

Chapter 1

Linear Equations

Section 1.1

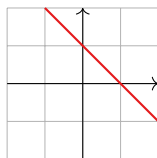
Systems of Linear Equations

One Linear Equation

What does the solution set of a linear equation look like?

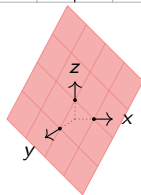
► $x + y = 1$

~~~~~> a line in the plane:  $y = 1 - x$



►  $x + y + z = 1$

~~~~~> a plane in space:  $z = 1 - x - y$



► $x + y + z + w = 1$

~~~~~> a "3-plane" in "4-space"...

[not pictured here]

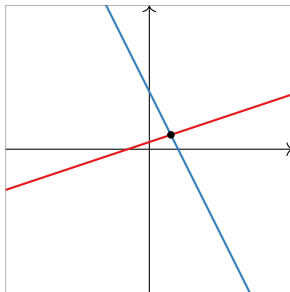
# Systems of Linear Equations

What does the solution set of a *system* of more than one linear equation look like?

$$x - 3y = -3$$

$$2x + y = 8$$

... is the *intersection* of two lines, which is a *point* in this case.



In general it's an intersection of lines, planes, etc.

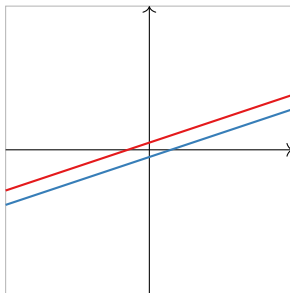
## Kinds of Solution Sets

In what other ways can two lines intersect?

$$x - 3y = -3$$

$$x - 3y = 3$$

has no solution: the lines are *parallel*.



A system of equations with no solutions is called **inconsistent**.

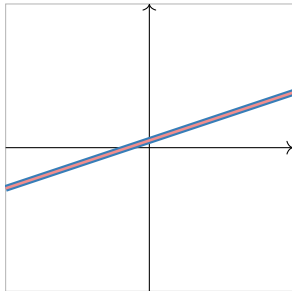
## Kinds of Solution Sets

In what other ways can two lines intersect?

$$x - 3y = -3$$

$$2x - 6y = -6$$

has infinitely many solutions:  
they are the *same line*.



Note that multiplying an equation by a nonzero number gives the *same solution set*. In other words, they are *equivalent* (systems of) equations.

What about in three variables?

Poll

In how many different ways can three planes intersect in space?

- A. One
- B. Two
- C. Three
- D. Four
- E. Five
- F. Six
- G. Seven

# Solving Systems of Equations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

This is the kind of problem we'll talk about for the first half of the course.

- ▶ A **solution** is a list of numbers  $x, y, z, \dots$  that make *all* of the equations true.
- ▶ The **solution set** is the collection of all solutions.
- ▶ **Solving** the system means finding the solution set.

What is a *systematic* way to solve a system of equations?



# Solving Systems of Equations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

What strategies do you know?

- ▶ Substitution
- ▶ Elimination

Both are perfectly valid, but only elimination scales well to large numbers of equations.

# Solving Systems of Equations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

**Elimination method:** in what ways can you manipulate the equations?

- ▶ Multiply an equation by a nonzero number.
- ▶ Add a multiple of one equation to another.
- ▶ Swap two equations.

(scale)

(replacement)

(swap)

# Solving Systems of Equations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Multiply first by  $-3$

