

Review for the Final Exam

Selected Topics

Orthogonal Sets

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Example: $\mathcal{B}_3 = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is orthonormal.

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To go from an orthogonal set $\{u_1, u_2, \dots, u_m\}$ to an orthonormal set, replace each u_i with $u_i/\|u_i\|$.

Theorem

An orthogonal set is linearly independent. In particular, it is a basis for its span.

Orthogonal Projection

Let W be a subspace of \mathbf{R}^n , and let $\mathcal{B} = \{u_1, u_2, \dots, u_m\}$ be an *orthogonal* basis for W . The **orthogonal projection** of a vector x onto W is

$$\text{proj}_W(x) \stackrel{\text{def}}{=} \sum_{i=1}^m \frac{x \cdot u_i}{u_i \cdot u_i} u_i = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m} u_m.$$

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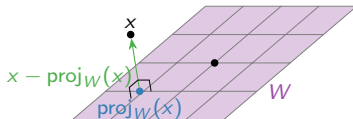
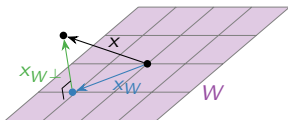
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So x_W is in W , x_{W^\perp} is in W^\perp , and $x = x_W + x_{W^\perp}$.



Orthogonal Projection

Special cases

Special case: If x is in W , then $x = \text{proj}_W(x)$, so

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m} u_m.$$

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In other words, the \mathcal{B} -coordinates of x are

$$\left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_1 \cdot u_2}, \dots, \frac{x \cdot u_m}{u_1 \cdot u_m} \right),$$

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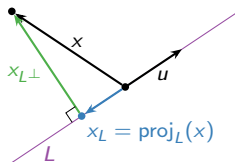
In other words, the \mathcal{B} -coordinates of x are

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Special case: If $W = L$ is a line, then $L = \text{Span}\{u\}$ for some nonzero vector u , and

$$\text{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u$$



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And matrices

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The 1-eigenspace of A is W , and the 0-eigenspace is W^\perp .

The Gram–Schmidt Process

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QR Factorization

QR Factorization Theorem

Let A be a matrix with linearly independent columns. Then

$$A = QR$$

where Q has orthonormal columns and R is upper-triangular with positive diagonal entries.

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Step 1: Let v_1, v_2, \dots, v_m be the columns of A . Run Gram–Schmidt on $\{v_1, v_2, \dots, v_m\}$ to get an orthogonal basis $\{u_1, u_2, \dots, u_m\}$, and solve for each v_i in terms of u_1, u_2, \dots, u_i .

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Step 2: Put the resulting equations in matrix form to get $A = \hat{Q}\hat{R}$ where

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix} \quad \hat{Q} = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{pmatrix}$$

and \hat{R} contains the coefficients from $v_i =$ (linear combination of u_1, u_2, \dots, u_{i-1}) in the columns.

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Step 3: Scale each column of \hat{Q} by its length to get a matrix with orthonormal columns, and scale each row of \hat{R} by the opposite factor to get Q and R , respectively.

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Example

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Step 1: Let v_1, v_2, v_3 be the columns. Run Gram–Schmidt and solve for v_1, v_2, v_3 in terms of u_1, u_2, u_3 :

QR Factorization

Example, continued

$$v_1 = 1 u_1 \quad v_2 = \frac{3}{2} u_1 + 1 u_2 \quad v_3 = 0 u_1 - \frac{4}{5} u_2 + 1 u_3$$

Step 2: write $A = \hat{Q}\hat{R}$, where \hat{Q} has *orthogonal* columns u_1, u_2, u_3 and \hat{R} is upper-triangular with 1s on the diagonal.

QR Factorization

Example, continued

$$A = \hat{Q}\hat{R} \quad \hat{Q} = \begin{pmatrix} 1 & -5/2 & 2 \\ 1 & 5/2 & 0 \\ 1 & 5/2 & 0 \\ 1 & -5/2 & -2 \end{pmatrix} \quad \hat{R} = \begin{pmatrix} 1 & 3/2 & 0 \\ 0 & 1 & -4/5 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 3: normalize the columns of \hat{Q} and the rows of \hat{R} to get Q and R :

QR Factorization

Example, continued

$$A = \hat{Q}\hat{R} \quad \hat{Q} = \begin{pmatrix} 1 & -5/2 & 2 \\ 1 & 5/2 & 0 \\ 1 & 5/2 & 0 \\ 1 & -5/2 & -2 \end{pmatrix} \quad \hat{R} = \begin{pmatrix} 1 & 3/2 & 0 \\ 0 & 1 & -4/5 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 3: normalize the columns of \hat{Q} and the rows of \hat{R} to get Q and R :

The final QR decomposition is

$$A = QR \quad Q = \begin{pmatrix} 1/2 & -1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & -1/\sqrt{2} \end{pmatrix} \quad R = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 5 & -4 \\ 0 & 0 & 2\sqrt{2} \end{pmatrix}.$$

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Then $A = PBP^{-1}$. B acts on the usual coordinates by scaling the first coordinate by 2, and the second by $1/2$:

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The unit coordinate vectors are eigenvectors: e_1 has eigenvalue 2, and e_2 has eigenvalue

Similarity

Geometric meaning

Let $A = PBP^{-1}$, and let v_1, v_2, \dots, v_n be the columns of P . These form a basis \mathcal{B} for \mathbf{R}^n because P is invertible. *Key relation:* for any vector x in \mathbf{R}^n ,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

This says:

A acts on the usual coordinates of x
in the same way that
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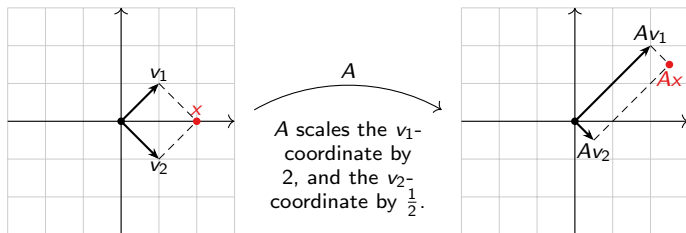
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Important

$Ax = b$ is consistent if and only if b is in $\text{Col } A$.

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Theorem

The least-squares solutions to $Ax = b$ are the solutions to

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If A has *orthogonal* columns u_1, u_2, \dots, u_n , then the least-squares solution is

$$\hat{x} = \left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2}, \dots, \frac{x \cdot u_m}{u_m \cdot u_m} \right)$$

because

$$A\hat{x} = \hat{b} = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \dots + \frac{x \cdot u_m}{u_m \cdot u_m} u_m.$$