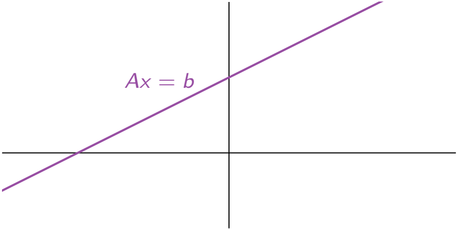
Section 1.5

Solution Sets of Linear Systems

Plan For Today

Today we will learn to describe and draw the solution set of an arbitrary system of linear equations $Ax = b$, using spans.



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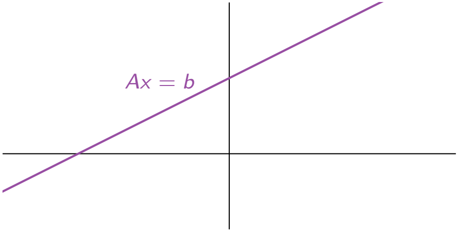

Recall: the **solution set** is the collection of all vectors x such that $Ax = b$ is true.

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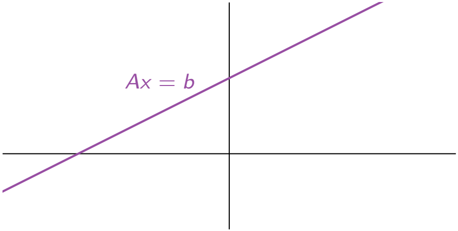

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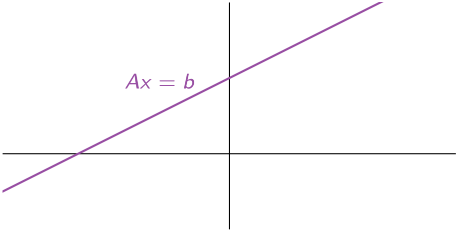

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Last time we discussed the set of vectors b for which $Ax = b$ has a solution.

We also described this set using spans, but it was a *different problem*.

Homogeneous Systems

Everything is easier when $b = 0$, so we start with this case.

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$Ax = 0$ has a nontrivial solution

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$Ax = 0$ has a nontrivial solution

$\iff$ there is a free variable

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Observation

$Ax = 0$ has a nontrivial solution

$\iff$ there is a free variable

$\iff$ A has a column with no pivot.

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

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We know how to do this: first form an augmented matrix and row reduce.

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Since the last column (everything to the right of the $=$) was zero to begin, it will always stay zero!

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Observation

Since the last column (everything to the right of the $=$) was zero to begin, it will always stay zero! So it's not really necessary to write augmented matrices in the homogeneous case.

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It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.

Homogeneous Systems

Example, continued

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Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for any x_2 in $\mathbf{R}$.

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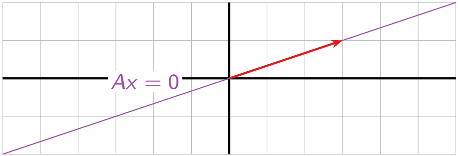

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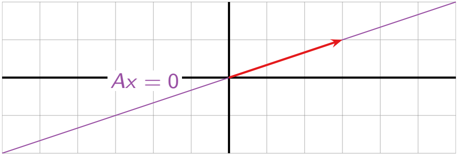

Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^2$ ($2 = \#$ variables $= \#$ columns).

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Example, continued

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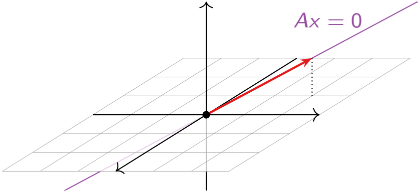

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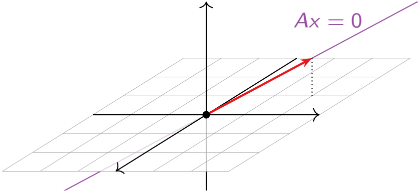

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Homogeneous Systems

Example, continued

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[not pictured here]

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[not pictured here]

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^4$ ($4 = \#$ variables $= \#$ columns).

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Parametric Vector Form

Homogeneous systems

Let A be an $m \times n$ matrix. Suppose that the free variables in the homogeneous equation $Ax = 0$ are $x_i, x_j, x_k, \ldots$

Then the solutions to $Ax = 0$ can be written in the form

$$x = x_i v_i + x_j v_j + x_k v_k + \cdots$$

for some vectors $v_i, v_j, v_k, \ldots$ in $\mathbf{R}^{-}$,

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The equation above is called the **parametric vector form** of the solution.

Poll

Poll

How many solutions can there be to a homogeneous system with more equations than variables?

