Section 1.3

Vector Equations

Motivation

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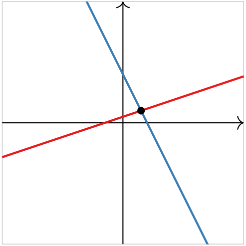

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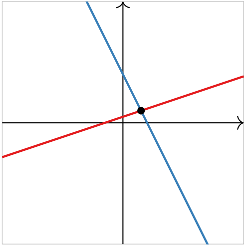

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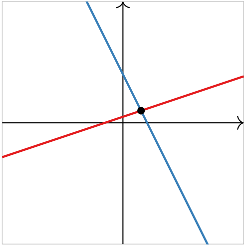

This will give us better insight into the properties of systems of equations and their solution sets.

To do this, we need to introduce n-dimensional space $\mathbf{R}^n$, and **vectors** inside it.

Line, Plane, Space, ...

Recall that **R** denotes the collection of all real numbers, i.e. the number line.

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Definition

Let n be a positive whole number. We define

$$\mathbf{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \ldots, x_n).$$

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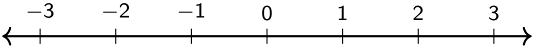
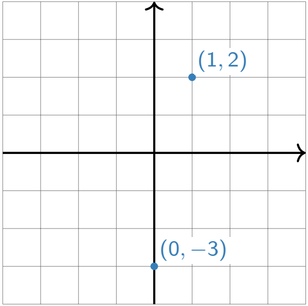
Example

When $n = 1$, we just get **R** back: $\mathbf{R}^1 = \mathbf{R}$. Geometrically, this is the *number line*.



Example

When $n = 2$, we can think of $\mathbf{R}^2$ as the *plane*. This is because every point on the plane can be represented by an ordered pair of real numbers, namely, its x- and y-coordinates.



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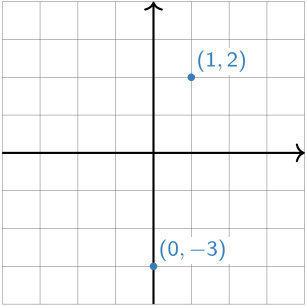
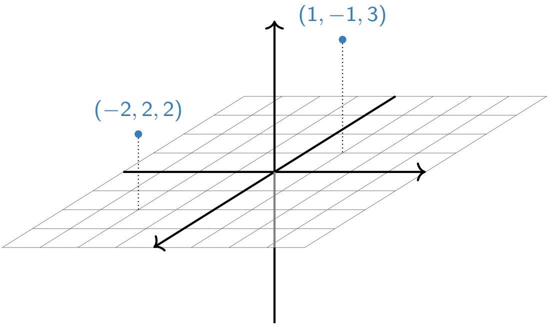


We can use the elements of $\mathbf{R}^2$ to *label* points on the plane, but $\mathbf{R}^2$ is not defined to be the plane!

Example

When $n = 3$, we can think of $\mathbf{R}^3$ as the *space* we (appear to) live in. This is because every point in space can be represented by an ordered triple of real numbers, namely, its x-, y-, and z-coordinates.



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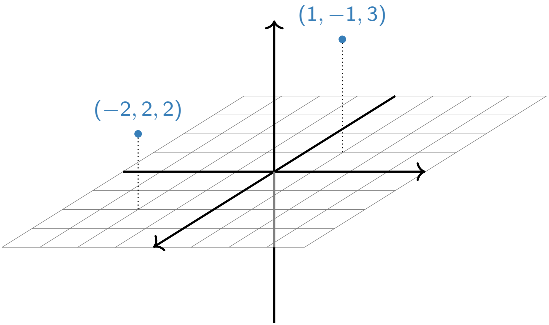

Again, we can use the elements of $\mathbf{R}^3$ to *label* points in space, but $\mathbf{R}^3$ is not defined to be space!

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Continued

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We'll make definitions and state theorems that apply to any $\mathbf{R}^n$, but we'll only draw pictures for $\mathbf{R}^2$ and $\mathbf{R}^3$.

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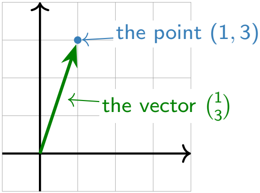

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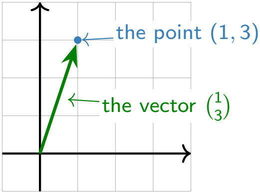

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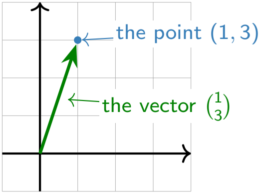

So the vector points *horizontally* in the amount of its x-coordinate, and *vertically* in the amount of its y-coordinate.

