Section 1.2

Row Reduction and Echelon Forms

Let's come up with an *algorithm* for turning an arbitrary matrix into a "solved" matrix. What do we mean by "solved"?

> A matrix is in **row echelon form** if
> 1. All zero rows are at the bottom.
> 2. Each leading nonzero entry of a row is to the *right* of the leading entry of the row above.
> 3. Below a leading entry of a row, all entries are *zero*.

Picture:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star$ = any number

$\boxed{\star}$ = any nonzero number

Definition
A **pivot** $\boxed{\star}$ is the first nonzero entry of a row of a matrix in row echelon form.

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,

4. The pivot in each nonzero row is equal to 1.
5. Each pivot is the only nonzero entry in its column.

Picture:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star$ = any number
1 = pivot

Note: Echelon forms do not care whether or not a column is augmented. Just ignore the vertical line.

Question

Can every matrix be put into reduced row echelon form only using row operations?

Answer: Yes! Stay tuned.

Why is this the "solved" version of the matrix?

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

is in reduced row echelon form. It translates into

$$
\begin{aligned}
x &= 1 \\
y &= -2 \\
z &= 3,
\end{aligned}
$$

which is clearly the solution.

But what happens if there are fewer pivots than rows? ... parametrized solution set (later).

Poll

Which of the following matrices are in reduced row echelon form?

A. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ D. $\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$ E. $\begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$

F. $\left(\begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 0 & 1 \end{array} \right)$

Theorem

Every matrix is row equivalent to one and only one matrix in reduced row echelon form.

We'll give an algorithm, called **row reduction**, which demonstrates that every matrix is row equivalent to *at least one* matrix in reduced row echelon form.

Note: Like echelon forms, the row reduction algorithm does not care if a column is augmented: ignore the vertical line when row reducing.

The uniqueness statement is interesting—it means that, nomatter *how* you row reduce, you *always* get the same matrix in reduced row echelon form. (Assuming you only do the three legal row operations.) (And you don't make any arithmetic errors.)

Maybe you can figure out why it's true!

Row Reduction Algorithm

Step 1a Swap the 1st row with a lower one so a leftmost nonzero entry is in 1st row (if necessary).

Step 1b Scale 1st row so that its leading entry is equal to 1.

Step 1c Use row replacement so all entries above and below this 1 are 0.

Step 2a Cover the first row, swap the 2nd row with a lower one so that the leftmost nonzero (uncovered) entry is in 2nd row; uncover 1st row.

Step 2b Scale 2nd row so that its leading entry is equal to 1.

Step 2c Use row replacement so all entries above and below this 1 are 0.

Step 3a Cover the first two rows, swap the 3rd row with a lower one so that the leftmost nonzero (uncovered) entry is in 3rd row; uncover first two rows.
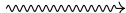
etc.

Example

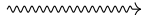$$\left(\begin{array}{ccc|c} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$
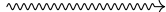
Row Reduction

Example

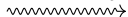$$\left(\begin{array}{ccc|c} \boxed{0} & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Step 1a: Row swap to make this nonzero.

$$\xrightarrow{R_1 \longleftrightarrow R_2}$$

$$\left(\begin{array}{ccc|c} \boxed{2} & 4 & 6 & 12 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Step 1b: Scale to make this 1.

$$\xrightarrow{R_1 = R_1 \div 2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ \boxed{3} & 1 & -1 & -2 \end{array}\right)$$

Step 1c: Subtract a multiple of the first row to clear this.

$$\xrightarrow{R_3 = R_3 - 3R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

Optional: swap rows 2 and 3 to make Step 2b easier later on.

$$\xrightarrow{R_2 \longleftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{pmatrix}$$

Step 2a: This is already nonzero.
Step 2b: Scale to make this 1.

(There are no fractions because of the optional step before.)

$$R_2 = R_2 \div -5$$

$$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{pmatrix}$$

Step 2c: Add multiples of the second row to clear these.

$$R_1 = R_1 - 2R_2$$

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{pmatrix}$$

$$R_3 = R_3 + 7R_2$$

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{pmatrix}$$

Note: Step 2 never messes up the first (nonzero) column of the matrix, because it looks like this:

"Active" row $\longrightarrow$ $\begin{pmatrix} 1 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 \div 10}$$

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Step 3a: This is already nonzero.
Step 3b: Scale to make this 1.

Step 3c: Add multiples of
the third row to clear these.

$$\xrightarrow{R_1 = R_1 + R_3}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - 2R_3}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Note: Step 3 never messes up the columns to the left.

Success! The reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \implies \begin{cases} x & & = 1 \\ & y & = -2 \\ & & z = 3 \end{cases}$$

Step 4: profit?

