Review for Midterm 2

Selected Topics

Matrix Multiplication

Method 1: Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2 \ldots, v_p$:

$$B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

The **product** AB is the $m \times p$ matrix with columns $Av_1, Av_2, \ldots, Av_p$:

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Method 2: The ij entry of $C = AB$ is the ith row of A times the jth column of B:

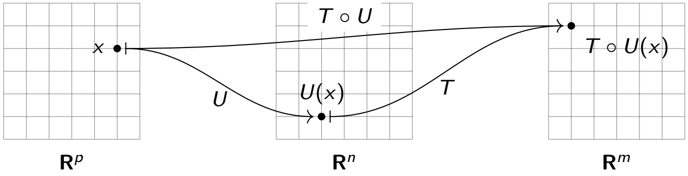
$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$\begin{array}{cc} \begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} & i\text{th row} \end{array} \cdot \underset{j\text{th column}}{\begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix}} = \underset{ij \text{ entry}}{\begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}}$$

Matrix Multiplication/Inversion and Linear Transformations

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ and $U\colon \mathbf{R}^p \to \mathbf{R}^n$ be linear transformations with matrices A and B. The **composition** is the linear transformation

$$T \circ U\colon \mathbf{R}^p \to \mathbf{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).$$



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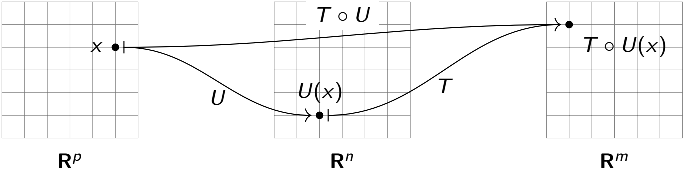

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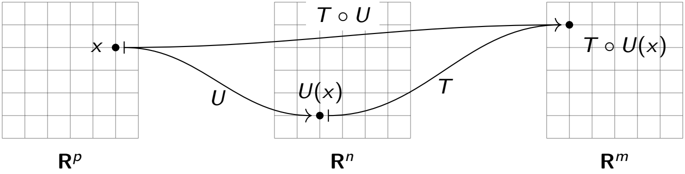

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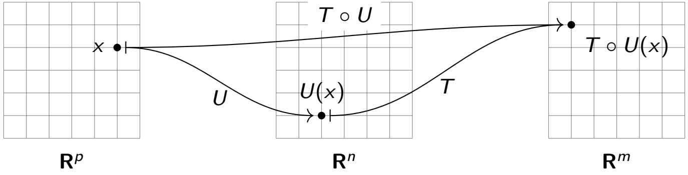

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Now let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be an *invertible* linear transformation. This means there is a linear transformation $T^{-1}\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $T \circ T^{-1}(x) = x$ for all x in $\mathbf{R}^n$.

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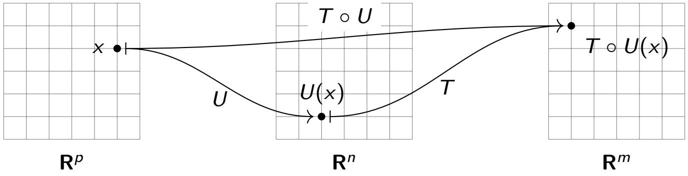

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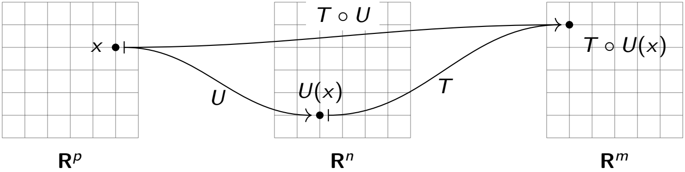

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Fact: If A is the matrix for T, then A^{-1} is the matrix for T^{-1}.

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ scale the x-axis by 2, and let $U\colon \mathbf{R}^2 \to \mathbf{R}^2$ be counterclockwise rotation by 90°.

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2×2 case:

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If A is invertible, then $Ax = b$ has exactly one solution for any b, namely, $x = A^{-1}b$.

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Example

Solve $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

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$$B = EA \quad \text{where} \quad E = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad \begin{pmatrix} \text{subtract } 2\times \text{ the first row} \\ \text{of } I_2 \text{ from the second row} \end{pmatrix}.$$

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The Invertible Matrix Theorem

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Let A be a square $n \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.
14. The columns of A form a basis for $\mathbf{R}^n$.
15. $\operatorname{Col} A = \mathbf{R}^n$.
16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
18. $\operatorname{Nul} A = \{0\}$.
19. $\dim \operatorname{Nul} A = 0$.

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16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
18. $\operatorname{Nul} A = \{0\}$.
19. $\dim \operatorname{Nul} A = 0$.

Learn it!

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
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> If V can be written in any of the above ways, then it is automatically a subspace: you're done!

Example

Is $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + y = 0 \right\}$ a subspace?

Subspaces
Example, continued

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Since conditions (1), (2), and (3) hold, V is a subspace.

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Since condition (3) fails, V is not a subspace.

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Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

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So if you *already know the dimension* of V, you only have to check *one*.

Verify that $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + y = 0 \right\}$.

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If we knew a priori that $\dim V = 2$, then we would only have to check 0, then 1 *or* 2.

Bases of Col A and Nul A

$$A = \begin{pmatrix} \boxed{1} & \boxed{2} & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis $\leftarrow$ pivot columns in rref

Rank Theorem

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If A is an $m \times n$ matrix, then

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In this case, $\operatorname{rank} A = 2$ and $\dim \operatorname{Nul} A = 2$, and $2 + 2 = 4$, which is the number of columns of A.

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Determinants

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5. Cofactor expansion and any other of the above. (The cofactor formula is recursive.)

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with the following **defining properties**:

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Determinants and Linear Transformations

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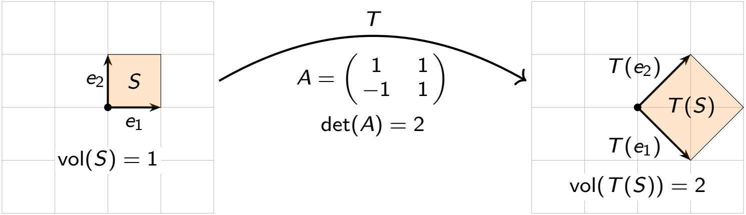

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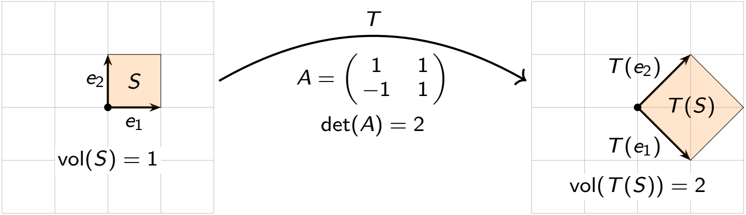

For curvy shapes, you break S up into a bunch of tiny cubes. Each one is scaled by $|\det(A)|$; then you use *calculus* to reduce to the previous situation!