Section 1.9

The Matrix of a Linear Transformation

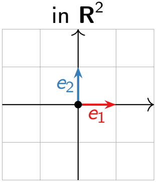
Unit Coordinate Vectors

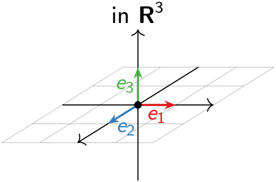
Definition

The **unit coordinate vectors** in $\mathbf{R}^n$ are

This is what $e_1, e_2, \ldots$ mean, *for the rest of the class.*

$$\boxed{e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.}$$





Note: if A is an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: multiplying a matrix by e_i gives you the ith column.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

Linear Transformations are Matrix Transformations

Recall: A matrix A defines a linear transformation T by $T(x) = Ax$.

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an $m \times n$ matrix, and T is the matrix transformation for A: $T(x) = Ax$.

The matrix A is called the **standard matrix** for T.

Take-Away

Linear transformations are the same as matrix transformations.

Dictionary

Linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$	$\rightsquigarrow$	$m \times n$ matrix $A = \begin{pmatrix} \vert & \vert & & \vert \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ \vert & \vert & & \vert \end{pmatrix}$
$T(x) = Ax$ $T\colon \mathbf{R}^n \to \mathbf{R}^m$	$\leftsquigarrow$	$m \times n$ matrix A

Linear Transformations are Matrix Transformations

Continued

Why is a linear transformation a matrix transformation?

Suppose for simplicity that $T\colon \mathbf{R}^3 \to \mathbf{R}^2$.

$$
\begin{aligned}
T\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= T\left(x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\
&= T\big(xe_1 + ye_2 + ze_3\big) \\
&= xT(e_1) + yT(e_2) + zT(e_3) \\
&= \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= A\begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\end{aligned}
$$

Before, we defined a **dilation** transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. What is its standard matrix?

$$\left.\begin{aligned} T(e_1) &= 1.5e_1 = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \\ T(e_2) &= 1.5e_2 = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5x \\ 1.5y \end{pmatrix} = 1.5\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix}.$$

Linear Transformations are Matrix Transformations

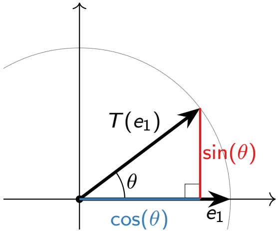
Example

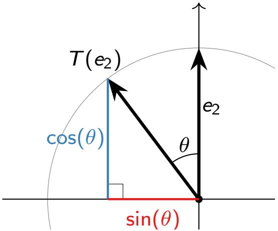
Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$T(x) = x \text{ rotated counterclockwise by an angle } \theta?$$

(Check linearity...)

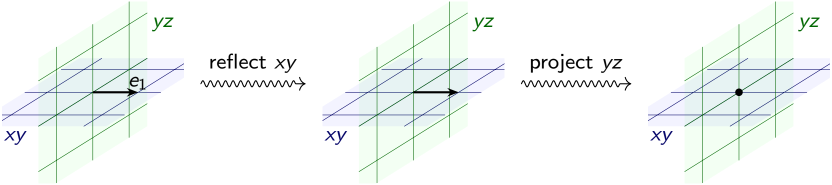




$$\left.\begin{aligned} T(e_1) &= \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \\ T(e_2) &= \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad \left(\begin{array}{c} \theta = 90^\circ \implies \\ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \text{(from before)} \end{array} \right)$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?



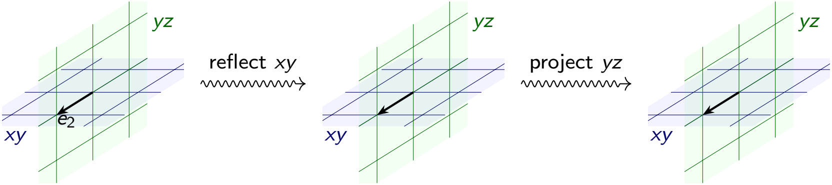
$$T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Linear Transformations are Matrix Transformations

Example, continued

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?