When we think of an element of $\mathbf{R}^n$ as a vector, we write it as a matrix with n rows and one column:

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

We'll see why this is useful later.

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Points and Vectors

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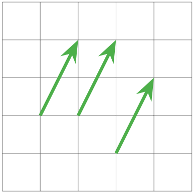

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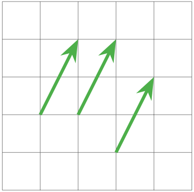

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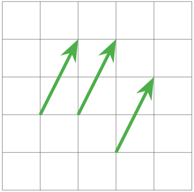

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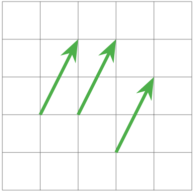

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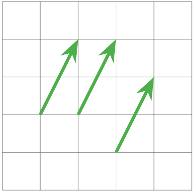

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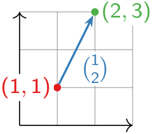
For instance, $\begin{pmatrix}1\\2\end{pmatrix}$ is the arrow from $(1,1)$ to $(2,3)$.



Vector Algebra

Definition

- We can add two vectors together:

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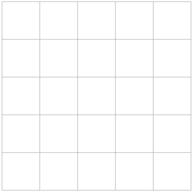
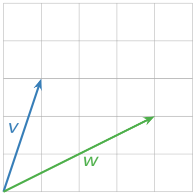
Vector Addition and Subtraction: Geometry



The parallelogram law for vector addition



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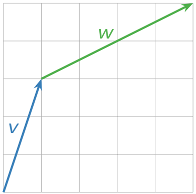

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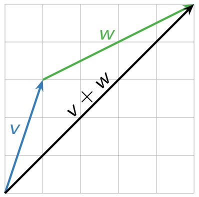

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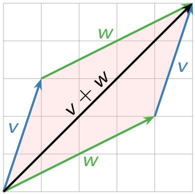

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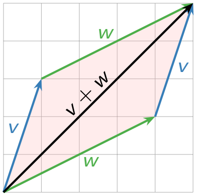

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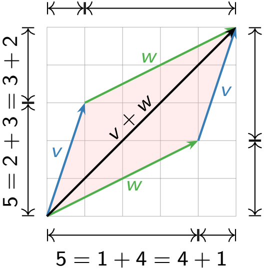

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Why? The width of $v + w$ is the sum of the widths, and likewise with the heights.

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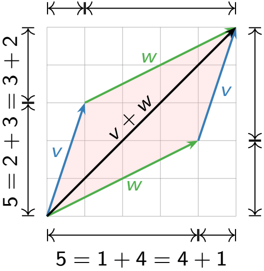

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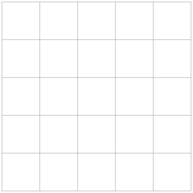
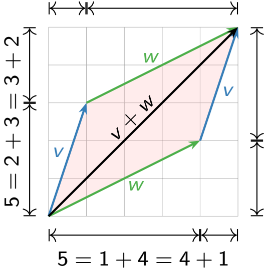


Vector subtraction



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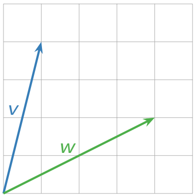

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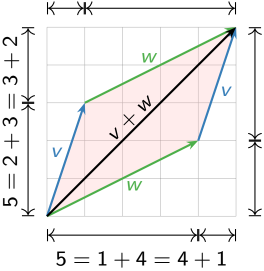

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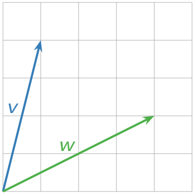

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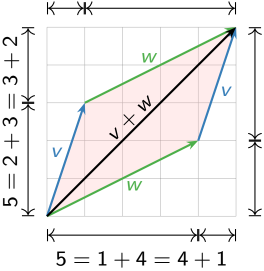

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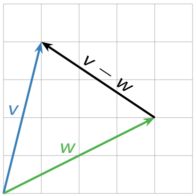

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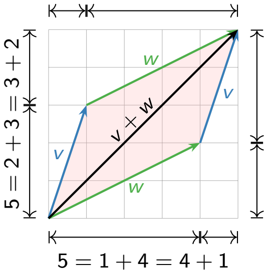

