Integration in Multiple Dimensions: Answer Key

Quantitative Engineering Analysis

January 28, 2016

1 Review of Single-Variable Calculus

1.Create a table of five fundamental functions x^n, sin(x), cos(x), exp(x), and ln(x). List both their derivatives and their anti-derivatives. Include in your table at least one other example.

nx^{n-1}	cos(x)	-sin(x)	e^x	$\frac{1}{x}$	???
x^n	sin(x)	cos(x)	e^x	ln(x)	????
$\frac{x^{n+1}}{n+1}+c$	-cos(x)+c	sin(x)+c	e^x+c	$xlnx-x+c$	?????

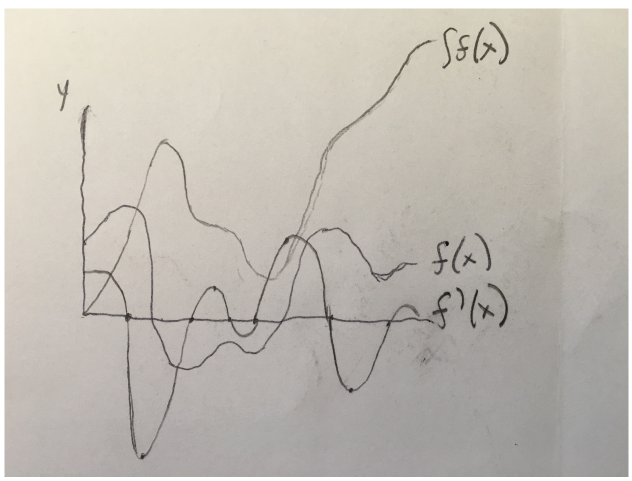

Figure 1: The derivative and anti-derivative.

2.Consider the sketch of the function below. Now try to sketch the derivative and an anti-derivative.

3. Make a visual argument about why these properties are true. Below is a visual argument for $(f'+g')=f'+g'$. Students should have some visual representation for each of the four properties.

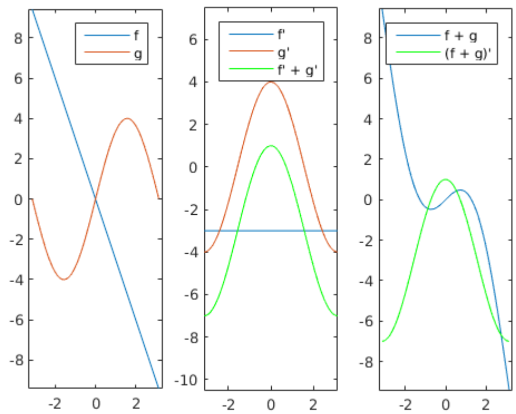

Figure 2: How cool!

4. Use your table of fundamental functions and the chain rule to determine the derivative of $(x^3-1)^100$.

$$\begin{aligned} u &= x^3 - 1 \\ \frac{d}{dx}u^{100} &= \frac{d}{du}u^{100}\frac{du}{dx} \\ \frac{du}{dx} &= 3x^2 \\ \frac{d}{du}u^{100} &= 100u^{99} \\ \frac{d}{dx}(x^3-1)^{100} &= 100(x^3-1)^{99}(3x^2) \\ &= 300x^2(x^3-1)^{99} \end{aligned}$$

5. Use your table of fundamental functions and the substitution rule to evaluate $\int_0^4 \sqrt{2x+1}\,dx$

$$\begin{aligned} u &= 2x + 1 \\ \int u^{\frac{1}{2}}\,dx &= \int \frac{u^{\frac{1}{2}}\,du}{\frac{du}{dx}} \\ &= \frac{1}{2}\int u^{\frac{1}{2}}\,du \\ &= \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right) \\ &= \frac{1}{3}(2x+1)^{\frac{3}{2}} \\ &= \frac{26}{3} \end{aligned}$$

6. Use your table of fundamental functions and the product rule to determine the derivative of $\sqrt{x}(1-x)$.

$$\begin{aligned} &= \sqrt{x} * (-1) + (1-x) * \left(\frac{1}{2}x^{\frac{-1}{2}}\right) \\ &= \frac{-3\sqrt{x}}{2} + \frac{1}{2\sqrt{x}} \end{aligned}$$

7. Use your table of fundamental functions and integration by parts to determine $\int_1^2 x \exp(-x)\, dx$.

$$
\begin{aligned}
\int f\, dg &= fg - \int g\, df \\
f = x,&\ dg = \mathrm{e}^{-x} dx \\
df = dx,&\ g = -\mathrm{e}^{-x} \\
fg - \int g\, df &= -x\mathrm{e}^{-x} - \int -\mathrm{e}^{-x}\, dx \\
&= -\mathrm{e}^{-x}(x+1) \\
&= \frac{2\mathrm{e} - 3}{\mathrm{e}^2}
\end{aligned}
$$

