

Section 6.3

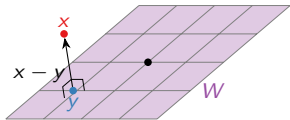
Orthogonal Projections

Idea Behind Orthogonal Projections

If x is not in a subspace W , then y in W is the closest to x if

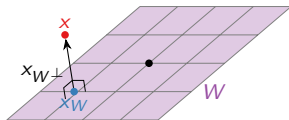
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If x is not in a subspace W , then y in W is the closest to x if $x - y$ is in W^\perp :



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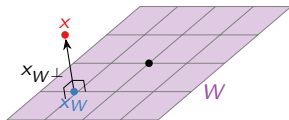
Reformulation: Every vector x can be decomposed uniquely as

$$x = x_W + x_{W^\perp}$$

where $x_W = y$ is the closest vector to x in W , and $x_{W^\perp} = x - y$ is in W^\perp .

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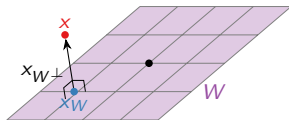
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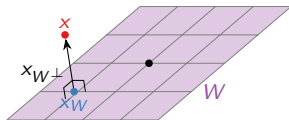
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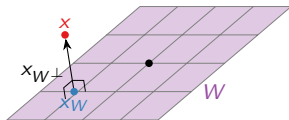
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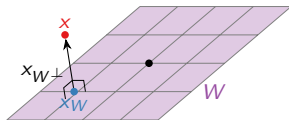
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$$x_L = \text{proj}_L(x) = -\frac{10}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad x_{L^\perp} =$$

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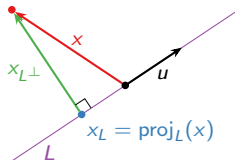
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where $x_W = y$ is the closest vector to x in W , and $x_{W^\perp} = x - y$ is in W^\perp .

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$$x_L = \text{proj}_L(x) = -\frac{10}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad x_{L^\perp} = x - \text{proj}_L(x) = \begin{pmatrix} -6 \\ 4 \end{pmatrix} + \frac{10}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$



Orthogonal Projections

Definition

Let W be a subspace of \mathbf{R}^n , and let $\{u_1, u_2, \dots, u_m\}$ be an *orthogonal* basis for W . The **orthogonal projection** of a vector x onto W is

$$\text{proj}_W(x) \stackrel{\text{def}}{=} \sum_{i=1}^m \frac{x \cdot u_i}{u_i \cdot u_i} u_i.$$

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Theorem

Let W be a subspace of \mathbf{R}^n , and let x be a vector in \mathbf{R}^n . Then $\text{proj}_W(x)$ is the closest point to x in W .

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Theorem

Let W be a subspace of \mathbf{R}^n , and let x be a vector in \mathbf{R}^n . Then $\text{proj}_W(x)$ is the closest point to x in W . Therefore

$$x_W = \text{proj}_W(x) \quad x_{W^\perp} = x - \text{proj}_W(x).$$

Orthogonal Projections

Easy example

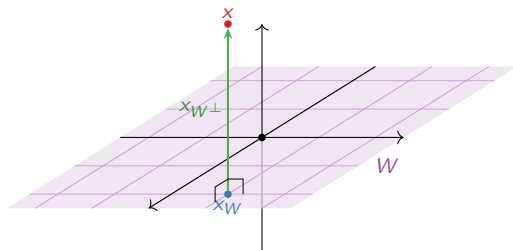
What is the projection of $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ onto the xy -plane?

Orthogonal Projections

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So this is the same projection as before.



Orthogonal Projections

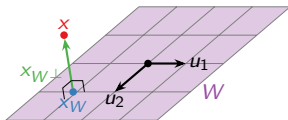
More complicated example

What is the projection of $x = \begin{pmatrix} -1.1 \\ 1.4 \\ 1.45 \end{pmatrix}$ onto $W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.1 \\ -0.2 \end{pmatrix} \right\}$?

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Orthogonal Projections

Picture

Let W be a subspace of \mathbf{R}^n , and let $\{u_1, u_2, \dots, u_m\}$ be an orthogonal basis for W . Let $L_i = \text{Span}\{u_i\}$. Then

$$\text{proj}_W(x) = \sum_{i=1}^m \frac{x \cdot u_i}{u_i \cdot u_i} u_i = \sum_{i=1}^m \text{proj}_{L_i}(x).$$

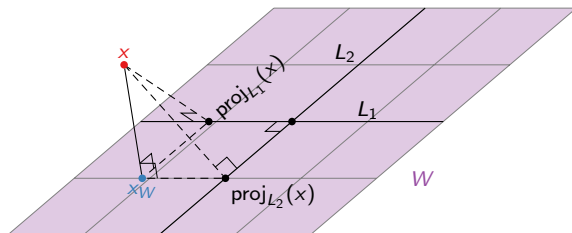
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So the orthogonal projection is formed by adding orthogonal projections onto perpendicular lines.



Orthogonal Projections

Properties

First we restate the property we've been using all along.

Best Approximation Theorem

Let W be a subspace of \mathbf{R}^n , and let x be a vector in \mathbf{R}^n . Then $y = \text{proj}_W(x)$ is the closest point in W to x , in the sense that

$$\text{dist}(x, y') \geq \text{dist}(x, y) \quad \text{for all } y' \text{ in } W.$$

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We can think of orthogonal projection as a *transformation*:

$$\text{proj}_W: \mathbf{R}^n \longrightarrow \mathbf{R}^n \quad x \mapsto \text{proj}_W(x).$$

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Let W be a subspace of \mathbf{R}^n .

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3. For every x in W^\perp , we have $\text{proj}_W(x) = 0$.
4. The range of proj_W is W .

Let W be a subspace of \mathbf{R}^n .

Poll

Let A be the matrix for proj_W . What is/are the eigenvalue(s) of A ?

- A. 0 B. 1 C. -1 D. 0, 1 E. 1, -1 F. 0, -1 G. -1 , 0, 1

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The 1-eigenspace is W .

The 0-eigenspace is W^\perp .

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The 1-eigenspace is W .

The 0-eigenspace is W^\perp .

We have $\dim W + \dim W^\perp = n$, so that gives n linearly independent eigenvectors already.

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So the answer is D.

Orthogonal Projections

Matrices

What is the matrix for $\text{proj}_W: \mathbf{R}^3 \rightarrow \mathbf{R}^3$, where

$$W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}?$$

Orthogonal Projections

Matrix facts

Let W be an m -dimensional subspace of \mathbf{R}^n , let $\text{proj}_W: \mathbf{R}^n \rightarrow W$ be the projection, and let A be the matrix for proj_L .

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Matrix facts

Let W be an m -dimensional subspace of \mathbf{R}^n , let $\text{proj}_W: \mathbf{R}^n \rightarrow W$ be the projection, and let A be the matrix for proj_L .

Fact 1: A is diagonalizable with eigenvalues 0 and 1; it is similar to the diagonal matrix with m ones and $n - m$ zeros on the diagonal.

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Matrix facts

Let W be an m -dimensional subspace of \mathbf{R}^n , let $\text{proj}_W: \mathbf{R}^n \rightarrow W$ be the projection, and let A be the matrix for proj_L .

Fact 1: A is diagonalizable with eigenvalues 0 and 1; it is similar to the diagonal matrix with m ones and $n - m$ zeros on the diagonal.

Fact 2: $A^2 = A$.

Orthogonal Projections

Minimum distance

What is the distance from e_1 to $W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$?

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