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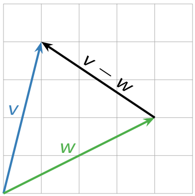

Vector subtraction

Geometrically, the difference of two vectors v, w is obtained as follows: place the tail of v and w at the same point. Then $v - w$ is the vector from the head of v to the head of w. For example,

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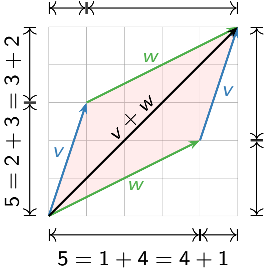
Vector Addition and Subtraction: Geometry



The parallelogram law for vector addition

Geometrically, the sum of two vectors v, w is obtained as follows: place the tail of w at the head of v. Then $v + w$ is the vector whose tail is the tail of v and whose head is the head of w. Doing this both ways creates a parallelogram. For example,

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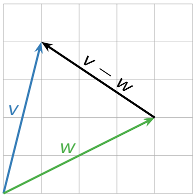

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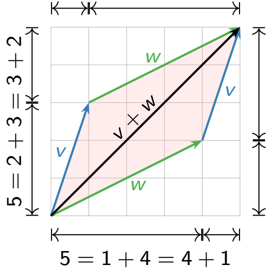

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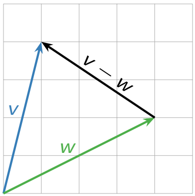

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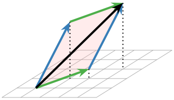
This works in higher dimensions too!

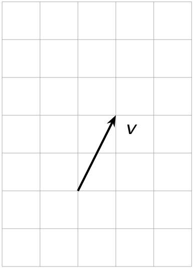


Scalar Multiplication: Geometry

Scalar multiples of a vector

These have the same *direction* but a different *length*.

Some multiples of v.



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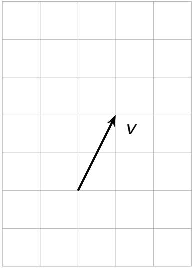

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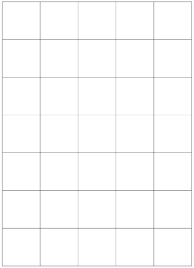

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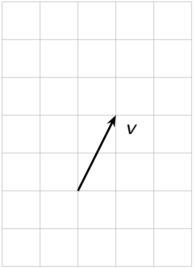

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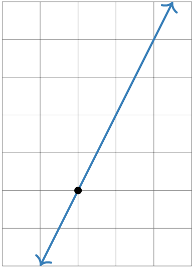

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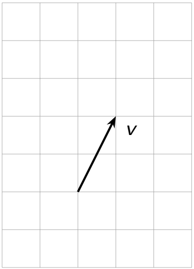

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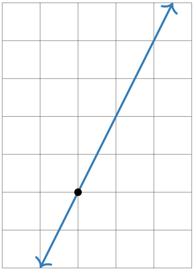

So the scalar multiples of v form a *line*.

Linear Combinations

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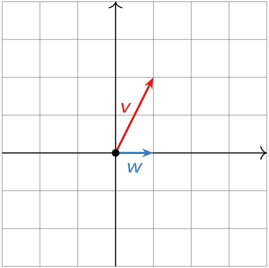
Example



Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of v and w?

- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

Poll

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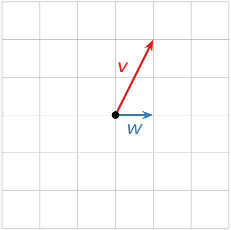
Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?



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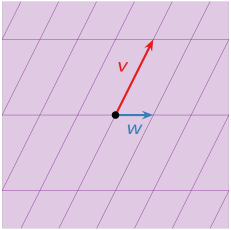

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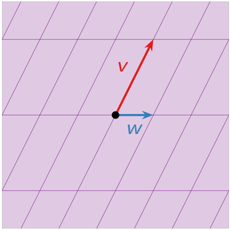
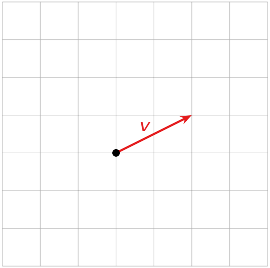


(The purple lines are to help measure *how much* of v and w you need to get to a given point.)

