

# Chapter 5

## Eigenvalues and Eigenvectors

# Section 5.1

## Eigenvectors and Eigenvalues

# A Biology Question

## Motivation

In a population of rabbits:

1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have 0, 6, 8 baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

$f_n$  = first-year rabbits in year  $n$

$s_n$  = second-year rabbits in year  $n$

$t_n$  = third-year rabbits in year  $n$

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let  $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$  and  $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$ . Then  $Av_n = v_{n+1}$ . ← difference equation

# A Biology Question

Continued

If you know  $v_0$ , what is  $v_{10}$ ?

$$v_{10} = Av_9 = AA v_8 = \cdots = A^{10} v_0.$$

This makes it easy to compute examples by computer:

$v_0$	$v_{10}$	$v_{11}$
$\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 30189 \\ 7761 \\ 1844 \end{pmatrix}$	$\begin{pmatrix} 61316 \\ 15095 \\ 3881 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 9459 \\ 2434 \\ 577 \end{pmatrix}$	$\begin{pmatrix} 19222 \\ 4729 \\ 1217 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 28856 \\ 7405 \\ 1765 \end{pmatrix}$	$\begin{pmatrix} 58550 \\ 14428 \\ 3703 \end{pmatrix}$

What do you notice about these numbers?

1. Eventually, each segment of the population doubles every year:  $Av_n = v_{n+1} = 2v_n$ .
2. The ratios get close to (16 : 4 : 1):

$$v_n = (\text{scalar}) \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}.$$

**Translation:** 2 is an eigenvalue, and  $\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$  is an eigenvector!

# Eigenvectors and Eigenvalues

## Definition

Let  $A$  be an  $n \times n$  matrix.

Eigenvalues and eigenvectors are only for square matrices.

1. An **eigenvector** of  $A$  is a *nonzero* vector  $v$  in  $\mathbf{R}^n$  such that  $Av = \lambda v$ , for some  $\lambda$  in  $\mathbf{R}$ . In other words,  $Av$  is a multiple of  $v$ .
2. An **eigenvalue** of  $A$  is a number  $\lambda$  in  $\mathbf{R}$  such that the equation  $Av = \lambda v$  has a *nontrivial* solution.

If  $Av = \lambda v$  for  $v \neq 0$ , we say  $\lambda$  is the **eigenvalue for**  $v$ , and  $v$  is an **eigenvector for**  $\lambda$ .

**Note:** Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

This is the most important definition in the course.

# Verifying Eigenvectors

## Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = 2v$$

Hence  $v$  is an eigenvector of  $A$ , with eigenvalue  $\lambda = 2$ .

## Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4v$$

Hence  $v$  is an eigenvector of  $A$ , with eigenvalue  $\lambda = 4$ .

## Poll

Which of the vectors

A.  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  B.  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  C.  $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$  D.  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$  E.  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

are eigenvectors of the matrix  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ ?

What are the eigenvalues?

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenvector with eigenvalue 2

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

eigenvector with eigenvalue 0

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

eigenvector with eigenvalue 0

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

not an eigenvector

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is never an eigenvector

## Verifying Eigenvalues

**Question:** Is  $\lambda = 3$  an eigenvalue of  $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$ ?

In other words, does  $Av = 3v$  have a nontrivial solution?

... does  $Av - 3v = 0$  have a nontrivial solution?

... does  $(A - 3I)v = 0$  have a nontrivial solution?

We know how to answer that! Row reduction!

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

Row reduce:

$$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

Parametric form:  $x = -4y$ ; parametric vector form:  $\begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ .

