# Section 2.3

## Characterization of Invertible Matrices

# Invertible Transformations

## Definition

A transformation $T \colon \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists another transformation $U \colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x$$

for all $x$ in $\mathbf{R}^n$.

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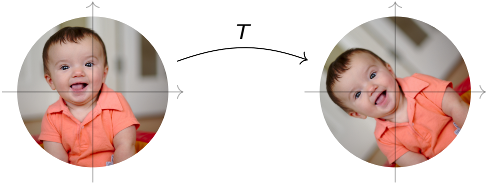
**Fact**

A transformation $T$ is invertible if and only if it is both one-to-one and onto.

Let $T =$ counterclockwise rotation in the plane by $45^\circ$. What is $T^{-1}$?



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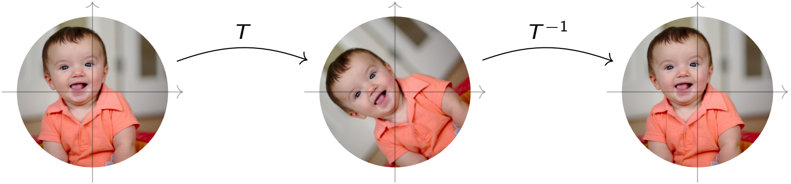


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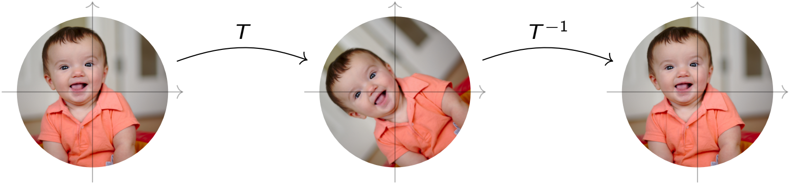


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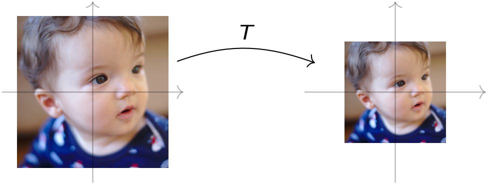

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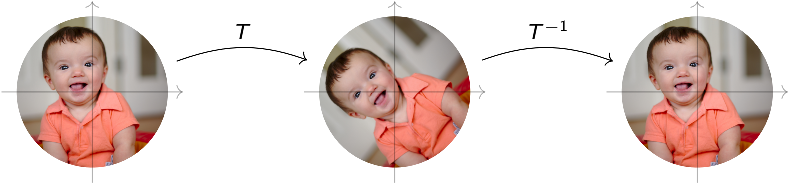


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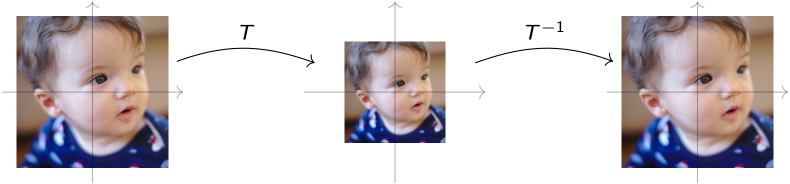


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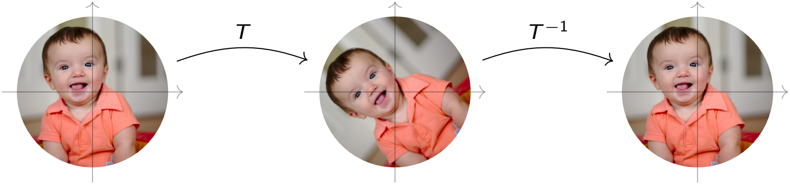


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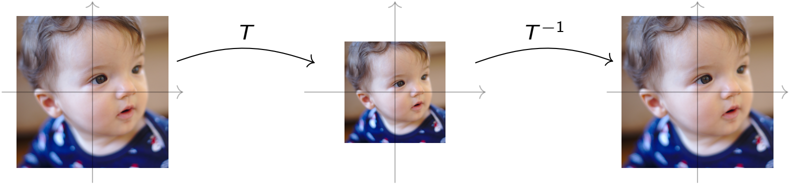


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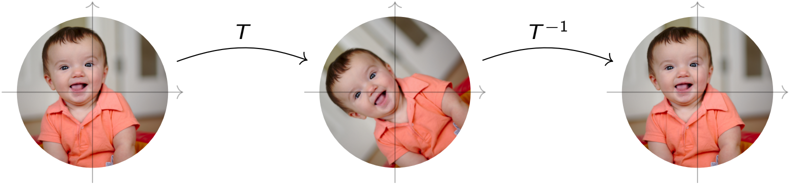


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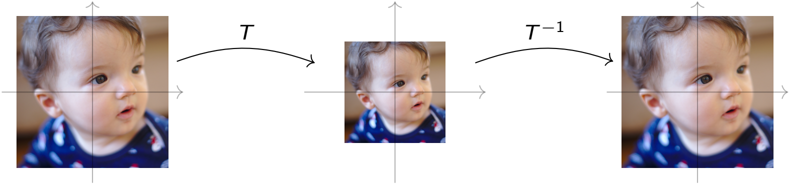


$T^{-1}$ is *stretching* by $3/2$.

Let $T$ = projection onto the $x$-axis. What is $T^{-1}$? It is not invertible: you can't undo it.

# Invertible Linear Transformations

If $T : \mathbf{R}^n \to \mathbf{R}^n$ is an invertible *linear* transformation with matrix $A$, then what is the matrix for $T^{-1}$?

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**Fact**

If $T$ is an invertible linear transformation with matrix $A$, then $T^{-1}$ is an invertible linear transformation with matrix $A^{-1}$.

Let $T$ = counterclockwise rotation in the plane by $45^\circ$. Its matrix is

Then $T^{-1}$ = counterclockwise rotation by $-45^\circ$. Its matrix is

Check:

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## The Invertible Matrix Theorem

Let $A$ be an $n \times n$ matrix, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. $A$ is invertible.

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A.K.A. The Really Big Theorem of Math 1553

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You know enough at this point to be able to reduce all of the statements to assertions about the pivots of a square matrix.