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Nonparametric estimation of reliability and survival function for continuous-time finite Markov processes

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Abstract

In this paper, problems of nonparametric statistical inference for reliability/survival, availability and failure rate functions of continuous-time Markov processes are discussed. We assume the state space to be finite. We shall discuss some results on the maximum likelihood estimator of generators for continuous-time Markov processes. The asymptotic properties of this estimator have been discussed and we present the extension of these results by obtaining the asymptotic properties for the estimators of transition matrix and of reliability/survival, availability and failure rate functions for a continuous-time Markov process. The confidence intervals for the proposed estimators are given. Finally we give a numerical example to illustrate the results.

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1. Introduction

Continuous-time finite state space Markov processes have been used extensively to construct stochastic models in a variety of disciplines such as biology, chemistry, electrical engineering, medicine, and physics. The most important problem here is the estimation of reliability and its measurements for these models. Many investigations involve the

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collection of data in the form of lifetimes, for example, of patients suffering from a particular disease, or of electrical components in a piece of machinery. In the analysis of such data, we want to find out how the lifetime is affected by various explanatory variables; for example, age and treatment in the case of disease, materials and operating environment in the case of machinery. In order to use Markov processes in stochastic modelling of real systems, we have to estimate first their characteristics.

In the literature concerning statistical inference of stochastic processes, two types of observational procedures are discussed. One either observes a single realization over the fixed time interval $[0, T]$, or observes $K > 1$ independent and identical copies of the process, each over the fixed duration $[0, T]$.

Corresponding to the two procedures of observation mentioned above, there are two methods of studying the asymptotic properties of procedures of statistical inference. In the case when a single realization of the process is observed, the asymptotic properties are obtained as the period T of observation becomes large. In the second case, T is kept fixed and the number K of observed realizations is allowed to grow to infinity.

The problems of statistical inference for a Markov process have been discussed by several authors, cf. Billingsley (1960, 1961), Basawa and Prakasa Rao (1980), Albert (1962), Grenander (1981), and Adke and Manjunath (1984). All previous works have been concerned with the statistical inference about the matrix of transition probabilities. Albert (1962) derived the maximum likelihood estimator for the infinitesimal generator of a continuous time, finite state space Markov process and investigated its asymptotic properties. Grenander (1981) discussed also the inference problems for the infinitesimal generator of a Markov process. Grenander used the transformation of the generator by the Jordan decomposition into diagonal form. So his results are a particular case of Albert's results when the generator matrix has simple eigenvalues.

In Sadek and Limnios (2002), we discussed the asymptotic properties for the maximum likelihood estimators for reliability, availability and failure rates for a discrete-time Markov process. Recently, there have been several works in the field of survival analysis using Markov model. For example, Chen and Sen (2001) used a multistate model to estimate the mean quality-adjusted survival based on the assumption that patient's health status is Markovian, which allows the estimator to accommodate a health status process with periodic observations. Andersen et al. (1993) studied a point process approach for Markov processes. Ouhbi and Limnios (1999) studied the empirical estimators of reliability and failure rate functions of a semi-Markov system and investigated their asymptotic properties. It should be mentioned here that a Markov process is a particular case of a semi-Markov process. So we can apply the results given in Ouhbi and Limnios (1999) to estimate the reliability and the failure rate functions for Markov systems. Nevertheless, we present here a more adequate and simpler approach for Markov processes. The principal reason is that there is no need here to estimate the Markov renewal function. We get a direct estimation of transition probabilities by the exponential formula. On the other hand, when we use the generator matrix, the transition rates are constant and it is easier to estimate them. In this paper, we present the maximum likelihood estimators for reliability and its measurements. Starting from Albert's results, the asymptotic properties for these estimators are investigated.

All the results given here for the reliability estimator concern equally *phase-type* distributions. The *phase-type* distribution is the distribution of the time until absorption in a

Markov process. For more details about these distributions, see, e.g., Neuts (1981). The *phase-type* distributions are widely used in the study of stochastic models, (see Cox, 1955; Limnios and Oprişan, 2001). Significant studies of the estimation problem for *phase-type* distribution are given by Baum et al. (1970), Bobbio and Telek (1992), Asmussen et al. (1996) and Olsson (1996).

The paper is organized as follows: in the next section, we present the maximum likelihood estimator of the infinitesimal generator of a continuous-time, finite state space Markov process. Asymptotic properties for this estimator are discussed. The estimator of the transition matrix and its asymptotic properties are discussed in Section 3. In Section 4, we present the principal results of this paper which are the asymptotic properties for availability, reliability and failure rate estimators of Markov processes. Using the asymptotic results we construct the confidence intervals for availability, reliability and failure rate. Finally, in Section 5, we give a numerical example.

2. Preliminaries

Let us give here some auxiliary results which are used in the following sections. Some definitions and basic concepts, such as availability, reliability/survival and failure rate functions and their maximum likelihood estimators for this model, are given.

2.1. The model

Let $\{X(t), t \geqslant 0\}$ be a time-homogeneous continuous-time Markov process with finite state space $E = \{1, 2, \dots, s\}$, and infinitesimal generator matrix $\mathbf{A} = (a_{ij})_{i,j \in E}$, where

$$a_{ij} \geqslant 0, \quad i \neq j \quad \text{and} \quad a_{ii} = -a_i = -\sum_{\substack{k \in E \\ i \neq k}} a_{ik}. \tag{1}$$

This process is a regular one, i.e., almost surely the paths are step functions and in any finite interval, the number of jumps is finite. We will suppose further that X is a càdlàg irreducible process.

Let $\alpha = (\alpha(i))_{i \in E}$ be the initial distribution of X, i.e., $\alpha(i) = \mathbb{P}[X(0) = i]$, $i \in E$ and let $P(t) = (p_{ij}(t))$, $t \geqslant 0$, be the transition function of X, i.e., for all $i, j \in E$, $t \geqslant 0$ and $h \geqslant 0$,

$$p_{ij}(t) = \mathbb{P}[X(t) = j | X(0) = i] = \mathbb{P}[X(t+h) = j | X(h) = i].$$

In this case, we have, as the solution of the Kolmogorov equation,

$$P(t) = \mathrm{e}^{t\mathbf{A}}. \tag{2}$$

Let us denote by $0 = T_0 < T_1 < \cdots < T_n < T_{n+1} < \cdots$ the jump times of the process, and let $W_n := T_{n+1} - T_n, n \geqslant 0$ be the sojourn times in successively visited states. The discrete-time process $\{X_n, \ n \geqslant 0\}$, is the so-called embedded Markov chain (EMC) of $X(t), t \geqslant 0$. That is,

$$X_n = X(T_n), \ n \geqslant 0 \quad \text{and} \quad X(t) = X_{N(t)}, \quad t \geqslant 0,$$

where $N(t) = \sup\{n \geqslant 0 \ : \ T_n \leqslant t\}$ is the number of jumps in the time interval $(0, t]$.

The transition probabilities matrix Q of the EMC $\{X_n\}$ is

$$Q(i,j) = \begin{cases} \dfrac{a_{ij}}{a_i} & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases} \tag{3}$$

A sample path of the Process $\{X(t), t \geqslant 0\}$ in $[0, T]$ can be represented as an ordered sequence

$$((X_0, W_0), (X_1, W_1), \ldots, (X_{N(T)-1}, W_{N(T)-1}), (X_{N(T)}, U_T)), \quad U_T = T - T_{N(T)}.$$

The path starts at state X_0 at time zero, remains in X_0 for W_0 units of time, makes a jump to X_1, remains in X_1 for W_1 units of time, …, jumps to $X_{N(T)-1}$, remains there for $W_{N(T)-1}$ units of time and then makes the final jump to $X_{N(T)}$ and remains there at least until time T.

