

Application

Stochastic Matrices and PageRank

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You'll be responsible for knowing basic facts about stochastic matrices and the Perron–Frobenius theorem, but we will not cover them in depth. These slides are the primary reference; see also §4.9 in Lay.

The specifics of the PageRank algorithm are just for fun.

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Stochastic Matrices and Difference Equations

If x_n, y_n, z_n are the numbers of movies in locations 1, 2, 3, respectively, on day n , and $v_n = (x_n, y_n, z_n)$, then:

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- ▶ v_n is the “state at time n ”,
- ▶ v_{n+1} is the “state at time $n + 1$ ”, and
- ▶ $v_{n+1} = Av_n$ means that A is the “change of state matrix.”

Eigenvalues of Stochastic Matrices

Fact: 1 is an eigenvalue of a stochastic matrix.

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Why? If A is stochastic, then 1 is an eigenvalue of A^T :

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Note: This doesn't give a new procedure for finding an eigenvector with eigenvalue 1; it only shows one exists.

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Fact: if λ is an eigenvalue of a stochastic matrix, then $|\lambda| \leq 1$. Hence 1 is the *largest* eigenvalue (in absolute value).

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Fact: if λ is an eigenvalue of a stochastic matrix, then $|\lambda| \leq 1$. Hence 1 is the *largest* eigenvalue (in absolute value).

Better fact: if $\lambda \neq 1$ is an eigenvalue of a *positive* stochastic matrix, then $|\lambda| < 1$.

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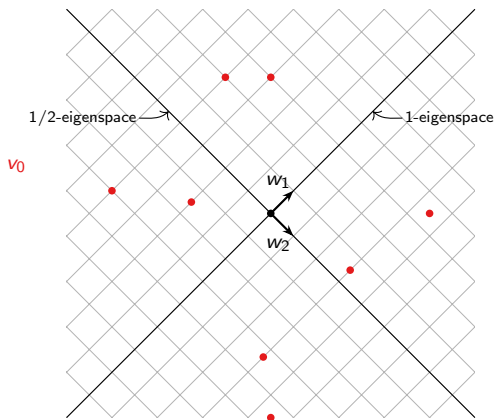
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So all vectors get “sucked into the 1-eigenspace,” which is spanned by $w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

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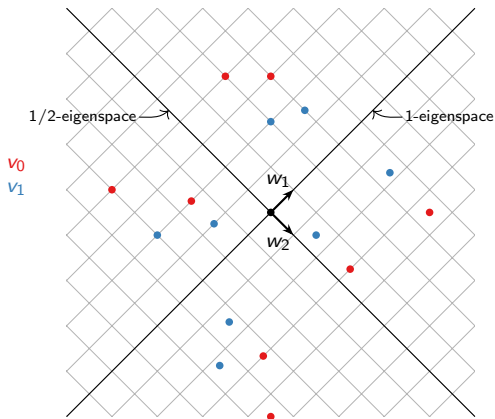
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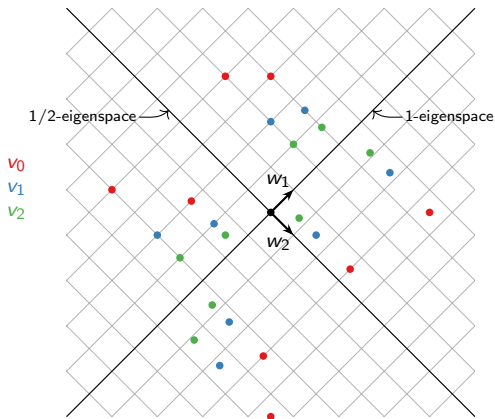
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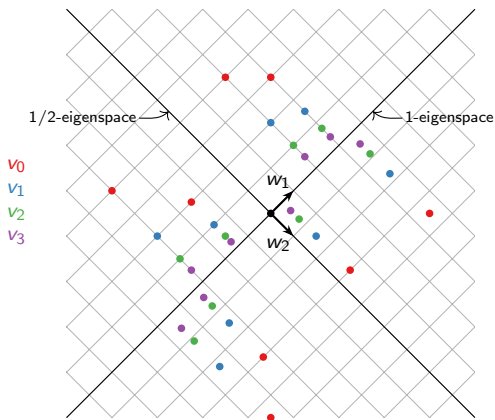
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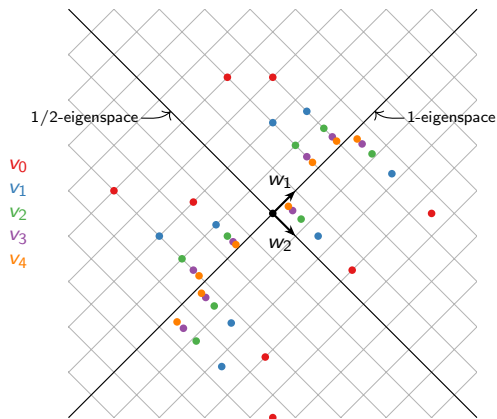
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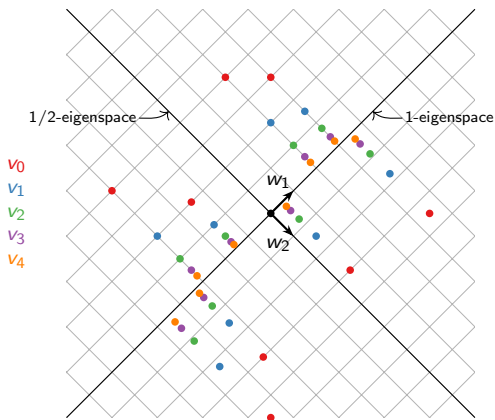
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$$A = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & -.2 \end{pmatrix} P^{-1} = PDP^{-1}.$$

