QF4102 Financial Modelling and Computation Assignment 3

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1 Transformed Black-Scholes PDE model

Consider the **transformed** Black-Scholes PDE model:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + (r - q - \frac{\sigma^2}{2})\frac{\partial u}{\partial x} - ru = 0, \quad x \in (-\infty, \infty), t \in [0, T) \\ u(x, T) = \varphi(x), \end{cases}$$

1.1 Derivation of fully implicit scheme

Evaluate partial derivatives at (x_n^i, t_n) where $t_n = n\Delta t, x_n^i = i\Delta x, n \in [0, \frac{T}{\Delta t}), i \in [-x_{max}, x_{max}], I_{max} = \frac{x_{max}}{\Delta x}$

$$\text{Use the forward time finite difference formulae : } \left.\frac{\partial u}{\partial t}\right|_{(x_n^i, t_n)} = \frac{u_{n+1}^i - u_n^i}{\Delta t} + O(\Delta t)$$

$$\text{Use the centred space finite difference formulae : } \left.\frac{\partial u}{\partial x}\right|_{(x_n^i, t_n)} = \frac{u_n^{i+1} - u_n^{i-1}}{2\Delta x} + O[(\Delta x)^2]$$

$$\text{Use the centred space finite difference formulae : } \left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_n^i, t_n)} = \frac{u_n^{i+1} - 2u_n^i + u_n^{i-1}}{(\Delta x)^2} + O[(\Delta x)^2]$$

The finite difference equation is hence :

$$\frac{u_{n+1}^i - u_n^i}{\Delta t} + O(\Delta t) + \frac{\sigma^2}{2}\frac{u_n^{i+1} - 2u_n^i + u_n^{i-1}}{(\Delta x)^2} + (r - q - \frac{\sigma^2}{2})\frac{u_n^{i+1} - u_n^{i-1}}{2\Delta x} + O[(\Delta x)^2] - ru_n^i = 0$$

$$\frac{u_{n+1}^i - u_n^i}{\Delta t} + O(\Delta t) + O[(\Delta x)^2] = -\frac{\sigma^2}{2}\frac{u_n^{i+1} - 2u_n^i + u_n^{i-1}}{(\Delta x)^2} - (r - q - \frac{\sigma^2}{2})\frac{u_n^{i+1} - u_n^{i-1}}{2\Delta x} + ru_n^i$$

$$\frac{U_{n+1}^i - U_n^i}{\Delta t} = rU_n^i - \frac{\sigma^2}{2}\frac{U_n^{i+1} - 2U_n^i + U_n^{i-1}}{(\Delta x)^2} - (r - q - \frac{\sigma^2}{2})\frac{U_n^{i+1} - U_n^{i-1}}{2\Delta x}$$

$$U_{n+1}^i = U_n^i + \Delta t[rU_n^i - \frac{\sigma^2}{2}\frac{U_n^{i+1} - 2U_n^i + U_n^{i-1}}{(\Delta x)^2} - (r - q - \frac{\sigma^2}{2})\frac{U_n^{i+1} - U_n^{i-1}}{2\Delta x}]$$

$$U_{n+1}^i = U_n^i(1 + r\Delta t) - \frac{\Delta t}{2(\Delta x)^2}[\sigma^2(U_n^{i+1} - 2U_n^i + U_n^{i-1}) + \Delta x(r - q - \frac{\sigma^2}{2})(U_n^{i+1} - U_n^{i-1})]$$

$$U_{n+1}^i = U_n^{i-1}[\frac{\Delta t(r - q - \frac{\sigma^2}{2})}{2\Delta x} - \frac{\sigma^2\Delta t}{2(\Delta x)^2}] + U_n^i[1 + r\Delta t + \frac{\sigma^2\Delta t}{(\Delta x)^2}] + U_n^{i+1}[-\frac{\Delta t(r - q - \frac{\sigma^2}{2})}{2\Delta x} - \frac{\sigma^2\Delta t}{2(\Delta x)^2}]$$

$$U_{n+1}^i = aU_n^{i-1} + bU_n^i + cU_n^{i+1}, \forall I_{min} + 1 \le i \le I_{max} - 1$$

where $a = \gamma - \frac{\alpha}{2}, b = \beta + \alpha, c = -\gamma - \frac{\alpha}{2}, \alpha = \frac{\sigma^2\Delta t}{(\Delta x)^2}, \beta = 1 + r\Delta t, \gamma = \frac{\Delta t(r-q-\frac{\sigma^2}{2})}{2\Delta x}$