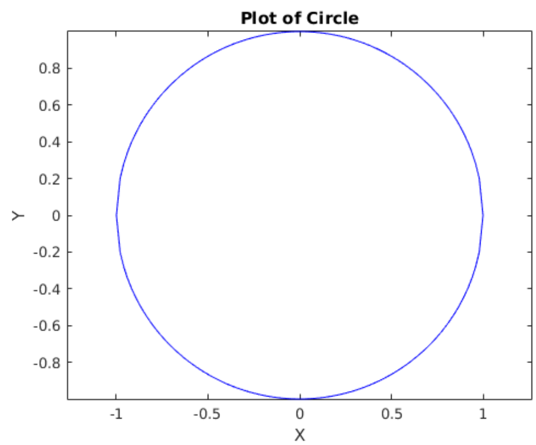

Figure 3: remember this?

8. Recall some of the explicit functions you used to describe curves generated by taking sections through fruit, vegetables, or manufactured objects. Propose an integral that would determine the cross-sectional area of the section, and evaluate it.

$$
\begin{aligned}
\text{Area} &= \int_{-1}^{1} \left(\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right)\, dx \\
&= x\sqrt{1-x^2} + \sin^{-1} x \Big|_{-1}^{1} \\
&= \pi
\end{aligned}
$$

9. Consider the family of functions defined by the power law, $y = x^n, n = 1, 2, 3, \ldots$. Propose an integral that would determine the area enclosed on the top by $y = d$, on the bottom by $y = x^n$, on the left by $x = 0$, and on the right by the intersection of the top and bottom functions. Evaluate it and graph the enclosed area as a function of the parameter d. Use a log-log plot and interpret the result.

$$
\begin{aligned}
\int_0^{\sqrt[n]{d}} (d - x^n)\, dx &= x\left(d - \frac{x^n}{n+1}\right)\Big|_0^{\sqrt[n]{d}} \\
&= \sqrt[n]{d}\left(d - \frac{d}{n+1}\right) \\
&= \sqrt[n]{d}\left(\frac{dn}{n+1}\right)
\end{aligned}
$$

```
function power_law()

    d = logspace(-2, 4);
    figure;

    for n=[1, 10, 100]
        area = nthroot(d, n) .* (d - d / (n + 1));
        loglog(d, area);
        hold on;
    end

    legend('n=1', 'n=10', 'n=100')
    xlabel('d');
    ylabel('area');

end
```

Listing 1: code to graph relationship between d n and area.

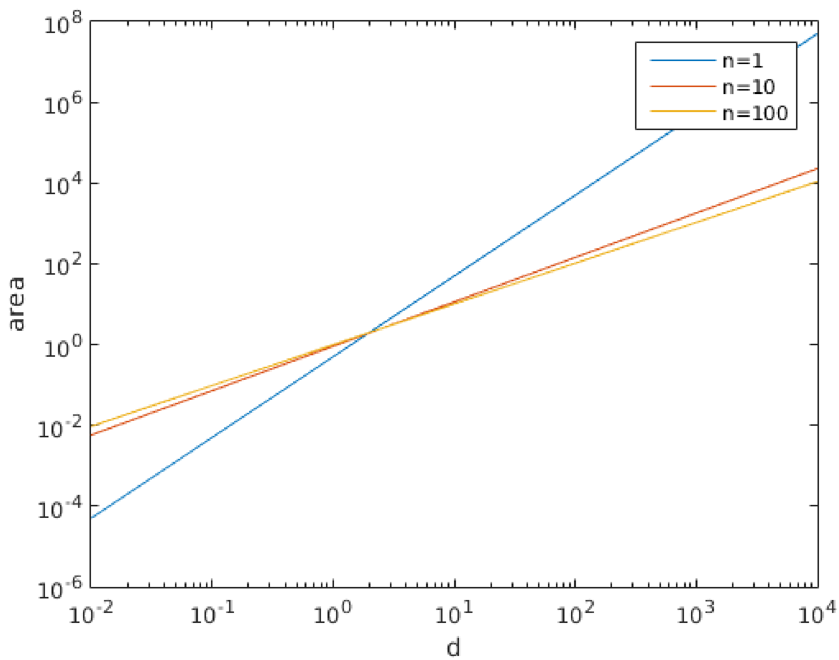

Figure 4: As d increases, the area accumulated increases exponentially. Also, the larger n, the smaller the area accumulated. Try to understand why!

10. Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = x^2 \sin(xy^2)$.

$$\frac{\partial f}{\partial x} = x^2y^2\cos(xy^2) + 2x\sin(xy^2)$$
$$\frac{\partial f}{\partial y} = 2x^3y\cos(xy^2)$$

11. Evaluate all four second-order derivatives of $f(x, y) = x^2 \sin(xy^2)$.

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} &= -2x^2 y \left(xy^2 \sin\left(xy^2\right) - 3\cos\left(xy^2\right)\right) \\
\frac{\partial^2 f}{\partial x^2} &= \left(2 - x^2 y^4\right)\sin(xy^2) + 4xy^2\cos(xy^2) \\
\frac{\partial^2 f}{\partial y^2} &= 2x^3\left(\cos(xy^2) - 2xy^2 sin(xy^2)\right)
\end{aligned}$$

12. Review the article on partial derivative at *http://mathworld.wolfram.com/PartialDerivative.html* **Under what conditions on the function f are the mixed partial derivatives equal?** The function's first derivatives must be differentiable. This can fail if the second partials are not continuous.

13. How many second-order derivatives are there of $f(x, y, z)$? 12.

14. Sketch the regions of integration and compute the area of the enclosed region by evaluating the double integral.

1. $\int_0^1 \int_0^x dydx$

$$\begin{aligned}
&= \int_0^1 xdx \\
&= \left.\frac{x^2}{2}\right|_0^1 \\
&= 1/2
\end{aligned}$$

2. $\int_0^{\pi/2} \int_0^{\sin x} dydx$

$$\begin{aligned}
&= \int_0^{\pi/2} \sin xdx \\
&= -\cos x\Big|_0^{\pi/2} \\
&= 1
\end{aligned}$$

3. $\int_1^2 \int_0^{\ln x} dydx$

$$\begin{aligned}
&= \int_1^2 \ln xdx \\
&= (x\ln x - x)\Big|_1^2 \\
&= 2\ln 2 - \ln 1 - 1 = 0.386
\end{aligned}$$

15. Sketch the regions of integration and compute the area of the enclosed region by evaluating the double integral.

1. $\int_0^1 \int_y^{2-y} dxdy$

$$= \int_1^2 2 - 2y dy$$
$$= 2y - y^2 \Big|_0^1$$
$$= 2 - 1 = 1$$

2. $\int_0^4 \int_{y/2}^2 dxdy$

$$= \int_0^4 2 - \frac{y}{2} dy$$
$$= 2y - \frac{y^2}{4} \Big|_0^4$$
$$= 8 - 4 = 4$$

3. $\int_0^1 \int_y^{\exp y} dxdy$

$$= \int_0^1 e^y - y dy$$
$$= e^y - \frac{y^2}{2} \Big|_0^1$$
$$= e - \frac{1}{2} - 1 = 1.22$$

16. Sketch the regions of integration and compute the area of the enclosed region by evaluating the double integral.

$$= \int_{-1}^1 \int_{-1}^{y^2} dxdy + \int_{-1}^1 \int_1^{1+x^2} dydx$$
$$= \int_{-1}^1 (y^2 + 1) dy + \int_{-1}^1 x^2 dx \qquad = \frac{10}{3}$$

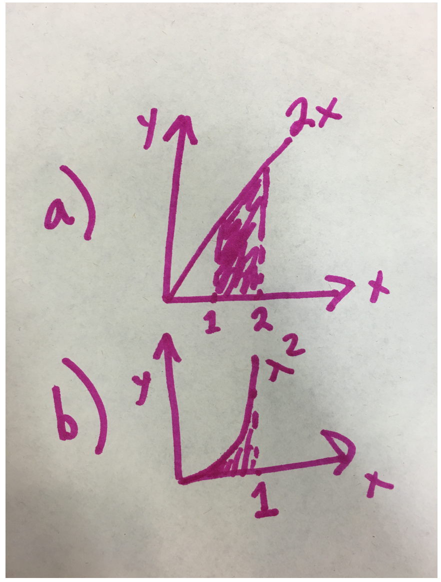

Figure 5: The regions of integration.

