Review for Midterm 2

Selected Topics

Method 1: Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2 \ldots, v_p$:

$$B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

The **product** AB is the $m \times p$ matrix with columns $Av_1, Av_2, \ldots, Av_p$:

$$AB \overset{\text{def}}{=} \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & & | \end{pmatrix}.$$

Method 2: The ij entry of $C = AB$ is the ith row of A times the jth column of B:

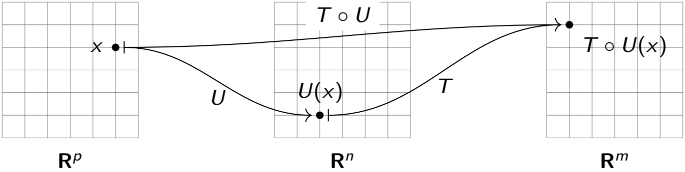
$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \begin{matrix} \\ \\ i\text{th row} \\ \\ \\ \end{matrix} \cdot \underset{j\text{th column}}{\begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix}} = \underset{ij \text{ entry}}{\begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}}$$

Matrix Multiplication/Inversion and Linear Transformations

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ and $U\colon \mathbf{R}^p \to \mathbf{R}^n$ be linear transformations with matrices A and B. The **composition** is the linear transformation

$$T \circ U\colon \mathbf{R}^p \to \mathbf{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).$$



Fact: The matrix for $T \circ U$ is AB.

Now let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be an *invertible* linear transformation. This means there is a linear transformation $T^{-1}\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $T \circ T^{-1}(x) = x$ for all x in $\mathbf{R}^n$. Equivalently, it means T is one-to-one and onto.

Fact: If A is the matrix for T, then A^{-1} is the matrix for T^{-1}.

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ scale the x-axis by 2, and let $U\colon \mathbf{R}^2 \to \mathbf{R}^2$ be counterclockwise rotation by 90°.

Their matrices are:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The composition $T \circ U$ is: first rotate counterclockwise by 90°, then scale the x-axis by 2. The matrix for $T \circ U$ is

$$AB = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

The inverse of U rotates *clockwise* by 90°. The matrix for U^{-1} is

$$B^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Matrix Inverses

The **inverse** of an $n \times n$ matrix A is a matrix A^{-1} such that $AA^{-1} = I_n$ (equivalently, $A^{-1}A = I_n$).

2×2 case:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Longrightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$n \times n$ case: Row reduce the augmented matrix $(\, A \mid I_n \,)$. If you get $(\, I_n \mid B \,)$, then $B = A^{-1}$. Otherwise, A is not invertible.

Solving linear systems by "dividing by A": If A is invertible, then

$$Ax = b \iff x = A^{-1}b.$$

Important

If A is invertible, then $Ax = b$ has exactly one solution for any b, namely, $x = A^{-1}b$.

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Example

Solve $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Answer:

$$x = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{2 \cdot 3 - 1 \cdot 1} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3b_1 - b_2 \\ -b_1 + 2b_2 \end{pmatrix}$$

Elementary Matrices

Definition

An **elementary matrix** is a square matrix E which differs from I_n by one row operation.

There are three kinds:

scaling	row replacement	swap
$(R_2 = 2R_2)$	$(R_2 = R_2 + 2R_1)$	$(R_1 \longleftrightarrow R_2)$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Fact: if E is the elementary matrix for a row operation, then EA differs from A by the same row operation.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \end{pmatrix} \rightsquigarrow B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 4 \end{pmatrix}$$

You get B by subtracting $2\times$ the first row of A from the second row.

$$B = EA \quad \text{where} \quad E = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad \begin{pmatrix} \text{subtract } 2\times \text{ the first row} \\ \text{of } I_2 \text{ from the second row} \end{pmatrix}.$$

The Inverse of an Elementary Matrix

Fact: the inverse of an elementary matrix E is the elementary matrix obtained by doing the opposite row operation to I_n.

$$\overset{R_2 = R_2 \times 2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}} = \overset{R_2 = R_2 \div 2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \qquad \overset{R_2 = R_2 + 2R_1}{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}} = \overset{R_2 = R_2 - 2R_1}{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\overset{R_1 \longleftrightarrow R_2}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}} = \overset{R_1 \longleftrightarrow R_2}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

If A is invertible, then there are a sequence of row operations taking A to I_n:

$$E_r E_{r-1} \cdots E_2 E_1 A = I_n.$$

Taking inverses (note the order!):

$$A = E_1^{-1} E_2^{-1} \cdots E_r^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_r^{-1}.$$

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.
14. The columns of A form a basis for $\mathbf{R}^n$.
15. $\operatorname{Col} A = \mathbf{R}^n$.
16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
18. $\operatorname{Nul} A = \{0\}$.
19. $\dim \operatorname{Nul} A = 0$.

