

Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

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A Biology Question

Motivation

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This makes it easy to compute examples by computer:

v_0	v_{10}	v_{11}
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$$v_n = (\text{scalar}) \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}.$$

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Translation: 2 is an eigenvalue, and $\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$ is an eigenvector!

Eigenvectors and Eigenvalues

Definition

Let A be an $n \times n$ matrix.

Eigenvalues and eigenvectors are only for square matrices.

1. An **eigenvector** of A is a nonzero vector v in \mathbf{R}^n such that $Av = \lambda v$, for some λ in \mathbf{R} .

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This is the most important definition in the course.

Verifying Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

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Multiply:

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Hence v is an eigenvector of A , with eigenvalue $\lambda = 4$.

Poll

Which of the vectors

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$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenvector with eigenvalue 2

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eigenvector with eigenvalue 0

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not an eigenvector

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is never an eigenvector

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

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$$A - 3I =$$

Eigenspaces

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A . The λ -**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ , plus the zero vector:

$$\lambda\text{-eigenspace} = \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\}$$

Eigenspaces

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A . The λ -**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ , plus the zero vector:

$$\begin{aligned}\lambda\text{-eigenspace} &= \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\} \\ &= \{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\}\end{aligned}$$

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How do you find a basis for the λ -eigenspace? Parametric vector form!

Eigenspaces

Example

Find a basis for the 2-eigenspace of

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

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Eigenspaces

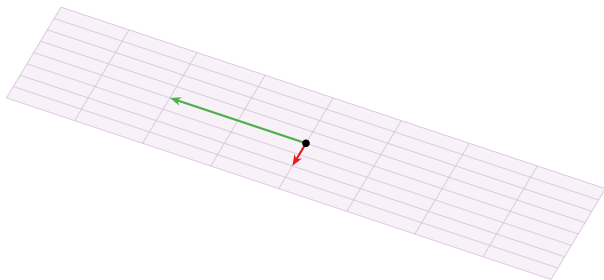
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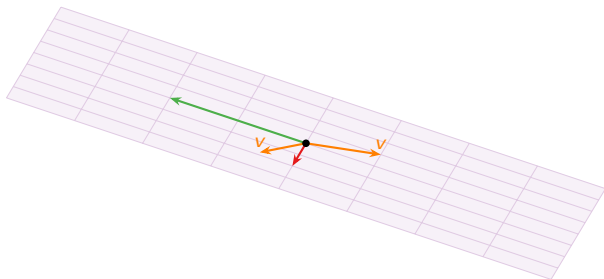
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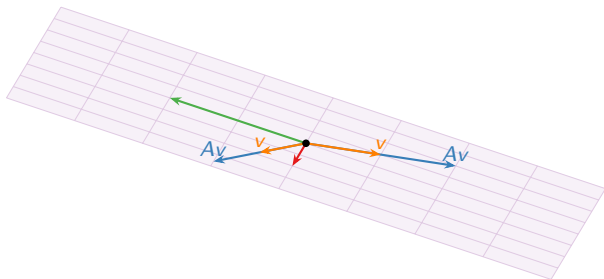


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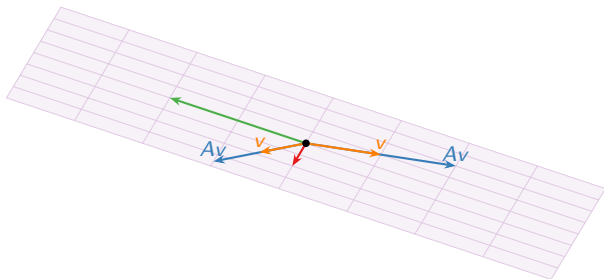


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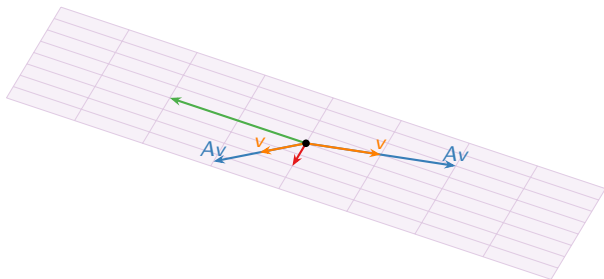


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For any v in the 2-eigenspace, $Av = 2v$ by definition. So A acts by *scaling by 2* on its 2-eigenspace. This is how eigenvalues and eigenvectors make matrices easier to understand.

