

QF4102 Financial Modelling and Computation Assignment 3

G01 Wang Zexin, Chen Penghao

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1 Transformed Black-Scholes PDE model

Consider the **transformed** Black-Scholes PDE model:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + (r - q - \frac{\sigma^2}{2}) \frac{\partial u}{\partial x} - ru = 0, & x \in (-\infty, \infty), t \in [0, T) \\ u(x, T) = \varphi(x), \end{cases}$$

1.1 Derivation of fully implicit scheme

Evaluate partial derivatives at (x_n^i, t_n) where $t_n = n\Delta t, x_n^i = i\Delta x, n \in [0, \frac{T}{\Delta t}), i \in [-x_{max}, x_{max}], I_{max} = \frac{x_{max}}{\Delta x}$

$$\text{Use the forward time finite difference formulae : } \left. \frac{\partial u}{\partial t} \right|_{(x_n^i, t_n)} = \frac{u_{n+1}^i - u_n^i}{\Delta t} + O(\Delta t)$$

$$\text{Use the centred space finite difference formulae : } \left. \frac{\partial u}{\partial x} \right|_{(x_n^i, t_n)} = \frac{u_n^{i+1} - u_n^{i-1}}{2\Delta x} + O[(\Delta x)^2]$$

$$\text{Use the centred space finite difference formulae : } \left. \frac{\partial^2 u}{\partial x^2} \right|_{(x_n^i, t_n)} = \frac{u_n^{i+1} - 2u_n^i + u_n^{i-1}}{(\Delta x)^2} + O[(\Delta x)^2]$$

The finite difference equation is hence :

$$\frac{u_{n+1}^i - u_n^i}{\Delta t} + O(\Delta t) + \frac{\sigma^2}{2} \frac{u_n^{i+1} - 2u_n^i + u_n^{i-1}}{(\Delta x)^2} + (r - q - \frac{\sigma^2}{2}) \frac{u_n^{i+1} - u_n^{i-1}}{2\Delta x} + O[(\Delta x)^2] - ru_n^i = 0$$

$$\frac{u_{n+1}^i - u_n^i}{\Delta t} + O(\Delta t) + O[(\Delta x)^2] = -\frac{\sigma^2}{2} \frac{u_n^{i+1} - 2u_n^i + u_n^{i-1}}{(\Delta x)^2} - (r - q - \frac{\sigma^2}{2}) \frac{u_n^{i+1} - u_n^{i-1}}{2\Delta x} + ru_n^i$$

$$\frac{U_{n+1}^i - U_n^i}{\Delta t} = rU_n^i - \frac{\sigma^2}{2} \frac{U_n^{i+1} - 2U_n^i + U_n^{i-1}}{(\Delta x)^2} - (r - q - \frac{\sigma^2}{2}) \frac{U_n^{i+1} - U_n^{i-1}}{2\Delta x}$$

$$U_{n+1}^i = U_n^i + \Delta t [rU_n^i - \frac{\sigma^2}{2} \frac{U_n^{i+1} - 2U_n^i + U_n^{i-1}}{(\Delta x)^2} - (r - q - \frac{\sigma^2}{2}) \frac{U_n^{i+1} - U_n^{i-1}}{2\Delta x}]$$

$$U_{n+1}^i = U_n^i (1 + r\Delta t) - \frac{\Delta t}{2(\Delta x)^2} [\sigma^2 (U_n^{i+1} - 2U_n^i + U_n^{i-1}) + \Delta x (r - q - \frac{\sigma^2}{2}) (U_n^{i+1} - U_n^{i-1})]$$

$$U_{n+1}^i = U_n^{i-1} [\frac{\Delta t (r - q - \frac{\sigma^2}{2})}{2\Delta x} - \frac{\sigma^2 \Delta t}{2(\Delta x)^2}] + U_n^i [1 + r\Delta t + \frac{\sigma^2 \Delta t}{(\Delta x)^2}] + U_n^{i+1} [-\frac{\Delta t (r - q - \frac{\sigma^2}{2})}{2\Delta x} - \frac{\sigma^2 \Delta t}{2(\Delta x)^2}]$$

$$U_{n+1}^i = aU_n^{i-1} + bU_n^i + cU_n^{i+1}, \forall I_{min} + 1 \leq i \leq I_{max} - 1$$

$$\text{where } a = \gamma - \frac{\alpha}{2}, b = \beta + \alpha, c = -\gamma - \frac{\alpha}{2}, \alpha = \frac{\sigma^2 \Delta t}{(\Delta x)^2}, \beta = 1 + r\Delta t, \gamma = \frac{\Delta t (r - q - \frac{\sigma^2}{2})}{2\Delta x}$$

