Application

Stochastic Matrices and PageRank

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You'll be responsible for knowing basic facts about stochastic matrices and the Perron–Frobenius theorem, but we will not cover them in depth. These slides are the primary reference; see also §4.9 in Lay.

The specifics of the PageRank algorithm are just for fun.

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If x_n, y_n, z_n are the numbers of movies in locations $1, 2, 3$, respectively, on day n, and $v_n = (x_n, y_n, z_n)$, then:

$$v_n = Av_{n-1} = A^2 v_{n-2} = \cdots = A^n v_0.$$

Stochastic Matrices and Difference Equations

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- v_{n+1} is the "state at time $n + 1$", and
- $v_{n+1} = Av_n$ means that A is the "change of state matrix."

Eigenvalues of Stochastic Matrices

Fact: 1 is an eigenvalue of a stochastic matrix.

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Why? If A is stochastic, then 1 is an eigenvalue of A^T:

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Note: This doesn't give a new procedure for finding an eigenvector with eigenvalue 1; it only shows one exists.

Fact: if λ is an eigenvalue of a stochastic matrix, then $|\lambda| \leq 1$. Hence 1 is the *largest* eigenvalue (in absolute value).

Eigenvalues of Stochastic Matrices

Continued

Fact: if λ is an eigenvalue of a stochastic matrix, then $|\lambda| \leq 1$. Hence 1 is the *largest* eigenvalue (in absolute value).

Better fact: if $\lambda \neq 1$ is an eigenvalue of a *positive* stochastic matrix, then $|\lambda| < 1$.

Diagonalizable Stochastic Matrices

Example from §5.3

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So all vectors get "sucked into the 1-eigenspace," which is spanned by $w_1 = \binom{1}{1}$.

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Recall: $A^n = PD^nP^{-1}$ acts on the usual coordinates of v_0 in the same way that D^n acts on the $\mathcal{B}$-coordinates, where $\mathcal{B} = \{w_1, w_2\}$.

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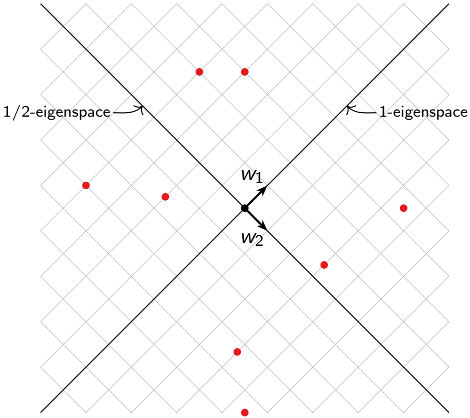

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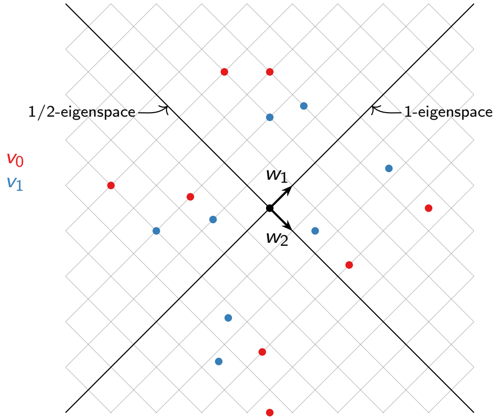

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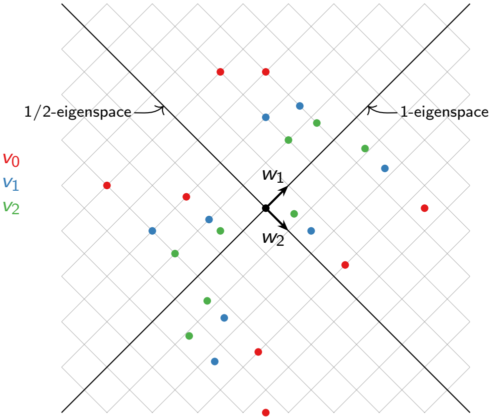

