Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

Eigenvectors and Eigenvalues

In a population of rabbits:

1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have $0, 6, 8$ baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

$$f_n = \text{first-year rabbits in year } n$$
$$s_n = \text{second-year rabbits in year } n$$
$$t_n = \text{third-year rabbits in year } n$$

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $\boxed{Av_n = v_{n+1}.}$ $\longleftarrow$ **difference equation**

A Biology Question

Continued

If you know v_0, what is v_{10}?

$$v_{10} = Av_9 = AAv_8 = \cdots = A^{10}v_0.$$

This makes it easy to compute examples by computer:

v_0	v_{10}	v_{11}
$\begin{pmatrix}3\\7\\9\end{pmatrix}$	$\begin{pmatrix}30189\\7761\\1844\end{pmatrix}$	$\begin{pmatrix}61316\\15095\\3881\end{pmatrix}$
$\begin{pmatrix}1\\2\\3\end{pmatrix}$	$\begin{pmatrix}9459\\2434\\577\end{pmatrix}$	$\begin{pmatrix}19222\\4729\\1217\end{pmatrix}$
$\begin{pmatrix}4\\7\\8\end{pmatrix}$	$\begin{pmatrix}28856\\7405\\1765\end{pmatrix}$	$\begin{pmatrix}58550\\14428\\3703\end{pmatrix}$

What do you notice about these numbers?

1. Eventually, each segment of the population doubles every year: $Av_n = v_{n+1} = 2v_n$.
2. The ratios get close to $(16 : 4 : 1)$:

$$v_n = (\text{scalar}) \cdot \begin{pmatrix}16\\4\\1\end{pmatrix}.$$

Translation: 2 is an eigenvalue, and $\begin{pmatrix}16\\4\\1\end{pmatrix}$ is an eigenvector!

Eigenvectors and Eigenvalues

Definition

Let A be an $n \times n$ matrix.

Eigenvalues and eigenvectors are only for square matrices.

1. An **eigenvector** of A is a *nonzero* vector v in $\mathbf{R}^n$ such that $Av = \lambda v$, for some λ in $\mathbf{R}$. In other words, Av is a multiple of v.
2. An **eigenvalue** of A is a number λ in $\mathbf{R}$ such that the equation $Av = \lambda v$ has a *nontrivial* solution.

If $Av = \lambda v$ for $v \neq 0$, we say λ is the **eigenvalue for** v, and v is an **eigenvector for** λ.

Note: Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

This is the most important definition in the course.

Verifying Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = 2v$$

Hence v is an eigenvector of A, with eigenvalue $\lambda = 2$.

Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4v$$

Hence v is an eigenvector of A, with eigenvalue $\lambda = 4$.

Poll

Poll

Which of the vectors

A. $\begin{pmatrix}1\\1\end{pmatrix}$ B. $\begin{pmatrix}1\\-1\end{pmatrix}$ C. $\begin{pmatrix}-1\\1\end{pmatrix}$ D. $\begin{pmatrix}2\\1\end{pmatrix}$ E. $\begin{pmatrix}0\\0\end{pmatrix}$

are eigenvectors of the matrix $\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$?

What are the eigenvalues?

$$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix}$$ eigenvector with eigenvalue 2

$$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = 0\begin{pmatrix}1\\-1\end{pmatrix}$$ eigenvector with eigenvalue 0

$$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = 0\begin{pmatrix}-1\\1\end{pmatrix}$$ eigenvector with eigenvalue 0

$$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$$ not an eigenvector

$$\begin{pmatrix}0\\0\end{pmatrix}$$ is never an eigenvector

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

In other words, does $Av = 3v$ have a nontrivial solution?
. . . does $Av - 3v = 0$ have a nontrivial solution?
. . . does $(A - 3I)v = 0$ have a nontrivial solution?
We know how to answer that! Row reduction!

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

Row reduce:

$$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

Parametric form: $x = -4y$; parametric vector form: $\begin{pmatrix} x \\ y \end{pmatrix} = y\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

Does there exist an eigenvector with eigenvalue $\lambda = 3$? Yes! Any nonzero multiple of $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Check:

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix} = 3\begin{pmatrix} -4 \\ 1 \end{pmatrix}. \quad ✔$$

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The λ-**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ, plus the zero vector:

$$\begin{aligned} \lambda\text{-eigenspace} &= \bigl\{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\bigr\} \\ &= \bigl\{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\bigr\} \\ &= \mathrm{Nul}\bigl(A - \lambda I\bigr). \end{aligned}$$

Since the λ-eigenspace is a null space, it is a *subspace* of $\mathbf{R}^n$.

