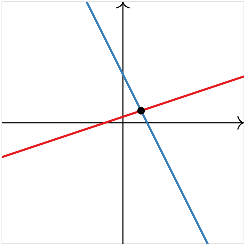
# Section 1.3

Vector Equations

# Motivation

We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

$$\begin{aligned} x - 3y &= -3 \\ 2x + y &= 8 \end{aligned} \quad \rightsquigarrow$$



This will give us better insight into the properties of systems of equations and their solution sets.

To do this, we need to introduce $n$-dimensional space $\mathbf{R}^n$, and **vectors** inside it.

# Line, Plane, Space, ...

Recall that **R** denotes the collection of all real numbers, i.e. the number line.

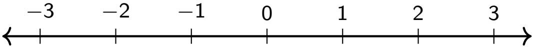
### Definition

Let $n$ be a positive whole number. We define

$$\mathbf{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \ldots, x_n).$$

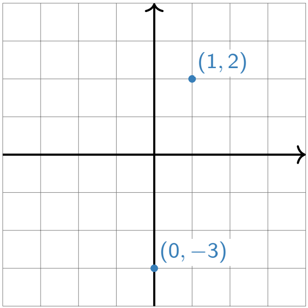
### Example

When $n = 1$, we just get $\mathbf{R}$ back: $\mathbf{R}^1 = \mathbf{R}$. Geometrically, this is the *number line*.



## Example

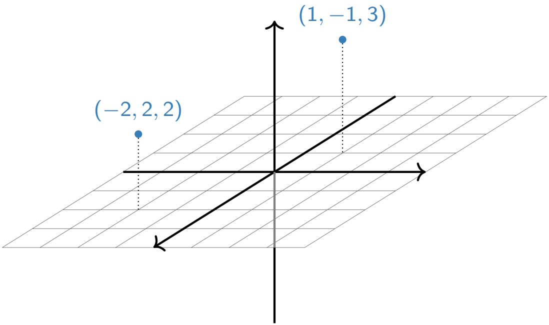
When $n = 2$, we can think of $\mathbf{R}^2$ as the *plane*. This is because every point on the plane can be represented by an ordered pair of real numbers, namely, its $x$- and $y$-coordinates.




We can use the elements of $\mathbf{R}^2$ to *label* points on the plane, but $\mathbf{R}^2$ is not defined to be the plane!

### Example

When $n = 3$, we can think of $\mathbf{R}^3$ as the *space* we (appear to) live in. This is because every point in space can be represented by an ordered triple of real numbers, namely, its $x$-, $y$-, and $z$-coordinates.




Again, we can use the elements of $\mathbf{R}^3$ to *label* points in space, but $\mathbf{R}^3$ is not defined to be space!

So what is $\mathbf{R}^4$? or $\mathbf{R}^5$? or $\mathbf{R}^n$?

...go back to the *definition*: ordered $n$-tuples of real numbers

$$(x_1, x_2, x_3, \ldots, x_n).$$

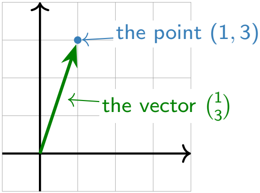
They're still "geometric" spaces, in the sense that our intuition for $\mathbf{R}^2$ and $\mathbf{R}^3$ sometimes extends to $\mathbf{R}^n$, but they're harder to visualize.

We'll make definitions and state theorems that apply to any $\mathbf{R}^n$, but we'll only draw pictures for $\mathbf{R}^2$ and $\mathbf{R}^3$.

# Vectors

In the previous slides, we were thinking of elements of $\mathbf{R}^n$ as **points**: in the line, plane, space, etc.

We can also think of them as **vectors**: arrows with a given length and direction.




So the vector points *horizontally* in the amount of its $x$-coordinate, and *vertically* in the amount of its $y$-coordinate.

When we think of an element of $\mathbf{R}^n$ as a vector, we write it as a matrix with $n$ rows and one column:

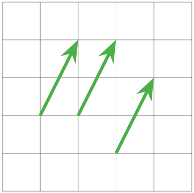
$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

We'll see why this is useful later.

# Points and Vectors

So what is the difference between a point and a vector?

A vector need not start at the origin: *it can be located anywhere*! In other words, an arrow is determined by its length and its direction, not by its location.



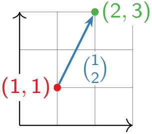
These arrows all represent the vector $\begin{pmatrix}1\\2\end{pmatrix}$.

