Section 6.3

Orthogonal Projections

Idea Behind Orthogonal Projections

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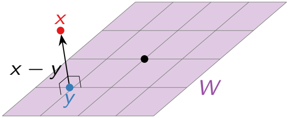

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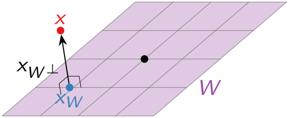

Reformulation: Every vector x can be decompsed uniquely as

$$x = x_W + x_{W^{\perp}}$$

where $x_W = y$ is the closest vector to x in W, and $x_{W^{\perp}} = x - y$ is in $W^{\perp}$.

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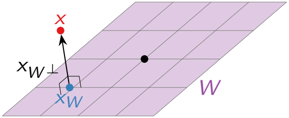

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Example: Let $u = \binom{3}{2}$ and let $L = \text{Span}\{u\}$. Let $x = \binom{-6}{4}$.

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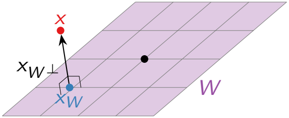

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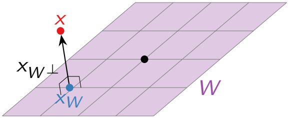

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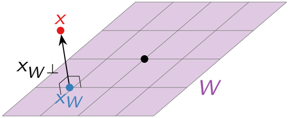

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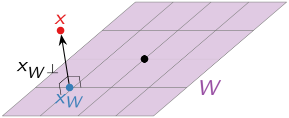

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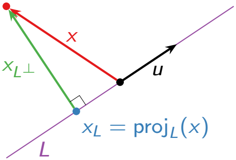
$$x_L = \mathrm{proj}_L(x) = -\frac{10}{13}\begin{pmatrix}3\\2\end{pmatrix} \qquad x_{L^\perp} = x - \mathrm{proj}_L(x) = \begin{pmatrix}-6\\4\end{pmatrix} + \frac{10}{13}\begin{pmatrix}3\\2\end{pmatrix}.$$



Orthogonal Projections

Definition

Let W be a subspace of $\mathbf{R}^n$, and let $\{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

$$\operatorname{proj}_W(x) \stackrel{\text{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i.$$

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Theorem

Let W be a subspace of $\mathbf{R}^n$, and let x be a vector in $\mathbf{R}^n$. Then $\mathrm{proj}_W(x)$ is the closest point to x in W. Therefore

$$\boxed{x_W = \mathrm{proj}_W(x) \qquad x_{W^\perp} = x - \mathrm{proj}_W(x).}$$

Orthogonal Projections

Easy example

What is the projection of $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ onto the xy-plane?

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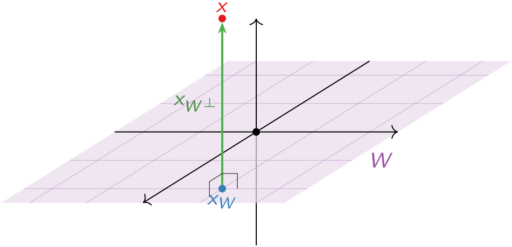
So this is the same projection as before.



What is the projection of $x = \begin{pmatrix} -1.1 \\ 1.4 \\ 1.45 \end{pmatrix}$ onto $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.1 \\ -.2 \end{pmatrix} \right\}$?

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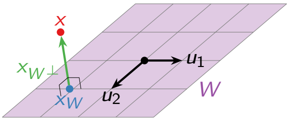
More complicated example

What is the projection of $x = \begin{pmatrix} -1.1 \\ 1.4 \\ 1.45 \end{pmatrix}$ onto $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.1 \\ -.2 \end{pmatrix} \right\}$?



Let W be a subspace of $\mathbf{R}^n$, and let $\{u_1, u_2, \ldots, u_m\}$ be an orthogonal basis for W. Let $L_i = \mathrm{Span}\{u_i\}$. Then

$$\mathrm{proj}_W(x) = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \sum_{i=1}^{m} \mathrm{proj}_{L_i}(x).$$

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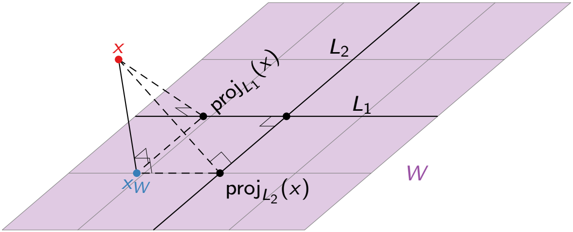
So the orthogonal projection is formed by adding orthogonal projections onto perpendicular lines.



First we restate the property we've been using all along.

Best Approximation Theorem

Let W be a subspace of $\mathbf{R}^n$, and let x be a vector in $\mathbf{R}^n$. Then $y = \mathrm{proj}_W(x)$ is the closest point in W to x, in the sense that

$$\mathrm{dist}(x, y') \geq \mathrm{dist}(x, y) \quad \text{for all} \quad y' \text{ in } W.$$

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We can think of orthogonal projection as a *transformation*:

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4. The range of proj_W is W.

Poll

Let W be a subspace of $\mathbf{R}^n$.

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Let A be the matrix for proj_W. What is/are the eigenvalue(s) of A?

A. 0 B. 1 C. -1 D. $0, 1$ E. $1, -1$ F. $0, -1$ G. $-1, 0, 1$

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We have $\dim W + \dim W^\perp = n$, so that gives n linearly independent eigenvectors already.

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So the answer is D.

What is the matrix for $\mathrm{proj}_W \colon \mathbf{R}^3 \to \mathbf{R}^3$, where

$$W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}?$$

Let W be an m-dimensional subspace of $\mathbf{R}^n$, let $\mathrm{proj}_W \colon \mathbf{R}^n \to W$ be the projection, and let A be the matrix for proj_L.

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Fact 1: A is diagonalizable with eigenvalues 0 and 1; it is similar to the diagonal matrix with m ones and $n - m$ zeros on the diagonal.

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Fact 1: A is diagonalizable with eigenvalues 0 and 1; it is similar to the diagonal matrix with m ones and $n-m$ zeros on the diagonal.

Fact 2: $A^2 = A$.

Orthogonal Projections

Minimum distance

What is the distance from e_1 to $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$?

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