

# State-space models

## FW 891

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# Purpose

- Goal
- Background
  - Some applications and history
- Dynamic linear model and some extensions
- Identifying nonidentifiability
- Future directions and other extensions
- R and Stan demo

# A key reference on ecological state space models:

## REVIEW

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### A guide to state-space modeling of ecological time series

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- Useful for modeling population dynamics, metapopulation dynamics, fisheries stock assessments, integrated population models, capture recapture data, animal movement, and biodiversity data
- Quite popular because they directly model temporal autocorrelation in a way that helps differentiate process variation vs. observation error.

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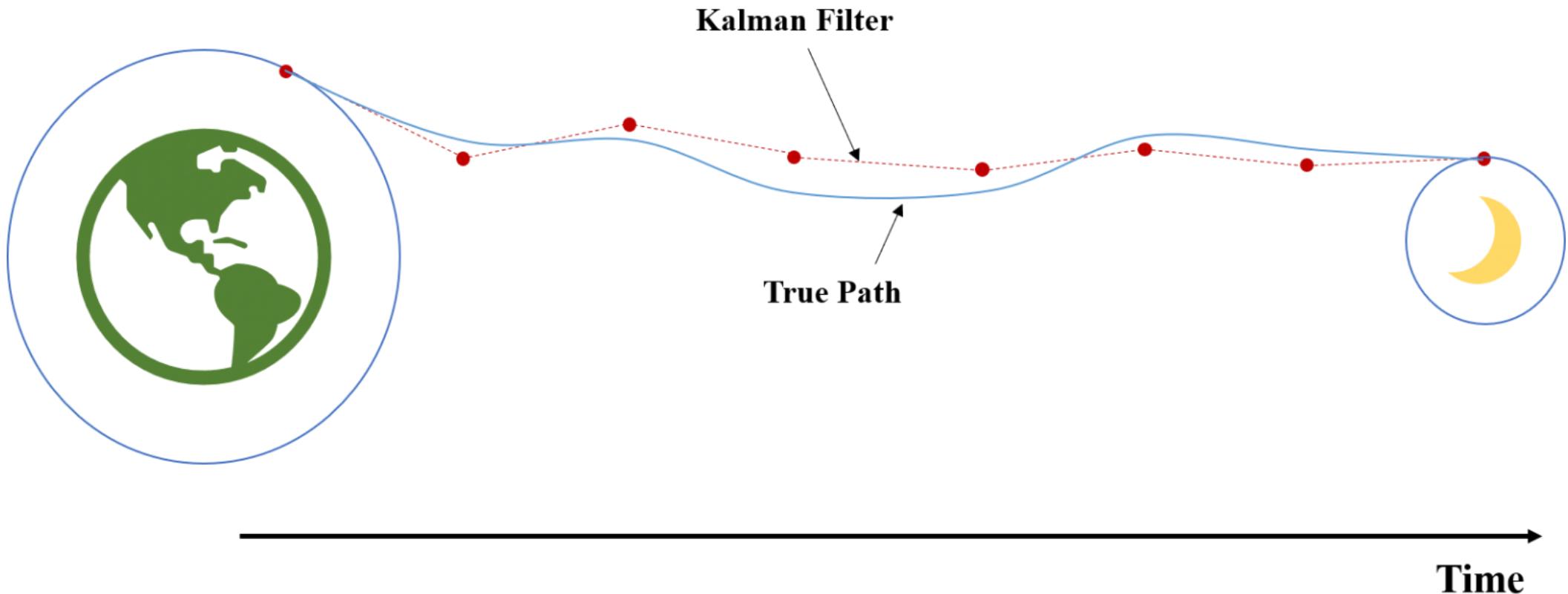
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  - Importance
  - Remarkably simple and elegant

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<https://www.lancaster.ac.uk/stor-i-student-sites/jack-trainer/how-nasa-used-the-kalman-filter-in-the-apollo-program/>