However, unless otherwise specified, we'll assume a vector starts at the origin: we'll usually be sloppy and identify the vector $\binom{1}{2}$ with the point $(1, 2)$.

This makes sense in the real world: many physical quantities, such as velocity, are represented as vectors. But it makes more sense to think of the velocity of a car as being located at the car.

Another way to think about it: a vector is a *difference* between two points, or the arrow from one point to another.

For instance, $\begin{pmatrix}1\\2\end{pmatrix}$ is the arrow from $(1, 1)$ to $(2, 3)$.



# Vector Algebra

## Definition

- We can add two vectors together:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}.$$

- We can multiply, or **scale**, a vector by a real number $c$:

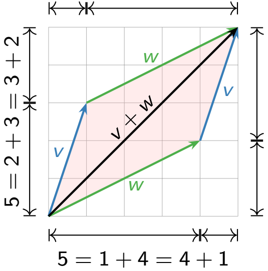
$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}.$$

  We call $c$ a **scalar** to distinguish it from a vector. If $v$ is a vector and $c$ is a scalar, $cv$ is called a **scalar multiple** of $v$.

(And likewise for vectors of length $n$.) For instance,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} \quad \text{and} \quad -2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}.$$

# Vector Addition and Subtraction: Geometry

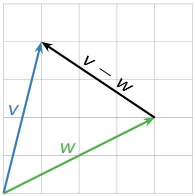



## The parallelogram law for vector addition

Geometrically, the sum of two vectors $v, w$ is obtained as follows: place the tail of $w$ at the head of $v$. Then $v + w$ is the vector whose tail is the tail of $v$ and whose head is the head of $w$. Doing this both ways creates a parallelogram. For example,

$$\begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix}.$$

Why? The width of $v + w$ is the sum of the widths, and likewise with the heights.



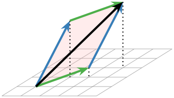

## Vector subtraction

Geometrically, the difference of two vectors $v, w$ is obtained as follows: place the tail of $v$ and $w$ at the same point. Then $v - w$ is the vector from the head of $v$ to the head of $w$. For example,

$$\begin{pmatrix}1\\4\end{pmatrix} - \begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}-3\\2\end{pmatrix}.$$

Why? If you add $v - w$ to $w$, you get $v$.

This works in higher dimensions too!

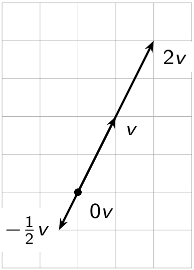


# Scalar Multiplication: Geometry

## Scalar multiples of a vector

These have the same *direction* but a different *length*.

Some multiples of $v$.



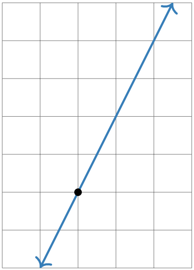

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$2v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$-\frac{1}{2}v = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}$$

$$0v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

All multiples of $v$.



So the scalar multiples of $v$ form a *line*.

# Linear Combinations

We can add and scalar multiply in the same equation:

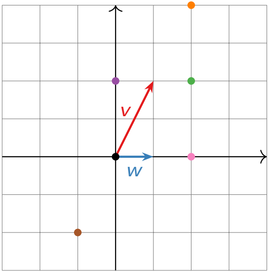
$$w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and $w$ is a vector in $\mathbf{R}^n$.

## Definition

We call $w$ a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

## Example




Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of $v$ and $w$?

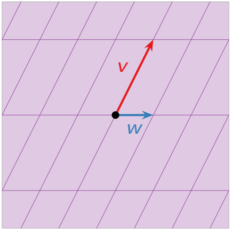
- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

# Poll

**Poll**

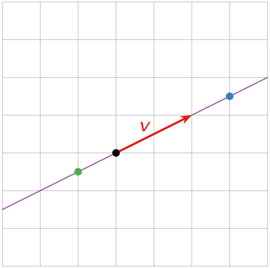
Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of $v$ and $w$?

No: in fact, *every* vector in $\mathbf{R}^2$ is a combination of $v$ and $w$.




(The purple lines are to help measure *how much* of $v$ and $w$ you need to get to a given point.)

## More Examples

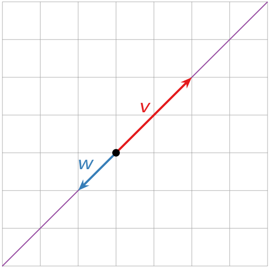



What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

- $\frac{3}{2}v$
- $-\frac{1}{2}v$
- ...

What are *all* linear combinations of $v$?

All vectors $cv$ for $c$ a real number. I.e., all *scalar multiples* of $v$. These form a *line*.