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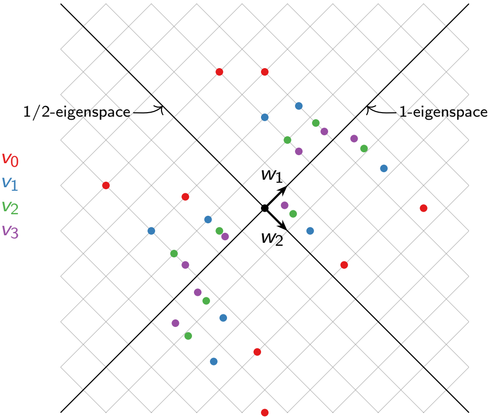

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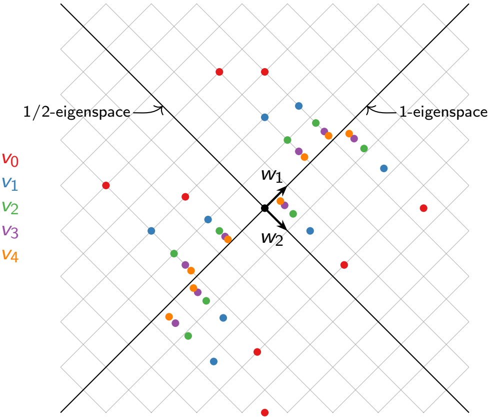

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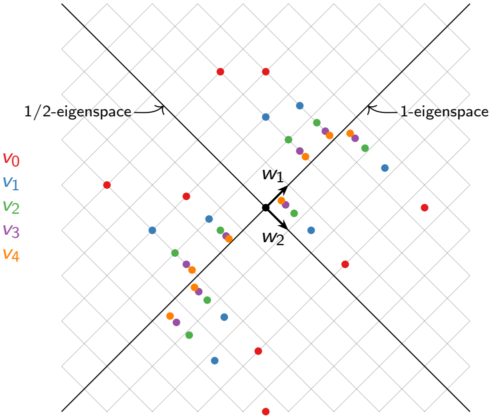

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The Red Box matrix $A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$ has characteristic polynomial

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Let w_1, w_2, w_3 be the columns of P, i.e. the eigenvectors of P with respective eigenvalues $1, .1, -.2$. Let $\mathcal{B} = \{w_1, w_2, w_3\}$.

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Let w_1, w_2, w_3 be the columns of P, i.e. the eigenvectors of P with respective eigenvalues $1, .1, -.2$. Let $\mathcal{B} = \{w_1, w_2, w_3\}$.

Recall: A^n acts on the usual coordinates of a vector in the same way that D acts on the $\mathcal{B}$-coordinates: $[A^n x]_{\mathcal{B}} = D^n [x]_{\mathcal{B}}$.

Diagonalizable Stochastic Matrices

Continued

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(We'll see in a moment why I chose that eigenvector.)

Diagonalizable Stochastic Matrices
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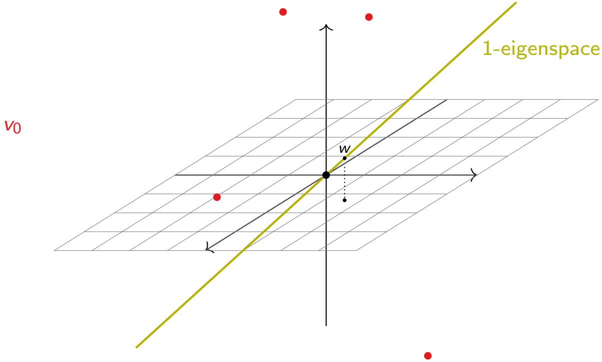
Start with a vector v_0 (the number of movies on the first day), let $v_1 = Av_0$ (the number of movies on the second day), let $v_2 = Av_1$, etc.



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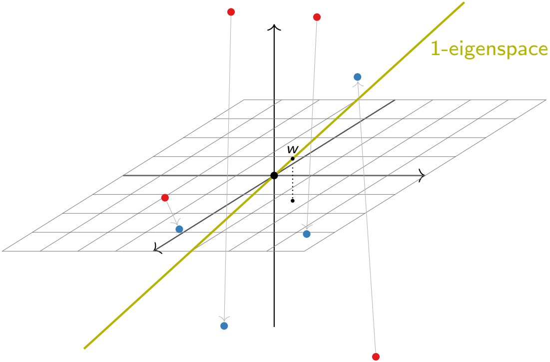

