Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

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Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $Av_n = v_{n+1}$.

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A Biology Question
Continued

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This makes it easy to compute examples by computer:

v_0	v_{10}	v_{11}
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Translation: 2 is an eigenvalue, and $\begin{pmatrix}16\\4\\1\end{pmatrix}$ is an eigenvector!

Eigenvectors and Eigenvalues

Definition

Let A be an $n \times n$ matrix.

Eigenvalues and eigenvectors are only for square matrices.

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This is the most important definition in the course.

Verifying Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

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Multiply:

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Hence v is an eigenvector of A, with eigenvalue $\lambda = 4$.

Poll

Poll

Which of the vectors

A. $\begin{pmatrix}1\\1\end{pmatrix}$ B. $\begin{pmatrix}1\\-1\end{pmatrix}$ C. $\begin{pmatrix}-1\\1\end{pmatrix}$ D. $\begin{pmatrix}2\\1\end{pmatrix}$ E. $\begin{pmatrix}0\\0\end{pmatrix}$

are eigenvectors of the matrix $\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$?

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$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenvector with eigenvalue 2

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$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ is never an eigenvector

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

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$\ldots$ does $Av - 3v = 0$ have a nontrivial solution?

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$$A - 3I =$$

Eigenspaces

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The λ-**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ, plus the zero vector:

$$\lambda\text{-eigenspace} = \left\{ v \text{ in } \mathbf{R}^n \mid Av = \lambda v \right\}$$

Eigenspaces

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Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The λ-**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ, plus the zero vector:

$$\begin{aligned} \lambda\text{-eigenspace} &= \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\} \\ &= \{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\} \end{aligned}$$

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How do you find a basis for the λ-eigenspace? Parametric vector form!

Find a basis for the 2-eigenspace of

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

Eigenspaces
Example

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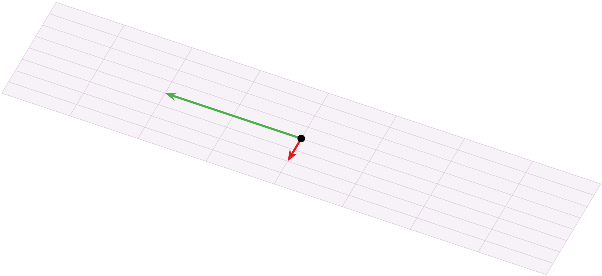

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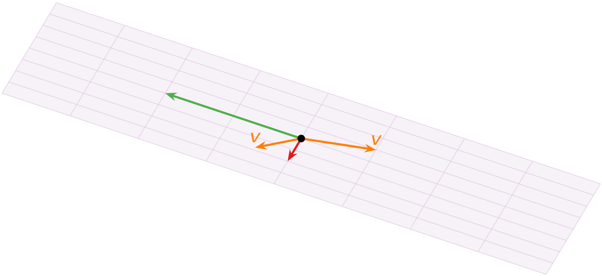

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Eigenspaces
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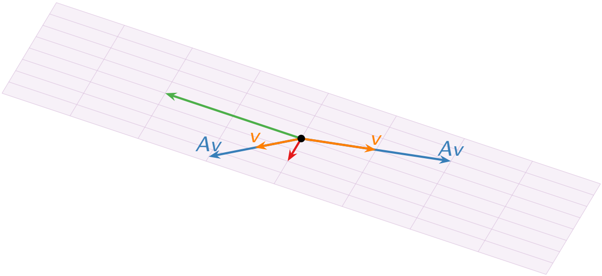

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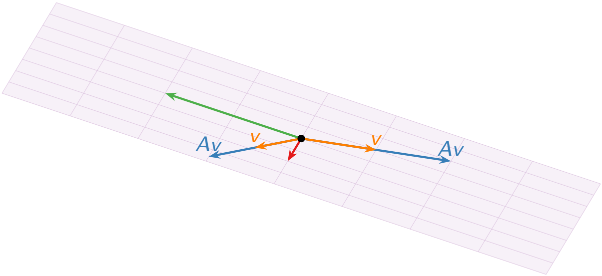

