# Section 1.9

## The Matrix of a Linear Transformation

# Unit Coordinate Vectors

## Definition

The **unit coordinate vectors** in $\mathbf{R}^n$ are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

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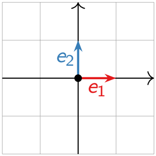

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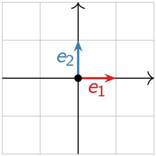


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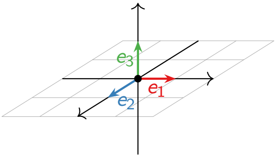

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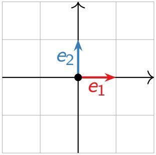


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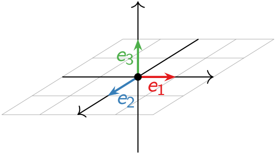


Note: if $A$ is an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$:

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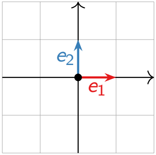


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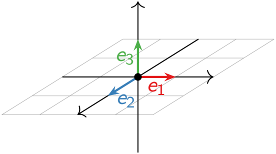


Note: if $A$ is an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: multiplying a matrix by $e_i$ gives you the $i$th column.

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Recall: A matrix $A$ defines a linear transformation $T$ by $T(x) = Ax$.

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**Theorem**

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an _____ matrix,

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# Linear Transformations are Matrix Transformations

## Continued

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Suppose for simplicity that $T\colon \mathbf{R}^3 \to \mathbf{R}^2$.

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(Check linearity. . . )

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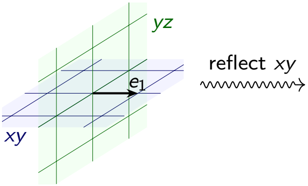

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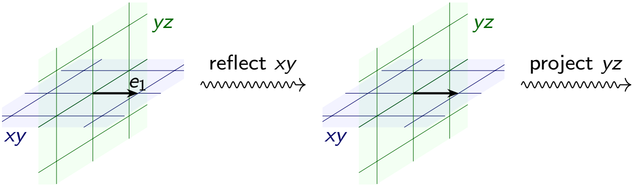

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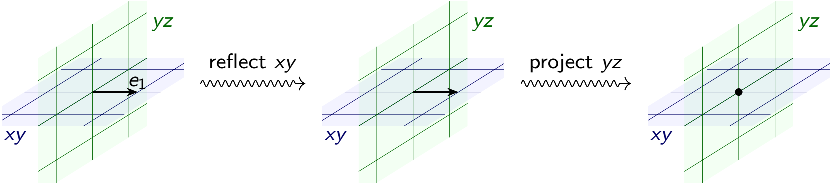

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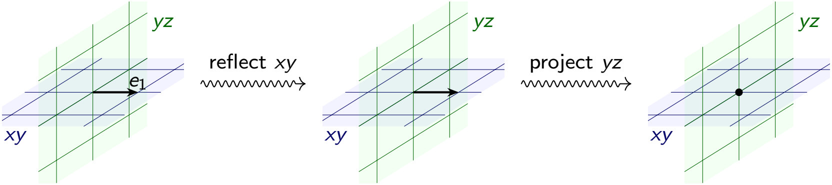


$$T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

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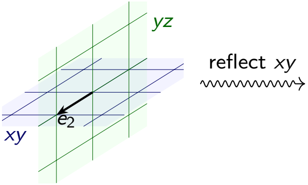

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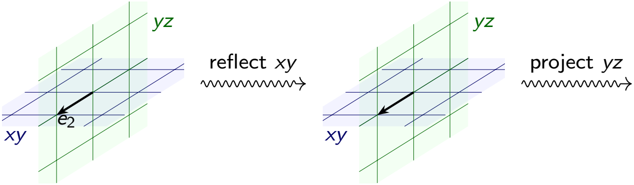

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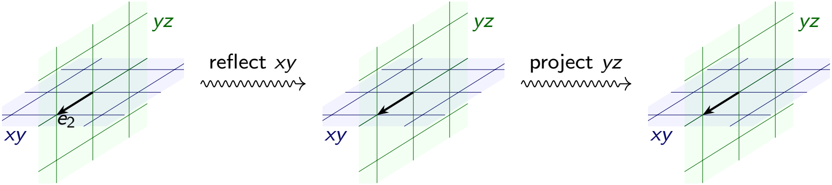

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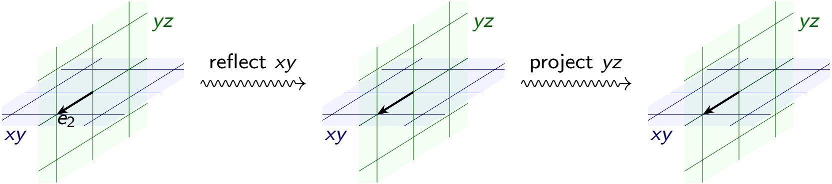


$$T(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

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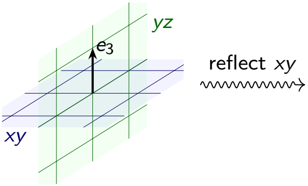

