

# Using the Kalman Filter to Estimate the State of a Maneuvering Aircraft

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**Abstract**—Using sensors that only measure the bearing angle and range of an aircraft, a Kalman filter is implemented to track the range, range rate, bearing, and bearing rate of a maneuvering aircraft with unknown varying accelerations. Simulations will demonstrate the tracking performance of the Kalman filter with single and multiple prediction steps between the measurement step. Then, a Kalman filter will be implemented with assumed correlated measurement and process noise.

**Keywords:** Kalman filter, Correlated noise, Estimation, Tracking

## I. INTRODUCTION

In applications such as target tracking, channel tracking in communications etc. It is required to measure and estimate unknown quantities. Within the significant toolbox of mathematical tools that can be used for stochastic estimation from noisy sensor measurements, one of the most often used tools is the Kalman filter. Typically, physical systems are described by a system model equation and a measurement equation. When the physical system equations are both linear, the Kalman filter can be used to estimate the state vector recursively. When the system equations are nonlinear, the physical system can be approximated by an extended Kalman filter (EKF). This paper presents a solution to a linear system using a Kalman filter. The physical system is an aircraft and the sensor used to measure its state is a radar. The remainder of this paper proceeds as follows: section 2 describes the Kalman filter when the measurement and process noise are correlated. Section 3 implements a Kalman filter to estimate the state of an aircraft when the received sensor measurements are range and bearing. Section 4 presents the simulated data from section 3. The simulated data demonstrates the accuracy of the Kalman filter in tracking performance. To enhance filter performance, extra prediction steps were run between measurement updates. These results given are given in section 4.

## II. KALMAN FILTER AND COVARIANCE MATRICES

The Kalman filter is a set of mathematical equations that provides an efficient computational (recursive) means to estimate the state of a process in a way that minimizes the mean of the squared error[7]. For a derivation of the Kalman filter see [1]. The discrete time Kalman filter equations are as follows,

System dynamic model:

$$\mathbf{x}_k = \Phi_{k-1}\mathbf{x}_{k-1} + \mathbf{w}_{k-1} \quad (1)$$

Measurement model:

$$\mathbf{z}_k = H_k\mathbf{x}_k + \mathbf{v}_k \quad (2)$$

State estimation:

$$\hat{\mathbf{x}}_k(-) = \Phi_{k-1}\hat{\mathbf{x}}_{k-1}(+) \quad (3)$$

Error covariance (a priori):

$$P_k(-) = \Phi_{k-1}P_{k-1}(+)\Phi_{k-1}^T + Q_{k-1} \quad (4)$$

Kalman Gain:

$$\bar{K}_k = P_k(-)H_k^T[H_kP_k(-)H_k^T + R_k]^{-1} \quad (5)$$

Error covariance update (a posteriori):

$$P_k(+) = [I - \bar{K}_kH_k]P_k(-) \quad (6)$$

State estimate update:

$$\hat{\mathbf{x}}_k(+) = \hat{\mathbf{x}}_k(-) + \bar{K}_k[\mathbf{z}_k - H_k\hat{\mathbf{x}}_k(-)] \quad (7)$$

Where  $\mathbf{x}_k$  denotes the state vector at time  $k$ , and  $\mathbf{z}_k$  denotes the corresponding measurement.  $\Phi_k$  represents the state transition matrix and  $H_k$  represents the measurement matrix.  $\mathbf{w}_k$  is the Gaussian zero-mean white process noise and  $\mathbf{v}_k$  is the Gaussian zero-mean white measurement noise. The covariance of  $\mathbf{w}(k)$  and  $\mathbf{v}(k)$  is given by,

$$E[\mathbf{w}(k)\mathbf{w}(j)^T] = Q_k\delta_{kj}$$

$$E[\mathbf{v}(k)\mathbf{v}(j)^T] = R_k\delta_{kj}$$

Where  $\delta$  denotes the Kronecker-delta function. Figure 1 shows each state before (a priori state estimate) and after (a posteriori state estimate) a measurement is received. From the cumulative knowledge of each past a priori and a posteriori state estimates, a Kalman filter can be implemented to estimate the future states.

