Review for the Final Exam

Selected Topics

Orthogonal Sets

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal.

Orthogonal Sets

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal. It is **orthonormal** if, in addition, each vector is a unit vector.

Orthogonal Sets

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal. It is **orthonormal** if, in addition, each vector is a unit vector.

Example: $\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is not orthogonal.

Orthogonal Sets

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal. It is **orthonormal** if, in addition, each vector is a unit vector.

Example: $\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is not orthogonal.

Example: $\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is orthogonal but not orthonormal.

Orthogonal Sets

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal. It is **orthonormal** if, in addition, each vector is a unit vector.

Example: $\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is not orthogonal.

Example: $\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is orthogonal but not orthonormal.

Example: $\mathcal{B}_3 = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is orthonormal.

Orthogonal Sets

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal. It is **orthonormal** if, in addition, each vector is a unit vector.

Example: $\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is not orthogonal.

Example: $\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is orthogonal but not orthonormal.

Example: $\mathcal{B}_3 = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is orthonormal.

To go from an orthogonal set $\{u_1, u_2, \ldots, u_m\}$ to an orthonormal set, replace each u_i with $u_i / \|u_i\|$.

Orthogonal Sets

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal. It is **orthonormal** if, in addition, each vector is a unit vector.

Example: $\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is not orthogonal.

Example: $\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is orthogonal but not orthonormal.

Example: $\mathcal{B}_3 = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is orthonormal.

To go from an orthogonal set $\{u_1, u_2, \ldots, u_m\}$ to an orthonormal set, replace each u_i with $u_i / \|u_i\|$.

Theorem

An orthogonal set is linearly independent. In particular, it is a basis for its span.

Orthogonal Projection

Let W be a subspace of $\mathbf{R}^n$, and let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

$$\mathrm{proj}_W(x) \overset{\mathrm{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

Orthogonal Projection

Let W be a subspace of $\mathbf{R}^n$, and let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

$$\mathrm{proj}_W(x) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

This is the closest vector to x that lies on W.

Orthogonal Projection

Let W be a subspace of $\mathbf{R}^n$, and let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

$$\mathrm{proj}_W(x) \stackrel{\text{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

This is the closest vector to x that lies on W. In other words, the difference $x - \mathrm{proj}_W(x)$ is perpendicular to W: it is in $W^\perp$.

Orthogonal Projection

Let W be a subspace of $\mathbf{R}^n$, and let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

$$\text{proj}_W(x) \stackrel{\text{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

This is the closest vector to x that lies on W. In other words, the difference $x - \text{proj}_W(x)$ is perpendicular to W: it is in $W^\perp$. Notation:

$$\boxed{x_W = \text{proj}_W(x) \qquad x_{W^\perp} = x - \text{proj}_W(x).}$$

Orthogonal Projection

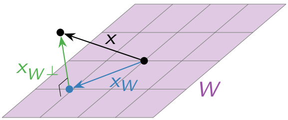
Let W be a subspace of $\mathbf{R}^n$, and let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

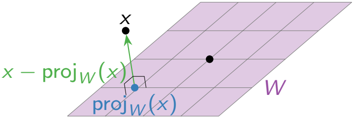
$$\mathrm{proj}_W(x) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

This is the closest vector to x that lies on W. In other words, the difference $x - \mathrm{proj}_W(x)$ is perpendicular to W: it is in $W^\perp$. Notation:

$$\boxed{x_W = \mathrm{proj}_W(x) \qquad x_{W^\perp} = x - \mathrm{proj}_W(x).}$$

So x_W is in W, $x_{W^\perp}$ is in $W^\perp$, and $x = x_W + x_{W^\perp}$.





Special case: If x is in W, then $x = \mathrm{proj}_W(x)$, so

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

Special case: If x is in W, then $x = \mathrm{proj}_W(x)$, so

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

In other words, the $\mathcal{B}$-coordinates of x are

$$\left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_1 \cdot u_2}, \ldots, \frac{x \cdot u_m}{u_1 \cdot u_m} \right),$$

where $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$, an orthogonal basis for W.

Orthogonal Projection

Special cases

Special case: If x is in W, then $x = \mathrm{proj}_W(x)$, so

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

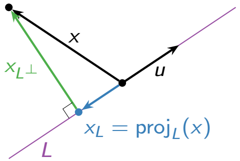
In other words, the $\mathcal{B}$-coordinates of x are

$$\left(\frac{x \cdot u_1}{u_1 \cdot u_1},\ \frac{x \cdot u_2}{u_1 \cdot u_2},\ \ldots,\ \frac{x \cdot u_m}{u_1 \cdot u_m} \right),$$

where $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$, an orthogonal basis for W.

Special case: If $W = L$ is a line, then $L = \mathrm{Span}\{u\}$ for some nonzero vector u, and

$$\mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u}\, u$$



Let W be a subspace of $\mathbf{R}^n$.

Theorem
The orthogonal projection proj_W is a *linear* transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$. Its range is W.

Let W be a subspace of $\mathbf{R}^n$.

Theorem

The orthogonal projection proj_W is a *linear* transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$. Its range is W.

If A is the matrix for proj_W, then $A^2 = A$ because projecting twice is the same as projecting once: $\mathrm{proj}_W \circ \mathrm{proj}_W = \mathrm{proj}_W$.

