Section 1.2

Row Reduction and Echelon Forms

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Picture:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} \star &= \text{any number} \\ \boxed{\star} &= \text{any nonzero number} \end{aligned}$$

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Definition

A **pivot** $\boxed{\star}$ is the first nonzero entry of a row of a matrix in row echelon form.

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$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Can every matrix be put into reduced row echelon form only using row operations?

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Answer: Yes! Stay tuned.

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Continued

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Poll

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Which of the following matrices are in reduced row echelon form?

$$\text{A. } \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad \text{B. } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{C. } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{D. } \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \qquad \text{E. } \begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$$

$$\text{F. } \left(\begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

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Maybe you can figure out why it's true!

Row Reduction Algorithm

Step 1a Swap the 1st row with a lower one so a leftmost nonzero entry is in 1st row (if necessary).

Example

$$\left(\begin{array}{ccc|c} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

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Step 2a Cover the first row, swap the 2nd row with a lower one so that the leftmost nonzero (uncovered) entry is in 2nd row; uncover 1st row.

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- **Step 3a** Cover the first two rows, swap the 3rd row with a lower one so that the leftmost nonzero (uncovered) entry is in 3rd row; uncover first two rows.

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Row Reduction

Example, continued

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

Row Reduction

Example, continued

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

Note: Step 2 never messes up the first (nonzero) column of the matrix, because it looks like this:

$$\text{“Active” row} \longrightarrow \left(\begin{array}{ccc|c} 1 & \star & \star & \star \\ \boxed{0} & \star & \star & \star \\ 0 & \star & \star & \star \end{array}\right)$$

Row Reduction

Example, continued

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

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$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

Note: Step 3 never messes up the columns to the left.

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Success! The reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad \Longrightarrow \quad \left\{\begin{array}{rrrl} x & & & = 1 \\ & y & & = -2 \\ & & z & = 3 \end{array}\right.$$

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$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \implies \begin{cases} x & & & = 1 \\ & y & & = -2 \\ & & z & = 3 \end{cases}$$

Step 4: profit?

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Another example

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The row reduced matrix

$$\left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

corresponds to the *inconsistent* system

$$\begin{aligned} x + 5y &= 0 \\ 0 &= 1. \end{aligned}$$

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Question

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Answer:

$$\left(\begin{array}{cccc|c} 1 & 0 & \star & \star & 0 \\ 0 & 1 & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Inconsistent Matrices

Question

What does an augmented matrix in reduced row echelon form look like, if its system of linear equations is inconsistent?

Answer:

$$\left(\begin{array}{cccc|c} 1 & 0 & \star & \star & \color{red}{0} \\ 0 & 1 & \star & \star & \color{red}{0} \\ 0 & 0 & 0 & 0 & \color{red}{1} \end{array}\right)$$

An augmented matrix corresponds to an inconsistent system of equations if and only if *the last* (i.e., the augmented) *column is a pivot column.*

Another Example

The linear system

$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned} \quad \text{gives rise to the matrix} \quad \left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right).$$

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The row reduced matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

corresponds to the linear system

$$\begin{cases} x + 5z = 1 \\ y + 2z = -1 \end{cases}$$

Another Example
Continued

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Yes! Rewrite:

$$\begin{aligned} x &= 1 - 5z \\ y &= -1 - 2z \end{aligned}$$

For any value of z, there is exactly one value of x and y that makes the equations true. But z can be *anything we want*!

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So we have found the solution set: it is all values x, y, z where

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This is called the **parametric form** for the solution.

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Definition

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the free variables are x_2 and x_4.

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Definition

Consider a *consistent* linear system of equations in the variables $x_1, \ldots, x_n$. Let A be a row echelon form of the matrix for this system.

We say that x_i is a **free variable** if its corresponding column in A is *not* a pivot column.

Important

1. You can choose *any value* for the free variables in a (consistent) linear system.
2. Free variables come from *columns without pivots* in a matrix in row echelon form.

In the previous example, z was free because the reduced row echelon form matrix was

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right).$$

In this matrix:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

the free variables are x_2 and x_4. (What about the last column?)

One More Example

The reduced row echelon form of the matrix for a linear system in x_1, x_2, x_3, x_4 is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right)$$

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$$\left\{\begin{array}{rcrcrcr} x_1 & & & + & 3x_4 & = & 2 \\ & & x_3 & + & x_4 & = & -1 \end{array}\right. \quad\Longrightarrow\quad \boxed{\begin{array}{rcrcl} x_1 & = & 2 & - & 3x_4 \\ x_3 & = & -1 & - & 4x_4 \end{array}}$$

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The boxed equation is called the **parametric form** of the general solution to the system of equations. It is obtained by moving all free variables to the right-hand side of the $=$.

Poll

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Is it possible for a system of linear equations to have exactly two solutions?

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3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:
   $$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$