2.2. Some known results

In this subsection, we present some known results that we need in the rest of this paper. Albert (1962) derived the maximum likelihood estimator of **A** and investigated its asymptotic properties. These results are the continuous-time versions of the results stated by Anderson and Goodman (1957) and by Billingsley (1961) who study the asymptotic behavior of the maximum likelihood estimator of the transition probability matrix of a discrete-time Markov chain.

- Maximum likelihood estimation of the generator.

Suppose K independent sample paths of $\{X(t),\ 0 \leqslant t \leqslant T\}$ are observed. Let $N_{ij}(T, K)$ be the total number of transitions from state i to state j observed during K trials in the time interval $[0, T]$ and let $V_i(T, K)$ be the total length of time that state i is occupied during K trials. Then we denote the maximum likelihood estimator for the generator matrix **A** by $\hat{A}(t; T, K) = (\hat{a}_{ij}(T, K))_{i,j \in E}$, where

$$\hat{a}_{ij}(T,K) = \begin{cases} \dfrac{N_{ij}(T,K)}{V_i(T,K)} & \text{if } i \neq j, V_i(T,K) \neq 0, \\ -\dfrac{\sum_{l \in E \setminus \{i\}} N_{il}(T,K)}{V_i(T,K)} & \text{if } i = j, V_i(T,K) \neq 0, \\ 0 & \text{if } V_i(T,K) = 0. \end{cases} \tag{4}$$

Remark. If $V_i(T, K) = 0$, the maximum likelihood estimator does not exist and we adopt the convention that

$$\hat{a}_{ij}(T,K) = 0 \quad \text{if } i \neq j \text{ and } V_i(T,K) = 0. \tag{5}$$

It is worth noting here that Albert (1962) does not mention the estimation of the diagonal components of **A**.

- Asymptotic properties.

Albert (1962) investigated the asymptotic properties of the vector $\{\hat{a}_{ij}(T, K), i, j \in E, j \neq i\}$ in two ways. A single realization of the processes can be observed over a long period of time. This corresponds to the investigation of $\hat{a}_{ij}(T, K)$, as $T \to \infty$, while K is fixed. We shall suppose that $K=1$ to simplify calculations. In this case we denote $\hat{a}_{ij}(T, 1)$ by $\hat{a}_{ij}(T)$. On the other hand, many independent realizations of the process $\{X(t), 0 \leqslant t < T\}$ could be observed. This corresponds to an investigation of the behavior $\hat{a}_{ij}(T, K)$, as $K \to \infty$, with T fixed. In the second case, we denote $\hat{a}_{ij}(T, K)$ by $\hat{a}_{ij}(K)$. Albert obtained results pertaining to the consistency and asymptotic normality of these estimators in both cases. We will present these results in the following theorems.

Theorem A (*Albert, 1962*). *For all $i, j \in E$, we have*

$$\lim_{T\to\infty} \hat{a}_{ij}(T) = a_{ij} \quad a.s. \tag{6}$$

and the distribution of the random vector

$$\{T^{1/2}(\hat{a}_{ij}(T) - a_{ij})\}_{i,j=1,j\neq i}^{s}$$

is asymptotically normal and independent components with zero mean and covariance matrix $H=(H(i, j; k, l))_{s(s-1)\times s(s-1)}$, where the element $H(i, j; k, l)$ is defined as follows:

$$H(i, j; k, l) = \delta(i, j; k, l)\ a_{ij}\ \rho/\mathbf{A}^{(i,i)}, \tag{7}$$

where

$$\delta(i, j; k, l) = \begin{cases} 1 & \text{if } i=k,\ j=l, \\ 0 & \text{otherwise}, \end{cases}$$

and ρ is the product of the non-zero eigenvalues of $\mathbf{A}$ *and* $\mathbf{A}^{(i,i)}$ *is the* (i, i)*th cofactor of* $\mathbf{A}$.

Theorem B (*Albert, 1962*). *If there is a positive probability that the ith visited state will be occupied at some time $t \geqslant 0$, then*

$$\lim_{K\to\infty} \hat{a}_{ij}(K) = a_{ij} \quad a.s. \tag{8}$$

and if every state has positive probability of being occupied, then the random vector

$$\{K^{1/2}(\hat{a}_{ij}(K) - a_{ij})\}_{i,j=1,j\neq i}^{s}$$

is asymptotically normal, as $K \to \infty$, with mean zero and covariance matrix $C = (C(i, j; k, l))_{s(s-1)\times s(s-1)}$, where the element $C(i, j; k, l)$ is defined as follows

$$C(i, j; k, l) = \delta(i, j; k, l) a_{ij}/v_i(T), \tag{9}$$

where $v_i(T) = \int_0^T \mathbb{P}[X(t) = i]\,\mathrm{d}t$.

Remark. The assumption of positive probability occupation is naturally fulfilled when the Markov process is irreducible (ergodic).

2.3. Reliability estimation and its measurements

In reliability and survival studies the state space E is partitioned into two sets: $U = \{1, \dots, r\}$ is the set of working states and $D = \{r+1, \dots, s\}$ is the set of down states (i.e., $E = U \cup D$, $U \cap D = \emptyset$ and $U \neq \emptyset$, $D \neq \emptyset$). According to this partition, we will partition the generator matrix $\mathbf{A}$ and the initial distribution α as follows:

$$\mathbf{A} = \begin{array}{cc} \begin{array}{cc} U & D \end{array} & \\ \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} & \begin{array}{c} U \\ D \end{array} \end{array}, \quad \alpha = [\overset{U}{\alpha_1} \;\; \overset{D}{\alpha_2}].$$

Let us denote by $\mathbf{1}_{s,r} = (1, \dots, 1, 0, \dots, 0)^\top$ the s-dimensional column vector whose r first elements are equal to 1 and the others are equal to zero. We note that $\mathbf{1}_r = \mathbf{1}_{r,r}$.

The definitions of availability, reliability/survival and failure rate functions for continuous-time Markov processes (see, e.g., Balakrishnan et al., 2001), can be expressed as follows:

- *Availability function*: In this work, we discuss the instantaneous availability at time t, $A(t)$. It is the probability that the system is up at moment t, i.e.,

$$A(t) = \mathbb{P}(X(t) \in U), \quad t \geqslant 0.$$

The instantaneous availability for a Markov process is given by

$$A(t) = \alpha \, \mathrm{e}^{t\mathbf{A}} \, \mathbf{1}_{s,r}, \quad t \geqslant 0. \tag{10}$$

We propose the following estimator for $A(t)$:

$$\hat{A}(t; T, K) = \alpha \, \mathrm{e}^{t\hat{\mathbf{A}}(T,K)} \, \mathbf{1}_{s,r}, \quad t \geqslant 0. \tag{11}$$

- *Reliability and survival function*: The reliability $R(t)$ is the probability of no failure in $(0, t]$, given $X(0) = i \in U$, i.e.,

$$R(t) = \mathbb{P}(\forall u \in [0, t], X(u) \in U), \quad t \geqslant 0. \tag{12}$$

The reliability for a Markov process is given by

$$R(t) = \alpha_1 \, \mathrm{e}^{t\mathbf{A}_{11}} \, \mathbf{1}_r. \tag{13}$$

The reliability estimator of this system is defined by

$$\hat{R}(t; T, K) = \alpha_1 \, \mathrm{e}^{t\hat{\mathbf{A}}_{11}(T,K)} \, \mathbf{1}_r. \tag{14}$$

Remark. In the case of survival analysis the survival function is usually denoted by $S(t)$ and is defined by (12) and is also expressed by formula (13) as reliability. Thus we will consider the same estimator (14) for the survival function $S(t)$. The distribution $1-\alpha_1\,\mathrm{e}^{t\mathbf{A}_{11}}\,\mathbf{1}_r$ on $\mathbb{R}_+$ is known as a phase type distribution.

