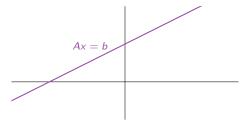
Section 1.5

Solution Sets of Linear Systems

Plan For Today

Today we will learn to describe and draw the solution set of an arbitrary system of linear equations Ax = b, using spans.



Recall: the **solution set** is the collection of all vectors x such that Ax = b is true.

Last time we discussed the set of vectors b for which Ax = b has a solution.

We also described this set using spans, but it was a different problem.

Homogeneous Systems

Everything is easier when b = 0, so we start with this case.

Definition

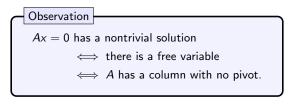
A system of linear equations of the form Ax = 0 is called **homogeneous.**

These are linear equations where everything to the right of the = is zero. The opposite is:

Definition

A system of linear equations of the form Ax = b with $b \neq 0$ is called **nonhomogeneous** or **inhomogeneous**.

A homogeneous system always has the solution x=0. This is called the **trivial solution**. The nonzero solutions are called **nontrivial**.



What is the solution set of Ax = 0, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$
?

We know how to do this: first form an augmented matrix and row reduce.

$$\begin{pmatrix} 1 & 3 & 4 & | & 0 \\ 2 & -1 & 2 & | & 0 \\ 1 & 0 & 1 & | & 0 \end{pmatrix} \quad \overset{\text{row reduce}}{\leftrightsquigarrow} \quad \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}.$$

The only solution is the trivial solution x = 0.

Observation

Since the last column (everything to the right of the =) was zero to begin, it will always stay zero! So it's not really necessary to write augmented matrices in the homogeneous case.

What is the solution set of Ax = 0, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equation}} \qquad x_1 - 3x_2 = 0$$

$$\xrightarrow{\text{parametric form}} \qquad \begin{cases} x_1 = 3x_2 \\ x_2 = x_2 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

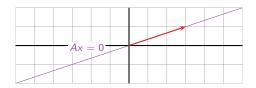
This last equation is called the parametric vector form of the solution.

It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.

What is the solution set of Ax = 0, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$$
?

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for any x_2 in **R**. The solution set is Span $\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$.



Note: one free variable means the solution set is a line in \mathbf{R}^2 (2 = # variables = # columns).

What is the solution set of Ax = 0, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \quad \begin{array}{c} \text{row reduce} \\ \text{volume} \\ \text{row reduce} \\ \text{or } 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\stackrel{\text{equations}}{\text{equations}} \quad \begin{cases} x_1 & -2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \quad \begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

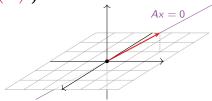
Homogeneous Systems Example, continued

Question

What is the solution set of Ax = 0, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix}$$
?

Answer: Span $\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$.



Note: *one* free variable means the solution set is a *line* in \mathbb{R}^3 (3 = # variables = # columns).

What is the solution set of Ax = 0, where A =

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 & -8x_3 - 7x_4 = 0 \\ x_2 + 4x_3 + 3x_4 = 0 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 8x_3 + 7x_4 \\ x_2 = -4x_3 - 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

parametric vector form
$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Homogeneous Systems

Example, continued

Question

What is the solution set of Ax = 0, where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}?$$

Answer: Span
$$\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$
.

[not pictured here]

Note: *two* free variables means the solution set is a *plane* in \mathbf{R}^4 (4 = # variables = # columns).

Parametric Vector Form

Homogeneous systems

Let A be an $m \times n$ matrix. Suppose that the free variables in the homogeneous equation Ax = 0 are x_i, x_j, x_k, \dots

Then the solutions to Ax = 0 can be written in the form

$$x = x_i v_i + x_i v_i + x_k v_k + \cdots$$

for some vectors v_i, v_j, v_k, \ldots in \mathbf{R}^n , and any scalars x_i, x_j, x_k, \ldots

The solution set is

$$Span\{v_i, v_j, v_k, \ldots\}.$$

The equation above is called the **parametric vector form** of the solution.

Poll

How many solutions can there be to a homogeneous system with more equations than variables?

- A. C
- B. 1
- **C**. ∞

The trivial solution is always a solution to a homogeneous system, so answer A is impossible.

