Estimation with Applications to Tracking and Navigation: Theory Algorithms and Software
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Chapter 6

ESTIMATION FOR KINEMATIC MODELS

6.1 INTRODUCTION

6.1.1 Outline

This chapter discusses a class of widely used models derived from simple equations of motion — constant velocity and constant acceleration. These models are more general in the sense that the corresponding (second- and third-order) derivatives of the position are not zero, but a zero-mean random process.

Section 6.2 presents the discrete-time kinematic model obtained by discretizing the continuous-time state space representation driven by white noise. The state model defined directly in discrete time using a piecewise constant white random sequence as process noise is presented in Section 6.3.

Section 6.4 presents explicit filters for noiseless kinematic models.

For noisy kinematic models, explicit steady-state filters are derived in Section 6.5. These filters corresponding to second- and third-order models are known as the α-β and α-β-γ filters, respectively, and their gains are expressed in terms of the ***target maneuvering index*** — the ratio of the motion and the observation uncertainties. Since the statistical characterization of the process noise is a key ***filter design parameter***, this is discussed in detail. Subsequently, one important aspect of tracking ***system design*** — selection of the ***sampling frequency***, i.e., the ***revisit rate*** — is discussed.

The models are presented for a single coordinate. For motion in several coordinates, it is customary to use such models assumed independent across coordinates — this leads to "decoupled" filtering. Finally, some intuitive insight into filter design is presented.

6.1.2 Kinematic Models — Summary of Objectives

Define the following kinematic models

- White noise acceleration (second-order model)
- Wiener process acceleration (third-order model)

Derive discrete-time kinematic models by

- Discretizing the continuous-time state space representation driven by white noise
- Directly defining the state model in discrete time using a piecewise-constant white random sequence as process noise

Present

- Explicit filters for noiseless kinematic models
- Explicit steady-state filters for noisy kinematic models
  - α-β
  - α-β-γ

  with their gains expressed in terms of the target maneuvering index.

Discuss

- Filter design and "noise reduction"
- System design — selection of the sampling frequency (revisit rate)

6.2 DISCRETIZED CONTINUOUS-TIME KINEMATIC MODELS

6.2.1 The Kinematic Models

Kinematic state models are defined by setting a certain derivative of the position to zero. In the absence of any random input they yield motion characterized by a polynomial in time. Such models are also called ***polynomial models***, and the corresponding state estimation filters are sometimes referred to as ***polynomial filters***.

Since it is not realistic to assume that there are no disturbances, one can model them as random inputs. One way of modeling this is via a *continuous-time white process noise*.

Since, in general, the state observations are done in discrete time, the corresponding discrete-time state equations are needed. Subsection 6.2.2 presents the ***white noise acceleration*** state model, which is two-dimensional per coordinate. The ***Wiener process acceleration*** state model, which is three-dimensional per coordinate, is presented in Subsection 6.2.3.

In many applications the same model is used for each coordinate. In some applications — for instance, in air traffic control — one can use two third-order models for the horizontal motion and a second-order model for the (more benign) vertical motion.

In general the motion along each coordinate is assumed "decoupled" from the other coordinates. The noises entering into the various coordinates are also assumed to be mutually independent with possibly different variances. The discussion in this section will deal with kinematic models in *one generic coordinate*.

6.2.2 Continuous White Noise Acceleration Model

A ***constant velocity object*** moving in a generic coordinate ξ is described by the equation

$$\ddot{\xi}(t) = 0 \tag{6.2.2-1}$$

Since the position $\xi(t)$ evolves, in the absence of noise, according to a polynomial in time (in this case, of second order), this model is also called a ***polynomial model.***

In practice, the velocity undergoes at least slight changes. This can be modeled by a continuous time zero-mean white noise $\tilde{v}$ as

$$\ddot{\xi}(t) = \tilde{v}(t) \tag{6.2.2-2}$$

where

$$E[\tilde{v}(t)] = 0 \tag{6.2.2-3}$$

$$E[\tilde{v}(t)\tilde{v}(\tau)] = \tilde{q}(t)\delta(t-\tau) \tag{6.2.2-4}$$

The continuous-time process noise intensity $\tilde{q}$, which is (when time-invariant) its power spectral density, is, in general, a design parameter for the estimation filter based on this model. This will be discussed later in more detail in Section 6.5.

The state vector corresponding to (6.2.2-2), which is two-dimensional per coordinate, is

$$x = \begin{bmatrix} \xi & \dot{\xi} \end{bmatrix}' \tag{6.2.2-5}$$

Thus, this model will be called the ***continuous white noise acceleration (CWNA) model*** or ***second-order kinematic model*** (double integrator). Note that the velocity in this model is a Wiener process — the integral of white noise.

The continuous-time state equation is

$$\dot{x}(t) = Ax(t) + D\tilde{v}(t) \tag{6.2.2-6}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{6.2.2-7}$$

$$D = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6.2.2-8}$$

The Discretized State Equation

The discrete-time state equation with sampling period T is

$$x(k+1) = Fx(k) + v(k) \tag{6.2.2-9}$$

where (see Subsection 4.3.1)

$$F = e^{AT} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{6.2.2-10}$$

and the discrete-time process noise relates to the continuous time one as

$$v(k) = \int_0^T e^{A(T-\tau)} D\tilde{v}(kT+\tau)\, d\tau \tag{6.2.2-11}$$

From the above, the covariance of the discrete-time process noise $v(k)$, assuming $\tilde{q}$ to be constant and using (6.2.2-4), is

$$\begin{aligned} Q &= E[v(k)v(k)'] = \int_0^T \begin{bmatrix} T-\tau \\ 1 \end{bmatrix} [T-\tau \quad 1]\, \tilde{q}\, d\tau \\ &= \begin{bmatrix} \frac{1}{3}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & T \end{bmatrix} \tilde{q} \end{aligned} \tag{6.2.2-12}$$

Guideline for Choice of Process Noise Intensity

The changes in the velocity over a sampling period T are of the order of

$$\sqrt{Q_{22}} = \sqrt{\tilde{q}T} \tag{6.2.2-13}$$

This can serve as a guideline for ***process noise intensity choice*** — the choice of the power spectral density $\tilde{q}$ of the process noise in this model.

Note that the physical dimension of $\tilde{q}$ is [length]^2/[time]^3.

A ***nearly constant velocity (NCV) model***[1] is obtained by the choice of a "small" intensity $\tilde{q}$ in the following sense: The changes in the velocity have to be small compared to the actual velocity.

6.2.3 Continuous Wiener Process Acceleration Model

The motion of a constant acceleration object for a generic coordinate ξ is described by the equation

$$\dddot{\xi}(t) = 0 \tag{6.2.3-1}$$

[1] The term ***constant velocity (CV) model*** is used sometimes in the literature, with some abuse of language, for this model regardless of the intensity of the process noise.

Similarly to (6.2.2-2), the acceleration is not exactly constant and its changes can be modeled by a continuous-time zero-mean white noise as

$$\dddot{\xi}(t) = \tilde{v}(t) \tag{6.2.3-2}$$

Note that in this case the acceleration is a Wiener process — hence the name ***continuous Wiener process acceleration (CWPA) model***. Since the derivative of the acceleration is the jerk, this model can also be called the ***white noise jerk model***.[2]

The state vector corresponding to the above is

$$x = \begin{bmatrix} \xi & \dot{\xi} & \ddot{\xi} \end{bmatrix}' \tag{6.2.3-3}$$

and its continuous-time state equation is

$$\dot{x}(t) = Ax(t) + D\tilde{v}(t) \tag{6.2.3-4}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.2.3-5}$$

This is a ***third-order model*** with three integrations: All three eigenvalues of A — the poles of the continuous time transfer function — are zero.

Remark

One can have other third-order models, for instance, with the acceleration having an exponentially decaying autocorrelation, rather than being a Wiener process. The ***exponentially autocorrelated acceleration model*** is presented later in Subsection 8.2.2.

The Discretized State Equation

The discrete-time state equation with sampling period T is

$$x(k+1) = Fx(k) + v(k) \tag{6.2.3-6}$$

with the transition matrix

$$F = \begin{bmatrix} 1 & T & \frac{1}{2}T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \tag{6.2.3-7}$$

[2] In view of this, one has the mathematical epithet "x triple dot."

and the covariance matrix of $v(k)$ given by

$$Q = E[v(k)v(k)'] = \begin{bmatrix} \frac{1}{20}T^5 & \frac{1}{8}T^4 & \frac{1}{6}T^3 \\ \frac{1}{8}T^4 & \frac{1}{3}T^3 & \frac{1}{2}T^2 \\ \frac{1}{6}T^3 & \frac{1}{2}T^2 & T \end{bmatrix} \tilde{q} \tag{6.2.3-8}$$

Note the three unity eigenvalues of the transition matrix F in (6.2.3-7) — the poles of the discrete-time transfer function — corresponding to the three integrations.

