Cosmologcial Perturbation Theory

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Preface

This reading note is mainly based on the perturbation part of Daniel Baumann's textbook *Cosmology* and Viatcheslav Mukhanov's textbook *Physical Foundation of Cosmology*

For convenience, We set

$$c = \hbar = k_B = 1$$

We use the metric with signature

$$(-,+,+,+)$$

We use the notation

$$\mu,\nu,\rho,\lambda,\cdots = 0,1,2,3$$
$$i,j,k,l,\cdots = 1,2,3$$

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1 The Homogeneous Universe

1.1 The FRW Metric

The FRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 (\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2) \tag{1.1}$$

The redshift parameter

$$z+1 = \frac{\lambda_0}{\lambda_1} = \frac{a(t_0)}{a(t_1)} \tag{1.2}$$

The conformal time

$$ad\eta = dt \tag{1.3}$$

1.2 Friedmann Equation

The Einstein equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.4}$$

The temporal component of the Einstein equation gives

$$(\frac{\dot{a}}{a})^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho \tag{1.5}$$

and the the spatial components give

$$2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{K}{a^2} = -8\pi G p$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{1.6}$$

1.3 Equation of State

Energy density ρ, pressure P

$$P = w\rho \tag{1.7}$$

- Dust: $w = \frac{1}{3}$
- Matter: $w = 0$
- Vacuum: $w = -1$

We assume that the fluid is at rest in the preferred frame of the universe, meaning that $U^\mu = (1, 0, 0, 0)$ in the FRW coordinates. And we have

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{1.8}$$

$$\frac{\dot{\rho}}{\rho} + 3(1+w)\frac{\dot{a}}{a} = 0 \tag{1.9}$$

Using $a_0 = 1$, we may solve it and get

$$\rho = \bar{\rho} a^{-3(1+w)} \tag{1.10}$$

1. For dust, $w = 0$ and so $\rho_m \propto a^{-3}$
2. For radiation, $w = \frac{1}{3}$ and so $\rho_m \propto a^{-4}$
3. For vacuum, dark energy or cosmologcial constant, $w = -1$ and so $\rho_\Lambda = \text{Const}$

1.4 Density Parameter

Set $\Lambda = 0$, $K = 0$ and solve the Friedmann equation (1.5), we have

$$\rho_c = \frac{3H^2}{8\pi G} \tag{1.11}$$

Define the density parameter

$$\Omega := \frac{\rho}{\rho_c} \tag{1.12}$$

Then we can rewrite the Friedmann equation (1.5) as

$$\Omega - 1 = \frac{K}{a^2 H^2} \tag{1.13}$$

We divide the density into its matter, radiation, and vacuum components $\rho = \rho_m + \rho_r + \rho_\Lambda$, and likewise for the density parameter

$$\Omega = \Omega_m + \Omega_r + \Omega_\Lambda \tag{1.14}$$

where

$$\Omega_m := \frac{\rho_m}{\rho_c} \qquad \Omega_r := \frac{\rho_r}{\rho_c} \qquad \Omega_\Lambda := \frac{\rho_\Lambda}{\rho_c} := \frac{\Lambda}{3H^2}$$

Define

$$\Omega_K = -\frac{K}{H^2} \tag{1.15}$$

and we may find

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_K = 1 \tag{1.16}$$

We can now write the Friedmann equation (1.5) as

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_\Lambda) - \frac{K}{a^2} \\ &= \frac{H_0^2}{\rho_c}(\rho_{m0}a^{-3} + \rho_{r0}a^{-4} + \rho_K a^{-2} + \rho_\Lambda) \\ &= H_0^2(\Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{K0}a^{-2} + \Omega_{\Lambda 0}) \end{aligned}$$

so that H can be evaluated as a function of a

$$H(a) = H_0\sqrt{\Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{K0}a^{-2} + \Omega_{\Lambda 0}} \tag{1.17}$$

2 Newtonian Perturbation Theory

Newtonian perturbation theory is used at sub-horizon scale, and relativistic perturbation theory is used at super-horizon and at-horizon scale.

2.1 Backgroud and Perturbations

Continuity equation,Euler equation, entropy conservation, poisson equation and the equation of state(ref:Mukhanov)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\nabla p}{\rho} + \nabla \phi = 0 \tag{2.2}$$

$$\frac{\partial S}{\partial t} + (\mathbf{v} \cdot \nabla) S = 0 \tag{2.3}$$

$$\nabla^2 \Phi = 4\pi G \rho \tag{2.4}$$

$$P = P(\rho, S) \tag{2.5}$$

Perturbation(Homogeneous + Perturbation)

$$\begin{aligned} \rho(t, \mathbf{r}) &= \bar{\rho}(t) + \delta\rho(t, \mathbf{r}) \\ \mathbf{v}(t, \mathbf{r}) &= \bar{\mathbf{v}}(t) + \delta\mathbf{v}(t, \mathbf{r}) = \delta\mathbf{v}(t, \mathbf{r}) \\ P(t, \mathbf{r}) &= \bar{P}(t) + \delta p(t, \mathbf{r}) \\ \Phi(t, \mathbf{r}) &= \bar{\Phi}(t) + \delta\Phi(t, \mathbf{r}) \\ S(t, \mathbf{r}) &= \bar{S}(t) + \delta S(t, \mathbf{r}) \end{aligned} \tag{2.6}$$

Subsituting the perturbations into the equation and keeping the linear terms only

$$\frac{\partial \delta\rho}{\partial t} + \bar{\rho} \nabla \cdot (\delta \mathbf{v}) = 0 \tag{2.7}$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} + \frac{1}{\bar{\rho}} \nabla \delta P + \nabla \delta \Phi = 0 \tag{2.8}$$

$$\frac{\partial \delta S}{\partial t} = 0 \tag{2.9}$$

$$\nabla^2 \delta\Phi = 4\pi G \delta\rho \tag{2.10}$$

$$\delta P = c_s^2 \delta\rho + \sigma \delta S \tag{2.11}$$

where c_s is the sound speed. All the five equations lead us to the linear equation of $\delta\rho$

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c_S^2 \nabla^2 \delta\rho - 4\pi G \bar{\rho} \delta\rho = \sigma \nabla^2 \delta S \tag{2.12}$$

For convenience, we can always analyse the perturbations in Fourier space

$$\delta\rho(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}} \int d^3k \, \delta\rho(\mathbf{k},t) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.13}$$

Adiabatic Perturbations

We assume that the entropy perturbations are absent, $\delta S = 0$. When we analyse the perturbations in Fourier space, (2.12) becomes

$$\ddot{\delta\rho}_{\mathbf{k}} + (k^2 c_S^2 - 4\pi G\bar{\rho})\delta\rho_{\mathbf{k}} = 0 \tag{2.14}$$

which have two independent solutions

$$\delta\rho_{\mathbf{k}} = Ae^{i\omega(k)t} + Ae^{-i\omega(k)t} \tag{2.15}$$

$$[\omega(k)]^2 = k^2 c_s^2 - 4\pi G\bar{\rho} \tag{2.16}$$

Defining the Jeans length as

$$\lambda_J = \frac{2\pi}{k_J} = c_S \left(\frac{\pi}{G\bar{\rho}}\right)^{1/2} \tag{2.17}$$

where $\omega(k_J) = 0$

If $\lambda < \lambda_J$, the solutions describe sound waves.

If $\lambda > \lambda_J$, one of these solutions describes the **exponentially fast** growth of inhomogeneities, while the other corresponds to a decaying mode. Obviously, in our universe, the bigger the lentgh, the less stable it will be against gravitational collapse.

Entropy Perturbations

In the presence of entropy inhomogeneities, the equation becomes

$$\ddot{\delta\rho}_{\mathbf{k}} + (k^2 c_S^2 - 4\pi G\bar{\rho})\delta\rho_{\mathbf{k}} = -\sigma k^2 \delta S_{\mathbf{k}} \tag{2.18}$$

The solution of this equation can be written as the sum of its particular solution and a general solution. The general solution is the solution of adiabatic perturbation, the particular solution is called the entropy perturbation

$$\delta\rho_{\mathbf{k}} = -\frac{\sigma k^2 \delta S_{\mathbf{k}}}{k^2 c_S^2 - 4\pi G\bar{\rho}} \tag{2.19}$$

Entropy perturbations can occur only in **multi-component fluids**(why?). For example, in a fluid consisting of baryons and radiation, the baryons can be distributed inhomogeneously on a homogeneous background of radiation.

Vector Perturbations

When $\delta\rho = 0$, $\delta S = 0$, we have

$$\frac{\partial \delta \mathbf{v}}{\partial t} = 0, \qquad \nabla \cdot \delta \mathbf{v} = 0 \tag{2.20}$$

$$\frac{\partial \delta \mathbf{v_k}}{\partial t} = 0, \qquad \mathbf{k} \cdot \delta \mathbf{v_k} = 0 \tag{2.21}$$

So, the vector perturbations describe shear motions of the media which do not disturb the energy density.

2.2 Adding Expansions

This dark energy changes the expansion rate and, as a result, influences the growth of inhomogeneities in the cold matter.

