Section 5.3

Diagonalization

Motivation

Difference equations

Many real-word linear algebra problems have the form:

$$v_1 = Av_0, \quad v_2 = Av_1 = A^2 v_0, \quad v_3 = Av_2 = A^3 v_0, \quad \dots \quad v_n = Av_{n-1} = A^n v_0.$$

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- Taking powers of diagonal matrices is easy!
- Taking powers of *diagonalizable* matrices is still easy!
- Diagonalizing a matrix is an eigenvalue problem.

Powers of Diagonal Matrices

If D is diagonal, then D^n is also diagonal; its diagonal entries are the nth powers of the diagonal entries of D:

Powers of Matrices that are Similar to Diagonal Ones

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In §5.2 lecture we saw that A is similar to a diagonal matrix:

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

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Therefore

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Important

If $A = PDP^{-1}$ for $D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$ then

$$A^k = PD^K P^{-1} = P \begin{pmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{pmatrix} P^{-1}.$$

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So diagonalizable matrices are easy to raise to any power.

Diagonalization

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

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In this case, $A = PDP^{-1}$ for

$$P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

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An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

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Corollary

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An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

The Corollary is true because eigenvectors with distinct eigenvalues are always linearly independent. We will see later that a diagonalizable matrix need not have n distinct eigenvalues though.

Problem: Diagonalize $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$.

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Note: In this case, there are three linearly independent eigenvectors, but only two distinct eigenvalues.

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Conclusion: A has only one linearly independent eigenvector, so by the "only if" part of the diagonalization theorem, A is not diagonalizable.

Poll

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Matrix D is already diagonal!

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3. If there are fewer than n total vectors in the union of all of the eigenspace bases $\mathcal{B}_\lambda$, then the matrix is not diagonalizable.
4. Otherwise, the n vectors $v_1, v_2, \ldots, v_n$ in your eigenspace bases are linearly independent, and $A = PDP^{-1}$ for

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where λ_i is the eigenvalue for v_i.

Why is the Diagonalization Theorem true?

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1. A is diagonalizable.
2. The sum of the geometric multiplicities of the eigenvalues of A equals n.
3. The sum of the algebraic multiplicities of the eigenvalues of A equals n, and *the geometric multiplicity equals the algebraic multiplicity* of each eigenvalue.

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Hence the geometric multiplicities add up to 3, so A is diagonalizable.

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Since the geometric multiplicity is smaller than the algebraic multiplicity, the matrix is *not* diagonalizable.

Applications to Difference Equations

Let $D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$.

Fix a vector v_0, and let $v_1 = Dv_0$, $v_2 = Dv_1$, etc., so $v_n = D^n v_0$.

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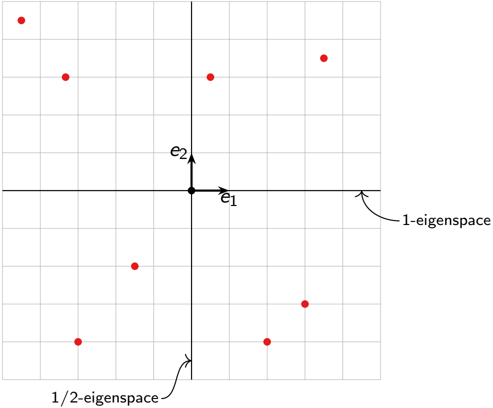
Question: What happens to the v_i's for different choices of v_0?

$$D\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1&0\\0&1/2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}a\\b/2\end{pmatrix}$$

v_0



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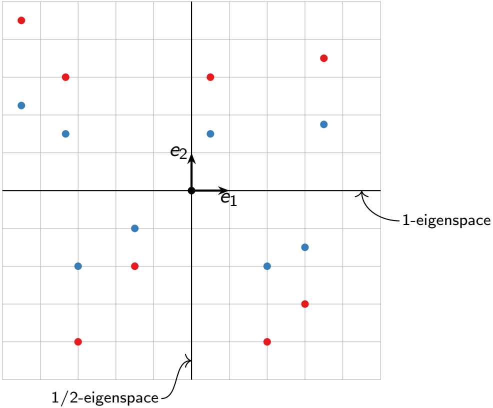