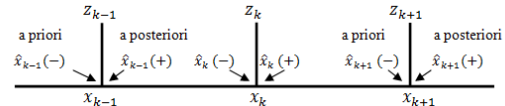


Figure 1. Each state with knowledge of measurement and estimation.

Figure 2 shows a typical time sequence of values assumed by the  $i^{th}$  component of the estimated state vector (plotted with solid circles) and its corresponding variance of estimation uncertainty (plotted with open circles). The arrows show the successive value assumed by the variables, with the annotation

(in parentheses) on the arrow indicating which input variables define the indicated transitions. Note that each variable assumes two distinct values at each discrete time: it's a priori value corresponding to the value before the information in the measurement is used, and the a posteriori value corresponding to the value after information is used.[1]

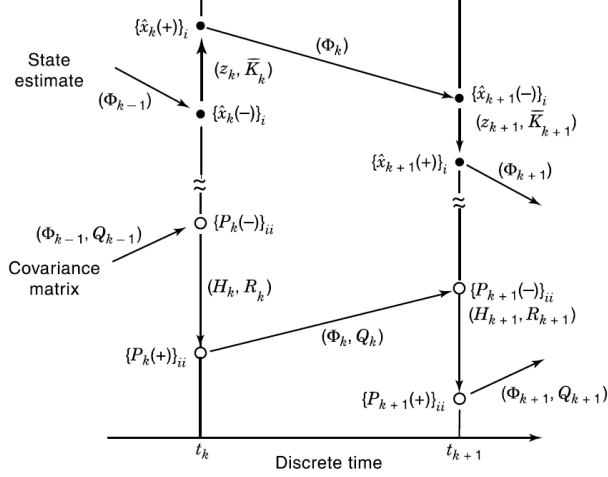


Figure 2. General diagram to show how each step is calculated

Thus in general, a recursive algorithm is formed. If you have  $\hat{x}_k(-)$  you can calculate the a priori error covariance matrix before the sensor update. With the a priori error covariance matrix, the Kalman gain can be computed, from the Kalman gain, the a posteriori error covariance matrix,  $(P_k(+))$ , can be calculated. Once  $P_k(+)$  is calculated it is easy to calculate a posteriori estimate of  $x_k$ , and repeat the process (see figure 3).

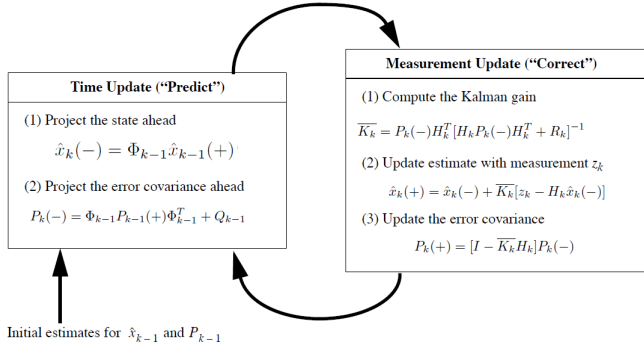


Figure 3. The Kalman filter recursive equation

The previous discussion only involved uncorrelated process and measurement noise. If the process and measurement noise are correlated the Kalman filter equations are modified as follows. Let the correlation between the measurement noise and process noise be given by,

$$E[w(k)v(j)^T] = C_k \delta_{kj} \quad (8)$$

Then the discrete-time estimate equations have the same initial conditions and state estimate extrapolation and error

covariance extrapolation equations. However, the measurement equations are modified as,

$$\bar{K}_k = [P_k(-)H_k^T + C_k][H_k P_k(-)H_k^T + R_k + H_k C_k + C_k^T H^T]^{-1}$$

$$P_k(+) = [P_k(-) - \bar{K}_k [H_k P_k(-) + C_k^T]]$$

$$\hat{x}_k(+) = \hat{x}_k(-) + \bar{K}_k [z_k - H_k \hat{x}_k(-)]$$

### III. BEARING AND RANGE

Radars are used to track and control aircraft. As a radar rotates, it continuously sends out pulses of electromagnetic radiation. Pulses intercepted by objects are reflected back and intercepted by the radar. The time delay from when the pulse is transmitted to when the pulse is received by the radar is used to calculate the range from the radar to the aircraft. To calculate the bearing angle, a straight line is drawn from the aircraft to the radar. The angle between the straight line and true north is the bearing angle. Figure 4 shows the range and bearing angle.

