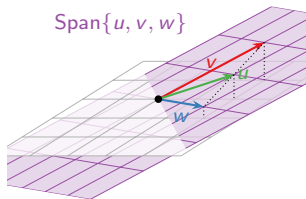
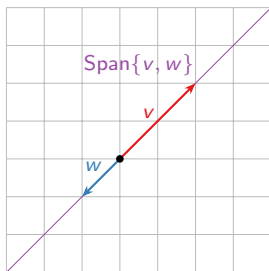


Section 1.7

Linear Independence

Motivation

Sometimes the span of a set of vectors is “smaller” than you expect from the number of vectors.



This can mean many things. For example, it can mean you're using too many vectors to write your solution set.

Notice in each case that one vector in the set is already in the span of the others—so it doesn't make the span bigger.

Today we will formalize this idea in the concept of *linear (in)dependence*.

Linear Independence

Definition

A set of vectors $\{v_1, v_2, \dots, v_p\}$ in \mathbf{R}^n is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \dots = x_p = 0$. The set $\{v_1, v_2, \dots, v_p\}$ is **linearly dependent** otherwise.

In other words, $\{v_1, v_2, \dots, v_p\}$ is linearly dependent if there exist numbers x_1, x_2, \dots, x_p , not all equal to zero, such that

$$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = 0.$$

This is called a **linear dependence relation**.

Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

Linear Independence

Definition

A set of vectors $\{v_1, v_2, \dots, v_p\}$ in \mathbf{R}^n is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \dots = x_p = 0$. The set $\{v_1, v_2, \dots, v_p\}$ is **linearly dependent** otherwise.

Note that linear (in)dependence is a notion that applies to a *collection of vectors*, not to a single vector, or to one vector in the presence of some others.

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

have a nontrivial solution? How do we solve this kind of vector equation?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So $x = -2z$ and $y = -z$. So the vectors are linearly dependent, and an equation of linear dependence is (taking $z = 1$)

$$-2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

have a nontrivial solution?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The trivial solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the unique solution. So the vectors are linearly *independent*.

Linear Independence and Matrix Columns

In general, $\{v_1, v_2, \dots, v_p\}$ is linearly independent if and only if the vector equation

$$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = 0$$

has only the trivial solution, if and only if the matrix equation

$$Ax = 0$$

has only the trivial solution, where A is the matrix with columns v_1, v_2, \dots, v_p :

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & \cdots & | \end{pmatrix}.$$

This is true if and only if the matrix A has a pivot in each column.

Important

- ▶ The vectors v_1, v_2, \dots, v_p are linearly independent if and only if the matrix with columns v_1, v_2, \dots, v_p has a pivot in each column.
- ▶ Solving the matrix equation $Ax = 0$ will either verify that the columns v_1, v_2, \dots, v_p of A are linearly independent, or will produce a linear dependence relation.

Linear Independence

Criterion

Suppose that one of the vectors $\{v_1, v_2, \dots, v_p\}$ is a linear combination of the other ones (that is, it is in the span of the other ones):

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Then the vectors are linearly *dependent*:

$$2v_1 - \frac{1}{2}v_2 - v_3 + 6v_4 = 0.$$

Conversely, if the vectors are linearly dependent

$$2v_1 - \frac{1}{2}v_2 + 6v_4 = 0.$$

then one vector is a linear combination of (in the span of) the other ones:

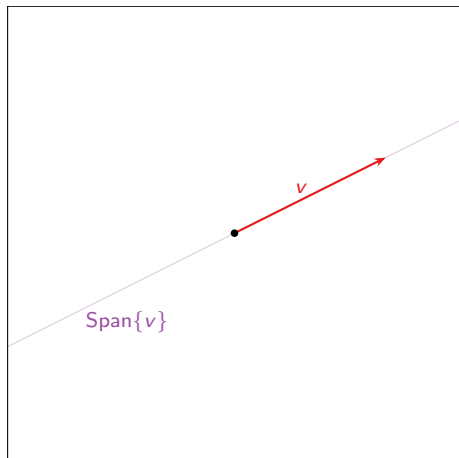
$$v_2 = 4v_1 + 12v_4.$$

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly *dependent* if and only if one of the vectors is in the span of the other ones.