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Rudolf Kalman receiving the National Medal of Science from Barack Obama: Photo NAE  
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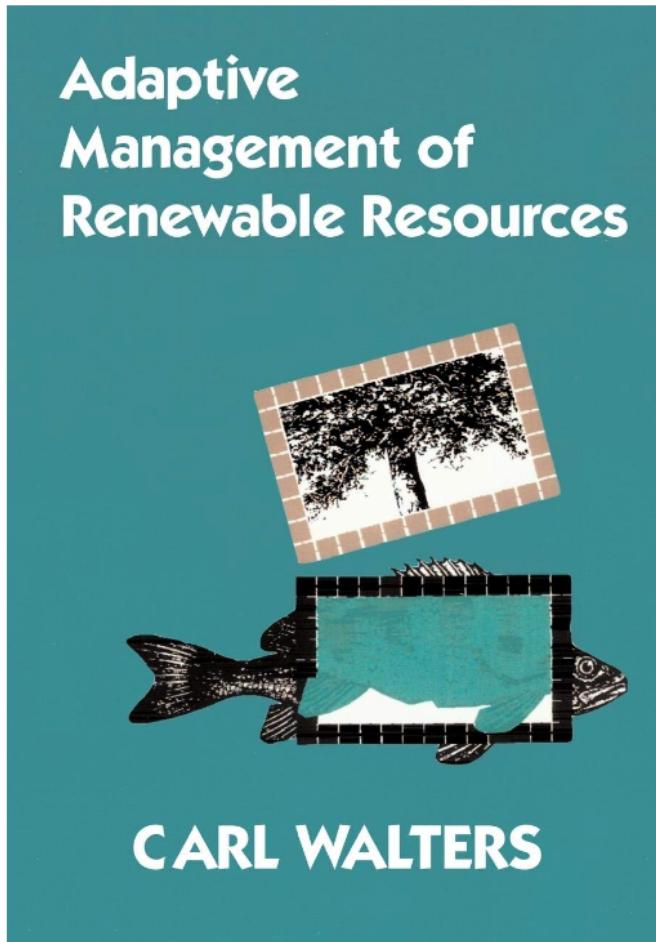
- Hint

# Other applications in ecology and resource management?



Walters 1986; Photo: Wikipedia

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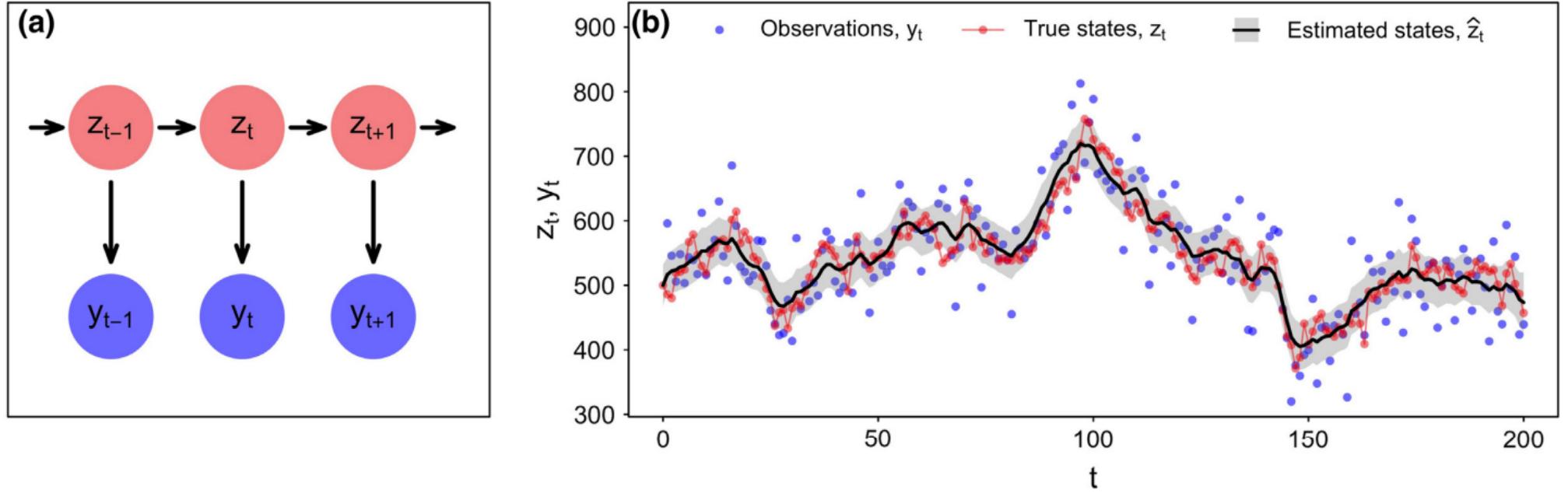


FIG. 1. The dependence structure and evolution of the two time series comprising a simple univariate state-space model. (a) Dependence relationships with arrows, demonstrating that once the dependence of the observations  $y_t$  on the states  $z_t$  is accounted for, the observations are assumed independent. (b) Our toy model (Eqs. 1, 2). The blue and red dots are the simulated observations and states, respectively. The black line and gray band are the estimated states and associated 95% confidence intervals. The true states, but not the observations, usually fall in the 95% confidence intervals. This demonstrates that the state estimates can be a closer approximation of the truth than the observations.