~~~~~→

$$-3x - 6y - 9z = -18$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Add first to third

~~~~~→

$$-3x - 6y - 9z = -18$$

$$2x - 3y + 2z = 14$$

$$-5y - 10z = -20$$

Now I've eliminated  $x$  from the last equation!

...but there's a long way to go still. Can we make our lives easier?

# Solving Systems of Equations

Better notation

It sure is a pain to have to write  $x, y, z$ , and  $=$  over and over again.

**Matrix notation:** write just the numbers, in a box, instead!

$$\begin{array}{rcl} x + 2y + 3z & = & 6 \\ 2x - 3y + 2z & = & 14 \\ 3x + y - z & = & -2 \end{array} \quad \begin{array}{c} \text{becomes} \\ \text{~~~~~} \end{array} \quad \left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- ▶ Multiply all entries in a row by a nonzero number. (scale)
- ▶ Add a multiple of each entry of one row to the corresponding entry in another. (row replacement)
- ▶ Swap two rows. (swap)

# Row Operations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Start:

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

**Goal:** we want our elimination method to eventually produce a system of equations like

$$\begin{array}{rcl} x & = & A \\ y & = & B \\ z & = & C \end{array} \quad \text{or in matrix form,} \quad \left( \begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array} \right)$$

So we need to do row operations that make the start matrix look like the end one.

**Strategy:** fiddle with it so we only have ones and zeros.

# Row Operations

Continued

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

We want these to be zero.  
So we subtract multiples of the first row.

$$\begin{array}{l} R_2 = R_2 - 2R_1 \\ \hline \end{array}$$

$$\begin{array}{l} R_3 = R_3 - 3R_1 \\ \hline \end{array}$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array} \right)$$

We want these to be zero.

It would be nice if this were a 1.  
We could divide by  $-7$ , but that  
would produce ugly fractions.

Let's swap the last two rows first.

$$\begin{array}{l} R_2 \longleftrightarrow R_3 \\ \hline \end{array}$$

$$\begin{array}{l} R_2 = R_2 \div -5 \\ \hline \end{array}$$

$$\begin{array}{l} R_1 = R_1 - 2R_2 \\ \hline \end{array}$$

$$\begin{array}{l} R_3 = R_3 + 7R_2 \\ \hline \end{array}$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$

# Row Operations

Continued

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$

We want these to be zero.

Let's make this a 1 first.

$$\begin{array}{l} R_3 = R_3 \div 10 \\ \hline \end{array}$$

$$\begin{array}{l} R_1 = R_1 + R_3 \\ \hline \end{array}$$

$$\begin{array}{l} R_2 = R_2 - 2R_3 \\ \hline \end{array}$$

translates into  
 $\hline$

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\begin{array}{rcl} x & = & 1 \\ y & = & -2 \\ z & = & 3 \end{array}$$

Success!

Check:

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

substitute solution  
 $\hline$

$$1 + 2 \cdot (-2) + 3 \cdot 3 = 6$$

$$2 \cdot 1 - 3 \cdot (-2) + 2 \cdot 3 = 14$$

$$3 \cdot 1 + (-2) - 3 = -2$$



# Row Equivalence

## Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

## Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

So the linear equations of row-equivalent matrices have the *same solution set*.



# A Bad Example

## Example

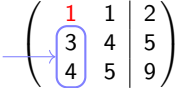
Solve the system of equations

$$x + y = 2$$

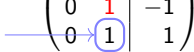
$$3x + 4y = 5$$

$$4x + 5y = 9$$

Let's try doing row operations:

First clear these by subtracting multiples of the first row. 

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array} \right) \xrightarrow{R_2 = R_2 - 3R_1} \left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 5 & 9 \end{array} \right)$$
$$\xrightarrow{R_3 = R_3 - 4R_1} \left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array} \right)$$

Now clear this by subtracting the second row. 

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array} \right) \xrightarrow{R_3 = R_3 - R_2} \left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array} \right)$$

# A Bad Example

Continued

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array} \right) \xrightarrow{\text{translates into}} \begin{array}{l} x + y = 2 \\ y = -1 \\ 0 = 2 \end{array}$$

In other words, the original equations

$$\begin{array}{lll} x + y = 2 & & x + y = 2 \\ 3x + 4y = 5 & \text{have the same solutions as} & y = -1 \\ 4x + 5y = 9 & & 0 = 2 \end{array}$$

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

## Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.