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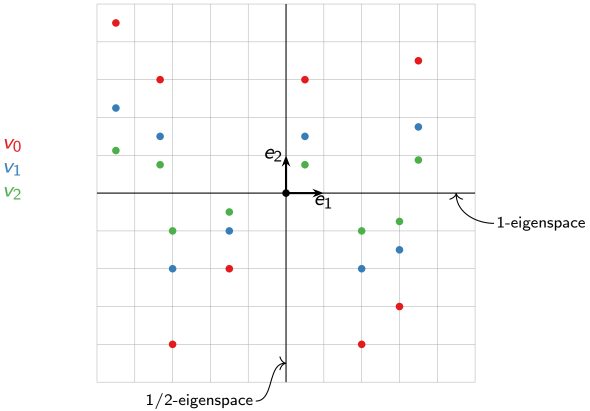

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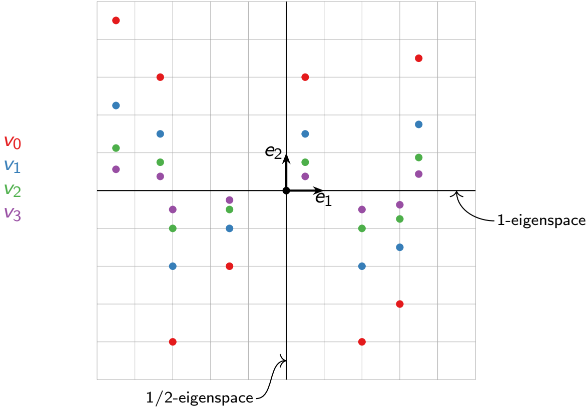

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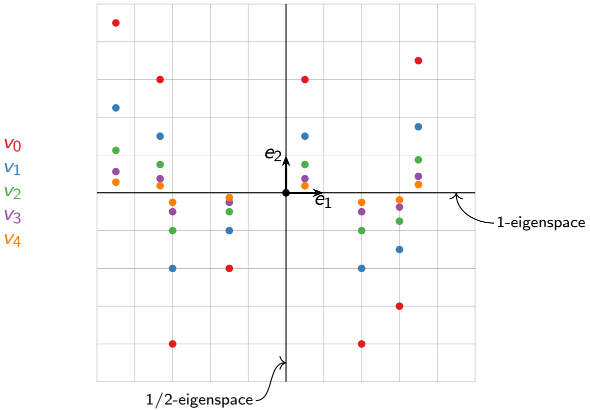

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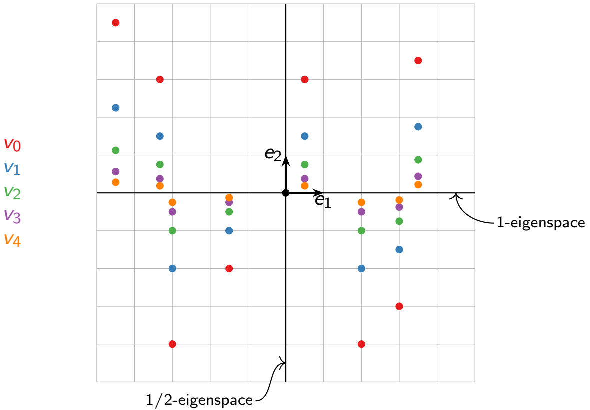

So all vectors get "sucked into the x-axis," which is the 1-eigenspace.

Let $A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$.

Fix a vector v_0, and let $v_1 = Av_0$, $v_2 = Av_1$, etc., so $v_n = A^n v_0$.

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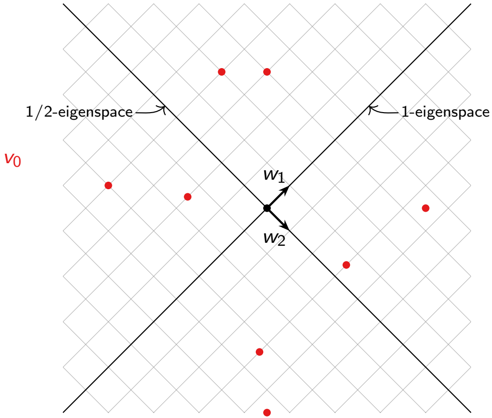
Question: What happens to the v_i's for different choices of v_0?

