Section 2.2

The Inverse of a Matrix

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Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

identity matrix

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Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

The 2×2 case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A is the number

$$\boxed{\det(A) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.}$$

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1. If $\det(A) \neq 0$, then A is invertible and $\boxed{A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.}$

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Example

Solve the system

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x \phantom{{}+3y} + 3z &= 1 \\ 2x + 2y + 3z &= 1 \end{aligned} \qquad \text{using} \qquad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}.$$

Answer:

Some Facts

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If A, B, C are invertible $n \times n$ matrices, what is the inverse of ABC?

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In general, a product of invertible matrices is invertible, and the inverse is the product of the inverses, in the *reverse order*.

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Let A be an $n \times n$ matrix. Here's how to compute A^{-1}.

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3. Otherwise, A is not invertible.

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Check:

Why Does This Work?

First answer: We can think of the algorithm as simultaneously solving the equations

$$Ax_1 = e_1 : \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

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Second answer: Elementary matrices.

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Example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 10 \\ 0 & -3 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} \xrightarrow{R_2 = R_2 + 2R_1} \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 10 \\ 0 & -3 & -4 \end{pmatrix}$$

Elementary Matrices

Continued

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Consequence

Elementary matrices are invertible, and the inverse is the elementary matrix which un-does the row operation.

Why Does The Inversion Algorithm Work?
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Theorem

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This means if you do these same row operations to A and to I_n, you'll end up with I_n and A^{-1}.

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This means if you do these same row operations to A and to I_n, you'll end up with I_n and A^{-1}. This is what you do when you row reduce the augmented matrix:

$$\left(\begin{array}{c|c} A & I_n \end{array}\right) \rightsquigarrow \left(\begin{array}{c|c} I_n & A^{-1} \end{array}\right)$$