# Section 6.4

## The Gram–Schmidt Process

## Motivation

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- Finding the orthogonal projection of a vector $x$ onto the span $W$ of $u_1, u_2, \ldots, u_m$:

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Problem: What if your basis isn't orthogonal?

Solution: The Gram–Schmidt process: take any basis and make it orthogonal.

## The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace $W$ of $\mathbf{R}^n$. Define:

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Then $\{u_1, u_2, \ldots, u_m\}$ is an *orthogonal* basis for the same subspace $W$.

## The Gram–Schmidt Process

Let $\{v_1, v_2, \dots, v_m\}$ be a basis for a subspace $W$ of $\mathbf{R}^n$. Define:

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Then $\{u_1, u_2, \dots, u_m\}$ is an *orthogonal* basis for the same subspace $W$.

## Remark

In fact, for every $i$ between 1 and $n$, the set $\{u_1, u_2, \dots, u_i\}$ is an orthogonal basis for $\text{Span}\{v_1, v_2, \dots, v_i\}$.

Find an orthogonal basis $\{u_1, u_2\}$ for $W = \mathrm{Span}\{v_1, v_2\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

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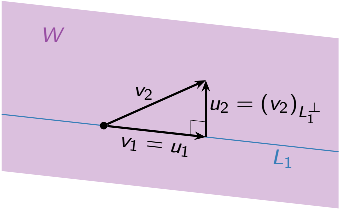

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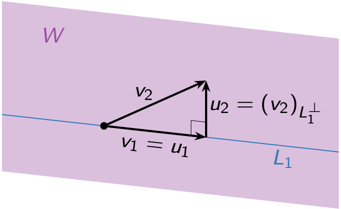


Important: $\text{Span}\{u_1, u_2\} = \text{Span}\{v_1, v_2\} = W$: this is an *orthogonal* basis for the *same* subspace.

# The Gram–Schmidt Process

## Three vectors

Find an orthogonal basis $\{u_1, u_2, u_3\}$ for $W = \mathrm{Span}\{v_1, v_2, v_3\} = \mathbf{R}^3$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

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$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix},\ v_2 = \begin{pmatrix}1\\1\\1\end{pmatrix},\ v_3 = \begin{pmatrix}3\\1\\1\end{pmatrix} \xrightarrow{\text{G–S}} u_1 = \begin{pmatrix}1\\1\\0\end{pmatrix},\ u_2 = \begin{pmatrix}0\\0\\1\end{pmatrix},\ u_3 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

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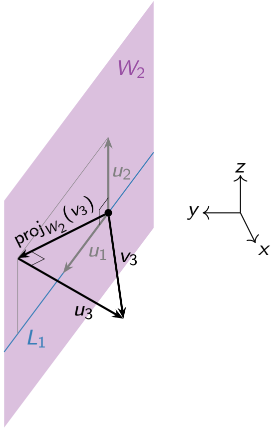

# The Gram–Schmidt Process

## Three vectors in $\mathbf{R}^4$

Find an orthogonal basis $\{u_1, u_2, u_3\}$ for $W = \mathrm{Span}\{v_1, v_2, v_3\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 4 \\ -2 \\ -2 \\ 0 \end{pmatrix}.$$

# Poll

Poll

What happens if you try to run Gram–Schmidt on a linearly *dependent* set of vectors $\{v_1, v_2, \dots, v_m\}$?

A. You get an inconsistent equation.

B. For some $i$ you get $u_i = u_{i-1}$.

C. For some $i$ you get $u_i = 0$.

D. You create a rift in the space-time continuum.

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If $\{v_1, v_2, \ldots, v_m\}$ is linearly dependent, then some $v_i$ is in $\mathrm{Span}\{v_1, v_2, \ldots, v_{i-1}\} = \mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}$.

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This means

$$v_i = \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}}(v_i)$$

$$\implies u_i = v_i - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}}(v_i) = 0.$$

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In this case, you can simply discard $u_i$ and $v_i$ and continue: so Gram–Schmidt produces an orthogonal basis from any spanning set!

### $QR$ Factorization Theorem

Let $A$ be a matrix with linearly independent columns. Then

$$A = QR$$

where $Q$ has orthonormal columns and $R$ is upper-triangular with positive diagonal entries.

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Recall: A set of vectors $\{v_1, v_2, \ldots, v_m\}$ is **orthonormal** if they are orthogonal unit vectors: $v_i \cdot v_j = 0$ when $i \neq j$, and $v_i \cdot v_i = 1$.

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Check: A matrix $Q$ has orthonormal columns if and only if $Q^T Q = I$.

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Here is the procedure for producing a $QR$ factorization.

# $QR$ Factorization

## Example

Find the $QR$ factorization of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

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Step 1: Run Gram–Schmidt

$$u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1 = v_2 - 1\, u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} u_3 &= v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2 \\ &= v_3 - 2\, u_1 - 1\, u_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \end{aligned}$$

Find the $QR$ factorization of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

(The columns of $A$ are the vectors $v_1, v_2, v_3$ from a previous example.)