- *Failure rate function*: In the continuous time case, the failure rate function is defined by

$$\lambda(t)=\lim_{h\downarrow 0}\frac{1}{h}\,\mathbb{P}(t<T\leqslant t+h \mid T>t),$$

where T is the hitting time of D, i.e.,

$$T=\inf\{t\geqslant 0 : X(t)\in D\}.$$

This is the so-called lifetime of the system.

The failure rate function for a Markov process is

$$\lambda(t)=\begin{cases}-\dfrac{\frac{\mathrm{d}R}{\mathrm{d}}t(t)}{R(t)}=-\dfrac{\alpha_1\,\mathrm{e}^{t\mathbf{A}_{11}}\mathbf{A}_{11}\mathbf{1}_r}{\alpha_1\,\mathrm{e}^{t\mathbf{A}_{11}}\mathbf{1}_r}, & \text{if } R(t)>0,\\ 0 & \text{if } R(t)=0.\end{cases} \tag{15}$$

Let us denote by $\mathbf{r}(t)=\alpha_1\,\mathrm{e}^{t\mathbf{A}_{11}}\mathbf{A}_{11}\mathbf{1}_r$.

If we replace $\mathbf{A}_{11}$ by its estimator we obtain an estimator of the failure rate for a continuous Markov system which is given by

$$\hat{\lambda}(t;T,K)=\begin{cases}-\dfrac{\hat{\mathbf{r}}(t;T,K)}{\hat{R}(t;T,K)}=-\dfrac{\alpha_1\,\mathrm{e}^{t\hat{\mathbf{A}}_{11}(T,K)}\hat{\mathbf{A}}_{11}(T,K)\mathbf{1}_r}{\alpha_1\,\mathrm{e}^{t\hat{\mathbf{A}}_{11}(T,K)}\mathbf{1}_r}, & \text{if } \hat{R}(t;T,K)>0,\\ 0 & \text{if } \hat{R}(t;T,K)=0.\end{cases} \tag{16}$$

Remark. In the proposed estimators, we suppose that the initial distribution is known. If the initial distribution is unknown we can estimate it as follows:

$$\hat{\alpha}(i)=\frac{N_i(0)}{K},\quad N_i(0)=\sum_{l=1}^{K}\mathbf{1}_{(x_0^l=i)},\quad K\geqslant 1.$$

3. Asymptotic properties for transition probabilities matrix estimator

In this section, we propose an estimator of the transition probabilities matrix for a continuous-time finite state Markov process and we investigate the asymptotic properties (consistency and asymptotic normality) of this estimator in both frameworks, as $T\to\infty$, while K is fixed, and as $K\to\infty$, with T fixed.

If we replace $\mathbf{A}$ in (2) by its estimator we obtain an estimator of the transition matrix for a continuous Markov process which is given by

$$\hat{P}(t;T,K)=\exp\{t\hat{\mathbf{A}}(T,K)\}. \tag{17}$$

We will now investigate the asymptotic properties for the transition matrix estimator $\hat{P}(t;T,K)$. For one trajectory, i.e. $K=1$, we write $\hat{P}(t;T)$ instead of $\hat{P}(t;T,K)$ and when T is fixed, we denote $\hat{P}(t;T,K)$ by $\hat{P}(t;K)$.

Theorem 1. (a) *For $K=1$, the estimator $\hat{p}_{ij}(t;T)$, for all $i,j\in E$, is uniformly strongly consistent, as $T\to\infty$, on compact $[0,L]$, for $L\in\mathbb{R}_+$, in the sense that*

$$\sup_{0<t\leqslant L} \mid \hat{p}_{ij}(t;T)-p_{ij}(t)\mid \underset{T\to\infty}{\overset{a.s.}{\to}} 0, \tag{18}$$

and the random vector

$$(T^{1/2}(\hat{p}_{ij}(t;T)-p_{ij}(t)))_{i,j\in E}$$

is asymptotically normal, as $T\to\infty$, with zero mean and covariance matrix $\Lambda(t)=(\Lambda(i,j;k,l;t))_{s^2\times s^2}$, where

$$\begin{aligned}\Lambda(i,j;k,l;t)&\\ =\sum_{\substack{u,v=1\\ u\neq v}}^{s}&\left(\left[\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{h=1}^{n}(-a_{iu}^{(h-1)}\,a_{uj}^{(n-h)}+a_{iu}^{(h-1)}\,a_{vj}^{(n-h)})\right]\right.\\ &\left.\times\left[\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{h=1}^{n}(-a_{ku}^{(h-1)}a_{ul}^{(n-h)}+a_{ku}^{(h-1)}a_{vl}^{(n-h)})\right]\left[\frac{a_{uv}\rho}{\mathbf{A}^{(u,u)}}\right]\right),\end{aligned} \tag{19}$$

and $a_{ij}^{(n)}$ is the (i,j) entry of the matrix $\mathbf{A}^n$.

(b) *For fixed T, the estimator $\hat{p}_{ij}(t;K)$, for all $i,j\in E$, is uniformly strongly consistent as, $K\to\infty$, on compact $[0,L]$, for $L\in\mathbb{R}_+$, in the sense that*

$$\sup_{0<t\leqslant L} \mid \hat{p}_{ij}(t;K)-p_{ij}(t)\mid \underset{K\to\infty}{\overset{a.s.}{\to}} 0, \tag{20}$$

and the random vector

$$(K^{1/2}(\hat{p}_{ij}(t;K)-p_{ij}(t)))_{i,j\in E}$$

is asymptotically normal, as $K\to\infty$, with mean zero and covariance matrix $\Gamma(t)=(\Gamma(i,j;k,l;t))_{s^2\times s^2}$, where

$$\begin{aligned}\Gamma(i,j;k,l;t)=\sum_{\substack{u,v=1\\ u\neq v}}^{s}&\left(\left[\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{h=1}^{n}(-a_{iu}^{(h-1)}\,a_{uj}^{(n-h)}+a_{iu}^{(h-1)}\,a_{vj}^{(n-h)})\right]\right.\\ &\left.\times\left[\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{h=1}^{n}(-a_{ku}^{(h-1)}a_{ul}^{(n-h)}+a_{ku}^{(h-1)}a_{vl}^{(n-h)})\right][a_{uv}/v_i(T)]\right).\end{aligned} \tag{21}$$

Proof. Consistency. From the consistency of the generator estimators $\hat{a}_{ij}(T, K)$ when $T \to \infty$ (see relation (6)) and when $K \to \infty$ (see relation (8)), we obtain the consistency of the random vector $(\hat{a}_{ij}(T, K))_{i,j\in E}$. We can write every transition $\hat{p}_{ij}(t; T, K)$ as a continuous function of this vector. Then the strong consistency for $\hat{p}_{ij}(t; T)$ and $\hat{p}_{ij}(t; K)$, are valid. To prove that these estimators are strongly consistent uniformly on compact $[0, L]$, for $L \in \mathbb{R}_+$, observe that $\hat{p}_{ij}(t; T, K)$ is monotone and continuous, so, relations (18) and (20) are valid.