Backgroud

In an expanding homogeneous and isotropic universe, the only difference is that the background velocities is not zero but obey the Hubble law

$$\bar{\mathbf{v}} = H\mathbf{r} \tag{2.22}$$

Substituting these expressions into (2.1) we have

$$\dot{\bar{\rho}} + 3H\bar{\rho} = 0 \tag{2.23}$$

into (2.2,2.4) we have

$$\dot{H} + H^2 = -\frac{3\pi G}{4}\bar{\rho} \tag{2.24}$$

Perturbations

We consider only adiabatic perturbations

$$\begin{aligned}
\rho(t,\mathbf{r}) &= \bar{\rho}(t) + \delta\rho(t,\mathbf{r}) \\
\mathbf{v}(t,\mathbf{r}) &= \bar{\mathbf{v}}(t) + \delta\mathbf{v}(t,\mathbf{r}) = H\mathbf{r} + \delta\mathbf{v}(t,\mathbf{r}) \\
P(t,\mathbf{r}) &= \bar{P}(t) + \delta P(t,\mathbf{r}) = \bar{P}(t) + c_S^2\delta\rho(t,\mathbf{r}) \\
\Phi(t,\mathbf{r}) &= \bar{\Phi}(t) + \delta\Phi(t,\mathbf{r})
\end{aligned} \tag{2.25}$$

then we have the linear equations

$$\frac{\partial \delta\rho}{\partial t} + \bar{\rho}\nabla\cdot(\delta\mathbf{v}) + \delta\rho\nabla\cdot\bar{\mathbf{v}} = 0 \tag{2.26}$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} + (\bar{\mathbf{v}}\cdot\nabla)\delta\mathbf{v} + (\delta\mathbf{v}\cdot\nabla)\bar{\mathbf{v}} + \frac{c_S^2}{\bar{\rho}}\nabla\delta P + \nabla\delta\Phi = 0 \tag{2.27}$$

$$\nabla^2\delta\Phi = 4\pi G a^2\delta\rho \tag{2.28}$$

Comiving Coordinates

The coordinates changes as

$$t' = t \qquad \mathbf{r} = a(t)\mathbf{x} \tag{2.29}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - H\mathbf{x}\cdot\nabla_{\mathbf{x}} \tag{2.30}$$

$$\nabla_{\mathbf{r}} = \frac{1}{a}\nabla_{\mathbf{x}} \tag{2.31}$$

We finally obtain our equation in comiving coordinates

$$\frac{\partial \delta\rho}{\partial t} + 3H\delta\rho + \frac{\bar{\rho}}{a}\nabla\delta\mathbf{v} = 0 \tag{2.32}$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} + H\delta\mathbf{v} + \frac{c_S^2}{a\bar{\rho}}\nabla\delta\rho + \frac{1}{a}\nabla\phi = 0 \tag{2.33}$$

$$\nabla^2\phi = 4\pi G a^2\delta\rho \tag{2.34}$$

which lead to

$$\ddot{\delta} + 2H\dot{\delta} - (\frac{c_S^2}{a^2}\nabla^2 + 4\pi G\bar{\rho})\delta = 0 \tag{2.35}$$

which describes gravitational instability in an expanding universe. Then we can analyse the equation in Fourier space like before

$$\ddot{\delta}_{\mathbf{k}} + 2H\dot{\delta}_{\mathbf{k}} + (\frac{c_S^2 k^2}{a^2} - 4\pi G\bar{\rho})\delta_{\mathbf{k}} = 0 \tag{2.36}$$

The second term is damping term, and the third term Discribes the contribution of gavity. In the k-mode, we have $\delta = \delta_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}$, and the Jeans length in comiving coordinates is

$$\lambda_J = \frac{2\pi}{k_J} = \frac{c_S}{a}\left(\frac{\pi}{G\bar{\rho}}\right)^{1/2} \tag{2.37}$$

1. For scales much smaller than the Jeans length, $k \gg k_J$, we have

$$\ddot{\delta}_{\mathbf{k}} + 2H\dot{\delta}_{\mathbf{k}} + \frac{c_S^2 k^2}{a^2}\delta_{\mathbf{k}} = 0 \tag{2.38}$$

$$\delta_{\mathbf{k}} \propto \frac{1}{\sqrt{c_s a}}\exp\left(\pm ik\int\frac{c_s}{a}dt\right) \tag{2.39}$$

2. For scales much larger than the Jeans length, $k \ll k_J$, we have

$$\ddot{\delta}_{\mathbf{k}} + 2H\dot{\delta}_{\mathbf{k}} - 4\pi G\bar{\rho}\delta_{\mathbf{k}} = 0 \tag{2.40}$$

Vector Perturbations

When $\delta = 0$, we have the vector perturbation

$$\frac{\partial \delta \mathbf{v}}{\partial t} + H\delta \mathbf{v} = 0, \qquad \nabla \cdot \delta \mathbf{v} = 0 \tag{2.41}$$

Obviously, the vector perturbations decay as the universe expands.

In an inflationary universe there is no room for such large primordial vector perturbations and they do not play any role in the formation of the large-scale structure of the universe. However, they can be generated at late times, after nonlinear structure has been formed, and can explain the rotation of galaxies.

2.3 Growth of Matter Perturbation

$$\ddot{\delta}_{\mathbf{k}} + 2H\dot{\delta}_{\mathbf{k}} - 4\pi G\bar{\rho}\delta_{\mathbf{k}} = 0 \tag{2.42}$$

Matter Dominated Universe

In a flat matter-dominated universe we have:

$$a \propto t^{2/3} \qquad H = \frac{2}{3t} \qquad 4\pi G\bar{\rho}_m = \frac{3}{2}H^2 = \frac{2}{3t^2} \tag{2.43}$$

then

$$\ddot{\delta}_{\mathbf{k}} + \frac{4}{3t}\dot{\delta}_{\mathbf{k}} - \frac{2}{3t^2}\delta_{\mathbf{k}} = 0 \tag{2.44}$$

$$\delta_{\mathbf{k}} = C_1 t^{3/2} + C_2 t^{-1} = C_1' a + C_2' a^{-3/2} \tag{2.45}$$

In addiction, using the Piosson euqation $\nabla^2\delta\Phi = 4\pi G a^2 \delta\rho$ we found that

$$\Phi = \text{Const} \tag{2.46}$$

in the matter dominated era

Radiation Dominated Universe

In a flat radiation-dominated universe we have:

$$a \propto t^{1/2} \qquad H = \frac{1}{2t} \qquad \bar{\rho}_m \ll \bar{\rho}_r \tag{2.47}$$

then

$$\ddot{\delta}_{\mathbf{k}} + \frac{1}{t}\dot{\delta}_{\mathbf{k}} - 4\pi G(\delta_m \delta_{m,\mathbf{k}} + \delta_r \delta_{r,\mathbf{k}}) = 0 \tag{2.48}$$

Since the radiation fluid has a large sound speed, we expect its fluctuations to oscillate on scales smaller than the horizon. Furthermore, $\rho_r \gg \rho_m$ in radiation era, we can neglect the last term and the equation becomes

$$\ddot{\delta}_{\mathbf{k}} + \frac{1}{t}\dot{\delta}_{\mathbf{k}} = 0 \tag{2.49}$$

$$\delta_{\mathbf{k}} = C_1 \ln t + C_2 = C_1' \ln a + C_2' \tag{2.50}$$

We can find that

$$\Phi \propto a^{-2} \tag{2.51}$$

in the radiation dominated era

Dark Energy Dominated Universe

In a flat Λ-dominated universe we have:

$$H = \text{Const} \qquad 4\pi G\bar{\rho}_m \ll H^2 \tag{2.52}$$

Since $4\pi G\bar{\rho}_m \ll H^2$ we can drop the last term, and

$$\ddot{\delta}_{\mathbf{k}} + 2H\dot{\delta}_{\mathbf{k}} = 0 \tag{2.53}$$

$$\delta_{\mathbf{k}} = C_1 + C_2 a^{-2} \tag{2.54}$$

We can find that

$$\Phi \propto a^{-1} \tag{2.55}$$

in the dakr energy era.

2.4 Summary

Table 1: Summary of the evolution of perturbations

	Radiation Era	Matter Era	Dark Energy era
Φ	a^{-2}	Const	a^{-1}
δ_m	$\ln a$	a	Const

3 Relativistic Perturbation Theory

The FRW metric and energy momentum tensor in the form of conformal time

$$d^2s = a^2(\eta)(-d^2\eta + d^2x) \qquad T_{\mu\nu} = \mathrm{diag}(\rho, \mathbf{p}) \tag{3.1}$$

Perturbations

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \tag{3.2}$$

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \tag{3.3}$$

3.1 Metric Perturbations

SVT Decomposition

Counting the number of independent functions used to form $h_{\mu\nu}$, we find

1. Four functions for the scalar perturbations
2. Four functions for the vector perturbations (two 3-vectors with one constraint each)
3. Two functions for the tensor perturbations (a symmetric 3-tensor has six independent components and there are four constraints).

