Chapter 1

Linear Equations

Section 1.1

Systems of Linear Equations

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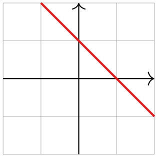

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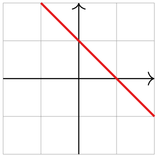

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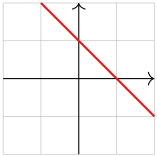

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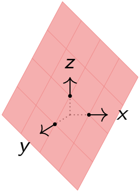

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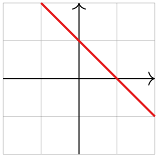

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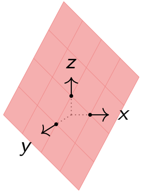

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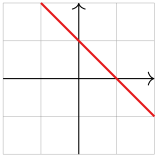

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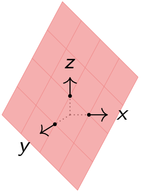

- $x + y + z + w = 1$

  $\rightsquigarrow$ a "3-plane" in "4-space"...

[not pictured here]

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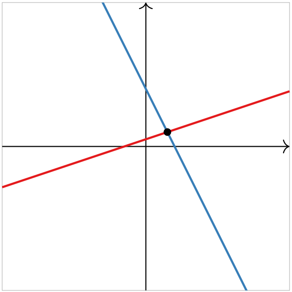

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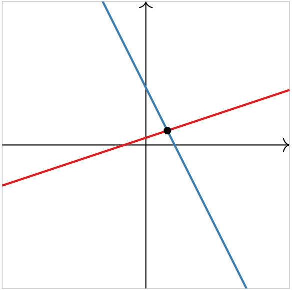

In general it's an intersection of lines, planes, etc.

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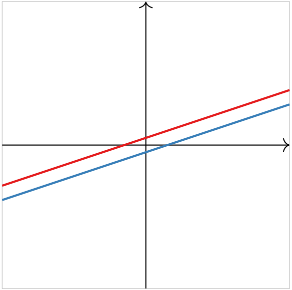

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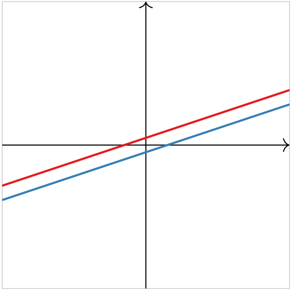

A system of equations with no solutions is called **inconsistent**.

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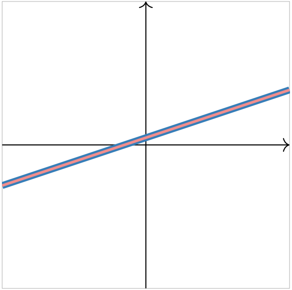

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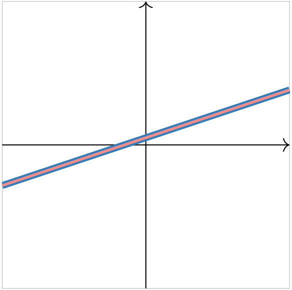

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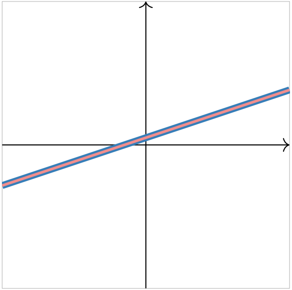

Note that multiplying an equation by a nonzero number gives the *same solution set*. In other words, they are *equivalent* (systems of) equations.

Poll

What about in three variables?

Poll

What about in three variables?

Poll

In how many different ways can three planes intersect in space?

A. One
B. Two
C. Three
D. Four
E. Five
F. Six
G. Seven

Solving Systems of Equations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

This is the kind of problem we'll talk about for the first half of the course.

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What is a *systematic* way to solve a system of equations?

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What strategies do you know?

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Elimination method: in what ways can you manipulate the equations?

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...but there's a long way to go still. Can we make our lives easier?

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Matrix notation: write just the numbers, in a box, instead!

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This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

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- Swap two rows. **(swap)**

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Strategy: fiddle with it so we only have ones and zeros.

Row Operations
Continued

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$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

We want these to be zero.

Row Operations
Continued

$$\left(\begin{array}{rrr|r} \color{red}{1} & 2 & 3 & 6 \\ \boxed{\begin{matrix}2\\3\end{matrix}} & -3 & 2 & 14 \\ & 1 & -1 & -2 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

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$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

Row Operations

Continued

$$\left(\begin{array}{ccc|c} \color{red}{1} & 2 & 3 & 6 \\ \boxed{\begin{matrix}2\\3\end{matrix}} & \begin{matrix}-3\\1\end{matrix} & \begin{matrix}2\\-1\end{matrix} & \begin{matrix}14\\-2\end{matrix} \end{array}\right)$$

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Let's swap the last two rows first.

Row Operations

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$$\left(\begin{array}{ccr|r} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

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We want these to be zero.

Let's make this a 1 first.

Success!

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

We want these to be zero.

Let's make this a 1 first.

Success!

Check:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \quad \xrightarrow{\text{substitute solution}}$$

Row Equivalence

Important

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Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

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So the linear equations of row-equivalent matrices have the *same solution set*.

A Bad Example

Example

Solve the system of equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

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First clear these by subtracting multiples of the first row.

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Now clear this by subtracting the second row.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & \boxed{1} & 1 \end{array}\right)$$

A Bad Example

Continued

$$\left(\begin{array}{cc|r} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right) \xrightsquigarrow{\text{translates into}}$$

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In other words, the original equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned} \qquad \text{have the same solutions as} \qquad \begin{aligned} x + y &= 2 \\ y &= -1 \\ 0 &= 2 \end{aligned}$$

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But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

A Bad Example

Continued

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right) \xrightarrow{\text{translates into}}$$

In other words, the original equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned} \qquad \text{have the same solutions as} \qquad \begin{aligned} x + y &= 2 \\ y &= -1 \\ 0 &= 2 \end{aligned}$$

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.