

Section 2.9

Dimension and Rank

Coefficients of Basis Vectors

Recall: a **basis** of a subspace V is a set of vectors that *spans* V and is *linearly independent*.

Lemma  like a theorem, but less important

If $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$ is a basis for a subspace V , then any vector x in V can be written as a linear combination

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

for *unique* coefficients c_1, c_2, \dots, c_m .

We know x is a linear combination of the v_i because they span V . Suppose that we can write x as a linear combination with different coefficients:

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

$$x = c'_1 v_1 + c'_2 v_2 + \cdots + c'_m v_m$$

Subtracting:

$$0 = x - x = (c_1 - c'_1)v_1 + (c_2 - c'_2)v_2 + \cdots + (c_m - c'_m)v_m$$

Since v_1, v_2, \dots, v_m are linearly independent, they only have the trivial linear dependence relation. That means each $c_i - c'_i = 0$, or $c_i = c'_i$.

Bases as Coordinate Systems

The unit coordinate vectors e_1, e_2, \dots, e_n form a basis for \mathbf{R}^n . Any vector is a unique linear combination of the e_i :

$$v = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3e_1 + 5e_2 - 2e_3.$$

Observe: the *coordinates* of v are exactly the *coefficients* of e_1, e_2, e_3 .

We can go backwards: given any basis \mathcal{B} , we interpret the coefficients of a linear combination as “coordinates” with respect to \mathcal{B} .

Definition

Let $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$ be a basis of a subspace V . Any vector x in V can be written uniquely as a linear combination $x = c_1 v_1 + c_2 v_2 + \dots + c_m v_m$. The coefficients c_1, c_2, \dots, c_m are the **coordinates of x with respect to \mathcal{B}** . The **\mathcal{B} -coordinate vector of x** is the vector

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{in } \mathbf{R}^m.$$

Bases as Coordinate Systems

Example 1

Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathcal{B} = \{v_1, v_2\}$, $V = \text{Span}\{v_1, v_2\}$.

Verify that \mathcal{B} is a basis:

Span: by definition $V = \text{Span}\{v_1, v_2\}$.

Linearly independent: because they are not multiples of each other.

Question: If $[x]_{\mathcal{B}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, then what is x ?

$$[x]_{\mathcal{B}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \text{means} \quad x = 5v_1 + 2v_2 = 5 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 7 \end{pmatrix}.$$

Question: Find the \mathcal{B} -coordinates of $x = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$.

We have to solve the vector equation $x = c_1 v_1 + c_2 v_2$ in the unknowns c_1, c_2 .

$$\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{array} \right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right)$$

So $c_1 = 2$ and $c_2 = 3$, so $x = 2v_1 + 3v_2$ and $[x]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Bases as Coordinate Systems

Example 2

$$\text{Let } v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}, \quad V = \text{Span}\{v_1, v_2, v_3\}.$$

Question: Find a basis for V .

V is the column span of the matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & 8 \\ 2 & 1 & 6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for the column span is formed by the pivot columns: $\mathcal{B} = \{v_1, v_2\}$.

Question: Find the \mathcal{B} -coordinates of $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$.

We have to solve $x = c_1 v_1 + c_2 v_2$.

$$\left(\begin{array}{cc|c} 2 & -1 & 4 \\ 3 & 1 & 11 \\ 2 & 1 & 8 \end{array} \right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right)$$

So $x = 3v_1 + 2v_2$ and $[x]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Bases as Coordinate Systems

Summary

If $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$ is a basis for a subspace V and x is in V , then

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{means} \quad x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m.$$

Finding the \mathcal{B} -coordinates for x means solving the vector equation

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

in the unknowns c_1, c_2, \dots, c_m . This (usually) means row reducing the augmented matrix

$$\left(\begin{array}{c|c|c|c|c} | & | & & | & | \\ \hline v_1 & v_2 & \cdots & v_m & x \\ \hline | & | & & | & | \end{array} \right).$$

Question: What happens if you try to find the \mathcal{B} -coordinates of x *not* in V ? You end up with an inconsistent system: V is the span of v_1, v_2, \dots, v_m , and if x is not in the span, then $x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$ has no solution.