Eigenspaces

Geometry

Eigenvectors, geometrically

An eigenvector of a matrix A is a nonzero vector v such that:

Eigenspaces

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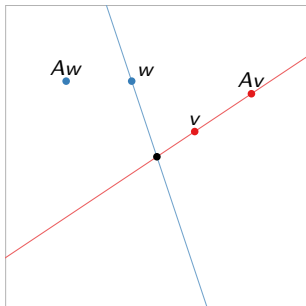
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v is an eigenvector

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Eigenspaces

Geometry; example

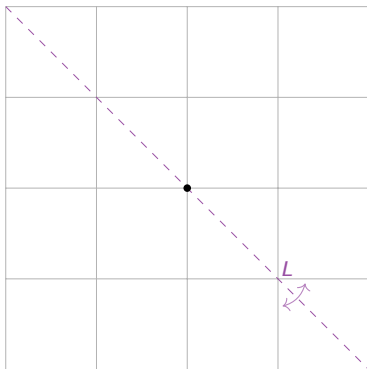
Let $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T .

Eigenspaces

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Let $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T .

Question: What are the eigenvalues and eigenspaces of A ? No computations!



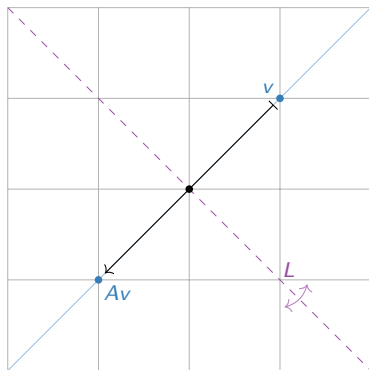
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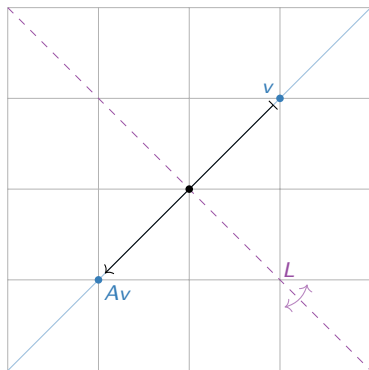
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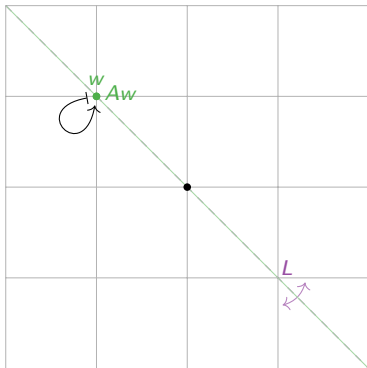
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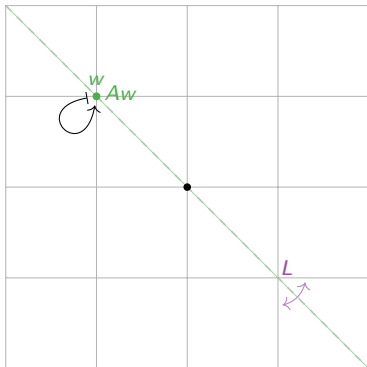
w is an eigenvector with eigenvalue $_$.

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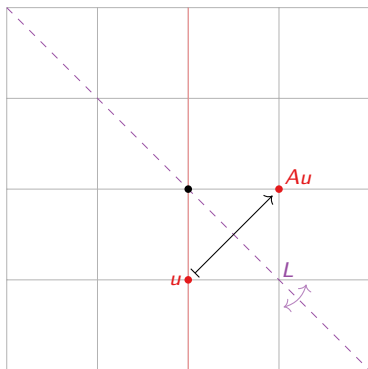
w is an eigenvector with eigenvalue 1.