Does there exist an eigenvector with eigenvalue  $\lambda = 3$ ? Yes! Any nonzero multiple of  $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ . Check:

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} -4 \\ 1 \end{pmatrix}. \quad \checkmark$$



# Eigenspaces

## Definition

Let  $A$  be an  $n \times n$  matrix and let  $\lambda$  be an eigenvalue of  $A$ . The  $\lambda$ -**eigenspace** of  $A$  is the set of all eigenvectors of  $A$  with eigenvalue  $\lambda$ , plus the zero vector:

$$\begin{aligned}\lambda\text{-eigenspace} &= \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\} \\ &= \{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\} \\ &= \text{Nul}(A - \lambda I).\end{aligned}$$

Since the  $\lambda$ -eigenspace is a null space, it is a *subspace* of  $\mathbf{R}^n$ .

How do you find a basis for the  $\lambda$ -eigenspace? Parametric vector form!

# Eigenspaces

## Example

Find a basis for the 2-eigenspace of

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

$$A - 2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric form}} x = \frac{1}{2}y - 3z$$

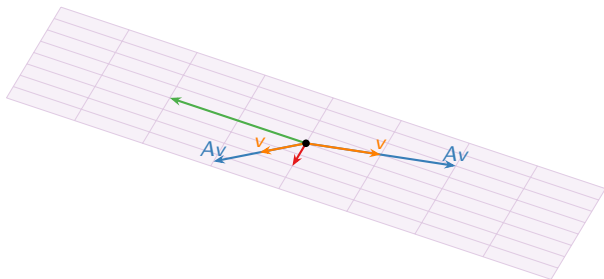
$$\xrightarrow{\text{parametric vector form}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\xrightarrow{\text{basis}} \left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

# Eigenspaces

Picture

A basis for the 2-eigenspace of  $\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$  is  $\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ . What does this look like?



For any  $v$  in the 2-eigenspace,  $Av = 2v$  by definition. So  $A$  acts by *scaling by 2* on its 2-eigenspace. This is how eigenvalues and eigenvectors make matrices easier to understand.

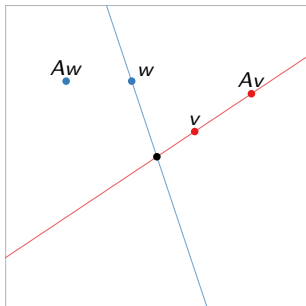
# Eigenspaces

## Geometry

### Eigenvectors, geometrically

An eigenvector of a matrix  $A$  is a nonzero vector  $v$  such that:

- ▶  $Av$  is a multiple of  $v$ , which means
- ▶  $Av$  is collinear with  $v$ , which means
- ▶  $Av$  and  $v$  are *on the same line*.



$v$  is an eigenvector

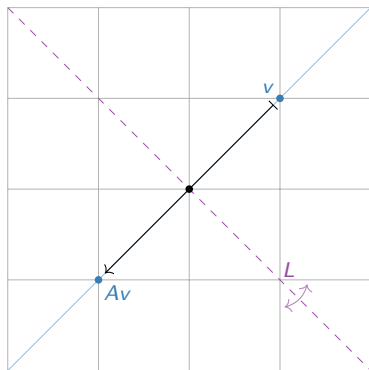
$w$  is not an eigenvector

# Eigenspaces

Geometry; example

Let  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be reflection over the line  $L$  defined by  $y = -x$ , and let  $A$  be the matrix for  $T$ .

**Question:** What are the eigenvalues and eigenspaces of  $A$ ? No computations!



Does anyone see any eigenvectors  
(vectors that don't move off their line)?

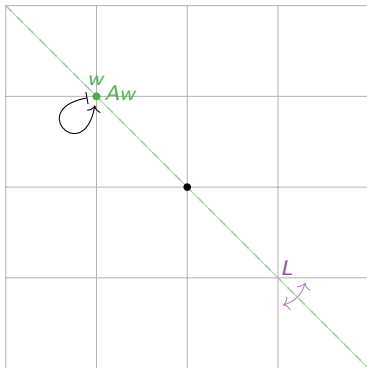
$v$  is an eigenvector with eigenvalue  $-1$ .