This matrix has only one solution to Ax = 0:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This matrix has infinitely many solutions to Ax = 0:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

What is the solution set of Ax = b, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -3 & | & -3 \\ 2 & -6 & | & -6 \end{pmatrix} \quad \overset{\text{row reduce}}{\longrightarrow} \quad \begin{pmatrix} 1 & -3 & | & -3 \\ 0 & 0 & | & 0 \end{pmatrix}$$

$$\stackrel{\text{equation}}{\longrightarrow} \quad x_1 - 3x_2 = -3$$

$$\stackrel{\text{parametric form}}{\longrightarrow} \quad \begin{cases} x_1 & = 3x_2 - 3 \\ x_2 & = x_2 + 0 \end{cases}$$

$$\stackrel{\text{parametric vector form}}{\longrightarrow} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

The only difference from the homogeneous case is the constant vector $p = \binom{-3}{0}$.

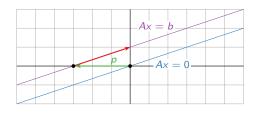
Note that p is itself a solution: take $x_2 = 0$.

What is the solution set of Ax = b, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

Answer:
$$x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$
 for any x_2 in **R**.

This is a *translate* of Span $\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$: it is the parallel line through $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$.



It can be written

$$\mathsf{Span}\!\left\{ \begin{pmatrix} \mathbf{3} \\ \mathbf{1} \end{pmatrix} \right\} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

What is the solution set of Ax = b, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -5 \\ -3 \\ -2 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & 3 & 1 & | & -5 \\ 2 & -1 & -5 & | & -3 \\ 1 & 0 & -2 & | & -2 \end{pmatrix} \quad \text{row reduce} \quad \begin{pmatrix} 1 & 0 & -2 & | & -2 \\ 0 & 1 & 1 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\stackrel{\text{equations}}{\underset{\text{average}}{\text{equations}}} \quad \begin{cases} x_1 & -2x_3 = -2 \\ x_2 + x_3 = -1 \end{cases}$$

$$\underset{\text{parametric form}}{\underset{\text{parametric vector form}}{\underset{\text{average}}{\text{equations}}} \quad \begin{cases} x_1 = 2x_3 - 2 \\ x_2 = -x_3 - 1 \\ x_3 = x_3 \end{cases}$$

$$\underset{\text{parametric vector form}}{\underset{\text{parametric vector form}}{\underset{\text{average}}{\text{equations}}} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.$$

Nonhomogeneous Systems

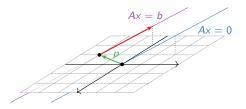
Example, continued

Question

What is the solution set of Ax = b, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -5 \\ -3 \\ -2 \end{pmatrix}?$$

Answer: Span
$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\} + \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$
.



The solution set is a translate of

Span
$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$$
:

it is the parallel line through

$$p = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.$$

Homogeneous vs. Nonhomogeneous Systems

Key Observation

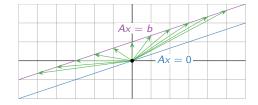
The set of solutions to Ax = b, if it is nonempty, is obtained by taking one **specific** or **particular solution** p to Ax = b, and adding all solutions to Ax = 0.

Why? If Ap = b and Ax = 0, then

$$A(p+x) = Ap + Ax = b + 0 = b,$$

so p + x is also a solution to Ax = b.

We know the solution set of Ax = 0 is a span. So the solution set of Ax = b is a *translate* of a span: it is *parallel* to a span. (Or it is empty.)

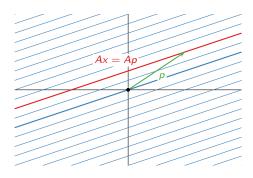


This works for *any* specific solution p: it doesn't have to be the one produced by finding the parametric vector form and setting the free variables all to zero, as we did before.

Homogeneous vs. Nonhomogeneous Systems

If we understand the solution set of Ax = 0, then we understand the solution set of Ax = b for all b: they are all translates (or empty).

For instance, if $A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$, then the solution sets for varying b look like this:



Which *b* gives the solution set Ax = b in red in the picture?

Choose p on the red line, and set b = Ap. Then p is a specific solution to Ax = b, so the solution set of Ax = b is the red line.

Note the cool optical illusion!

For a matrix equation Ax = b, you now know how to find which b's are possible, and what the solution set looks like for all b, both using spans.