Let W be a subspace of $\mathbf{R}^n$.

Theorem
The orthogonal projection proj_W is a *linear* transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$. Its range is W.

If A is the matrix for proj_W, then $A^2 = A$ because projecting twice is the same as projecting once: $\mathrm{proj}_W \circ \mathrm{proj}_W = \mathrm{proj}_W$.

Theorem
The only eigenvalues of A are

Orthogonal Projection
And matrices

Let W be a subspace of $\mathbf{R}^n$.

Theorem

The orthogonal projection proj_W is a *linear* transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$. Its range is W.

If A is the matrix for proj_W, then $A^2 = A$ because projecting twice is the same as projecting once: $\text{proj}_W \circ \text{proj}_W = \text{proj}_W$.

Theorem

The only eigenvalues of A are 1 and 0.

Why?

Let W be a subspace of $\mathbf{R}^n$.

Theorem
The orthogonal projection proj_W is a *linear* transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$. Its range is W.

If A is the matrix for proj_W, then $A^2 = A$ because projecting twice is the same as projecting once: $\mathrm{proj}_W \circ \mathrm{proj}_W = \mathrm{proj}_W$.

Theorem
The only eigenvalues of A are 1 and 0.

Why?

The 1-eigenspace of A is

Let W be a subspace of $\mathbf{R}^n$.

Theorem

The orthogonal projection proj_W is a *linear* transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$. Its range is W.

If A is the matrix for proj_W, then $A^2 = A$ because projecting twice is the same as projecting once: $\mathrm{proj}_W \circ \mathrm{proj}_W = \mathrm{proj}_W$.

Theorem

The only eigenvalues of A are 1 and 0.

Why?

The 1-eigenspace of A is W, and the 0-eigenspace is

Let W be a subspace of $\mathbf{R}^n$.

Theorem

The orthogonal projection proj_W is a *linear* transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$. Its range is W.

If A is the matrix for proj_W, then $A^2 = A$ because projecting twice is the same as projecting once: $\mathrm{proj}_W \circ \mathrm{proj}_W = \mathrm{proj}_W$.

Theorem

The only eigenvalues of A are 1 and 0.

Why?

The 1-eigenspace of A is W, and the 0-eigenspace is $W^\perp$.

The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace W of $\mathbf{R}^n$. Define:

1. $u_1 = v_1$

The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace W of $\mathbf{R}^n$. Define:

1. $u_1 = v_1$
2. $u_2 = v_2 - \mathrm{proj}_{\mathrm{Span}\{u_1\}}(v_2) \qquad = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1$

The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace W of $\mathbf{R}^n$. Define:

1. $u_1 = v_1$
2. $u_2 = v_2 - \mathrm{proj}_{\mathrm{Span}\{u_1\}}(v_2) \qquad = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1$
3. $u_3 = v_3 - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2\}}(v_3) \qquad = v_3 - \dfrac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \dfrac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2$

The Gram–Schmidt Process

The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace W of $\mathbf{R}^n$. Define:

1. $u_1 = v_1$
2. $u_2 = v_2 - \mathrm{proj}_{\mathrm{Span}\{u_1\}}(v_2) = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1$
3. $u_3 = v_3 - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2\}}(v_3) = v_3 - \dfrac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \dfrac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2$

$\vdots$

m. $u_m = v_m - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{m-1}\}}(v_m) = v_m - \displaystyle\sum_{i=1}^{m-1} \frac{v_m \cdot u_i}{u_i \cdot u_i}\, u_i$

The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace W of $\mathbf{R}^n$. Define:

1. $u_1 = v_1$
2. $u_2 = v_2 - \operatorname{proj}_{\operatorname{Span}\{u_1\}}(v_2) \quad = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1$
3. $u_3 = v_3 - \operatorname{proj}_{\operatorname{Span}\{u_1, u_2\}}(v_3) \quad = v_3 - \dfrac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \dfrac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2$

$\vdots$

m. $u_m = v_m - \operatorname{proj}_{\operatorname{Span}\{u_1, u_2, \ldots, u_{m-1}\}}(v_m) \quad = v_m - \displaystyle\sum_{i=1}^{m-1} \frac{v_m \cdot u_i}{u_i \cdot u_i}\, u_i$

Then $\{u_1, u_2, \ldots, u_m\}$ is an *orthogonal* basis for the same subspace W.

The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace W of $\mathbf{R}^n$. Define:

1. $u_1 = v_1$
2. $u_2 = v_2 - \mathrm{proj}_{\mathrm{Span}\{u_1\}}(v_2) \qquad = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1$
3. $u_3 = v_3 - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2\}}(v_3) \qquad = v_3 - \dfrac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \dfrac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2$

$\vdots$

m. $u_m = v_m - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{m-1}\}}(v_m) \;= v_m - \displaystyle\sum_{i=1}^{m-1} \frac{v_m \cdot u_i}{u_i \cdot u_i}\, u_i$

Then $\{u_1, u_2, \ldots, u_m\}$ is an *orthogonal* basis for the same subspace W.