More Examples



What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

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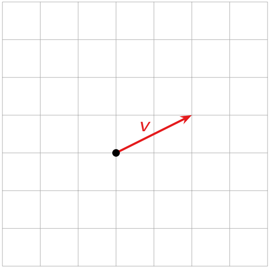

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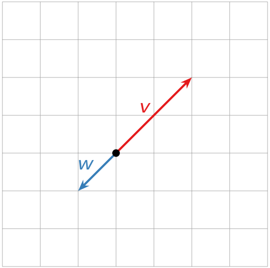

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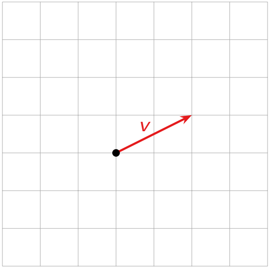

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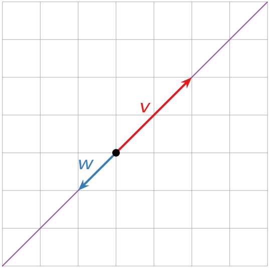

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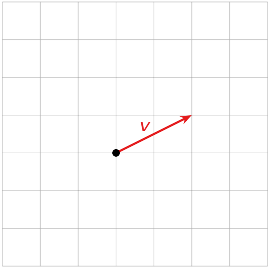

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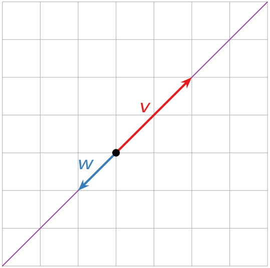

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What's different about this example and the one on the poll?

Question

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

Systems of Linear Equations

Continued

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3 \end{aligned}$$

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Shortcut: You can make the augmented matrix without writing down the system of linear equations first.

Summary

The vector equation

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where $v_1, v_2, \ldots, v_p, b$ are vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_p$ are scalars,

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$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array}\right),$$

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The last one is more geometric in nature.

Span

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Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$.

It is important to know what are *all* linear combinations of a set of vectors $v_1, v_2, \ldots, v_p$ in $\mathbf{R}^n$: it's exactly the collection of all b in $\mathbf{R}^n$ such that the vector equation (in the unknowns $x_1, x_2, \ldots, x_p$)

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This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

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Span
Continued

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Note: **equivalent** means that, for any given list of vectors $v_1, v_2, \ldots, v_p, b$, *either* all three statements are true, *or* all three statements are false.

Pictures of Span

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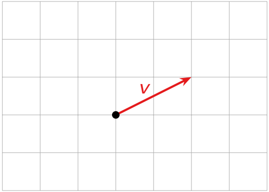

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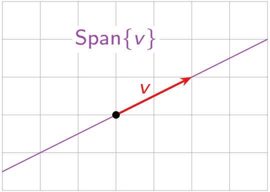

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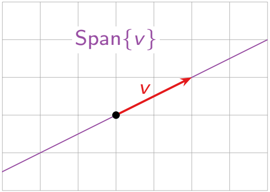

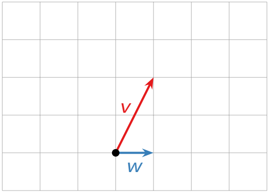

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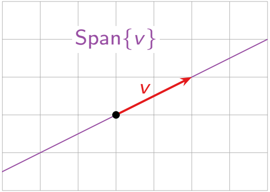

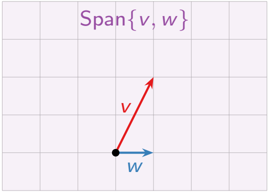

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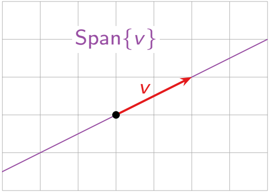

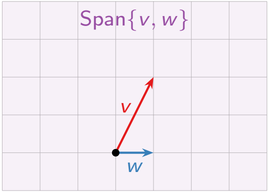

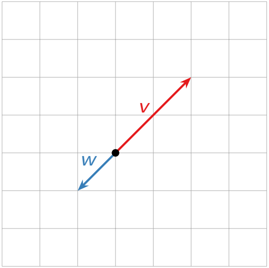

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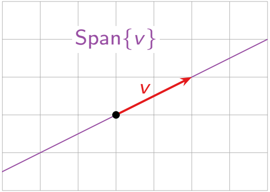