Guideline for Choice of Process Noise Intensity

The changes in the acceleration over a sampling period T are of the order of

$$\sqrt{Q_{33}} = \sqrt{\tilde{q}T} \tag{6.2.3-9}$$

This can serve as a guideline in the ***process noise intensity choice*** — the choice of the power spectral density $\tilde{q}$ of the continuous-time process noise $\tilde{v}$ for "tuning" this model to the actual motion of the object of interest.

A ***nearly constant acceleration (NCA) model***[3] is obtained by choosing a "small" intensity $\tilde{q}$ in the following sense: The changes in the acceleration, which are of the order of $\sqrt{\tilde{q}T}$ over an interval T, should be small relative to the actual acceleration levels.

6.3 DIRECT DISCRETE-TIME KINEMATIC MODELS

6.3.1 Introduction

The discrete time plant equation for the continuous time white noise acceleration and white noise jerk (Wiener process acceleration) were given in the previous section.

Another common kinematic model is directly defined in discrete time as follows. The discrete-time process noise $v(k)$ is a scalar-valued *zero-mean white sequence*

$$E[v(k)v(j)] = \sigma_v^2 \delta_{kj} \tag{6.3.1-1}$$

and enters into the dynamic equation as follows:

$$x(k+1) = Fx(k) + \Gamma v(k) \tag{6.3.1-2}$$

where the ***noise gain*** Γ is an n_x-dimensional vector.

[3]The term ***constant acceleration (CA) model*** is used sometimes in the literature, with some abuse of language, for this model regardless of the intensity of the process noise.

The assumption in the second-order model is that the object undergoes a *constant acceleration* during each sampling period (of length T)

$$\tilde{v}(t) = v(k) \qquad t \in [kT, (k+1)T) \tag{6.3.1-3}$$

and that these accelerations are *uncorrelated from period to period.* The above indicates a **piecewise constant acceleration.**

Remark

It is clear that if the above assumption is correct for a given sampling period T_1, then it cannot be correct for any other T_2 (except integer multiples of T_1). Neither this *piecewise constant white noise* assumption nor the *continuous-time white noise* (6.2.2-4) are completely realistic — both are approximations. Nevertheless, with judicious choices of their intensities, both can successfully model motion uncertainties over short to moderate intervals of time.

6.3.2 Discrete White Noise Acceleration Model

If $v(k)$ is the *constant acceleration* during the kth sampling period (of length T), the increment in the velocity during this period is $v(k)T$, while the effect of this acceleration on the position is $v(k)T^2/2$.

The state equation for the **piecewise constant white acceleration model,** or **discrete white noise acceleration (DWNA) model,** which is of second order, is

$$x(k+1) = Fx(k) + \Gamma v(k) \tag{6.3.2-1}$$

with the process noise $v(k)$ a *zero-mean white* acceleration sequence.

The transition matrix is

$$F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{6.3.2-2}$$

and the vector gain multiplying the scalar process noise is given, in view of the above discussion, by

$$\Gamma = \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix} \tag{6.3.2-3}$$

The covariance of the process noise multiplied by the gain, $\Gamma v(k)$, is

$$Q = E[\Gamma v(k) v(k) \Gamma'] = \Gamma \sigma_v^2 \Gamma' = \begin{bmatrix} \frac{1}{4}T^4 & \frac{1}{2}T^3 \\ \frac{1}{2}T^3 & T^2 \end{bmatrix} \sigma_v^2 \tag{6.3.2-4}$$

Note the difference between this and (6.2.2-12).

The physical dimension of v and σ_v is [length]/[time]^2, i.e., that of acceleration.

Guideline for Choice of Process Noise Variance

For this model, σ_v should be of the order of the maximum acceleration magnitude a_M. A practical range is $0.5a_M \le \sigma_v \le a_M$.

A ***nearly constant velocity (NCV) model***[4] is obtained by the choice of a "small" intensity q in the following sense: The changes in the velocity over a sampling interval, which are of the order of $\sigma_v T$, have to be small compared to the actual velocity.

Note on the Multidimensional Case

When motion is in several coordinates, then, with decoupled filtering across coordinates, (6.3.2-4) is a block of the overall Q, which is then block diagonal.

6.3.3 Discrete Wiener Process Acceleration Model

For the ***piecewise constant Wiener process acceleration model***, or ***discrete Wiener process acceleration (DWPA) model***, the (third order) state equation is

$$x(k+1) = Fx(k) + \Gamma v(k) \tag{6.3.3-1}$$

where

$$F = \begin{bmatrix} 1 & \mathrm{T} & \frac{1}{2}T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \tag{6.3.3-2}$$

$$\Gamma = \begin{bmatrix} \frac{1}{2}T^2 \\ T \\ 1 \end{bmatrix} \tag{6.3.3-3}$$

In this model, the white process noise $v(k)$ is the ***acceleration increment*** during the kth sampling period and it is assumed to be a *zero-mean white sequence* — the *acceleration is a discrete-time Wiener process*. The formulation in terms of acceleration increment is more convenient than the one in terms of the third-order derivative (jerk).

The covariance of the process noise multiplied by the gain Γ is

$$Q = \Gamma\sigma_v^2\Gamma' = \begin{bmatrix} \frac{1}{4}T^4 & \frac{1}{2}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^3 & T^2 & T \\ \frac{1}{2}T^2 & T & 1 \end{bmatrix}\sigma_v^2 \tag{6.3.3-4}$$

[4]As in the continuous-time case, the term ***constant velocity (CV) model*** is used sometimes in the literature, with some abuse of language, for this model regardless of the intensity of the process noise.

Guideline for Choice of Process Noise Variance

For this model, σ_v should be of the order of the magnitude of the maximum acceleration increment over a sampling period, Δa_M. A practical range is $0.5\Delta a_M \leq \sigma_v \leq \Delta a_M$.

The Process Noise Variance for Different Sampling Periods

Note that if the sampling period is changed, one has to carry out a ***rescaling of the variance of the process noise***. (See problems 6-1 through 6-4.)

6.3.4 Kinematic Models — Summary

Kinematic (polynomial) model of order n: The nth derivative of the position is equal to

- Zero — noiseless model
- White noise — noisy model

There are two major classes of noisy discrete time kinematic models:

1. Obtained from discretization of the continuous-time model, driven by *continuous-time white noise*, for a given sampling period.
2. Obtained by direct definition of the process noise in discrete time as a *piecewise constant white sequence* — the process noise is assumed to be constant over each sampling period and independent between periods.

Within each class the following models were discussed in detail:

- White noise acceleration (WNA) — second-order model (sometimes called CV)
- Wiener process acceleration (WPA) — third-order model (sometimes called CA)

The resulting process noise covariance matrices for the two classes of models are different in their dependence on the sampling period.

The process noise covariance matrices in the direct discrete time models are positive semidefinite of rank 1, while their counterparts from the discretized continuous time models are of full rank.

Both models are, obviously, approximations.

The more commonly used model is the one in item 2. Its advantage is that the process noise intensity in this case is easily related to physical characteristics of the motion (acceleration). The model from item 1 is more convenient when one deals with variable sampling intervals.

6.4 EXPLICIT FILTERS FOR NOISELESS KINEMATIC MODELS

6.4.1 LS Estimation for Noiseless Kinematic Models

Consider an object moving with constant velocity (without process noise). Such a case was treated in Subsection 3.5.1, where, using least squares, the estimate of the *initial position and the (constant) velocity* were obtained, both in batch and in recursive form.

Using these results, the recursion for the estimate of the *current state* $x(k) \triangleq x(t_k)$ of a second-order **noiseless kinematic model** — the (exact) **constant velocity (CV) model** — will be obtained.

Similarly to (6.2.2-9), but without the process noise, one has

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) \tag{6.4.1-1}$$

Assuming a uniform sampling rate with sampling period T, we obtain

$$\hat{x}(t_{k+1}|t_{k+1}) \triangleq \hat{x}(k+1|k+1) = \begin{bmatrix} 1 & (k+1)T \\ 0 & 1 \end{bmatrix} \hat{x}(0|k+1) \tag{6.4.1-2}$$

In the above equation $\hat{x}(0|k+1)$ is the estimate of the *initial state*, with the initial time 0, to which it pertains, now explicitly indicated. This estimate of the initial state was obtained via recursive LS in (3.5.1-8), where it was denoted, without the first time index, as $\hat{x}(k+1)$.