A. 0

B. 1

C. ∞

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This matrix has only one solution to $Ax = 0$:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

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This matrix has infinitely many solutions to $Ax = 0$:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

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Note that p is itself a solution: take $x_2 = 0$.

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This is a *translate* of $\text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$: it is the parallel line through $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

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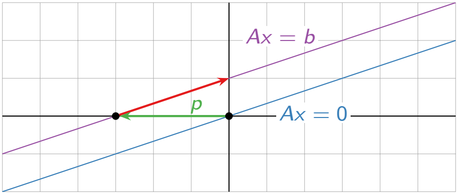

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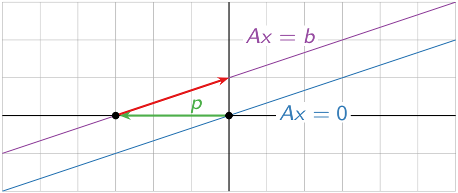

It can be written

$$\mathrm{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -5 \\ -3 \\ -2 \end{pmatrix}?$$

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Nonhomogeneous Systems

Example, continued

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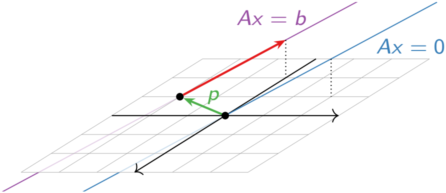

The solution set is a *translate* of

$$\operatorname{Span}\left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\}:$$

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Key Observation

The set of solutions to $Ax = b$, if it is nonempty, is obtained by taking one **specific** or **particular solution** p to $Ax = b$, and adding all solutions to $Ax = 0$.

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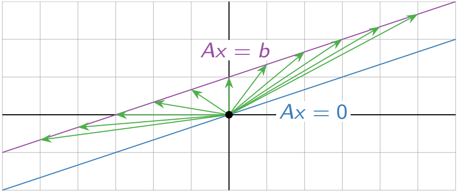

This works for *any* specific solution p: it doesn't have to be the one produced by finding the parametric vector form and setting the free variables all to zero, as we did before.

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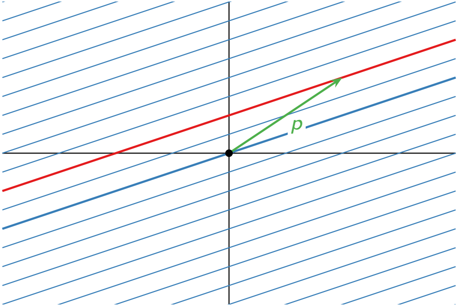
For instance, if $A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$, then the solution sets for varying b look like this:



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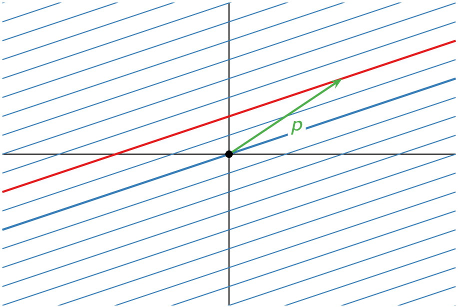

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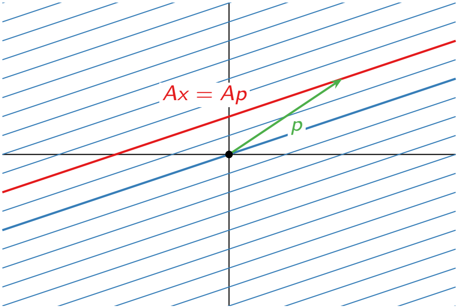

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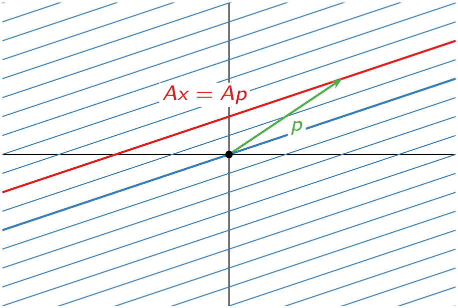

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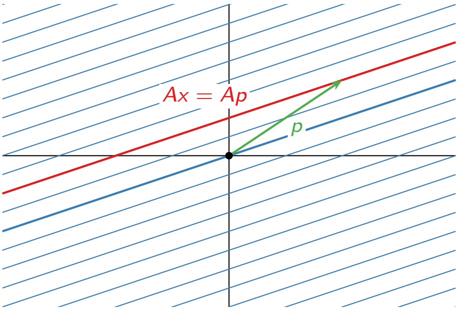

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For a matrix equation $Ax = b$, you now know how to find which b's are possible, and what the solution set looks like for all b, both using spans.