Section 5.5

Complex Eigenvalues

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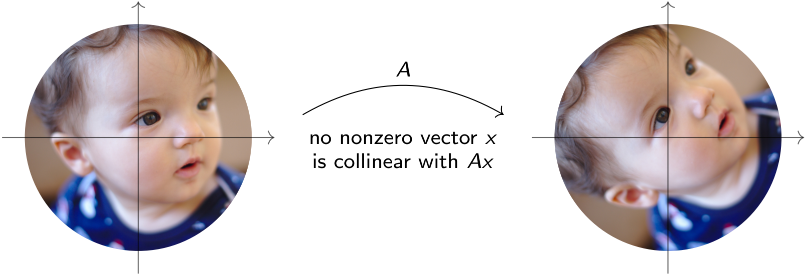

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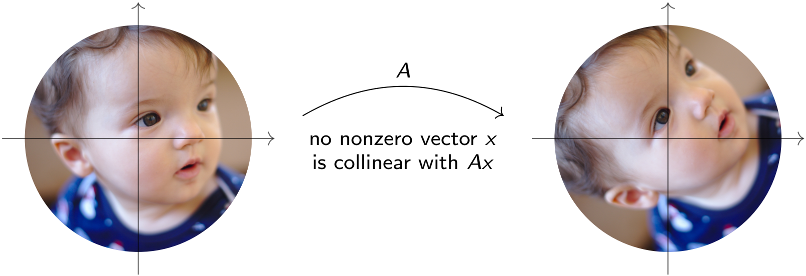

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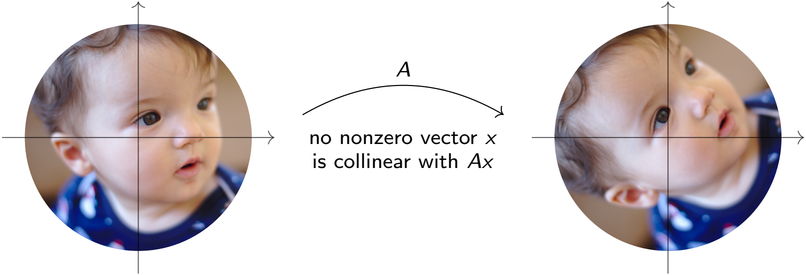

or algebraically:

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - \sqrt{2}\,\lambda + 1 \implies \lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2}.$$

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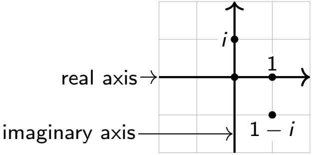

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So what's so strange about inventing a number i to solve the equation $x^2 + 1 = 0$?

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Real and imaginary part: $\mathrm{Re}(a+bi) = a \qquad \mathrm{Im}(a+bi) = b$.

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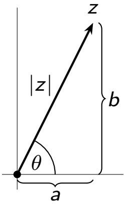
$$z = |z|(\cos\theta + i\sin\theta).$$

The angle θ is called the **argument** of z, and is denoted $\theta = \arg(z)$.



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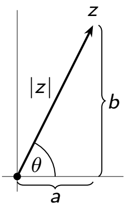

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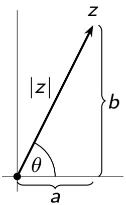
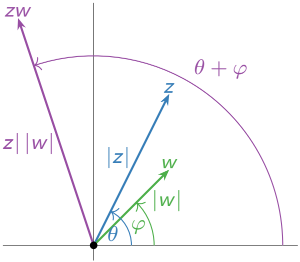


When you multiply complex numbers, you multiply the absolute values and add the arguments:

$$|zw| = |z|\,|w| \qquad \arg(zw) = \arg(z) + \arg(w).$$



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Equivalently, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then

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Important

If f is a polynomial with *real* coefficients, and if λ is a root of f, then so is $\overline{\lambda}$:

$$\begin{aligned} 0 = \overline{f(\lambda)} &= \overline{\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0} \\ &= \overline{\lambda}^n + a_{n-1}\overline{\lambda}^{n-1} + \cdots + a_1\overline{\lambda} + a_0 = f\left(\overline{\lambda}\right). \end{aligned}$$

The Fundamental Theorem of Algebra

The whole point of using complex numbers is to solve polynomial equations. It turns out that they are enough to find all solutions of all polynomial equations:

Fundamental Theorem of Algebra

Every polynomial of degree n has exactly n complex roots, counted with multiplicity.