Normality. To prove the normality for the transition probability matrix estimator, we observe that the transition matrix is the exponential function of the generator matrix $\mathbf{A}$. Let us define the continuous mapping $\Psi : \mathbb{R}^{s(s-1)} \to \mathbb{R}^{s^2}$, where $\Psi((a_{uv})_{u,v\in E,u\neq v}) = \mathrm{e}^{t\mathbf{A}}$. From Theorem A and Theorem B, applying the delta method (see, e.g., VanDerVaart, 2000, p. 25) we obtain that the random vector $(T^{1/2}(\hat{p}_{ij}(t; T) - p_{ij}(t)))_{i,j\in E}$ is asymptotically normal, as $T \to \infty$, with zero mean and covariance matrix $\Lambda = \Psi' \cdot H \cdot \Psi'^{\top}$. In the same way, the random vector $(K^{1/2}(\hat{p}_{ij}(t; K) - p_{ij}(t)))_{i,j\in E}$ is asymptotically normal, as $K \to \infty$, with zero mean and covariance matrix $\Gamma = \Psi' \cdot C \cdot \Psi'^{\top}$, where

$$\Psi' = (\Psi'_{12}, \ldots, \Psi'_{1s}, \ldots, \Psi'_{s1}, \ldots, \Psi'_{s(s-1)})$$

is an $s^2 \times s(s-1)$ matrix. Ψ'_{uv} is a s^2-column vector whose elements are taken row-wise reshaped from matrix $\tilde{\Psi}'_{uv}$, (i.e. rewriting the matrix $\tilde{\Psi}'_{uv}$ as a column vector), where

$$\tilde{\Psi}'_{uv} = \frac{\partial \mathrm{e}^{t\mathbf{A}}}{\partial a_{uv}} = \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} \mathbf{A}^{h-1} \frac{\partial \mathbf{A}}{\partial a_{uv}} \mathbf{A}^{n-h}, \quad u \neq v. \tag{22}$$

Denoting by $a_{ij}^{(n)}$ the (i, j) entry of the matrix $\mathbf{A}^n$, we can write the (i, j) entry of the matrix $\tilde{\Psi}'_{uv}$ as follows

$$\tilde{\Psi}'_{uv}(i, j) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} (-a_{iu}^{(h-1)} a_{uj}^{(n-h)} + a_{iu}^{(h-1)} a_{vj}^{(n-h)}).$$

Then, from relations (7) and (9), we can write the covariance matrix Λ as in relation (19) and the covariance matrix Γ as in relation (21).

Remark. Matrix (22) is well defined since $\| \sum_{h=1}^{n} \mathbf{A}^{h-1} \frac{\partial \mathbf{A}}{\partial a_{uv}} \mathbf{A}^{n-h} \| \leqslant 2n \| \mathbf{A}\|^{n-1}$, where $\| \mathbf{A} \| = \max_i \sum_j |a_{ij}|$, and so $\tilde{\Psi}'_{uv}$ is bounded in norm by $2t\,\mathrm{e}^{t\|\mathbf{A}\|}$.

4. Asymptotic properties for reliability and its measurements

Here we present the asymptotic properties of availability, reliability and failure rate estimators, when either we observe a single realization of the process over a long period of time or we observe K realizations over the same fixed period of time and we let K tend to infinity.

Theorem 2. *The maximum likelihood estimators of the availability, reliability and failure rate of a Markov process are strongly consistent uniformly on compact* $[0, L]$, *for* $L \in \mathbb{R}_+$, *as T tends to infinity. That is, for all* $t > 0$, *we have*:

$$\text{(i)} \qquad \sup_{0<t\leqslant L} |\hat{A}(t;T) - A(t)| \underset{T\to\infty}{\overset{a.s.}{\to}} 0,$$

$$\text{(ii)} \qquad \sup_{0<t\leqslant L} |\hat{R}(t;T) - R(t)| \underset{T\to\infty}{\overset{a.s.}{\to}} 0,$$

and,

$$\text{(iii)} \qquad \sup_{0<t\leqslant L} |\hat{\lambda}(t;T) - \lambda(t)| \underset{T\to\infty}{\overset{a.s.}{\to}} 0.$$

Proof. (i) From the definition of availability and its estimator (10) and (11), we get

$$\begin{aligned}\sup_{0<t\leqslant L} |\hat{A}(t,T) - A(t)| &= \sup_{0<t\leqslant L} \| \alpha\hat{P}(t,T)\, \mathbf{1}_{s,r} - \alpha P(t)\mathbf{1}_{s,r} \| \\ &\leqslant \|\alpha\| \sup_{0<t\leqslant L} \|\hat{P}(t,T) - P(t)\| \, \|\mathbf{1}_{s,r}\| .\end{aligned}$$

From (18), we see that $\hat{A}(t,T)$ converges strongly and uniformly to $A(t)$, as $T \to \infty$.

(ii) In the same way, on the basis of the definition of reliability and its estimator, we get the uniform strong consistency for reliability.

(iii) To prove the consistency for failure rate, from the definition of failure rate (15) and its estimator (16) we obtain that

$$\begin{aligned}&\sup_{0\leqslant t\leqslant L} |\hat{\lambda}(t,T) - \lambda(t)| \\ &\quad= \sup_{0\leqslant t\leqslant L} \left| -\frac{\hat{\mathbf{r}}(t,T)}{\hat{R}(t,T)} + \frac{\mathbf{r}(t)}{R(t)} \right| \\ &\quad= \sup_{0\leqslant t\leqslant L} \left| \frac{\hat{\mathbf{r}}(t,T) - \mathbf{r}(t)}{\hat{R}(t,T)} - \frac{\mathbf{r}(t)\hat{R}(t,T) - \mathbf{r}(t)R(t)}{\hat{R}(t,T)R(t)} \right| \\ &\quad= \sup_{0\leqslant t\leqslant L} \left| \frac{\hat{\mathbf{r}}(t,T) - \mathbf{r}(t)}{\hat{R}(t,T)} - \frac{\mathbf{r}(t)}{R(t)} \, \frac{\hat{R}(t,T) - R(t)}{\hat{R}(t,T)} \right| \\ &\quad\leqslant \sup_{0\leqslant t\leqslant L} \left| \frac{\hat{\mathbf{r}}(t,T) - \mathbf{r}(t)}{\hat{R}(t,T)} \right| + \sup_{0\leqslant t\leqslant L} \left| \frac{\mathbf{r}(t)}{R(t)} \, \frac{\hat{R}(t,T) - R(t)}{\hat{R}(t,T)} \right| .\end{aligned} \tag{23}$$