17. Sketch the following regions of integration in the plane, and evaluate the double integral.

1. $\iint\limits_D \frac{4y}{x^3+2}\,dA,\ D = \{(x, y)|1 \leq x \leq 2, 0 \leq y \leq 2x\}$

$$\int_1^2 \int_0^{2x} \frac{4y}{x^3+2} dydx = \int_1^2 \frac{2y^2}{x^3+2}\bigg|_0^{2x} dydx$$
$$= \int_1^2 \frac{8x^2}{x^3+2} dx$$
$$u = x^3 + 2, dx = \frac{du}{3x^2}$$
$$\int_1^2 \frac{8x^2}{x^3+2} dx = \int_3^{10} \frac{8}{3u} du$$
$$= \frac{8}{3} \ln \frac{10}{3}$$

2. $\iint\limits_D x \cos y\,dA,\quad$ D is bounded by $y = 0, y = x^2, x = 1$

$$\int_0^1 \int_0^{x^2} x \cos y\,dydx = \int_0^1 xsiny\bigg|_0^{x^2} dx$$
$$= \int_0^1 x \sin x^2\,dx$$
$$u = x^2, dx = \frac{du}{2x}$$
$$\int_0^1 x \sin x^2\,dx = \int_0^1 \frac{1}{2} \sin u\,du$$
$$= -\frac{1}{2} \cos u\bigg|_0^1$$
$$= \frac{1 - \cos(1)}{2}$$

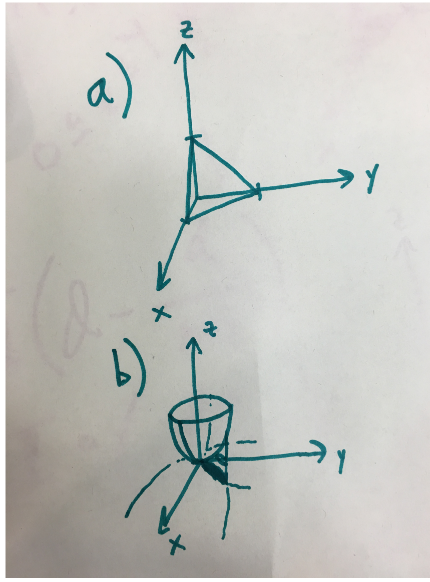

Figure 6: The solids.

18. Visualize the following solids and compute their volume using a double integral.

1. Bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

$$\begin{aligned}
\int_0^1 \int_0^{1-x} (1 - x - y)dydx &= \int_0^1 y(1-x) - \frac{1}{2}y^2 \Big|_0^{1-x} dx \\
&= \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2\right) dx \\
&= \frac{1}{2}\int_0^1 (x^2 - 2x + 1)dx \\
&= \frac{1}{2}\left(\frac{1}{3}x^3 - x^2 + x\right)\Big|_0^1 \\
&= \frac{1}{6}
\end{aligned}$$

2. Under the paraboloid $z = x^2 + y^2$ and above the region bounded by $y = x^2$ and $x = y^2$.

$$\begin{aligned}
\int_0^1 \int_{x^2}^{\sqrt{x}} \left(x^2 + y^2\right) dydx &= \int_0^1 yx^2 + \frac{1}{3}y^3 \Big|_{x^2}^{\sqrt{x}} dx \\
&= \int_0^1 \left(x^{\frac{5}{2}} + \frac{1}{3}x^{\frac{3}{2}} - x^4 - \frac{1}{3}x^6\right) dx \\
&= \left(\frac{2}{7}x^{\frac{7}{2}} + \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{5}x^5 - \frac{1}{21}x^7\right)\Big|_0^1 \\
&= \frac{6}{35}
\end{aligned}$$

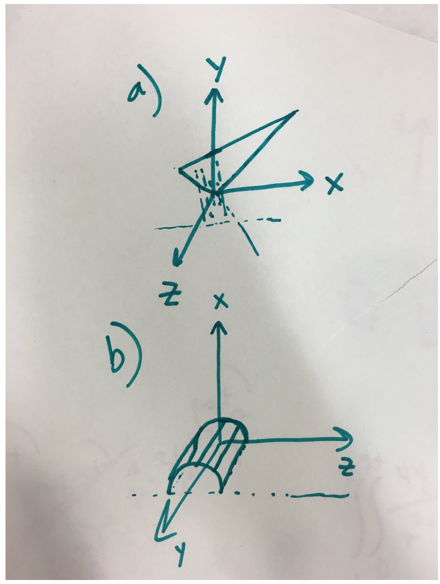

Figure 7: The solids.

