Section 1.7

Linear Independence

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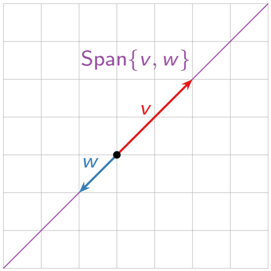

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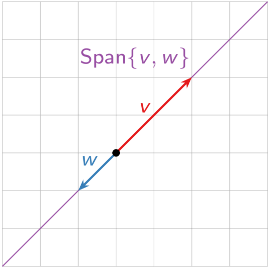

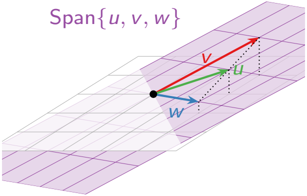

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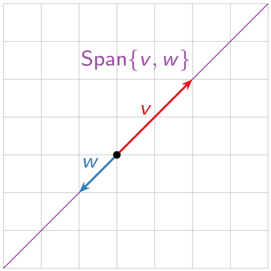

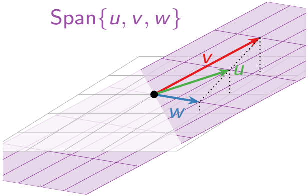

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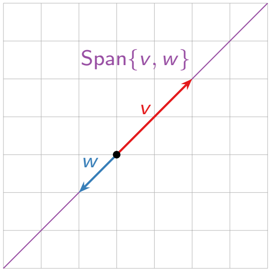

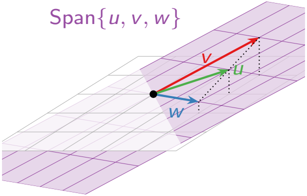

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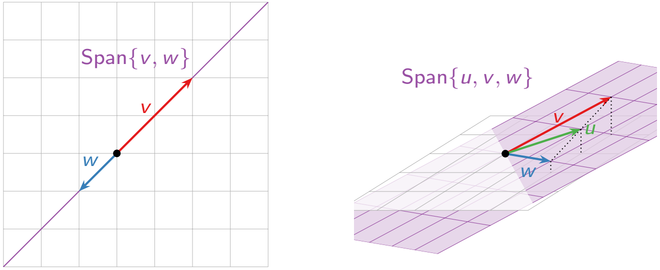

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Notice in each case that one vector in the set is already in the span of the others—so it doesn’t make the span bigger.

Today we will formalize this idea in the concept of *linear (in)dependence*.

Linear Independence

Definition

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$.

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This is called a **linear dependence relation**.

Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

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Note that linear (in)dependence is a notion that applies to a *collection of vectors*, not to a single vector, or to one vector in the presence of some others.

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

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Important

- The vectors $v_1, v_2, \ldots, v_p$ are linearly independent if and only if the matrix with columns $v_1, v_2, \ldots, v_p$ has a pivot in each column.
- Solving the matrix equation $Ax = 0$ will either verify that the columns $v_1, v_2, \ldots, v_p$ of A are linearly independent, or will produce a linear dependence relation.

Suppose that one of the vectors $\{v_1, v_2, \ldots, v_p\}$ is a linear combination of the other ones (that is, it is in the span of the other ones):

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Then the vectors are linearly *de*pendent:

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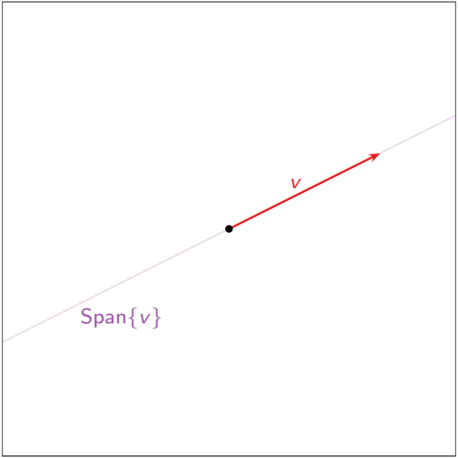
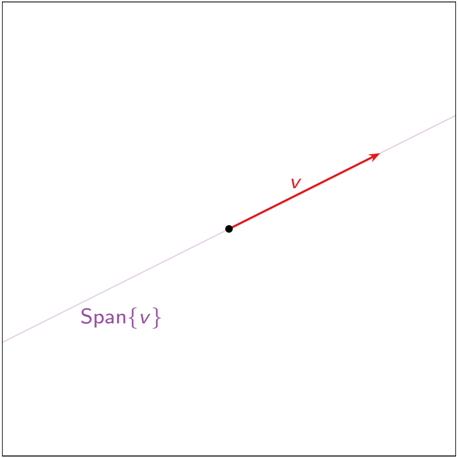