How do you find a basis for the λ-eigenspace? Parametric vector form!

Eigenspaces

Example

Find a basis for the 2-eigenspace of

λ

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

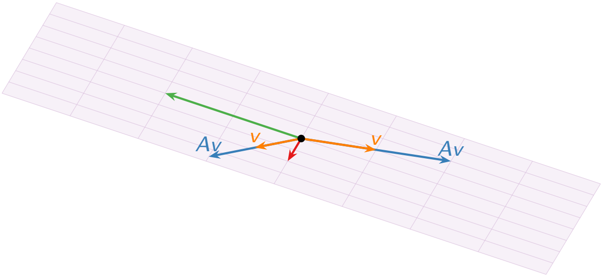
$$A - 2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\substack{\text{parametric}\\ \text{form}}} x = \frac{1}{2}y - 3z$$

$$\xrightarrow{\substack{\text{parametric vector}\\ \text{form}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\xrightarrow{\text{basis}} \left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

A basis for the 2-eigenspace of $\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$ is $\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$. What does this look like?

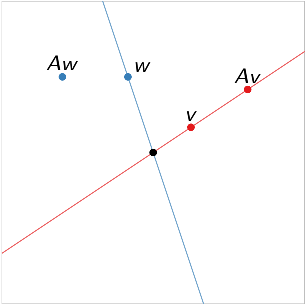


For any v in the 2-eigenspace, $Av = 2v$ by definition. So A acts by *scaling by* 2 on its 2-eigenspace. This is how eigenvalues and eigenvectors make matrices easier to understand.

Eigenvectors, geometrically

An eigenvector of a matrix A is a nonzero vector v such that:

- Av is a multiple of v, which means
- Av is collinear with v, which means
- Av and v are *on the same line*.

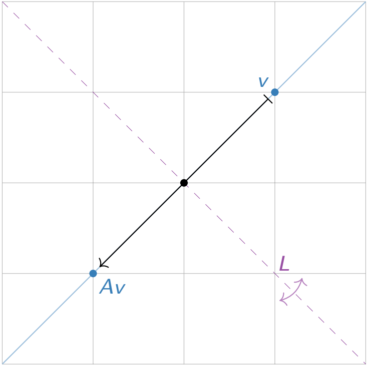


v is an eigenvector

w is not an eigenvector

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

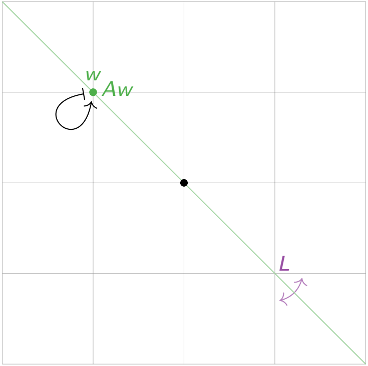


Does anyone see any eigenvectors (vectors that don't move off their line)?

v is an eigenvector with eigenvalue -1.

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

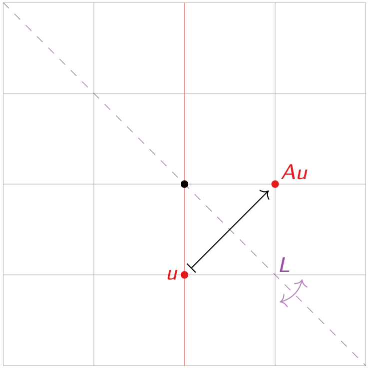


Does anyone see any eigenvectors (vectors that don't move off their line)?

w is an eigenvector with eigenvalue 1.

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

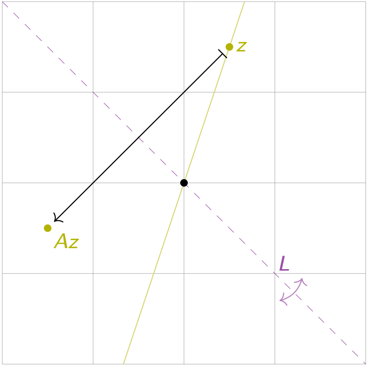


Does anyone see any eigenvectors (vectors that don't move off their line)?

u is *not* an eigenvector.

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

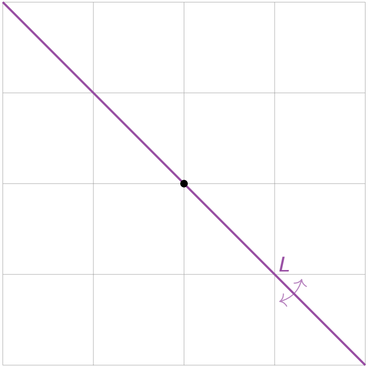


Does anyone see any eigenvectors (vectors that don't move off their line)?

