Section 3.2

Properties of Determinants

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The determinant is one of the most amazing functions ever devised. Today is about beginning to understand why.

The Determinant is a Function

We can think of the determinant as a function of the entries of a matrix:

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The determinant function is characterized by how it is changed by row operations.

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4. If you scale one coordinate by k, the volume is multiplied by k.

Properties of the Determinant

2×2 matrix

Since an elementary matrix differs from the identity matrix by one row operation, and since $\det(I_n) = 1$, it is easy to calulate the determinant of an elementary matrix:

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$$\det \begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} =$$

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

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(Cofactor expansion is $O(n!) \sim O(n^n\sqrt{n})$, row reduction is $O(n^3)$.)

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Suppose that A is a 4×4 matrix satisfying

$$Ae_1 = e_2 \quad Ae_2 = e_3 \quad Ae_3 = e_4 \quad Ae_4 = e_1.$$

What is $\det(A)$?

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So $\det(A) = (-1)^3 = -1$.

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Here is a summary of the magical properties of the determinant. Prof. Margalit's notes (on the website) have very understandable proofs.

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Why is Property 5 true?

Multiplicativity of the Determinant

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Why is Property 5 true? In Lay, there's a proof using elementary matrices. Here's a better one.

Determinants and Linear Transformations

Why is Property 8 true?

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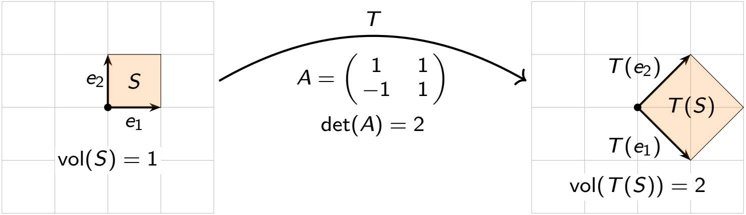

Determinants and Linear Transformations

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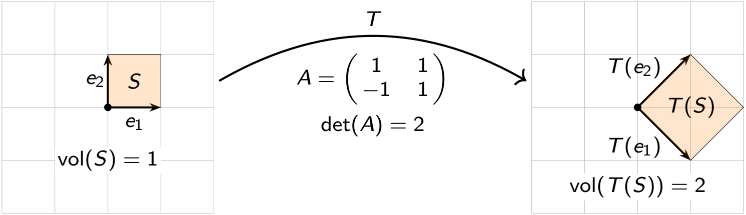
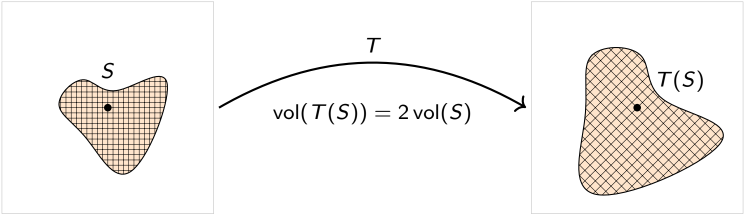


For curvy shapes, you break S up into a bunch of tiny cubes. Each one is scaled by $|\det(A)|$; then you use *calculus* to reduce to the previous situation!



Multi-Linearity of the Determinant

We can also think of det as a function of the columns (or the rows) of an $n \times n$ matrix:

$$\det\colon \underbrace{\mathbf{R}^n \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n}_{n \text{ times}} \longrightarrow \mathbf{R}$$

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Property 9 says that for any i and any vectors $v_1, v_2, \ldots, v_n$ and v_i' and any scalar c,

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Multi-Linearity of the Determinant

We can also think of det as a function of the columns (or the rows) of an $n \times n$ matrix:

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Proof: just expand cofactors along column i.