### Question

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

Answer: The line which contains both vectors.

What's different about this example and the one on the poll?

Question

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

This means: can we solve the equation

$$x \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

where $x$ and $y$ are the unknowns (the coefficients)? Rewrite:

$$\begin{pmatrix} x \\ 2x \\ 6x \end{pmatrix} + \begin{pmatrix} -y \\ -2y \\ -y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

This is just a system of linear equations:

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3. \end{aligned}$$

# Systems of Linear Equations

## Continued

$$\begin{aligned} x - \phantom{2}y &= \phantom{1}8 \\ 2x - 2y &= 16 \\ 6x - \phantom{2}y &= \phantom{1}3 \end{aligned} \quad \xrightarrow{\text{matrix form}} \quad \left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{row reduce}} \quad \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{solution}} \quad \begin{aligned} x &= -1 \\ y &= -9 \end{aligned}$$

Conclusion:

$$-\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

What is the relationship between the original vectors and the matrix form of the linear equation? They have the same columns!

Shortcut: You can make the augmented matrix without writing down the system of linear equations first.

**Summary**

The vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b,$$

where $v_1, v_2, \ldots, v_p, b$ are vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_p$ are scalars, has the same solution set as the linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right),$$

where the $v_i$'s and $b$ are the columns of the matrix.

So we now have (at least) *two* equivalent ways of thinking about linear systems of equations:

1. Augmented matrices.
2. Linear combinations of vectors (vector equations).

The last one is more geometric in nature.

It is important to know what are *all* linear combinations of a set of vectors $v_1, v_2, \ldots, v_p$ in $\mathbf{R}^n$: it's exactly the collection of all $b$ in $\mathbf{R}^n$ such that the vector equation (in the unknowns $x_1, x_2, \ldots, x_p$)

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

has a solution (i.e., is consistent).

## Definition

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

"the set of" "such that"

$$\boxed{\mathrm{Span}\{v_1, v_2, \ldots, v_p\} = \big\{ x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \;\big|\; x_1, x_2, \ldots, x_p \text{ in } \mathbf{R} \big\}.}$$

Synonyms: $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

## Span

### Continued

Now we have several equivalent ways of making the same statement:

1. A vector $b$ is in the span of $v_1, v_2, \ldots, v_p$.
2. The linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right)$$

   is consistent.
3. The vector equation

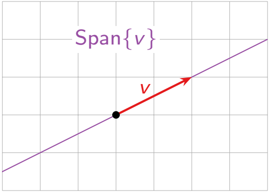
$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

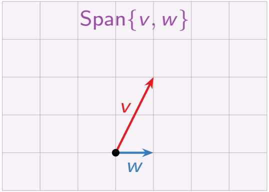
   has a solution.

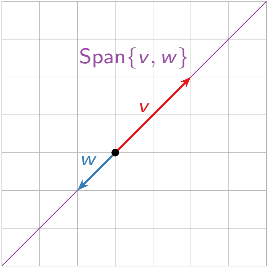
Note: **equivalent** means that, for any given list of vectors $v_1, v_2, \ldots, v_p, b$, *either* all three statements are true, *or* all three statements are false.

# Pictures of Span

Drawing a picture of Span$\{v_1, v_2, \ldots, v_p\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_p$.

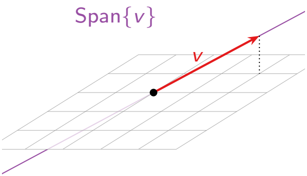


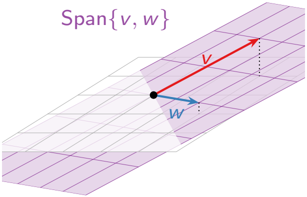



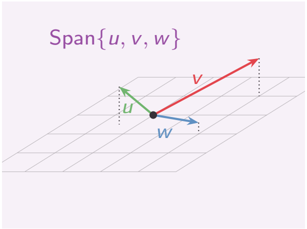



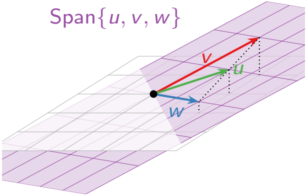
# Pictures of Span

## In $\mathbf{R}^3$



# Poll

**Poll**

How many vectors are in $\mathrm{Span}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$?

A. Zero

B. One

C. Infinity

In general, it appears that $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the smallest "linear space" (line, plane, etc.) containing the origin and all of the vectors $v_1, v_2, \ldots, v_p$.

We will make this precise later.