Linear Independence

Pictures in \mathbb{R}^2



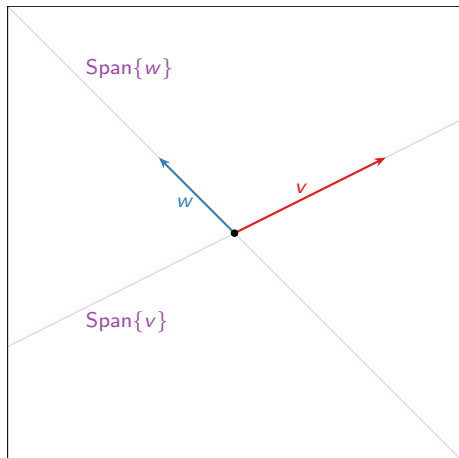
In this picture

One vector $\{v\}$:

Linearly independent if $v \neq 0$.

Linear Independence

Pictures in \mathbb{R}^2



In this picture

One vector $\{v\}$:

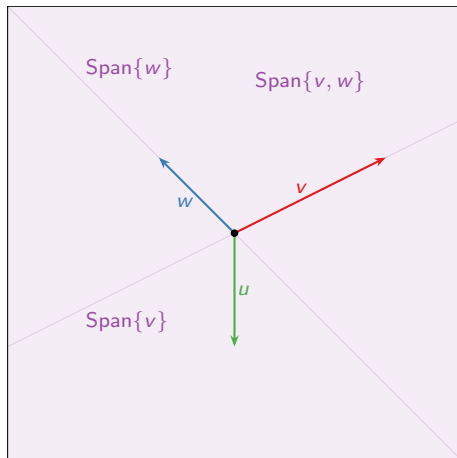
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:

Linearly independent: neither is in the span of the other.

Linear Independence

Pictures in \mathbb{R}^2



In this picture

One vector $\{v\}$:

Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:

Linearly independent: neither is in the span of the other.

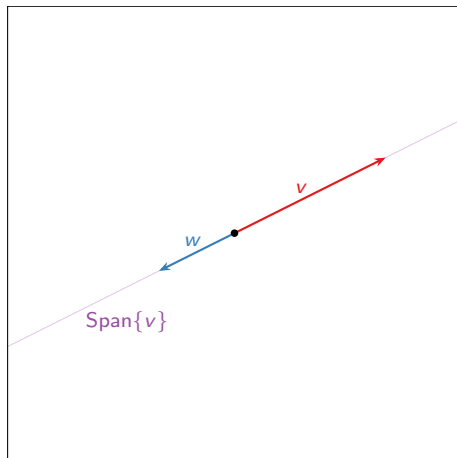
Three vectors $\{v, w, u\}$:

Linearly dependent: u is in $\text{Span}\{v, w\}$.

Also v is in $\text{Span}\{u, w\}$ and w is in $\text{Span}\{u, v\}$.

Linear Independence

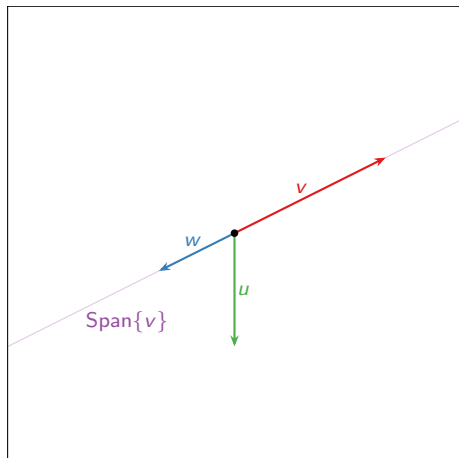
Pictures in \mathbb{R}^2



Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in
 $\text{Span}\{v\}$ (and vice-versa).

Linear Independence

Pictures in \mathbb{R}^2



Two collinear vectors $\{v, w\}$:

Linearly dependent: w is in $\text{Span}\{v\}$ (and vice-versa).

Observe: Two vectors are linearly *dependent* if and only if they are *collinear*.

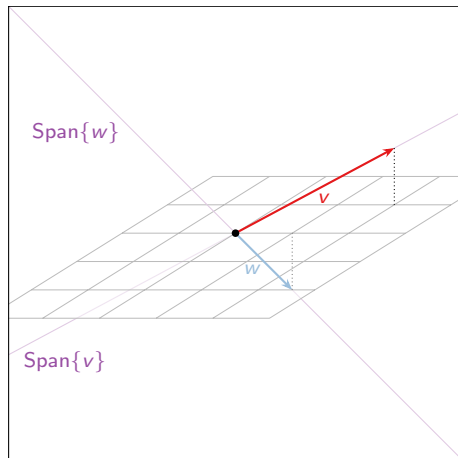
Three vectors $\{v, w, u\}$:

Linearly dependent: w is in $\text{Span}\{v\}$ (and vice-versa).