Neither is z.

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

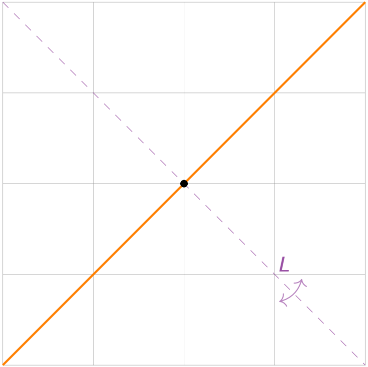
The 1-eigenspace is L
(all the vectors x where $Ax = x$).

Eigenspaces

Geometry; example

Let $T \colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

The (-1)-eigenspace is the line $y = x$
(all the vectors x where $Ax = -x$).

Let A be an $n \times n$ matrix and let λ be a number.

1. λ is an eigenvalue of A if and only if $(A - \lambda I)x = 0$ has a nontrivial solution, if and only if $\mathrm{Nul}(A - \lambda I) \neq \{0\}$.
2. In this case, finding a basis for the λ-eigenspace of A means finding a basis for $\mathrm{Nul}(A - \lambda I)$ as usual, i.e. by finding the parametric vector form for the general solution to $(A - \lambda I)x = 0$.
3. The eigenvectors with eigenvalue λ are the nonzero elements of $\mathrm{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$.

The Eigenvalues of a Triangular Matrix are the Diagonal Entries

We've seen that finding eigenvectors for a given eigenvalue is a row reduction problem.

Finding all of the eigenvalues of a matrix *is not a row reduction problem!* We'll see how to do it in general next time. For now:

Fact: The eigenvalues of a triangular matrix are the diagonal entries.

Why? $\mathrm{Nul}(A - \lambda I) \neq \{0\}$ if and only if $A - \lambda I$ is not invertible, if and only if $\det(A - \lambda I) = 0$.

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 0 & -1 & -2 & 7 \\ 0 & 0 & 8 & 12 \\ 0 & 0 & 0 & -3 \end{pmatrix} - \lambda I_4 = \begin{pmatrix} 3-\lambda & 4 & 1 & 2 \\ 0 & -1-\lambda & -2 & 7 \\ 0 & 0 & 8-\lambda & 12 \\ 0 & 0 & 0 & -3-\lambda \end{pmatrix}.$$

The determinant is $(3-\lambda)(-1-\lambda)(8-\lambda)(-3-\lambda)$, which is zero exactly when $\lambda = 3, -1, 8$, or -3.

A Matrix is Invertible if and only if Zero is not an Eigenvalue

Fact: A is invertible if and only if 0 is not an eigenvalue of A.

Why?

$$\begin{aligned} 0 \text{ is an eigenvalue of } A &\iff Ax = 0x \text{ has a nontrivial solution} \\ &\iff Ax = 0 \text{ has a nontrivial solution} \\ &\iff A \text{ is not invertible.} \end{aligned}$$

invertible matrix theorem

Eigenvectors with Distinct Eigenvalues are Linearly Independent

Fact: If $v_1, v_2, \ldots, v_k$ are eigenvectors of A with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Why? If $k = 2$, this says v_2 can't lie on the line through v_1.

But the line through v_1 is contained in the λ_1-eigenspace, and v_2 does not have eigenvalue λ_1.

In general: see Lay, Theorem 2 in §5.1 (or work it out for yourself; it's not too hard).

Consequence: An $n \times n$ matrix has at most n distinct eigenvalues.

Let A be an $n \times n$ matrix. Suppose we want to solve $Av_n = v_{n+1}$ for all n. In other words, we want vectors $v_0, v_1, v_2, \ldots$, such that

$$Av_0 = v_1 \qquad Av_1 = v_2 \qquad Av_2 = v_3 \qquad \ldots$$

We saw before that $v_n = A^n v_0$. But it is inefficient to multiply by A each time.

If v_0 is an *eigenvector* with eigenvalue λ, then

$$v_1 = Av_0 = \lambda v_0 \qquad v_2 = Av_1 = \lambda v_1 = \lambda^2 v_0 \qquad v_3 = Av_2 = \lambda v_2 = \lambda^3 v_0.$$

In general, $v_n = \lambda^n v_0$. This is *much easier* to compute.

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v_0 = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \qquad Av_0 = 2v_0.$$

So if you start with 16 baby rabbits, 4 first-year rabbits, and 1 second-year rabbit, then the population will exactly double every year. In year n, you will have $2^n \cdot 16$ baby rabbits, $2^n \cdot 4$ first-year rabbits, and 2^n second-year rabbits.