In this picture

One vector $\{v\}$:

Linear Independence

Pictures in $\mathbf{R}^2$



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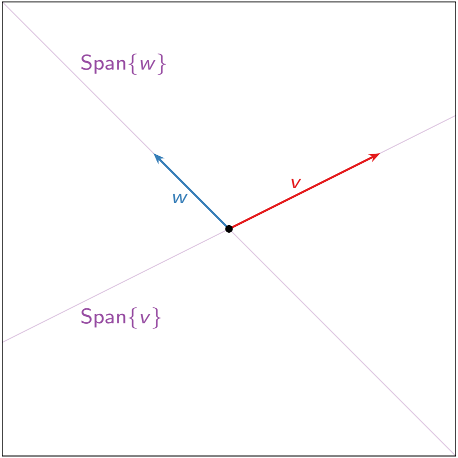

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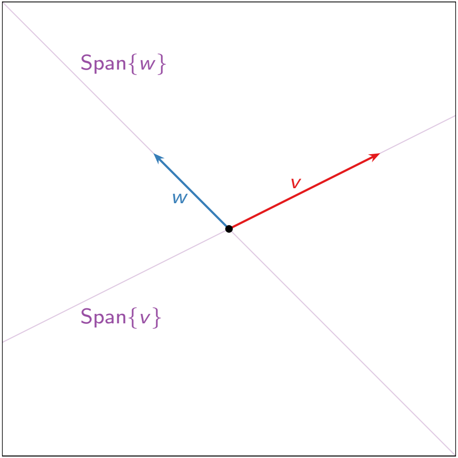

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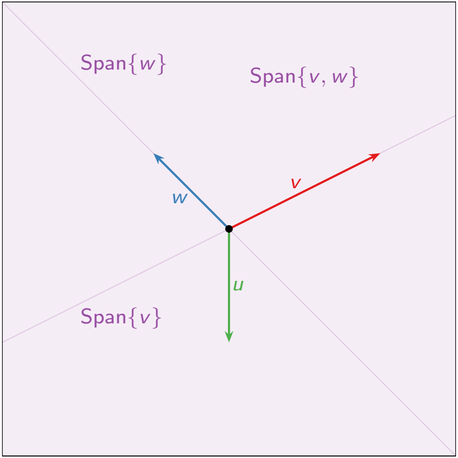

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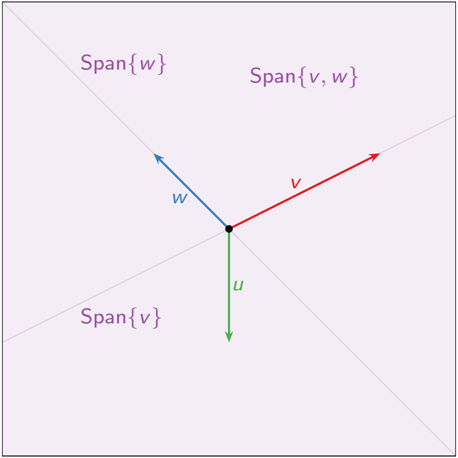

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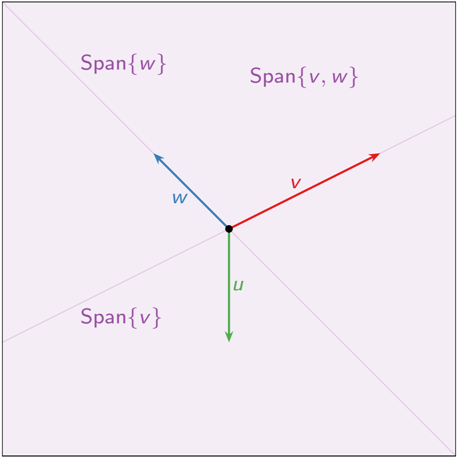

