Chapter 6

Orthogonality and Least Squares

Section 6.1

Inner Product, Length, and Orthogonality

Orientation

Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$

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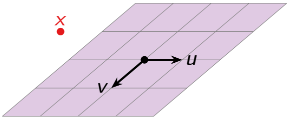

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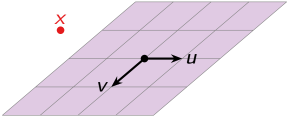

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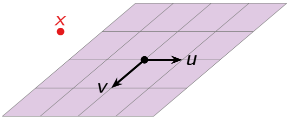

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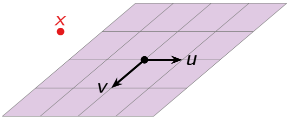

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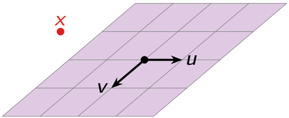

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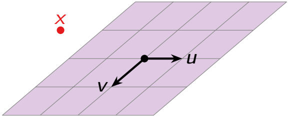

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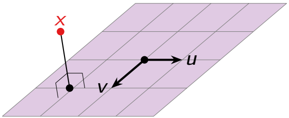

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Example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} =$$

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The Dot Product and Length

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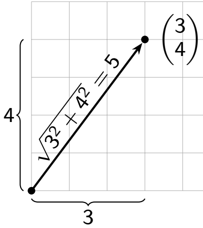

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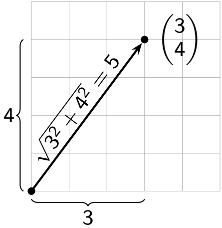

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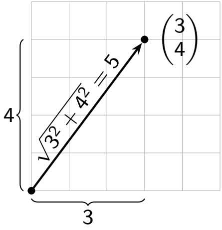

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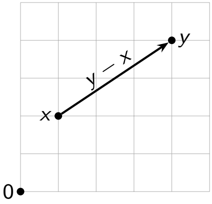

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This is in fact a unit vector:

scalar

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Example

What is the unit vector in the direction of $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$?

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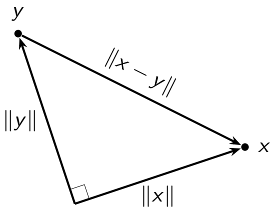

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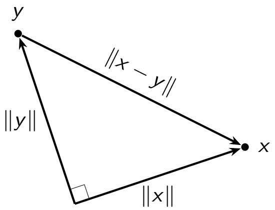

Fact: $x \perp y \iff \|x - y\|^2 = \|x\|^2 + \|y\|^2$

Problem: Find *all* vectors orthogonal to $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

Orthogonality
Example

Problem: Find *all* vectors orthogonal to both $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

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In particular, this set is a subspace!

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Let W be a subspace of $\mathbf{R}^n$. Its **orthogonal complement** is

$$W^\perp = \{ v \text{ in } \mathbf{R}^n \mid v \cdot w = 0 \text{ for all } w \text{ in } W \} \qquad \text{read "}W\text{ perp".}$$

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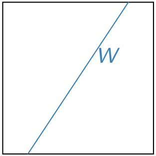
Pictures:

The orthogonal complement of a line in $\mathbf{R}^2$ is



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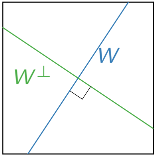
Pictures:

The orthogonal complement of a line in $\mathbf{R}^2$ is the perpendicular line.



Orthogonal Complements

Definition

Let W be a subspace of $\mathbf{R}^n$. Its **orthogonal complement** is

$$W^{\perp} = \{ v \text{ in } \mathbf{R}^n \mid v \cdot w = 0 \text{ for all } w \text{ in } W \} \qquad \text{read "}W\text{ perp".}$$

$W^{\perp}$ is orthogonal complement
A^{T} is transpose

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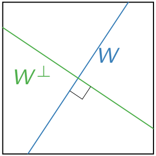

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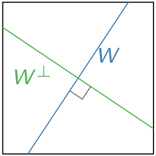
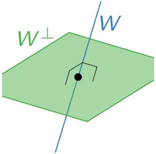


The orthogonal complement of a line in $\mathbf{R}^3$ is the perpendicular plane.



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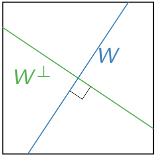

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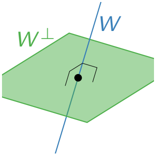

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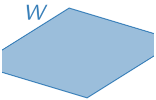

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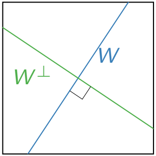

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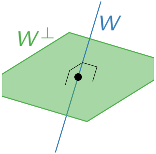

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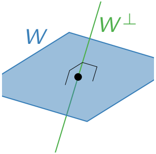

Poll

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Let W be a plane in $\mathbf{R}^4$. How would you describe $W^\perp$?

A. The zero space $\{0\}$.
B. A line in $\mathbf{R}^4$.
C. A plane in $\mathbf{R}^4$.
D. A 3-dimensional space in $\mathbf{R}^4$.
E. All of $\mathbf{R}^4$.

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Orthogonal Complements

Basic properties

Let W be a subspace of $\mathbf{R}^n$.

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1. $W^\perp$ is also a subspace of $\mathbf{R}^n$
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1. $W^\perp$ is also a subspace of $\mathbf{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, then

$$\begin{aligned} W^\perp &= \text{all vectors orthogonal to each } v_1, v_2, \ldots, v_m \\ &= \bigl\{x \text{ in } \mathbf{R}^n \mid x \cdot v_i = 0 \text{ for all } i = 1, 2, \ldots, m\bigr\} \\ &= \mathrm{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}. \end{aligned}$$

Orthogonal Complements

Computation

Problem: if $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, compute $W^{\perp}$.

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Using property 2 and taking the orthogonal complements of both sides, we get:

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Using property 2 and taking the orthogonal complements of both sides, we get:

Fact: $(\text{Nul}\, A)^\perp = \text{Row}\, A$ and $\text{Col}\, A = (\text{Nul}\, A^T)^\perp$.

Orthogonal Complements of Most of the Subspaces We've Seen

For any vectors $v_1, v_2, \ldots, v_m$:

$$\mathrm{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \mathrm{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}$$

For any matrix A:

$$\mathrm{Row}\,A = \mathrm{Col}\,A^T$$

and

$$(\mathrm{Row}\,A)^\perp = \mathrm{Nul}\,A \qquad \mathrm{Row}\,A = (\mathrm{Nul}\,A)^\perp$$

$$(\mathrm{Col}\,A)^\perp = \mathrm{Nul}\,A^T \qquad \mathrm{Col}\,A = (\mathrm{Nul}\,A^T)^\perp$$