# Section 5.2

## The Characteristic Equation

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### The Invertible Matrix Theorem

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1. $A$ is invertible.
2. $T$ is invertible.
3. $A$ is row equivalent to $I_n$.
4. $A$ has $n$ pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of $A$ are linearly independent.
7. $T$ is one-to-one.
8. $Ax = b$ is consistent for all $b$ in $\mathbf{R}^n$.
9. The columns of $A$ span $\mathbf{R}^n$.
10. $T$ is onto.
11. $A$ has a left inverse (there exists $B$ such that $BA = I_n$).
12. $A$ has a right inverse (there exists $B$ such that $AB = I_n$).
13. $A^T$ is invertible.
14. The columns of $A$ form a basis for $\mathbf{R}^n$.
15. $\operatorname{Col} A = \mathbf{R}^n$.
16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
18. $\operatorname{Nul} A = \{0\}$.
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20\. The number 0 is *not* an eigenvalue of $A$.

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**Important**

The eigenvalues of $A$ are the roots of the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$.

Question: What are the eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}?$$

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**Shortcut**

The characteristic polynomial of a $2 \times 2$ matrix $A$ is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A).$$

Question: What are the eigenvalues of the rabbit population matrix

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}?$$

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Fact: If $A$ is an $n \times n$ matrix, the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I)$$

turns out to be a polynomial of degree $n$, and its roots are the eigenvalues of $A$:

$$f(\lambda) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0.$$

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**Poll**

If you count the eigenvalues of $A$, with their algebraic multiplicities, you will get:

A. Always $n$.

B. Always at most $n$, but sometimes less.

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## What does this mean?

$A$ acts on the standard coordinates of $x$ in the same way that $B$ acts on the $\mathcal{B}$-coordinates of $x$: $B[x]_{\mathcal{B}} = [Ax]_{\mathcal{B}}$.

# Similarity

## Example

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \Longrightarrow \quad A = CBC^{-1}.$$

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So $A$ does to the standard coordinates what $B$ does to the $\mathcal{B}$-coordinates, where

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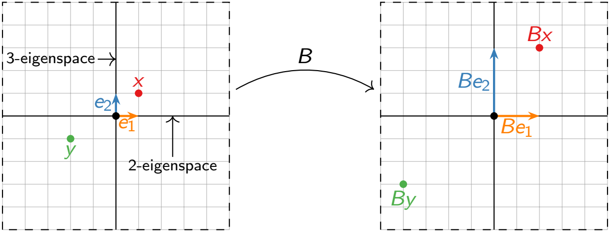
$B$ acting on the usual coordinates



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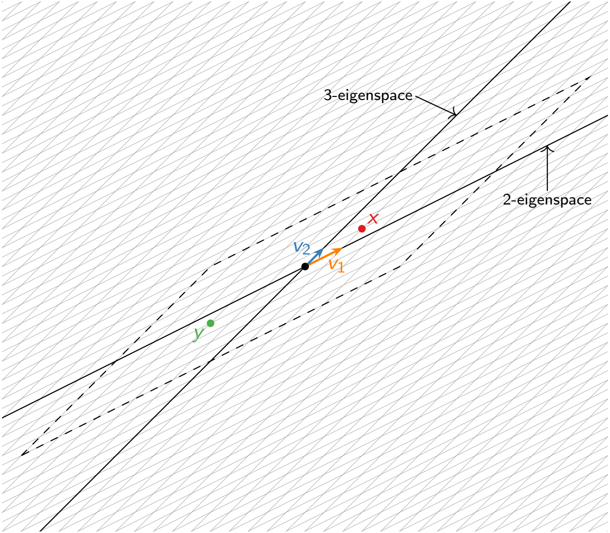


$$\left.\begin{aligned} v_1 &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ v_2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}\right\} \text{vectors in } \mathcal{B}$$

$$[x]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x =$$

$$[y]_{\mathcal{B}} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$y =$$

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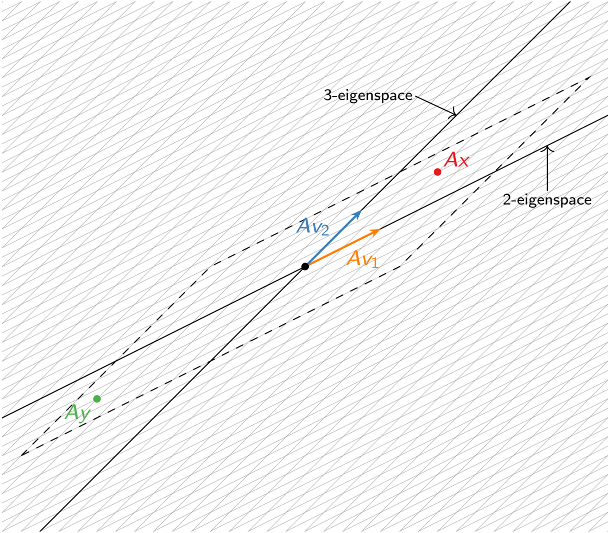


$Av_1 =$

$Av_2 =$

$B[x]_{\mathcal{B}} = \qquad = [Ax]_{\mathcal{B}}$

$Ax =$

$B[y]_{\mathcal{B}} = \qquad = [Ay]_{\mathcal{B}}$

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Check:

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Why? Suppose $A = CBC^{-1}$.

Conseqence: similar matrices have the same eigenvalues!
(But different eigenvectors in general.)

Warning

1. Matrices with the same eigenvalues need not be similar. For instance,

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

both only have the eigenvalue 2, but they are not similar.

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2. Similarity has nothing to do with row equivalence. For instance,

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are row equivalent, but they have different eigenvalues.