In fact, for each i,

$$\mathrm{Span}\{u_1, u_2, \ldots, u_i\} = \mathrm{Span}\{v_1, v_2, \ldots, v_i\}.$$

The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace W of $\mathbf{R}^n$. Define:

1. $u_1 = v_1$
2. $u_2 = v_2 - \mathrm{proj}_{\mathrm{Span}\{u_1\}}(v_2) \qquad = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1$
3. $u_3 = v_3 - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2\}}(v_3) \qquad = v_3 - \dfrac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \dfrac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2$

$\vdots$

m. $u_m = v_m - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{m-1}\}}(v_m) = v_m - \displaystyle\sum_{i=1}^{m-1} \frac{v_m \cdot u_i}{u_i \cdot u_i}\, u_i$

Then $\{u_1, u_2, \ldots, u_m\}$ is an *orthogonal* basis for the same subspace W.

In fact, for each i,

$$\mathrm{Span}\{u_1, u_2, \ldots, u_i\} = \mathrm{Span}\{v_1, v_2, \ldots, v_i\}.$$

Note if v_i is in $\mathrm{Span}\{v_1, v_2, \ldots, v_{i-1}\} = \mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}$, then $v_i = \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}}(v_i)$, so $u_i = 0$.

The Gram–Schmidt Process

The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace W of $\mathbf{R}^n$. Define:

1. $u_1 = v_1$
2. $u_2 = v_2 - \operatorname{proj}_{\operatorname{Span}\{u_1\}}(v_2) \qquad = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1$
3. $u_3 = v_3 - \operatorname{proj}_{\operatorname{Span}\{u_1, u_2\}}(v_3) \qquad = v_3 - \dfrac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \dfrac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2$

$\vdots$

m. $u_m = v_m - \operatorname{proj}_{\operatorname{Span}\{u_1, u_2, \ldots, u_{m-1}\}}(v_m) \; = v_m - \displaystyle\sum_{i=1}^{m-1} \frac{v_m \cdot u_i}{u_i \cdot u_i}\, u_i$

Then $\{u_1, u_2, \ldots, u_m\}$ is an *orthogonal* basis for the same subspace W.

In fact, for each i,

$$\operatorname{Span}\{u_1, u_2, \ldots, u_i\} = \operatorname{Span}\{v_1, v_2, \ldots, v_i\}.$$

Note if v_i is in $\operatorname{Span}\{v_1, v_2, \ldots, v_{i-1}\} = \operatorname{Span}\{u_1, u_2, \ldots, u_{i-1}\}$, then $v_i = \operatorname{proj}_{\operatorname{Span}\{u_1, u_2, \ldots, u_{i-1}\}}(v_i)$, so $u_i = 0$. So this also detects linear dependence.

QR Factorization Theorem

Let A be a matrix with linearly independent columns. Then

$$A = QR$$

where Q has orthonormal columns and R is upper-triangular with positive diagonal entries.

QR Factorization Theorem

Let A be a matrix with linearly independent columns. Then

$$A = QR$$

where Q has orthonormal columns and R is upper-triangular with positive diagonal entries.

Step 1: Let $v_1, v_2, \ldots, v_m$ be the columns of A. Run Gram–Schmidt on $\{v_1, v_2, \ldots, v_m\}$ to get an orthogonal basis $\{u_1, u_2, \ldots, u_m\}$, and solve for each v_i in terms of $u_1, u_2, \ldots, u_i$.

QR Factorization

QR Factorization Theorem

Let A be a matrix with linearly independent columns. Then

$$A = QR$$

where Q has orthonormal columns and R is upper-triangular with positive diagonal entries.

Step 1: Let $v_1, v_2, \ldots, v_m$ be the columns of A. Run Gram–Schmidt on $\{v_1, v_2, \ldots, v_m\}$ to get an orthogonal basis $\{u_1, u_2, \ldots, u_m\}$, and solve for each v_i in terms of $u_1, u_2, \ldots, u_i$.

Step 2: Put the resulting equations in matrix form to get $A = \widehat{Q}\widehat{R}$ where

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix} \qquad \widehat{Q} = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{pmatrix}$$

and $\widehat{R}$ contains the coefficients from $v_i =$ (linear combination of $u_1, u_2, \ldots, u_{i-1}$) in the columns.

QR Factorization Theorem

Let A be a matrix with linearly independent columns. Then

$$A = QR$$

where Q has orthonormal columns and R is upper-triangular with positive diagonal entries.

Step 1: Let $v_1, v_2, \ldots, v_m$ be the columns of A. Run Gram–Schmidt on $\{v_1, v_2, \ldots, v_m\}$ to get an orthogonal basis $\{u_1, u_2, \ldots, u_m\}$, and solve for each v_i in terms of $u_1, u_2, \ldots, u_i$.

Step 2: Put the resulting equations in matrix form to get $A = \widehat{Q}\widehat{R}$ where

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix} \qquad \widehat{Q} = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{pmatrix}$$

and $\widehat{R}$ contains the coefficients from $v_i =$ (linear combination of $u_1, u_2, \ldots, u_{i-1}$) in the columns.

Step 3: Scale each column of $\widehat{Q}$ by its length to get a matrix with orthonormal columns, and scale each row of $\widehat{R}$ by the opposite factor to get Q and R, respectively.

Find the QR factorization of $A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & -2 \\ 1 & -1 & 0 \end{pmatrix}$.

Find the QR factorization of $A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & -2 \\ 1 & -1 & 0 \end{pmatrix}$.