Equivalently, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then

$$f(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$

for (not necessarily distinct) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

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Therefore complex roots of real polynomials come in *conjugate pairs*.

Degree 2: The quadratic formula gives you the (real or complex) roots of any degree-2 polynomial:

$$f(x) = x^2 + bx + c \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

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Note the roots are complex conjugates if b, c are real.

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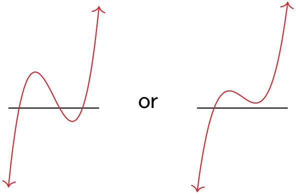

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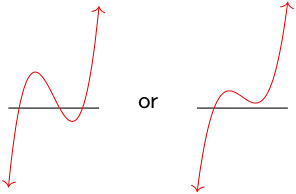

respectively.

Example: let $f(\lambda) = 5\lambda^3 - 18\lambda^2 + 21\lambda - 10$.

Poll

The characteristic polynomial of

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

is $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$. This has two complex roots $(1 \pm i)/\sqrt{2}$.

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Poll

Does A have any eigen*vectors*? If so, what are they?

A Matrix *with* an Eigenvector

Every matrix is guaranteed to have *complex* eigenvalues and eigenvectors.

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So is $i\begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ (you can scale by *complex* numbers).

A Trick for Computing Eigenvectors of 2×2 Matrices

Very useful for complex eigenvalues

Let A be a 2×2 matrix, and let λ be an eigenvalue of A.

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Example:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \lambda = \frac{1-i}{\sqrt{2}}.$$

Conjugate Eigenvectors

For $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$,

$$\text{the eigenvalue } \frac{1+i}{\sqrt{2}} \text{ has eigenvector } \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

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Both eigenvalues and eigenvectors of real square matrices occur in conjugate pairs.

A 3×3 Example

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

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We eyeball an eigenvector with eigenvalue 2 as $(0, 0, 1)$.

A 3×3 Example
Continued

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

To find the other eigenvectors, we row reduce:

Theorem

Let A be a 2×2 matrix with complex (non-real) eigenvalue λ, and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \operatorname{Re} v & \operatorname{Im} v \\ | & | \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \operatorname{Re} \lambda & \operatorname{Im} \lambda \\ -\operatorname{Im} \lambda & \operatorname{Re} \lambda \end{pmatrix}.$$

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The matrix C is a composition of rotation by $-\arg(\lambda)$ and scaling by $|\lambda|$:

$$C = \begin{pmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{pmatrix} \begin{pmatrix} \cos(-\arg(\lambda)) & -\sin(-\arg(\lambda)) \\ \sin(-\arg(\lambda)) & \cos(-\arg(\lambda)) \end{pmatrix}.$$

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A 2×2 matrix with complex eigenvalue λ is similar to (rotation by the argument of $\overline{\lambda}$) composed with (scaling by $|\lambda|$). This is multiplication by $\overline{\lambda}$ in $\mathbf{C} \sim \mathbf{R}^2$.

What does $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ do geometrically?

Geometric Interpretation of Complex Eigenvalues

2×2 example, continued

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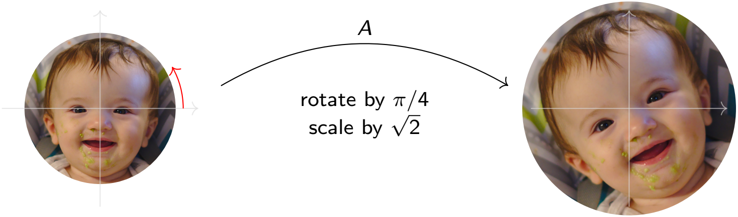

Geometric Interpretation of Complex Eigenvalues

Another 2×2 example

What does $A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$ do geometrically?