Now, we get

$$\begin{aligned}
\sup_{0\leqslant t\leqslant L} |\hat{\mathbf{r}}(t,T)-\mathbf{r}(t)| &= \sup_{0\leqslant t\leqslant L} |\alpha_1\, \mathrm{e}^{t\hat{\mathbf{A}}_{11}(T)}\hat{\mathbf{A}}_{11}(T)\mathbf{1}_r - \alpha_1\, \mathrm{e}^{t\mathbf{A}_{11}}\mathbf{A}_{11}\mathbf{1}_r| \\
&\leqslant ||\alpha_1|| \sup_{0\leqslant t\leqslant L} ||\, \mathrm{e}^{t\hat{\mathbf{A}}_{11}(T)}\hat{\mathbf{A}}_{11}(T) - \mathrm{e}^{t\mathbf{A}_{11}}\mathbf{A}_{11}||\, ||\mathbf{1}_r|| \\
&\leqslant ||\alpha_1|| \sup_{0\leqslant t\leqslant L} ||(\mathrm{e}^{t\hat{\mathbf{A}}_{11}(T)} - \mathrm{e}^{t\mathbf{A}_{11}})\hat{\mathbf{A}}_{11} \\
&\quad + \mathrm{e}^{t\mathbf{A}_{11}(T)}(\hat{\mathbf{A}}_{11}(T)-\mathbf{A}_{11})||\, ||\mathbf{1}_r|| \\
&\leqslant ||\alpha_1|| \sup_{0\leqslant t\leqslant L} ||(\mathrm{e}^{t\hat{\mathbf{A}}_{11}(T)} - \mathrm{e}^{t\mathbf{A}_{11}})\hat{\mathbf{A}}_{11}||\, ||\mathbf{1}_r|| \\
&\quad + ||\alpha_1|| \sup_{0\leqslant t\leqslant L} ||\mathrm{e}^{t\mathbf{A}_{11}(T)}(\hat{\mathbf{A}}_{11}(T)-\mathbf{A}_{11})||\, ||\mathbf{1}_r||.
\end{aligned}$$

From the strong consistency of the generator estimator $\hat{\mathbf{A}}(T)$ and the uniform strong consistency of $\mathrm{e}^{t\hat{\mathbf{A}}(T)}$, we conclude that the right-hand side term of the last equation above converges almost surely to zero as T tends to infinity. Then the estimator $\hat{\mathbf{r}}(t,T)$ is uniformly strongly consistent and from the uniform strong consistency for the reliability estimator $\hat{R}(t,T)$ we obtain that both terms on the right-hand side of (23) converge almost surely to zero as T tends to infinity.

In the sequel of this section, we will give asymptotic normality results for the estimators $\hat{A}$, $\hat{R}$, and $\hat{\lambda}$.

Theorem 3. *For any fixed $t \geqslant 0$, $\sqrt{T}(\hat{A}(t,T)-A(t))$ converges in distribution, as T tends to infinity, to a centered at its mean normal random variable with variance $\sigma_A^2(t)$, where*

$$\begin{aligned}
\sigma_A^2(t) &= \sum_{i,k=1}^{s}\sum_{j,l=1}^{r} \alpha(i)\,\alpha(k) \\
&\times \left\{ \sum_{\substack{u,v=1\\ u\neq v}}^{s} \left(\left[\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} (-a_{iu}^{(h-1)}\, a_{uj}^{(n-h)} + a_{iu}^{(h-1)}\, a_{vj}^{(n-h)}) \right] \right.\right. \\
&\times \left.\left. \left[\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} (-a_{ku}^{(h-1)}\, a_{ul}^{(n-h)} + a_{ku}^{(h-1)}\, a_{vl}^{(n-h)}) \right] \left[\frac{a_{uv}\rho}{\mathbf{A}^{(u,u)}} \right] \right) \right\}. \qquad (24)
\end{aligned}$$

Proof. From the definition of availability and its estimator in (10) and (11), respectively, we get

$$\sqrt{T}(\hat{A}(t,T)-A(t)) = \sqrt{T}\sum_{i=1}^{s}\sum_{j=1}^{r} \alpha(i)(\mathrm{e}^{t\hat{\mathbf{A}}(T)}(i,j) - \mathrm{e}^{t\mathbf{A}}(i,j)),$$

where $\mathrm{e}^{t\mathbf{A}}(i,j)$ is the (i,j) entry of the matrix $\mathrm{e}^{t\mathbf{A}}$ and $\mathrm{e}^{t\hat{\mathbf{A}}(T)}(i,j)$ is the entry (i,j) of the matrix $\mathrm{e}^{t\hat{\mathbf{A}}(T)}$.

We observe that $A(t)$ can be written as a continuous function of the vector $(p_{ij}(t))_{i,j\in E}$. Let us define the continuous mapping $\Phi : \mathbb{R}^{s^2} \to \mathbb{R}$, by

$$\Phi((p_{ij}(t))_{i,j\in E}) = \sum_{i=1}^{s}\sum_{j=1}^{r} \alpha(i)\,p_{ij}(t).$$

From Theorem 1and the delta method, we obtain that the random variable $\sqrt{T}(\hat{A}(t,T) - A(t))$ converges in distribution, as T tends to infinity, to a zero mean normal random variable with variance

$$\sigma_A^2(t) = \Phi' \cdot \Lambda(t) \cdot \Phi'^{\top}, \tag{25}$$

where Φ' is the row vector of first order derivatives of Φ with respect to $p_{ij}(t)$, for all $i, j \in E$, i.e.,

$$\Phi' = \left[\, \underbrace{\alpha(1)\cdots\alpha(1)}_{r}\, \underbrace{0\cdots 0}_{s-r},\, \underbrace{\alpha(2)\cdots\alpha(2)}_{r}\, \underbrace{0\cdots 0}_{s-r},\, \ldots,\, \underbrace{\alpha(s)\cdots\alpha(s)}_{r}\, \underbrace{0\cdots 0}_{s-r} \,\right].$$

Thus, from relations (25) and (19), we obtain the desired result (24).

Theorem 4. *For any fixed* $t \geqslant 0$, $\sqrt{T}(\hat{R}(t,T) - R(t))$ *converges in distribution, as* T *tends to infinity, to a centered at its expectation normal random variable with variance* $\sigma_R^2(t)$, *where*

$$\begin{aligned}\sigma_R^2(t) = &\sum_{i,k=1}^{r}\sum_{j,l=1}^{r} \alpha(i)\,\alpha(k)\\ &\times \left\{ \sum_{\substack{u\in U, v\in E\\ u\neq v}} \left(\left[\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{h=1}^{n}(-a_{iu}^{(h-1)}\,a_{uj}^{(n-h)} + a_{iu}^{(h-1)}\,a_{vj}^{(n-h)}) \right]\right.\right.\\ &\times \left.\left. \left[\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{h=1}^{n}(-a_{ku}^{(h-1)}\,a_{ul}^{(n-h)} + a_{ku}^{(h-1)}\,a_{vl}^{(n-h)}) \right] \left[\frac{a_{uv}\,\rho}{\mathbf{A}^{(u,u)}}\right] \right) \right\}.\end{aligned} \tag{26}$$

Proof. We follow the same steps as in the preceding proof, but with the definition of reliability and its estimator as is given in (13) and (14). We get that the random variable $\sqrt{T}(\hat{R}(t,T) - R(t))$ converges in distribution, as T tends to infinity, to a zero mean normal random variable with variance

$$\sigma_R^2(t) = \Phi_1' \cdot \Lambda_{11}(t) \cdot \Phi_1'^{\top}, \tag{27}$$

where Φ_1' is a row vector of first order derivatives of $\Phi_1 = \sum_{i,j=1}^{r} \alpha(i)\,p_{ij}(t)$ with respect to $p_{ij}(t)$ for all $i, j \in U$, and $\Lambda_{11}(t)$ is the limit of covariance matrix for the random vector

$(T^{1/2}(\hat{p}_{ij}(t;T) - p_{ij}(t)))_{i,j\in U}$, as T tends to infinity. From the proof of Theorem 1, we have

$$\Lambda_{11}(t) = \sum_{\substack{u\in U, v\in E\\ u\neq v}} \left(\left[\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} (-a_{iu}^{(h-1)}\, a_{uj}^{(n-h)} + a_{iu}^{(h-1)}\, a_{vj}^{(n-h)}) \right] \right.$$
$$\left. \times \left[\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} (-a_{ku}^{(h-1)}\, a_{ul}^{(n-h)} + a_{ku}^{(h-1)}\, a_{vl}^{(n-h)}) \right] \left[\frac{a_{uv}\,\rho}{\mathbf{A}^{(u,u)}} \right] \right). \tag{28}$$

From relations (27) and (28), we get the desired result.