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- In a pop dy context, this could be interpreted to mean that  $y_t$  values are autocorrelated because the underlying process driving them is autocorrelated through time

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- We'll set  $z_{t=0} = z_0$ , where  $z_0$  is an estimated parameter

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- stochastic logistic model

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$$w_t = w_{t-1} + \beta_0 + \beta_1 \exp(w_{t-1}) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_p^2) \quad [\text{process eq.}]$$

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where  $w_t = \log(z_t)$

- playing with math and modeling population on logarithmic scale

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$$z_t = z_{t-1} \exp(\beta_0 + \beta_1 \log(z_{t-1}) + \varepsilon_t), \quad \varepsilon_t \sim N(0, \sigma_p^2) \quad [\text{process eq.}]$$

- Gompertz model, which assumes that per-unit-abundance growth rate depends on log abundance

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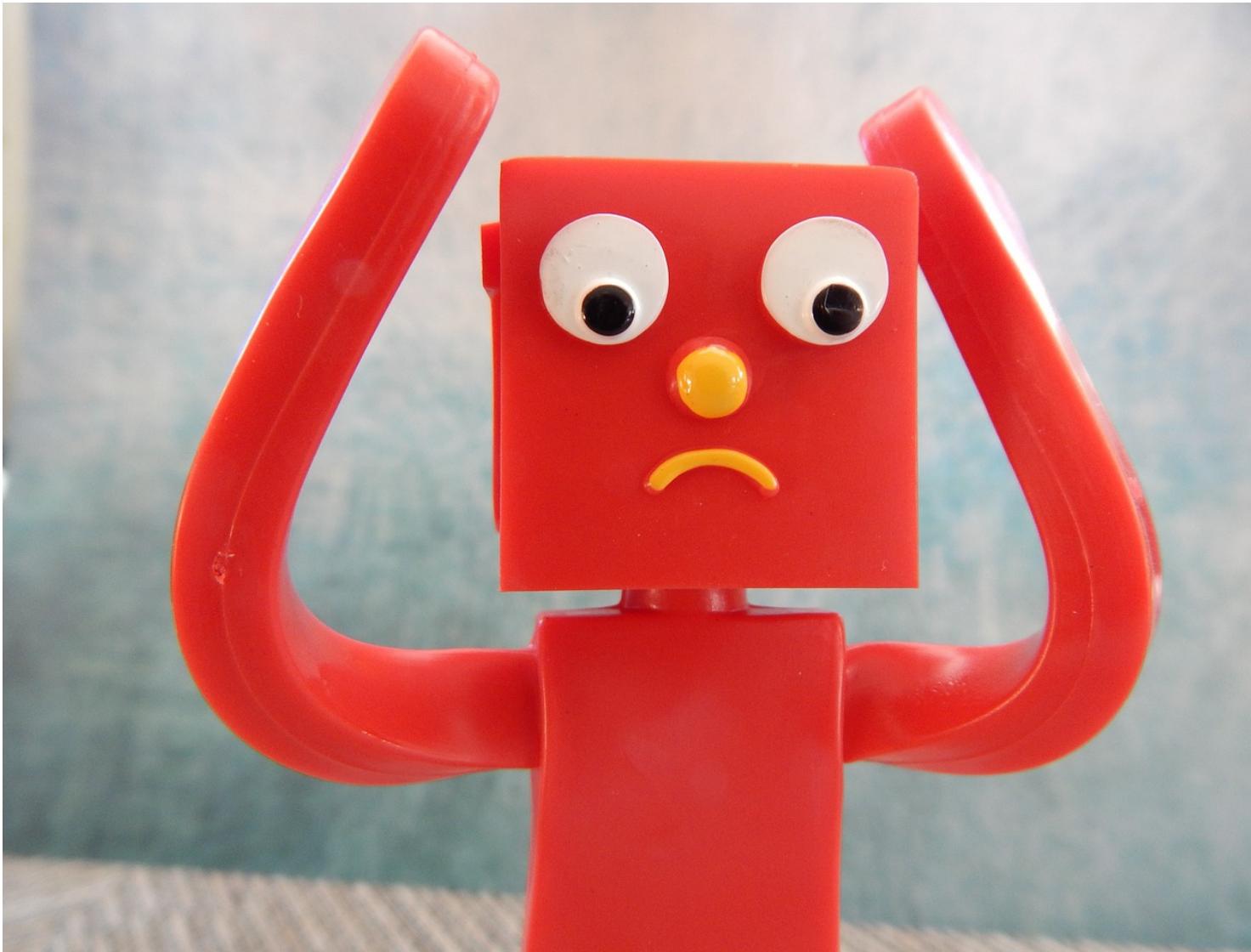
$$w_t = \beta_0 + (1 + \beta_1) w_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_p^2), \quad [\text{process equation}]$$

