

# Chapter 6

## Orthogonality and Least Squares

# Section 6.1

Inner Product, Length, and Orthogonality

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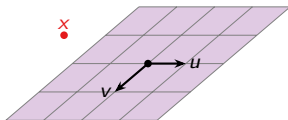
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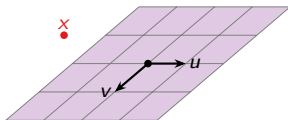
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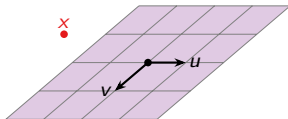
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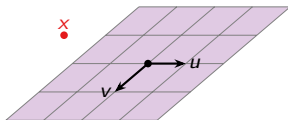
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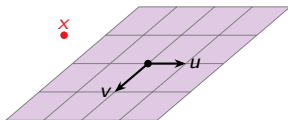
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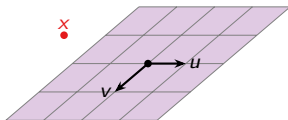
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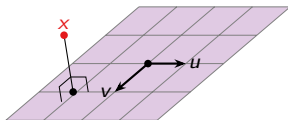
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## Example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} =$$

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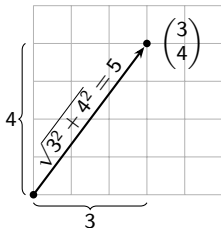
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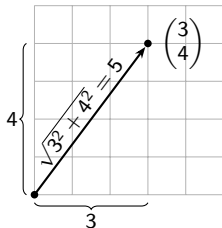
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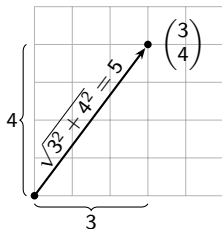
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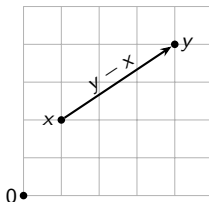
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This is in fact a unit vector:

scalar

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# Unit Vectors

## Example

### Example

What is the unit vector in the direction of  $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ ?

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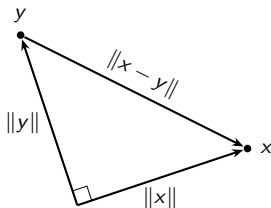
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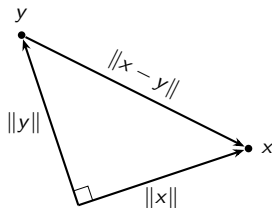
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**Fact:**  $x \perp y \iff \|x - y\|^2 = \|x\|^2 + \|y\|^2$

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**Problem:** Find *all* vectors orthogonal to both  $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$  and  $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .

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### Important

The set of all vectors orthogonal to some vectors  $v_1, v_2, \dots, v_m$  in  $\mathbf{R}^n$  is the *null space* of the  $m \times n$  matrix

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The set of all vectors orthogonal to some vectors  $v_1, v_2, \dots, v_m$  in  $\mathbf{R}^n$  is the *null space* of the  $m \times n$  matrix

$$\begin{pmatrix} -v_1^T- \\ -v_2^T- \\ \vdots \\ -v_m^T- \end{pmatrix}.$$

In particular, this set is a subspace!



# Orthogonal Complements

## Definition

Let  $W$  be a subspace of  $\mathbf{R}^n$ . Its **orthogonal complement** is

$$W^\perp = \{v \text{ in } \mathbf{R}^n \mid v \cdot w = 0 \text{ for all } w \text{ in } W\} \quad \text{read “} W \text{ perp”}.$$

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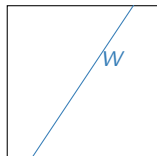
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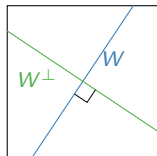
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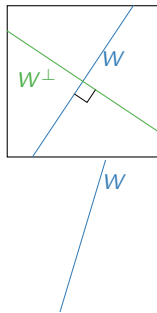
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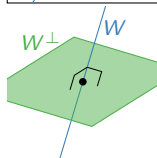
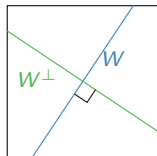
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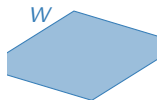
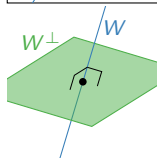
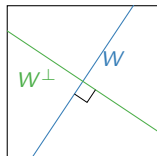
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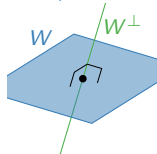
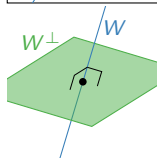
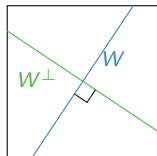
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## Poll

Let  $W$  be a plane in  $\mathbf{R}^4$ . How would you describe  $W^\perp$ ?

- A. The zero space  $\{0\}$ .
- B. A line in  $\mathbf{R}^4$ .
- C. A plane in  $\mathbf{R}^4$ .
- D. A 3-dimensional space in  $\mathbf{R}^4$ .
- E. All of  $\mathbf{R}^4$ .

# Orthogonal Complements

## Basic properties

Let  $W$  be a subspace of  $\mathbf{R}^n$ .

Facts:

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3.  $\dim W + \dim W^\perp = n$
4. If  $W = \text{Span}\{v_1, v_2, \dots, v_m\}$ , then

$$\begin{aligned} W^\perp &= \text{all vectors orthogonal to each } v_1, v_2, \dots, v_m \\ &= \{x \text{ in } \mathbf{R}^n \mid x \cdot v_i = 0 \text{ for all } i = 1, 2, \dots, m\} \\ &= \text{Nul} \begin{pmatrix} -v_1^T & - \\ -v_2^T & - \\ \vdots & \\ -v_m^T & - \end{pmatrix}. \end{aligned}$$

# Orthogonal Complements

## Computation

**Problem:** if  $W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ , compute  $W^\perp$ .

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# Orthogonal Complements

Row space, column space, null space

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**Fact:**  $(\text{Nul } A)^\perp = \text{Row } A$  and  $\text{Col } A = (\text{Nul } A^T)^\perp$ .

# Orthogonal Complements

## Reference sheet

### Orthogonal Complements of Most of the Subspaces We've Seen

For any vectors  $v_1, v_2, \dots, v_m$ :

$$\text{Span}\{v_1, v_2, \dots, v_m\}^\perp = \text{Nul} \begin{pmatrix} -v_1^T & - \\ -v_2^T & - \\ \vdots & \\ -v_m^T & - \end{pmatrix}$$

For any matrix  $A$ :

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and

$$\begin{aligned} (\text{Row } A)^\perp &= \text{Nul } A & \text{Row } A &= (\text{Nul } A)^\perp \\ (\text{Col } A)^\perp &= \text{Nul } A^T & \text{Col } A &= (\text{Nul } A^T)^\perp \end{aligned}$$