6.4.2 The KF for Noiseless Kinematic Models

The recursion for the estimate of the current state is

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + W(k+1)[z(k+1) - \hat{z}(k+1|k)] \tag{6.4.2-1}$$

where

$$\hat{x}(k+1|k) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \hat{x}(k|k) \tag{6.4.2-2}$$

$$\hat{z}(k+1|k) = [1 \quad 0]\, \hat{x}(k+1|k) \tag{6.4.2-3}$$

Note that (6.4.2-1) is actually a recursive LS estimator "disguised" as a Kalman filter. In view of this, the gain $W(k)$ in (6.4.2-1) can be obtained directly from the gain (3.5.1-18), denoted now as $W_0(k)$ (since it pertains to the initial state), by multiplying it with the same matrix as in (6.4.1-2)

$$\begin{aligned} W(k+1) &= \begin{bmatrix} 1 & (k+1)T \\ 0 & 1 \end{bmatrix} W_0(k+1) \\ &= \begin{bmatrix} 1 & (k+1)T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{(k+1)} \\ \frac{6}{(k+1)(k+2)T} \end{bmatrix} = \begin{bmatrix} \frac{4k+2}{(k+1)(k+2)} \\ \frac{6}{(k+1)(k+2)T} \end{bmatrix} \end{aligned} \tag{6.4.2-4}$$

It can be easily shown that the covariance associated with the estimate (6.4.1-2) is, using (3.5.1-19), given by

$$\begin{aligned} P(k+1|k+1) &= \begin{bmatrix} 1 & (k+1)T \\ 0 & 1 \end{bmatrix} P(k+1) \begin{bmatrix} 1 & (k+1)T \\ 0 & 1 \end{bmatrix}' \\ &= \frac{2\sigma^2}{(k+1)(k+2)} \begin{bmatrix} 2k+1 & \dfrac{3}{T} \\ \dfrac{3}{T} & \dfrac{6}{kT^2} \end{bmatrix} \end{aligned} \tag{6.4.2-5}$$

Summary

For noiseless kinematic (polynomial) models, one can obtain explicit expressions of the Kalman filter gain and the state covariance.

Similar results can be obtained for the (exact) ***constant acceleration (CA) model.***

Due to the absence of process noise, the state covariance converges to zero and so does the filter gain.

6.5 STEADY-STATE FILTERS FOR NOISY KINEMATIC MODELS

6.5.1 The Problem

As indicated in Subsection 5.2.5, the state estimation covariance for a time-invariant system (with constant coefficients in the state and measurement equations) will converge under suitable conditions to a steady-state value.

These conditions are satisfied for the kinematic models described in Sections 6.2 and 6.3. Furthermore, *explicit expressions of the steady-state covariance and filter gain* can be obtained.

It will be assumed that only position measurements are available, that is,

$$z(k) = Hx(k) + w(k) \tag{6.5.1-1}$$

where for the white noise acceleration (second-order) model

$$H = [1 \quad 0] \tag{6.5.1-2}$$

and for the Wiener process acceleration (third-order) model

$$H = [1 \quad 0 \quad 0] \tag{6.5.1-3}$$

The measurement noise autocorrelation function is

$$E[w(k)w(j)] = R\delta_{kj} = \sigma_w^2 \delta_{kj} \tag{6.5.1-4}$$

The resulting ***steady-state filters for noisy kinematic models*** are known as ***alpha-beta*** and ***alpha-beta-gamma*** filters[5] for the second and third-order models, respectively. The coefficients α, β, and γ yield the filter steady-state gain vector components.

Subsection 6.5.2 presents the methodology of the derivation of the α-β filter. This is used for the direct discrete time and the discretized continuous-time second-order models in Subsections 6.5.3 and 6.5.4, respectively. Subsection 6.5.5 presents the α-β-γ filter for the direct discrete time third-order model.

6.5.2 Derivation Methodology for the Alpha-Beta Filter

The steady-state filter for the two-dimensional kinematic model is obtained as follows. The plant equation is

$$x(k+1) = Fx(k) + v(k) \tag{6.5.2-1}$$

where

$$F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{6.5.2-2}$$

and the (vector-valued) process noise has the autocorrelation function

$$E[v(k)v(j)'] = Q\delta_{kj} \tag{6.5.2-3}$$

The measurement is given by (6.5.1-1), (6.5.1-2), and (6.5.1-4). The variance of the (scalar) measurement noise will be denoted as $\sigma_w^2 \triangleq R$.

The steady-state values of the components of the state estimation covariance matrix will be denoted as

$$\lim_{k\to\infty} P(k|k) = [p_{ij}] \tag{6.5.2-4}$$

The components of the one-step prediction covariance are denoted as

$$\lim_{k\to\infty} P(k+1|k) = [m_{ij}] \tag{6.5.2-5}$$

while for the ***alpha-beta filter gain*** the notation will be

$$\boxed{\lim_{k\to\infty} W(k) \triangleq [g_1 \;\; g_2]' \triangleq \begin{bmatrix} \alpha & \dfrac{\beta}{T} \end{bmatrix}'} \tag{6.5.2-6}$$

Note that, as defined, α and β are dimensionless.

[5]Also called f-g and f-g-h filters in the literature [Brookner98].

Note

The existence, uniqueness and positive definiteness of (6.5.2-4) are guaranteed since the required observability and controllability conditions are satisfied (see problem 6-6).

The expression of the innovation covariance (5.2.3-9) yields

$$S = H \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} H' + R = m_{11} + \sigma_w^2 \tag{6.5.2-7}$$

where the notation $\sigma_w^2 = R$ is now used.

The filter gain given by (5.2.3-11) becomes

$$W = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} H'S^{-1} = \begin{bmatrix} \frac{m_{11}}{m_{11}+\sigma_w^2} & \frac{m_{12}}{m_{11}+\sigma_w^2} \end{bmatrix}' \tag{6.5.2-8}$$

From (6.5.2-6) and (6.5.2-8) it follows that

$$g_1 = \frac{m_{11}}{m_{11}+\sigma_w^2} \tag{6.5.2-9}$$

$$g_2 = \frac{m_{12}}{m_{11}+\sigma_w^2} = g_1 \frac{m_{12}}{m_{11}} \tag{6.5.2-10}$$

The covariance update (5.2.3-15) becomes, using (6.5.2-8) to (6.5.2-10),

$$\begin{aligned} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} &= (I - WH) \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \\ &= \begin{bmatrix} (1-g_1)m_{11} & (1-g_1)m_{12} \\ (1-g_1)m_{12} & m_{22} - g_2 m_{12} \end{bmatrix} \end{aligned} \tag{6.5.2-11}$$

The covariance prediction equation (5.2.3-5) is rewritten as follows:

$$P(k|k) = F^{-1}[P(k+1|k) - Q](F^{-1})' \tag{6.5.2-12}$$

where, from (6.5.2-2), one has

$$F^{-1} = \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix} \tag{6.5.2-13}$$

The steady-state solution for the covariance and gains is obtained from the set of nonlinear equations (6.5.2-9) to (6.5.2-12) using the suitable expression of the process noise covariance Q in (6.5.2-12). The expression for the direct discrete time model is (6.3.2-4), while for the discretized continuous-time model it is (6.2.2-12).

6.5.3 The Alpha-Beta Filter for the DWNA Model

Using the process noise covariance (6.3.2-4), which corresponds to a ***discrete white noise acceleration (DWNA) model*** in the form of a *piecewise constant white process noise* — accelerations that are constant over each sampling period and uncorrelated from period to period — yields in (6.5.2-12)

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} m_{11} - 2Tm_{12} + T^2 m_{22} - \frac{1}{4}T^4\sigma_v^2 & m_{12} - Tm_{22} + \frac{1}{2}T^3\sigma_v^2 \\ m_{12} - Tm_{22} + \frac{1}{2}T^3\sigma_v^2 & m_{22} - T^2\sigma_v^2 \end{bmatrix} \tag{6.5.3-1}$$

Equating the terms of (6.5.2-11) and (6.5.3-1) yields, after some cancellations,

$$g_1 m_{11} = 2Tm_{12} - T^2 m_{22} + \frac{T^4}{4}\sigma_v^2 \tag{6.5.3-2}$$

$$g_1 m_{12} = Tm_{22} - \frac{T^3}{2}\sigma_v^2 \tag{6.5.3-3}$$

$$g_2 m_{12} = T^2\sigma_v^2 \tag{6.5.3-4}$$

Equations (6.5.2-9), (6.5.2-10), and (6.5.3-2) to (6.5.3-4) with the five unknowns g_1, g_2, m_{11}, m_{12}, and m_{22} are solved next.