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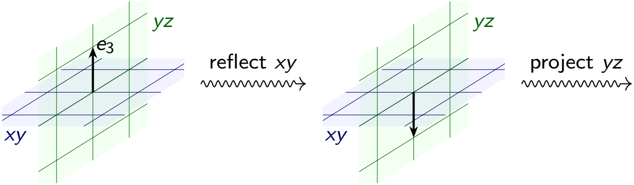

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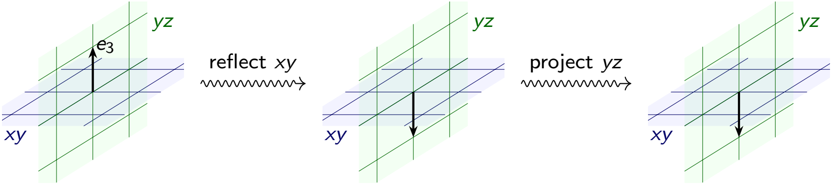

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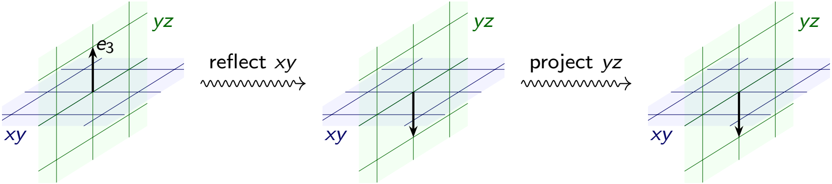


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# Other Geometric Transformations

There is a long list of geometric transformations of $\mathbf{R}^2$ in §1.9 of Lay. (Reflections over the diagonal, contractions and expansions along different axes, shears, projections, ...) Please look them over.

# Onto Transformations

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A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **onto** (or **surjective**) if the range of $T$ is equal to $\mathbf{R}^m$ (its codomain).

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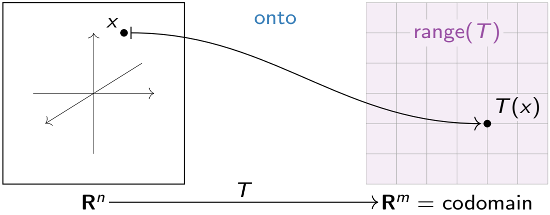

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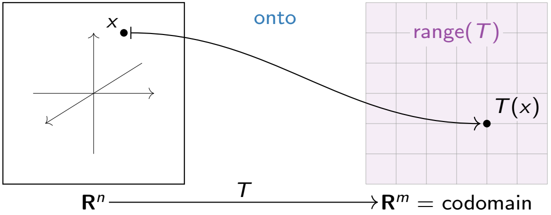


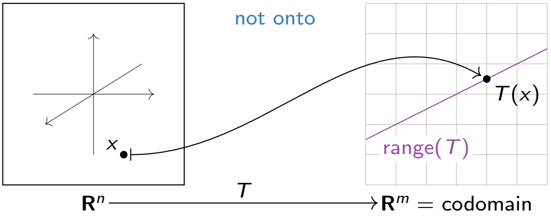

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For instance, $\mathbf{R}^2$ is "too small" to map *onto* $\mathbf{R}^3$.

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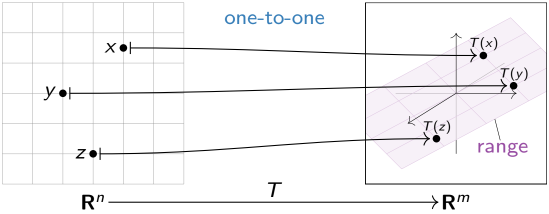

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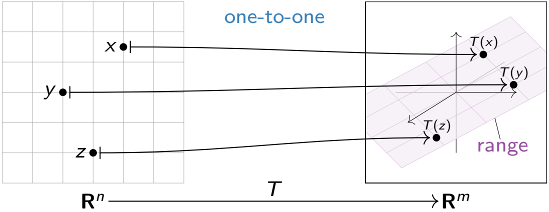


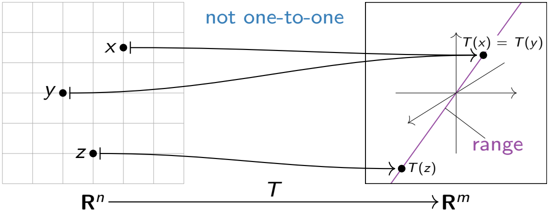

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For instance, $\mathbf{R}^3$ is "too big" to map *into* $\mathbf{R}^2$.