Section 6.4

The Gram–Schmidt Process

Motivation

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- Finding the orthogonal projection of a vector x onto the span W of $u_1, u_2, \ldots, u_m$:

$$\mathrm{proj}_W(x) = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i.$$

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Problem: What if your basis isn't orthogonal?

Solution: The Gram–Schmidt process: take any basis and make it orthogonal.

The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace W of $\mathbf{R}^n$. Define:

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3. $u_3 = v_3 - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2\}}(v_3) \qquad = v_3 - \dfrac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \dfrac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2$

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$\vdots$

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Then $\{u_1, u_2, \ldots, u_m\}$ is an *orthogonal* basis for the same subspace W.

The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace W of $\mathbf{R}^n$. Define:

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Then $\{u_1, u_2, \ldots, u_m\}$ is an *orthogonal* basis for the same subspace W.

Remark

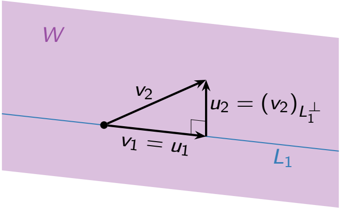
In fact, for every i between 1 and n, the set $\{u_1, u_2, \ldots, u_i\}$ is an orthogonal basis for $\mathrm{Span}\{v_1, v_2, \ldots, v_i\}$.

Find an orthogonal basis $\{u_1, u_2\}$ for $W = \mathrm{Span}\{v_1, v_2\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find an orthogonal basis $\{u_1, u_2\}$ for $W = \mathrm{Span}\{v_1, v_2\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

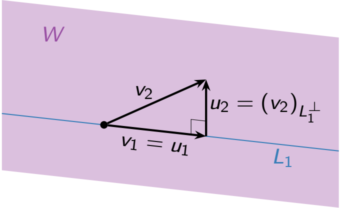


The Gram–Schmidt Process

Two vectors

Find an orthogonal basis $\{u_1, u_2\}$ for $W = \text{Span}\{v_1, v_2\}$, where

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}.$$



Important: $\text{Span}\{u_1, u_2\} = \text{Span}\{v_1, v_2\} = W$: this is an *orthogonal* basis for the *same* subspace.

Find an orthogonal basis $\{u_1, u_2, u_3\}$ for $W = \mathrm{Span}\{v_1, v_2, v_3\} = \mathbf{R}^3$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

The Gram–Schmidt Process

Three vectors

Find an orthogonal basis $\{u_1, u_2, u_3\}$ for $W = \mathrm{Span}\{v_1, v_2, v_3\} = \mathbf{R}^3$, where

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The Gram–Schmidt Process

Three vectors, continued

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \; v_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \; v_3 = \begin{pmatrix}3\\1\\1\end{pmatrix} \xrightarrow{\text{G–S}} u_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \; u_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \; u_3 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

The Gram–Schmidt Process

Three vectors, continued

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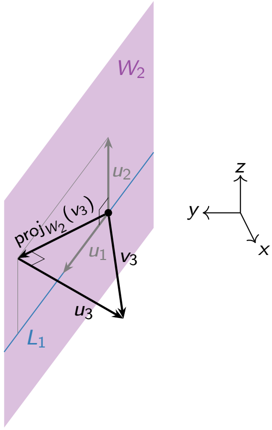

The Gram–Schmidt Process

Three vectors in $\mathbf{R}^4$

Find an orthogonal basis $\{u_1, u_2, u_3\}$ for $W = \mathrm{Span}\{v_1, v_2, v_3\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 4 \\ -2 \\ -2 \\ 0 \end{pmatrix}.$$

Poll

Poll

What happens if you try to run Gram–Schmidt on a linearly *dependent* set of vectors $\{v_1, v_2, \ldots, v_m\}$?

A. You get an inconsistent equation.

B. For some i you get $u_i = u_{i-1}$.

C. For some i you get $u_i = 0$.

D. You create a rift in the space-time continuum.

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If $\{v_1, v_2, \ldots, v_m\}$ is linearly dependent, then some v_i is in $\mathrm{Span}\{v_1, v_2, \ldots, v_{i-1}\} = \mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}$.

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This means

$$v_i = \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}}(v_i)$$
$$\implies u_i = v_i - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}}(v_i) = 0.$$

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In this case, you can simply discard u_i and v_i and continue: so Gram–Schmidt produces an orthogonal basis from any spanning set!

QR Factorization

QR Factorization Theorem

Let A be a matrix with linearly independent columns. Then

$$A = QR$$

where Q has orthonormal columns and R is upper-triangular with positive diagonal entries.

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Recall: A set of vectors $\{v_1, v_2, \ldots, v_m\}$ is **orthonormal** if they are orthogonal unit vectors: $v_i \cdot v_j = 0$ when $i \neq j$, and $v_i \cdot v_i = 1$.

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The columns of A are a basis for $W = \operatorname{Col} A$. The columns of Q come from Gram–Schmidt as applied to the columns of A, after normalizing to unit vectors.

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Here is the procedure for producing a QR factorization.