# Eigenspaces

Geometry; example

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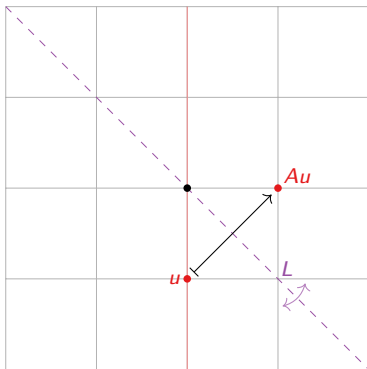
$w$  is an eigenvector with eigenvalue 1.

# Eigenspaces

Geometry; example

Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be reflection over the line  $L$  defined by  $y = -x$ , and let  $A$  be the matrix for  $T$ .

**Question:** What are the eigenvalues and eigenspaces of  $A$ ? No computations!



Does anyone see any eigenvectors  
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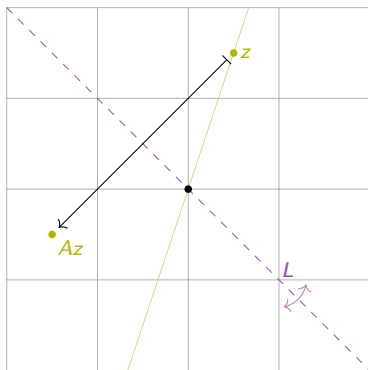
$u$  is *not* an eigenvector.

# Eigenspaces

Geometry; example

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**Question:** What are the eigenvalues and eigenspaces of  $A$ ? No computations!



Does anyone see any eigenvectors  
(vectors that don't move off their line)?

Neither is  $z$ .

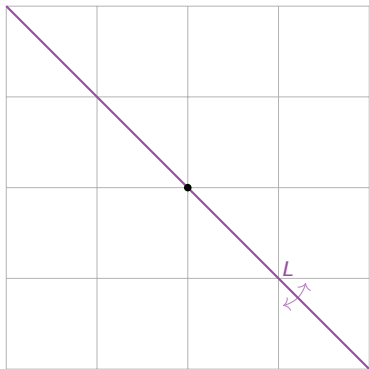


# Eigenspaces

Geometry; example

Let  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be reflection over the line  $L$  defined by  $y = -x$ , and let  $A$  be the matrix for  $T$ .

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Does anyone see any eigenvectors  
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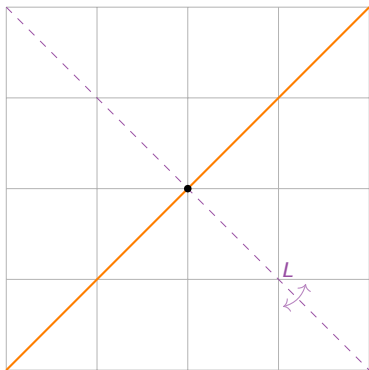
The 1-eigenspace is  $L$   
(all the vectors  $x$  where  $Ax = x$ ).

# Eigenspaces

Geometry; example

Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be reflection over the line  $L$  defined by  $y = -x$ , and let  $A$  be the matrix for  $T$ .

**Question:** What are the eigenvalues and eigenspaces of  $A$ ? No computations!



Does anyone see any eigenvectors  
(vectors that don't move off their line)?

The  $(-1)$ -eigenspace is **the line  $y = x$**   
(all the vectors  $x$  where  $Ax = -x$ ).

# Eigenspaces

## Summary

Let  $A$  be an  $n \times n$  matrix and let  $\lambda$  be a number.

1.  $\lambda$  is an eigenvalue of  $A$  if and only if  $(A - \lambda I)x = 0$  has a nontrivial solution, if and only if  $\text{Nul}(A - \lambda I) \neq \{0\}$ .
2. In this case, finding a basis for the  $\lambda$ -eigenspace of  $A$  means finding a basis for  $\text{Nul}(A - \lambda I)$  as usual, i.e. by finding the parametric vector form for the general solution to  $(A - \lambda I)x = 0$ .
3. The eigenvectors with eigenvalue  $\lambda$  are the nonzero elements of  $\text{Nul}(A - \lambda I)$ , i.e. the nontrivial solutions to  $(A - \lambda I)x = 0$ .

