Section 2.9

Dimension and Rank

Coefficients of Basis Vectors

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If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V, then any vector x in V can be written as a linear combination

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

for *unique* coefficients $c_1, c_2, \ldots, c_m$.

Bases as Coordinate Systems

The unit coordinate vectors $e_1, e_2, \ldots, e_n$ form a basis for $\mathbf{R}^n$. Any vector is a unique linear combination of the e_i:

$$v = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3e_1 + 5e_2 - 2e_3.$$

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We can go backwards: given any basis $\mathcal{B}$, we interpret the coefficients of a linear combination as "coordinates" with respect to $\mathcal{B}$.

Definition

Let $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ be a basis of a subspace V. Any vector x in V can be written uniquely as a linear combination $x = c_1v_1 + c_2v_2 + \cdots + c_mv_m$. The coefficients $c_1, c_2, \ldots, c_m$ are the **coordinates of x with respect to $\mathcal{B}$**. The **$\mathcal{B}$-coordinate vector of x** is the vector

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{in } \mathbf{R}^m.$$

Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\quad \mathcal{B} = \{v_1, v_2\}, \quad V = \mathrm{Span}\{v_1, v_2\}.$

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Verify that $\mathcal{B}$ is a basis:

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Question: Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$.

Bases as Coordinate Systems

Example 2

$$\text{Let } v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \; v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \; v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}, \quad V = \mathrm{Span}\{v_1, v_2, v_3\}.$$

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Question: Find a basis for V.

Question: Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$.

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V and x is in V, then

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{means} \quad x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m.$$

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$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_m & x \\ | & | & & | & | \end{array}\right).$$

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Question: What happens if you try to find the $\mathcal{B}$-coordinates of x *not* in V?

Bases as Coordinate Systems

Picture

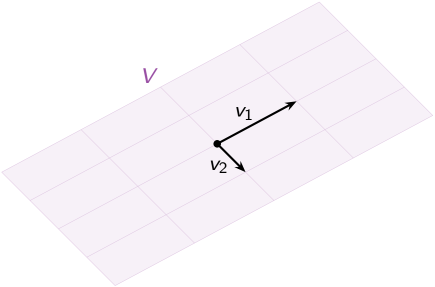
Let

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

These form a basis $\mathcal{B}$ for the plane

$$V = \mathrm{Span}\{v_1, v_2\}$$

in $\mathbf{R}^3$.



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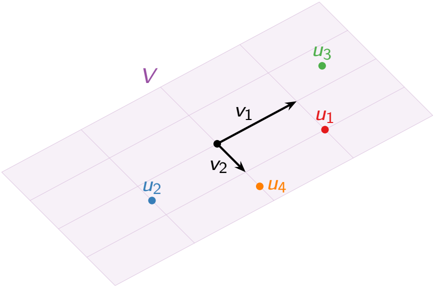

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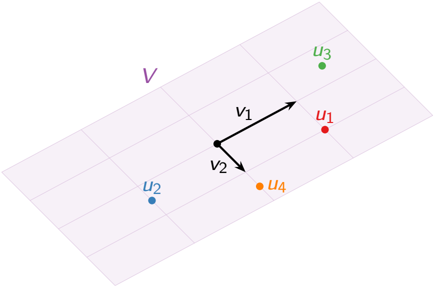

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Remark

Many of you want to think of a plane in $\mathbf{R}^3$ as "being" $\mathbf{R}^2$. Choosing a basis $\mathcal{B}$ and using $\mathcal{B}$-coordinates is one way to make sense of that. But remember that the coordinates are the coefficients of a linear combination of the basis vectors.

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Recall:

- The **dimension** of a subspace V is the number of vectors in a basis for V.

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Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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free variables

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But $\dim \operatorname{Nul} A = 1$ and $\dim \operatorname{Col} B = 2$, and a 1-dimensional space can't contain a 2-dimensional space.

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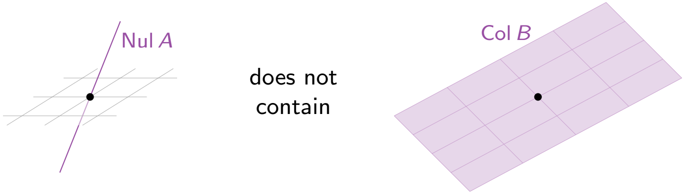

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If you *already* know that $\dim V = m$, and you have m vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in V, then you only have to check *one* of

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in order for $\mathcal{B}$ to be a basis.

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

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3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.

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13. A^T is invertible.
14. The columns of A form a basis for $\mathbf{R}^n$.

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The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
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These are equivalent to the previous conditions by the Rank Theorem and the Basis Theorem.