So under the SVT(scalar-vector-tensor) decomposition the spacetime can be written as

$$ds^2 = a^2(\eta)[-(1+2A)d\eta^2 + 2B_i dx^i d\eta + (\delta_{ij} + 2D_{ij})dx^i dx^j] \tag{3.4}$$

where the three-vectors B_i can be split into the gradient of a scalar B and a divergenceless vector $\hat{B}_i$

$$B_i = \partial_i B + \hat{B}_i \tag{3.5}$$

and any rank-2 symmetric tensor can be written into

$$D_{ij} = C\delta_{ij} + \partial_{\langle i}\partial_{j\rangle} D + \partial_{(i}\hat{D}_{j)} + \hat{D}_{ij} \tag{3.6}$$

where $\hat{D}_{ij}$ are traceless and transverse, and

$$\partial_{\langle i}\partial_{j\rangle} D = (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)D \tag{3.7}$$

$$\partial_{(i}\hat{D}_{j)} = \frac{1}{2}(\partial_i\hat{D}_j + \partial_j\hat{D}_i) \tag{3.8}$$

So we can concluded that

1. **Scalar perturbations.** Scalar perturbations are characterized by the four scalar functions A, B, C, D. They are induced by energy density inhomogeneities. They exhibit gravitational instability and may lead to the formation of structure in the universe.
2. **Vector perturbations.** Described by the two vectors $\hat{B}_i$ and $\hat{D}_i$, vector perturbations are related to the rotational motions of the fluid. As in Newtonian theory, they decay very quickly. and are not very interesting from the point of view of cosmology.
3. **Tensor perturbations.** Tensor perturbations $\hat{D}_{ij}$ have no analog in Newtonian theory. They describe gravitational waves.

Now scalar, vector and tensor perturbations are decoupled and thus can be studied separately.

Coordinate Transformations

Consider an infinitesimally small coordinate transformation

$$x^\mu \to \widetilde{x}^\mu = x^\mu + \xi^\mu \tag{3.9}$$

Our metric will also transformation

$$\widetilde{g}_{\mu\nu}(\widetilde{x}) = \frac{\partial x^\alpha}{\partial \widetilde{x}^\mu}\frac{\partial x^\beta}{\partial \widetilde{x}^\nu} g_{\alpha\beta}(x) \tag{3.10}$$

And we can expand $\widetilde{g}_{\mu\nu}(\widetilde{x})$ near x

$$\widetilde{g}_{\mu\nu}(\widetilde{x}) = \widetilde{g}_{\mu\nu}(x) + \frac{\partial \widetilde{g}_{\mu\nu}}{\partial x^\alpha}\xi^\alpha + \cdots \tag{3.11}$$

If we give a perturbation to the metric

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{3.12}$$

and (3.10) and (3.11) will becomes

$$\widetilde{g}_{\mu\nu}(\widetilde{x}) = \bar{g}_{\mu\nu} + h_{\mu\nu} - \bar{g}_{\alpha\nu}\partial_\mu \xi^\alpha - \bar{g}_{\alpha\mu}\partial_\nu \xi^\alpha \tag{3.13}$$

$$\widetilde{g}_{\mu\nu}(\widetilde{x}) = \widetilde{g}_{\mu\nu}(x) + \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\xi^\alpha + \cdots \tag{3.14}$$

It is more convenient to work with so-called gauge transformations, which affect only the field perturbations. For this purpose, after making the coordinate transformation (3.9), we relabel coordinates by dropping the prime on the coordinate argument, and we attribute the whole change in $g_{\mu\nu}(x)$ to a change in the perturbation $h_{\mu\nu}(x)$. The field equations should thus be invariant under the gauge transformation. (Weinberg. Cosmology)

$$h_{\mu\nu} \to h_{\mu\nu} - \bar{g}_{\alpha\nu}\partial_\mu \xi^\alpha - \bar{g}_{\alpha\mu}\partial_\nu \xi^\alpha - \partial_\alpha \bar{g}_{\mu\nu}\xi^\alpha \tag{3.15}$$

Similarly, for scalars and vectors we have

$$\delta q \rightarrow \delta q - \partial_\mu \bar{q} \xi^\mu \tag{3.16}$$

$$\delta u_\mu \rightarrow \delta u_\mu - \bar{u}_\alpha \partial_\mu \xi^\alpha - \bar{u}_\mu \xi^\alpha \tag{3.17}$$

For Perturbations under the FRW metirc, $\xi^\mu = (T, S_i)$ and $S_i = \partial_i S + \hat{S}_i$ we have

$$\begin{aligned}
\widetilde{h}_{00} &= h_{00} + 2a(aT)' = h_{00} + 2a^2(\mathcal{H}T + T') \\
\widetilde{h}_{i0} &= h_{i0} + a^2[(T - S')_{,i} - \hat{S}'_i] \\
\widetilde{h}_{ij} &= h_{ij} - a^2 \left(2\mathcal{H}T\delta_{ij} + 2S_{,ij} + \hat{S}_{i,j} + \hat{S}_{j,i}\right)
\end{aligned} \tag{3.18}$$

By applying SVT decomposition we have

$$\widetilde{A} = A - \mathcal{H}T - T'$$

$$\widetilde{B} = B + T - S' \qquad \widetilde{\hat{B}}_i = \hat{B}_i - \hat{S}'_i$$

$$\widetilde{C} = C - \mathcal{H}T - \frac{1}{3}\nabla^2 S$$

$$\widetilde{D} = D - S \qquad \widetilde{\hat{D}}_i = \hat{D}_i - \hat{S}_i \qquad \widetilde{\hat{D}}_{ij} = \hat{D}_{ij}$$

Gauge-Invariant Variables

One way to avoid the gauge problem is to define special combinations of the metric perturbations that do not transform under a change of coordinates. These are the so-called Bardeen variables

$$\Psi = A + \mathcal{H}(B - D') + (B - D')' \tag{3.19}$$

$$\Phi = -C + \frac{1}{3}\nabla^2 D - \mathcal{H}(B - D') \tag{3.20}$$

Choosing a Gauge

1. **Newtonian gauge**

$$B = D = 0 \tag{3.21}$$

$$ds^2 = a^2(\eta)[-(1 + 2\Psi)d\eta^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j] \tag{3.22}$$

2. **Synchronous gauge**

$$A = B = 0 \tag{3.23}$$

$$ds^2 = a^2(\eta)[-d\eta^2 + (\delta_{ij} + 2D_{ij})dx^i dx^j] \tag{3.24}$$

3.2 Energy momentum Tensor Perturbations

Energy Momentum Tensor

For perfect fluid we have

$$T_{\mu\nu} = P\delta_{\mu\nu} + (P+\rho)U_\mu U_\nu \tag{3.25}$$

with $\bar{U}^\mu = a^{-1}(1,0)$ and $g_{\mu\nu}U^\mu U^\nu = 1$ we have $U^\mu = a^{-1}(1-A, v^i)$

Analogous to the condiction of metric perturbation, the perturbation of energy momentum tensor is

$$\begin{aligned} T_{00} &= \bar{\rho} + \delta\rho \\ T_{i0} &= -(\bar{\rho}+\bar{P})v_i = -q_i \\ T_{ij} &= (\bar{P}+\delta P)\delta_{ij} + \Pi_{ij} \end{aligned} \tag{3.26}$$

where

$$v_i = \partial_i v + \hat{v}_i \tag{3.27}$$

$$\Pi_{ij} = \partial_{\langle i}\partial_{j\rangle}\Pi + \partial_{(i}\hat{\Pi}_{j)} + \hat{\Pi}_{ij} \tag{3.28}$$

Coordinate Transformation

Consider an infinitesimally small coordinate transformation

$$x^\mu \to \widetilde{x}^\mu = x^\mu + \xi^\mu \quad \text{where} \;\; \xi^\mu = (T, S^i) \tag{3.29}$$

$$\widetilde{T}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \widetilde{x}^\mu}\frac{\partial x^\beta}{\partial \widetilde{x}^\nu} T_{\alpha\beta} \tag{3.30}$$

We get

$$\begin{aligned} \delta\widetilde{\rho} &= \delta\rho - \bar{\rho}' T & \qquad \widetilde{q}_i &= q_i + (\bar{\rho}+\bar{P})S_i' & & \\ \delta\widetilde{P} &= \delta P - \bar{P}' T & \qquad \widetilde{v}_i &= v_i + S_i' & \qquad \widetilde{\Pi}_{ij} &= \Pi_{ij} \end{aligned}$$

Gauge-Invariant Variables

As before, we can define specific combinations of variables for which these transformations cancel. There are various gauge-invariant quantities that can be formed from the metric and matter variables

1. **comoving density contrast Δ**

$$\bar{\rho}\Delta = \delta\rho + \bar{\rho}'(v+B) \tag{3.31}$$

2. **curvature perturbations** $\mathcal{R}$ and ζ

$$\mathcal{R} = -C + \frac{1}{3}\nabla^2 D + \mathcal{H}\frac{\delta\rho}{\bar{\rho}'} \tag{3.32}$$

$$\zeta = -C + \frac{1}{3}\nabla^2 D - \mathcal{H}(v + B) \tag{3.33}$$

Choosing a Gauge

1. **Uniform density gauge**

$$\delta\rho = 0 \tag{3.34}$$

The main scalar perturbation in this gauge is the curvature perturbation $\delta g_{ij} = a^2(1-2\zeta)\delta_{ij}$

2. **Comoving gauge**

$$v + B = 0 \tag{3.35}$$

The main scalar perturbation in this gauge is the curvature perturbation $\delta g_{ij} = a^2(1 - 2\mathcal{R})\delta_{ij}$

3.3 The Evolution Equations of Perturbations

We defined the perturbations of the metric and the energy-momentum tensor, and discussed their gauge dependence. Our next task is to derive the evolution equations for these perturbations. The evolution of the metric is governed by the Einstein equation, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, while the evolution of the matter perturbations follows from the conservation of the energy momentum tensor, $\nabla_\mu T^\mu{}_\nu = 0$.