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Another 2×2 example: picture

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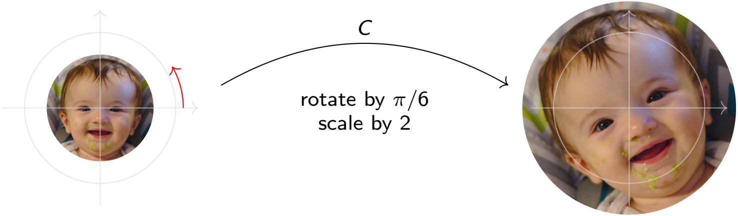

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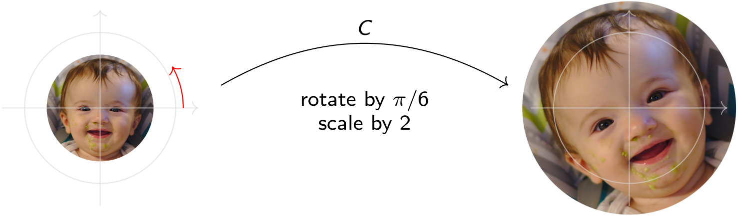
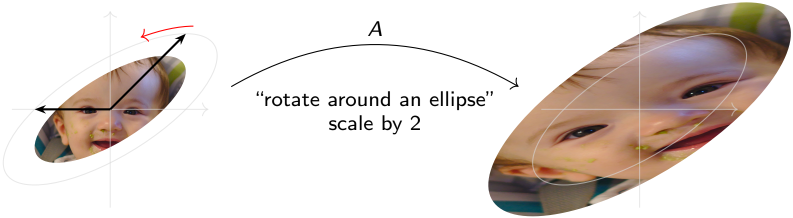


$A = PCP^{-1}$ does the same thing, but with respect to the basis $\mathcal{B} = \left\{ \binom{1}{1}, \binom{-1}{0} \right\}$ of columns of P:



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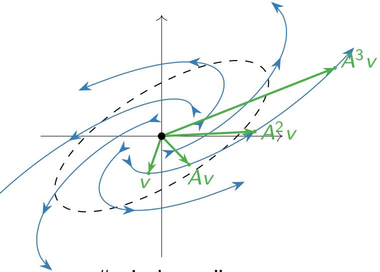

"spirals out"

Classification of 2×2 Matrices with a Complex Eigenvalue

Triptych

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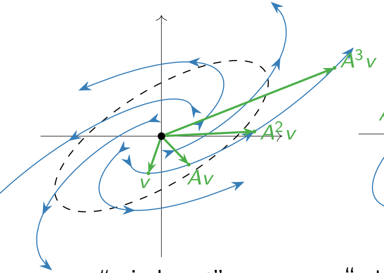

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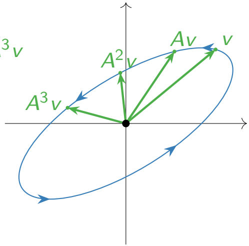

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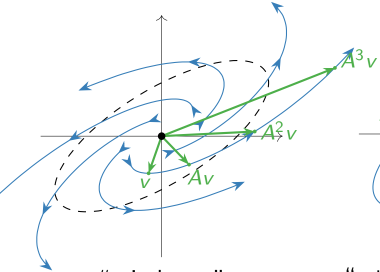

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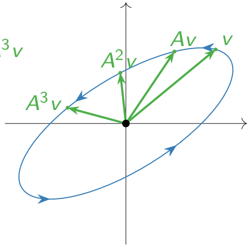

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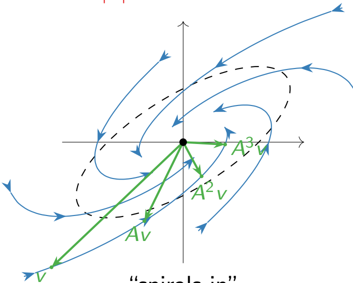

“spirals in”

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Let A be a 2×2 matrix with linearly independent eigenvectors v_1, v_2 and associated eigenvalues λ_1, λ_2. Then

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scale x-axis by λ_1
scale y-axis by λ_2

Picture with 2 Real Eigenvalues

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Therefore, $A = PDP^{-1}$ with

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Picture with 2 Real Eigenvalues

We can draw analogous pictures for a matrix with 2 real eigenvalues.

Example: Let $A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$.

This has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \frac{1}{2}$, with eigenvectors

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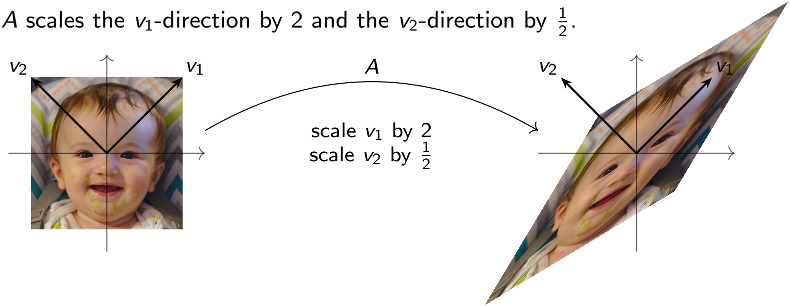
So A scales the v_1-direction by 2 and the v_2-direction by $\frac{1}{2}$.