Recall: $A^n = PD^nP^{-1}$ acts on the usual coordinates of v_0 in the same way that D^n acts on the $\mathcal{B}$-coordinates, where $\mathcal{B} = \{w_1, w_2\}$.



Applications to Difference Equations

Picture of the more complicated example

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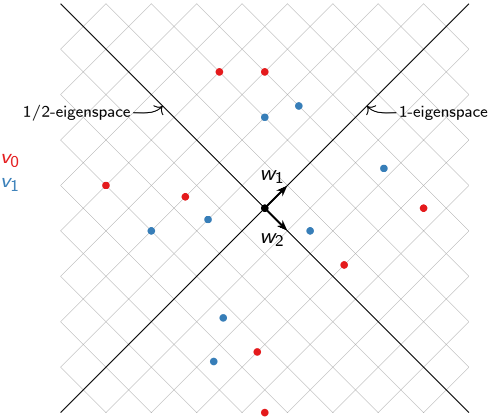

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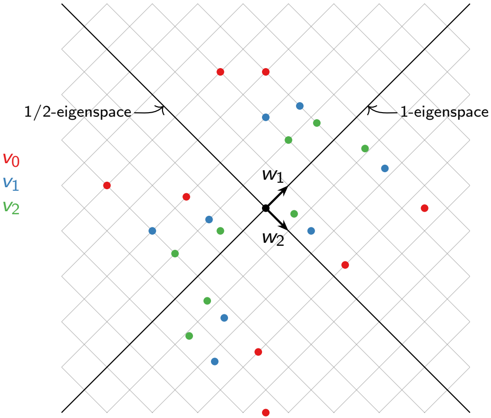

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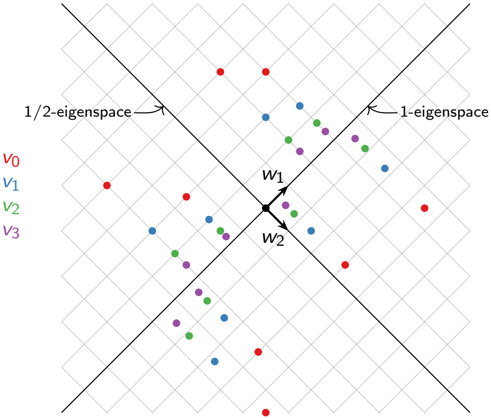

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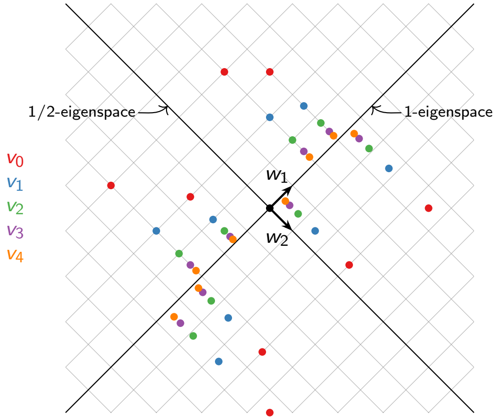

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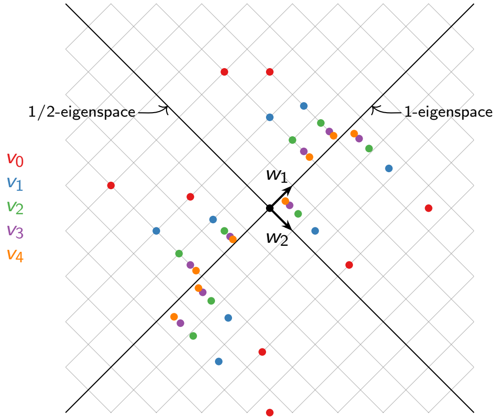

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We will study such matrices in detail next time.