From (6.5.2-9) and (6.5.2-10) one has

$$m_{11} = \frac{g_1}{1-g_1}\sigma_w^2 \tag{6.5.3-5}$$

$$m_{12} = \frac{g_2}{1-g_1}\sigma_w^2 \tag{6.5.3-6}$$

From (6.5.3-3) and (6.5.3-4) one obtains

$$m_{22} = \frac{g_1 m_{12}}{T} + \frac{T^2}{2}\sigma_v^2 = \left(\frac{g_1}{T} + \frac{g_2}{2}\right) m_{12} \tag{6.5.3-7}$$

Using (6.5.3-4) to (6.5.3-7) in (6.5.3-2) yields

$$\frac{g_1^2}{1-g_1}\sigma_w^2 = 2T\frac{g_2}{1-g_1}\sigma_w^2 - T^2\left(\frac{g_1}{T} + \frac{g_2}{2}\right)\frac{g_2}{1-g_1}\sigma_w^2 + \frac{T^2}{4}\frac{g_2^2}{1-g_1}\sigma_w^2 \tag{6.5.3-8}$$

which, after cancellations, becomes

$$g_1^2 - 2Tg_2 + Tg_1g_2 + \frac{T^2}{4}g_2^2 = 0 \tag{6.5.3-9}$$

With the dimensionless variables α and β one has

$$\alpha^2 - 2\beta + \alpha\beta + \frac{\beta^2}{4} = 0 \tag{6.5.3-10}$$

which yields the first equation for α and β as

$$\alpha = \sqrt{2\beta} - \frac{\beta}{2} \tag{6.5.3-11}$$

The second equation for α and β follows immediately from (6.5.3-4) and (6.5.3-6) as

$$m_{12} = \frac{T^2\sigma_v^2}{\beta/T} = \frac{\beta/T}{1-\alpha}\sigma_w^2 \tag{6.5.3-12}$$

or

$$\frac{\beta^2}{1-\alpha} = \frac{T^4\sigma_v^2}{\sigma_w^2} \triangleq \lambda^2 \tag{6.5.3-13}$$

The quantity

$$\boxed{\lambda \triangleq \frac{\sigma_v T^2}{\sigma_w}} \tag{6.5.3-14}$$

is called the ***target maneuvering index*** (also called ***target tracking index***) since it is proportional to the ratio of

- The ***motion uncertainty*** — the RMS value of the process noise (acceleration) effect on the position over one period, which is $\sigma_v T^2/2$ — see (6.3.2-4).
- The ***observation uncertainty*** — the measurement noise RMS value σ_w.

Eliminating α from (6.5.3-13) with (6.5.3-11) yields

$$\frac{\beta^2}{1-\alpha} = \frac{\beta^2}{1-\sqrt{2\beta}+\beta/2} = \frac{\beta^2}{\left(1-\sqrt{\beta/2}\right)^2} = \lambda^2 \tag{6.5.3-15}$$

or

$$\beta + \frac{\lambda}{\sqrt{2}}\sqrt{\beta} - \lambda = 0 \tag{6.5.3-16}$$

The positive solution for $\sqrt{\beta}$ from the above is

$$\sqrt{\beta} = \frac{1}{2\sqrt{2}}\left(-\lambda + \sqrt{\lambda^2+8\lambda}\right) \tag{6.5.3-17}$$

The expression of the ***velocity gain coefficient*** β in terms of λ is

$$\boxed{\beta = \frac{1}{4}\left(\lambda^2 + 4\lambda - \lambda\sqrt{\lambda^2+8\lambda}\right)} \tag{6.5.3-18}$$

Using (6.5.3-18) in (6.5.3-11) gives the ***position gain*** α in terms of λ as

$$\boxed{\alpha = -\frac{1}{8}\left(\lambda^2 + 8\lambda - (\lambda+4)\sqrt{\lambda^2+8\lambda}\right)} \tag{6.5.3-19}$$

The elements of the state estimation covariance matrix are, using (6.5.2-11),

$$p_{11} = (1 - g_1)m_{11} = g_1\sigma_w^2 \tag{6.5.3-20}$$

$$p_{12} = (1 - g_1)m_{12} = g_2\sigma_w^2 \tag{6.5.3-21}$$

$$p_{22} = \left(\frac{g_1}{T} + \frac{g_2}{2}\right)m_{12} - g_2 m_{12} = \left(\frac{g_1}{T} - \frac{g_2}{2}\right)m_{12} \tag{6.5.3-22}$$

The expressions of these (steady-state) error covariance matrix elements can be rewritten using (6.5.3-5) to (6.5.3-7) as

$$\boxed{p_{11} = \alpha\sigma_w^2} \tag{6.5.3-23}$$

$$\boxed{p_{12} = \frac{\beta}{T}\sigma_w^2} \tag{6.5.3-24}$$

$$\boxed{p_{22} = \frac{\beta}{T^2}\frac{\alpha - \beta/2}{1 - \alpha}\sigma_w^2} \tag{6.5.3-25}$$

The (MS) ***position estimation improvement***, or the ***noise reduction factor***, with respect to a single observation is seen from (6.5.3-23) to be α ($0 \le \alpha \le 1$) — that is, the same as the *optimal* position gain of the filter.

The innovation variance is, in terms of the position prediction variance and the measurement noise variance,

$$s = m_{11} + \sigma_w^2 \tag{6.5.3-26}$$

Using (6.5.3-5) with g_1 replaced by α yields

$$\boxed{s = \frac{\sigma_w^2}{1 - \alpha}} \tag{6.5.3-27}$$

Figure 6.5.3-1 presents the ***alpha-beta filter*** gain coefficients α and β as a function of the maneuvering index λ in semilog and log-log scales.

The ***velocity estimation improvement*** — compared to the differencing of two adjacent observations — is

$$\boxed{\eta \triangleq \frac{p_{22}}{2\sigma_w^2/T^2} = \frac{\beta}{2}\frac{\alpha - \beta/2}{1 - \alpha}} \tag{6.5.3-28}$$

Note that this ignores the process noise — it is not meaningful for significant levels of the maneuvering index.

Figure 6.5.3-2 presents the velocity estimation improvement factor given above.

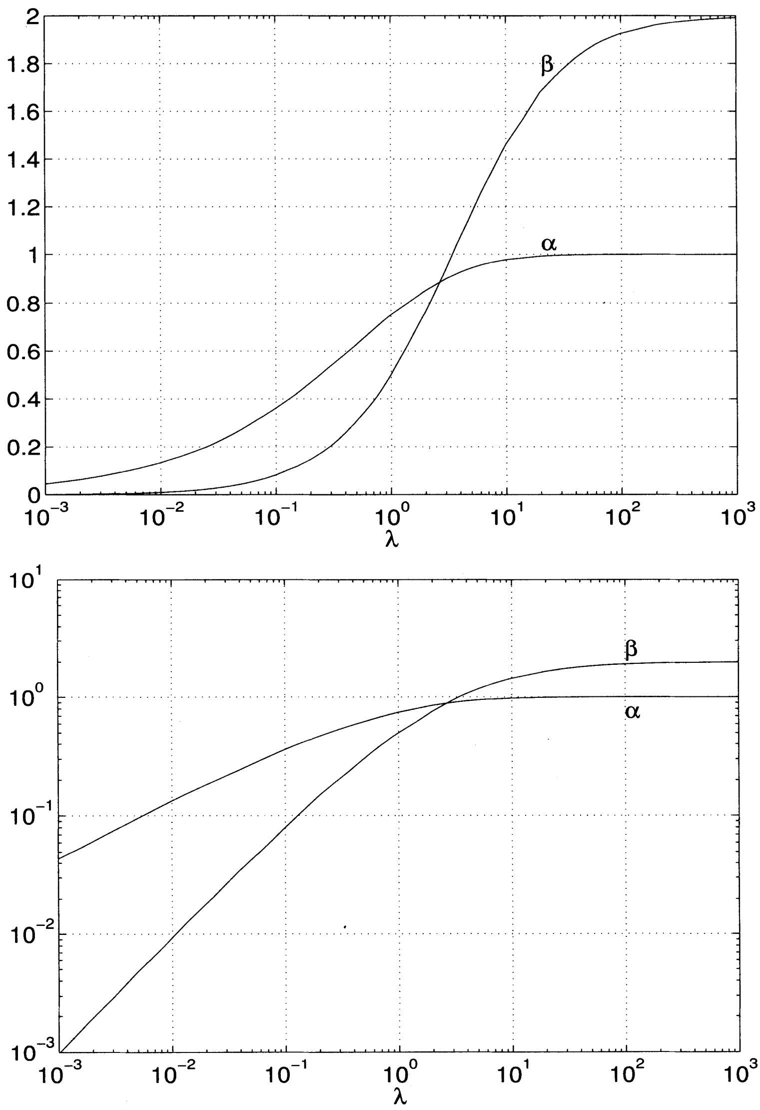

Figure 6.5.3-1: Steady-state filter gain coefficients for the piecewise constant white acceleration model.

Simplified Expressions for Low Maneuvering Index

Note that for small λ (up to about 0.1) one has the following simplified expressions of the gains and the velocity estimation improvement factor

$$\alpha \approx \sqrt{2\lambda} \tag{6.5.3-29}$$

$$\beta \approx \lambda \tag{6.5.3-30}$$

$$\eta \approx \frac{1}{\sqrt{2}} \lambda^{1.5} \tag{6.5.3-31}$$

Remarks

A high value of the process noise variance relative to the measurement noise variance — that is, a large maneuvering index λ — yields a high position gain α and the filter will give large weight to the latest measurement and consequently little weight to the past data, resulting in less noise reduction.