19. Visualize the solids defined by the limits of integration, and evaluate their volume.

1. $\int_0^1 \int_0^z \int_0^{x+z} \, dydxdz$

$$\begin{aligned}
&= \int_0^1 \int_0^z (x+z)\, dxdz \\
&= \int_0^1 \frac{1}{2}x^2 + zx\Big|_0^z dz \\
&= \int_0^1 \left(\frac{1}{2}z^2 + z^2\right)\, dz \\
&= \frac{1}{2}z^3\Big|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

2. $\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} \, dxdzdy$

$$\begin{aligned}
\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} \, dxdzdy &= \int_0^3 \int_0^1 \sqrt{1-z^2}\, dzdy \\
&= \int_0^3 \frac{1}{2}\left(z\sqrt{1-z^2} + \sin^{-1} z\right)\Big|_0^1 dy \\
&= \frac{1}{2}\int_0^3 \frac{\pi}{2} dy \\
&= \frac{3\pi}{4}
\end{aligned}$$

20. Visualize the solid defined by the limits of integration, and figure out the five other triple integrals that are the same.

1. $\int_0^1 \int_y^1 \int_0^y \, dzdxdy$
   Below is the visualization of the volume bounded by the integral

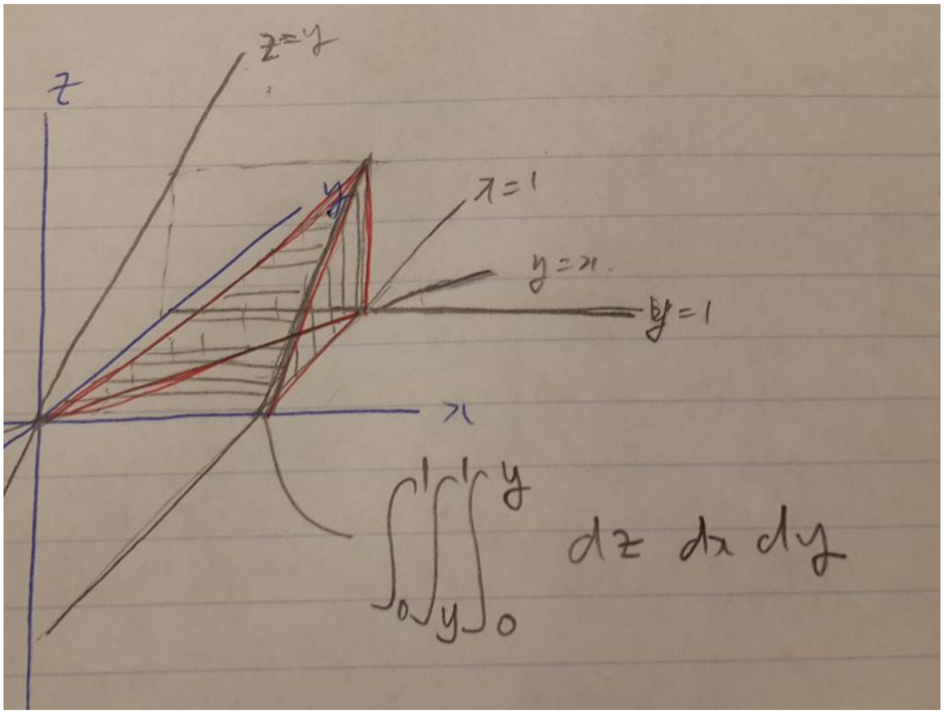

Figure 8: Volume bounded by the integral

Here are the five equivalent integrals:

(a) $\int_0^1 \int_0^x \int_0^y \ dzdydx$

(b) $\int_0^1 \int_z^1 \int_y^1 \ dxdydz$

(c) $\int_0^1 \int_0^z \int_0^x \ dydxdz$

(d) $\int_0^1 \int_0^y \int_y^1 \ dxdzdy$

(e) $\int_0^1 \int_0^x \int_0^x \ dydzdx$

2. $\int_0^1 \int_0^{1-x^2} \int_0^{1-x} \ dydzdx$
Below is the visualization of the volume bounded by the integral

