- The *range* of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is the *column span* of A. It is a span of vectors in the codomain.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T \colon \mathbf{R}^2 \to \mathbf{R}^3$.

- If $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ then $T(u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$.
- Let $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$. Find v in $\mathbf{R}^2$ such that $T(v) = b$. Is there more than one?

  We want to find v such that $T(v) = Av = b$. We know how to do that:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} v = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix} \xrightarrow[\text{matrix}]{\text{augmented}} \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array}\right) \xrightarrow[\text{reduce}]{\text{row}} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right).$$

  This gives $x = 2$ and $y = 5$, or $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ (unique). In other words,

$$T(v) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}.$$

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$.

- Is there any c in $\mathbf{R}^3$ such that there is more than one v in $\mathbf{R}^2$ with $T(v) = c$?

  Translation: is there any c in $\mathbf{R}^3$ such that the solution set of $Ax = c$ has more than one vector v in it?

  The solution set of $Ax = c$ is a translate of the solution set of $Ax = b$ (from before), which has one vector in it. So the solution set to $Ax = c$ has only one vector. So no!

- Find c such that there is *no* v with $T(v) = c$.

  Translation: Find c such that $Ax = c$ is inconsistent.

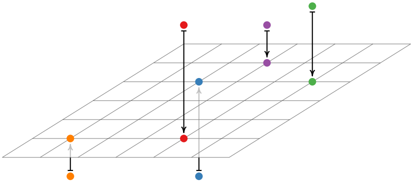
  Translation: Find c not in the column span of A (i.e., the range of T).

  We could draw a picture, or notice: $a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ b \\ a+b \end{pmatrix}$. So anything in the column span has the same first and last coordinate. So $c = \left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right)$ is not in the column span (for example).

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^3 \to \mathbf{R}^3$. Then

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

This is *projection onto the xy-axis*. Picture:



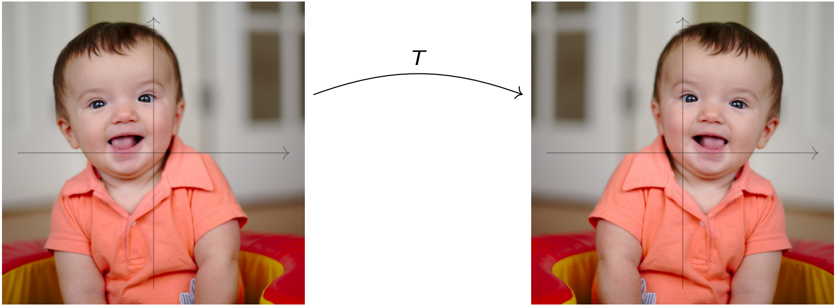
Matrix Transformations

Geometric example

Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T \colon \mathbf{R}^2 \to \mathbf{R}^2$. Then

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}.$$

This is *reflection over the y-axis*. Picture:



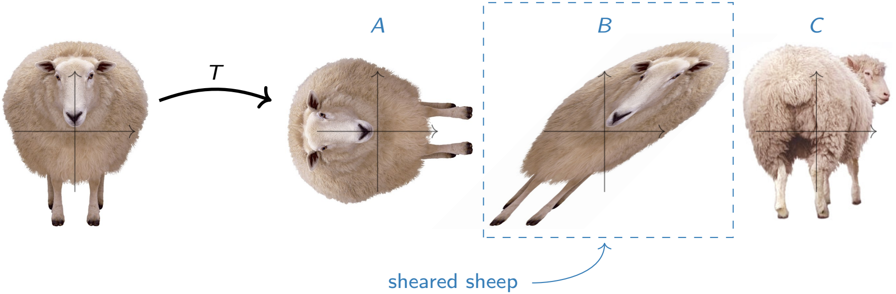
Poll

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^2$. ($T$ is called a **shear**.)

Poll

What does T do to this sheep?

Hint: first draw a picture what it does to the box *around* the sheep.



Linear Transformations

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

$$A(u + v) = Au + Av \qquad A(cv) = cAv.$$

So if $T(x) = Ax$ is a matrix transformation then,

$$T(u + v) = T(u) + T(v) \qquad T(cv) = cT(v).$$

This property is so special that it has its own name.

Definition

A transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the above equations for all vectors u, v in $\mathbf{R}^n$ and all scalars c.

In other words, T "respects" addition and scalar multiplication.

Check: if T is linear, then

$$T(0) = 0 \qquad T(cu + dv) = cT(u) + dT(v)$$

for all vectors u, v and scalars c, d. More generally,

$$T\left(c_1v_1 + c_2v_2 + \cdots + c_nv_n\right) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

In engineering this is called **superposition**.

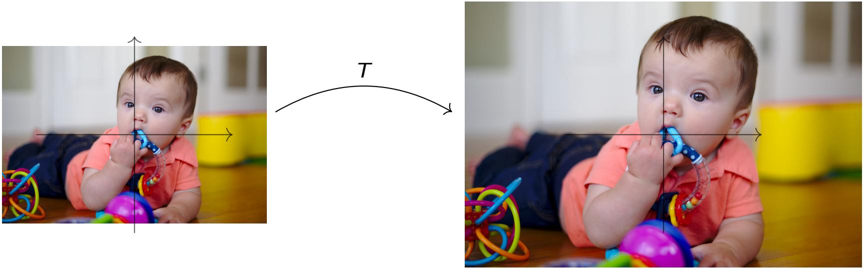
Linear Transformations
Dilation

Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. Is T linear? Check:

$$T(u+v) = 1.5(u+v) = 1.5u + 1.5v = T(u) + T(v)$$
$$T(cv) = 1.5(cv) = c(1.5v) = c(Tv).$$

So T satisfies the two equations, hence T is linear.

This is called **dilation** or **scaling** (by a factor of 1.5). Picture:



Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Is T linear? Check:

$$T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} + \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} -(u_2 + v_2) \\ (u_1 + v_1) \end{pmatrix} = T\begin{pmatrix} u_1 + u_2 \\ v_1 + v_2 \end{pmatrix}$$

$$T\left(c\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = T\begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix} = \begin{pmatrix} -cv_2 \\ cv_1 \end{pmatrix} = c\begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = cT\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

So T satisfies the two equations, hence T is linear. This is called **rotation** (by 90°). Picture:

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$