Technical Report

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1 Proof

1.1 Proof of Theorem 1

Proof. Let $\Delta\mathbf{x} = \tilde{\mathbf{x}}_j^i - \mathbf{x}_j^i = \text{clamp}(\tilde{\mathbf{y}}_j^i, 0, t) - \text{ReLU}(\mathbf{y}_j^i)$ be the difference of two neurons in QNN and DNN after applying the activation function. Note that, here we use $\tilde{\mathbf{y}}_j^i$ (resp. $\mathbf{y}_j^i$) to denote the value of neuron before an activation function in QNN (resp. DNN). Let $\Delta = \tilde{\mathbf{y}}_j^i - \mathbf{y}_j^i$.

First, considering **Case 1** ($\text{UB}(S^{in}(\mathbf{x}_j^i)) \leq 0$), the neuron of DNN is always deactivated as 0. Hence, the output difference $\delta_{i,j} = S(\tilde{\mathbf{x}}_j^i)$.

Next, we consider following cases when the neuron in DNN is always activated, i.e., $\text{LB}(S^{in}(\mathbf{x}_j^i)) > 0$:

- **Case 2-1&2-2** ($\text{UB}(S^{in}(\tilde{\mathbf{x}}_j^i)) \leq 0$ or $\text{LB}(S^{in}(\tilde{\mathbf{x}}_j^i)) \geq t$): Since the neuron in QNN is always deactivated or clampped to t, we can get $\delta_{i,j} = -S^{in}(\mathbf{x}_j^i)$ or $\delta_{i,j} = t - S^{in}(\mathbf{x}_j^i)$.
- **Case 2-3** ($\text{LB}(S^{in}(\tilde{\mathbf{x}}_j^i)) \geq 0$ and $\text{UB}(S^{in}(\tilde{\mathbf{x}}_j^i)) \leq t$): Both of two neurons in QNN and DNN are activated and without clampped. Therefore, we have $\delta_{i,j} = \delta_{i,j}^{in}$.
- **Case 2-4** ($\text{LB}(S^{in}(\tilde{\mathbf{x}}_j^i)) < 0$ and $0 \leq \text{UB}(S^{in}(\tilde{\mathbf{x}}_j^i)) \leq t$): Since $\tilde{\mathbf{y}}_j^i$ is always smaller than t, we have $\Delta\mathbf{x} = \max(\tilde{\mathbf{y}}_j^i, 0) - \mathbf{y}_j^i = \max(\tilde{\mathbf{y}}_j^i - \mathbf{y}_j^i, -\mathbf{y}_j^i)$. Then, $\delta = \max(\delta_{i,j}^{in}, -S^{in}(\mathbf{x}_j^i))$.
- **Case 2-5** ($0 \leq \text{LB}(S^{in}(\tilde{\mathbf{x}}_j^i)) < t$ and $\text{UB}(S^{in}(\tilde{\mathbf{x}}_j^i)) > t$): Since $\tilde{\mathbf{y}}_j^i$ is always larger than 0, we have $\Delta\mathbf{x} = \min(\tilde{\mathbf{y}}_j^i, t) - \mathbf{y}_j^i = \min(\tilde{\mathbf{y}}_j^i - \mathbf{y}_j^i, t - \mathbf{y}_j^i)$. Hence, we have $\delta_{i,j} = \min(\delta_{i,j}^{in}, t - S^{in}(\mathbf{x}_j^i))$.
- **Case 2-6** (Otherwise): By case 2-4 & 2-5, we have $\Delta\mathbf{x} = \max(\min(\tilde{\mathbf{y}}_j^i, t), 0) - \mathbf{y}_j^i = \max(\min(\Delta, t - \mathbf{y}_j^i), -\mathbf{y}_j^i)$. Hence, we have $\delta_{i,j} = \max(\min(\delta_{i,j}^{in}, t - S^{in}(\mathbf{x}_j^i)), -S^{in}(\mathbf{x}_j^i))$.