It will be convenient to perform this analysis in a fixed gauge and we will take this to be the Newtonian gauge

$$ds^2 = a^2(\eta)[-(1+2\Psi)d\eta^2 + (1-2\Phi)\delta_{ij}dx^i dx^j] \tag{3.36}$$

Conservation Equations

The conservation equations

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu{}_{\alpha\mu} T^\alpha{}_\nu - \Gamma^\alpha{}_{\nu\mu} T^\mu{}_\alpha = 0 \tag{3.37}$$

The temporal part gives the continuity equation

$$\delta\rho' = -3\mathcal{H}(\delta\rho + \delta P) - \partial_i q^i + 3\Phi'(\bar{\rho} + \bar{P}) \tag{3.38}$$

$$\delta' = -\left(1 + \frac{\bar{P}}{\bar{\rho}}\right)(\nabla \cdot \mathbf{v} - 3\Phi') - 3\mathcal{H}\left(\frac{\delta P}{\delta \rho} - \frac{\bar{P}}{\bar{\rho}}\right) \tag{3.39}$$

and the spatial part gives the Euler equations

$$q_i' = -4\mathcal{H}q_i - (\partial_i \Psi)(\bar{\rho} + \bar{P}) - \partial_i \delta P - \partial^j \Pi_{ij} \tag{3.40}$$

$$v_i' = -\left(\mathcal{H} + \frac{\bar{P}}{\bar{\rho} + \bar{P}}\right) v_i - \frac{1}{\bar{\rho} + \bar{P}}(\partial_i \delta P - \partial^j \Pi_{ij}) - (\partial_i \Psi) \tag{3.41}$$

Consider a non-relativistic fluid with $P_m = 0$ and $\Pi_{ij} = 0$

$$\delta_m' = -\nabla \cdot \mathbf{v}_m + 3\Phi' \tag{3.42}$$

$$\mathbf{v}_m' = -\mathcal{H}\mathbf{v}_m - \nabla \Psi \tag{3.43}$$

and

$$\delta_m'' + \mathcal{H}\delta_m' = \nabla^2 \Phi + 3(\Phi'' + \mathcal{H}\Phi') \tag{3.44}$$

For a relativistic perfect fluid with $P_r = \rho_r/3$ and $\Pi_{ij} = 0$

$$\delta_r' = -\frac{4}{3}\nabla \cdot \mathbf{v}_r + 4\Phi' \tag{3.45}$$

$$\mathbf{v}_r' = -\frac{1}{4}\nabla \delta_r - \nabla \Psi \tag{3.46}$$

and

$$\delta_r'' - \frac{1}{3}\nabla^2 \delta_m = \frac{4}{3}\nabla^2 \Psi + 4\Phi'' \tag{3.47}$$

Einstein Euqations

The temporal part of Einstein equations gives

$$\nabla^2 \Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Psi) = 4\pi G a^2 \delta \rho \tag{3.48}$$

the space-time component gives

$$-(\Phi' + \mathcal{H}\Psi) = 4\pi G a^2 q \tag{3.49}$$

with this the two equations (3.48) and (3.48) can be combine into the Poisson equation

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho} \Delta \tag{3.50}$$

The spatial part gives

$$\partial_{\langle i}\partial_{j\rangle}(\Phi - \Psi) = 4\pi G a^2 \Pi_{ij} \tag{3.51}$$

Finally we look at the trace of the space-space component of Einstein equations, which gives

$$\Phi'' + \mathcal{H}\Psi' + 2\mathcal{H}\Phi' + \frac{1}{3}\nabla^2(\Psi - \Phi) + (2\mathcal{H}' + \mathcal{H}^2)\Psi = 4\pi G a^2 \delta P \tag{3.52}$$

If $\Pi_{ij} = 0$ we have $\Phi \approx \Psi$, then the equations above can be written as

$$\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \delta\rho \tag{3.53}$$

$$-(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 q \tag{3.54}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P \tag{3.55}$$

For the equation of state $\delta P = c_s^2 \delta\rho + \sigma\delta S$, we have

$$\Phi'' + 3\mathcal{H}[\Phi' + c_s^2(\Phi' + \mathcal{H}\Psi)] + (2\mathcal{H}' + \mathcal{H}^2)\Phi - c_s^2\nabla^2\Phi = 4\pi G a^2 \sigma\delta S \tag{3.56}$$

4 Initial Conditions

4.1 Super Horizon Limit

At sufficiently early times, all scales of interest to current observations were outside of the Hubble radius. On such superhorizon scales, the evolution of the perturbations becomes very simple.

Consider the superhorizon limit of the continuity equations (6.85) and (6.88) for matter and radiation

$$\delta_m' = 3\Phi' \tag{4.1}$$
$$\delta_r' = 4\Phi' \tag{4.2}$$

we have

$$\begin{aligned}
\delta_\gamma &= 4\Phi + C_\gamma & \delta_\gamma &= 4\Phi + C_\gamma \\
\delta_\nu &= 4\Phi + C_\nu & \delta_\nu &= \delta_\gamma + S_\nu \\
\delta_c &= 3\Phi + C_c & \delta_c &= \frac{3}{4}\delta_\gamma + S_c \\
\delta_b &= 3\Phi + C_b & \delta_b &= \frac{3}{4}\delta_\gamma + S_b
\end{aligned} \tag{4.3}$$

The parameters S_ν, S_c and S_b are called Isocurvature modes. And if $S_\nu = S_c = S_b = 0$ and $C_\gamma \neq 0$, we call such kind of perturbation adiabatic modes. The Einstein equation will become

$$\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Psi) = 4\pi G a^2 \sum_{\alpha=\gamma,\nu,c,b} \rho_\alpha \delta_\alpha = 4\pi G a^2 \rho_{\text{tot}} \sum_{\alpha=\gamma,\nu,c,b} f_\alpha \delta_\alpha \tag{4.4}$$
$$f_\gamma + f_\nu + f_c + f_b = 1 \tag{4.5}$$

If we ignore the anisotropic stress and neglect the fisrt term for superhorizon modes

$$-3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \rho_{\text{tot}} \sum_{\alpha=\gamma,\nu,c,b} f_\alpha \delta_\alpha \tag{4.6}$$

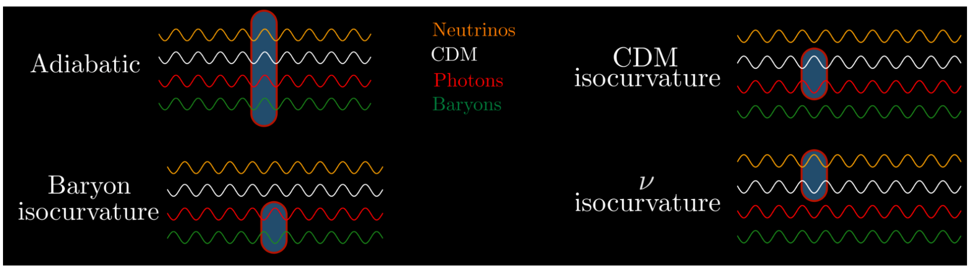

Figure 1: Perturbation modes(Daniel Grin)

4.2 Adiabatic Perturbations

For adiabatic perturbations, we have

$$\delta_\gamma = \delta_\nu = \frac{4}{3}\delta_c = \frac{4}{3}\delta_b = -2\Phi_{\text{ini}} \tag{4.7}$$

Proof. For the gravitational potentials,

$$-3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \rho_{\text{tot}}(f_\gamma \delta_\gamma + f_\nu \delta_\nu) \tag{4.8}$$

where we have neglected the matter contributions at early times in the radiation era. Specializing to adiabatic initial conditions, we have $\delta_\nu = \delta_\gamma$ and $4\pi G a^2(\bar{\rho}_\gamma + \bar{\rho}_\nu) = 3\mathcal{H}^2/2$. Substituting $\mathcal{H} = 1/\eta$, we then get

$$\eta\Phi' + \Phi = -\frac{1}{2}\delta_\gamma \tag{4.9}$$

$$\Rightarrow \eta\Phi'' + 4\Phi' = 0 \tag{4.10}$$

$$\Rightarrow \Phi = C_1 + C_2\frac{1}{\eta^3} \tag{4.11}$$

Focusing on the growing mode, $\Phi = \Phi_{\text{ini}}$ we have $\delta_\gamma = -2\Phi_{\text{ini}}$ which completes the proof.

Why adiabatic? From the thermodynamical relation:

$$T\delta S = \delta U + P\delta V \tag{4.12}$$

we can write

$$\begin{aligned}
\frac{T\delta S}{V} &= \delta\rho + (\bar{\rho} + \bar{P})\frac{\delta V}{V} = \delta\rho + (\bar{\rho} + \bar{P})\frac{\delta n}{n} \\
&= \sum_{\alpha=\gamma,\nu,c,b}\left(\delta\rho_\alpha + \frac{\bar{\rho} + \bar{P}}{\bar{n}}\delta n_\alpha\right) \\
&= \left(\rho_\nu + n_\nu\frac{\rho_{\text{tot}} - P_{\text{tot}}}{n_{\text{tot}}}\right)(\delta_\nu - \delta_\gamma) + \sum_{\alpha=c,b}\left(\rho_\alpha + n_\alpha\frac{\rho_{\text{tot}} - P_{\text{tot}}}{n_{\text{tot}}}\right)\left(\delta_\alpha - \frac{3}{4}\delta_\gamma\right)
\end{aligned}$$

So $S_\nu = S_c = S_b = 0$, implies $\delta S = 0$.