A small λ yields a lower α and more noise reduction. However, a small α will *not* yield more noise reduction unless it has been *optimally determined* based on λ and *all the modeling assumptions hold.*

The two gains α and β *cannot be chosen independently* — they are both determined by the maneuvering index λ.

Example

The example of Section 5.3, which dealt with what now is called a discrete white noise acceleration (DWNA) model, is reconsidered. The closed-form expressions for the steady-state gain and covariances developed above will be used and compared with the results of the covariance equation iterations that are plotted in Figs. 5.3.2-2 and 5.3.2-3.

The two cases of interest are those with nonzero process noise: $q = 1$ and $q = 9$ (the case with $q = 0$ leads to zero variances and gain in steady state). The corresponding maneuvering indices are, using (6.5.3-14) with $T = 1$, $\sigma_v = \sqrt{q}$ and $\sigma_w = 1$, $\lambda = 1$ and $\lambda = 3$, respectively.

Table 6.5.3-1 shows the steady-state values of the gain coefficients as well as the position and velocity variances plotted in Figures 5.3.2-2 and 5.3.2-3, parts (b) and (c), respectively. The innovation variance (6.5.3-27) is also shown.

The updated variances, obtained from (6.5.3-23) and (6.5.3-25), and the predicted variances, obtained from (6.5.3-5) and (6.5.3-7), are seen to match the values plotted in the above figures.

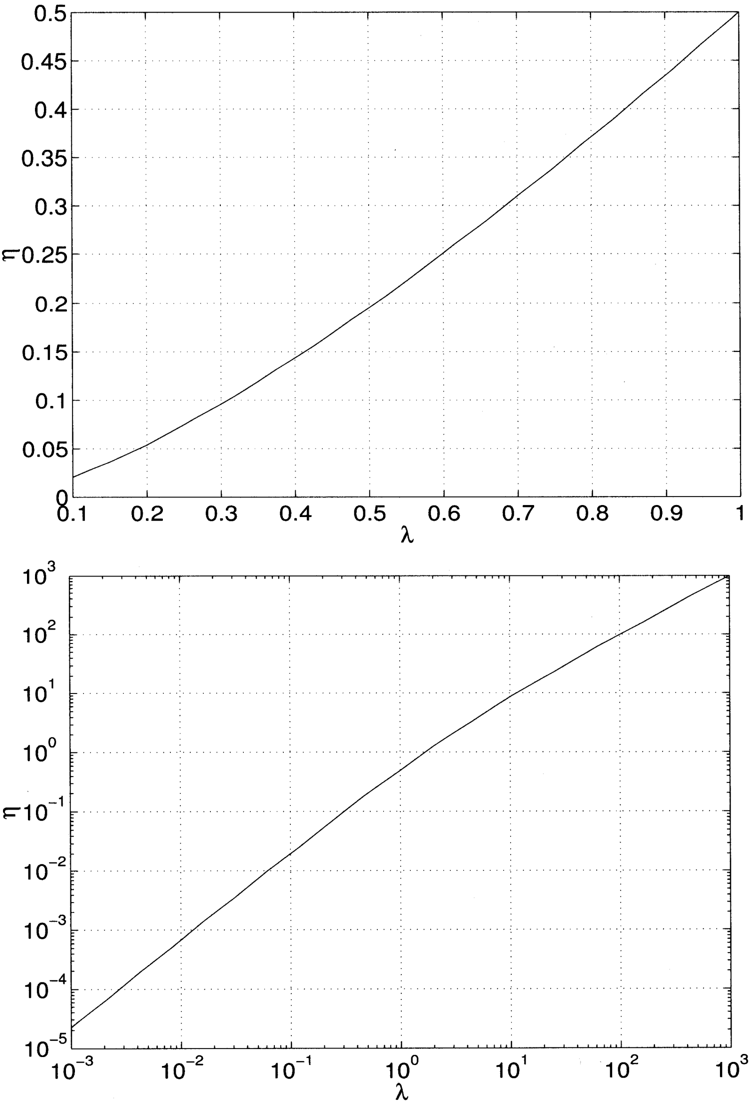

Figure 6.5.3-2: Steady-state filter velocity estimation improvement over two-point differencing for the piecewise constant white acceleration model.

Table 6.5.3-1: Steady-state gains and variances for two maneuvering indices.

Maneuvering Index	Position Gain	Velocity Gain	Position Variance Updated	Position Variance Predicted	Velocity Variance Updated	Velocity Variance Predicted	Innovation Variance
λ	α	β	p_{11}	m_{11}	p_{22}	m_{22}	s
1	0.75	0.50	0.75	3	1	2	4
3	0.90	0.94	0.90	9.15	4.12	13.12	10.15

6.5.4 The Alpha-Beta Filter for the Discretized CWNA Model

Next, the ***alpha-beta filter*** for the ***discretized continuous-time white noise acceleration (CWNA) model*** is derived using the process noise covariance matrix (6.2.2-12) instead of (6.3.2-4). Since almost all the equations stay the same as in Subsection 6.5.3, they will not be repeated and only those which are different will be indicated.

Equating the terms in the updated covariance expressions (6.5.2-11) and (6.5.2-12) with Q given by (6.2.2-12) yields

$$g_1 m_{11} = 2T m_{12} - T^2 m_{22} + \frac{T^3}{3}\tilde{q} \tag{6.5.4-1}$$

$$g_1 m_{12} = T m_{22} - \frac{T^2}{2}\tilde{q} \tag{6.5.4-2}$$

$$g_2 m_{12} = T\tilde{q} \tag{6.5.4-3}$$

To eliminate m_{22}, one has

$$m_{22} = \frac{g_1 m_{12}}{T} + \frac{T}{2}\tilde{q} = \left(\frac{g_1}{T} + \frac{g_2}{2}\right) m_{12} \tag{6.5.4-4}$$

The counterpart of (6.5.3-8) is

$$\frac{g_1^2}{1-g_1}\sigma_w^2 = 2T\frac{g_2}{1-g_1}\sigma_w^2 - T^2\left(\frac{g_1}{T} + \frac{g_2}{2}\right)\frac{g_2}{1-g_1}\sigma_w^2 + \frac{T^2}{3}\frac{g_2^2}{1-g_1}\sigma_w^2 \tag{6.5.4-5}$$

which becomes

$$g_1^2 - 2Tg_2 + Tg_1g_2 + \frac{T^2}{6}g_2^2 = 0 \tag{6.5.4-6}$$

In terms of α and β, the above can be written as

$$\alpha^2 - 2\beta + \alpha\beta + \frac{\beta^2}{6} = 0 \tag{6.5.4-7}$$

or

$$\alpha = \sqrt{2\beta + \frac{\beta^2}{12}} - \frac{\beta}{2} \tag{6.5.4-8}$$

The above equation differs from (6.5.3-11) by an extra term.

Equating m_{12} from (6.5.4-3) and (6.5.3-6) yields

$$m_{12} = \frac{T\tilde{q}}{\beta/T} = \frac{\beta/T}{1-\alpha}\sigma_w^2 \tag{6.5.4-9}$$

which results in

$$\frac{\beta^2}{1-\alpha} = \frac{T^3\tilde{q}}{\sigma_w^2} \triangleq \lambda_c^2 \tag{6.5.4-10}$$

where λ_c is the **maneuvering index** for this *discretized continuous-time system* and has a similar interpretation as λ in (6.5.3-14).

The equation for β becomes

$$\frac{\beta^2}{1-\alpha} = \frac{\beta^2}{1-\sqrt{2\beta+\beta^2/12}+\beta/2} = \lambda_c^2 \tag{6.5.4-11}$$

which can be solved explicitly as follows.

Let

$$u \triangleq \frac{\alpha^2}{\beta^2} \tag{6.5.4-12}$$

Using (6.5.4-12) in (6.5.4-8) yields

$$\beta = \frac{12}{6(u+\sqrt{u})+1} \tag{6.5.4-13}$$

and, thus,

$$\alpha = \beta\sqrt{u} = \frac{12\sqrt{u}}{6(u+\sqrt{u})+1} \tag{6.5.4-14}$$

Substituting (6.5.4-13) and (6.5.4-14) in (6.5.4-10) yields a quadratic equation for u with positive solution

$$u = \frac{1}{2} + \sqrt{\frac{1}{12} + \frac{4}{\lambda_c^2}} \tag{6.5.4-15}$$

which, when inserted in (6.5.4-13) and (6.5.4-14), provides the explicit solution for the gains of the alpha-beta filter for the discretized CWNA model.

Figure 6.5.4-1 presents the ***alpha-beta filter*** gain coefficients α and β for the discretized CWNA model as a function of the maneuvering index λ_c in semilog and log-log scales.

The equations for the updated covariance terms (6.5.3-23) to (6.5.3-25) stay the same.