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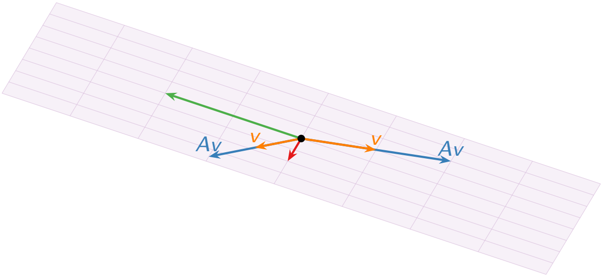

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Eigenvectors, geometrically

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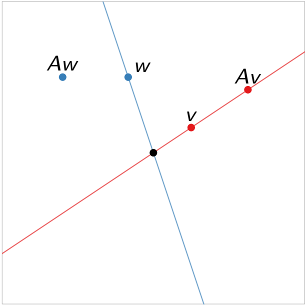

v is an eigenvector

w is not an eigenvector

Eigenspaces

Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

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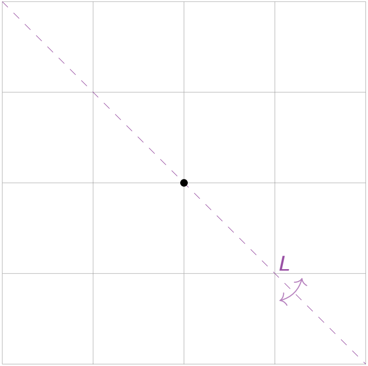

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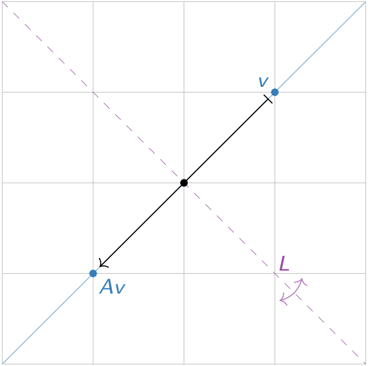

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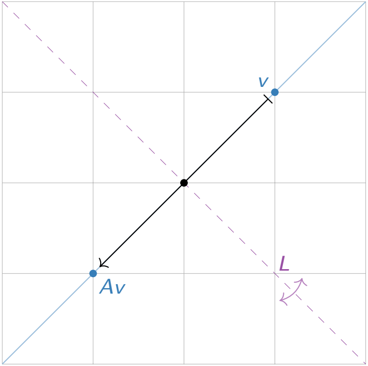

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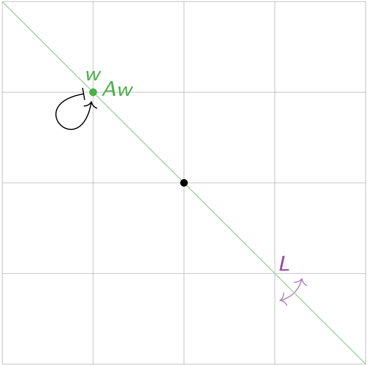

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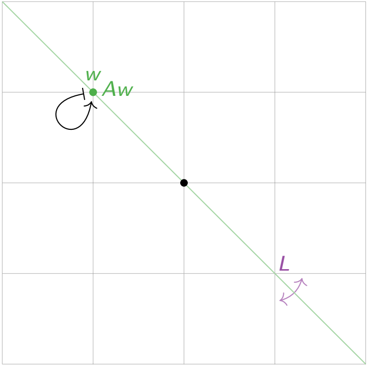

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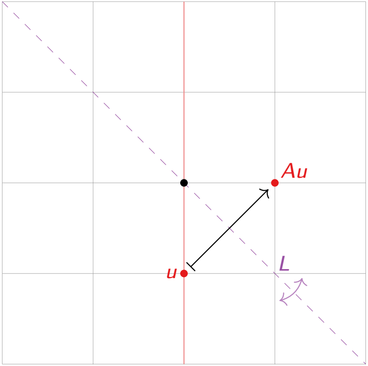

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u is *not* an eigenvector.