Hence it is easy to compute the powers of A :

$$A^n = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & (.1)^n & 0 \\ 0 & 0 & (-.2)^n \end{pmatrix} P^{-1} = PD^nP^{-1}.$$

Let w_1, w_2, w_3 be the columns of P , i.e. the eigenvectors of P with respective eigenvalues 1, .1, $-.2$. Let $\mathcal{B} = \{w_1, w_2, w_3\}$.

Recall: A^n acts on the usual coordinates of a vector in the same way that D acts on the \mathcal{B} -coordinates: $[A^n x]_{\mathcal{B}} = D^n [x]_{\mathcal{B}}$.

Diagonalizable Stochastic Matrices

Continued

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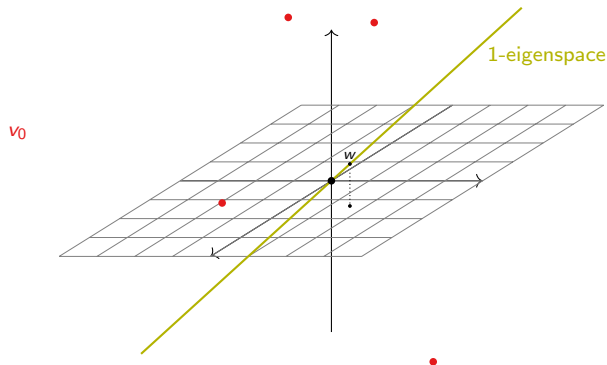
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(We’ll see in a moment why I chose that eigenvector.)