Learn it!

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Examples:

- Any $\text{Span}\{v_1, v_2, \ldots, v_m\}$.
- The *column space* of a matrix: $\text{Col}\, A = \text{Span}\{\text{columns of } A\}$.
- The *null space* of a matrix: $\text{Nul}\, A = \{x \mid Ax = 0\}$.
- $\mathbf{R}^n$ and $\{0\}$

If V can be written in any of the above ways, then it is automatically a subspace: you're done!

Example

Is $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + y = 0 \right\}$ a subspace?

1. Since $0 + 0 = 0$, the zero vector is in V.
2. Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ be arbitrary vectors in V.
   - This means $x + y = 0$ and $x' + y' = 0$.
   - We have to check if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ z + z' \end{pmatrix}$ is in V.
   - This means $(x + x') + (y + y') = 0$.

   Indeed:

   $$(x + x') + (y + y') = (x + y) + (x' + y') = 0 + 0 = 0,$$

   so condition (2) holds.

Example

Is $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + y = 0 \right\}$ a subspace?

3. Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be in V and let c be a scalar.
   - This means $x + y = 0$.
   - We have to check if $c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix}$ is in V.
   - This means $cx + cy = 0$.

   Indeed:
   $$cx + cy = c(x + y) = c \cdot 0 = 0.$$
   So condition (3) holds.

Since conditions (1), (2), and (3) hold, V is a subspace.

Example

Is $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; \sin(x) = 0 \right\}$ a subspace?

1. Since $\sin(0) = 0$, the zero vector is in V.
3. Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be in V and let c be a scalar.
   - This means $\sin(x) = 0$.
   - We have to check if $c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix}$ is in V.
   - This means $\sin(cx) = 0$.

   This is not true in general: take $x = \pi$ and $c = \frac{1}{2}$. Then $\sin(cx) = \sin(\pi/2) = 1$. So $\begin{pmatrix} \pi \\ 0 \\ 0 \end{pmatrix}$ is in V but $\dfrac{1}{2}\begin{pmatrix} \pi \\ 0 \\ 0 \end{pmatrix}$ is not.

Since condition (3) fails, V is not a subspace.

Basis of a Subspace

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

> To check that $\mathcal{B}$ is a basis for V, you have to check two things:
> 1. $\mathcal{B}$ spans V.
> 2. $\mathcal{B}$ is linearly independent.
>
> This is what it means to justify the statement "$\mathcal{B}$ is a basis for V."

Basis Theorem

Let V be a subspace of dimension m. Then:

- Any m linearly independent vectors in V form a basis for V.
- Any m vectors that span V form a basis for V.

So if you *already know the dimension* of V, you only have to check *one*.

Verify that $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + y = 0 \right\}$.

0. In V: both are in V because $1 + (-1) = 0$ and $0 + 0 = 0$.
1. Span: If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in V, then $y = -x$, so we can write it as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -x \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

2. Linearly independent:

$$x \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \implies \begin{pmatrix} x \\ -x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies x = y = 0.$$

If we knew a priori that $\dim V = 2$, then we would only have to check 0, then 1 *or* 2.

Bases of Col A and Nul A

$$A = \begin{pmatrix} \boxed{\begin{matrix}1\\-2\\2\end{matrix}} & \boxed{\begin{matrix}2\\-3\\4\end{matrix}} & \begin{matrix}0\\4\\0\end{matrix} & \begin{matrix}-1\\5\\-2\end{matrix} \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis $\longleftarrow$ pivot columns in rref

So a basis for Col A is $\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}$. A vector in Col A : $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

Parametric vector form for solutions to $Ax = 0$:

$$x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\substack{\text{basis of} \\ \text{Nul}\,A}} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

A vector in Nul A: any solution to $Ax = 0$, e.g., $x = \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}$.