Step 1: Run Gram–Schmidt and solve for $v_1, v_2, v_3$ in terms of $u_1, u_2, u_3$.

$$u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_1 = u_1$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1 = v_2 - 1\, u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = u_1 + u_2$$

$$\begin{aligned} u_3 &= v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2 \\ &= v_3 - 2\, u_1 - 1\, u_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \end{aligned} \qquad v_3 = 2u_1 + u_2 + u_3$$

$$v_1 = 1\,u_1 \qquad v_2 = 1\,u_1 + 1\,u_2 \qquad v_3 = 2\,u_1 + 1\,u_2 + 1\,u_3$$

Step 2: Write $A = \widehat{Q}\widehat{R}$, where $\widehat{Q}$ has *orthogonal* columns $u_1, u_2, u_3$ and $\widehat{R}$ is upper-triangular with 1s on the diagonal.

# $QR$ Factorization

## Example, continued

$$v_1 = 1\,u_1 \qquad v_2 = 1\,u_1 + 1\,u_2 \qquad v_3 = 2\,u_1 + 1\,u_2 + 1\,u_3$$

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Do this by putting the above equations in matrix form:

$$A \longrightarrow \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \underbrace{\begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix}}_{\widehat{Q}} \underbrace{\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{\widehat{R}}$$

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$$\text{first column of } A = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 u_1 = v_1$$

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Example, continued

$$v_1 = 1\,u_1 \qquad v_2 = 1\,u_1 + 1\,u_2 \qquad v_3 = 2\,u_1 + 1\,u_2 + 1\,u_3$$

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# *QR* Factorization

## Example, continued

$$v_1 = 1u_1 \qquad v_2 = 1u_1 + 1u_2 \qquad v_3 = 2u_1 + 1u_2 + 1u_3$$

Step 2: Write $A = \widehat{Q}\widehat{R}$, where $\widehat{Q}$ has *orthogonal* columns $u_1, u_2, u_3$ and $\widehat{R}$ is upper-triangular with 1s on the diagonal.

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$$\text{third column of } A = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 2u_1 + 1u_2 + 1u_3 = v_3$$

# $QR$ Factorization

## Example, continued

$$A = \widehat{Q}\widehat{R} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 3: Scale the columns of $\widehat{Q}$ to get unit vectors, and scale the rows of $\widehat{R}$ by the opposite factor, to get $Q$ and $R$.

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## Example, continued

$$A = \widehat{Q}\widehat{R} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 3: Scale the columns of $\widehat{Q}$ to get unit vectors, and scale the rows of $\widehat{R}$ by the opposite factor, to get $Q$ and $R$.

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# $QR$ Factorization

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$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1 \\ 1/\sqrt{2} & 0 & -1 \\ 0/\sqrt{2} & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \cdot \sqrt{2} & 1 \cdot \sqrt{2} & 2 \cdot \sqrt{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Note that the entries in the $i$th column of $Q$ multiply by the entries in the $i$th row of $R$, so this doesn't change the product.

# $QR$ Factorization

## Example, continued

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The final $QR$ decomposition is:

$$A = QR \qquad Q = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} \sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

Find the $QR$ factorization of $A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & -2 \\ 1 & -1 & 0 \end{pmatrix}$.

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(The columns are vectors from a previous example.)

Step 1: Run Gram–Schmidt and solve for $v_1, v_2, v_3$ in terms of $u_1, u_2, u_3$:

# $QR$ Factorization

## Another example, continued

$$v_1 = 1\, u_1 \qquad v_2 = \frac{3}{2}\, u_1 + 1\, u_2 \qquad v_3 = 0\, u_1 - \frac{4}{5}\, u_2 + 1\, u_3$$

Step 2: Write $A = \widehat{Q}\widehat{R}$, where $\widehat{Q}$ has *orthogonal* columns $u_1, u_2, u_3$ and $\widehat{R}$ is upper-triangular with 1s on the diagonal.

$$A = \widehat{Q}\widehat{R} \qquad \widehat{Q} = \begin{pmatrix} 1 & -5/2 & 2 \\ 1 & 5/2 & 0 \\ 1 & 5/2 & 0 \\ 1 & -5/2 & -2 \end{pmatrix} \qquad \widehat{R} = \begin{pmatrix} 1 & 3/2 & 0 \\ 0 & 1 & -4/5 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 3: Normalize the columns of $\widehat{Q}$ and the rows of $\widehat{R}$ to get $Q$ and $R$:

# $QR$ Factorization

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In fact, if $v_1, v_2, \ldots, v_n$ are the columns of $A$, and $u_1, u_2, \ldots, u_n$ are the vectors you obtain by applying Gram–Schmidt, then the $(i, i)$ entry of $R$ is $\|u_i\|$, so

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So you can use Gram–Schmidt to compute determinants (up to sign)!

Let $A$ be an $n \times n$ matrix with real eigenvalues. Here is an algorithm:

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**Theorem**

The matrices $A_k$ converge to an upper triangular matrix, and the diagonal entries converge (quickly!) to the eigenvalues of $A$.

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### Theorem
The matrices $A_k$ converge to an upper triangular matrix, and the diagonal entries converge (quickly!) to the eigenvalues of $A$.

This gives a computationally efficient way (called the $QR$ algorithm) to find the eigenvalues of a matrix.