Review for Midterm 3

Selected Topics

Eigenvectors and Eigenvalues

Definition

Let A be an $n \times n$ matrix.

1. An **eigenvector** of A is a nonzero vector v in $\mathbf{R}^n$ such that $Av = \lambda v$, for some λ in $\mathbf{R}$. In other words, Av is a multiple of v.
2. An **eigenvalue** of A is a number λ in $\mathbf{R}$ such that the equation $Av = \lambda v$ has a nontrivial solution.

If $Av = \lambda v$ for $v \neq 0$, we say λ is the **eigenvalue for** v, and v is an **eigenvector for** λ.

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Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The λ-**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ, plus the zero vector:

$$\begin{aligned} \lambda\text{-eigenspace} &= \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\} \\ &= \{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\} \\ &= \operatorname{Nul}(A - \lambda I). \end{aligned}$$

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You find a basis for the λ-eigenspace by finding the parametric vector form for the general solution to $(A - \lambda I)x = 0$ using row reduction.

The Characteristic Polynomial

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Definition

The **algebraic multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

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1. Matrices with the same characteristic polynomial need not be similar.
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3. Similar matrices usually do not have the same eigenvectors.

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Example:

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

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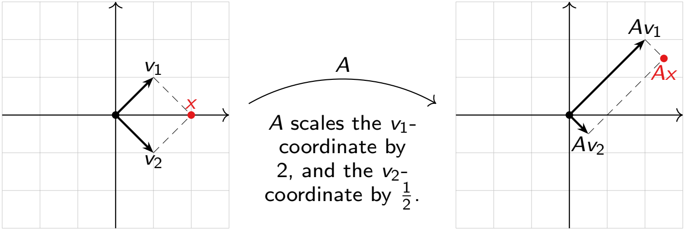
Picture:



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Corollary

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

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Let A be a square matrix with eigenvalue λ. The **geometric multiplicity** of λ is the dimension of the λ-eigenspace.

Theorem

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if, for every eigenvalue λ, the algebraic multiplicity of λ is equal to the geometric multiplicity.

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Notes:

- The algebraic and geometric multiplicities are both whole numbers ≥ 1, and the algebraic multiplicity is always greater than or equal to the geometric multiplicity. In particular, they're equal if the algebraic multiplicity is 1.
- Equivalently, A is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues is n.

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Example

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

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Perron–Frobenius Theorem

If A is a positive stochastic matrix, then it admits a unique steady state vector w, which spans the 1-eigenspace.

Moreover, for any vector v_0 with entries summing to some number c, the iterates $v_1 = Av_0$, $v_2 = Av_1$, ..., $v_n = Av_{n-1}$, ..., approach cw as n gets large.

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Computing the Steady State

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Let A be a 2×2 matrix with complex (non-real) eigenvalue λ, and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \operatorname{Re} v & \operatorname{Im} v \\ | & | \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix}.$$

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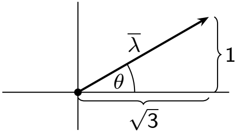

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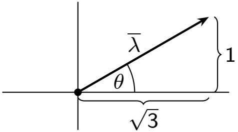

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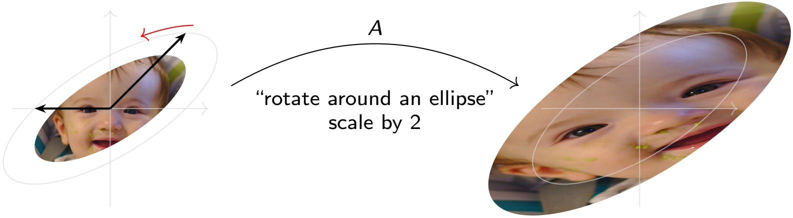

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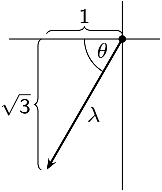
Anyway that's the number your calculator will give you.

You have to *draw a picture*:



$$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\text{argument} = \theta + \pi = \frac{4\pi}{3}$$