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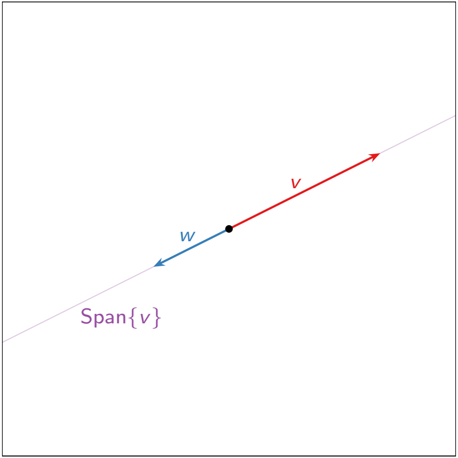
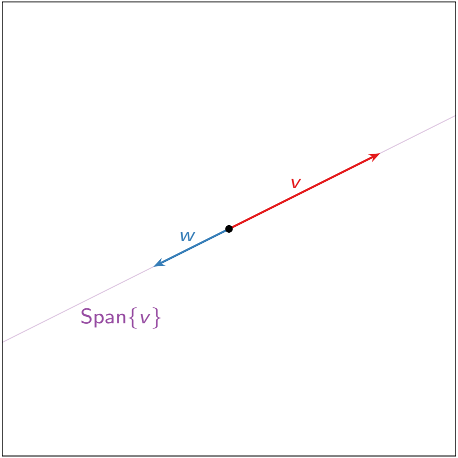
Also v is in Span$\{u, w\}$ and w is in Span$\{u, v\}$.



Two collinear vectors $\{v, w\}$:



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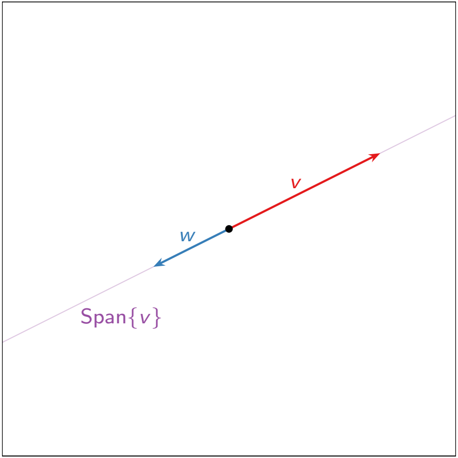

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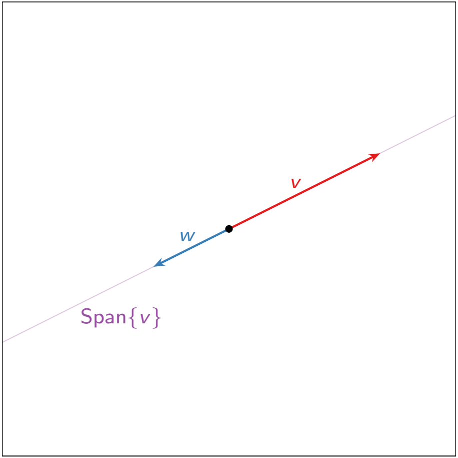
Observe: *Two* vectors are linearly *de*pendent if and only if



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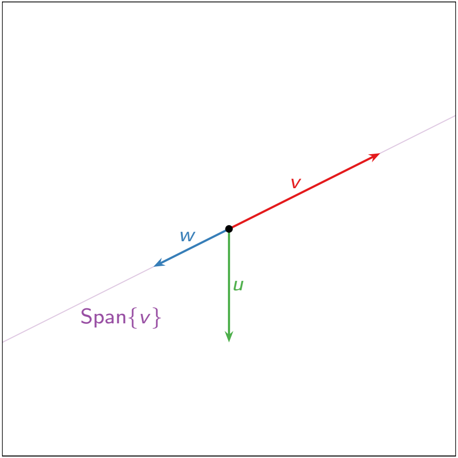

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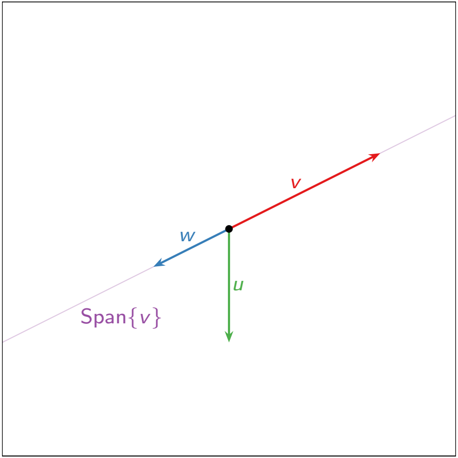