Step 1: Let v_1, v_2, v_3 be the columns. Run Gram–Schmidt and solve for v_1, v_2, v_3 in terms of u_1, u_2, u_3:

$$v_1 = 1\, u_1 \qquad v_2 = \frac{3}{2}\, u_1 + 1\, u_2 \qquad v_3 = 0\, u_1 - \frac{4}{5}\, u_2 + 1\, u_3$$

Step 2: write $A = \widehat{Q}\widehat{R}$, where $\widehat{Q}$ has *orthogonal* columns u_1, u_2, u_3 and $\widehat{R}$ is upper-triangular with 1s on the diagonal.

QR Factorization

Example, continued

$$A = \widehat{Q}\widehat{R} \qquad \widehat{Q} = \begin{pmatrix} 1 & -5/2 & 2 \\ 1 & 5/2 & 0 \\ 1 & 5/2 & 0 \\ 1 & -5/2 & -2 \end{pmatrix} \qquad \widehat{R} = \begin{pmatrix} 1 & 3/2 & 0 \\ 0 & 1 & -4/5 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 3: normalize the columns of $\widehat{Q}$ and the rows of $\widehat{R}$ to get Q and R:

$$A = \widehat{Q}\widehat{R} \qquad \widehat{Q} = \begin{pmatrix} 1 & -5/2 & 2 \\ 1 & 5/2 & 0 \\ 1 & 5/2 & 0 \\ 1 & -5/2 & -2 \end{pmatrix} \qquad \widehat{R} = \begin{pmatrix} 1 & 3/2 & 0 \\ 0 & 1 & -4/5 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 3: normalize the columns of $\widehat{Q}$ and the rows of $\widehat{R}$ to get Q and R:

The final QR decomposition is

$$A = QR \qquad Q = \begin{pmatrix} 1/2 & -1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & -1/\sqrt{2} \end{pmatrix} \qquad R = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 5 & -4 \\ 0 & 0 & 2\sqrt{2} \end{pmatrix}.$$

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Examples:

- Any $\mathrm{Span}\{v_1, v_2, \ldots, v_m\}$.

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Examples:

- Any $\mathrm{Span}\{v_1, v_2, \ldots, v_m\}$.
- The *column space* of a matrix: $\mathrm{Col}\, A = \mathrm{Span}\{\text{columns of } A\}$.

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Examples:

- Any $\text{Span}\{v_1, v_2, \ldots, v_m\}$.
- The *column space* of a matrix: $\text{Col}\, A = \text{Span}\{\text{columns of } A\}$.
- The range of a linear transformation (same as above).

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Examples:

- Any $\mathrm{Span}\{v_1, v_2, \ldots, v_m\}$.
- The *column space* of a matrix: $\mathrm{Col}\, A = \mathrm{Span}\{\text{columns of } A\}$.
- The range of a linear transformation (same as above).
- The *null space* of a matrix: $\mathrm{Nul}\, A = \{x \mid Ax = 0\}$.

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Examples:

- Any $\mathrm{Span}\{v_1, v_2, \ldots, v_m\}$.
- The *column space* of a matrix: $\mathrm{Col}\, A = \mathrm{Span}\{\text{columns of } A\}$.
- The range of a linear transformation (same as above).
- The *null space* of a matrix: $\mathrm{Nul}\, A = \{x \mid Ax = 0\}$.
- The *row space* of a matrix: $\mathrm{Row}\, A = \mathrm{Span}\{\text{rows of } A\}$.

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u+v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Examples:

- Any $\mathrm{Span}\{v_1, v_2, \ldots, v_m\}$.
- The *column space* of a matrix: $\mathrm{Col}\,A = \mathrm{Span}\{\text{columns of } A\}$.
- The range of a linear transformation (same as above).
- The *null space* of a matrix: $\mathrm{Nul}\,A = \{x \mid Ax = 0\}$.
- The *row space* of a matrix: $\mathrm{Row}\,A = \mathrm{Span}\{\text{rows of } A\}$.
- The λ-eigenspace of a matrix, where λ is an eigenvalue.

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Examples:

- Any $\mathrm{Span}\{v_1, v_2, \ldots, v_m\}$.
- The *column space* of a matrix: $\mathrm{Col}\, A = \mathrm{Span}\{\text{columns of } A\}$.
- The range of a linear transformation (same as above).
- The *null space* of a matrix: $\mathrm{Nul}\, A = \{x \mid Ax = 0\}$.
- The *row space* of a matrix: $\mathrm{Row}\, A = \mathrm{Span}\{\text{rows of } A\}$.
- The λ-eigenspace of a matrix, where λ is an eigenvalue.
- The orthogonal complement $W^\perp$ of a subspace W.

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Examples:

- Any $\mathrm{Span}\{v_1, v_2, \ldots, v_m\}$.
- The *column space* of a matrix: $\mathrm{Col}\, A = \mathrm{Span}\{\text{columns of } A\}$.
- The range of a linear transformation (same as above).
- The *null space* of a matrix: $\mathrm{Nul}\, A = \{x \mid Ax = 0\}$.
- The *row space* of a matrix: $\mathrm{Row}\, A = \mathrm{Span}\{\text{rows of } A\}$.
- The λ-eigenspace of a matrix, where λ is an eigenvalue.
- The orthogonal complement $W^\perp$ of a subspace W.
- The zero subspace $\{0\}$.

