Section 2.8

Subspaces of $\mathbf{R}^n$

Motivation

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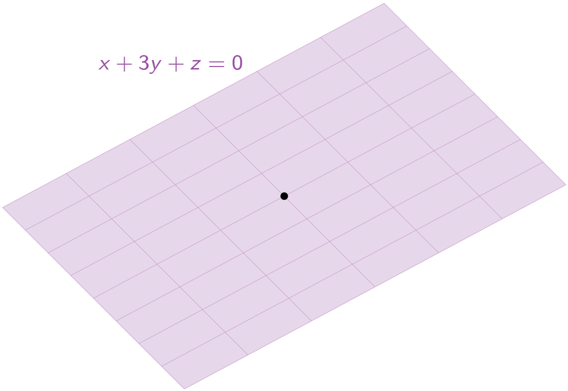
A subspace turns out to be the same as a span, except we don't know *which* vectors it's the span of.

This arises naturally when you have, say, a plane through the origin in $\mathbf{R}^3$ which is *not* defined (a priori) as a span, but you still want to say something about it.



Definition of Subspace

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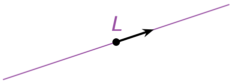
A subspace V contains the span of any set of vectors in V.

Example

A line L through the origin: this contains the span of any vector in L.



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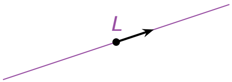

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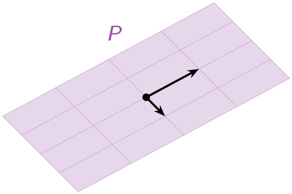
A plane P through the origin: this contains the span of any vectors in P.



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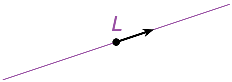

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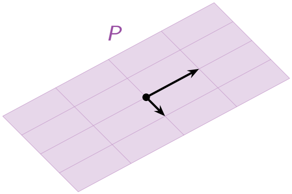

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All of $\mathbf{R}^n$: this contains 0, and is closed under addition and scalar multiplication.

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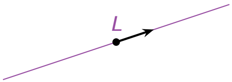

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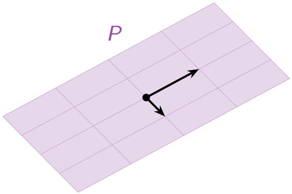

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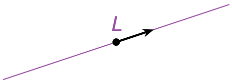

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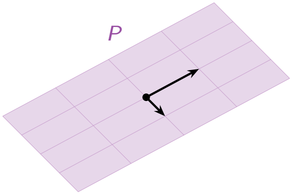

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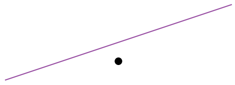
Note these are all pictures of spans! (Line, plane, space, etc.)

Non-Examples

Non-Example

A line L (or any other set) that doesn't contain the origin is not a subspace. Fails:



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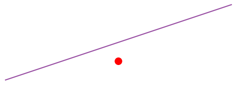

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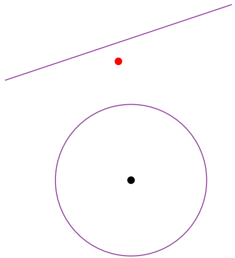

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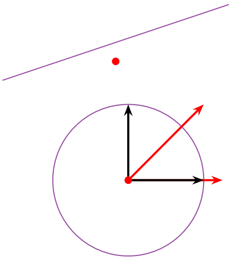

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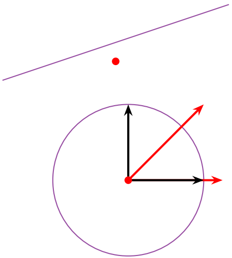
A circle C is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."



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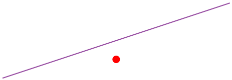

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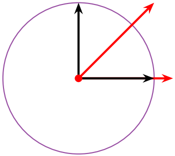

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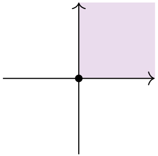
The first quadrant in $\mathbf{R}^2$ is not a subspace. Fails:



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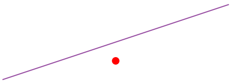

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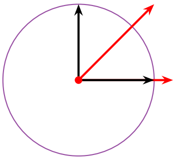

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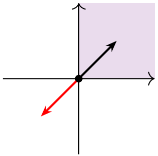
The first quadrant in $\mathbf{R}^2$ is not a subspace. Fails: 3 only.



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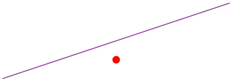

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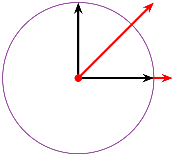

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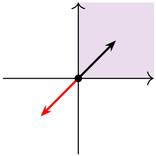
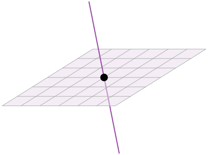


Non-Example

A line union a plane in $\mathbf{R}^3$ is not a subspace. Fails:



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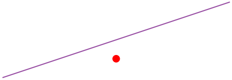

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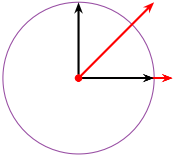

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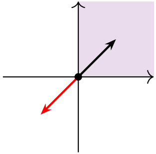

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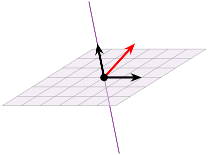

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They aren't the same thing

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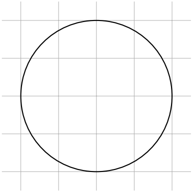

Subset: *yes*
Subspace: *no*

Spans are Subspaces

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Definition

If $V = \mathrm{Span}\{v_1, v_2, \ldots, v_n\}$, we say that V is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_n$.

