

Section 1.8

Introduction to Linear Transformations

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This is also a way to understand the *geometry of matrices*.

Transformations

Definition

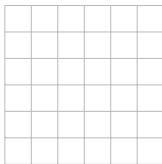
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- ▶ \mathbf{R}^n is called the **domain** of T (the inputs).



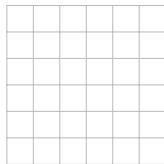
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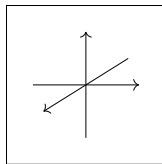
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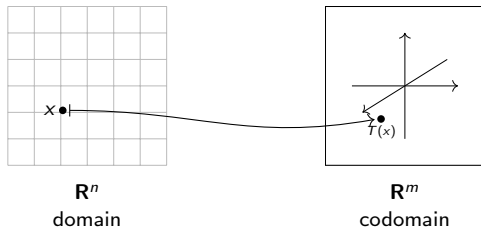
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Notation: $x \mapsto T(x)$.



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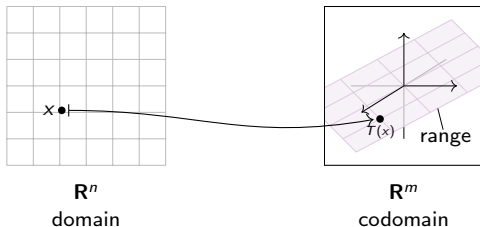
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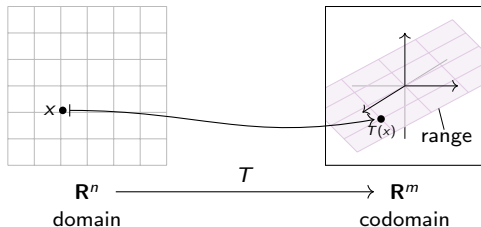
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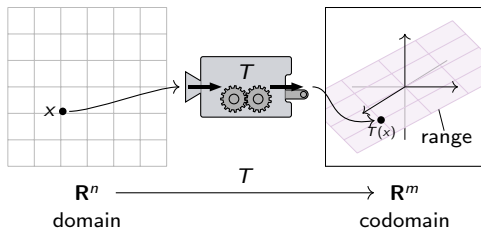
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It may help to think of T as a “machine” that takes x as an input, and gives you $T(x)$ as the output.

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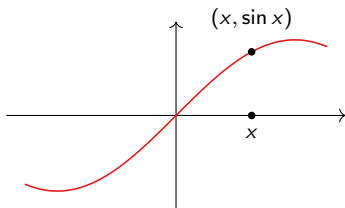
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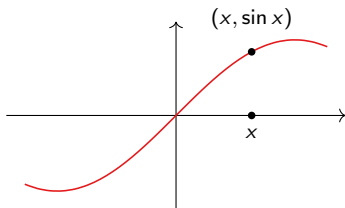
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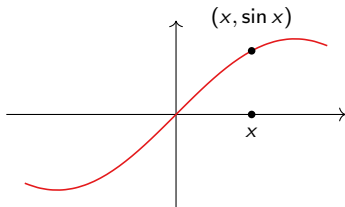
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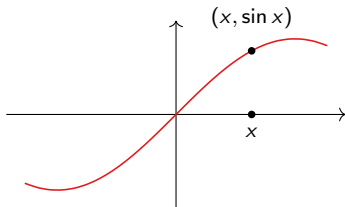
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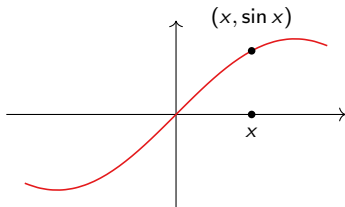
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Your life will be much easier
if you just remember these.

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The solution set of $Ax = c$ is a translate of the solution set of $Ax = b$ (from before), which has ___ vector in it.

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- Is there any c in \mathbf{R}^3 such that there is more than one v in \mathbf{R}^2 with $T(v) = c$?

Translation: is there any c in \mathbf{R}^3 such that the solution set of $Ax = c$ has more than one vector v in it?