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Examples:

- Any $\mathrm{Span}\{v_1, v_2, \ldots, v_m\}$.
- The *column space* of a matrix: $\mathrm{Col}\, A = \mathrm{Span}\{\text{columns of } A\}$.
- The range of a linear transformation (same as above).
- The *null space* of a matrix: $\mathrm{Nul}\, A = \{x \mid Ax = 0\}$.
- The *row space* of a matrix: $\mathrm{Row}\, A = \mathrm{Span}\{\text{rows of } A\}$.
- The λ-eigenspace of a matrix, where λ is an eigenvalue.
- The orthogonal complement $W^\perp$ of a subspace W.
- The zero subspace $\{0\}$.
- All of $\mathbf{R}^n$.

Subspaces and Bases

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \dots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \dots, v_m\}$, and
2. $\{v_1, v_2, \dots, v_m\}$ is linearly independent.

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Every subspace has a basis, so every subspace is a span.

Subspaces and Bases

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Every subspace has a basis, so every subspace is a span. But subspaces have many different bases, and some might be better than others.

Subspaces and Bases

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Every subspace has a basis, so every subspace is a span. But subspaces have many different bases, and some might be better than others. For instance, Gram–Schmidt takes a basis and produces an *orthogonal* basis.

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Every subspace has a basis, so every subspace is a span. But subspaces have many different bases, and some might be better than others. For instance, Gram–Schmidt takes a basis and produces an *orthogonal* basis. Or, diagonalization produces a basis of *eigenvectors* of a matrix.

Subspaces and Bases

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Every subspace has a basis, so every subspace is a span. But subspaces have many different bases, and some might be better than others. For instance, Gram–Schmidt takes a basis and produces an *orthogonal* basis. Or, diagonalization produces a basis of *eigenvectors* of a matrix.

How do I know if a subset V is a subspace or not?

Subspaces and Bases

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Every subspace has a basis, so every subspace is a span. But subspaces have many different bases, and some might be better than others. For instance, Gram–Schmidt takes a basis and produces an *orthogonal* basis. Or, diagonalization produces a basis of *eigenvectors* of a matrix.

How do I know if a subset V is a subspace or not?

- Can you write V as one of the examples on the previous slide?

Subspaces and Bases

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Every subspace has a basis, so every subspace is a span. But subspaces have many different bases, and some might be better than others. For instance, Gram–Schmidt takes a basis and produces an *orthogonal* basis. Or, diagonalization produces a basis of *eigenvectors* of a matrix.

How do I know if a subset V is a subspace or not?

- Can you write V as one of the examples on the previous slide?
- If not, does it satisfy the three defining properties?

Subspaces and Bases

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Every subspace has a basis, so every subspace is a span. But subspaces have many different bases, and some might be better than others. For instance, Gram–Schmidt takes a basis and produces an *orthogonal* basis. Or, diagonalization produces a basis of *eigenvectors* of a matrix.

How do I know if a subset V is a subspace or not?

- Can you write V as one of the examples on the previous slide?
- If not, does it satisfy the three defining properties?

Note on subspaces versus subsets: A **subset** of $\mathbf{R}^n$ is any collection of vectors whatsoever.

Subspaces and Bases

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Every subspace has a basis, so every subspace is a span. But subspaces have many different bases, and some might be better than others. For instance, Gram–Schmidt takes a basis and produces an *orthogonal* basis. Or, diagonalization produces a basis of *eigenvectors* of a matrix.

How do I know if a subset V is a subspace or not?

- Can you write V as one of the examples on the previous slide?
- If not, does it satisfy the three defining properties?

Note on subspaces versus subsets: A **subset** of $\mathbf{R}^n$ is any collection of vectors whatsoever. Like, the unit circle in $\mathbf{R}^2$, or all vectors with whole-number coefficients.

Subspaces and Bases

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Every subspace has a basis, so every subspace is a span. But subspaces have many different bases, and some might be better than others. For instance, Gram–Schmidt takes a basis and produces an *orthogonal* basis. Or, diagonalization produces a basis of *eigenvectors* of a matrix.

How do I know if a subset V is a subspace or not?

- Can you write V as one of the examples on the previous slide?
- If not, does it satisfy the three defining properties?

Note on subspaces versus subsets: A **subset** of $\mathbf{R}^n$ is any collection of vectors whatsoever. Like, the unit circle in $\mathbf{R}^2$, or all vectors with whole-number coefficients. A *subspace* is a subset that satisfies three additional properties.

Subspaces and Bases

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \dots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \dots, v_m\}$, and
2. $\{v_1, v_2, \dots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Every subspace has a basis, so every subspace is a span. But subspaces have many different bases, and some might be better than others. For instance, Gram–Schmidt takes a basis and produces an *orthogonal* basis. Or, diagonalization produces a basis of *eigenvectors* of a matrix.

How do I know if a subset V is a subspace or not?

- Can you write V as one of the examples on the previous slide?
- If not, does it satisfy the three defining properties?

Note on subspaces versus subsets: A **subset** of $\mathbf{R}^n$ is any collection of vectors whatsoever. Like, the unit circle in $\mathbf{R}^2$, or all vectors with whole-number coefficients. A *subspace* is a subset that satisfies three additional properties. Most subsets are not subspaces.