The boundary conditions are as follows:

$$U_n^{I_{max}} = e^{-q(T-n\Delta t)} \exp(I_{max}\Delta x) - e^{-r(T-n\Delta t)} X, \text{ when the underlying value is very large at } \exp(I_{max}\Delta x)$$

$$U_n^{I_{min}} = 0, \text{ when the underlying value is very small at } \exp(I_{min}\Delta x)$$

With the values of $U_n^{I_{min}}$ and $U_n^{I_{max}}$ specified, we can express the FDE into matrix form.

$$\begin{bmatrix} b & c & \dots & \dots & \dots & \dots & \dots \\ a & b & c & \dots & \dots & \dots & \dots \\ \dots & a & b & c & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & a & b & c & \dots \\ \dots & \dots & \dots & \dots & a & b & c \\ \dots & \dots & \dots & \dots & \dots & a & b \end{bmatrix} \begin{bmatrix} U_n^{I_{min}+1} \\ U_n^{I_{min}+2} \\ U_n^{I_{min}+3} \\ \dots \\ U_n^{I_{max}-3} \\ U_n^{I_{max}-2} \\ U_n^{I_{max}-1} \end{bmatrix} = \begin{bmatrix} U_{n+1}^{I_{min}+1} \\ U_{n+1}^{I_{min}+2} \\ U_{n+1}^{I_{min}+3} \\ \dots \\ U_{n+1}^{I_{max}-3} \\ U_{n+1}^{I_{max}-2} \\ U_{n+1}^{I_{max}-1} \end{bmatrix} + \begin{bmatrix} -aU_n^{I_{min}} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -cU_n^{I_{max}} \end{bmatrix}$$

More concisely, we can name the tridiagonal matrix A and the right hand side vector F to express the FDE in this form: $AU_n = U_{n+1} + F$, $U_n = A^{-1}(U_{n+1} + F)$

1.2 Finite Difference Scheme Algorithm on fully implicit scheme

Data: $S_0, X, r, T, \sigma, I, N, x_{max}$

Result: c_{IDS} , Option Premium

$$\Delta t = \frac{T}{N}, \Delta x = \frac{x_{max}}{I};$$

$$\alpha = \frac{\Delta t(r - q - \frac{\sigma^2}{2})}{2\Delta x};$$

$$\beta = 1 + r\Delta t;$$

$$\gamma = \frac{\sigma^2 \Delta t}{2(\Delta x)^2};$$

$$a = \alpha - \gamma;$$

$$b = \beta + \alpha;$$

$$c = -\alpha - \gamma;$$

for $i = -I + 1, -I + 2, \dots, I - 2, I - 1$ **do**

$$U_N^i = \max(\exp(i\Delta x) - X, 0);$$

end

Generate a tridiagonal matrix A of dimension $(2I - 1) * (2I - 1)$,

with $A_i i = b, \forall i = 1, 2, \dots, 2I - 1, A_i i - 1 = b, \forall i = 2, \dots, 2I - 1, A_i i + 1 = b, \forall i = 1, 2, \dots, 2I - 2$.

for $j = N - 1, N - 2, \dots, 0$ **do**

$$\begin{array}{|l} \text{Generate a vector } F \text{ of length } (2I - 1), \text{ with } F_{2I-1} = c \exp(-r(T - j\Delta t))(S_{max} - X), F_i = 0 \\ \text{otherwise;} \\ U_j = A^{-1}(U_{j+1} + F); \end{array}$$

end

$$i_0 = \text{round}\left(\frac{\ln S_0}{\Delta x}\right);$$

$$c_{IDS} = U_0^{i_0};$$

2 Valuation of digital call option

2.1 Algo