Finally, we consider following cases when the neuron in DNN can be either activated or deactivated:

- **Case 3-1&3-2** ($\text{UB}(S^{in}(\tilde{\mathbf{x}}_j^i)) \leq 0$ or $\text{LB}(S^{in}(\tilde{\mathbf{x}}_j^i)) \geq t$): Similar to above, we have $\delta_{i,j} = -S(\mathbf{x}_j^i)$ or $t - S(\mathbf{x}_j^i)$ directly.
- **Case 3-3** ($\text{LB}(S^{in}(\tilde{\mathbf{x}}_j^i)) \geq 0$ and $\text{UB}(S^{in}(\tilde{\mathbf{x}}_j^i)) \leq t$): $\Delta\mathbf{x}_j^i = \tilde{\mathbf{y}}_j^i - \max(\mathbf{y}_j^i, 0) = \tilde{\mathbf{y}}_j^i + \min(-\mathbf{y}_j^i, 0) = \min(\tilde{\mathbf{y}}_j^i, \Delta)$. Then, we have $\delta_{i,j} = \min(S^{in}(\tilde{\mathbf{x}}_j^i), \delta_{i,j}^{in})$.
- **Case 3-4** ($\text{LB}(S^{in}(\tilde{\mathbf{x}}_j^i)) < 0$ and $0 \leq \text{UB}(S^{in}(\tilde{\mathbf{x}}_j^i)) \leq t$): We rewrite $\Delta\mathbf{x}$ as $\max(\tilde{\mathbf{y}}_j^i) - \max(\mathbf{y}_j^i)$:

- If $\Delta \leq 0$, $\Delta\mathbf{x} = \max(\mathbf{y}_j^i + \Delta) - \max(\mathbf{y}_j^i) \leq 0$. Then, we have $\Delta\mathbf{x} = 0$ when $\mathbf{y}_j^i \leq 0$, and $\Delta\mathbf{x} = \max(\mathbf{y}_j^i + \Delta, 0) - \mathbf{y}_j^i = \max(\Delta, -\mathbf{y}_j^i) \leq 0$ when $\mathbf{y}_j^i \geq 0$. Therefore, we have $\max(\mathrm{LB}(\delta_{i,j}^{in}), -\mathrm{UB}(S^{in}(\mathbf{x}_j^i))) \leq \Delta\mathbf{x} \leq 0$
- If $\Delta \geq 0$, $\Delta\mathbf{x} = \max(\tilde{\mathbf{y}}_j^i) - \max(\tilde{\mathbf{y}}_j^i - \Delta) \geq 0$. Then, we have $\Delta\mathbf{x} = 0$ when $\tilde{\mathbf{y}}_j^i \leq 0$, and $\Delta\mathbf{x} = \tilde{\mathbf{y}}_j^i - \max(\tilde{\mathbf{y}}_j^i - \Delta, 0) = \tilde{\mathbf{y}}_j^i + \min(\Delta - \tilde{\mathbf{y}}_j^i, 0) = \min(\Delta, \tilde{\mathbf{y}}_j^i) \geq 0$ when $\mathbf{y}_j^i \geq 0$. Therefore, we have $0 \leq \Delta\mathbf{x} \leq \min(\mathrm{UB}(\delta_{i,j}^{in}), \mathrm{UB}(S^{in}(\tilde{\mathbf{x}}_j^i)))$

Therefore, we have $\mathrm{LB}(\delta_{i,j}) = 0$ if $\mathrm{LB}(\delta_{i,j}^{in}) \geq 0$, and $\max(\mathrm{LB}(\delta_{i,j}^{in}), -\mathrm{UB}(S^{in}(\mathbf{x}_j^i)))$ otherwise. $\mathrm{UB}(\delta_{i,j}) = 0$ if $\mathrm{UB}(\delta_{i,j}^{in}) \leq 0$, and $\min(\mathrm{UB}(\delta_{i,j}^{in}), \mathrm{UB}(S^{in}(\tilde{\mathbf{x}}_j^i)))$ otherwise.