Theorem 5. *For any fixed $t \geqslant 0$, $\sqrt{T}(\hat{\lambda}(t,T) - \lambda(t))$ converges in distribution, as T tends to infinity, to a centered normal random variable with variance $\sigma_\lambda^2(t)$, defined as follows*

$$\sigma_\lambda^2(t) = \sum_{\substack{u,v=1\\ u\neq v}}^{r} (Z'_{uv}(t))^2\, a_{uv}\,\rho/\mathbf{A}^{(u,u)},$$

where

$$Z'_{uv}(t) = \frac{1}{R(t)^2}$$
$$\times \left\{ \mathbf{r}(t) \left[\sum_{i,j=1}^{r} \alpha(i) \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} (-a_{iu}^{(h-1)}\, a_{uj}^{(n-h)} + a_{iu}^{(h-1)}\, a_{vj}^{(n-h)}) \right] \right.$$
$$\left. - R(t) \left[\sum_{i,j=1}^{r} \alpha(i) \sum_{k=1}^{r} \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} (-a_{iu}^{(h-1)} a_{uj}^{(n-h)} + a_{iu}^{(h-1)} a_{vk}^{(n-h)}) \right) a_{kj} \right] \right\}. \tag{29}$$

Proof. From the definition of the failure rate and its estimator in (15) and (16), we have

$$\sqrt{T}(\hat{\lambda}(t,T) - \lambda(t)) = \sqrt{T} \left(-\frac{\hat{\mathbf{r}}(t,T)}{\hat{R}(t,T)} + \frac{\mathbf{r}(t)}{R(t)} \right).$$

We write $-\frac{\mathbf{r}(t)}{R(t)}$ as a function of the row random vector $((a_{uv})_{u,v\in U,u\neq v})$, as follows

$$Z((a_{uv})_{u,v\in U}) = -\frac{\mathbf{r}(t)}{R(t)} = -\frac{\alpha_1\, \mathrm{e}^{t\mathbf{A}_{11}}\mathbf{A}_{11}\, \mathbf{1}_r}{\alpha_1\, \mathrm{e}^{t\mathbf{A}_{11}}\, \mathbf{1}_r}.$$

By computing the first order derivatives of this function with respect to the vector $((a_{uv})_{v,v\in U,u\neq v})$, we obtain

$$Z'(t) = \frac{\mathbf{r}(t)R'(t) - R(t)\mathbf{r}'(t)}{R(t)^2},$$

where $R'(t) = ((R'_{uv}(t))_{u,v\in U, u\neq v})$ and $R'_{uv}(t)$ denotes the first derivative of R with respect to a_{uv},

$$\begin{aligned} R'_{uv}(t) = \frac{\partial R(t)}{\partial a_{uv}} &= \sum_{i,j=1}^{r} \alpha(i) \left(\frac{\partial \mathrm{e}^{t\mathbf{A}_{11}}}{\partial a_{uv}} \right)_{ij} \\ &= \sum_{i,j=1}^{r} \alpha(i) \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} (-a_{iu}^{(h-1)}\, a_{uj}^{(n-h)} + a_{iu}^{(h-1)}\, a_{vj}^{(n-h)}), \end{aligned}$$

and

$$\begin{aligned} \mathbf{r}'_{uv}(t) = \frac{\partial \mathbf{r}(t)}{\partial a_{uv}} &= \sum_{i,j=1}^{r} \alpha(i) \left(\frac{\partial\, \mathrm{e}^{t\mathbf{A}_{11}} \mathbf{A}_{11}}{\partial a_{uv}} \right)_{ij} \\ &= \sum_{i,j=1}^{r} \alpha(i) \left[\sum_{k=1}^{r} \left(\frac{\partial \mathrm{e}^{t\mathbf{A}_{11}}}{\partial a_{uv}} \right)_{ik} a_{kj} + \sum_{k=1}^{r} (\mathrm{e}^{t\mathbf{A}_{11}})_{ik} \left(\frac{\partial \mathbf{A}_{11}}{\partial a_{uv}} \right)_{kj} \right], \end{aligned}$$

where

$$\left(\frac{\partial \mathbf{A}_{11}}{\partial a_{uv}} \right)_{ij} = \begin{cases} 1 & \text{if } i = u,\, v = j \\ -1 & \text{if } i = j = u = v \\ 0 & \text{otherwise.} \end{cases}$$

We can easily see that $\sum_{i,j=1}^{r} \alpha(i)\, (\sum_{k=1}^{r} (\mathrm{e}^{t\mathbf{A}_{11}})_{ik} (\frac{\partial \mathbf{A}_{11}}{\partial a_{uv}})_{kj}) = 0$. Then we obtain $Z'_{uv}(t)$ as in relation (29).

All the first order derivatives of Z with respect to a_{uv} for all $u, v \in U$, $u \neq v$ exist and are continuous. Thus Z is differentiable and we note from Theorem A that the vector function $\{T^{1/2}(\hat{a}_{uv}(T) - a_{uv})\}_{u,v=1,u\neq v}^{r}$ converges, as T tends to infinity, to the centered normal distribution with covariance matrix $H_1 = (H(u,v;k,l))_{r(r-1)\times r(r-1)}$, where $H(u,v;k,l)$ is defined in relation (7). Then, using the delta method, we obtain, as $T \to \infty$,

$$\sqrt{T}\,(Z((\hat{a}_{uv}(T))_{\substack{u,v\in U\\ u\neq v}}) - Z((a_{ij})_{\substack{u,v\in U\\ u\neq v}})) \xrightarrow{d} \mathcal{N}\ (0,\ Z'\, H_1\, Z'^{\top}).$$

Using the same technique, we obtain the consistency and the asymptotic normality for these estimators when we observe many independent realizations of the process, as K (the number of realizations) tends to infinity. We present these results in the next theorem. The proofs of these results are the same as above, when we use Theorem B instead of Theorem A.

Theorem 6. (i) *The maximum likelihood estimators of availability, reliability and failure rate functions of a Markov process are uniformly strongly consistent, as K tends to infinity.*

(ii) *For any fixed T and* $t \geqslant 0$, $\sqrt{K}(\hat{A}(t,K) - A(t))$ *converges in distribution, as K tends to infinity, to a centered at expectation normal random variable with variance* $\sigma^2_{A_1}(t)$, *where*

$$\begin{aligned}\sigma^2_{A_1}(t) = & \sum_{i,k=1}^{s} \sum_{j,l=1}^{r} \alpha(i)\,\alpha(k) \\ & \times \left\{ \sum_{\substack{u,v=1 \\ u \neq v}}^{s} \left(\left[\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} (-a^{(h-1)}_{iu}\, a^{(n-h)}_{uj} + a^{(h-1)}_{iu}\, a^{(n-h)}_{vj}) \right] \right. \right. \\ & \left. \left. \times \left[\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} (-a^{(h-1)}_{ku} a^{(n-h)}_{ul} + a^{(h-1)}_{ku} a^{(n-h)}_{vl}) \right] \left[\frac{a_{uv}}{v_u(T)} \right] \right) \right\}. \qquad (30)\end{aligned}$$