The solution set of $Ax = c$ is a translate of the solution set of $Ax = b$ (from before), which has one vector in it.

Matrix Transformations

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anything in the column span has the same first and last coordinate. So $c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is not in the column span (for example).

Matrix Transformations

Geometric example

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbf{R}^3 \rightarrow \mathbf{R}^3$.

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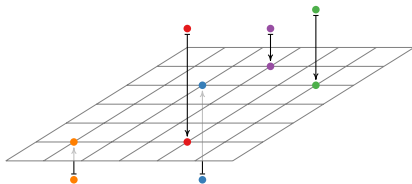
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This is *projection onto the xy -axis*. Picture:



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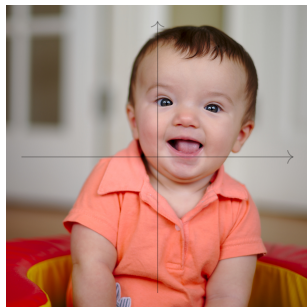
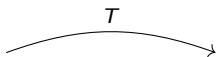
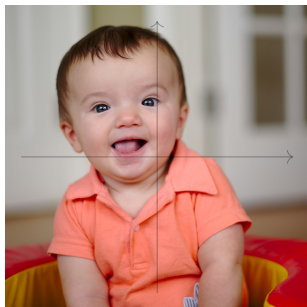
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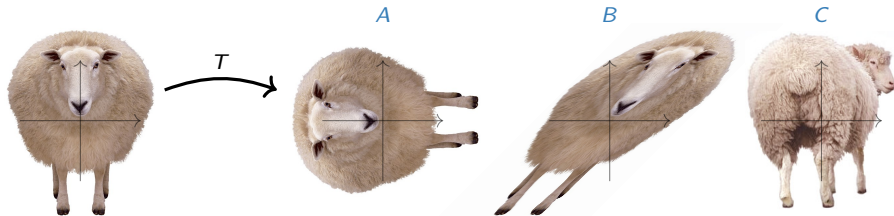


Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$. (T is called a **shear**.)

Poll

What does T do to this sheep?

Hint: first draw a picture what it does to the box *around* the sheep.

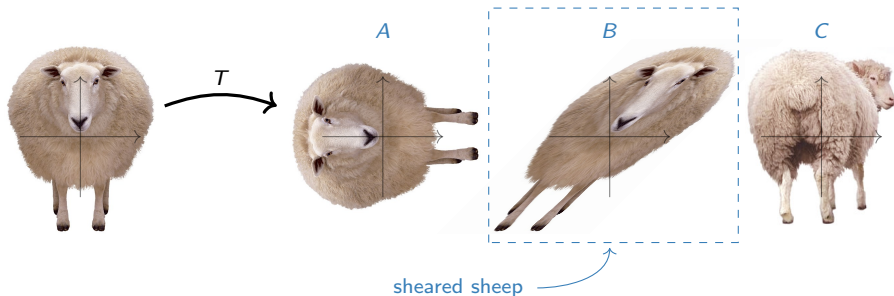


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Linear Transformations

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

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In engineering this is called **superposition**.

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Dilation

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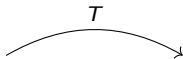
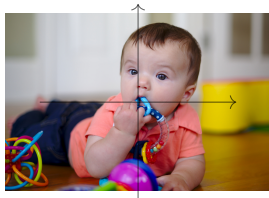
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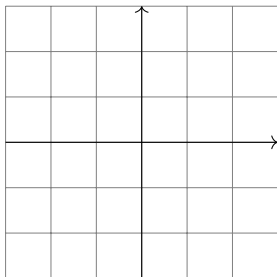
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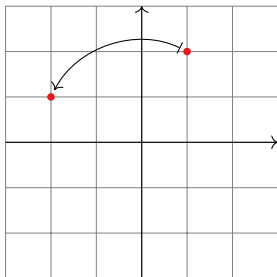
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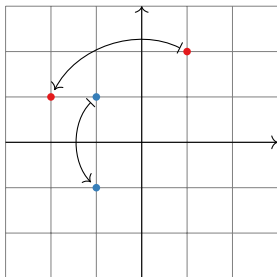
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$$T \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

