Section 2.3

Characterization of Invertible Matrices

Invertible Transformations

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists another transformation $U\colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x$$

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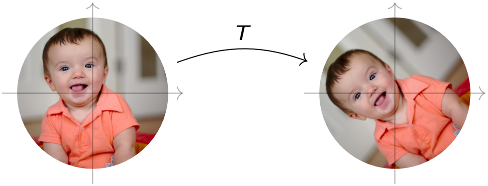
Fact

A transformation T is invertible if and only if it is both one-to-one and onto.

Invertible Transformations

Examples

Let T = counterclockwise rotation in the plane by 45°. What is T^{-1}?



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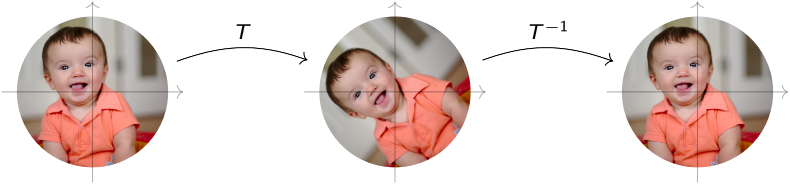

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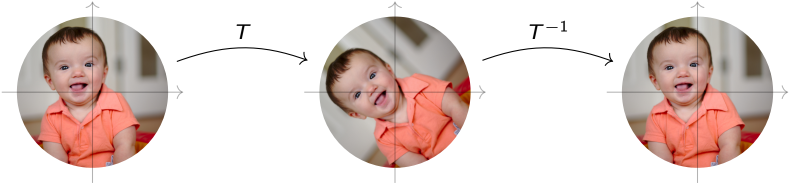

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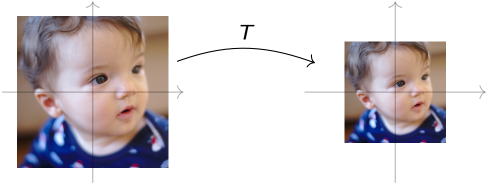
Let T = shrinking by a factor of $2/3$ in the plane. What is T^{-1}?



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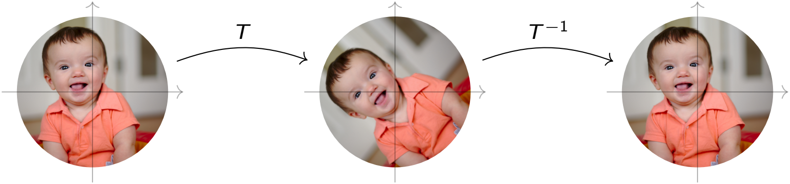

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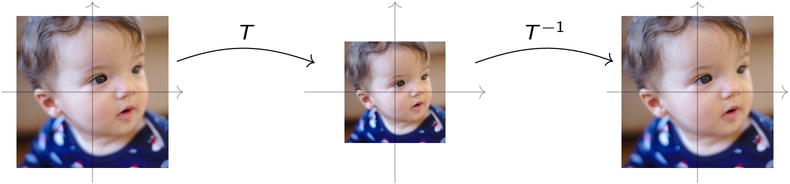

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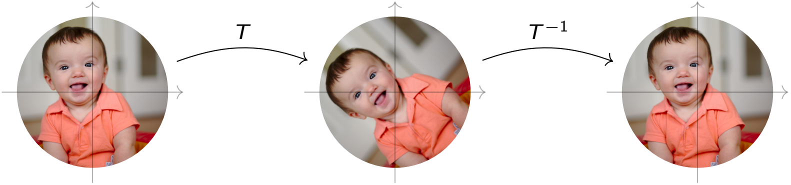

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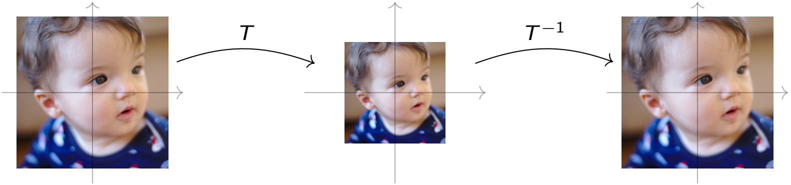

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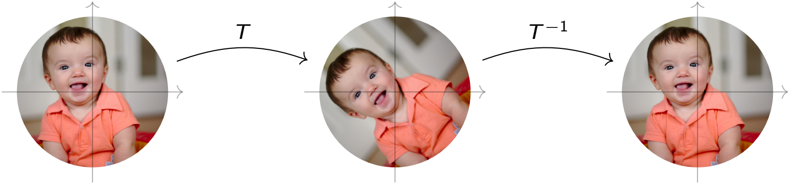

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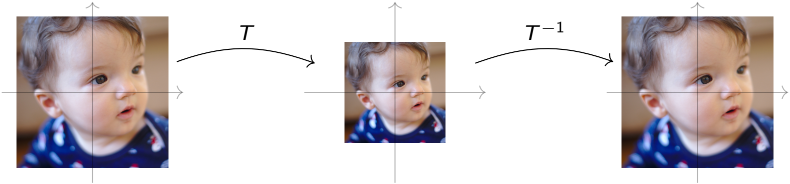

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Let $T =$ projection onto the x-axis. What is T^{-1}? It is not invertible: you can't undo it.

Invertible Linear Transformations

If $T : \mathbf{R}^n \to \mathbf{R}^n$ is an invertible *linear* transformation with matrix A, then what is the matrix for T^{-1}?

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Fact

If T is an invertible linear transformation with matrix A, then T^{-1} is an invertible linear transformation with matrix A^{-1}.

Let $T =$ counterclockwise rotation in the plane by 45°. Its matrix is

Then $T^{-1} =$ counterclockwise rotation by -45°. Its matrix is

Check:

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The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

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A.K.A. The Really Big Theorem of Math 1553

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You know enough at this point to be able to reduce all of the statements to assertions about the pivots of a square matrix.