– **Case 3-5** ($0 \leq \mathrm{LB}(S^{in}(\tilde{\mathbf{x}}_j^i)) \leq t$ and $\mathrm{UB}(S^{in}(\tilde{\mathbf{x}}_j^i)) > t$): We rewrite $\Delta\mathbf{x}$ as $\min(\tilde{\mathbf{y}}_j^i, t) - \max(\mathbf{y}_j^i, 0)$. Then, $\Delta\mathbf{x} = t - \max(\mathbf{y}_j^i, 0) = \min(t - \mathbf{y}_j^i, t)$ when $\tilde{\mathbf{y}}_j^i \geq t$, and $\Delta\mathbf{x} = \tilde{\mathbf{y}}_j^i - \max(\tilde{\mathbf{y}}_j^i - \Delta, 0) = \tilde{\mathbf{y}}_j^i + \min(\Delta - \tilde{\mathbf{y}}_j^i, 0) = \min(\Delta, \tilde{\mathbf{y}}_j^i)$ when $\tilde{\mathbf{y}}_j^i \leq t$. Specifically, when $\tilde{\mathbf{y}}_j^i \geq t$:
  - If $\Delta \leq t$, then $\mathbf{y}_j^i = \tilde{\mathbf{y}}_j^i - \Delta \geq 0$, and we will have $\Delta\mathbf{x} = t - \mathbf{y}_j^i = t - \tilde{\mathbf{y}}_j^i + \Delta \leq \Delta \leq t$;
  - If $\Delta \geq t$, then $\Delta\mathbf{x} = t - \max(\mathbf{y}_j^i, 0) \leq t \leq \Delta$.

  Then, $t - \mathrm{UB}(S^{in}(\mathbf{x}_j^i)) \leq \Delta\mathbf{x} \leq \{\Delta, t\}$ for $\tilde{\mathbf{y}}_j^i \leq t$, and $\min(\mathrm{LB}(\delta_{i,j}^{in}), \mathrm{LB}(S^{in}(\tilde{\mathbf{x}}_j^i))) \leq \Delta\mathbf{x} \leq \{\Delta, t\}$ for $\tilde{\mathbf{y}}_j^i \leq t$. Finally, $\mathrm{LB}(\delta_{i,j}) = \min\big(\mathrm{LB}(\delta_{i,j}^{in}), \mathrm{LB}(S^{in}(\tilde{\mathbf{x}}_j^i)), t - \mathrm{UB}(S^{in}(\mathbf{x}_j^i))\big)$ and $\mathrm{UB}(\delta_{i,j}) = \min\big(\mathrm{UB}(\delta_{i,j}^{in}), t\big)$.

– **Case 3-6** (Otherwise):
  - If $\tilde{\mathbf{y}}_j^i < 0$, $\mathrm{UB}(\delta_{i,j}) = 0$ by case 3-1.
  - If $\tilde{\mathbf{y}}_j^i \geq 0$, $\mathrm{UB}(\delta_{i,j}) = \min\big(\mathrm{UB}(\delta_{i,j}^{in}), t\big)$ by case 3-5.
  - If $\tilde{\mathbf{y}}_j^i \geq t$, $\mathrm{LB}(\delta_{i,j}) = t - \mathrm{UB}(S^{in}(\mathbf{x}_j^i))$ by case 3-2;
  - If $\tilde{\mathbf{y}}_j^i \leq t$, $\mathrm{LB}(\delta_{i,j}) = 0$ if $\mathrm{LB}(\delta_{i,j}^{in}) \geq 0$, and $\max(\mathrm{LB}(\delta_{i,j}^{in}), -\mathrm{UB}(S^{in}(\mathbf{x}_j^i)))$ otherwise by case 3-4.

  Then, we get the lower bound as

$$\mathrm{LB}(\delta_{i,j}) = \min\big(t - \mathrm{UB}(S^{in}(\mathbf{x}_j^i)), 0, \max(\mathrm{LB}(\delta_{i,j}^{in}), -\mathrm{UB}(S^{in}(\mathbf{x}_j^i)))\big),$$

  and upper bound as

$$\mathrm{UB}(\delta_{i,j}) = \max\big(\min(\mathrm{UB}(\delta_{i,j}^{in}), t), 0\big) = \mathrm{clamp}(\mathrm{UB}(\delta_{i,j}^{in}), 0, t).$$