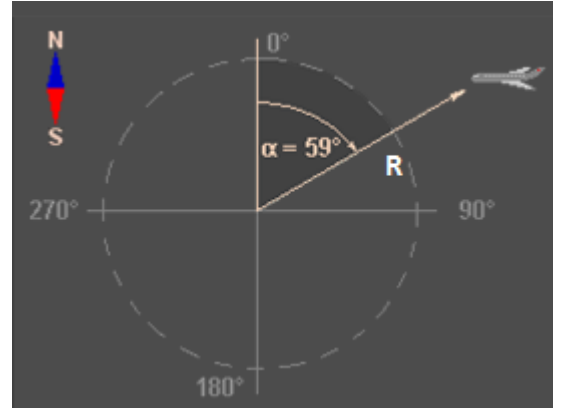


Figure 4. The range and bearing angle of an aircraft

For control purposes, it is desired to track the range and bearing angle of an object as it navigates. In tracking, it is desired to have an improved sensor measurement of the bearing angle and range at regular sample intervals and to have a predicted estimate of the objects navigation route. Furthermore, if it assumed the only measurements received from the radar are range and bearing information, it is desired to accurately reconstruct the range rate and bearing rate even if the aircraft acceleration is unknown. By assuming reasonable flight characteristics, the appropriate Kalman filter can be constructed to handle these conditions. Let  $r(k)$  denote the range,  $\dot{r}(k)$  denote the range rate,  $u_1(k)$  denote the range acceleration,  $\theta(k)$  denote the bearing angle,  $\dot{\theta}(k)$  denote the bearing angle rate,  $u_2(k)$  denote the bearing acceleration, and  $w_1(k)$ , and  $w_2(k)$  denote the zero-mean white process noise with variance  $\sigma_w^2$  at time sample k. Then, using a general model of the singer equations[6], The state equation for time  $k + 1$  can be written as,

$$\mathbf{x}(k+1) = \Phi \mathbf{x}(k) + \mathbf{w}(k)$$

where

$$\mathbf{x}(k) = \begin{bmatrix} r(k) \\ \dot{r}(k) \\ u_1(k) \\ \theta(k) \\ \dot{\theta}(k) \\ u_2(k) \end{bmatrix} \quad \mathbf{w}(k) = \begin{bmatrix} 0 \\ 0 \\ w_1(k) \\ 0 \\ 0 \\ w_2(2) \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 1 & T & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & T & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \rho \end{bmatrix}$$

The variable  $\rho$  is defined in [2]. It is a value to account for the correlated acceleration between measurements and is given by,

$$\rho = \begin{cases} 1 - \lambda T & T \leq 1/\lambda \\ 0 & T \geq 1/\lambda \end{cases}$$

where  $\lambda$  is the inverse of the average maneuver duration and  $T$  is the time between radar measurements.

To calculate the sensor measurements, let  $z_1(k)$  denote the range sensor measurement,  $z_2(k)$  denote the bearing measurement, and  $v_1(k)$  and  $v_2(k)$  denote the zero-mean white sensor noise measurements with variance  $\sigma_r^2$  and  $\sigma_\theta^2$  respectively at time  $k$ , then the sensor measurements are given by,

$$\mathbf{z}(k) = H\mathbf{x}(k) + \mathbf{v}(k)$$

where

$$\mathbf{z}(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{v}(k) = \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix}$$

By assuming the sensor noise is uncorrelated, The noise covariance matrix,  $R_k$ , is given by,

$$R_k = E[\mathbf{v}(k)\mathbf{v}(k)^T] = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix}$$