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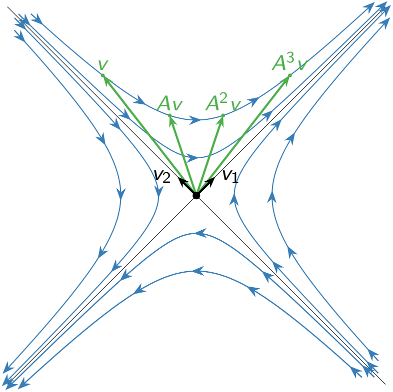

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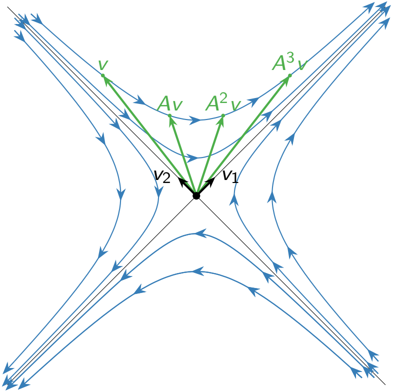

Exercise: Draw analogous pictures when $|\lambda_1|, |\lambda_2|$ are any combination of $< 1, = 1, > 1$.

Theorem

Let A be a real $n \times n$ matrix. Suppose that for each (real or complex) eigenvalue, the dimension of the eigenspace equals the algebraic multiplicity. Then $A = PCP^{-1}$, where P and C are as follows:

The Higher-Dimensional Case

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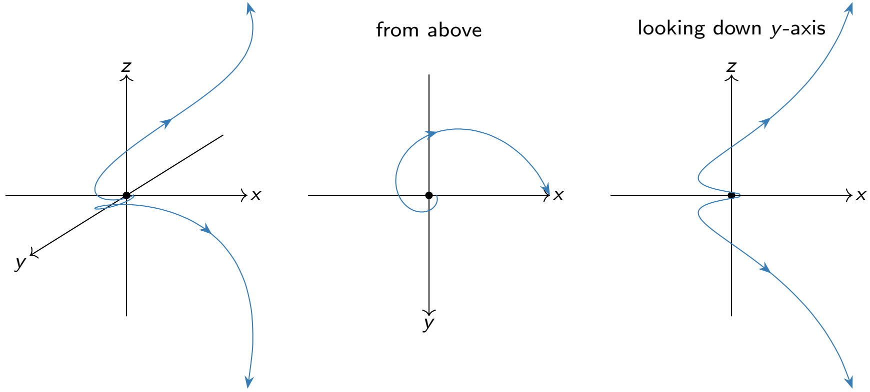

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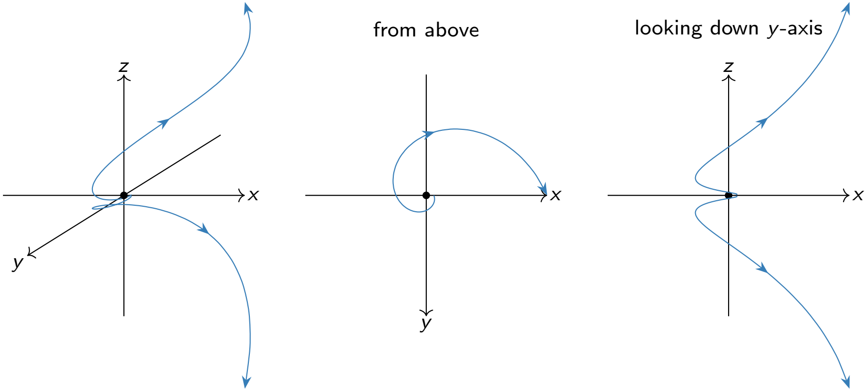

Remember, in general $A = PCP^{-1}$ is only *similar* to such a matrix C: so the x, y, z axes have to be replaced by the columns of P.