

## Section 2.2

### The Inverse of a Matrix

# The Definition of Inverse

**Recall:** The multiplicative inverse (or reciprocal) of a nonzero number  $a$  is the number  $b$  such that  $ab = 1$ . We define the inverse of a matrix in almost the same way.

## Definition

Let  $A$  be an  $n \times n$  square matrix. We say  $A$  is **invertible** (or **nonsingular**) if there is a matrix  $B$  of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In this case,  $B$  is the **inverse** of  $A$ , and is written  $A^{-1}$ .

## Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim  $B = A^{-1}$ . Check:

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$



## The $2 \times 2$ case

Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . The **determinant** of  $A$  is the number

$$\det(A) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Facts:

1. If  $\det(A) \neq 0$ , then  $A$  is invertible and  $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ .
2. If  $\det(A) = 0$ , then  $A$  is not invertible.

Why 1?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

So we get the identity by dividing by  $ad - bc$ .

Example

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2 \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

# Solving Linear Systems via Inverses

Solving  $Ax = b$  by "dividing by  $A$ "

## Theorem

If  $A$  is invertible, then  $Ax = b$  has exactly one solution for every  $b$ , namely:

$$x = A^{-1}b.$$

Why? Divide by  $A$ !

$$\begin{aligned} Ax = b &\rightsquigarrow A^{-1}(Ax) = A^{-1}b \rightsquigarrow (A^{-1}A)x = A^{-1}b \\ &\rightsquigarrow I_n x = A^{-1}b \rightsquigarrow x = A^{-1}b. \end{aligned}$$

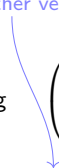
$I_n x = x$  for every  $x$  

## Example

Solve the system

$$\begin{array}{rcl} 2x + 3y + 2z & = & 1 \\ x & + & 3z = 1 \\ 2x + 2y + 3z & = & 1 \end{array}$$

using  $\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}.$

*could be any other vector* 

Answer:  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$

## Some Facts

Say  $A$  and  $B$  are invertible  $n \times n$  matrices.

1.  $A^{-1}$  is invertible and its inverse is  $(A^{-1})^{-1} = A$ .
2.  $AB$  is invertible and its inverse is  $(AB)^{-1} = \cancel{A^{-1}B^{-1}} B^{-1}A^{-1}$ .

Why?  $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}I_n B = B^{-1}B = I_n$ .

3.  $A^T$  is invertible and  $(A^T)^{-1} = (A^{-1})^T$ .

Why?  $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$ .

### Poll

If  $A, B, C$  are invertible  $n \times n$  matrices, what is the inverse of  $ABC$ ?

- i.  $A^{-1}B^{-1}C^{-1}$    ii.  $B^{-1}A^{-1}C^{-1}$    iii.  $C^{-1}B^{-1}A^{-1}$    iv.  $C^{-1}A^{-1}B^{-1}$

It's (iii):

$$\begin{aligned}(ABC)(C^{-1}B^{-1}A^{-1}) &= AB(CC^{-1})B^{-1}A^{-1} = A(BB^{-1})A^{-1} \\ &= AA^{-1} = I_n.\end{aligned}$$

In general, a product of invertible matrices is invertible, and the inverse is the product of the inverses, in the *reverse order*.

## Computing $A^{-1}$

Let  $A$  be an  $n \times n$  matrix. Here's how to compute  $A^{-1}$ .

1. Row reduce the augmented matrix  $(A \mid I_n)$ .
2. If the result has the form  $(I_n \mid B)$ , then  $A$  is invertible and  $B = A^{-1}$ .
3. Otherwise,  $A$  is not invertible.

Example


$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}$$

# Computing $A^{-1}$

Example

$$\begin{pmatrix} 1 & 0 & 4 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & -3 & -4 & | & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} R_3 = R_3 + 3R_2 \\ \hline R_1 = R_1 - 2R_3 \\ R_2 = R_2 - R_3 \\ \hline R_3 = R_3 \div 2 \\ \hline \end{array} \begin{pmatrix} 1 & 0 & 4 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & 0 & 1 & 0 \\ 0 & 0 & 2 & | & 0 & 3 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & | & 1 & -6 & -2 \\ 0 & 1 & 0 & | & 0 & -2 & -1 \\ 0 & 0 & 2 & | & 0 & 3 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & | & 1 & -6 & -2 \\ 0 & 1 & 0 & | & 0 & -2 & -1 \\ 0 & 0 & 1 & | & 0 & 3/2 & 1/2 \end{pmatrix}$$

$$\text{So } \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix}.$$

Check:  $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} \begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  

## Why Does This Work?

**First answer:** We can think of the algorithm as simultaneously solving the equations

$$Ax_1 = e_1 : \left( \begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array} \right)$$

$$Ax_2 = e_2 : \left( \begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array} \right)$$

$$Ax_3 = e_3 : \left( \begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array} \right)$$

Now note  $A^{-1}e_i = A^{-1}(Ax_i) = x_i$ , and  $x_i$  is the  $i$ th column in the augmented part. Also  $A^{-1}e_i$  is the  $i$ th column of  $A^{-1}$ .

**Second answer:** Elementary matrices.



# Elementary Matrices

## Definition

An **elementary matrix** is a square matrix  $E$  which differs from  $I_n$  by one row operation.

There are three kinds, corresponding to the three elementary row operations:

scaling  
( $R_2 = 2R_2$ )

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

row replacement  
( $R_2 = R_2 + 2R_1$ )

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

swap  
( $R_1 \longleftrightarrow R_2$ )

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**Fact:** if  $E$  is the elementary matrix for a row operation, then  $EA$  differs from  $A$  by the same row operation.

**Example:**

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 10 \\ 0 & -3 & -4 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} \xrightarrow{R_2 = R_2 + 2R_1} \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 10 \\ 0 & -3 & -4 \end{pmatrix}$$

# Elementary Matrices

Continued

**Fact:** if  $E$  is the elementary matrix for a row operation, then  $EA$  differs from  $A$  by the same row operation.

## Consequence

Elementary matrices are invertible, and the inverse is the elementary matrix which un-does the row operation.

$$\begin{array}{cccc} R_2 = R_2 \times 2 & R_2 = R_2 \div 2 & R_2 = R_2 + 2R_1 & R_2 = R_2 - 2R_1 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} & = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} & = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} R_1 \longleftrightarrow R_2 & R_1 \longleftrightarrow R_2 \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} & = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

# Why Does The Inversion Algorithm Work?

Second answer

## Theorem

An  $n \times n$  matrix  $A$  is invertible if and only if it is row equivalent to  $I_n$ . In this case, the sequence of row operations taking  $A$  to  $I_n$  also takes  $I_n$  to  $A^{-1}$ .

**Why?** Say the row operations taking  $A$  to  $I_n$  have elementary matrices  $E_1, E_2, \dots, E_k$ . So

$$\begin{aligned} \text{note the order!} \longrightarrow E_k E_{k-1} \cdots E_2 E_1 A &= I_n \\ \implies E_k E_{k-1} \cdots E_2 E_1 A A^{-1} &= A^{-1} \\ \implies E_k E_{k-1} \cdots E_2 E_1 I_n &= A^{-1}. \end{aligned}$$

This means if you do these same row operations to  $A$  and to  $I_n$ , you'll end up with  $I_n$  and  $A^{-1}$ . This is what you do when you row reduce the augmented matrix:

$$(A \mid I_n) \rightsquigarrow (I_n \mid A^{-1})$$