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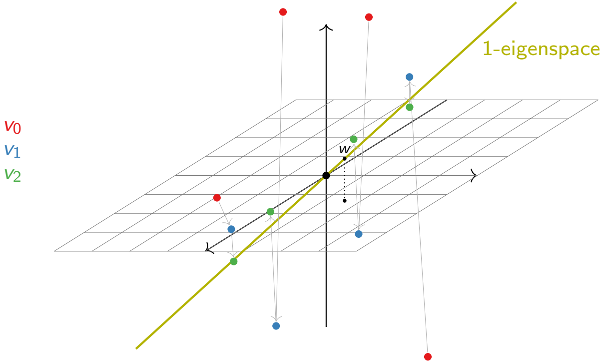

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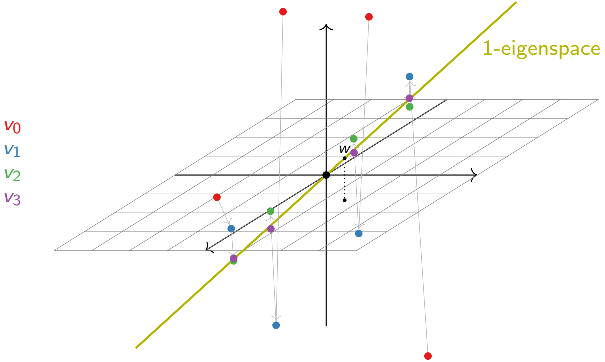

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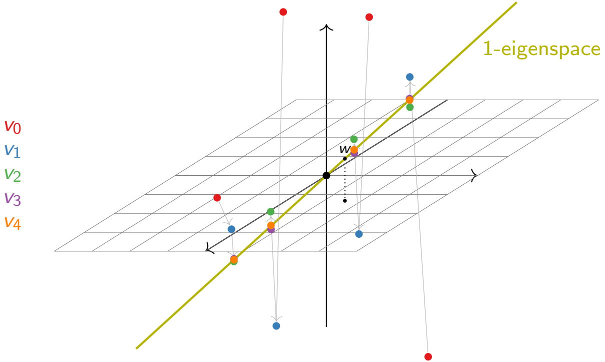

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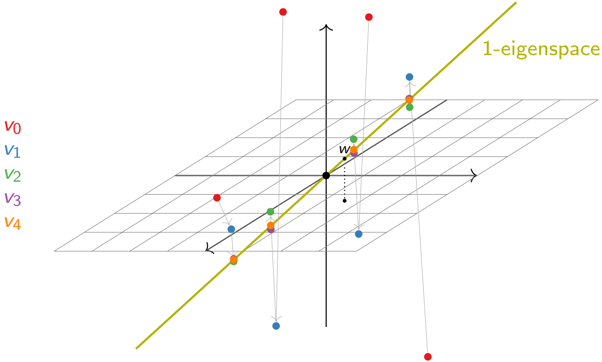

We see that v_n approaches an eigenvector with eigenvalue 1 as n gets large: all vectors get "sucked into the 1-eigenspace."

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Presumably, Red Box really does have to do this kind of analysis to determine how many videos to put in each box.

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- The sum c of the entries of v_0 is the total number of movies; eventually, the movies arrange themselves according to the steady state percentage, i.e., $v_n \to cw$.

Steady State

Red Box example

Consider the Red Box matrix $A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$.

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The Perron–Frobenius Theorem says that our analysis of the Red Box matrix works for *any* positive stochastic matrix—whether or not it is diagonalizable!

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Reference:

`http://www.math.cornell.edu/~mec/Winter2009/RalucaRemus/Lecture3/lecture3.html`

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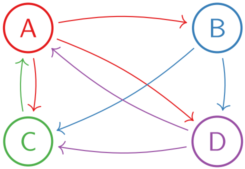

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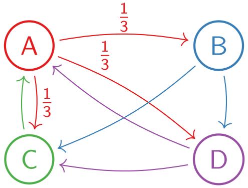

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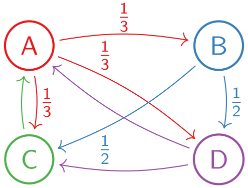

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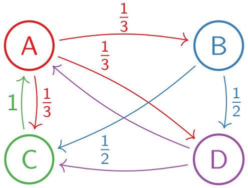