## The Eigenvalues of a Triangular Matrix are the Diagonal Entries

We've seen that finding eigenvectors for a given eigenvalue is a row reduction problem.

Finding all of the eigenvalues of a matrix *is not a row reduction problem!* We'll see how to do it in general next time. For now:

**Fact:** The eigenvalues of a triangular matrix are the diagonal entries.

**Why?**  $\text{Nul}(A - \lambda I) \neq \{0\}$  if and only if  $A - \lambda I$  is not invertible, if and only if  $\det(A - \lambda I) = 0$ .

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 0 & -1 & -2 & 7 \\ 0 & 0 & 8 & 12 \\ 0 & 0 & 0 & -3 \end{pmatrix} - \lambda I_4 = \begin{pmatrix} 3 - \lambda & 4 & 1 & 2 \\ 0 & -1 - \lambda & -2 & 7 \\ 0 & 0 & 8 - \lambda & 12 \\ 0 & 0 & 0 & -3 - \lambda \end{pmatrix}.$$

The determinant is  $(3 - \lambda)(-1 - \lambda)(8 - \lambda)(-3 - \lambda)$ , which is zero exactly when  $\lambda = 3, -1, 8$ , or  $-3$ .

# A Matrix is Invertible if and only if Zero is not an Eigenvalue

**Fact:**  $A$  is invertible if and only if 0 is not an eigenvalue of  $A$ .

Why?

0 is an eigenvalue of  $A \iff Ax = 0x$  has a nontrivial solution

$\iff Ax = 0$  has a nontrivial solution

$\iff A$  is not invertible.

invertible matrix theorem



## Eigenvectors with Distinct Eigenvalues are Linearly Independent

**Fact:** If  $v_1, v_2, \dots, v_k$  are eigenvectors of  $A$  with *distinct* eigenvalues  $\lambda_1, \dots, \lambda_k$ , then  $\{v_1, v_2, \dots, v_k\}$  is linearly independent.

**Why?** If  $k = 2$ , this says  $v_2$  can't lie on the line through  $v_1$ .

But the line through  $v_1$  is contained in the  $\lambda_1$ -eigenspace, and  $v_2$  does not have eigenvalue  $\lambda_1$ .

**In general:** see Lay, Theorem 2 in §5.1 (or work it out for yourself; it's not too hard).

**Consequence:** An  $n \times n$  matrix has at most  $n$  distinct eigenvalues.

# Difference Equations

## Preview

Let  $A$  be an  $n \times n$  matrix. Suppose we want to solve  $Av_n = v_{n+1}$  for all  $n$ . In other words, we want vectors  $v_0, v_1, v_2, \dots$ , such that

$$Av_0 = v_1 \quad Av_1 = v_2 \quad Av_2 = v_3 \quad \dots$$

We saw before that  $v_n = A^n v_0$ . But it is inefficient to multiply by  $A$  each time.

If  $v_0$  is an *eigenvector* with eigenvalue  $\lambda$ , then

$$v_1 = Av_0 = \lambda v_0 \quad v_2 = Av_1 = \lambda v_1 = \lambda^2 v_0 \quad v_3 = Av_2 = \lambda v_2 = \lambda^3 v_0.$$

In general,  $v_n = \lambda^n v_0$ . This is *much easier* to compute.

## Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad v_0 = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \quad Av_0 = 2v_0.$$

So if you start with 16 baby rabbits, 4 first-year rabbits, and 1 second-year rabbit, then the population will exactly double every year. In year  $n$ , you will have  $2^n \cdot 16$  baby rabbits,  $2^n \cdot 4$  first-year rabbits, and  $2^n$  second-year rabbits.