Observe: If a set of vectors is linearly dependent, then so is any larger set of vectors!

Linear Independence

Pictures in \mathbb{R}^3



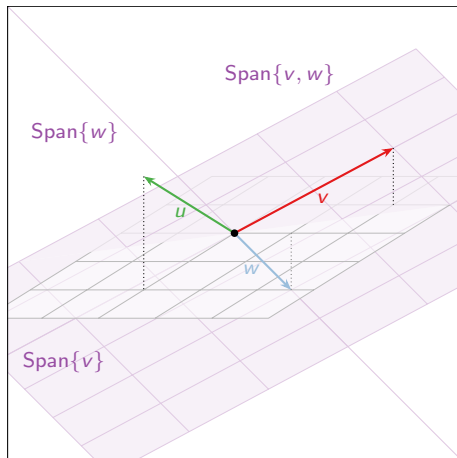
In this picture

Two vectors $\{v, w\}$:

Linearly independent: neither is in the span of the other.

Linear Independence

Pictures in \mathbb{R}^3



In this picture

Two vectors $\{v, w\}$:

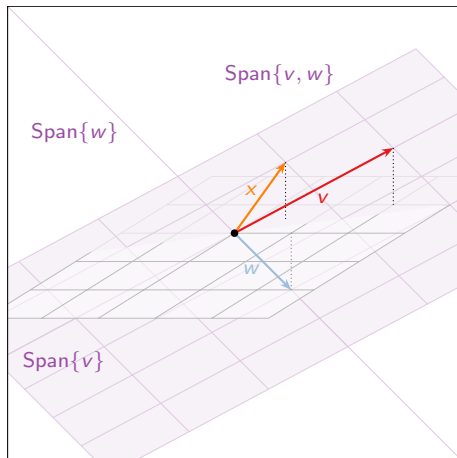
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, u\}$:

Linearly independent: no one is in the span of the other two.

Linear Independence

Pictures in \mathbb{R}^3



In this picture

Two vectors $\{v, w\}$:

Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, x\}$:

Linearly dependent: x is in $\text{Span}\{v, w\}$.

Poll

Are there four vectors u, v, w, x in \mathbf{R}^3 which are linearly dependent, but such that u is *not* a linear combination of v, w, x ? If so, draw a picture; if not, give an argument.

Yes: actually the pictures on the previous slides provide such an example.

Linear dependence of $\{v_1, \dots, v_p\}$ means *some* v_i is a linear combination of the others, not *any*.

Linear Independence

Stronger criterion

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly *dependent* if and only if one of the vectors is in the span of the other ones.

Take the largest j such that v_j is in the span of the others. Then v_j is in the span of v_1, v_2, \dots, v_{j-1} . Why? If not ($j = 3$):

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Rearrange:

$$v_4 = -\frac{1}{6} \left(2v_1 - \frac{1}{2}v_2 - v_3 \right)$$

so v_4 works as well, but v_3 was supposed to be the last one that was in the span of the others.

Better Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly dependent if and only if there is some j such that v_j is in $\text{Span}\{v_1, v_2, \dots, v_{j-1}\}$.

Linear Independence

Increasing span criterion

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly dependent if and only if there is some j such that v_j is in $\text{Span}\{v_1, v_2, \dots, v_{j-1}\}$.

Equivalently, $\{v_1, v_2, \dots, v_p\}$ is linearly *independent* if for every j , the vector v_j is not in $\text{Span}\{v_1, v_2, \dots, v_{j-1}\}$.

This means $\text{Span}\{v_1, v_2, \dots, v_j\}$ is *bigger* than $\text{Span}\{v_1, v_2, \dots, v_{j-1}\}$.

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly independent if and only if, for every j , the span of v_1, v_2, \dots, v_j is strictly larger than the span of v_1, v_2, \dots, v_{j-1} .