By assuming the process noise is uncorrelated, The process-noise covariance matrix is given by,

$$U_k = E[\mathbf{w}(k)\mathbf{w}(k)^T] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_2^2 \end{bmatrix}$$

The derivation used to calculate  $\sigma_1^2$  and  $\sigma_2^2$  is given in [2]. Using that model, the acceleration of the aircraft is assumed to be a uniformly distributed Random Variable with maximum acceleration,  $A$ , and variance,  $\sigma_u^2$ , given by,

$$\sigma_u^2 = \frac{A^2}{3}(1 + 4P_1 + P_2)$$

where  $P_1$  is the probability it will accelerate at the maximum acceleration, and  $P_2$  is the probability it will undergo zero acceleration.

From  $\sigma_u^2$ , the values of  $\sigma_1^2$  and  $\sigma_2^2$  are given by,

$$\sigma_1^2 = \sigma_u^2 T^2 = \frac{A^2 T^2}{3}(1 + 4P_1 - P_2)$$

$$\sigma_2^2 = \sigma_u^2 \frac{T^2}{R^2} = \frac{A^2 T^2}{3R^2}(1 + 4P_1 - P_2)$$

#### IV. SIMULATIONS

The figures below demonstrate the tracking performance of the Kalman filter. To produce these results, the following values were assumed:  $A = 2.1$ ,  $P_1 = 0$ ,  $P_2 = 0$ ,  $T = 3$ ,  $R = 1.6e5$ ,  $\sigma_r = 1000$ ,  $\sigma_\theta = 1$ , and  $\frac{1}{\lambda} = 30$ . From these values,  $\sigma_1^2 = 13.23$ ,  $\sigma_2^2 = 5.17e - 10$ , and  $\rho = .9$ . The left side of figure 4 and the left side of figure 5 show the error covariance of the range and bearing before and after a measurement is received. The right half of figure 4 shows the Kalman gain for the range and bearing. The remaining figures demonstrate the ability of the Kalman filter to track a maneuvering aircraft with unknown varying accelerations. The figures display the true, estimated, and measured range and bearing of the aircraft; the range and bearing error of the aircraft; and, the true and estimated bearing and range rate.

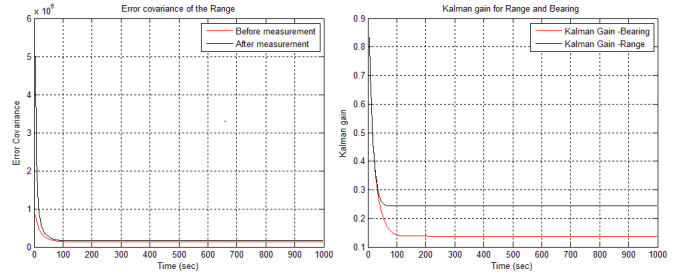


Figure 5. The error covariance of the range and the Kalman gain for the Bearing and Range as a function of time

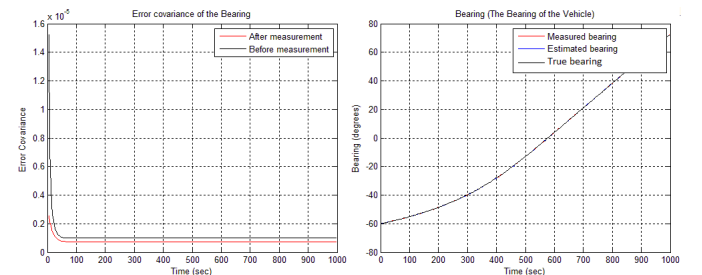


Figure 6. The error covariance of the bearing and the true, measured, and estimated bearing

