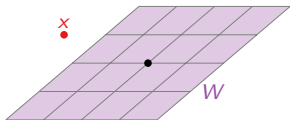


Section 6.2

Orthogonal Sets

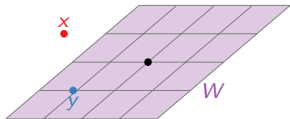
Best Approximation

Suppose you measure a data point x which you know for theoretical reasons must lie on a subspace W .



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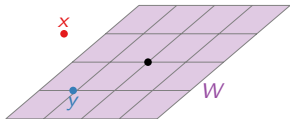
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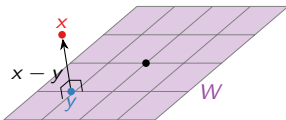


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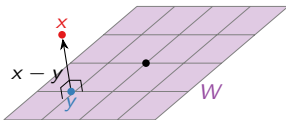


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Orthogonal Projection onto a Line

Theorem

Let $L = \text{Span}\{u\}$ be a line in \mathbf{R}^n , and let x be in \mathbf{R}^n . The closest point to x on L is the point

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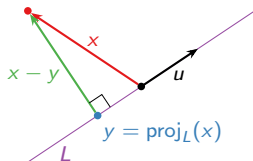
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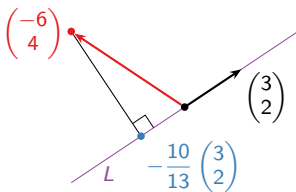
Compute the orthogonal projection of $x = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ onto the line L spanned by

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Lemma

An orthogonal set of vectors is linearly independent.

Orthogonal Bases

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Theorem

Let $\mathcal{B} = \{u_1, u_2, \dots, u_m\}$ be an orthogonal set, and let x be a vector in $W = \text{Span } \mathcal{B}$. Then

$$x = \sum_{i=1}^m \frac{x \cdot u_i}{u_i \cdot u_i} u_i = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m} u_m.$$

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In other words, the \mathcal{B} -coordinates of x are $\left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2}, \dots, \frac{x \cdot u_m}{u_m \cdot u_m} \right)$.


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If L_i is the line spanned by u_i , then this says

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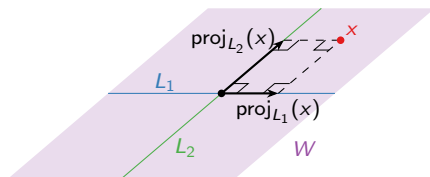
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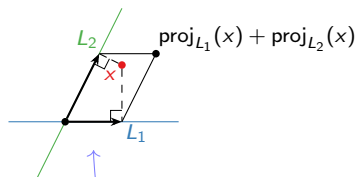
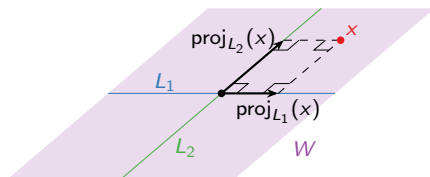
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Warning: This only works for an *orthogonal* basis.

Orthogonal Bases

Example

Problem: Find the \mathcal{B} -coordinates of $x = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, where

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$$\left(\begin{array}{cc|c} 1 & -4 & 0 \\ 2 & 2 & 3 \end{array} \right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 6/5 \\ 0 & 1 & 6/20 \end{array} \right) \implies [x]_{\mathcal{B}} = \begin{pmatrix} 6/5 \\ 6/20 \end{pmatrix}.$$

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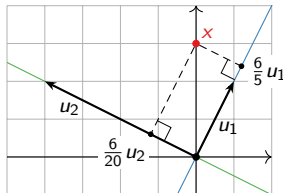
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