Diagonalizable Stochastic Matrices

Picture

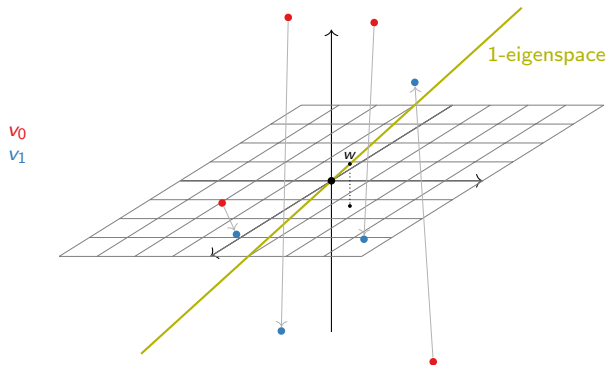
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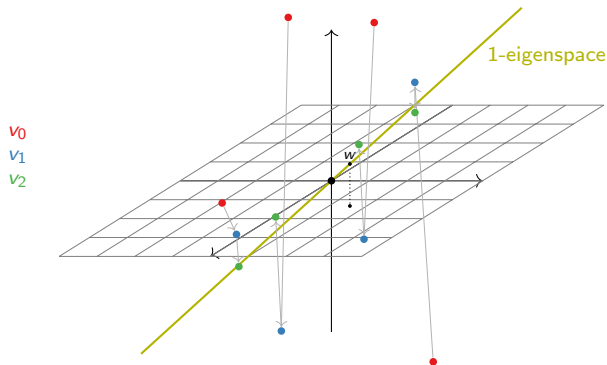
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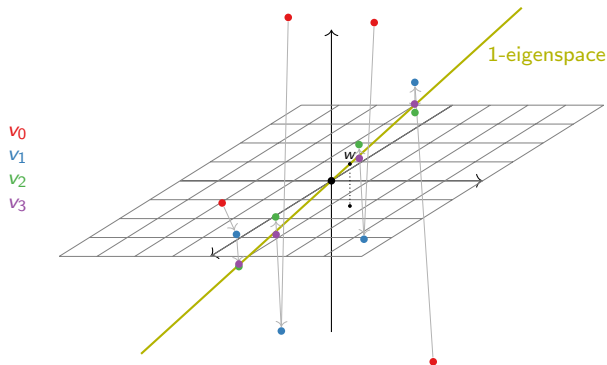
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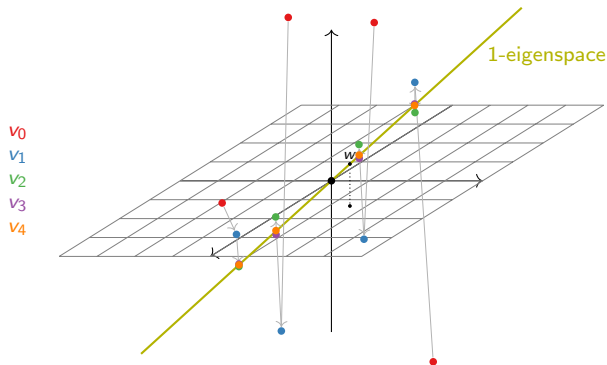
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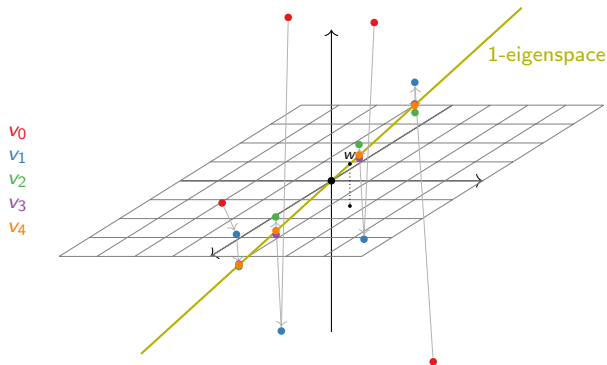
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We see that v_n approaches an eigenvector with eigenvalue 1 as n gets large: all vectors get “sucked into the 1-eigenspace.”

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If A is the Red Box matrix, and v_n is the vector representing the number of movies in the three locations on day n , then

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Presumably, Red Box really does have to do this kind of analysis to determine how many videos to put in each box.

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- ▶ The sum c of the entries of v_0 is the total number of movies; eventually, the movies arrange themselves according to the steady state percentage, i.e., $v_n \rightarrow cw$.

Steady State

Red Box example

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The Perron–Frobenius Theorem says that our analysis of the Red Box matrix works for *any* positive stochastic matrix—whether or not it is diagonalizable!