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Important Facts:

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Important Facts:

1. Similar matrices have the same characteristic polynomial.

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Important Facts:

1. Similar matrices have the same characteristic polynomial.
2. It follows that similar matrices have the same eigenvalues.

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Important Facts:

1. Similar matrices have the same characteristic polynomial.
2. It follows that similar matrices have the same eigenvalues.
3. If A is similar to B and B is similar to C, then A is similar to C.

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Important Facts:

1. Similar matrices have the same characteristic polynomial.
2. It follows that similar matrices have the same eigenvalues.
3. If A is similar to B and B is similar to C, then A is similar to C.

Caveats:

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Important Facts:

1. Similar matrices have the same characteristic polynomial.
2. It follows that similar matrices have the same eigenvalues.
3. If A is similar to B and B is similar to C, then A is similar to C.

Caveats:

1. Matrices with the same characteristic polynomial need not be similar.

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Important Facts:

1. Similar matrices have the same characteristic polynomial.
2. It follows that similar matrices have the same eigenvalues.
3. If A is similar to B and B is similar to C, then A is similar to C.

Caveats:

1. Matrices with the same characteristic polynomial need not be similar.
2. Similarity has nothing to do with row equivalence.

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Important Facts:

1. Similar matrices have the same characteristic polynomial.
2. It follows that similar matrices have the same eigenvalues.
3. If A is similar to B and B is similar to C, then A is similar to C.

Caveats:

1. Matrices with the same characteristic polynomial need not be similar.
2. Similarity has nothing to do with row equivalence.
3. Similar matrices usually do not have the same eigenvectors.

Similarity

Geometric meaning

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P.

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P. These form a basis $\mathcal{B}$ for $\mathbf{R}^n$ because P is invertible.

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P. These form a basis $\mathcal{B}$ for $\mathbf{R}^n$ because P is invertible. *Key relation:* for any vector x in $\mathbf{R}^n$,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P. These form a basis $\mathcal{B}$ for $\mathbf{R}^n$ because P is invertible. *Key relation:* for any vector x in $\mathbf{R}^n$,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

This says:

A acts on the usual coordinates of x
in the same way that
B acts on the $\mathcal{B}$-coordinates of x.

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P. These form a basis $\mathcal{B}$ for $\mathbf{R}^n$ because P is invertible. *Key relation:* for any vector x in $\mathbf{R}^n$,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

This says:

A acts on the usual coordinates of x
in the same way that
B acts on the $\mathcal{B}$-coordinates of x.

Example:

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P. These form a basis $\mathcal{B}$ for $\mathbf{R}^n$ because P is invertible. *Key relation:* for any vector x in $\mathbf{R}^n$,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

This says:

A acts on the usual coordinates of x
in the same way that
B acts on the $\mathcal{B}$-coordinates of x.

Example:

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $A = PBP^{-1}$.

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P. These form a basis $\mathcal{B}$ for $\mathbf{R}^n$ because P is invertible. *Key relation:* for any vector x in $\mathbf{R}^n$,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

This says:

A acts on the usual coordinates of x
in the same way that
B acts on the $\mathcal{B}$-coordinates of x.

Example:

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $A = PBP^{-1}$. B acts on the usual coordinates by

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P. These form a basis $\mathcal{B}$ for $\mathbf{R}^n$ because P is invertible. *Key relation:* for any vector x in $\mathbf{R}^n$,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

This says:

A acts on the usual coordinates of x
in the same way that
B acts on the $\mathcal{B}$-coordinates of x.

Example:

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $A = PBP^{-1}$. B acts on the usual coordinates by scaling the first coordinate by 2, and the second by $1/2$:

$$B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_2/2 \end{pmatrix}.$$

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P. These form a basis $\mathcal{B}$ for $\mathbf{R}^n$ because P is invertible. *Key relation:* for any vector x in $\mathbf{R}^n$,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

This says:

A acts on the usual coordinates of x
in the same way that
B acts on the $\mathcal{B}$-coordinates of x.

Example:

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $A = PBP^{-1}$. B acts on the usual coordinates by scaling the first coordinate by 2, and the second by 1/2:

$$B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_2/2 \end{pmatrix}.$$

The unit coordinate vectors are eigenvectors: e_1 has eigenvalue

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P. These form a basis $\mathcal{B}$ for $\mathbf{R}^n$ because P is invertible. *Key relation:* for any vector x in $\mathbf{R}^n$,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

This says:

A acts on the usual coordinates of x
in the same way that
B acts on the $\mathcal{B}$-coordinates of x.

Example:

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $A = PBP^{-1}$. B acts on the usual coordinates by scaling the first coordinate by 2, and the second by $1/2$:

$$B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_2/2 \end{pmatrix}.$$

The unit coordinate vectors are eigenvectors: e_1 has eigenvalue 2, and e_2 has eigenvalue

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P. These form a basis $\mathcal{B}$ for $\mathbf{R}^n$ because P is invertible. *Key relation:* for any vector x in $\mathbf{R}^n$,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

This says:

A acts on the usual coordinates of x
in the same way that
B acts on the $\mathcal{B}$-coordinates of x.