Furthermore, we can get more details in Prof. Mukhanov's textbook.

4.3 Curvature Perturbations

In Newtonian gauge we have

$$\mathcal{R} = \Phi - \mathcal{H}v \qquad \zeta = \Phi + \mathcal{H}\frac{\delta\rho}{\rho'} = \Phi - \frac{\delta\rho}{3(\rho + P)} \tag{4.13}$$

Consider a universe dominated by a fluid with constant equation of state, the superhorizon limit of the curvature perturbation is

$$\mathcal{R} = \frac{5 + 3w}{3 + 3w}\Phi \tag{4.14}$$

4.4 Entropy Modes and Isocurvature Perturbations

The name “isocurvature” comes from the fact that when these modes are present it is possible to have $\mathcal{R} = 0$ or $\zeta = 0$.

where we have used the continuity equation to replace $\bar{\rho}'$. In early times we have

$$\mathcal{R} = \zeta = \Phi + \frac{1}{2}\Psi \tag{4.15}$$

We can see that the isocurvature perturbation always turn into curvature perturbations.

5 Scalar Perturbation

In this section, we consider the adiabatic scalar perturbation, mainly on superhorizon scale. We will use the Einstein equations

$$\Phi'' + 3\mathcal{H}[\Phi' + c_s^2(\Phi' + \mathcal{H}\Psi)] + (2\mathcal{H}' + \mathcal{H}^2)\Phi - c_s^2\nabla^2\Phi = 4\pi G a^2 \sigma \delta S \tag{5.1}$$

$$\nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \delta\rho \tag{5.2}$$

where evolution of perturbations. The former one is used for the evolution of potential Φ and the latter one is used for the evolution of energy density perturbation in its dominated era. And the evolution of energy density perturbation in its undominated era should be derive from the continuity equation and Euler equations.

5.1 Evolution of the Potential

Matter era

In a flat matter-dominated universe ($P = 0$), for adiabatic modes

$$c_s^2 = 0 \qquad \sigma = 0 \qquad a \propto \eta^2 \qquad \mathcal{H} = \frac{2}{\eta} \tag{5.3}$$

In this case, for both superhorizon and subhorizon perturbations, and (5.1) simplifies to

$$\Phi'' + 3\mathcal{H}\Phi' = \Phi'' + \frac{6}{\eta}\Phi' = 0 \tag{5.4}$$

$$\Phi_{\mathbf{k}} = C_1 + \frac{C_2}{\eta^5} \tag{5.5}$$

Radiation era

For the case $P = w\rho$ we have:

$$c_s^2 = w \qquad \sigma = 0 \qquad a \propto \eta^{\frac{2}{1+3w}} \qquad \mathcal{H} = \frac{2}{1+3w}\frac{1}{\eta} \tag{5.6}$$

In this case (5.1) simplifies to

$$\Phi'' + 3\mathcal{H}(1+w)\Phi' - w\nabla^2\Phi = 0 \tag{5.7}$$

$$\Phi_{\mathbf{k}}'' + \frac{1+w}{1+3w}\frac{6}{\eta}\Phi_{\mathbf{k}}' + wk^2\Phi_{\mathbf{k}} = 0 \tag{5.8}$$

has the solution

$$\Phi_{\mathbf{k}} = \frac{1}{\eta^\nu}\left[C_1 J_\nu(\sqrt{w}k\eta) + C_2 Y_\nu(\sqrt{w}k\eta)\right] \qquad \nu = \frac{1}{2}\frac{5+3w}{1+3w} \tag{5.9}$$

where J_ν and Y_ν are Bessel functions. For radiation we have $w = 1/3$

$$\Phi_{\mathbf{k}} = C_1 \frac{\sin\varphi - \varphi\cos\varphi}{\varphi^3} \qquad (\varphi = k\eta/\sqrt{3}) \tag{5.10}$$

- For superhorizon mode ($k\eta \ll 1$)

$$\delta_{\mathbf{k}} = \frac{1}{3}C_1 \tag{5.11}$$

- For subhorizon mode ($k\eta \gg 1$)

$$\delta_{\mathbf{k}} = \frac{C_1}{\varphi^2}\cos\varphi \tag{5.12}$$

Evolution through radiation-matter equality

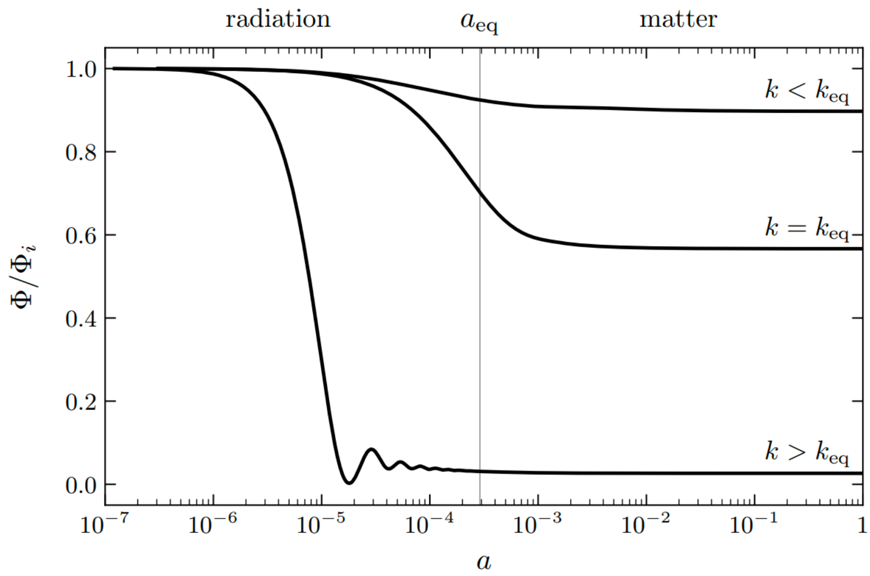

Figure 2: Potential evolution

Consider the important case of a flat universe with a mixture of matter and radiation. The energy density of matter decreases as $1/a^3$ while that of radiation decays as $1/a^4$. Therefore, we have

$$\bar\rho = \bar\rho_m + \bar\rho_r = \frac{\bar\rho_{\rm eq}}{2}\left[\left(\frac{a_{\rm eq}}{a}\right)^3 + \left(\frac{a_{\rm eq}}{a}\right)^4\right] \tag{5.13}$$

Define

$$\frac{\bar\rho_m}{\bar\rho_r} = \frac{a}{a_{\rm eq}} = y \tag{5.14}$$

Assuming adiabatic perturbations, the superhorizon evolution of the potential in a universe with matter and radiation can be derive from the Einstein euqation

$$
\begin{aligned}
&\nabla^2\Phi - 3\mathcal{H}(\Phi' - \mathcal{H}\Phi) = 4\pi G a^2(\bar{\rho}_m\delta_m + \bar{\rho}_r\delta_r) \\
\Rightarrow & -3\mathcal{H}(\Phi' - \mathcal{H}\Phi) = 4\pi G a^2\bar{\rho}\left(\frac{y}{1+y} + \frac{4}{3}\frac{1}{1+y}\right)\delta_m \\
\Rightarrow & -3\mathcal{H}(\Phi' - \mathcal{H}\Phi) = \frac{3}{2}\mathcal{H}^2\left(\frac{y}{1+y} + \frac{4}{3}\frac{1}{1+y}\right)\delta_m \\
\Rightarrow & \; y\frac{d\Phi}{dy} + \Phi = -\frac{1}{6}\frac{3y+4}{y+1}\delta_m
\end{aligned}
$$

We get

$$
y\frac{d\Phi}{dy} + \Phi = -\frac{1}{6}\frac{3y+4}{y+1}\delta_m \tag{5.15}
$$

Using $\delta_m' = 3\Phi'$, we have $\frac{d\delta_m}{dy} = 3\frac{d\Phi}{dy}$, and the mma gives us

$$
2y(1+y)(4+3y)\frac{d^2\Phi}{dy^2} + (32 + 3y(18+7y))\frac{d\Phi}{dy} + 2\Phi = 0 \tag{5.16}
$$

$$
\Phi = C_1\frac{\sqrt{1+y}}{y^3} + C_2\frac{2(-16 - 8y + 2y^2 + 9y^3)}{15y^3} \tag{5.17}
$$

Adding initial conditions: solution reduces to a constant at early times, we must have

$$
\Phi = \frac{\Phi_{\text{ini}}}{10y^3}[16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16] \tag{5.18}
$$

Note that for $y \to 0$ gives $\Phi \to \Phi_{\text{ini}}$ and $y \to \infty$ gives $\Phi \to \frac{9}{10}\Phi_{\text{ini}}$

Dark energy era

In a flat Λ-dominated universe we have

$$
H = \text{Const} \qquad a = \frac{1}{1 + H(\eta_0 - \eta)} \qquad 4\pi G\rho \ll H^2 \tag{5.19}
$$

which means $\delta\rho \approx \delta P \approx 0$, then

$$
\begin{aligned}
&\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2\delta P \\
\Rightarrow & \frac{d^2\Phi}{da^2} + \frac{5}{a}\frac{d\Phi}{da} + \frac{3}{a^2}\Phi = 0 \\
\Rightarrow & \Phi = C_1 a^{-1} + C_2 a^{-3}
\end{aligned}
$$

