Chapter 2

Matrix Algebra

Section 2.1

Matrix Operations

Motivation

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Today we'll study *matrix algebra*: adding and multiplying matrices.

More Notation for Matrices

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We write a_{ij} for the entry in the ith row and the jth column. It is called the ij**th entry** of the matrix.

$$\begin{array}{cc} \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} & i\text{th row} \\ j\text{th column} & \end{array}$$

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The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

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A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$
\begin{pmatrix}
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The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = \_$ for *all* v in $\mathbf{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

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The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$\underset{A}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}} \rightsquigarrow \underset{A^T}{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}}$$

flip

Addition and Scalar Multiplication

You add two matrices component by component, like with vectors.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{pmatrix}$$

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Note you can only add two matrices *of the same size*.

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You multiply a matrix by a scalar by multiplying each component, like with vectors.

$$c \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}.$$

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These satisfy the expected rules, like with vectors:

Matrix Multiplication

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Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2 \ldots, v_p$:

$$B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

The **product** AB is the $m \times p$ matrix with columns $Av_1, Av_2, \ldots, Av_p$:

The equality is a definition

$$AB \stackrel{\text{def}}{=} \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & & | \end{pmatrix}.$$

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Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} =$$

Composition of Transformations

Why is this the correct definition of matrix multiplication?

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Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ and $U\colon \mathbf{R}^p \to \mathbf{R}^n$ be transformations. The **composition** is the transformation

$$T \circ U\colon \mathbf{R}^p \to \mathbf{R}^m \qquad \text{defined by} \qquad T \circ U(x) = T(U(x)).$$

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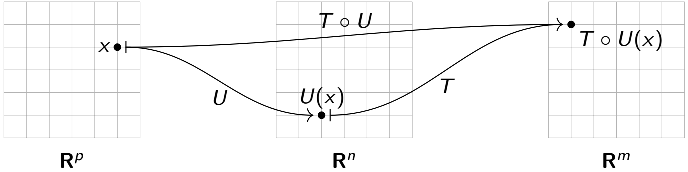

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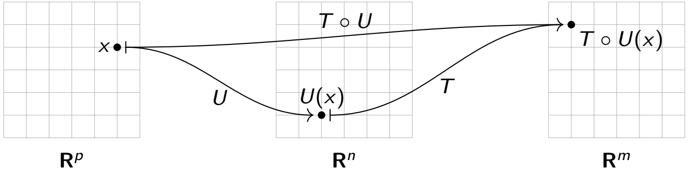

Fact: If T and U are linear then so is $T \circ U$.

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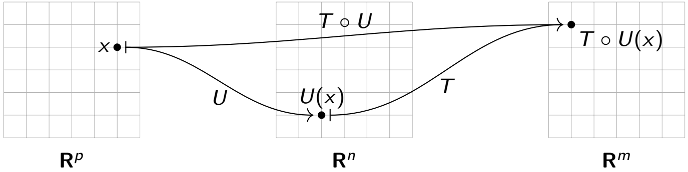

Fact: If T and U are linear then so is $T \circ U$.

Guess: If A is the matrix for T, and B is the matrix for U, what is the matrix for $T \circ U$?

Composition of Linear Transformations

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$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix} \quad B = \begin{pmatrix} | & | & & | \\ U(e_1) & U(e_2) & \cdots & U(e_p) \\ | & | & & | \end{pmatrix}$$

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The matrix of the composition is the product of the matrices!

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be rotation by 45°, and let $U\colon \mathbf{R}^2 \to \mathbf{R}^2$ be projection onto the x-axis. Let's compute their standard matrices A and B:

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$$\implies \quad A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

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Check:

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$$\Longrightarrow \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \checkmark$$

Composition of Linear Transformations

Another example

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix}.$$

Let $T(x) = Ax$ and $U(y) = By$, so

$$T\colon \mathbf{R}^{-} \longrightarrow \mathbf{R}^{-} \qquad U\colon \mathbf{R}^{-} \longrightarrow \mathbf{R}^{-} \qquad T\circ U\colon \mathbf{R}^{-} \longrightarrow \mathbf{R}^{-}.$$

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Let's find the matrix for $T \circ U$:

$$T \circ U(e_1) =$$

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Before we computed $AB = \begin{pmatrix} 14 & -10 \\ 32 & -28 \end{pmatrix}$, so AB is the matrix of $T \circ U$.

Poll

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Do there exist *nonzero* matrices A and B with $AB = 0$?

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Do there exist *nonzero* matrices A and B with $AB = 0$?

Yes! Here's an example:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The Row-Column Rule for Matrix Multiplication

Recall: A row vector of length n times a column vector of length n is a scalar:

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n.$$

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Another way of multiplying a matrix by a vector is:

$$Ax = \begin{pmatrix} \text{—} \; r_1 \; \text{—} \\ \vdots \\ \text{—} \; r_m \; \text{—} \end{pmatrix} x = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix}.$$

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$$Ax = \begin{pmatrix} — r_1 — \\ \vdots \\ — r_m — \end{pmatrix} x = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix}.$$

On the other hand, you multiply two matrices by

$$AB = A \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ac_1 & \cdots & Ac_p \\ | & & | \end{pmatrix}.$$

The Row-Column Rule for Matrix Multiplication

Recall: A row vector of length n times a column vector of length n is a scalar:

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It follows that

$$AB = \begin{pmatrix} — r_1 — \\ \vdots \\ — r_m — \end{pmatrix} \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} r_1 c_1 & r_1 c_2 & \cdots & r_1 c_p \\ r_2 c_1 & r_2 c_2 & \cdots & r_2 c_p \\ \vdots & \vdots & & \vdots \\ r_m c_1 & r_m c_2 & \cdots & r_m c_p \end{pmatrix}$$

The Row-Column Rule for Matrix Multiplication

The ij entry of $C = AB$ is the ith row of A times the jth column of B:

$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

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$$\begin{array}{cc} \begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{l} \\ \\ i\text{th row} \\ \\ \\ \end{array} \end{array} \cdot \begin{array}{c} \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} \\ j\text{th column} \end{array} = \begin{array}{c} \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix} \\ ij \text{ entry} \end{array}$$

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Example

Properties of Matrix Multiplication

Mostly matrix multiplication works like you'd expect. Suppose A has size $m \times n$, and that the other matrices below have the right size to make multiplication work.

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Recommended: Try to verify all of them on your own.

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- $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

- $AB = 0$ does not imply $A = 0$ or $B = 0$.

Other Reading

Read about powers of a matrix and multiplication of transposes in §2.1.