Example:

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $A = PBP^{-1}$. B acts on the usual coordinates by scaling the first coordinate by 2, and the second by $1/2$:

$$B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_2/2 \end{pmatrix}.$$

The unit coordinate vectors are eigenvectors: e_1 has eigenvalue 2, and e_2 has eigenvalue $1/2$.

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

In this case, $\mathcal{B} = \left\{ \binom{1}{1}, \binom{1}{-1} \right\}$. Let $v_1 = \binom{1}{1}$ and $v_2 = \binom{1}{-1}$.

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

In this case, $\mathcal{B} = \left\{ \binom{1}{1}, \binom{1}{-1} \right\}$. Let $v_1 = \binom{1}{1}$ and $v_2 = \binom{1}{-1}$.

To compute $y = Ax$:

Say $x = \binom{2}{0}$.

Similarity

Example

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

In this case, $\mathcal{B} = \left\{ \binom{1}{1}, \binom{1}{-1} \right\}$. Let $v_1 = \binom{1}{1}$ and $v_2 = \binom{1}{-1}$.

To compute $y = Ax$: Say $x = \binom{2}{0}$.

1. Find $[x]_{\mathcal{B}}$.

Similarity
Example

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

In this case, $\mathcal{B} = \left\{ \binom{1}{1}, \binom{1}{-1} \right\}$. Let $v_1 = \binom{1}{1}$ and $v_2 = \binom{1}{-1}$.

To compute $y = Ax$:

1. Find $[x]_{\mathcal{B}}$.

Say $x = \binom{2}{0}$.

1. $x = v_1 + v_2$ so $[x]_{\mathcal{B}} = \binom{1}{1}$.

Similarity

Example

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

In this case, $\mathcal{B} = \left\{ \binom{1}{1}, \binom{1}{-1} \right\}$. Let $v_1 = \binom{1}{1}$ and $v_2 = \binom{1}{-1}$.

To compute $y = Ax$:

1. Find $[x]_{\mathcal{B}}$.
2. $[y]_{\mathcal{B}} = B[x]_{\mathcal{B}}$.

Say $x = \binom{2}{0}$.

1. $x = v_1 + v_2$ so $[x]_{\mathcal{B}} = \binom{1}{1}$.

Similarity
Example

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

In this case, $\mathcal{B} = \left\{ \binom{1}{1}, \binom{1}{-1} \right\}$. Let $v_1 = \binom{1}{1}$ and $v_2 = \binom{1}{-1}$.

To compute $y = Ax$:

1. Find $[x]_{\mathcal{B}}$.
2. $[y]_{\mathcal{B}} = B[x]_{\mathcal{B}}$.

Say $x = \binom{2}{0}$.

1. $x = v_1 + v_2$ so $[x]_{\mathcal{B}} = \binom{1}{1}$.
2. $[y]_{\mathcal{B}} = B\binom{1}{1} = \binom{2}{1/2}$.

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

In this case, $\mathcal{B} = \left\{ \binom{1}{1}, \binom{1}{-1} \right\}$. Let $v_1 = \binom{1}{1}$ and $v_2 = \binom{1}{-1}$.

To compute $y = Ax$:

1. Find $[x]_{\mathcal{B}}$.
2. $[y]_{\mathcal{B}} = B[x]_{\mathcal{B}}$.
3. Compute y from $[y]_{\mathcal{B}}$.

Say $x = \binom{2}{0}$.

1. $x = v_1 + v_2$ so $[x]_{\mathcal{B}} = \binom{1}{1}$.
2. $[y]_{\mathcal{B}} = B\binom{1}{1} = \binom{2}{1/2}$.

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

In this case, $\mathcal{B} = \left\{\binom{1}{1}, \binom{1}{-1}\right\}$. Let $v_1 = \binom{1}{1}$ and $v_2 = \binom{1}{-1}$.

To compute $y = Ax$:

1. Find $[x]_{\mathcal{B}}$.
2. $[y]_{\mathcal{B}} = B[x]_{\mathcal{B}}$.
3. Compute y from $[y]_{\mathcal{B}}$.

Say $x = \binom{2}{0}$.

1. $x = v_1 + v_2$ so $[x]_{\mathcal{B}} = \binom{1}{1}$.
2. $[y]_{\mathcal{B}} = B\binom{1}{1} = \binom{2}{1/2}$.
3. $y = 2v_1 + \frac{1}{2}v_2 = \binom{5/2}{3/2}$.

Similarity

Example

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

In this case, $\mathcal{B} = \left\{\binom{1}{1}, \binom{1}{-1}\right\}$. Let $v_1 = \binom{1}{1}$ and $v_2 = \binom{1}{-1}$.

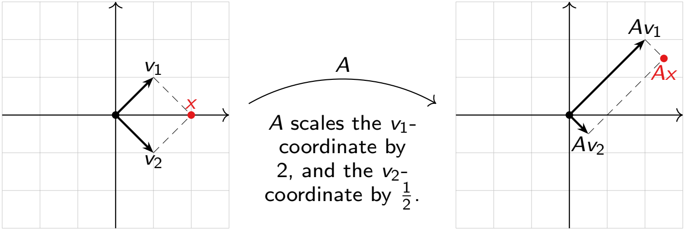
To compute $y = Ax$:

1. Find $[x]_{\mathcal{B}}$.
2. $[y]_{\mathcal{B}} = B[x]_{\mathcal{B}}$.
3. Compute y from $[y]_{\mathcal{B}}$.

Say $x = \binom{2}{0}$.