(iii) *For any fixed T and* $t \geqslant 0$, $\sqrt{K}(\hat{R}(t,K) - R(t))$ *converges in distribution, as K tends to infinity, to a centered at expectation normal random variable with variance* $\sigma^2_{R_1}(t)$, *where*

$$\begin{aligned}\sigma^2_{R_1}(t) = & \sum_{i,k=1}^{r} \sum_{j,l=1}^{r} \alpha(i)\alpha(k) \\ & \times \left\{ \sum_{\substack{u \in U, v \in E \\ u \neq v}} \left(\left[\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} (-a^{(h-1)}_{iu} a^{(n-h)}_{uj} + a^{(h-1)}_{iu} a^{(n-h)}_{vj}) \right] \right. \right. \\ & \left. \left. \times \left[\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{h=1}^{n} (-a^{(h-1)}_{ku} a^{(n-h)}_{ul} + a^{(h-1)}_{ku} a^{(n-h)}_{vl}) \right] \left[\frac{a_{uv}}{v_u(T)} \right] \right) \right\}. \qquad (31)\end{aligned}$$

(iv) *For any fixed T and* $t \geqslant 0$, $\sqrt{K}(\hat{\lambda}(t,K) - \lambda(t))$ *converges in distribution, as K tends to infinity, to a centered at expectation normal random variable with variance* $\sigma^2_{\lambda_1}(t)$, *where*

$$\sigma^2_{\lambda_1}(t) = \sum_{\substack{u,v=1 \\ u \neq v}}^{r} (Z'_{uv}(t))^2\, a_{uv}/v_u(T).$$

Using the asymptotic results for availability, reliability and failure rate estimators of a Markov process we can construct confidence intervals for these estimators. In order to obtain the confidence intervals of availability, we estimate the variance, $\sigma_A(t)$, by replacing all the elements of the generator **A** with their estimators in relation (24). The resulting confidence intervals at level $100(1-\gamma)\%$, $\gamma \in]0,1[$ is given by

$$\hat{A}_T(t) - z_{\gamma/2} \frac{\hat{\sigma}_A(t)}{\sqrt{T}} \leqslant A(t) \leqslant \hat{A}_T(t) + z_{\gamma/2} \frac{\hat{\sigma}_A(t)}{\sqrt{T}} \ ,$$

where $z_{\gamma/2}$ is given by $\frac{1}{\sqrt{2\pi}} \int_{-z_{\gamma/2}}^{z_{\gamma/2}} \mathrm{e}^{-\frac{t^2}{2}}\, \mathrm{d}t = 1 - \gamma$.

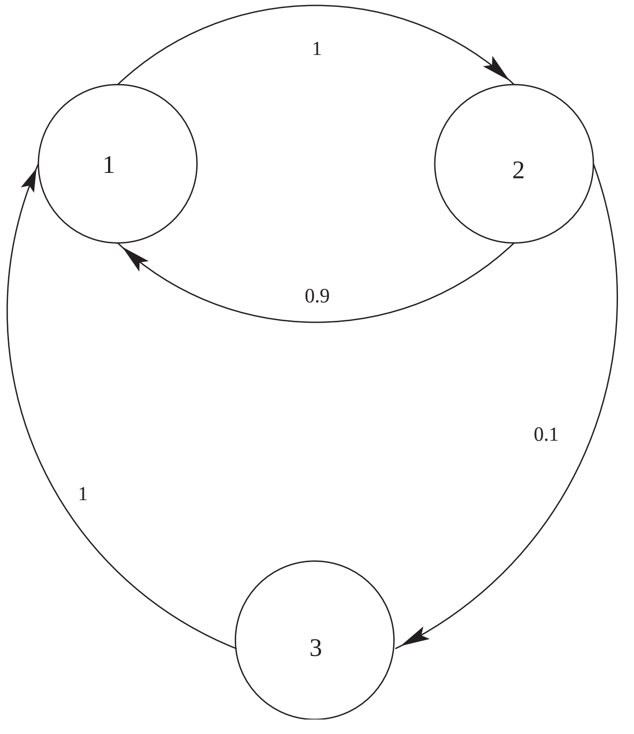

Fig. 1. Graph of states.

Using the same technique we can obtain confidence intervals of reliability and failure rate.

5. Numerical example

Let us consider a Markov system whose graph of states is given by Fig. 1. This system has three states: states 1 and 2 are working states and state 3 is a failure state. Thus $U = \{1, 2\}$ and $D = \{3\}$. The generator matrix $\mathbf{A}$ for this system is the following

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 & 0 \\ 0.027 & -0.03 & 0.003 \\ 0.01 & 0 & -0.01 \end{bmatrix}.$$

The sojourn times in the different states are independent and follow exponential distributions with parameters 0.02, 0.03, and 0.01, and we suppose that the initial distribution is known. The numerical values of transition rates between states are similar to those encountered in a reliability study of production systems.

We generate one sample trajectory of this system in the time interval $[0, T]$. We use the embedded Markov chain method for the simulation of this system. This method consists

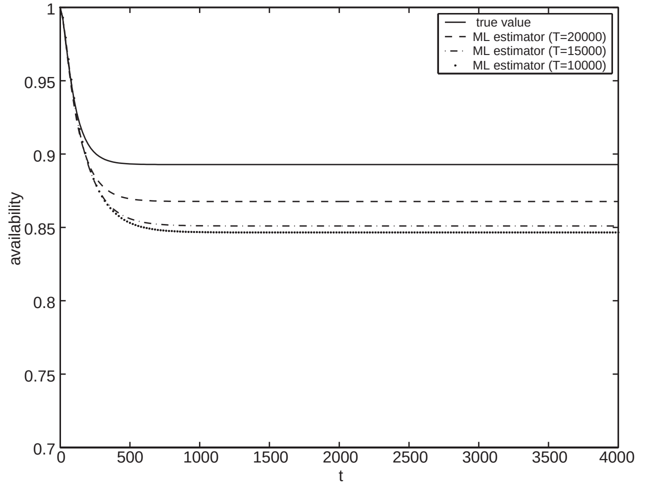

Fig. 2. Maximum likelihood estimator of availability.

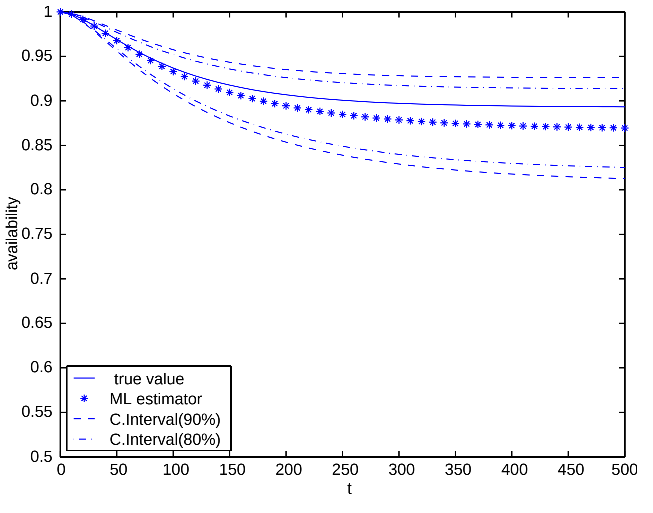

Fig. 3. Confidence intervals for availability at levels 90% and 80%.

simply in using the transition probabilities of the embedded Markov chain in order to find the next state of the system and in generating the corresponding holding times to the visited states by using the exponential distribution functions.