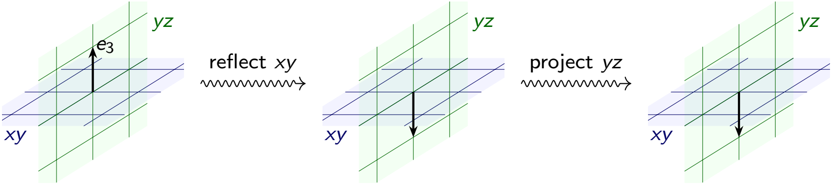
$$T(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Linear Transformations are Matrix Transformations

Example, continued

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?



$$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

$$\left.\begin{aligned} T(e_1) &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ T(e_2) &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ T(e_1) &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

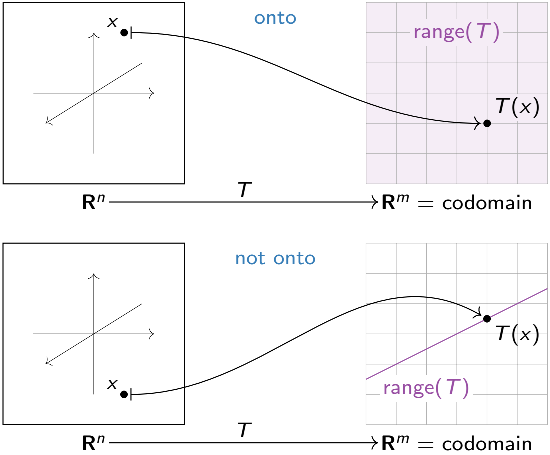
Other Geometric Transformations

There is a long list of geometric transformations of $\mathbf{R}^2$ in §1.9 of Lay. (Reflections over the diagonal, contractions and expansions along different axes, shears, projections, ...) Please look them over.

Onto Transformations

Definition

A transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is **onto** (or **surjective**) if the range of T is equal to $\mathbf{R}^m$ (its codomain). In other words, each b in $\mathbf{R}^m$ is the image of *at least one* x in $\mathbf{R}^n$: every possible output has an input. Note that *not* onto means there is some b in $\mathbf{R}^m$ which is not the image of any x in $\mathbf{R}^n$.



Characterization of Onto Transformations

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. Then the following are equivalent:

- T is onto
- $T(x) = b$ has a solution for every b in $\mathbf{R}^m$
- $Ax = b$ is consistent for every b in $\mathbf{R}^m$
- The columns of A span $\mathbf{R}^m$
- A has a pivot in every row

Question

If $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is onto, what can we say about the relative sizes of n and m?

Answer: T corresponds to an $m \times n$ matrix A. In order for A to have a pivot in every row, it must have *at least as many* columns as rows: $m \le n$.

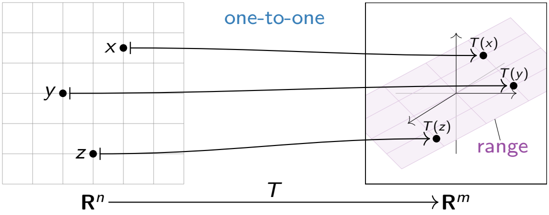
$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix}$$

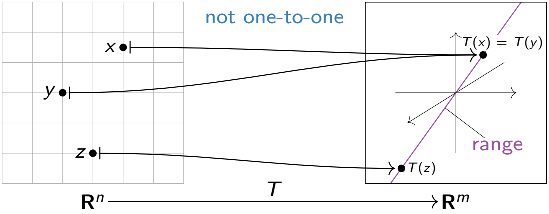
For instance, $\mathbf{R}^2$ is "too small" to map *onto* $\mathbf{R}^3$.

One-to-one Transformations

Definition

A transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is **one-to-one** (or **into**, or **injective**) if different vectors in $\mathbf{R}^n$ map to different vectors in $\mathbf{R}^m$. In other words, each b in $\mathbf{R}^m$ is the image of *at most one* x in $\mathbf{R}^n$: different inputs have different outputs. Note that *not* one-to-one means different vectors in $\mathbf{R}^n$ have the same image.





Characterization of One-to-One Transformations

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. Then the following are equivalent:

- T is one-to-one
- $T(x) = b$ has one or zero solutions for every b in $\mathbf{R}^m$
- $Ax = b$ has a unique solution or is inconsistent for every b in $\mathbf{R}^m$
- $Ax = 0$ has a unique solution
- The columns of A are linearly independent
- A has a pivot in every column.

Question

If $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is one-to-one, what can we say about the relative sizes of n and m?

Answer: T corresponds to an $m \times n$ matrix A. In order for A to have a pivot in every column, it must have *at least as many* rows as columns: $n \leq m$.

$$\begin{pmatrix} \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & 0 \\ 0 & 0 & \color{red}{1} \\ 0 & 0 & 0 \end{pmatrix}$$

For instance, $\mathbf{R}^3$ is "too big" to map *into* $\mathbf{R}^2$.