Chapter 3

Determinants

Section 3.1

Introduction to Determinants

Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$
  We've said most of what we'll say about this topic now.
- Solve the matrix equation $Ax = \lambda x$ (eigenvalue problem)
  We are now aiming at this.
- Almost solve the equation $Ax = b$
  This will happen later.

The next topic is *determinants*.

Dan Margalit has written some notes which, in my opinion, explain the topic in a much better way than Lay does. (Both cover the same material.)

Prof. Margalit's notes are the primary reference for Chapter 3.

Let A be an $n \times n$ matrix. Determinants are only for square matrices.

The columns $v_1, v_2, \ldots, v_n$ give you n vectors in $\mathbf{R}^n$. These determine a **parallelepiped** P.
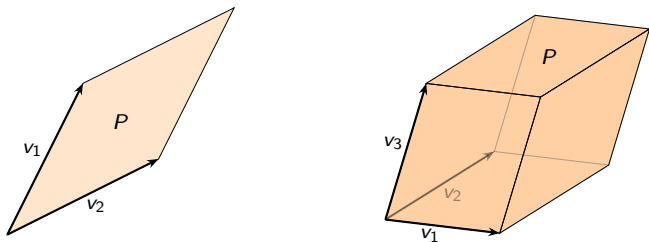
Observation: the volume of P is zero $\iff$ the columns are *linearly dependent* (P is "flat") $\iff$ the matrix A is not invertible.

The **determinant** of A will be a number $\det(A)$ whose absolute value is the volume of P. In particular, $\det(A) \neq 0 \iff A$ is invertible.
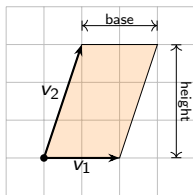
We already have a formula in the 2 × 2 case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

What does this have to do with volumes?

$$v_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The area of the parallelogram is

$$\text{base} \times \text{height} = 2 \cdot 3 = \left| \det \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \right|.$$

The area of the parallelogram is always $|ad - bc|$. If v_1 is not on the x-axis: it's a fun geometry problem!

Note: this shows $\det(A) \neq 0 \iff A$ is invertible in this case. (The volume is zero if and only if the columns are collinear.)

Question: What does the sign of the determinant mean?

Here's the formula:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ \quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{array}$$

How on earth do you remember this? Draw a bigger matrix, repeating the first two columns to the right:

$$+ \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$$

Then add the products of the downward diagonals, and subtract the product of the upward diagonals. For example,

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = \begin{vmatrix} 5 & 1 & 0 & 5 & 1 \\ -1 & 3 & 2 & -1 & 3 \\ 4 & 0 & -1 & 4 & 0 \end{vmatrix} = -15 + 8 + 0 - 0 - 0 - 1 = -8$$

What does this have to do with volumes? Next time.

A Formula for the Determinant

When $n \geq 4$, the determinant isn't just a sum of products of diagonals. The formula is *recursive*: you compute a larger determinant in terms of smaller ones.

First some notation. Let A be an $n \times n$ matrix.

$A_{ij} = ij$th **minor** of A

$\quad = (n-1) \times (n-1)$ matrix you get by deleting the ith row and jth column

$C_{ij} = (-1)^{i+j} \det A_{ij}$

$\quad = ij$th **cofactor** of A

The signs of the cofactors follow a checkerboard pattern:

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix} \qquad \pm \text{ in the } ij \text{ entry is the sign of } C_{ij}$$

Definition

The **determinant** of an $n \times n$ matrix A is

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

This formula is called **cofactor expansion along the first row**.

A Formula for the Determinant

This is the beginning of the recursion.

$$\det(\, a_{11}\,) = a_{11}.$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The minors are:

$$A_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} \end{pmatrix} \qquad A_{12} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{21} \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{12} \end{pmatrix} \qquad A_{22} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} \end{pmatrix}$$

The cofactors are

$$C_{11} = +\det A_{11} = a_{22} \qquad\qquad C_{12} = -\det A_{12} = -a_{21}$$
$$C_{21} = -\det A_{21} = -a_{12} \qquad\qquad C_{22} = +\det A_{22} = a_{11}$$