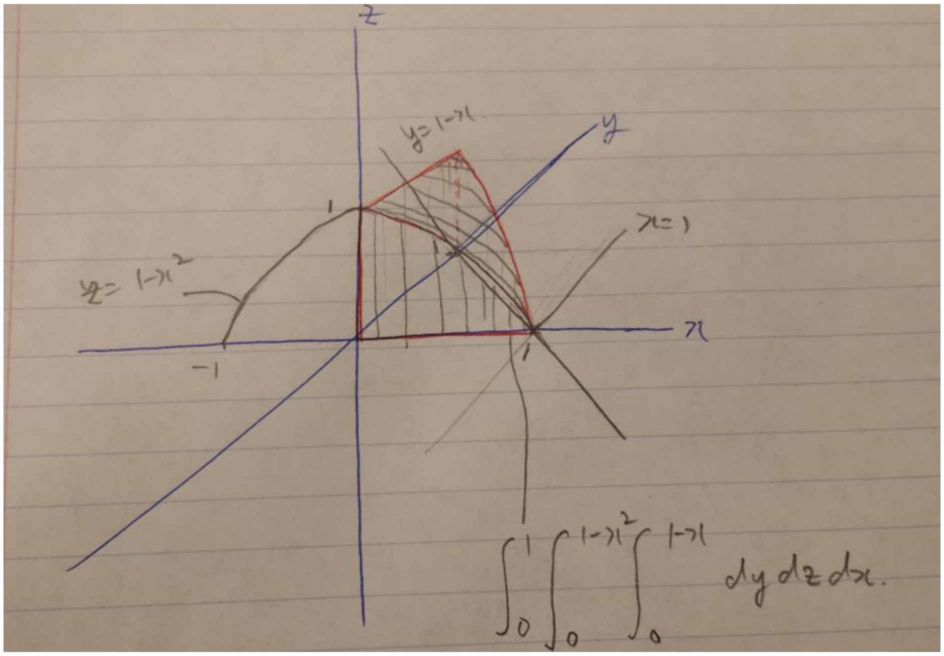

Figure 9: Volume bounded by the integral

Here are the five equivalent integrals:

(a) $\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} \; dydxdz$

(b) $\int_0^1 \int_0^{1-x} \int_0^{1-x^2} \; dzdydx$

(c) $\int_0^1 \int_0^{1-y} \int_0^{1-x2} \; dzdxdy$

(d) $\int_0^1 \int_0^{1-x} \int_0^{\sqrt{1-z}} \; dxdydz$

(e) $\int_0^1 \int_0^{1-x^2} \int_0^{1-y} \; dxdzdy$

However, (d) and (e) are not allowed since their limits of integration include a variable that is integrated beforehands.

You can think of them as a piece wise function where from 0 to y in z axis, the surface function is 1-x, and from y to 1 in z axis, the surface function is 1-x^2.

So (d) and (e) becomes:

(a) $\int_0^1 \int_z^1 \int_0^{1-y} \; dxdydz + \int_0^1 \int_0^z \int_0^{\sqrt{1-z}} \; dxdydz$

(b) $\int_0^1 \int_y^1 \int_0^{\sqrt{1-z}} \; dxdzdy + \int_0^1 \int_0^y \int_0^{1-y} \; dxdzdy$

21. Visualize the solids defined by the limits of integration, and evaluate the triple integrals.

1. $\int_0^1 \int_x^{2x} \int_0^y 2xyz \; dzdydx$
   Here is the region bounded by the integration ranges.