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Is the empty set $\{\}$ a subspace? If not, which property(ies) does it fail?

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Question: What is the difference between $\{\}$ and $\{0\}$?

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; ab = 0 \right\}$. Let's check if V is a subspace or not.

Subspaces
Verification

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; ab = 0 \right\}$. Let's check if V is a subspace or not.

We conclude that V is *not* a subspace.

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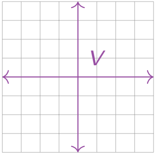

We conclude that V is *not* a subspace. A picture is above. (It doesn't look like a span.)

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$$\operatorname{Nul} A = \{x \text{ in } \mathbf{R}^{-} \mid Ax = 0\}.$$

  This is a subspace of $\mathbf{R}^{-}$.

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Check that the null space is a subspace:

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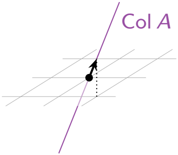

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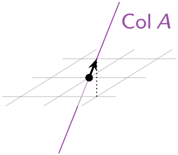

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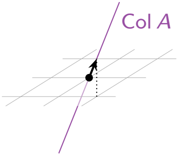

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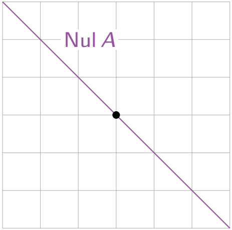

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Answer: Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

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Refer back to the slides for §1.5 (Solution Sets).

Note: It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.

The Null Space is a Span

Example, revisited

Find vector(s) that span the null space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

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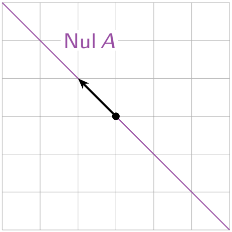

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- Can you verify directly that it satisfies the three defining properties?

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Important

A subspace has *many different* bases, but they all have the same number of vectors (see the exercises in §2.9).

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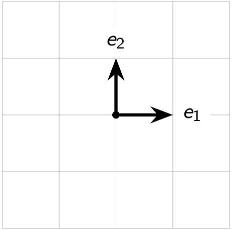

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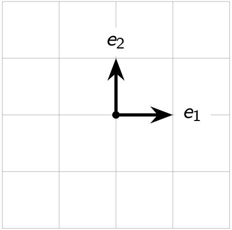

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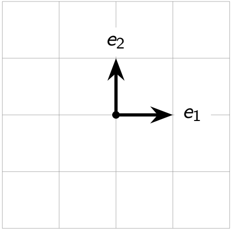

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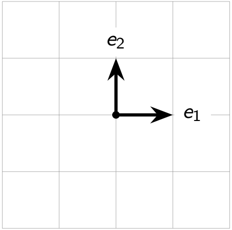

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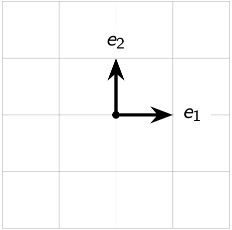

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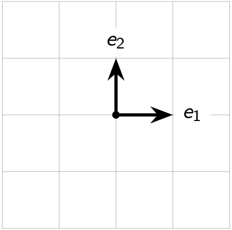

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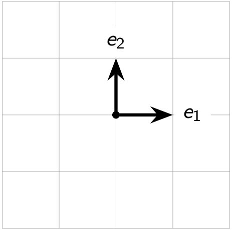

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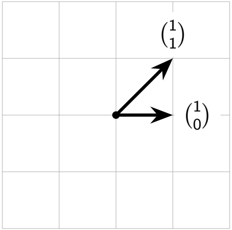

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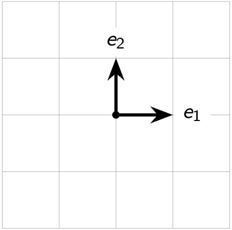

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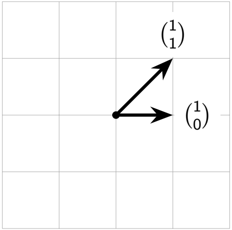

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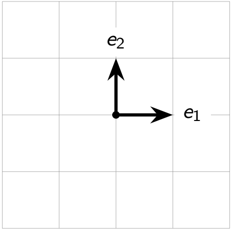

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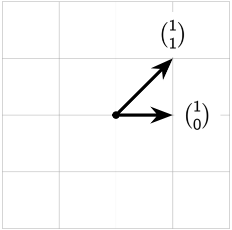

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The unit coordinate vectors

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Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + 3y + z = 0 \right\} \qquad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that $\mathcal{B}$ is a basis for V.

Basis for Nul A

Fact

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul A.

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Example

Basis for Col A

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The *pivot columns* of A always form a basis for Col A.

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Example

Why? End of §2.8, or ask in office hours.