Translation

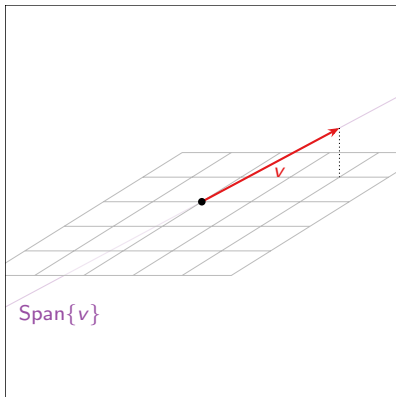
A set of vectors is linearly independent if and only if, every time you add another vector to the set, the span gets bigger.

Linear Independence

Increasing span criterion: pictures

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly independent if and only if, for every j , the span of v_1, v_2, \dots, v_j is strictly larger than the span of v_1, v_2, \dots, v_{j-1} .



One vector $\{v\}$:

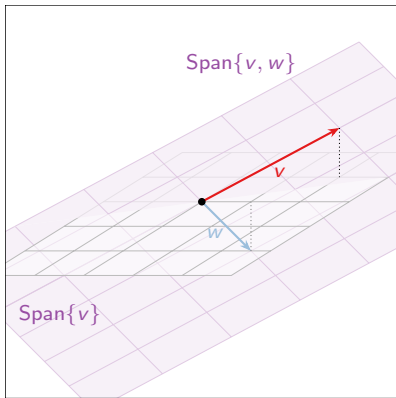
Linearly independent: span got bigger (than $\{0\}$).

Linear Independence

Increasing span criterion: pictures

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly independent if and only if, for every j , the span of v_1, v_2, \dots, v_j is strictly larger than the span of v_1, v_2, \dots, v_{j-1} .



One vector $\{v\}$:

Linearly independent: span got bigger (than $\{0\}$).

Two vectors $\{v, w\}$:

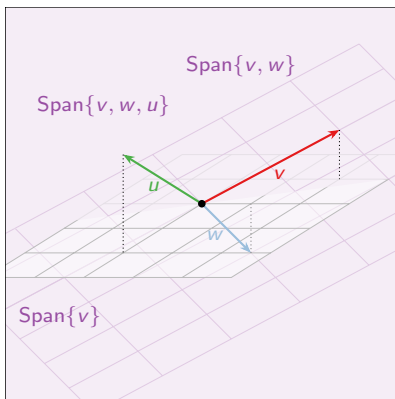
Linearly independent: span got bigger.

Linear Independence

Increasing span criterion: pictures

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly independent if and only if, for every j , the span of v_1, v_2, \dots, v_j is strictly larger than the span of v_1, v_2, \dots, v_{j-1} .



One vector $\{v\}$:

Linearly independent: span got bigger (than $\{0\}$).

Two vectors $\{v, w\}$:

Linearly independent: span got bigger.

Three vectors $\{v, w, u\}$:

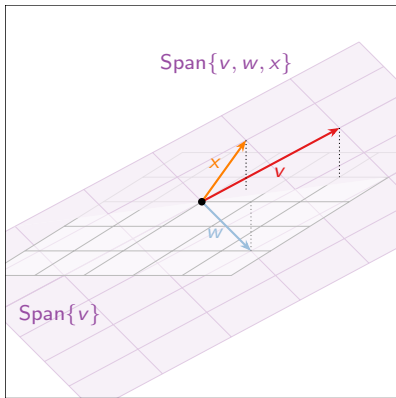
Linearly independent: span got bigger.

Linear Independence

Increasing span criterion: pictures

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly independent if and only if, for every j , the span of v_1, v_2, \dots, v_j is strictly larger than the span of v_1, v_2, \dots, v_{j-1} .



One vector $\{v\}$:

Linearly independent: span got bigger (than $\{0\}$).

Two vectors $\{v, w\}$:

Linearly independent: span got bigger.

Three vectors $\{v, w, x\}$:

Linearly dependent: span didn't get bigger.

Linear Independence

Two more facts

Fact 1: Say v_1, v_2, \dots, v_n are in \mathbf{R}^m . If $n > m$ then $\{v_1, v_2, \dots, v_n\}$ is linearly dependent: the matrix

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}.$$

cannot have a pivot in each column (it is too wide).

This says you can't have 4 linearly independent vectors in \mathbf{R}^3 , for instance.

A wide matrix can't have linearly independent columns.

Fact 2: If one of v_1, v_2, \dots, v_n is zero, then $\{v_1, v_2, \dots, v_n\}$ is linearly dependent. For instance, if $v_1 = 0$, then

$$1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + \cdots + 0 \cdot v_n = 0$$

is a linear dependence relation.

A set containing the zero vector is linearly dependent.