The boundary conditions are as follows:

$U_n^{I_{max}} = e^{-q(T-n\Delta t)}\exp(I_{max}\Delta x) - e^{-r(T-n\Delta t)}X$, when the underlying value is very large at $\exp(I_{max}\Delta x)$

$U_n^{I_{min}} = 0$, when the underlying value is very small at $\exp(I_{min}\Delta x)$

With the values of $U_n^{I_{min}}$ and $U_n^{I_{max}}$ specified, we can express the FDE into matrix form.

$$\begin{bmatrix} b & c & \dots & \dots & \dots & \dots & \dots \\ a & b & c & \dots & \dots & \dots & \dots \\ \dots & a & b & c & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & a & b & c & \dots \\ \dots & \dots & \dots & \dots & a & b & c \\ \dots & \dots & \dots & \dots & \dots & a & b \end{bmatrix} \begin{bmatrix} U_n^{I_{min}+1} \\ U_n^{I_{min}+2} \\ U_n^{I_{min}+3} \\ \dots \\ U_n^{I_{max}-3} \\ U_n^{I_{max}-2} \\ U_n^{I_{max}-1} \end{bmatrix} = \begin{bmatrix} U_{n+1}^{I_{min}+1} \\ U_{n+1}^{I_{min}+2} \\ U_{n+1}^{I_{min}+3} \\ \dots \\ U_{n+1}^{I_{max}-3} \\ U_{n+1}^{I_{max}-2} \\ U_{n+1}^{I_{max}-1} \end{bmatrix} + \begin{bmatrix} -aU_n^{I_{min}} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -cU_n^{I_{max}} \end{bmatrix}$$

More concisely, we can name the tridiagonal matrix A and the right hand side vector F to express the FDE in this form: $AU_n = U_{n+1} + F \rightarrow U_n = A^{-1}(U_{n+1} + F)$

1.2 Finite Difference Scheme Algorithm on fully implicit scheme

Data: S_0, X, r, T, σ, I, N, x_{max}

Result: c_{IDS}, Option Premium

$\Delta t = \dfrac{T}{N}$, $\Delta x = \dfrac{x_{max}}{I}$;

$\alpha = \frac{\Delta t(r-q-\frac{\sigma^2}{2})}{2\Delta x}$;

$\beta = 1 + r\Delta t$;

$\gamma = \frac{\sigma^2 \Delta t}{2(\Delta x)^2}$;

$a = \alpha - \gamma$;

$b = \beta + \alpha$;

$c = -\alpha - \gamma$;

for $i = -I+1, -I+2, \dots, I-2, I-1$ **do**

  $U_N^i = max(\exp(i\Delta x) - X, 0)$;

end

Generate a tridiagonal matrix A of dimension $(2I-1) * (2I-1)$,

with $A_{i,i} = b\,\forall i = 1, 2, \dots 2I-1$,$A_{i,i-1} = b\,\forall i = 2, \dots 2I-1$, $A_{i,i+1} = b\,\forall i = 1, 2, \dots 2I-2$.

for $j = N-1, N-2, \dots, 0$ **do**

  Generate a vector F of length $(2I-1)$, with $F_{2I-1} = c\exp(-r(T-j\Delta t))(S_{max} - X)$, $F_i = 0$ otherwise;

  $U_j = A^{-1}(U_{j+1} + F)$;

end

$i_0 = round\left(\dfrac{\ln S_0}{\Delta x}\right)$;

$c_{\text{IDS}} = U_0^{i_0}$;

2 Valuation of digital call option

2.1 Algo