Review for Midterm 1

Selected Topics

Linear Equations

We have *four* equivalent ways of writing (and thinking about) linear systems:

1. As a system of equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$

2. As an augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$$

3. As a vector equation ($x_1 v_1 + \cdots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation ($Ax = b$):

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

In particular, *all four have the same solution set*.

Number of Solutions

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.
   There are *zero* solutions, i.e. the solution set is *empty*. In this case, the system is called **inconsistent**. Picture:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   In this case, the system has a *unique solution*. Picture:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

Span

The **span** of vectors $v_1, v_2, \ldots, v_n$ is the set of all linear combinations of these vectors:

$$\mathrm{Span}\{v_1, v_2, \ldots, v_n\} = \big\{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \mid a_1, a_2, \ldots, a_n \text{ in } \mathbf{R}\big\}.$$

Theorem

Let $v_1, v_2, \ldots, v_n$, and b be vectors in $\mathbf{R}^m$, and let A be the $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$. The following are equivalent:

either they're all true, or they're all false, for the given vectors

1. $Ax = b$ is consistent.
2. $(\, A \mid b \,)$ does not have a pivot in the last column.
3. b is in $\mathrm{Span}\{v_1, v_2, \ldots, v_n\}$ (the span of the columns of A).

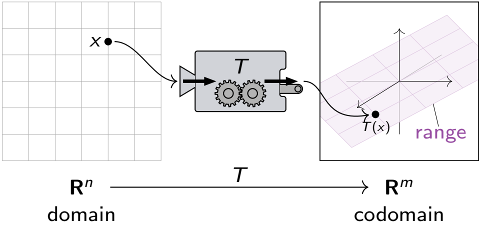
In this case, a solution to the matrix equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b \qquad \text{gives the linear combination} \qquad x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b.$$

Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

Picture and vocabulary words:



It is **one-to-one** if different vectors in the domain go to different vectors in the codomain: $x \neq y \implies T(x) \neq T(y)$.

It is **onto** if every vector in the codomain is $T(x)$ for some x. In other words, the range equals the codomain.

Linear Transformations

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies:

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cu) = cT(u)$$

for every u, v in $\mathbf{R}^n$ and every c in $\mathbf{R}$.

Linear transformations are the same as matrix transformations.

Dictionary

Linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ $\rightsquigarrow$ $m \times n$ matrix $A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$

$T(x) = Ax$, $T\colon \mathbf{R}^n \to \mathbf{R}^m$ $\leftsquigarrow$ $m \times n$ matrix A

As always, $e_1, e_2, \ldots, e_n$ are the **unit coordinate vectors**

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Let A be an $m \times n$ matrix and let T be the linear transformation $T(x) = Ax$.

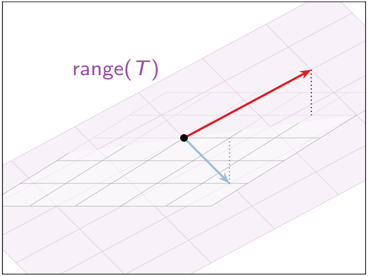
- The domain of T is $\mathbf{R}^n$. (Inputs are vectors with n entries.)
- The codomain of T is $\mathbf{R}^m$. (Outputs are vectors with m entries.)
- The range of T is span of the columns of A.
  (This is the set of all b in $\mathbf{R}^m$ such that $Ax = b$ has a solution.)

Example

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} \qquad T(x) = Ax$$

- The domain of T is $\mathbf{R}^2$.
- The codomain of T is $\mathbf{R}^3$.
- The range of T is

$$\operatorname{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$



$\mathbf{R}^3 = \mathrm{codomain}(T)$

When the Span is Everything

Theorem

Let A be an $m \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be the linear transformation $T(x) = Ax$. The following are equivalent:

1. T is onto.
2. $T(x) = b$ has a solution for every b in $\mathbf{R}^m$.
3. $Ax = b$ is consistent for every b in $\mathbf{R}^m$.
4. The columns of A span $\mathbf{R}^m$.
5. A has a pivot in each *row*.

Moral: If A has a pivot in each row then its reduced row echelon form looks like this:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix} \quad \text{and } (\, A \mid b \,) \text{ reduces to this:} \quad \left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & \star \\ 0 & 1 & \star & 0 & \star & \star \\ 0 & 0 & 0 & 1 & \star & \star \end{array}\right).$$

There's no b that makes it inconsistent, so there's always a solution.