1. $x = v_1 + v_2$ so $[x]_{\mathcal{B}} = \binom{1}{1}$.
2. $[y]_{\mathcal{B}} = B\binom{1}{1} = \binom{2}{1/2}$.
3. $y = 2v_1 + \frac{1}{2}v_2 = \binom{5/2}{3/2}$.

Picture:



Consistent and Inconsistent Systems

Definition

A matrix equation $Ax = b$ is **consistent** if it has a solution, and **inconsistent** otherwise.

Definition

A matrix equation $Ax = b$ is **consistent** if it has a solution, and **inconsistent** otherwise.

If A has columns $v_1, v_2, \ldots, v_n$, then

$$b = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

Consistent and Inconsistent Systems

Definition

A matrix equation $Ax = b$ is **consistent** if it has a solution, and **inconsistent** otherwise.

If A has columns $v_1, v_2, \ldots, v_n$, then

$$b = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

So if $Ax = b$ has a solution, then b is a linear combination of $v_1, v_2, \ldots, v_n$, and conversely.

Consistent and Inconsistent Systems

Definition

A matrix equation $Ax = b$ is **consistent** if it has a solution, and **inconsistent** otherwise.

If A has columns $v_1, v_2, \ldots, v_n$, then

$$b = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

So if $Ax = b$ has a solution, then b is a linear combination of $v_1, v_2, \ldots, v_n$, and conversely. Equivalently, b is in $\mathrm{Span}\{v_1, v_2, \ldots, v_n\} = \mathrm{Col}\, A$.

Consistent and Inconsistent Systems

Definition

A matrix equation $Ax = b$ is **consistent** if it has a solution, and **inconsistent** otherwise.

If A has columns $v_1, v_2, \ldots, v_n$, then

$$b = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

So if $Ax = b$ has a solution, then b is a linear combination of $v_1, v_2, \ldots, v_n$, and conversely. Equivalently, b is in $\mathrm{Span}\{v_1, v_2, \ldots, v_n\} = \mathrm{Col}\, A$.

Important

$Ax = b$ is consistent if and only if b is in $\mathrm{Col}\, A$.

Least-Squares Solutions

Suppose that $Ax = b$ is *in*consistent.

Least-Squares Solutions

Suppose that $Ax = b$ is *in*consistent. Let $\widehat{b} = \mathrm{proj}_{\mathrm{Col}\,A}(b)$ be the closest vector for which $A\widehat{x} = \widehat{b}$ *does* have a solution.

Least-Squares Solutions

Suppose that $Ax = b$ is *in*consistent. Let $\widehat{b} = \mathrm{proj}_{\mathrm{Col}\,A}(b)$ be the closest vector for which $A\widehat{x} = \widehat{b}$ *does* have a solution.

Definition

A solution to $A\widehat{x} = \widehat{b}$ is a **least squares solution** to $Ax = b$.

Least-Squares Solutions

Suppose that $Ax = b$ is *in*consistent. Let $\widehat{b} = \mathrm{proj}_{\mathrm{Col}\,A}(b)$ be the closest vector for which $A\widehat{x} = \widehat{b}$ *does* have a solution.

Definition

A solution to $A\widehat{x} = \widehat{b}$ is a **least squares solution** to $Ax = b$. This is the solution $\widehat{x}$ for which $A\widehat{x}$ is *closest* to b (with respect to the usual notion of distance in $\mathbf{R}^n$).

Least-Squares Solutions

Suppose that $Ax = b$ is *in*consistent. Let $\widehat{b} = \mathrm{proj}_{\mathrm{Col}\,A}(b)$ be the closest vector for which $A\widehat{x} = \widehat{b}$ *does* have a solution.

Definition

A solution to $A\widehat{x} = \widehat{b}$ is a **least squares solution** to $Ax = b$. This is the solution $\widehat{x}$ for which $A\widehat{x}$ is *closest* to b (with respect to the usual notion of distance in $\mathbf{R}^n$).

Theorem

The least-squares solutions to $Ax = b$ are the solutions to

$$A^T A\widehat{x} = A^T b.$$

Least-Squares Solutions

Suppose that $Ax = b$ is *in*consistent. Let $\widehat{b} = \mathrm{proj}_{\mathrm{Col}\,A}(b)$ be the closest vector for which $A\widehat{x} = \widehat{b}$ *does* have a solution.

Definition

A solution to $A\widehat{x} = \widehat{b}$ is a **least squares solution** to $Ax = b$. This is the solution $\widehat{x}$ for which $A\widehat{x}$ is *closest* to b (with respect to the usual notion of distance in $\mathbf{R}^n$).

Theorem

The least-squares solutions to $Ax = b$ are the solutions to

$$A^T A\widehat{x} = A^T b.$$

If A has *orthogonal* columns $u_1, u_2, \ldots, u_n$, then the least-squares solution is

$$\widehat{x} = \left(\frac{x \cdot u_1}{u_1 \cdot u_1},\ \frac{x \cdot u_2}{u_2 \cdot u_2},\ \cdots,\ \frac{x \cdot u_m}{u_m \cdot u_m}\right)$$

because

$$A\widehat{x} = \widehat{b} = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$