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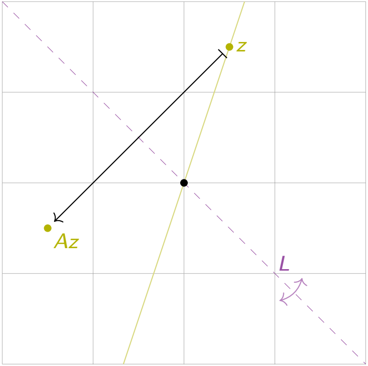

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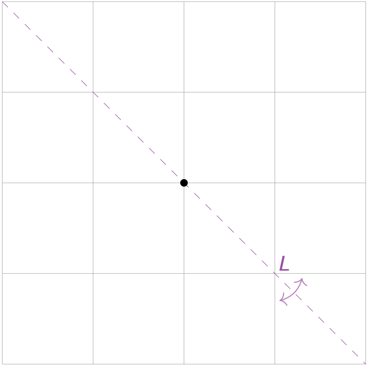

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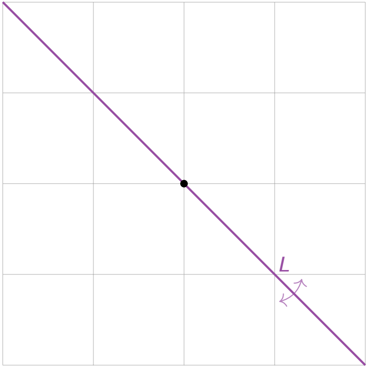

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(all the vectors x where $Ax = x$).

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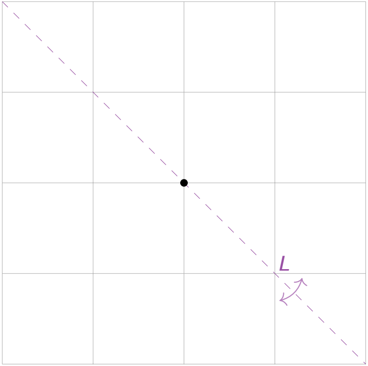

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The (-1)-eigenspace is ____________

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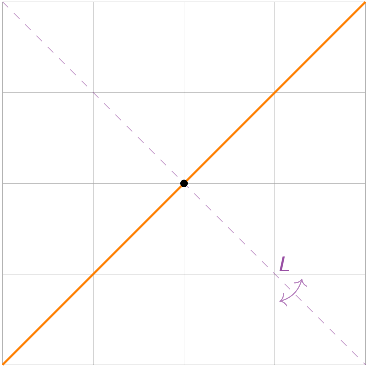

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3. The eigenvectors with eigenvalue λ are the nonzero elements of $\mathrm{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$.

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Fact: The eigenvalues of a triangular matrix are the diagonal entries.

A Matrix is Invertible if and only if Zero is not an Eigenvalue

Fact: A is invertible if and only if 0 is not an eigenvalue of A.

Eigenvectors with Distinct Eigenvalues are Linearly Independent

Fact: If $v_1, v_2, \ldots, v_k$ are eigenvectors of A with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

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Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v_0 = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \qquad Av_0 = 2v_0.$$

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If v_0 is an *eigenvector* with eigenvalue λ, then

$$v_1 = Av_0 = \lambda v_0 \qquad v_2 = Av_1 = \lambda v_1 = \lambda^2 v_0 \qquad v_3 = Av_2 = \lambda v_2 = \lambda^3 v_0.$$

In general, $v_n = \lambda^n v_0$. This is *much easier* to compute.

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v_0 = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \qquad Av_0 = 2v_0.$$

So if you start with 16 baby rabbits, 4 first-year rabbits, and 1 second-year rabbit, then the population will exactly double every year.

Let A be an $n \times n$ matrix. Suppose we want to solve $Av_n = v_{n+1}$ for all n. In other words, we want vectors $v_0, v_1, v_2, \ldots$, such that

$$Av_0 = v_1 \qquad Av_1 = v_2 \qquad Av_2 = v_3 \qquad \ldots$$

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So if you start with 16 baby rabbits, 4 first-year rabbits, and 1 second-year rabbit, then the population will exactly double every year. In year n, you will have $2^n \cdot 16$ baby rabbits, $2^n \cdot 4$ first-year rabbits, and 2^n second-year rabbits.