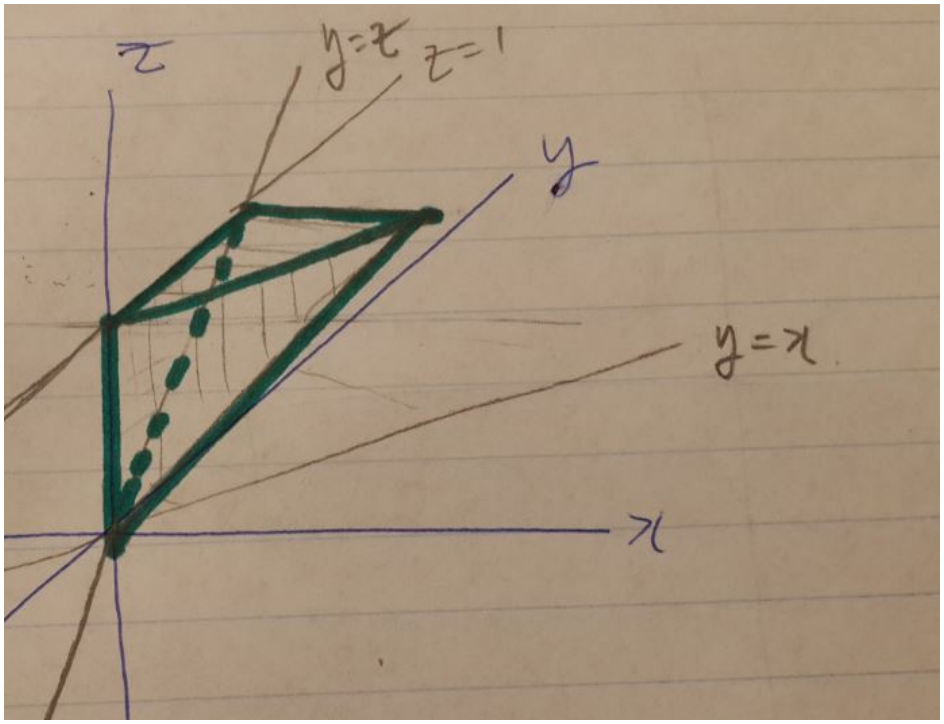

Figure 10: Region bounded by the integration ranges

$$\begin{aligned}
\int_0^1 \int_x^{2x} \int_0^y 2xyz \; dzdydx &= \int_0^1 \int_x^{2x} xyz^2 \Big|_0^y \; dydx \\
&= \int_0^1 \int_x^{2x} xy^3 \; dydx \\
&= \int_0^1 x\frac{1}{4}y^4 \Big|_x^{2x} \; dx \\
&= \int_0^1 \frac{1}{4}x(16x^4 - x^4) \; dx \\
&= \int_0^1 \frac{15}{4}x^5 \; dx \\
&= \frac{15}{24}x^6 \Big|_0^1 \\
&= \frac{5}{8}
\end{aligned}$$

2. $\int_0^1 \int_0^z \int_0^y ze^{-y^2} \; dxdydz$
   Here is the region bounded by the integration ranges.

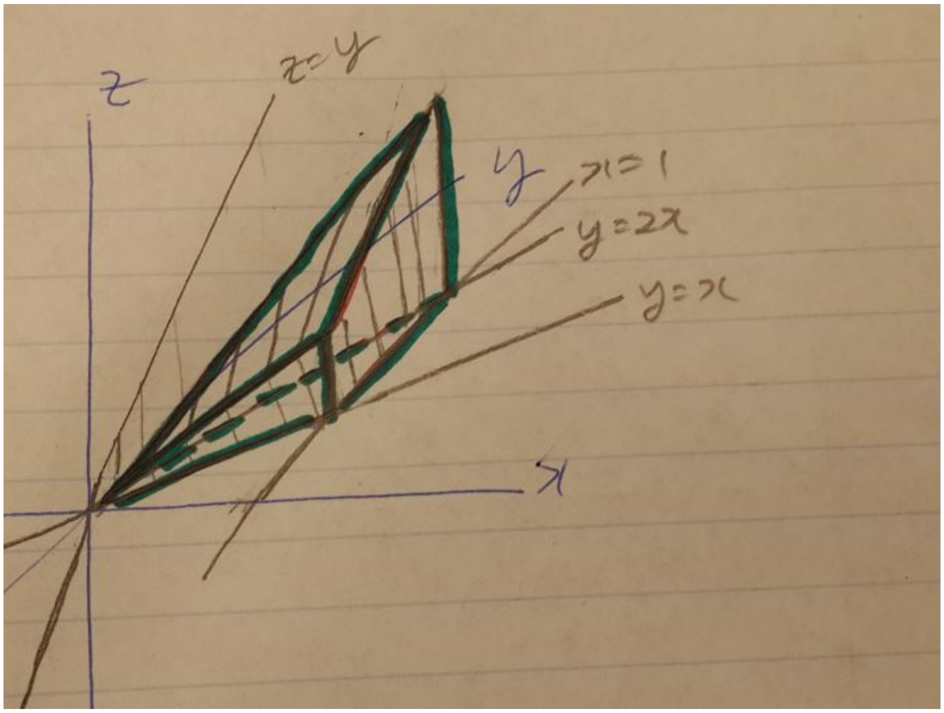

Figure 11: Region bounded by the integration ranges

$$
\begin{aligned}
\int_0^1 \int_0^z \int_0^y ze^{-y^2}\ dxdydz &= \int_0^1 \int_0^z ze^{-y^2}x\Big|_0^y\ dydz \\
&= \int_0^1 \int_0^z zye^{-y^2}\ dydz \\
&= \int_0^1 z\frac{-1}{2}e^{-y^2}\Big|_0^z\ dz \\
&= \int_0^1 \frac{-1}{2}ze^{-z^2}\ dz \\
&= \frac{1}{4}e^{-z^2}\Big|_0^1 \\
&= \frac{1}{4e} - \frac{1}{4}
\end{aligned}
$$

22. Find the total mass of the following plates and solids.

1. The plate bounded by the parabola $y = 9 - x^2$ and the x-axis; mass (or charge) density $\rho(x, y) = y$. Here is the region bounded by the integration ranges.