$$g_t = w_t + \eta_t, \quad \eta_t \sim N(0, \sigma_o^2), \quad [\text{observation equation}]$$

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- Linearized Gompertz model

# What's the point?



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  - SSMs superior to simpler models when both the process variance and the observation error are large (e.g., de Valpine and Hastings 2002)

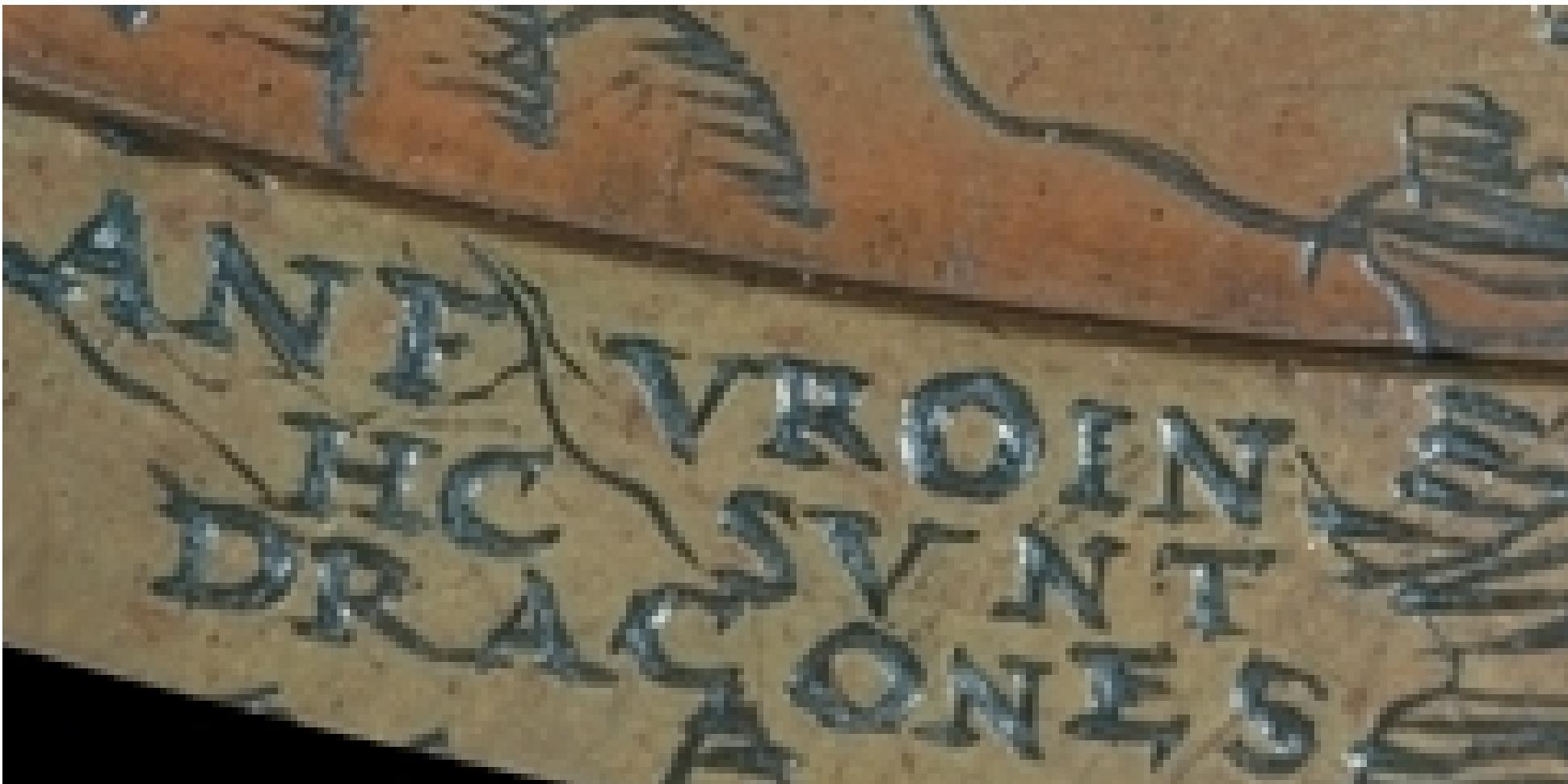
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  - Many studies have shown that SSMs provided better inference than simpler models
- Should SSMs be the default for many ecological time series?

