Chapter 3

Determinants

Section 3.1

Introduction to Determinants

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Prof. Margalit's notes are the primary reference for Chapter 3.

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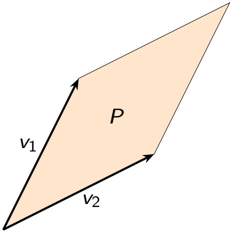

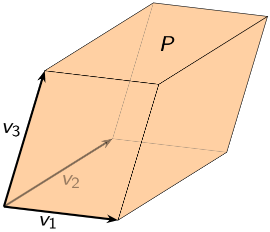

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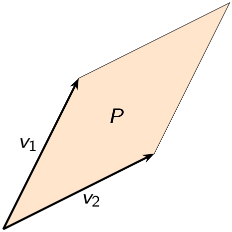

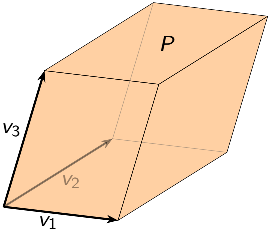

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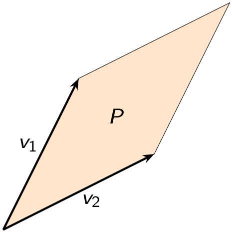

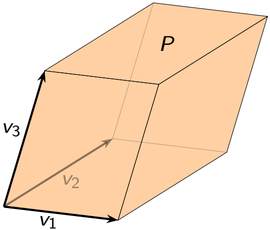

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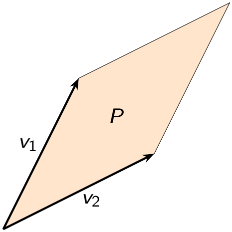

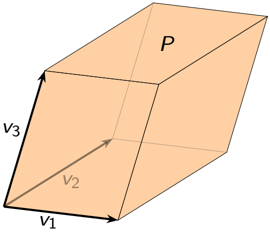

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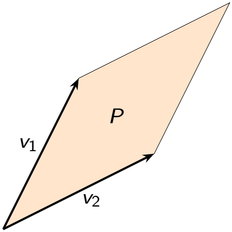

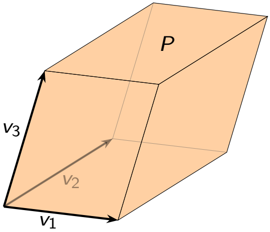

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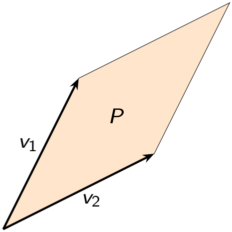

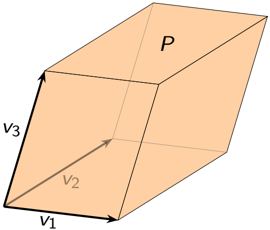

Observation: the volume of P is zero $\iff$ the columns are *linearly dependent* (P is "flat") $\iff$ the matrix A is not invertible.

The **determinant** of A will be a number $\det(A)$ whose absolute value is the volume of P. In particular, $\det(A) \neq 0 \iff A$ is invertible.

We already have a formula in the 2×2 case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Determinants of 2×2 Matrices

Revisited

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Question: What does the sign of the determinant mean?

Determinants of 3×3 Matrices

Here's the formula:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{aligned} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & \quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{aligned}$$

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The signs of the cofactors follow a checkerboard pattern:

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The **determinant** of an $n \times n$ matrix A is

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This formula is called **cofactor expansion along the first row**.

This is the beginning of the recursion.

A Formula for the Determinant

1×1 Matrices

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$$\det(\, a_{11} \,) = a_{11}.$$

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2×2 Matrices

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$$A_{21} = \qquad\qquad A_{22} =$$

The cofactors are

$$C_{11} = \qquad\qquad C_{12} =$$

$$C_{21} = \qquad\qquad C_{22} =$$

The determinant is

$$\det A = a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} - a_{12}a_{21}.$$

A Formula for the Determinant

3×3 Matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

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The determinant is the same formula as before (as it turns out):

$$\begin{aligned} \det A &= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} \\ &= a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \end{aligned}$$

A Formula for the Determinant

Example

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} =$$

$2n - 1$ More Formulas for the Determinant

Recall: the formula

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

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Try this with a row or a column with a lot of zeros.

Cofactor Expansion
Example

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Cofactor Expansion

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It looks easiest to expand along the third column:

$$\det A =$$

Poll

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$$\det \begin{pmatrix} 1 & 7 & -5 & 14 & 3 & 22 \\ 0 & -2 & -3 & 13 & 11 & 1 \\ 0 & 0 & -1 & -9 & 7 & 18 \\ 0 & 0 & 0 & 3 & 6 & -8 \\ 0 & 0 & 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} = ?$$

A. -6 B. -3 C. -2 D. -1 E. 1 F. 2 G. 3 H. 6

Poll

Poll

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If you expand repeatedly along the first column, you get

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The Determinant of an Upper-Triangular Matrix

The computation in the poll works for any matrix that is *upper-triangular* (all entries below the main diagonal are zero).

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The same is true for lower-triangular matrices. (Repeatedly expand along the first row.)

A Formula for the Inverse

For fun—from §3.3

For 2×2 matrices we had a nice formula for the inverse:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

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The proof uses Cramer's rule. See Dan Margalit's notes on the website for a nice explanation.

Compute A^{-1}, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

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Example, continued

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