Review for Midterm 1

Selected Topics

Linear Equations

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In particular, *all four have the same solution set*.

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2. Every column except the last column is a pivot column.
   In this case, the system has a *unique solution*. Picture:

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$$\left(\begin{array}{cc|c} 1 & 0 & \color{red}{0} \\ 0 & 1 & \color{red}{0} \\ 0 & 0 & \color{red}{1} \end{array}\right)$$

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3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:

$$\left(\begin{array}{cccc|c} 1 & \color{red}{\star} & 0 & \color{blue}{\star} & \star \\ 0 & \color{red}{0} & 1 & \color{blue}{\star} & \star \end{array}\right)$$

Span

The **span** of vectors $v_1, v_2, \ldots, v_n$ is the set of all linear combinations of these vectors:

$$\mathrm{Span}\{v_1, v_2, \ldots, v_n\} = \left\{ a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \mid a_1, a_2, \ldots, a_n \text{ in } \mathbf{R} \right\}.$$

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Theorem

Let $v_1, v_2, \ldots, v_n$, and b be vectors in $\mathbf{R}^m$, and let A be the $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$. The following are equivalent:

either they're all true, or they're all false, for the given vectors

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In this case, a solution to the matrix equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b \qquad \text{gives the linear combination} \qquad x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b.$$

Transformations

Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

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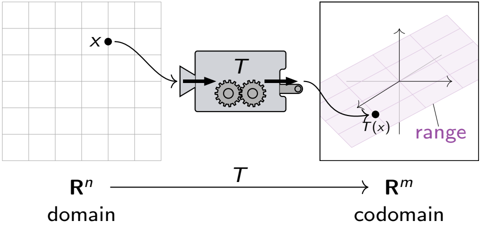

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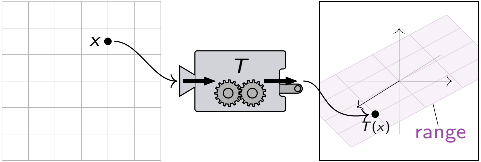

$$\underset{\text{domain}}{\mathbf{R}^n} \xrightarrow{\quad\quad T \quad\quad} \underset{\text{codomain}}{\mathbf{R}^m}$$

It is **one-to-one** if different vectors in the domain go to different vectors in the codomain: $x \neq y \implies T(x) \neq T(y)$.

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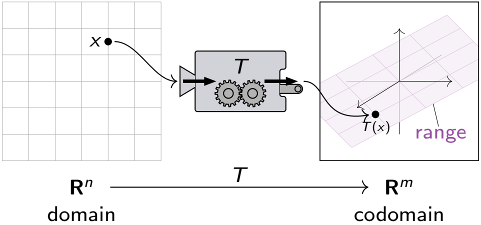

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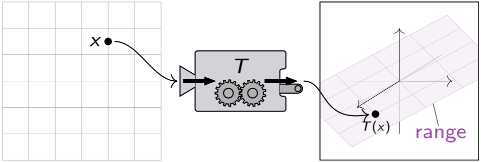

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It is **onto** if every vector in the codomain is $T(x)$ for some x. In other words, the range equals the codomain.

Linear Transformations

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies:

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cu) = cT(u)$$

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$T(x) = Ax$, $T\colon \mathbf{R}^n \to \mathbf{R}^m$ $\leftsquigarrow$ $m \times n$ matrix A

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As always, $e_1, e_2, \ldots, e_n$ are the **unit coordinate vectors**

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

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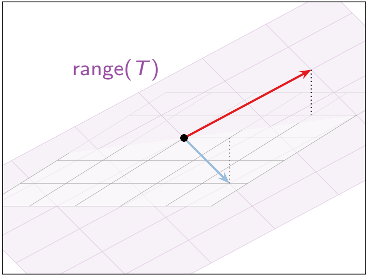
$$\text{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$



$\mathbf{R}^3 = \mathrm{codomain}(T)$

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3. $Ax = b$ is consistent for every b in $\mathbf{R}^m$.

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4. The columns of A span $\mathbf{R}^m$.

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If the vectors are linearly *de*pendent, a nontrivial solution to the matrix equation

$$A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0 \quad \begin{array}{c}\text{gives the linear} \\ \text{dependence relation}\end{array} \quad x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0.$$

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Move the free variables to the other side, get the *parametric form*:

$$\begin{aligned} x_1 &= \;\;2 - 3x_2 - x_4 \\ x_3 &= \;\;3 \qquad\quad + x_4 \\ x_5 &= -7 \end{aligned}$$

This is a solution for every value of x_3 and x_4.

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Now collect all of the equations into a vector equation:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ -7 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

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$$(\text{solution set}) = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ -7 \end{pmatrix} + \mathrm{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

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$$(\text{solutions to } Ax = 0) = \operatorname{Span}\{v_1, v_2, \ldots, v_r\}$$

where r is the number of free variables. The solution set to a consistent non-homogeneous equation is

$$(\text{solutions to } Ax = b) = p + \operatorname{Span}\{v_1, v_2, \ldots, v_r\}$$

where p is a **specific solution** (i.e. some vector such that $Ap = b$), and $\operatorname{Span}\{v_1, \ldots, v_r\}$ is the solution set to the homogeneous equation $Ax = 0$.

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Both expressions can be read off from the parametric vector form.