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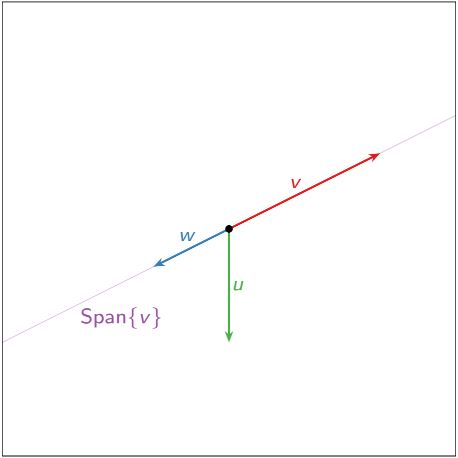

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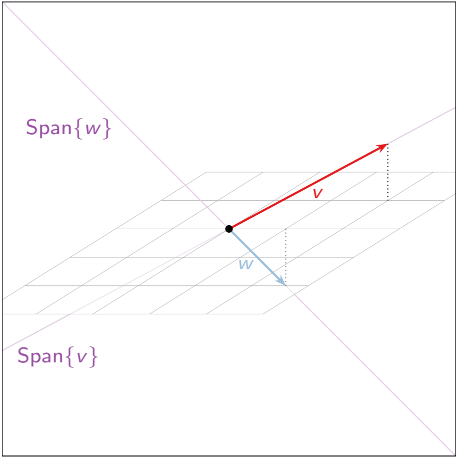
Observe: If a set of vectors is linearly dependent, then so is any larger set of vectors!



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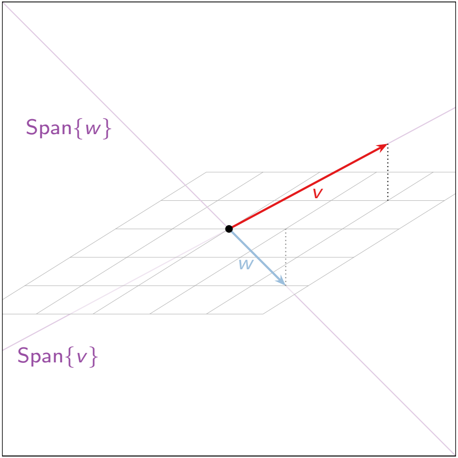

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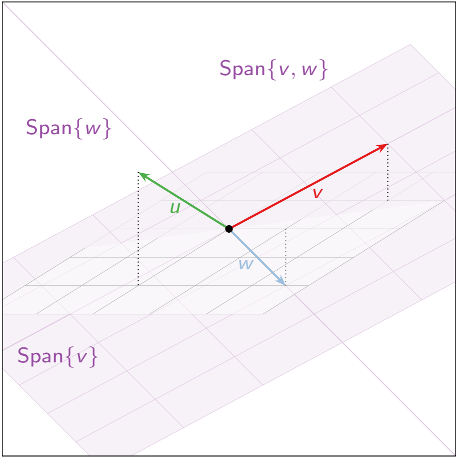

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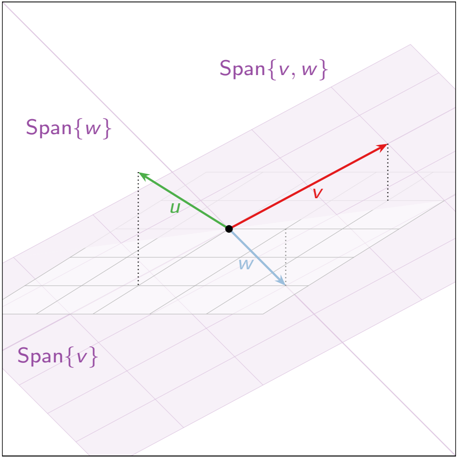

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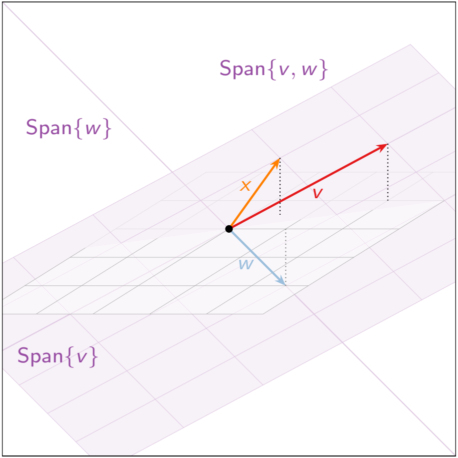
Three vectors $\{v, w, u\}$:
Linearly independent: no one is in the span of the other two.