Remark

This discretized continuous-time model is somewhat less common in use than the direct discrete-time piecewise constant acceleration model.

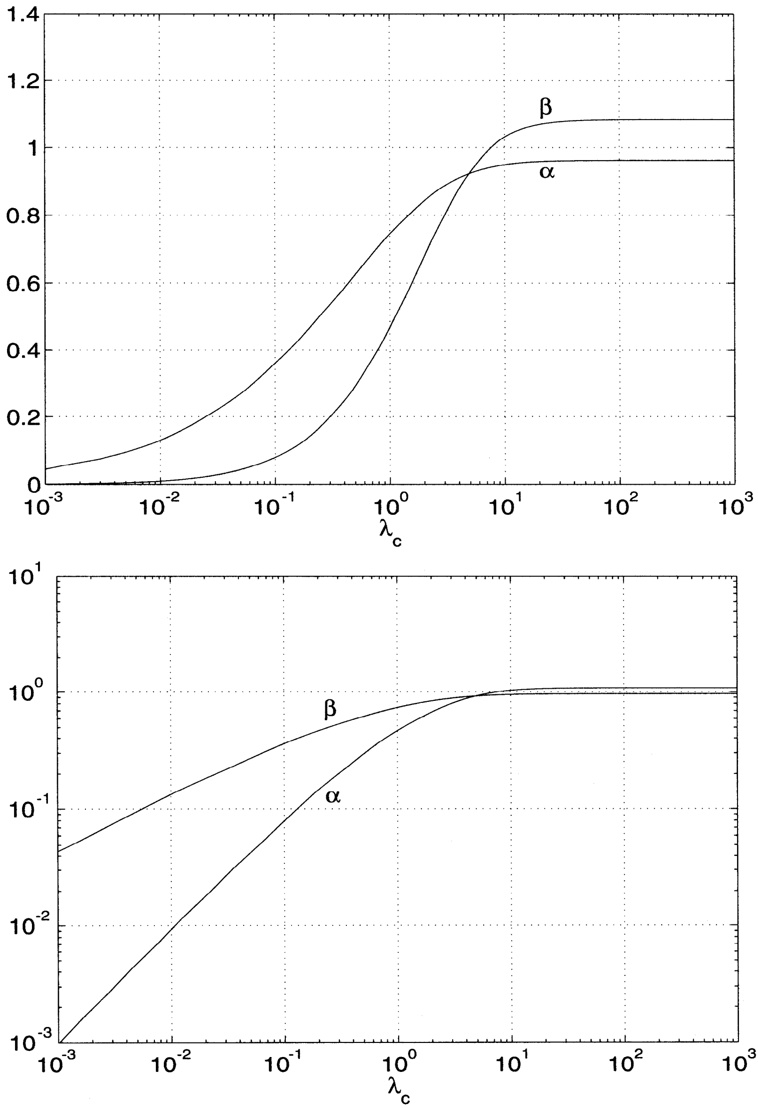

Figure 6.5.4-1: Steady-state filter gain coefficients for the discretized continuous-time white acceleration model.

6.5.5 The Alpha-Beta-Gamma Filter for the DWPA Model

The ***piecewise constant Wiener process acceleration model*** is the third-order system (6.3.3-1) with the *zero-mean white* process noise, assumed to be the ***acceleration increment over a sampling period***, with variance σ_v^2.

The target maneuvering index is defined in the same manner as for the second-order model in (6.5.3-14), that is,

$$\lambda = \frac{\sigma_v T^2}{\sigma_w} \tag{6.5.5-1}$$

The steady-state gain for the resulting filter — the ***alpha-beta-gamma filter*** — is

$$\boxed{\lim_{k\to\infty} W(k) \triangleq [g_1 \;\; g_2 \;\; g_3]' \triangleq \begin{bmatrix} \alpha & \dfrac{\beta}{T} & \dfrac{\gamma}{2T^2} \end{bmatrix}'} \tag{6.5.5-2}$$

It can be shown that the three equations that yield the optimal steady-state filter ***gain coefficients*** are

$$\frac{\gamma^2}{4(1-\alpha)} = \lambda^2 \tag{6.5.5-3}$$

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \tag{6.5.5-4}$$

or

$$\alpha = \sqrt{2\beta} - \frac{\beta}{2} \tag{6.5.5-5}$$

and

$$\gamma = \frac{\beta^2}{\alpha} \tag{6.5.5-6}$$

The relationship between α and β in (6.5.5-4) is the same as (6.5.3-11).

The explicit solution for this system of three nonlinear equations that yields the three gain coefficients from (6.5.5-2) in terms of λ is given at the end of this subsection.

Figure 6.5.5-1 shows the gain coefficients for this filter as a function of the maneuvering index λ in semilog and log-log scale.

Similarly to the second-order system, it can be shown that the corresponding updated state covariance expressions (in steady state) are

$$p_{11} = \alpha\sigma_w^2 \qquad p_{12} = \frac{\beta}{T}\sigma_w^2 \qquad p_{13} = \frac{\gamma}{2T^2}\sigma_w^2 \tag{6.5.5-7}$$

$$p_{22} = \frac{8\alpha\beta + \gamma(\beta - 2\alpha - 4)}{8T^2(1-\alpha)}\sigma_w^2 \tag{6.5.5-8}$$

$$p_{23} = \frac{\beta(2\beta-\gamma)}{4T^3(1-\alpha)}\sigma_w^2 \qquad p_{33} = \frac{\gamma(2\beta-\gamma)}{4T^4(1-\alpha)}\sigma_w^2 \tag{6.5.5-9}$$

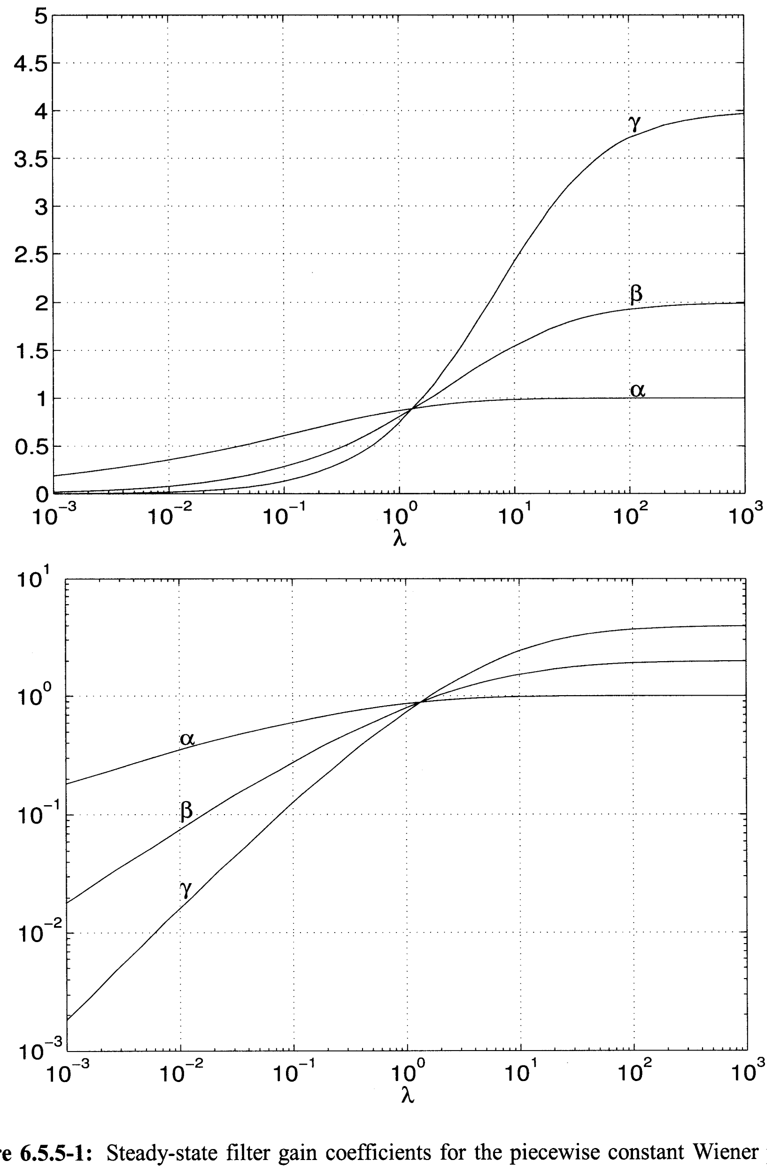

Figure 6.5.5-1: Steady-state filter gain coefficients for the piecewise constant Wiener process acceleration model.

The Solution for the Gain Coefficients

Substituting

$$\alpha = 1 - s^2 \tag{6.5.5-10}$$

in (6.5.5-4) yields

$$\beta = 2(1-s)^2 \tag{6.5.5-11}$$

Rewriting (6.5.5-3) with (6.5.5-10) yields

$$\gamma = 2\lambda\sqrt{1-\alpha} = 2\lambda s \tag{6.5.5-12}$$

Equations (6.5.5-10) to (6.5.5-12) provide the *explicit solution for the gain coefficients* in terms of the new variable s for which a cubic equation is obtained next.

