Section 1.8

Introduction to Linear Transformations

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This is also a way to understand the *geometry* of *matrices*.

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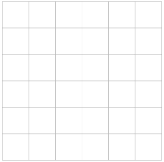

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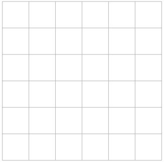

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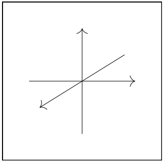

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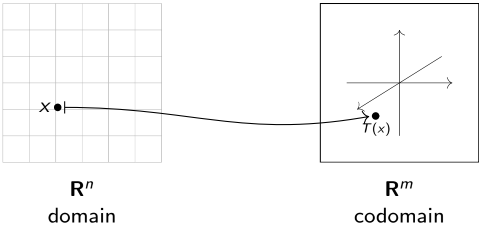

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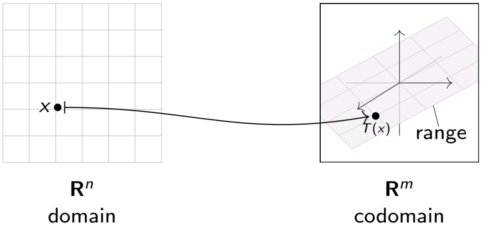

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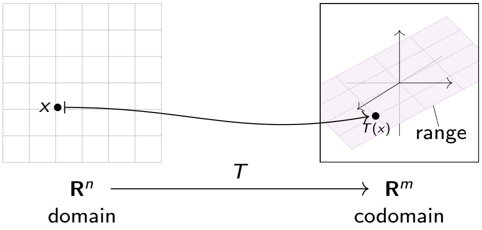

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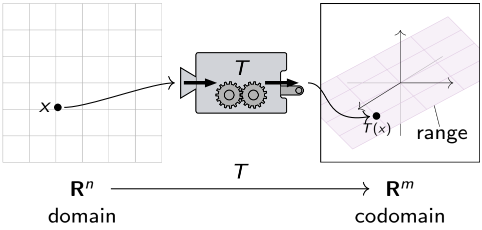

It may help to think of T as a "machine" that takes x as an input, and gives you $T(x)$ as the output.

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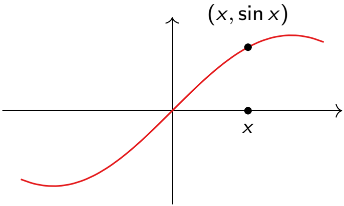

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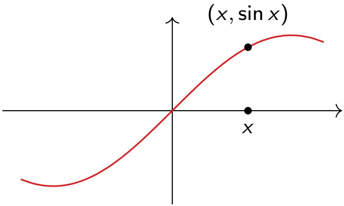

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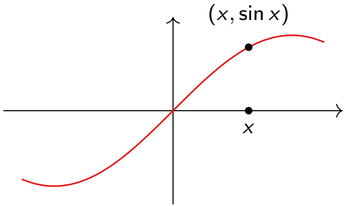

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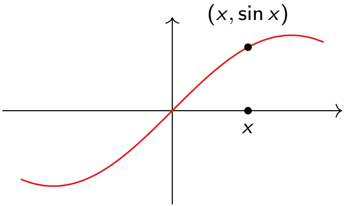

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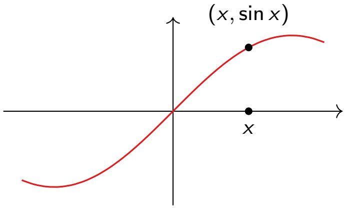

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$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

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Your life will be much easier if you just remember these.

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- Let $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$. Find v in $\mathbf{R}^{-}$ such that $T(v) = b$.

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  Translation: is there any c in $\mathbf{R}^3$ such that the solution set of $Ax = c$ has more than one vector v in it?

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- Is there any c in $\mathbf{R}^3$ such that there is more than one v in $\mathbf{R}^2$ with $T(v) = c$?

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  The solution set of $Ax = c$ is a translate of the solution set of $Ax = b$ (from before), which has ___ vector in it.

Matrix Transformations

Example, continued

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Matrix Transformations
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Matrix Transformations

Geometric example

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^{-} \to \mathbf{R}^{-}$.