Fig. 2 presents the estimation of availability of this system for different $T = 10000$, 15000, 20000. We remark here the consistency of the availability estimator when the total time of observation T becomes large. It should be mentioned here that the substantial constant bias for $t > 500$ is due to the ergodicity of the system, i.e., $\lim_{t\to\infty} p_{ij}(t) = \pi_j$. In Fig. 3,

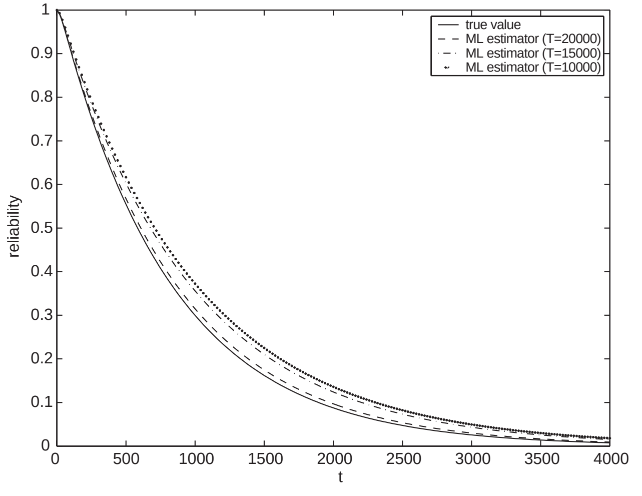

Fig. 4. Maximum likelihood estimator of reliability.

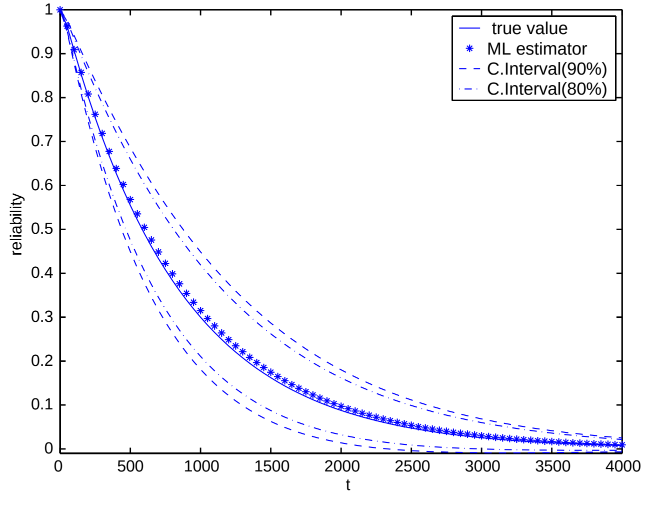

Fig. 5. Confidence intervals for reliability at levels 90% and 80%.

we present the confidence intervals at levels 80% and 90%, for the availability when the total time $T = 20,000$. Figs. 4 and 6 present the consistency for the reliability estimator and the failure rate estimator, respectively. In Figs. 5 and 7 we present the confidence intervals, at levels 80% and 90%, for the reliability and the failure rate when the total time $T = 20,000$. We remark that the confidence intervals for these estimators contain all the true value.

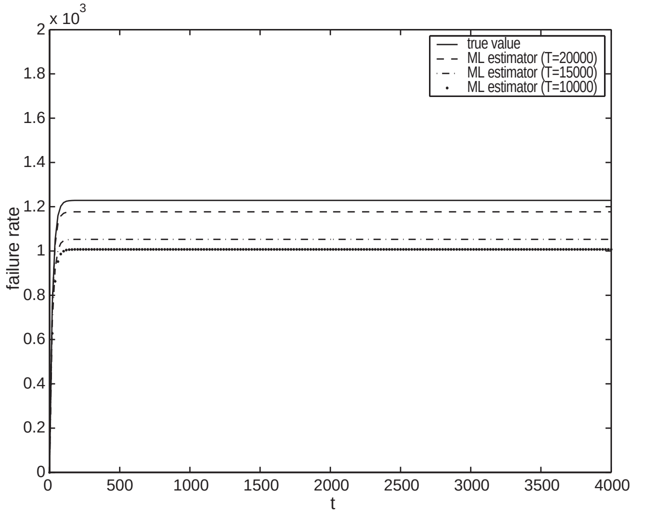

Fig. 6. Maximum likelihood estimator of failure rate.

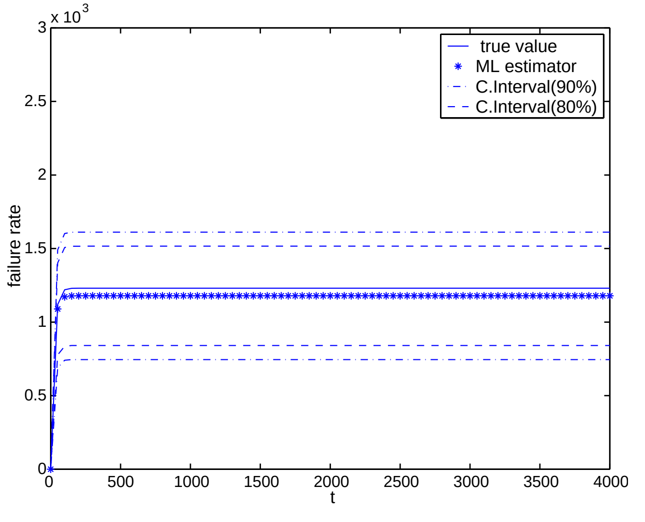

Fig. 7. Confidence intervals for failure rate at levels 90% and 80%.

Let us compare the maximum likelihood estimator for reliability with the usual empirical estimator. We use the same numerical example for comparing these estimators, but in this case we use the stationary distribution to calculate the initial distribution. From the relation, $\pi\mathbf{A}=\underline{0}$, we can calculate the stationary distribution π. Then we generate a trajectory using

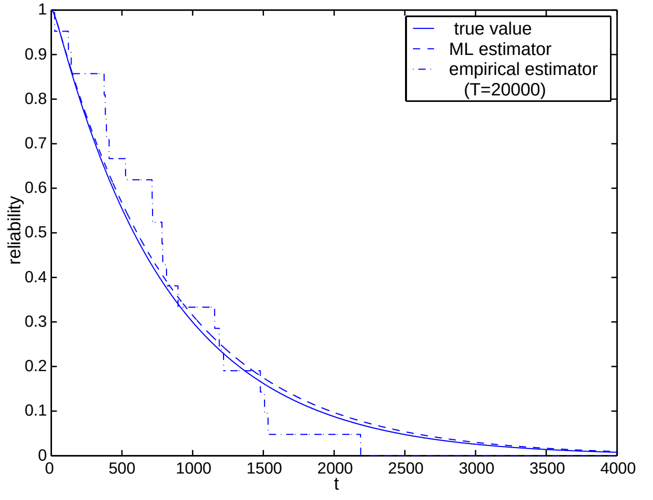

Fig. 8. Reliability estimators comparison.

the following initial distribution,

$$\alpha(1) = \frac{\pi(1)}{\pi(1) + \pi(2)} \quad \text{and} \quad \alpha(2) = \frac{\pi(2)}{\pi(1) + \pi(2)}.$$

From this trajectory, we calculate the maximum likelihood estimator as above and the empirical estimator for reliability.

Let $L_1, \ldots, L_n$ be the continuous random variables of successive sojourn times in working states U when transitions are performed from the working state to the failure state.

Then the empirical reliability in this case is defined by

$$R_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{L_i > t\}}, \quad t \in \mathbb{R}_+.$$

In Fig. 8 we present the comparison between the empirical and maximum likelihood estimators of reliability. We notice that the maximum likelihood estimator is closer to the true value than the empirical estimator.

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