# Nonidentifiability



The Hunt-Lenox Globe: here be dragons

# Nonidentifiability

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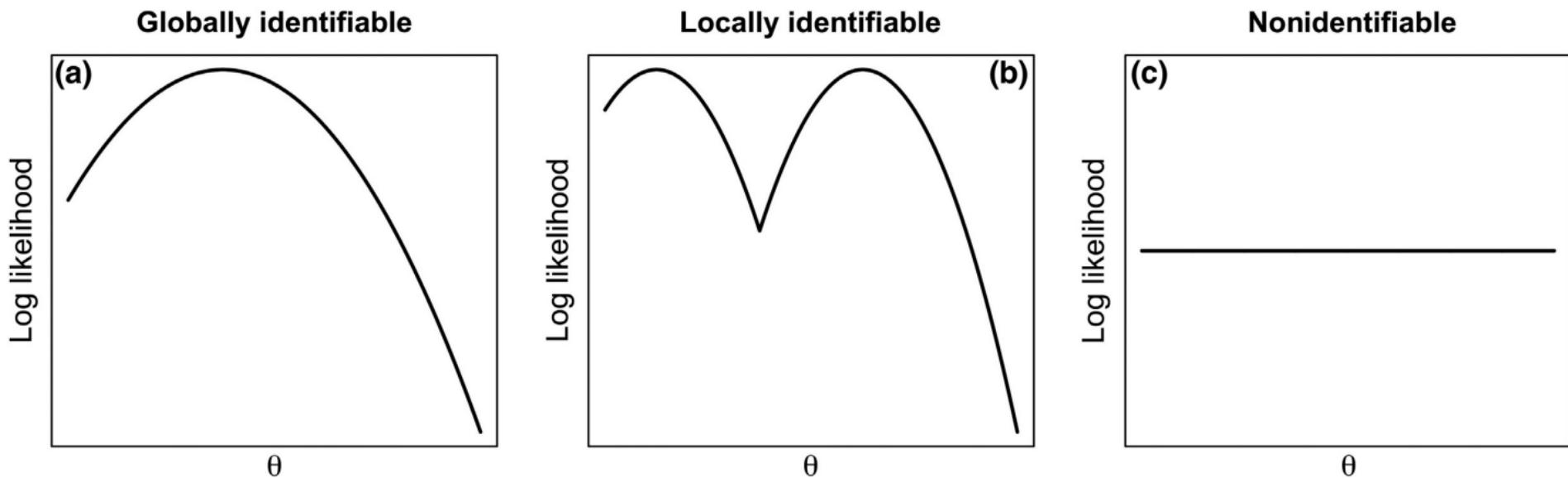


FIG. 3. Examples of log-likelihood profiles for a parameter  $\theta$  under various identifiability scenarios: (a) globally identifiable, (b) locally identifiable, and (c) nonidentifiable model.

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    - If your model is non-identifiable, parameters are usually biased with large variances
- *One should choose priors with great care for SSMs*

# Remedies for nonidentifiability

- Reformulate the model
- Make simplifying assumptions when data are limited
- Estimate measurement errors externally
- Integrate additional data
- Try to use replicated observations
- Match temporal resolution between states and observations
  - If one has a data set with locations every 8 h, it would be challenging to estimate behavioral states lasting <16–24 h

# Wrap up

- SSMs are flexible models for time series that can be used to answer a broad range of ecological questions
- They can be used to model univariate or multivariate time series
- SSMs can be linear or nonlinear, and have discrete or continuous time steps
- Key point:

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- They can be used to model univariate or multivariate time series
- SSMs can be linear or nonlinear, and have discrete or continuous time steps
- Key point:
  - Recognize that this is a crash course, and there are a great many extensions of SSMs in ecology

# To the example

# References

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