5.2 Evolution of Matter Perturbation

Matter era

In Fourier space, equation (5.2) becomes

$$-k^2\Phi_{\mathbf{k}} - 3\mathcal{H}(\Phi'_{\mathbf{k}} + \mathcal{H}\Phi_{\mathbf{k}}) = 4\pi G a^2 \delta\rho_{\mathbf{k}} \tag{5.20}$$

$$\Phi_{\mathbf{k}} = C_1 + \frac{C_2}{\eta^5} \qquad \mathcal{H}^2 = \frac{8\pi G}{3}\bar{\rho}_{\mathbf{k}} \qquad \mathcal{H} = \frac{2}{\eta} \tag{5.21}$$

and

$$\delta_{\mathbf{k}} = -\frac{\eta^2 k^2}{6}\Phi_{\mathbf{k}} - 2\Phi_{\mathbf{k}} - \eta\Phi'_{\mathbf{k}} \tag{5.22}$$

- For superhorizon mode ($k\eta \ll 1$)

$$\delta_{\mathbf{k}} = -2C_1 + \frac{3C_2}{\eta^5} \tag{5.23}$$

  Neglecting the decaying mode we have

$$\delta_{\mathbf{k}} \approx -2\Phi_{\mathbf{k}} = \text{Const} \tag{5.24}$$

- For subhorizon mode ($k\eta \gg 1$)

$$\delta_{\mathbf{k}} = -\frac{k^2}{6}\left(C_1\eta^2 + \frac{C_2}{\eta^3}\right) \tag{5.25}$$

Therefore, δ_m is constant on super-horizon scales, but when a scale crosses the horizon it starts to grow as $\delta_m \propto \eta^2 \propto a$

Radiation era

During the radiation era, matter is a subdominant component and we cannot use the above trick to determine the evolution of matter perturbations from the Einstein equations for the gravitational potential. Instead, we must work with the continuity and Euler equations.

$$\delta''_m + \mathcal{H}\delta'_m = \nabla^2\Phi + 3(\Phi'' + \mathcal{H}\Phi') \tag{5.26}$$

where $\Phi = \Phi_r + \Phi_m$ is sourced by both radiation and matter. The contribution from the radiation, Φ_r, is rapidly oscillating on subhorizon scales, while the contribution from matter, Φ_m, is a constant. The solution δ_m therefore inherits a "fast mode" sourced by Φ_r and a "slow mode" sourced by Φ_m. It turns out that the fast mode is suppressed by a factor of $(H/k)^2$ relative to the slow mode (Weinberg). This reflects the fact that the matter can't react to the fast change in the gravitational potential and effectively only evolves in response to the time-averaged potential. As a result, δ_m is sourced by Φ_m even deep in the radiation era.

$$\delta''_m + \mathcal{H}\delta'_m = 4\pi G a^2 \bar{\rho}_m \delta_m \tag{5.27}$$

Mészáros equation

$$\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(y+1)}\frac{d\delta_m}{dy} - \frac{3}{2y(y+1)}\delta_m = 0 \tag{5.28}$$

whose solutions are

$$\delta_m \propto \begin{cases} 1+\frac{3}{2}y \\ \left(1+\frac{3}{2}y\right)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 3\sqrt{1+y} \end{cases} \qquad y = a/a_{\text{eq}} \tag{5.29}$$

Dark energy era

5.3 Evolution of Radiation Perturbation

Radiation era

In radiation era, equation (5.2) becomes

$$-k^2\Phi_{\mathbf{k}} - 3\mathcal{H}(\Phi'_{\mathbf{k}} + \mathcal{H}\Phi_{\mathbf{k}}) = 4\pi G a^2 \delta\rho_{\mathbf{k}} \tag{5.30}$$

$$\Phi_{\mathbf{k}} = C_1 \frac{\sin\varphi - \cos\varphi}{\varphi^3} \qquad \mathcal{H}^2 = \frac{8\pi G}{3}\bar{\rho}_{\mathbf{k}} \qquad \mathcal{H} = \frac{1}{\eta} \tag{5.31}$$

and

$$\delta_{\mathbf{k}} = -\frac{2\eta^2 k^2}{3}\Phi_{\mathbf{k}} - 2\Phi_{\mathbf{k}} - 2\eta\Phi'_{\mathbf{k}} \tag{5.32}$$

- For superhorizon mode ($k\eta \ll 1$)

$$\delta_{\mathbf{k}} = -\frac{2}{3}C_1 \tag{5.33}$$

- For subhorizon mode ($k\eta \gg 1$)

$$\delta_{\mathbf{k}} = -2\cos\varphi \tag{5.34}$$

Matter era

Since the radiation fluctuations are subdominant, we must use the continuity and Euler equations to follow their evolution.

$$\delta''_r - \frac{1}{3}\nabla^2\delta_r = \frac{4}{3}\nabla^2\Phi + 4\Phi'' = \frac{4}{3}\nabla^2\Phi \tag{5.35}$$

- For superhorizon mode ($k\eta \ll 1$)

$$\delta_{\mathbf{k}} = -4\Phi_{\mathbf{k}} = \text{Const} \tag{5.36}$$

- For subhorizon mode ($k\eta \gg 1$)

$$\delta_{\mathbf{k}} = C_1\cos\varphi + C_2\sin\varphi - 4\Phi_{\mathbf{k}} \tag{5.37}$$

5.4 Summary

The evolution of perturbation of some situations have been give in Tab.2 , in which the other cases will be filled in at a later date.

Table 2: Summary of the evolution of adiabatic perturbations

	Scales	Radiation Era	Matter Era	DE Era
Φ	$k < \mathcal{H}$	Const	Const	a^{-1}
	$k > \mathcal{H}$	$a^{-2}\cos(k\eta/\sqrt{3})$	Const	a^{-1}
δ_m	$k < \mathcal{H}$	Const	Const	
	$k > \mathcal{H}$	$\ln a$	a	
Δ_m	$k < \mathcal{H}$	a^2	a	
	$k > \mathcal{H}$	$\ln a$	a	
δ_r	$k < \mathcal{H}$	Const	Const	
	$k > \mathcal{H}$	$\cos(k\eta/\sqrt{3})$	$\cos(k\eta/\sqrt{3})$+Const	
Δ_r	$k < \mathcal{H}$	a^2	a	
	$k > \mathcal{H}$	$\cos(k\eta/\sqrt{3})$	$\cos(k\eta/\sqrt{3})$+Const	

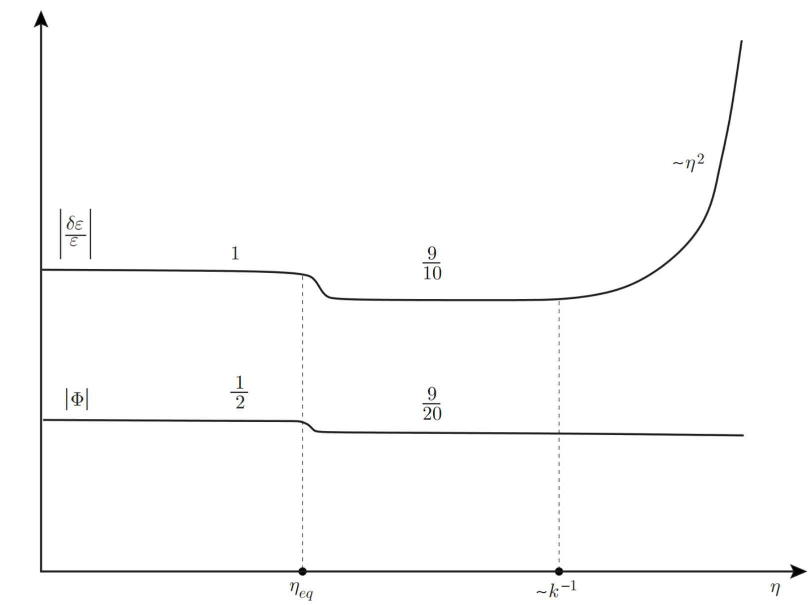

Figure 3: Adiabatic perturbation evolution (Mukhanov)

6 Tensor Perturbations

For the tensor parts we have

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla'^2 h_{ij} = 16\pi G\hat{\Phi}_{ij} \tag{6.1}$$

where $\hat{\Pi}_{ij}$ is energy-momentum tensor which has the same structural form as h_{ij}. we can write the metric in the form of (the gravitational wave propagates in the z direction)

$$h_{ij} = \begin{bmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.2}$$

Considering the perturbations in vacuum, $\delta^{(T)}\pi_{ij} = 0$, in Fourier space we have

$$h''_{\mathbf{k}} + 2\mathcal{H}h'_{\mathbf{k}} + k^2 h_{\mathbf{k}} = 0 \tag{6.3}$$