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Reference:

<http://www.math.cornell.edu/~mec/Winter2009/RalucaRemus/Lecture3/lecture3.html>

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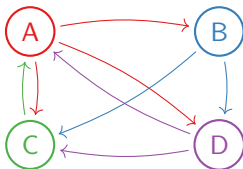
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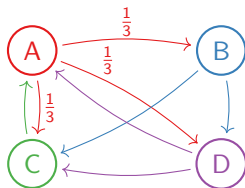
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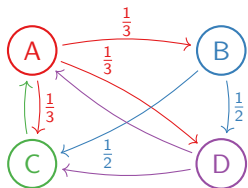
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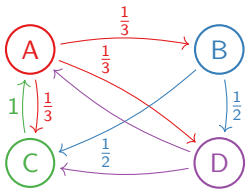


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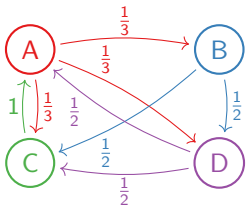
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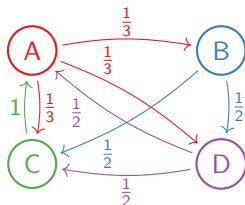
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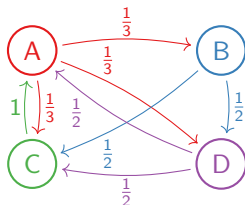
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In terms of matrices, if $v = (a, b, c, d)$ is the vector containing the ranks a, b, c, d of the pages **A**, **B**, **C**, **D**, then

$$\begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ 3 & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} c + \frac{1}{2}d \\ \frac{1}{3}a \\ \frac{1}{3}a + \frac{1}{2}b + \frac{1}{2}d \\ \frac{1}{3}a + \frac{1}{2}b \end{pmatrix}$$

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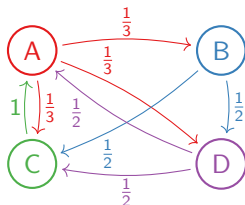
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$$\begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ 3 & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} c + \frac{1}{2}d \\ \frac{1}{3}a \\ \frac{1}{3}a + \frac{1}{2}b + \frac{1}{2}d \\ \frac{1}{3}a + \frac{1}{2}b \end{pmatrix} \stackrel{\text{Importance Rule}}{=} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

The Importance Matrix

Consider the following Internet with only four pages. Links are indicated by arrows.



Page **A** has 3 links, so it passes $\frac{1}{3}$ of its importance to pages **B**, **C**, **D**.

Page **B** has 2 links, so it passes $\frac{1}{2}$ of its importance to pages **C**, **D**.

Page **C** has one link, so it passes all of its importance to page **A**.

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Observations:

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So, the important (high-ranked) pages are those where a random surfer will end up most often.

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Dangling pages

Observation: the importance matrix is *not* positive: it's only nonnegative.

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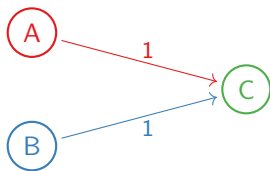
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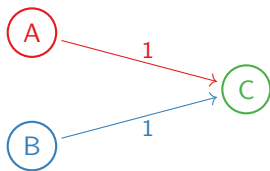


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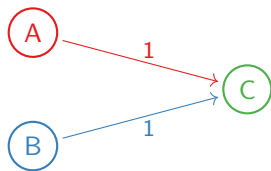
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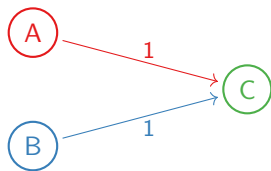
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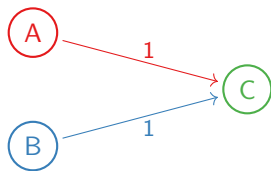
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So 1 is not an eigenvalue at all: there is no rank vector! (It is not stochastic.)

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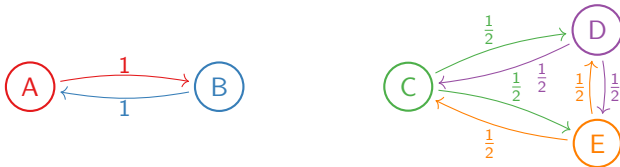
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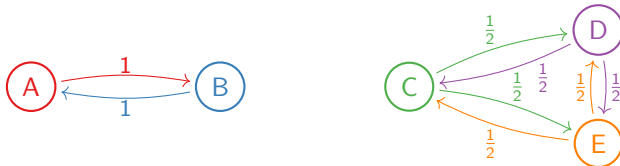
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$$M = (1 - p) \cdot A + p \cdot B \quad \text{where} \quad B = \frac{1}{N} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

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