Find the QR factorization of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

QR Factorization

Example

Find the QR factorization of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

(The columns of A are the vectors v_1, v_2, v_3 from a previous example.)

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Step 1: Run Gram–Schmidt

$$u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1 = v_2 - 1\, u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} u_3 &= v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2 \\ &= v_3 - 2\, u_1 - 1\, u_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \end{aligned}$$

Find the QR factorization of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

(The columns of A are the vectors v_1, v_2, v_3 from a previous example.)

Step 1: Run Gram–Schmidt and solve for v_1, v_2, v_3 in terms of u_1, u_2, u_3.

$$u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_1 = u_1$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1 = v_2 - 1\, u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = u_1 + u_2$$

$$\begin{aligned} u_3 &= v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2 \\ &= v_3 - 2\, u_1 - 1\, u_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \end{aligned} \qquad v_3 = 2u_1 + u_2 + u_3$$

QR Factorization

Example, continued

$$v_1 = 1\,u_1 \qquad v_2 = 1\,u_1 + 1\,u_2 \qquad v_3 = 2\,u_1 + 1\,u_2 + 1\,u_3$$

Step 2: Write $A = \widehat{Q}\widehat{R}$, where $\widehat{Q}$ has *orthogonal* columns u_1, u_2, u_3 and $\widehat{R}$ is upper-triangular with 1s on the diagonal.

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Do this by putting the above equations in matrix form:

$$A \longrightarrow \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \underbrace{\begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix}}_{\widehat{Q}} \underbrace{\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{\widehat{R}}$$

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$$\text{first column of } A = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 u_1 = v_1$$

$$v_1 = 1\,u_1 \qquad v_2 = 1\,u_1 + 1\,u_2 \qquad v_3 = 2\,u_1 + 1\,u_2 + 1\,u_3$$

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QR Factorization

Example, continued

$$v_1 = 1 u_1 \qquad v_2 = 1 u_1 + 1 u_2 \qquad v_3 = 2 u_1 + 1 u_2 + 1 u_3$$

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$$\text{third column of } A = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 2u_1 + 1u_2 + 1u_3 = v_3$$

$$A = \widehat{Q}\widehat{R} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 3: Scale the columns of $\widehat{Q}$ to get unit vectors, and scale the rows of $\widehat{R}$ by the opposite factor, to get Q and R.

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QR Factorization

Example, continued

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The final QR decomposition is:

$$A = QR \qquad Q = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} \sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

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(The columns are vectors from a previous example.)

Step 1: Run Gram–Schmidt and solve for v_1, v_2, v_3 in terms of u_1, u_2, u_3:

$$v_1 = 1\, u_1 \qquad v_2 = \frac{3}{2}\, u_1 + 1\, u_2 \qquad v_3 = 0\, u_1 - \frac{4}{5}\, u_2 + 1\, u_3$$

Step 2: Write $A = \widehat{Q}\widehat{R}$, where $\widehat{Q}$ has *orthogonal* columns u_1, u_2, u_3 and $\widehat{R}$ is upper-triangular with 1s on the diagonal.

$$A = \widehat{Q}\widehat{R} \qquad \widehat{Q} = \begin{pmatrix} 1 & -5/2 & 2 \\ 1 & 5/2 & 0 \\ 1 & 5/2 & 0 \\ 1 & -5/2 & -2 \end{pmatrix} \qquad \widehat{R} = \begin{pmatrix} 1 & 3/2 & 0 \\ 0 & 1 & -4/5 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 3: Normalize the columns of $\widehat{Q}$ and the rows of $\widehat{R}$ to get Q and R:

QR Factorization

Another example, continued

$$A = \widehat{Q}\widehat{R} \qquad \widehat{Q} = \begin{pmatrix} 1 & -5/2 & 2 \\ 1 & 5/2 & 0 \\ 1 & 5/2 & 0 \\ 1 & -5/2 & -2 \end{pmatrix} \qquad \widehat{R} = \begin{pmatrix} 1 & 3/2 & 0 \\ 0 & 1 & -4/5 \\ 0 & 0 & 1 \end{pmatrix}$$

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The final QR decomposition is

$$A = QR \qquad Q = \begin{pmatrix} 1/2 & -1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & -1/\sqrt{2} \end{pmatrix} \qquad R = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 5 & -4 \\ 0 & 0 & 2\sqrt{2} \end{pmatrix}.$$

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In fact, if $v_1, v_2, \ldots, v_n$ are the columns of A, and $u_1, u_2, \ldots, u_n$ are the vectors you obtain by applying Gram–Schmidt, then the (i, i) entry of R is $\|u_i\|$, so

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So you can use Gram–Schmidt to compute determinants (up to sign)!

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Theorem

The matrices A_k converge to an upper triangular matrix, and the diagonal entries converge (quickly!) to the eigenvalues of A.

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Theorem

The matrices A_k converge to an upper triangular matrix, and the diagonal entries converge (quickly!) to the eigenvalues of A.

This gives a computationally efficient way (called the QR algorithm) to find the eigenvalues of a matrix.