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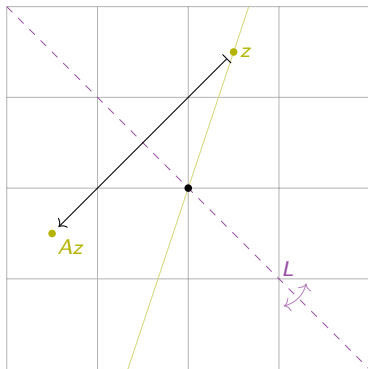
u is *not* an eigenvector.

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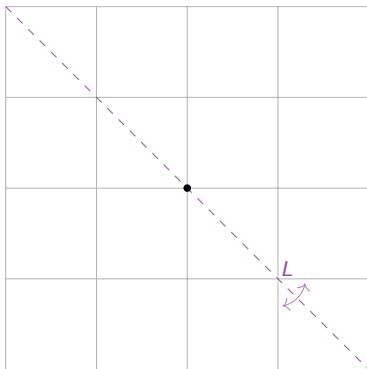
Neither is z .

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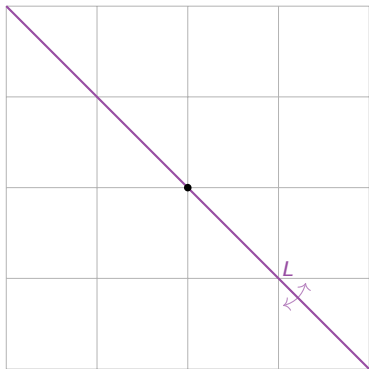
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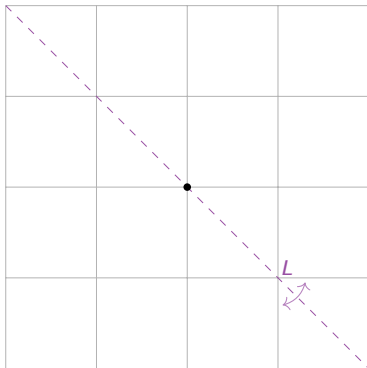
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Eigenspaces

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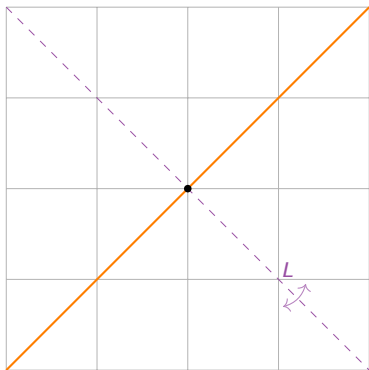
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Eigenspaces

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3. The eigenvectors with eigenvalue λ are the nonzero elements of $\text{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$.

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Fact: The eigenvalues of a triangular matrix are the diagonal entries.

A Matrix is Invertible if and only if Zero is not an Eigenvalue

Fact: A is invertible if and only if 0 is not an eigenvalue of A .

Eigenvectors with Distinct Eigenvalues are Linearly Independent

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Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad v_0 = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \quad Av_0 = 2v_0.$$

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$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad v_0 = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \quad Av_0 = 2v_0.$$

So if you start with 16 baby rabbits, 4 first-year rabbits, and 1 second-year rabbit, then the population will exactly double every year.

Difference Equations

Preview

Let A be an $n \times n$ matrix. Suppose we want to solve $Av_n = v_{n+1}$ for all n . In other words, we want vectors v_0, v_1, v_2, \dots , such that

$$Av_0 = v_1 \quad Av_1 = v_2 \quad Av_2 = v_3 \quad \dots$$

We saw before that $v_n = A^n v_0$. But it is inefficient to multiply by A each time.

If v_0 is an *eigenvector* with eigenvalue λ , then

$$v_1 = Av_0 = \lambda v_0 \quad v_2 = Av_1 = \lambda v_1 = \lambda^2 v_0 \quad v_3 = Av_2 = \lambda v_2 = \lambda^3 v_0.$$

In general, $v_n = \lambda^n v_0$. This is *much easier* to compute.

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad v_0 = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \quad Av_0 = 2v_0.$$

So if you start with 16 baby rabbits, 4 first-year rabbits, and 1 second-year rabbit, then the population will exactly double every year. In year n , you will have $2^n \cdot 16$ baby rabbits, $2^n \cdot 4$ first-year rabbits, and 2^n second-year rabbits.