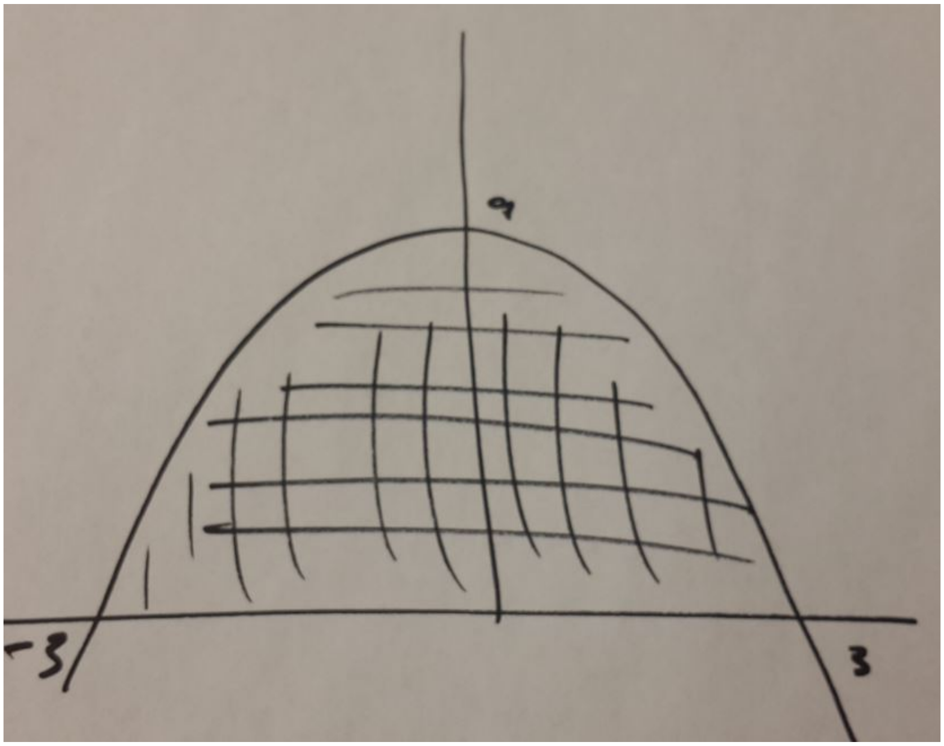

Figure 12: Region bounded by the integration ranges

$$\begin{aligned}
\int_{-3}^{3}\int_{0}^{9-x^2} y\, dydx &= \int_{-3}^{3} 0.5y^2\Big|_0^{9-x^2}\, dx \\
&= \int_{-3}^{3} 81 - 18x^2 + x^4\, dx \\
&= 81x - 6x^3 + \frac{1}{5}x^5\Big|_{-3}^{3} \\
&= 129.6
\end{aligned}$$

2. The tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$; mass density $\rho(x, y, z) = y$.
Here is the region bounded by the integration ranges.

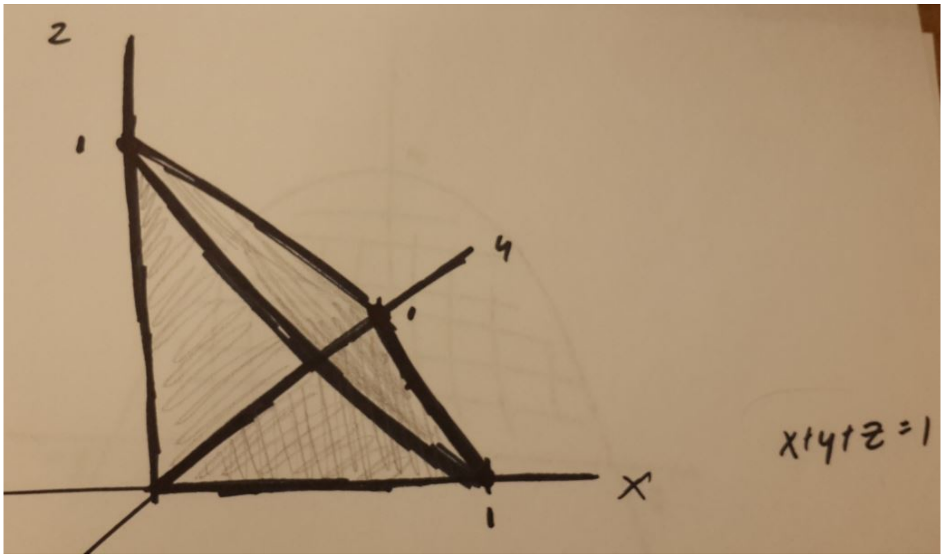

Figure 13: Region bounded by the integration ranges

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} y \; dzdydx$$

$$= \frac{1}{24}$$