Bases as Coordinate Systems

Picture

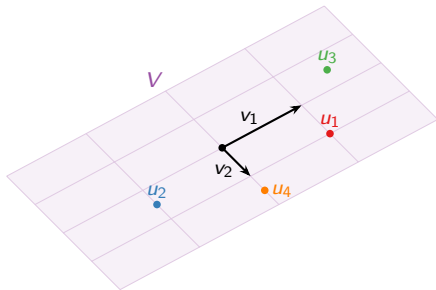
Let

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

These form a basis \mathcal{B} for the plane

$$V = \text{Span}\{v_1, v_2\}$$

in \mathbf{R}^3 .



Question: Estimate the \mathcal{B} -coordinates of these vectors:

$$[u_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad [u_2]_{\mathcal{B}} = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix} \quad [u_3]_{\mathcal{B}} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} \quad [u_4]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}$$

Remark

Many of you want to think of a plane in \mathbf{R}^3 as “being” \mathbf{R}^2 . Choosing a basis \mathcal{B} and using \mathcal{B} -coordinates is one way to make sense of that. But remember that the coordinates are the coefficients of a linear combination of the basis vectors.

The Rank Theorem

Recall:

- ▶ The **dimension** of a subspace V is the number of vectors in a basis for V .
- ▶ A basis for the column space of a matrix A is given by the pivot columns.
- ▶ A basis for the null space of A is given by the vectors attached to the free variables in the parametric vector form.

Definition

The **rank** of a matrix A , written $\text{rank } A$, is the dimension of the column space $\text{Col } A$.

Observe:

$$\begin{aligned}\text{rank } A &= \dim \text{Col } A = \text{the number of columns with pivots} \\ \dim \text{Nul } A &= \text{the number of free variables} \\ &= \text{the number of columns without pivots.}\end{aligned}$$

Rank Theorem



If A is an $m \times n$ matrix, then

$$\text{rank } A + \dim \text{Nul } A = n = \text{the number of columns of } A.$$

The Rank Theorem

Example

$$A = \begin{pmatrix} \boxed{1} & \boxed{2} & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & \boxed{-8} & \boxed{-7} \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for Col A is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\},$$

so $\text{rank } A = \dim \text{Col } A = 2$.

Since there are two free variables x_3, x_4 , the parametric vector form for the solutions to $Ax = 0$ is

$$x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis for Nul } A} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus $\dim \text{Nul } A = 2$.

The Rank Theorem says $2 + 2 = 4$.

Poll

Let A and B be 3×3 matrices. Suppose that $\text{rank}(A) = 2$ and $\text{rank}(B) = 2$. Is it possible that $AB = 0$? Why or why not?

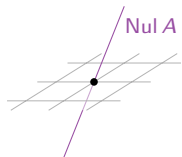
If $AB = 0$, then $ABx = 0$ for every x in \mathbf{R}^3 .

This means $A(Bx) = 0$, so Bx is in $\text{Nul } A$.

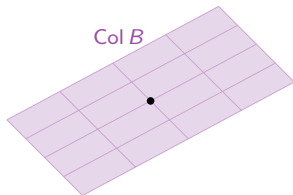
This is true for every x , so $\text{Col } B$ is contained in $\text{Nul } A$.

But $\dim \text{Nul } A = 1$ and $\dim \text{Col } B = 2$, and a 1-dimensional space can't contain a 2-dimensional space.

Hence it can't happen.



does not
contain



The Basis Theorem

Basis Theorem

Let V be a subspace of dimension m . Then:

- ▶ Any m linearly independent vectors in V form a basis for V .
- ▶ Any m vectors that span V form a basis for V .

Upshot

If you *already* know that $\dim V = m$, and you have m vectors $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$ in V , then you only have to check *one* of

1. \mathcal{B} is linearly independent, *or*
2. \mathcal{B} spans V

in order for \mathcal{B} to be a basis.

The Invertible Matrix Theorem

Addenda

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. A is row equivalent to I_n .
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in \mathbf{R}^n .
9. The columns of A span \mathbf{R}^n .
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.
14. The columns of A form a basis for \mathbf{R}^n .
15. $\text{Col } A = \mathbf{R}^n$.
16. $\dim \text{Col } A = n$.
17. $\text{rank } A = n$.
18. $\text{Nul } A = \{0\}$.
19. $\dim \text{Nul } A = 0$.

These are equivalent to the previous conditions by the Rank Theorem and the Basis Theorem.