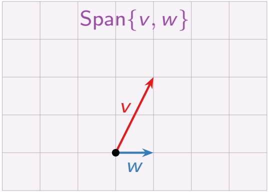

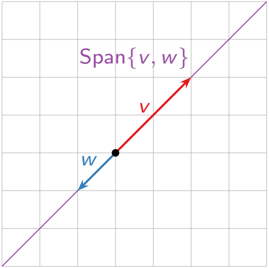

Pictures of Span

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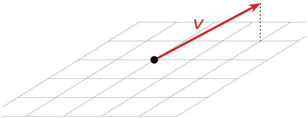

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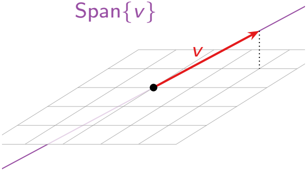

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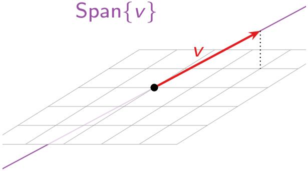

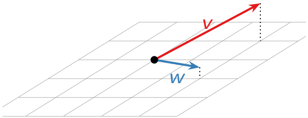

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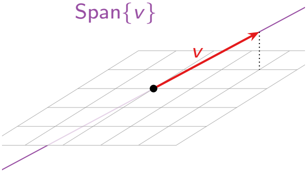

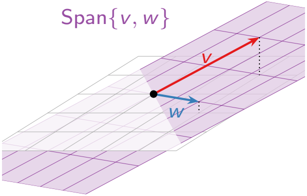

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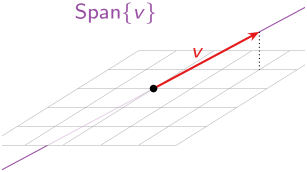

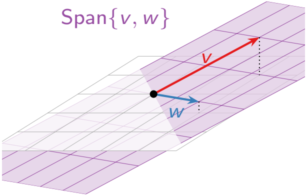

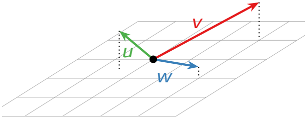

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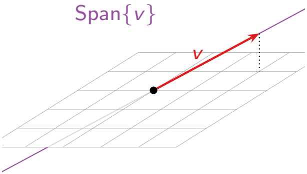

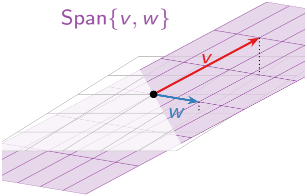

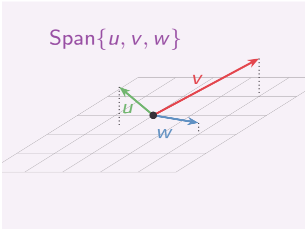

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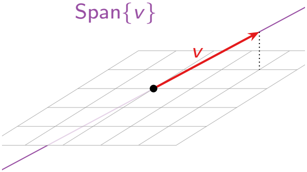

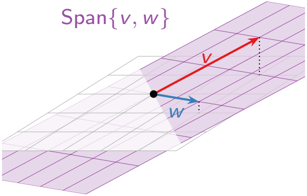

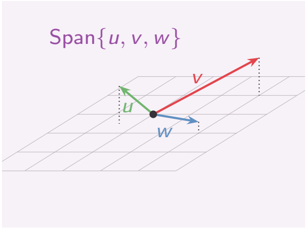

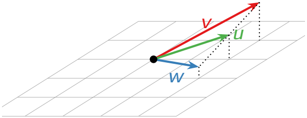

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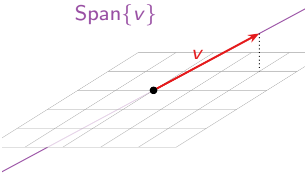

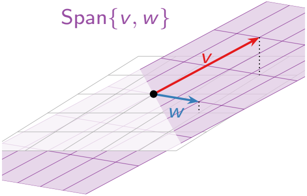

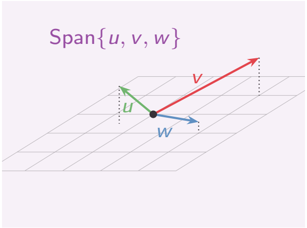

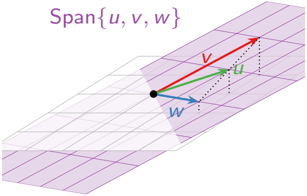

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A. Zero

B. One

C. Infinity

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We will make this precise later.