The determinant is

$$\det A = a_{11} C_{11} + a_{12} C_{12} = a_{11} a_{22} - a_{12} a_{21}.$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The top row minors and cofactors are:

$$A_{11} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \qquad C_{11} = + \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \qquad C_{12} = - \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \qquad C_{13} = + \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

The determinant is the same formula as before (as it turns out):

$$\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = 5 \cdot \det \begin{pmatrix} 5 & 1 & 0 \\ -1 & -3 & 2 \\ 4 & 0 & -1 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$+ 0 \cdot \det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$= 5 \cdot \det \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix} + 0 \cdot \det \begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix}$$

$$= 5 \cdot (-3 - 0) - 1 \cdot (1 - 8)$$

$$= -15 + 7 = -8$$

Recall: the formula

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

is called **cofactor expansion along the first row.** Actually, you can expand cofactors along any row or column you like!

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{for any fixed } i$$

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{for any fixed } j$$

Try this with a row or a column with a lot of zeros.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

It looks easiest to expand along the third column:

$$\det A = 0 \cdot \det \begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix} - 0 \cdot \det \begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 2 - 1 = 1$$

Poll

$$\det \begin{pmatrix} 1 & 7 & -5 & 14 & 3 & 22 \\ 0 & -2 & -3 & 13 & 11 & 1 \\ 0 & 0 & -1 & -9 & 7 & 18 \\ 0 & 0 & 0 & 3 & 6 & -8 \\ 0 & 0 & 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} = ?$$

A. −6   B. −3   C. −2   D. −1   E. 1   F. 2   G. 3   H. 6

If you expand repeatedly along the first column, you get

$$1 \cdot \det \begin{pmatrix} -2 & -3 & 13 & 11 & 1 \\ 0 & -1 & -9 & 7 & 18 \\ 0 & 0 & 3 & 6 & -8 \\ 0 & 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = 1 \cdot (-2) \cdot \det \begin{pmatrix} -1 & -9 & 7 & -18 \\ 0 & 3 & 6 & -8 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= 1 \cdot (-2) \cdot (-1) \cdot \det \begin{pmatrix} 3 & 6 & -8 \\ 0 & 1 & -11 \\ 0 & 0 & -1 \end{pmatrix} = 1 \cdot (-2) \cdot (-1) \cdot 3 \cdot \det \begin{pmatrix} 1 & -11 \\ 0 & -1 \end{pmatrix}$$

$$= 1 \cdot (-2) \cdot (-1) \cdot 3 \cdot 1 \cdot (-1) = -6$$

The computation in the poll only uses that the matrix is *upper-triangular* (all entries below the main diagonal are zero).

Theorem

The determinant of an upper-triangular matrix is the product of the diagonal entries:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11} a_{22} a_{33} \cdots a_{nn}.$$

The same is true for lower-triangular matrices. (Repeatedly expand along the first row.)

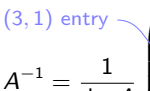For 2×2 matrices we had a nice formula for the inverse:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix}.$$

Theorem

This last formula works for any $n \times n$ invertible matrix A:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \cdots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \cdots & C_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & C_{3n} & \cdots & C_{nn} \end{pmatrix} = \frac{1}{\det A} \left(C_{ij} \right)^{T}$$

$(3, 1)$ entry — C_{13}

Note that the cofactors are "transposed": the (i, j) entry of the matrix is C_{ji}.

The proof uses Cramer's rule. See Dan Margalit's notes on the website for a nice explanation.

Compute A^{-1}, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The minors are:

$$A_{11} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{13} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{23} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$A_{31} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad A_{32} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A_{33} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The cofactors are (don't forget to multiply by $(-1)^{i+j}$):

$$
\begin{array}{lll}
C_{11} = -1 & C_{12} = 1 & C_{13} = -1 \\
C_{21} = 1 & C_{22} = -1 & C_{23} = -1 \\
C_{31} = -1 & C_{32} = -1 & C_{33} = 1
\end{array}
$$

The determinant is (expanding along the first row):

$$\det A = 1 \cdot C_{11} + 0 \cdot C_{12} + 1 \cdot C_{13} = -2$$

Compute A^{-1}, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The inverse is

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot -\frac{1}{2} \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \checkmark$$