Rank Theorem

Rank Theorem

If A is an $m \times n$ matrix, then

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n = \text{the number of columns of } A.$$

$$A = \begin{pmatrix} \underbrace{\boxed{\begin{matrix} 1 \\ -2 \\ 2 \end{matrix}} \; \boxed{\begin{matrix} 2 \\ -3 \\ 4 \end{matrix}}}_{\text{basis of Col } A} & \begin{matrix} 0 \\ 4 \\ 0 \end{matrix} & \begin{matrix} -1 \\ 5 \\ -2 \end{matrix} \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \underbrace{\boxed{\begin{matrix} -8 \\ 4 \\ 0 \end{matrix}} \; \boxed{\begin{matrix} -7 \\ 3 \\ 0 \end{matrix}}}_{\text{free variables}} \end{pmatrix}$$

In this case, $\operatorname{rank} A = 2$ and $\dim \operatorname{Nul} A = 2$, and $2 + 2 = 4$, which is the number of columns of A.

1. Special formulas for 2×2 and 3×3 matrices.
2. For [upper or lower] triangular matrices:

$$\det A = (\text{product of diagonal entries}).$$

3. Cofactor expansion along any row or column:

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \text{ for any fixed } i$$

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} \text{ for any fixed } j$$

   Start here for matrices with a row or column with lots of zeros.

4. By row reduction without scaling:

$$\det(A) = (-1)^{\#\text{swaps}} \big(\text{product of diagonal entries in REF}\big)$$

   This is fastest for big and complicated matrices.

5. Cofactor expansion and any other of the above. (The cofactor formula is recursive.)

Definition

The **determinant** is a function

$$\det\colon \{\text{square matrices}\} \longrightarrow \mathbf{R}$$

with the following **defining properties**:

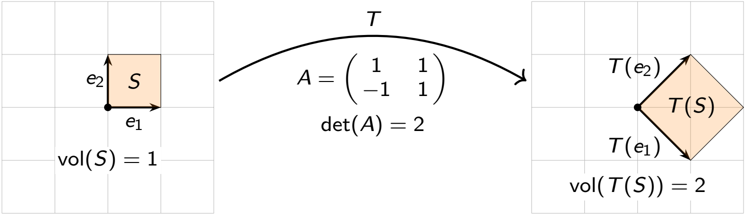
1. $\det(I_n) = 1$
2. If we do a row replacement on a matrix (add a multiple of one row to another), the determinant does not change.
3. If we swap two rows of a matrix, the determinant scales by -1.
4. If we scale a row of a matrix by k, the determinant scales by k.

When computing a determinant via row reduction, try to only use *row replacement* and *row swaps*. Then you never have to worry about scaling by the inverse.

1. There is one and only one function $\det\colon \{\text{square matrices}\} \to \mathbf{R}$ satisfying the defining properties (1)–(4).
2. A is invertible if and only if $\det(A) \neq 0$.
3. If we row reduce A without row scaling, then
$$\det(A) = (-1)^{\#\text{swaps}}\bigl(\text{product of diagonal entries in REF}\bigr).$$
4. The determinant can be computed using any of the $2n$ cofactor expansions.
5. $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \det(A)^{-1}$.
6. $\det(A) = \det(A^T)$.
7. $|\det(A)|$ is the volume of the parallelepiped defined by the columns of A.
8. If A is an $n \times n$ matrix with transformation $T(x) = Ax$, and S is a subset of $\mathbf{R}^n$, then the volume of $T(S)$ is $|\det(A)|$ times the volume of S. (Even for curvy shapes S.)
9. The determinant is multi-linear.

Determinants and Linear Transformations

Why is Property 8 true? For instance, if S is the unit cube, then $T(S)$ is the parallelepiped defined by the columns of A, since the columns of A are $T(e_1), T(e_2), \ldots, T(e_n)$. In this case, Property 8 is the same as Property 7.



For curvy shapes, you break S up into a bunch of tiny cubes. Each one is scaled by $|\det(A)|$; then you use *calculus* to reduce to the previous situation!