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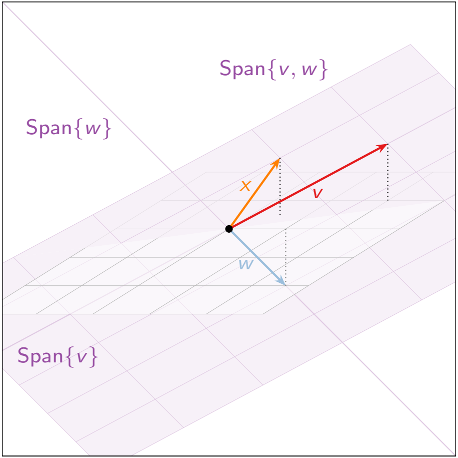

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Two vectors $\{v, w\}$:
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Three vectors $\{v, w, x\}$:
Linearly dependent: x is in $\mathrm{Span}\{v, w\}$.

Poll

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Are there four vectors u, v, w, x in $\mathbf{R}^3$ which are linearly dependent, but such that u is *not* a linear combination of v, w, x? If so, draw a picture; if not, give an argument.

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Linear dependence of $\{v_1, \ldots, v_p\}$ means *some* v_i is a linear combination of the others, not *any*.

Theorem

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Rearrange:

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A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.

Take the largest j such that v_j is in the span of the others. Then v_j is in the span of $v_1, v_2, \ldots, v_{j-1}$. Why? If not ($j = 3$):

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Better Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if and only if there is some j such that v_j is in $\mathrm{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

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Linear Independence

Increasing span criterion

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A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly independent if and only if, for every j, the span of $v_1, v_2, \ldots, v_j$ is strictly larger than the span of $v_1, v_2, \ldots, v_{j-1}$.

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Translation

A set of vectors is linearly independent if and only if, every time you add another vector to the set, the span gets bigger.

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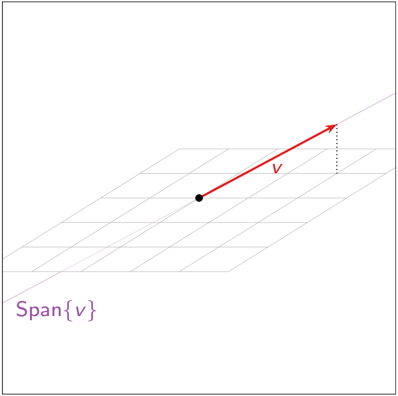

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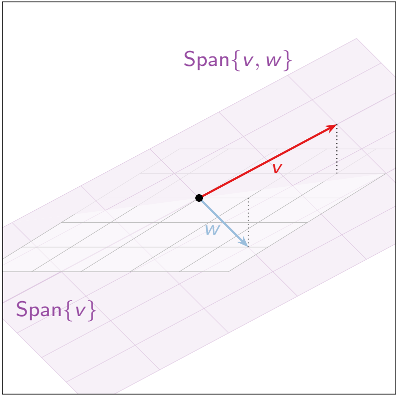

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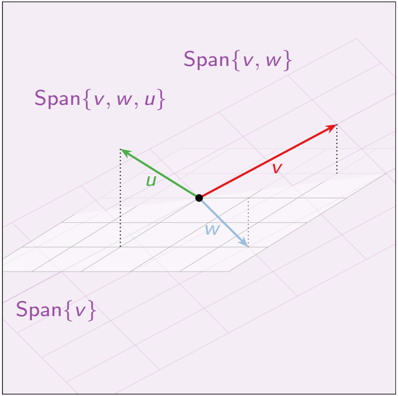

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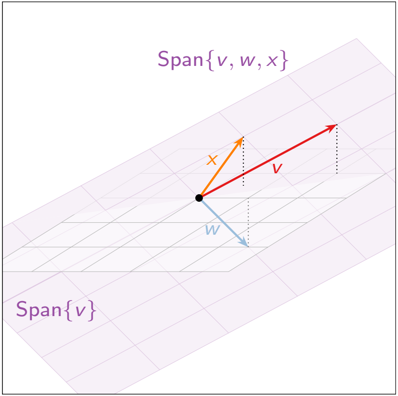

One vector $\{v\}$:
Linearly independent: span got bigger (than $\{0\}$).

Two vectors $\{v, w\}$:
Linearly independent: span got bigger.

Three vectors $\{v, w, x\}$:
Linearly dependent: span didn't get bigger.

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Linear Independence

Two more facts

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> A set containing the zero vector is linearly dependent.