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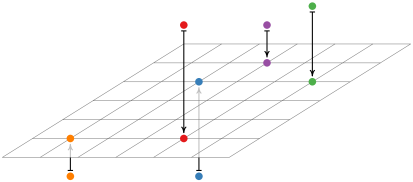
This is *projection onto the xy-axis*. Picture:



Matrix Transformations
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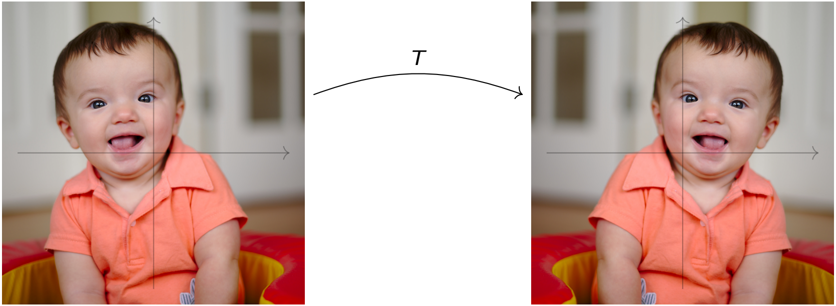


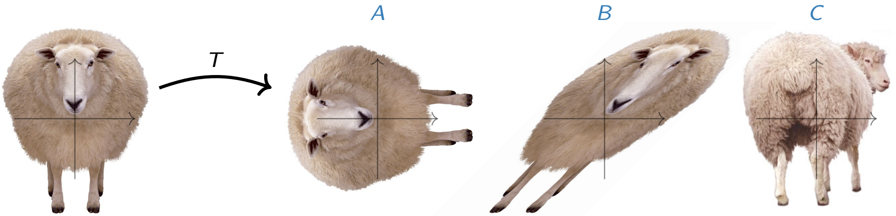
Poll

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^2$. ($T$ is called a **shear**.)

Poll

What does T do to this sheep?

Hint: first draw a picture what it does to the box *around* the sheep.



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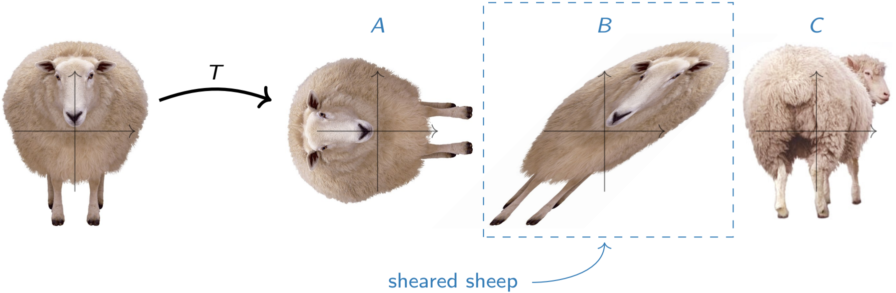

Linear Transformations

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

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In engineering this is called **superposition**.

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This is called **dilation** or **scaling** (by a factor of 1.5). Picture:

Linear Transformations
Dilation

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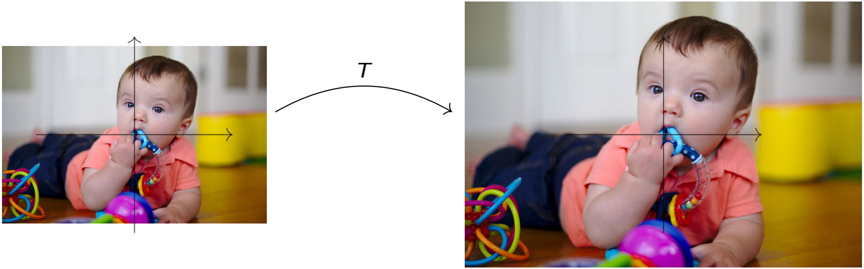

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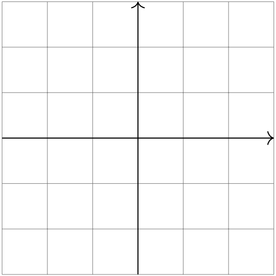

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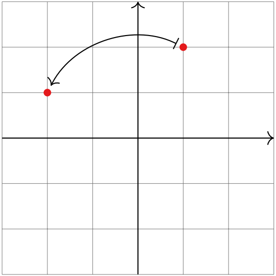

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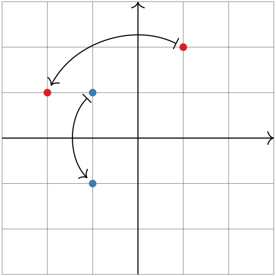

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