Refer: slides for §1.4 and §1.9.

A *set* of vectors $\{v_1, v_2, \ldots, v_n\}$ is **linearly independent** if

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0 \quad \text{only when} \quad a_1 = a_2 = \cdots = a_n = 0.$$

Otherwise they are **linearly dependent**, and an equation $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$ with some $a_i \neq 0$ is a **linear dependence relation**.

Theorem

Let $v_1, v_2, \ldots, v_n$ be vectors in $\mathbf{R}^m$, and let A be the $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$. The following are equivalent:

1. The set $\{v_1, v_2, \ldots, v_n\}$ is linearly independent.
2. For every i between 1 and n, v_i is not in $\mathrm{Span}\{v_1, v_2, \ldots, v_{i-1}\}$.
3. $Ax = 0$ only has the trivial solution.
4. A has a pivot in every column.

If the vectors are linearly *de*pendent, a nontrivial solution to the matrix equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0 \quad \text{gives the linear dependence relation} \quad x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0.$$

More Criteria for Linear Independence

Theorem

Let A be an $m \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be the linear transformation $T(x) = Ax$. The following are equivalent:

1. T is one-to-one.
2. $T(x) = b$ has one or zero solutions for every b in $\mathbf{R}^m$.
3. $Ax = b$ has a unique solution or is inconsistent for every b in $\mathbf{R}^m$.
4. $Ax = 0$ has a unique solution.
5. The columns of A are linearly independent.
6. A has a pivot in each *column*.

Moral: If A has a pivot in each column then its reduced row echelon form looks like this:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and } (\,A \mid b\,) \text{ reduces to this: } \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \\ 0 & 0 & 0 & \star \end{array}\right).$$

This can be inconsistent, but if it is consistent, it has a unique solution.

Refer: slides for §1.4, §1.8, §1.9.

Parametric Form of Solution Sets

To find the solution set to $Ax = b$, first form the augmented matrix $(\,A \mid b\,)$, then row reduce.

$$\left(\begin{array}{ccccc|c} 1 & 3 & 0 & 4 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -7 \end{array}\right)$$

This translates into

$$\begin{aligned} x_1 + 3x_2 \qquad\quad + x_4 \qquad &= 2 \\ x_3 - x_4 \qquad &= 3 \\ x_5 &= -7 \end{aligned}$$

The variables correspond to the non-augmented columns of the matrix.
The *free variables* correspond to the non-augmented columns *without pivots*.

Move the free variables to the other side, get the *parametric form*:

$$\begin{aligned} x_1 &= 2 - 3x_2 - x_4 \\ x_3 &= 3 \qquad\quad + x_4 \\ x_5 &= -7 \end{aligned}$$

This is a solution for every value of x_3 and x_4.

Parametric Vector Form of Solution Sets

Parametric form:

$$\begin{aligned} x_1 &= 2 - 3x_2 - x_4 \\ x_3 &= 3 + x_4 \\ x_5 &= -7 \end{aligned} \quad \xrightarrow{\text{add free variables}} \quad \begin{aligned} x_1 &= 2 - 3x_2 - x_4 \\ x_2 &= x_2 \\ x_3 &= 3 + x_4 \\ x_4 &= x_4 \\ x_5 &= -7 \end{aligned}$$

Now collect all of the equations into a vector equation:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ -7 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

This is the **parametric vector form** of the solution set. This means that the

$$(\text{solution set}) = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ -7 \end{pmatrix} + \mathrm{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Homogeneous and Non-Homogeneous Equations

The equation $Ax = b$ is called **homogeneous** if $b = 0$, and **non-homogeneous** otherwise. A homogeneous equation always has the **trivial solution** $x = 0$:

$$A0 = 0.$$

The solution set to a homogeneous equation is always a span:

$$(\text{solutions to } Ax = 0) = \text{Span}\{v_1, v_2, \ldots, v_r\}$$

where r is the number of free variables. The solution set to a consistent non-homogeneous equation is

$$(\text{solutions to } Ax = b) = p + \text{Span}\{v_1, v_2, \ldots, v_r\}$$

where p is a **specific solution** (i.e. some vector such that $Ap = b$), and $\text{Span}\{v_1, \ldots, v_r\}$ is the solution set to the homogeneous equation $Ax = 0$. This is a *translate of a span*.

Both expressions can be read off from the parametric vector form.