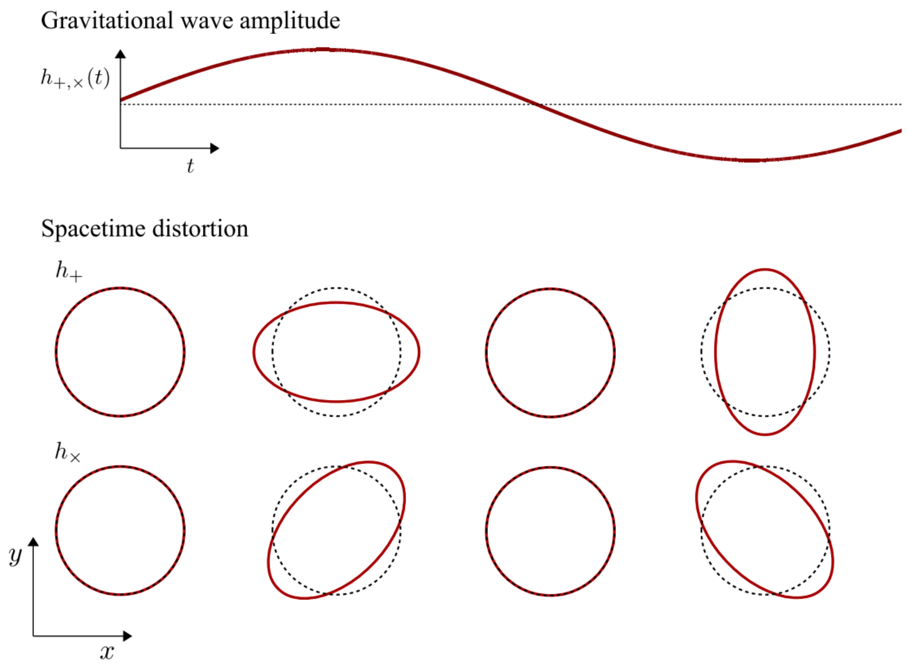

Figure 4: Gravitational Wave(Scott Dodelson, Fabian Schmidt. Modern Cosmology. 2020)

7 Isocurvature Modes

The evolution of isocurvature perturbation are much more complicated. Fig.5 has shown the evolution of isocurvature perturbation from radiation era to the matter era, in which the mode comes into horizon in the matter era.

Maybe I will add more details of isocurvature perturbation someday in the future.

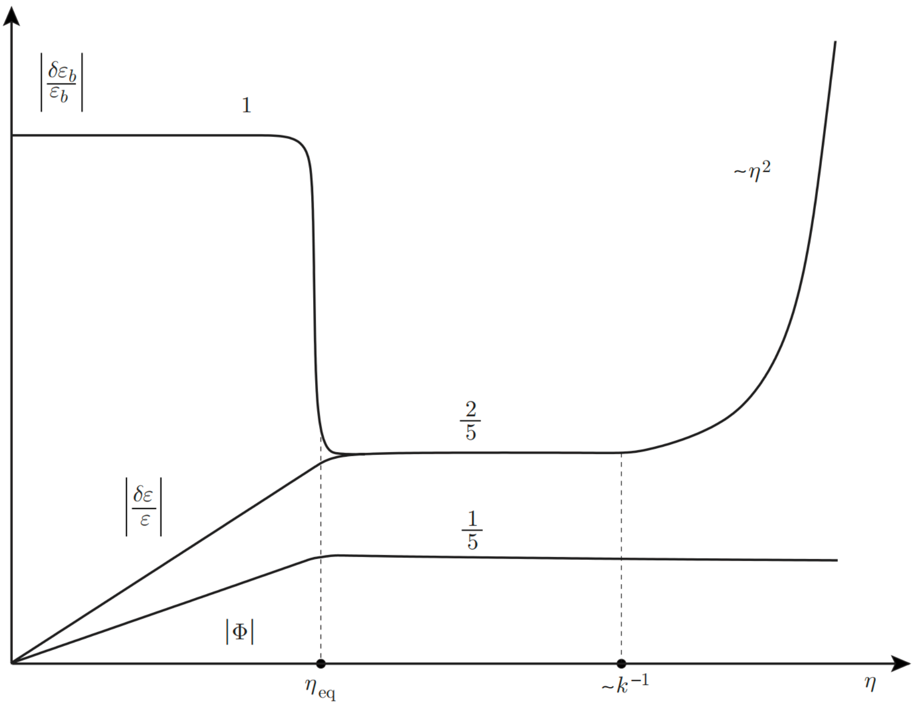

Figure 5: Isocurvature modes evolution (Mukhanov)

8 Boltzmann Equation

8.1 Boltzmann Equation

Boltzmann Equation

Boltzmann equation is very simple to write:

$$\frac{df}{dt} = C[f] \tag{8.1}$$

where f is the one-particle distribution function and $C[f]$ is the collisional term, i.e. a functional of f describing the interactions among the particles constituting the system under investigation. The one-particle distribution is a function of time t, of the particle position x and of the particle momentum p Therefore, the total time derivative can be written as:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p^i}\frac{dp^i}{dt} = \hat{L}f \tag{8.2}$$

The operator $\hat{L}$ acting on f is called Liouville operator.

Remarks:

1. If interactions are absent, then
$$\frac{df}{dt} = 0 \tag{8.3}$$
which is called the collisionless Boltzmann equation, or Vlasov equation.
2.

Liouville Theorem

8.2 Boltzmann Equation in General Relativity and Cosmology

In GR the distribution function must be expressed covariantly as $f = f(x^\mu, P^\mu)$, and the total derivative of f should be taken with respect to an affine parameter λ, as follows:

$$\frac{df}{dt} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} + \frac{\partial f}{\partial P^\mu}\frac{dP^\mu}{d\lambda} \tag{8.4}$$

The geometry enters through the derivative of the four-momentum, which can be expressed via the geodesic equation:

$$\frac{dP^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta}P^\alpha P\beta = 0 \tag{8.5}$$

so we have

$$\frac{df}{dt} = P^\mu\frac{\partial f}{\partial x^\mu} - \Gamma^\mu{}_{\alpha\beta}P^\alpha P\beta\frac{\partial f}{\partial P^\mu} = \hat{L}_{\mathrm{GR}}f \tag{8.6}$$

It might seem that in the relativistic case we have gained one variable P^0, but this is not so because P^0 is related to the spatial momentum P^i via the mass-shell relation $g_{\mu\nu}P^\mu P^\nu = -m^2c^2$. For this reason, we can reformulate the Liouville operator as follows

$$\frac{df}{dt} = P^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i{}_{\alpha\beta} P^\alpha P\beta \frac{\partial f}{\partial P^i} = \hat{L}_{\mathrm{GR}} f \tag{8.7}$$

In FLRW metric, we must take into account tha f cannot depend on the position x^i, because of homogeneity and isotropy.

$$\frac{df}{dt} = P^0 \frac{\partial f}{\partial x^0} - \Gamma^i{}_{\alpha\beta} P^\alpha P\beta \frac{\partial f}{\partial P^i} = \hat{L}_{\mathrm{GR}} f \tag{8.8}$$

Considering the spatially flat case $K = 0$, we can show that the above equation can be cast as follows

$$\frac{\partial f}{\partial t} - 2HP^i \frac{\partial f}{\partial P^i} = 0 \tag{8.9}$$

Again, because of isotropy, f cannot depend on the direction of P^i, but only on its modulus $P^2 = \delta_{ij} P^i P^j$, we have

$$\frac{\partial f}{\partial t} - 2HP \frac{\partial f}{\partial P} = 0 \tag{8.10}$$

8.3 Perturbed Boltzmann Equations

The distribution function f can also be split in a background contribution plus a perturbation

$$f = \bar{f} + \mathcal{F} \tag{8.11}$$

9 Power Spectrum

9.1 Power Spectrum

Because of the principle of cosmology, the spatial average of δ itself at a given time vanishes

$$\langle \delta(\mathbf{x}, t) \rangle = 0 \tag{9.1}$$

The first non-trivial information lies in the correlation function, defined by the spatial average

$$\xi(r, t) = \xi(\mathbf{x} - \mathbf{y}, t) = \langle \delta(\mathbf{x}, t)\delta(\mathbf{y}, t) \rangle \tag{9.2}$$

when $\mathbf{x} = \mathbf{y}$, which means $r = 0$, the correlation function becomes variance

$$\sigma^2(t) = \xi(0, t) = \langle \delta(\mathbf{x}, t)\delta(\mathbf{x}, t) \rangle \tag{9.3}$$

We have learned that the evolution of the density perturbations is best described in momentum space. The correlation function in momentum space is given by

$$\begin{aligned}
\langle \delta(\mathbf{k}, t)\delta^*(\mathbf{k}', t) \rangle &= \int d^3x d^3y \; e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{k}'\cdot\mathbf{y}} \, \langle \delta(\mathbf{x}, t)\delta(\mathbf{y}, t) \rangle \\
&= \int d^3r d^3y \; e^{-i\mathbf{k}\cdot\mathbf{r} - i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{y}} \xi(r, t) \\
&= (2\pi)^3 \delta_D^3(\mathbf{k} - \mathbf{k}') \int d^3r \; e^{-i\mathbf{k}\cdot\mathbf{r}} \xi(r, t) \\
&= (2\pi)^3 \delta_D^3(\mathbf{k} - \mathbf{k}') P(\mathbf{k}, t)
\end{aligned} \tag{9.4}$$

The $P(\mathbf{k}, t)$ is called the power spectrum. If we work in spherical polar coordinates, the power spectrum depends only on the magnitude of the wavevector due to the rotational invariance. We can derive

$$P(k, t) = \int \mathrm{d}^3 r e^{-i\mathbf{k}\cdot\mathbf{r}} \xi(r, t) = \frac{4\pi}{k} \int_0^\infty dr \; r \sin(kr) \xi(r, t) \tag{9.5}$$

$$\xi(r, t) = \int \mathrm{d}^3 k e^{i\mathbf{k}\cdot\mathbf{r}} P(k, t) = \frac{4\pi}{k} \int_0^\infty \frac{dk}{k} \frac{k^3}{2\pi^2} P(k, t) j_0(kr) \tag{9.6}$$