Substituting (6.5.5-12) and (6.5.5-10) into the second equation of (6.5.5-6) leads to

$$2\lambda s = \frac{4(1-s)^4}{1-s^2} \tag{6.5.5-13}$$

which can be rewritten as

$$s^3 + bs^2 + cs - 1 = 0 \tag{6.5.5-14}$$

where

$$b \triangleq \frac{\lambda}{2} - 3 \qquad\qquad c \triangleq \frac{\lambda}{2} + 3 \tag{6.5.5-15}$$

Substitute again

$$s = y - \frac{b}{3} \tag{6.5.5-16}$$

to obtain

$$y^3 + py + q = 0 \tag{6.5.5-17}$$

where

$$p \triangleq c - \frac{b^2}{3} \qquad\qquad q \triangleq \frac{2b^3}{27} - \frac{bc}{3} - 1 \tag{6.5.5-18}$$

Finally, one more substitution,

$$y = z - \frac{p}{3z} \tag{6.5.5-19}$$

yields

$$z^6 + qz^3 - \frac{p^2}{27} = 0 \tag{6.5.5-20}$$

which has the solution

$$z^3 = \frac{-q \pm \sqrt{q^2 + 4p^2/27}}{2} \tag{6.5.5-21}$$

from which the negative sign should be chosen.

Using (6.5.5-16) and (6.5.5-19) yields

$$s = z - \frac{p}{3z} - \frac{b}{3} \tag{6.5.5-22}$$

which can be used directly in (6.5.5-10) to (6.5.5-12) to obtain the gain coefficients α, β, and γ.

6.5.6 A System Design Example for Sampling Rate Selection

The following ***system design*** problem is considered:

- An object moves in one dimension and undergoes acceleration up to 1 m/s^2 and whose position is measured at intervals of T with an additive zero mean white noise with standard deviation $\sigma_w = 10\,\text{m}$.
- It is desired to determine the ***sampling rate***, i.e., the ***revisit rate*** $1/T$ such that the steady-state position estimation RMS error $\sqrt{p_{11}} = 0.5\,\text{m}$.

It will be assumed that the tracking of this moving object is based on a DWNA model, as discussed in Subsection 6.5.3, with process noise standard deviation $\sigma_v = 1\,\text{m/s}^2$.

Based on (6.5.3-23), the requirement on the position error translates into a noise reduction factor

$$\alpha = \frac{p_{11}}{\sigma_w^2} = \frac{0.25}{100} = \frac{1}{400} \tag{6.5.6-1}$$

Note that the required RMS reduction factor of 1/20 becomes 1/400 for the MSE.

Using (6.5.3-29) and the expression of the maneuvering index (6.5.3-14), one has

$$2\frac{\sigma_v T^2}{\sigma_w} = \alpha^2 \tag{6.5.6-2}$$

which yields

$$T = \alpha\sqrt{\frac{\sigma_w}{2\sigma_v}} = \frac{\sqrt{10}}{400\sqrt{2}}\,\text{s} \tag{6.5.6-3}$$

This corresponds to a sampling rate of nearly 200 Hz, i.e., extremely high. The reason for this is that the MS noise reduction factor of 1/400 is hard to achieve: Even for a static problem (i.e., an object at a fixed location) it would require 400 measurements of the object's fixed position to obtain this reduction factor.

In other words, one can improve the position accuracy over the sensor accuracy, but, typically, not by such a large factor. This illustrates the limitations faced when a certain performance has to be met.

If the desired accuracy is $\sqrt{p_{11}} = 5\,\text{m}$, then $\alpha = 1/4$ and the resulting sampling interval is $T = \sqrt{10}/(4\sqrt{2}) \approx 0.5\,\text{s}$, which amounts to a more reasonable revisit rate of about 2 Hz.

Remarks

A similar approach can be used if the system requirement is given for the velocity accuracy.

The same approach can be also used for a ***trade-off*** study between, for example, the sensor accuracy σ_w and T.

One has to keep in mind that the above relies on the DWNA model, which, like any model, is *only an approximation of the reality*. Alternatively, one can use another stochastic model like the discretized CWNA model (Subsection 6.5.4) or a deterministic model (e.g., as discussed in Section 1.1.4).

Note that, if the process noise assumption for the piecewise constant DWNA model (6.3.2-1) are valid for a given T, then, strictly speaking, they cannot be valid for any other T. On the other hand, while the discretized CWNA model (6.2.2-9) is valid for any T, its underlying continuous-time model (6.2.2-6) is an idealization. In other words, neither approach is perfect — this is why the designer has to have a *good understanding of the underlying modeling assumptions*.

6.5.7 Alpha-Beta and Alpha-Beta-Gamma Filters — Summary

For discrete-time kinematic models

- with zero-mean white process noise that models
  - the acceleration (second-order model) or
  - the acceleration increments (third-order model — Wiener process acceleration)
- with noisy position measurements,

one has explicit expressions of the *steady-state filter gain and the corresponding covariance*.

These filters, called alpha-beta and alpha-beta-gamma, respectively, are the simplest possible: They use fixed precomputed (steady-state) gains. Consequently, they are *not optimal* during the initial transient period or if the noises are nonstationary.

The gains of these filters depend *only* on the *target maneuvering index*.

The *target maneuvering index* is defined as the ratio between the standard deviations (RMS values) of the following two uncertainties:

- Motion uncertainty — the position displacement over one sampling period due to the process noise (multiplied by 2) and
- Observation uncertainty — the (position) measurement noise.

These filters are usually used independently for each coordinate; however, one can encounter instability under certain extreme circumstances due to the errors introduced by the decoupling [Rogers88].

Two classes of models were discussed:

- Discretized continuous-time models based on *continuous-time zero-mean white noise*,
- Direct discrete-time models based on *piecewise constant zero-mean white noise* — that is, a *zero-mean white sequence*.

These noises model the uncertainties of the motion — acceleration or acceleration increments. These two classes exhibit a different dependence on the sampling period of the effect of the noises on the motion.

None of these assumptions can model exactly target maneuvers, which are neither zero mean nor white — actually they are not even random, but the state models (which have to be Markov processes) require the specification of some randomness.

Nevertheless, they have been used extensively in real-time implementations of target tracking as well as for frequency estimation in speech recognition and in navigation systems. All the GPS (global positioning system) receivers use such filters to "smooth" the indicated position, speed and course (estimate them).

In particular, such fixed-gain filters have proven to be useful in implementations where their very modest computational and memory requirements were a major consideration.

One convenient application of these explicit results is to obtain quick (but possibly dirty) evaluations of achievable tracking performance — the quality of estimation, measured by the steady-state error variances. They can also be used for selection of system parameters, e.g., sampling interval.

These kinematic models can also be used as elements in a set of models describing different target behavior modes in the context of multiple model estimation algorithms, to be discussed in Chapter 11.

6.6 NOTES AND PROBLEMS

6.6.1 Bibliographical Notes

The kinematic (polynomial) models for filtering date back to [Sklansky57, Benedict62]. They have been extensively discussed in the literature, for example, [Wishner70], and several papers presented steady-state filters for them. In [Friedland73] analytical expressions of the position and velocity estimation accuracy with position measurements were given. The coupling between range and range-rate (Doppler) measurements is discussed in [Fitzgerald74]. Gain curves as a function of the maneuvering index were presented in [Fitzgerald80]. Closed-form solutions for the continuous-time and discrete-time filter with exponentially autocorrelated acceleration were presented in [Fitzgerald81]. Tracking accuracies with position and velocity measurements were derived in [Castella81]. Analytical solutions for the steady-state filter gain and covariance with position and velocity measurements were given in [Ekstrand83].

The explicit derivations presented in Section 6.5 using the target maneuvering index are based on [Kalata84]. Its generalization to the coordinate-coupled case can be found in [Li97]. The idea of target maneuvering index has been used in [Friedland73] and can be traced back to [Sittler64]. The derivation of the explicit solution for the alpha-beta-gamma filter is based on [Gray93], while the explicit solution of the gains for the discretized CWNA model is due to J. Gray.

Track initiation and simple approximations of the gains during the transient for kinematic models have been discussed in [Kalata84].

Frequency domain analysis of alpha-beta filters and the steady-state bias resulting from constant accelerations are discussed in [Farina85]. The equivalent bandwidth of polynomial filters has been presented in [Ng83]. The response of alpha-beta-gamma filters to step inputs can be found in [Navarro77].

A discussion of the possible unbounded errors in decoupled alpha-beta filters is presented in [Rogers88].