The linear system

$$2x + 10y = -1$$
$$3x + 15y = 2$$

gives rise to the matrix

$$\begin{pmatrix} 2 & 10 & \big| & -1 \\ 3 & 15 & \big| & 2 \end{pmatrix}.$$

Let's row reduce it:

$$\begin{pmatrix} 2 & 10 & \big| & -1 \\ 3 & 15 & \big| & 2 \end{pmatrix}$$

$R_1 = R_1 \div 2$ 〰〰〰〰〰〰〰⟿

$$\begin{pmatrix} 1 & 5 & \big| & -\frac{1}{2} \\ 3 & 15 & \big| & 2 \end{pmatrix}$$ (Step 1b)

$R_2 = R_2 - 3R_1$ 〰〰〰〰〰〰〰〰〰⟿

$$\begin{pmatrix} 1 & 5 & \big| & -\frac{1}{2} \\ 0 & 0 & \big| & \frac{7}{2} \end{pmatrix}$$ (Step 1c)

$R_2 = R_2 \times \frac{2}{7}$ 〰〰〰〰〰〰〰〰⟿

$$\begin{pmatrix} 1 & 5 & \big| & -\frac{1}{2} \\ 0 & 0 & \big| & 1 \end{pmatrix}$$ (Step 2b)

$R_1 = R_1 + \frac{1}{2}R_2$ 〰〰〰〰〰〰〰〰〰⟿

$$\begin{pmatrix} 1 & 5 & \big| & 0 \\ 0 & 0 & \big| & 1 \end{pmatrix}$$ (Step 2c)

The row reduced matrix

$$\begin{pmatrix} 1 & 5 & \big| & 0 \\ 0 & 0 & \big| & 1 \end{pmatrix}$$

corresponds to the *inconsistent* system

$$x + 5y = 0$$
$$0 = 1.$$

Question

What does an augmented matrix in reduced row echelon form look like, if its system of linear equations is inconsistent?

Answer:

$$\left(\begin{array}{cccc|c} 1 & 0 & \star & \star & 0 \\ 0 & 1 & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

An augmented matrix corresponds to an inconsistent system of equations if and only if *the last* (i.e., the augmented) *column is a pivot column.*

The linear system

$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned}$$

gives rise to the matrix

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right).$$

Let's row reduce it:

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right)$$

$$\xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 2 & 1 & 12 & 1 \end{array}\right) \qquad \text{(Optional)}$$

$$\xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & -3 & -6 & 3 \end{array}\right) \qquad \text{(Step 1c)}$$

$$\xrightarrow{R_2 = R_2 \div -3} \left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right) \qquad \text{(Step 2b)}$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \qquad \text{(Step 2c)}$$

The row reduced matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

corresponds to the linear system

$$\begin{cases} x + 5z = 1 \\ y + 2z = -1 \end{cases}$$

The system

$$x + 5z = \quad 1$$
$$y + 2z = -1$$

comes from a matrix in reduced row echelon form. Are we done? Is the system solved?

Yes! Rewrite:

$$x = \quad 1 - 5z$$
$$y = -1 - 2z$$

For any value of z, there is exactly one value of x and y that makes the equations true. But z can be *anything we want*!

So we have found the solution set: it is all values x, y, z where

$$x = \quad 1 - 5z$$
$$y = -1 - 2z \qquad \text{for } z \text{ any real number.}$$
$$(z = \qquad z)$$

This is called the **parametric form** for the solution.

For instance, $(1, -1, 0)$ and $(-4, -3, 1)$ are solutions.

Definition

Consider a *consistent* linear system of equations in the variables $x_1, \ldots, x_n$. Let A be a row echelon form of the matrix for this system.

We say that x_i is a **free variable** if its corresponding column in A is *not* a pivot column.

> **Important**
>
> 1. You can choose *any value* for the free variables in a (consistent) linear system.
> 2. Free variables come from *columns without pivots* in a matrix in row echelon form.

In the previous example, z was free because the reduced row echelon form matrix was

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 2 & -1 \end{array} \right).$$

In this matrix:

$$\left(\begin{array}{ccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array} \right)$$

the free variables are x_2 and x_4. (What about the last column?)

The reduced row echelon form of the matrix for a linear system in x_1, x_2, x_3, x_4 is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right)$$

The free variables are x_2 and x_4: they are the ones whose columns are *not* pivot columns.

This translates into the system of equations

$$\begin{cases} x_1 & + 3x_4 = & 2 \\ & x_3 + x_4 = & -1 \end{cases} \qquad \Longrightarrow \qquad \boxed{\begin{array}{l} x_1 = 2 - 3x_4 \\ x_3 = -1 - 4x_4 \end{array}}$$

What happened to x_2? What is it allowed to be? Anything! The general solution is

$$(x_1, x_2, x_3, x_4) = (2 - 3x_4, x_2, -1 - 4x_4, x_4)$$

for any values of x_2 and x_4. For instance, $(2, 0, -1, 0)$ is a solution ($x_2 = x_4 = 0$), and $(5, 1, 3, -1)$ is a solution ($x_2 = 1$, $x_4 = -1$).

> The boxed equation is called the **parametric form** of the general solution to the system of equations. It is obtained by moving all free variables to the right-hand side of the $=$.

Poll

Is it possible for a system of linear equations to have exactly two solutions?

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.
   In this case, the system is *inconsistent*. There are *zero* solutions, i.e. the solution set is *empty*. Picture:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Every column except the last column is a pivot column.
   In this case, the system has a *unique solution*. Picture:

$$\begin{pmatrix} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{pmatrix}$$

3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:

$$\begin{pmatrix} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{pmatrix}$$