Consider the variance

$$\sigma_\delta^2 = \xi(0) = \int \frac{dk}{k} \frac{k^3}{2\pi^2} P(k, t) = \int d \ln k \; \Delta^2(k, t) \tag{9.7}$$

where $\Delta^2(k, t) = \frac{k^3}{2\pi^2} P(k, t)$ is thedimensionless power spectrum (equivalently to the variance of the field per logarithmic range of k)

For an arbitrary scalar variable X in position space, we define the power spectrum and the dimensionless power spectrum in Fourier space by

$$\langle X(\mathbf{k},t)X^*(\mathbf{k}',t)\rangle = (2\pi)^3\delta_D^3(\mathbf{k}-\mathbf{k}')P_X(\mathbf{k},t) \tag{9.8}$$

$$\Delta_X^2(\mathbf{k},t) = 4\pi\left(\frac{k}{(2\pi)}\right)^3 P_X(\mathbf{k},t) \tag{9.9}$$

9.2 Angular Power Spectrum

Comparing the perturbations at two distinct points $\mathbf{n}$ and $\mathbf{n}$ gives the two-point correlation function

$$C(\theta,t) = \langle\delta(\mathbf{n},t)\delta(\mathbf{n}',t)\rangle \tag{9.10}$$

where $\cos\theta = \mathbf{n}\cdot\mathbf{n}'$.

Given that we observe fluctuations on the spherical surface, it is convenient to expand the temperature field in spherical harmonics

$$\delta(\mathbf{n},t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\mathbf{n}) \tag{9.11}$$

where the expansion coefficients a_{lm} are called multipole moments.

The two-point function of the multipole moments is defined as

$$\langle a_{lm}a^*_{l'm'}\rangle = C_l\delta_{ll'}\delta_{mm'} \tag{9.12}$$

where C_l is the angularpower spectrum and the Kronecker deltas are a consequence of statistical isotropy.

The angular power spectrum is the harmonic space equivalent of the two-point correlation function in real space.

$$\begin{aligned} C(\theta) &= \langle\delta(\mathbf{n},t)\delta(\mathbf{n}',t)\rangle \\ &= \sum_{lm}\sum_{l'm'}\langle a_{lm}a^*_{l'm'}\rangle\, Y_{lm}(\mathbf{n})Y^*_{l'm'}(\mathbf{n}') \\ &= \sum_{lm} C_l Y_{lm}(\mathbf{n})Y^*_{lm}(\mathbf{n}') \\ &= \sum_{lm}\frac{2l+1}{4\pi}C_l P_l(\cos\theta) \end{aligned} \tag{9.13}$$

we have

$$C_l = \frac{1}{2l+1}\sum_m |a_{lm}|^2 \tag{9.14}$$

9.3 Window Functions

> Mathematically, there is no difficulty in defining the density at a point $\mathbf{x}$. But how do we construct $\delta(\mathbf{x})$ from observations? In particular, what volume do we divide by? If we observe many galaxies, each localised at some point $\mathbf{x}_i$, then the universe looks far from homogeneous. The same is true for any fluid if we look closely enough. But our interest is in a more coarse-grained description.
>
> —-David Tong's Lecture

So we introduce a window function at scale R as $W_R(\mathbf{x})$. The purpose of this function is to provide a way to turn the observed density $\delta(\mathbf{x})$ into something that is **smooth**, and varies on length scales $\sim R$. We construct the smoothed density contrast as

$$\delta_R(\mathbf{x}) = \int d^3x W_R(\mathbf{x}-\mathbf{x}')\delta(\mathbf{x}) \tag{9.15}$$

In Fourier space, using the convolution theorem we have

$$\delta_R(\mathbf{k}) = \widetilde{W}_R(\mathbf{k})\delta(\mathbf{k}) \tag{9.16}$$

9.4 Transfer Functions

The evolution of a perturbation δ of a given wavevector k from an initial time η_i to the present can be distilled into a transfer function $\mathcal{T}(k,\eta)$, defined as

$$\delta(\mathbf{k},\eta) = \mathcal{T}_\delta(k,\eta)\delta(\mathbf{k},\eta_i) \tag{9.17}$$

For specific modes, every variable X can be determined via a deterministic transfer function by the initial condition for the curvature perturbation $\mathcal{R}_i$ or initial entropy S_i.

$$X(\mathbf{k},\eta) = \mathcal{T}_X^{\text{ad}}(k,\eta)\mathcal{R}_i \tag{9.18}$$

$$X(\mathbf{k},\eta) = \mathcal{T}_X^{\text{iso}}(k,\eta)S_i \tag{9.19}$$

9.5 Matter Power Spectrum

Matter Power Spectrum

The density power spectrum of matter fluctuations is defined as

$$\langle\delta_m(\mathbf{k},t)\delta_m^*(\mathbf{k}',t)\rangle = (2\pi)^3\delta_D^3(\mathbf{k}-\mathbf{k}')P_m(\mathbf{k},t) \tag{9.20}$$

we have

$$P_m(\mathbf{k},t) = \mathcal{T}^2(k,t)P_m(\mathbf{k},t_{\text{ini}}) \tag{9.21}$$

where $\mathcal{T}(k,t)$ is the transfer function of matter perturbation satisfied $\delta_m(\mathbf{k},t) = \mathcal{T}(k,t)\delta_m(\mathbf{k},t_{\text{ini}})$.

$$\mathcal{T}^2(k,t) \tag{9.22}$$

$P_m(k)$ is usually compared with the observed power spectrum of the galaxy distribution. This is clearly problematic, as it is by no means evident what the relation between these two spectra should be. This problem is known under the name of "bias" and it is very often simply assumed that the dark matter and galaxy power spectra differ only by a multiplicative factor. We assume bias to be linear and scale independent, so that $P_g(k) = b^2 P_m(k)$

Harrison-Zel'dovich Spectrum

The primordial power spectrum is often written as a power law

$$P_m(k,t_{\text{ini}}) = Ak^n \tag{9.23}$$

where A and n are constants. The exponent n (often also denoted by n_s) is called the spectral index. In 1970, well before inflation was introduced, it was argued by Harrison, Zel'dovich and Peebles that the initial perturbations of our universe are likely to have taken a power law form with spectral index $n \approx 1$[]. This is now called the Harrison-Zel'dovich spectrum.

We note that the Poisson equation we have $\Phi_k = -\frac{4\pi G a^2 \bar{\rho}\delta}{k^2}$,which implies the following relation between the power spectra of Φ and δ

$$P_\phi(k,t_{\text{ini}}) \propto k^4 P_m(k,t_{\text{ini}}) \propto k^{n-4} \tag{9.24}$$

and the dimensionless power spectrum yields

$$\Delta_\Phi^2 = \frac{k^3}{2\pi^2}P_\Phi(k,t_{\text{ini}}) \propto k^{n-1} \tag{9.25}$$

For $n = 1$, the dimensionless power spectrum of the gravitational potential therefore becomes k-independent, so that the variance receives equal contributions from every decade in k! This property of the field is called scale invariance.

Combining the transfer function with the primordial power spectrum, we predict that the late-time matter power spectrum should have the following asymptotic scalings:

$$P_m(k,t) \propto \begin{cases} k^n & k < k_{\text{eq}} \\ k^{n-4} & k > k_{\text{eq}} \end{cases} \tag{9.26}$$

9.6 CMB power Spectrum

For convenience, define

$$\Theta(\mathbf{n}) = \frac{\delta T(\mathbf{n})}{\bar{T}} \tag{9.27}$$

and the temperature fluctuations can be written as

$$\Theta(\mathbf{n}) = \left(\frac{1}{4}\delta_r + \Psi\right)_\star - (\mathbf{v}\cdot\mathbf{n})_\star + \int_{\eta_\star}^{\eta_0} (\Psi' + \Phi')d\eta \tag{9.28}$$

Remarks:

1. We must known that $\Theta(\mathbf{n})$ is observed in the cosmic rest frame but not our local group. If we want to derive such quantity observed by an observer o, we must consider the doppler effect of the velocity of the observer

$$\Theta(\mathbf{n}) = \left(\frac{1}{4}\delta_r + \Psi\right)_\star + [\mathbf{v}\cdot\mathbf{n}]_\star^o + \int_{\eta_\star}^{\eta_o} (\Psi' + \Phi')d\eta \tag{9.29}$$

2. A
   - The SW term: $\left(\frac{1}{4}\delta_r + \Psi\right)_\star$, which is caused by gravitational redshift occurring at the surface of last scattering.
   - The ISW term: $\int_{\eta_\star}^{\eta_o}(\Psi' + \Phi')d\eta$, which is also caused by gravitational redshift, but it occurs between the surface of last scattering and the Earth, so it is not part of the primordial CMB.
   - The doppler term: $[\mathbf{v}\cdot\mathbf{n}]_\star^o$
3. Since there has been no evolution on large scales, this limit of the CMB spectrum directly probes the initial conditions.
4. In the SW term, the gravitational redshift has won over the intrinsic temperature fluctuations

Large Scales: Sachs-Wolfe Effect

Small Scales: