Section 1.4

The Matrix Equation $Ax = b$

Matrix $\times$ Vector

Let A be an $m \times n$ matrix

Matrix × Vector

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The **product** of A with a vector x in $\mathbf{R}^n$ is the linear combination

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Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$$

Question

Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$.

Matrix Equations

An example

Question

Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

Matrix Equations

In general

Let $v_1, v_2, \ldots, v_n$, and b be vectors in $\mathbf{R}^m$.

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We will move back and forth freely between these over and over again, for the rest of the semester. Get comfortable with them now!

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Matrix $\times$ Vector

Another way

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$$Ax = \begin{pmatrix} — r_1 — \\ — r_2 — \\ \vdots \\ — r_m — \end{pmatrix} x = \begin{pmatrix} r_1 x \\ r_2 x \\ \vdots \\ r_m x \end{pmatrix}$$

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This is a vector in $\mathbf{R}^m$ (again).

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The second way is usually the most convenient, but we'll use both.

Spans and Solutions to Equations

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The last condition is geometric.

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

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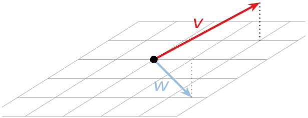

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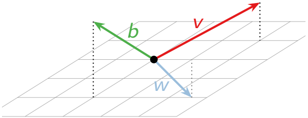

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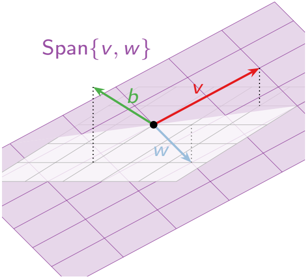

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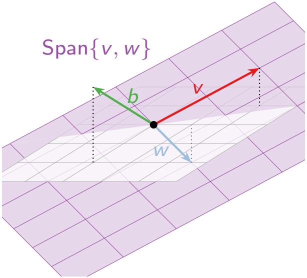

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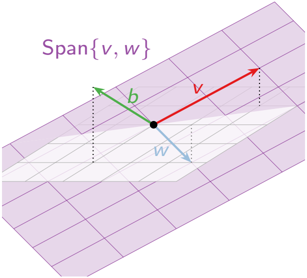

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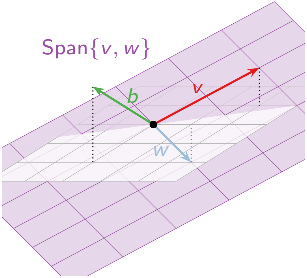

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Conclusion: $Ax = b$ is *inconsistent*.

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Answer: Let's check by solving the matrix equation using row reduction.

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In other words, the matrix equation

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

has no solution, as the picture shows.

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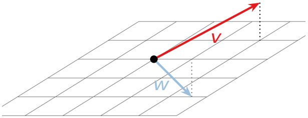

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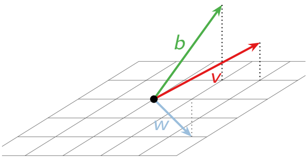

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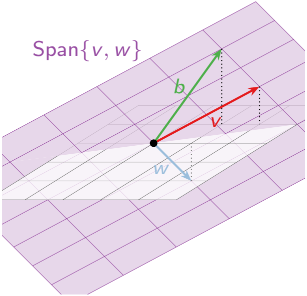

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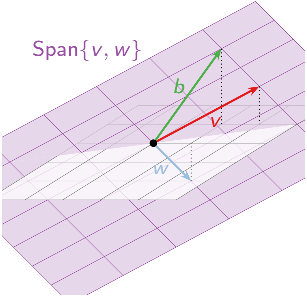

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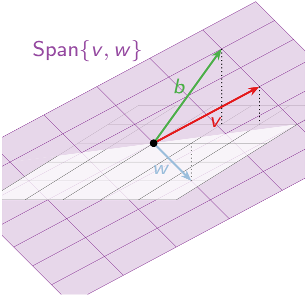

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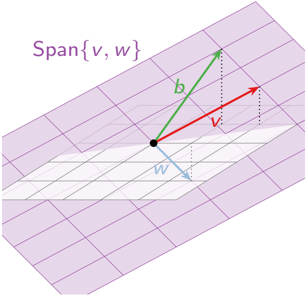

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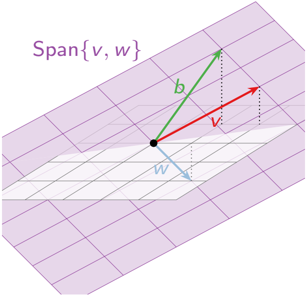

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Poll

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Which of the following true statements can be checked by eyeballing them, *without* row reduction?

A. $\begin{pmatrix}0\\1\\2\end{pmatrix}$ is in the span of $\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\-1\\-2\end{pmatrix}$.

B. $\begin{pmatrix}0\\1\\2\end{pmatrix}$ is in the span of $\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}, \begin{pmatrix}0\\6\\8\end{pmatrix}$.

C. $\begin{pmatrix}0\\1\\2\end{pmatrix}$ is in the span of $\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\sqrt{2}\end{pmatrix}$.

D. $\begin{pmatrix}0\\1\\2\end{pmatrix}$ is in the span of $\begin{pmatrix}5\\7\\0\end{pmatrix}, \begin{pmatrix}6\\8\\0\end{pmatrix}, \begin{pmatrix}3\\3\\4\end{pmatrix}$.

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Here are criteria for a linear system to always have a solution.

Theorem

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Let c be a scalar, u, v be vectors, and A a matrix.

Properties of the Matrix–Vector Product

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Important

The set of solutions to $Ax = 0$ is a span.