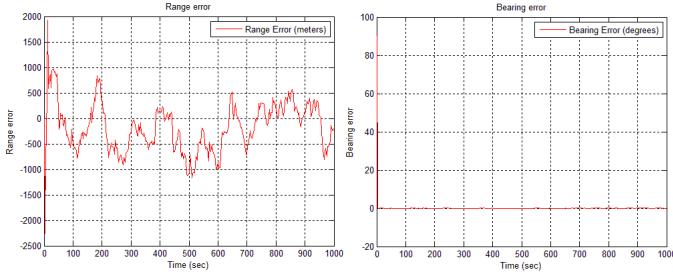


Figure 7. The range error and bearing error of the Kalman filter estimates

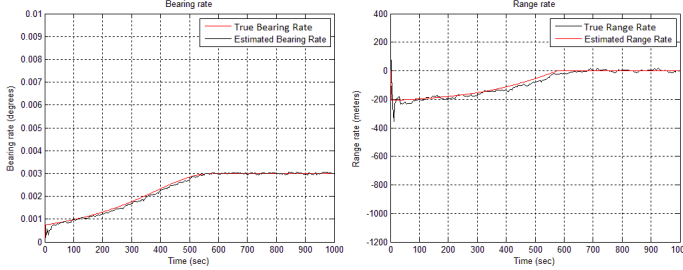


Figure 8. The true bearing rate and estimated bearing rate and the true range rate and estimated range rate of the Kalman filter

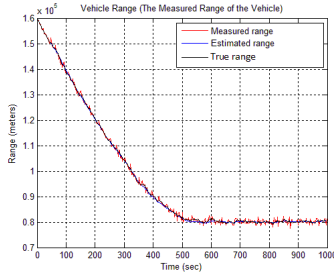


Figure 9. The measured, estimated, and true range of the aircraft

In performing these simulations, it was assumed that one prediction of the aircraft state was given for each received measurement. To improve the filter performance, a simulation was run that predicted the state of the aircraft multiple times between sensor measurements. The only noticeable improvement from this change was in the range rate and the range as seen in figure 10 and 11.

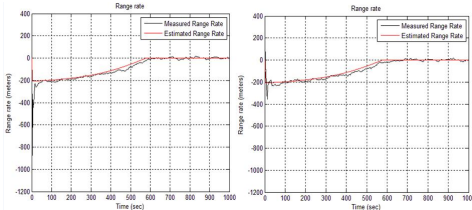


Figure 10. The range rate of the aircraft using single and multiple prediction estimates between sensor measurements. The figure on the left is the single prediction estimate.

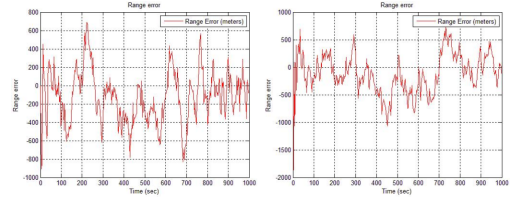


Figure 11. The Range Error of the aircraft using single and multiple prediction estimates between sensor measurements. The figure on the left is the single prediction estimate

The final simulation demonstrates when the process noise and measurement noise are Gaussian-correlated. The results are given in figure 12.

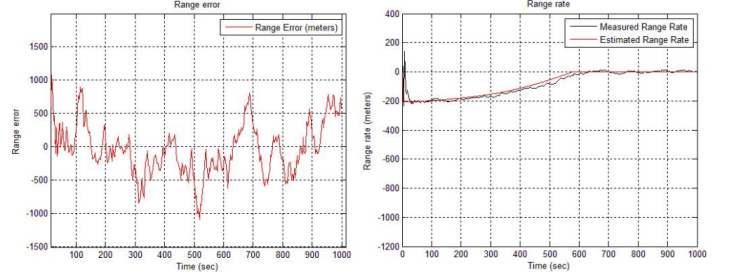


Figure 12. The range error and range rate using a Kalman filter with correlated process and measurement noise

## V. CONCLUSION

For a linear system, the Kalman filter is the best estimator of a state in the presence of zero-mean white Gaussian process and measurement noise. From the data, it is seen that the Kalman filter closely tracks the true state of the aircraft in the presence of unknown accelerations. If the filter is modified so that multiple prediction steps between measurements are made, then the Kalman filter performance is improved. Finally, if there are correlations between process noise and measurement noise, the Kalman filter becomes suboptimal, but it still tracks the aircraft with excellent accuracy.

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