Section 6.2

Orthogonal Sets

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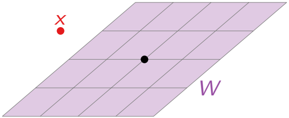
Suppose you measure a data point x which you know for theoretical reasons must lie on a subspace W.



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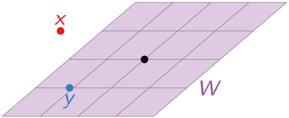

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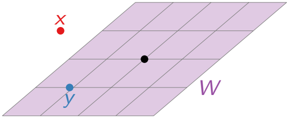

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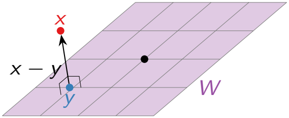

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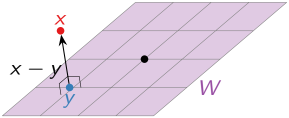

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How do you know that y is the closest point? The vector from y to x is orthogonal to W: it is in the *orthogonal complement* $W^{\perp}$.

Orthogonal Projection onto a Line

Theorem

Let $L = \mathrm{Span}\{u\}$ be a line in $\mathbf{R}^n$, and let x be in $\mathbf{R}^n$. The closest point to x on L is the point

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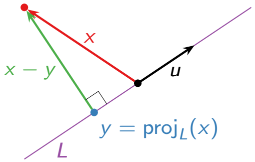

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Compute the orthogonal projection of $x = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ onto the line L spanned by $u = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

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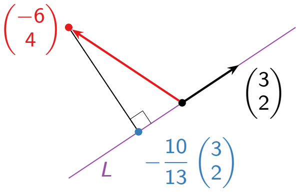

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Lemma

An orthogonal set of vectors is linearly independent.

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Theorem

Let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an orthogonal set, and let x be a vector in $W = \operatorname{Span} \mathcal{B}$. Then

$$x = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

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In other words, the $\mathcal{B}$-coordinates of x are $\left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2}, \ldots, \frac{x \cdot u_m}{u_m \cdot u_m}\right)$.

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If L_i is the line spanned by u_i, then this says

$$x = \mathrm{proj}_{L_1}(x) + \mathrm{proj}_{L_2}(x) + \cdots + \mathrm{proj}_{L_m}(x).$$

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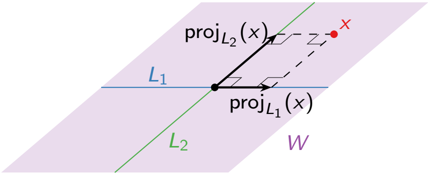

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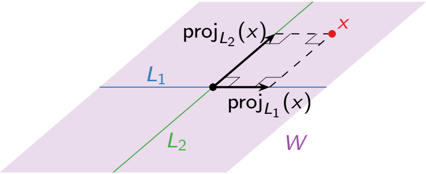

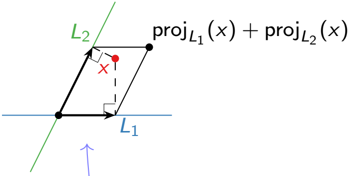

Warning: This only works for an *orthogonal* basis.

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$$\left(\begin{array}{cc|c} 1 & -4 & 0 \\ 2 & 2 & 3 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 6/5 \\ 0 & 1 & 6/20 \end{array}\right) \implies [x]_{\mathcal{B}} = \begin{pmatrix} 6/5 \\ 6/20 \end{pmatrix}.$$

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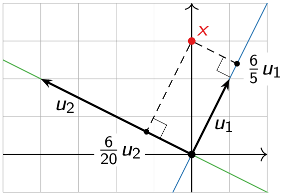

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