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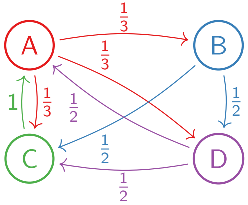

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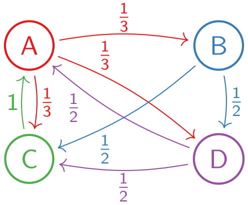

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In terms of matrices, if $v = (a, b, c, d)$ is the vector containing the ranks a, b, c, d of the pages A, B, C, D, then

$$\begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} c + \frac{1}{2}d \\ \frac{1}{3}a \\ \frac{1}{3}a + \frac{1}{2}b + \frac{1}{2}d \\ \frac{1}{3}a + \frac{1}{2}b \end{pmatrix}$$

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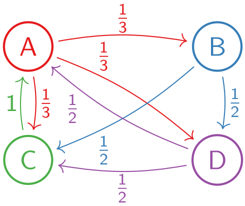

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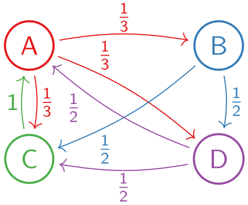
The Importance Matrix

Consider the following Internet with only four pages. Links are indicated by arrows.



Page A has 3 links, so it passes $\frac{1}{3}$ of its importance to pages B, C, D.
Page B has 2 links, so it passes $\frac{1}{2}$ of its importance to pages C, D.
Page C has one link, so it passes all of its importance to page A.
Page D has 2 links, so it passes $\frac{1}{2}$ of its importance to pages A, C.

In terms of matrices, if $v = (a, b, c, d)$ is the vector containing the ranks a, b, c, d of the pages A, B, C, D, then

importance matrix: *ij entry is importance page j passes to page i*

$$\begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} c + \frac{1}{2}d \\ \frac{1}{3}a \\ \frac{1}{3}a + \frac{1}{2}b + \frac{1}{2}d \\ \frac{1}{3}a + \frac{1}{2}b \end{pmatrix} \overset{\text{Importance Rule}}{=} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

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The rank vector is a *steady state* for the importance matrix: it's the probability vector (a, b, c, d) such that, after clicking on a random link, the random surfer will have the *same probability* of being on each page.

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So, the important (high-ranked) pages are those where a random surfer will end up most often.

Problems with the Importance Matrix

Dangling pages

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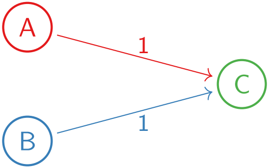

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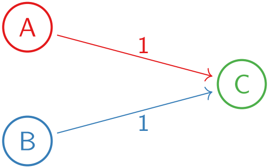

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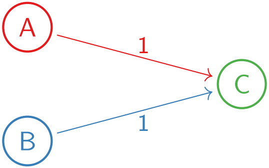

The importance matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$.

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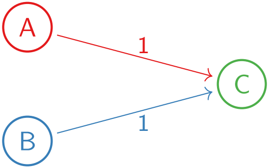

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$$f(\lambda) = \det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 1 & 1 & -\lambda \end{pmatrix} = -\lambda^3.$$

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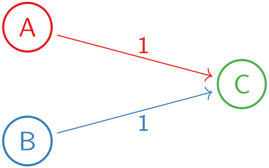

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So 1 is not an eigenvalue at all: there is no rank vector! (It is not stochastic.)

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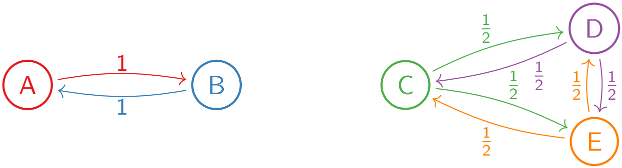

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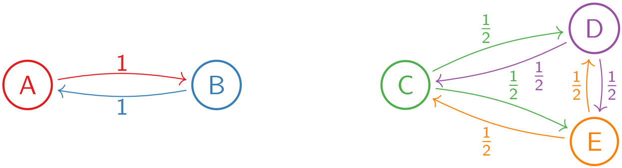

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Here is Page and Brin's solution.

The Google Matrix

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$$M = (1-p)\cdot A + p\cdot B \quad \text{where} \quad B = \frac{1}{N}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

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