Section 3.2

Properties of Determinants

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The determinant is one of the most amazing functions ever devised. Today is about beginning to understand why.

The Determinant is a Function

We can think of the determinant as a function of the entries of a matrix:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{aligned} &a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \end{aligned}$$

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The determinant function is characterized by how it is changed by row operations.

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4. If you scale one coordinate by k, the volume is multiplied by k.

Properties of the Determinant

2×2 matrix

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$$\det\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} =$$

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

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(Cofactor expansion is $O(n!) \sim O(n^n\sqrt{n})$, row reduction is $O(n^3)$.)

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Suppose that A is a 4×4 matrix satisfying

$$Ae_1 = e_2 \quad Ae_2 = e_3 \quad Ae_3 = e_4 \quad Ae_4 = e_1.$$

What is $\det(A)$?

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So $\det(A) = (-1)^3 = -1$.

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Here is a summary of the magical properties of the determinant. Prof. Margalit's notes (on the website) have very understandable proofs.

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Why is Property 5 true? In Lay, there's a proof using elementary matrices. Here's a better one.

Why is Property 8 true?

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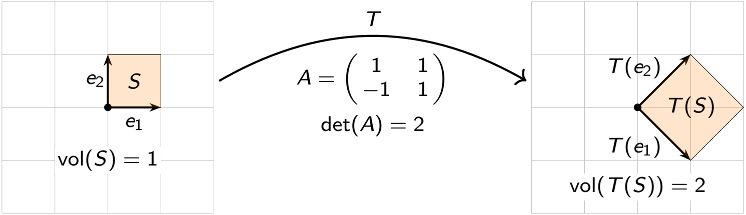

Determinants and Linear Transformations

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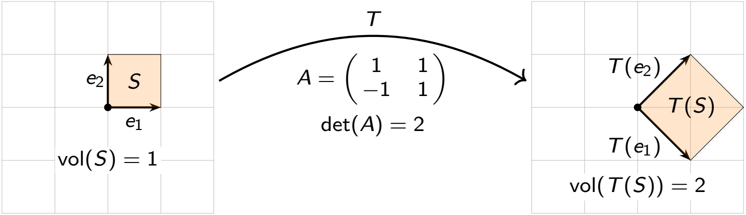
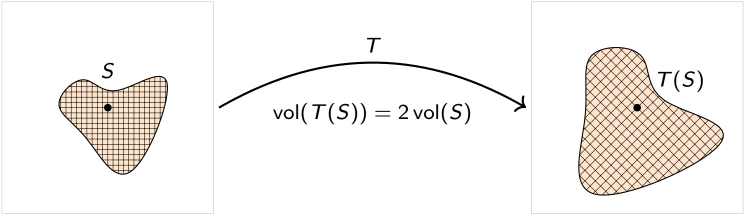


For curvy shapes, you break S up into a bunch of tiny cubes. Each one is scaled by $|\det(A)|$; then you use *calculus* to reduce to the previous situation!



Multi-Linearity of the Determinant

We can also think of det as a function of the columns (or the rows) of an $n \times n$ matrix:

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Property 9 says that for any i and any vectors $v_1, v_2, \ldots, v_n$ and v_i' and any scalar c,

$$\begin{aligned} \det(v_1, \ldots, v_i + v_i', \ldots, v_n) &= \det(v_1, \ldots, v_i, \ldots, v_n) + \det(v_1, \ldots, v_i', \ldots, v_n) \\ \det(v_1, \ldots, cv_i, \ldots, v_n) &= c\det(v_1, \ldots, v_i, \ldots, v_n). \end{aligned}$$

Multi-Linearity of the Determinant

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Proof: just expand cofactors along column i.