6.6.2 Problems

6-1 Simulated kinematic trajectory behavior. A target is simulated as having a nearly constant velocity motion with white noise acceleration, constant over the sampling period T, as in (6.3.2-1). The noise is $v(k) \sim \mathcal{N}(0, \sigma_v^2)$.

1. Find the prior pdf of the velocity at time k.

2. What is the range of the velocity k sampling periods after the initial time?

3. Assume that the initial velocity is 10, and that the process noise variance and the sampling time are both unity. If after $k = 25$ samples the velocity became zero, is this a sign that the random number generator is biased?

4. What conclusion can be drawn from the above about the behavior of the velocity for the third-order kinematic model (6.3.3-1) with process noise representing acceleration increments?

6-2 Process noise rescaling (for second-order direct discrete time kinematic model when sampling period is changed). A target is simulated according to the second-order kinematic model (6.3.2-1) with a sampling period T_1 with process noise $v(k)$, which represents the constant acceleration over a sampling period, with variance $\sigma_v^2(T_1)$. Subsequently, the sampling period is changed to T_2 and we want to preserve the statistical properties of the motion. What should be done?

6-3 Process noise rescaling (for third-order direct discrete time kinematic model when sampling period is changed). A target is simulated according to the third-order kinematic model (6.3.3-1) with a sampling period T_1 with process noise $v(k)$, which represents the acceleration increment over a sampling period, with variance $\sigma_v^2(T_1)$. Subsequently, the sampling period is changed to T_2 and we want to preserve the statistical properties of the motion. What should be done?

6-4 Simulated kinematic trajectory variability when sampling period is changed. A trajectory is generated as in problem 6-3 with $T_1 = 1$ and $\sigma_v^2(T_1) = 0.1$ for $N_1 = 20$ periods. Then, using the same random number generator seed, the same trajectory is generated with a higher sampling rate, $T_2 = 0.5$ for $N_2 = 40$ samples — that is, the same total time.

1. Can one expect the trajectories to be identical?

2. Find the range of the difference between the two trajectories' accelerations at the common final time t_F.

6-5 Alpha-beta filter design and evaluation.

1. Design an α-β filter for a data rate of 40 Hz, with maximum acceleration $|a_M| = 16g$ ($g \approx 10\,\mathrm{m/s^2}$), and measurement noise with $\sigma_w = 10\,\mathrm{m}$.

2. Calculate the (steady-state) error covariance matrix elements and the position and velocity RMS errors.

3. Calculate the RMS position prediction error for a prediction time of $t = 2\,\mathrm{s}$ under the assumptions of the filter.

4. Assume that the target has a constant acceleration of $16g$ during this prediction time. Calculate the position prediction error due to this and compare it with the result from item 3.

6-6 Existence of steady-state filter for a kinematic model. Consider the system

$$x(k+1) = Fx(k) + \Gamma v(k)$$

$$z(k) = Hx(k) + w(k)$$

with

$$F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \qquad H = [1 \;\; 0]$$

and the (scalar) process and measurement noise sequences are zero mean white with variances q and r, respectively.

1. State the condition for stability (existence of steady state) of the Kalman filter for this system in terms of F and H. Prove that this holds for the above system.

2. State the additional condition that the steady-state filter covariance is positive definite and unique. Prove that this also holds.

6-7 Alpha filter. Consider the scalar system

$$x(k+1) = x(k) + \frac{T^2}{2} v(k)$$

$$z(k) = x(k) + w(k)$$

with the two noise sequences mutually uncorrelated, zero mean, white, and with variances σ_v^2 and σ_w^2, respectively. Let

$$\lambda \triangleq \frac{T^2 \sigma_v}{\sigma_w}$$

Find

1. The steady-state Kalman filter gain α in terms of λ.

2. The noise reduction factor

$$\frac{P_{xx}}{\sigma_w^2}$$

in terms of α, where P_{xx} is the steady-state variance of the estimate of x from the KF.

6-8 Alpha filter and suboptimality. Consider the scalar system

$$x(k+1) = x(k) + v(k)$$

with $v(k)$ zero-mean, white with variance $q = 1$ and the observations

$$z(k) = x(k) + w(k)$$

with $w(k)$ zero-mean, white with variance $r = 1$, with the two sequences and the initial error mutually uncorrelated.

1. Find the optimal steady-state Kalman filter gain.

2. Find the MS and RMS noise reduction factors.

3. Assume the filter has the arbitrary gain $W = 0.1$. Find the steady state MS estimation error. How much is now the "noise reduction factor"?

4. Find the range of values of W for which the filter is stable.

5. What happens if $W = 2$?

6-9 Bias and total MS error in the alpha filter. Consider the estimator (alpha filter)

$$\hat{x}(k+1) = \hat{x}(k) + \alpha[z(k+1) - \hat{x}(k)]$$

with $0 < \alpha < 1$, for the scalar system

$$x(k+1) = x(k) + v$$

where v is a fixed bias, with measurements

$$z(k) = x(k) + w(k)$$

where $w(k)$ is a zero-mean white noise with known variance r.

1. Assume $v = 0$. Find the steady-state MSE

$$P = E[\tilde{x}(k)^2]$$

(due to the measurement noise only).

2. Assume $v \neq 0$. Find the steady-state bias

$$b = E[\tilde{x}(k)]$$

due to v.

3. Find the total MSE (due to both v and the measurement noise).

6-10 Filter with position/velocity measurement. Consider a DWNA model (with piecewise constant accelerations) for one coordinate with the measurement being a linear combination of the position ξ and velocity $\dot{\xi}$, namely,

$$z(k) = [1 \ a]\begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix} + w(k)$$

Assume one has the state covariance $P(1|0) = I$. Furthermore, let the sampling interval be $T = 1$, the process noise variance $\sigma_v^2 = 0$ and the measurement noise variance $\sigma_w^2 = 1$.

1. Find P(2—1).
2. If the choices for a are $a > 0$ or $a, 0$, which one would you pick and why?
3. Find the optimal value of a for minimum position variance in $P(2|1)$.
4. Set up the equations for the steady state filter gains.
5. Solve them.
6. Find the optimal value of a for minimum position variance component in $P(k|k)$ in steady state.

6-11 Bandwidth of an alpha filter. Consider the alpha filter for a scalar system with state x and observation y

$$\hat{x}(k+1) = a\,\hat{x}(k) + \alpha\,[\hat{y}(k+1) - a\,\hat{x}(k)]$$

1. Find the transfer function

$$H(z) = \frac{\hat{X}(z)}{\hat{Y}(z)}$$

2. With the sampling interval T and

$$z = e^{j\omega T} = e^{j2\pi fT} = e^{j2\pi f/f_0}$$

find the bandwidth of the filter, i.e., the frequency f (with $|f| < f_0/2$) at which

$$|H(f)/H(f=0)|^2 = 1/2$$

Evaluate for $a = 1$, $\alpha = 0.5$. You can assume $\cos^{-1} 0.75 \approx 45^o$. Express the resulting f_{BW} in terms of $f_0 = 1/T$.

6-12 Bias and total MS error in the alpha-beta filter. The state x of a target, consisting of range r and range rate $\dot{r}$, is observed with range measurements and is estimated with an α-β filter at intervals T.

The filter has been designed for zero-mean white noises $v(k)$, $w(k)$. However, there is a constant input (bias) target acceleration u. The goal is to find the total MSE in the estimates.

The dynamic equation is

$$x(k) = Fx(k-1) + G[u + v(k-1)]$$

where

$$F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

The measurement is

$$z(k) = [1\ \ 0]x(k) + w(k) = Hx(k) + w(k)$$

The filter gain is

$$W = \begin{bmatrix} \alpha \\ \frac{\beta}{T} \end{bmatrix}$$

1. Show that the mean of the estimation error $\bar{\tilde{x}}$ obeys a dynamic equation

$$\bar{\tilde{x}}(k) = \tilde{F}\bar{\tilde{x}}(k-1) + \tilde{G}u$$

Find $\tilde{F}$ and $\tilde{G}$.

2. Find the expressions of the components of the steady-state mean error in terms of T, α, β and u.

3. Evaluate the above for $T = 0.25\,\mathrm{s}$, $\alpha = 0.75$, $\beta = 0.5$ and $u = 5\,\mathrm{m/s^2}$.

4. Find the total estimation MSE (in s.s.) assuming $\sigma_w = 2\,\mathrm{m}$.

6-13 Position and velocity estimation for known acceleration objects. Consider an object that moves in one dimension with an exactly known acceleration. Show that if a large number m of position measurements are made on this object, with errors that are zero-mean, white and with variance σ^2, at intervals T (covering a total time span $T_T \approx mT$), then the RMS accuracies of its position and velocity estimates at the end of the interval are given by $2\sigma/\sqrt{m}$ and $2\sqrt{3}\sigma/(\sqrt{m}T_T)$, respectively.

Hint: Recast the problem, with appropriate justification, so you can use the results from Subsection 6.4.2.