State-space models

FW 891

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Christopher Cahill

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Quantitative Fisheries Center
MICHIGAN STATE UNIVERSITY

Purpose

- Goal
- Background
  - Some applications and history
- Dynamic linear model and some extensions
- Identifying nonidentifiability
- Future directions and other extensions
- R and Stan demo

A key reference on ecological state space models:

REVIEW

Ecological Monographs, 91(4), 2021, e01470

A guide to state–space modeling of ecological time series

Marie Auger-Méthé,[1,2,13] Ken Newman,[3,4] Diana Cole,[5] Fanny Empacher,[6] Rowenna Gryba,[1,2] Aaron A. King,[7] Vianey Leos-Barajas,[8,9] Joanna Mills Flemming,[10] Anders Nielsen,[11] Giovanni Petris,[12] and Len Thomas[6]

What is a state space model

- State space models are a popular modeling framework for analyzing time-series data

Auger-Methe et al. 2021

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- Useful for modeling population dynamics, metapopulation dynamics, fisheries stock assessments, integrated population models, capture recapture data, animal movement, and biodiversity data

Auger-Methe et al. 2021

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- Useful for modeling population dynamics, metapopulation dynamics, fisheries stock assessments, integrated population models, capture recapture data, animal movement, and biodiversity data
- Quite popular because they directly model temporal autocorrelation in a way that helps differentiate process variation vs. observation error.

Auger-Methe et al. 2021

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- An example

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A historical aside

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  - Remarkably simple and elegant

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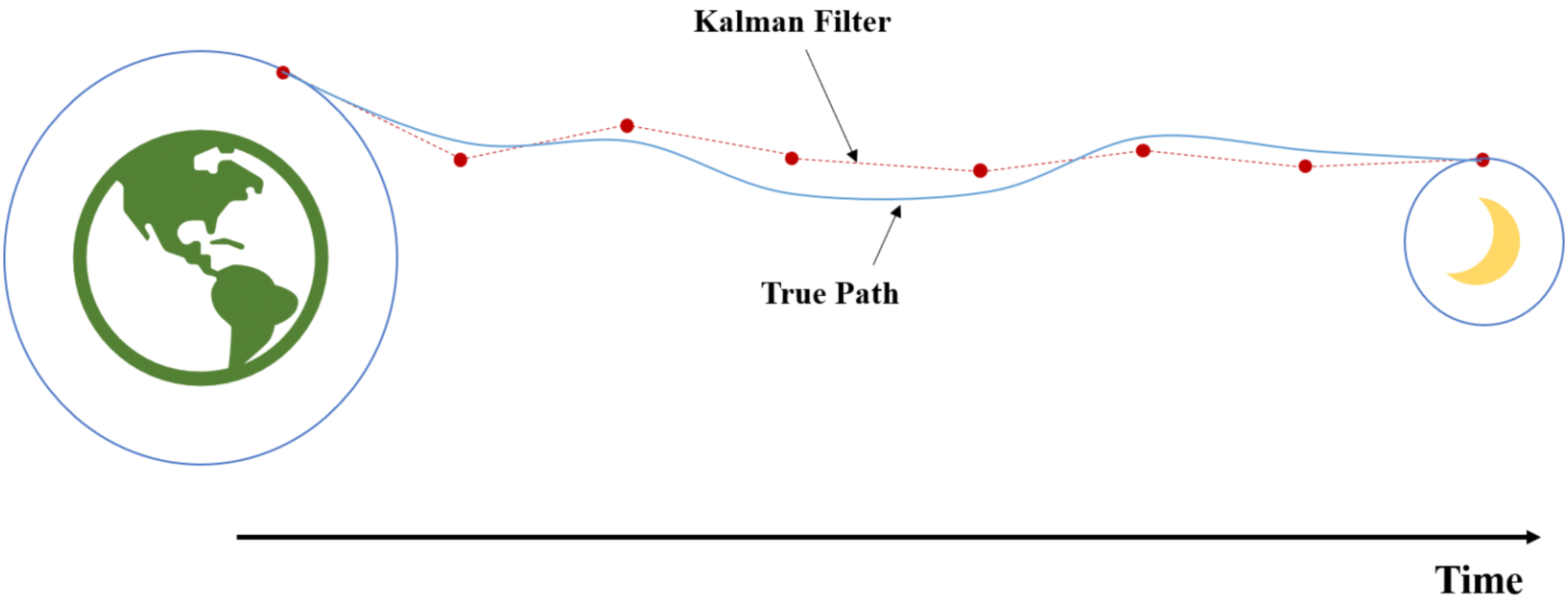

https://www.lancaster.ac.uk/stor-i-student-sites/jack-trainer/how-nasa-used-the-kalman-filter-in-the-apollo-program/

A historical aside

Rudolf Kalman receiving the National Medal of Science from Barack Obama: Photo NAE

Other applications in ecology and resource management?

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- Hint

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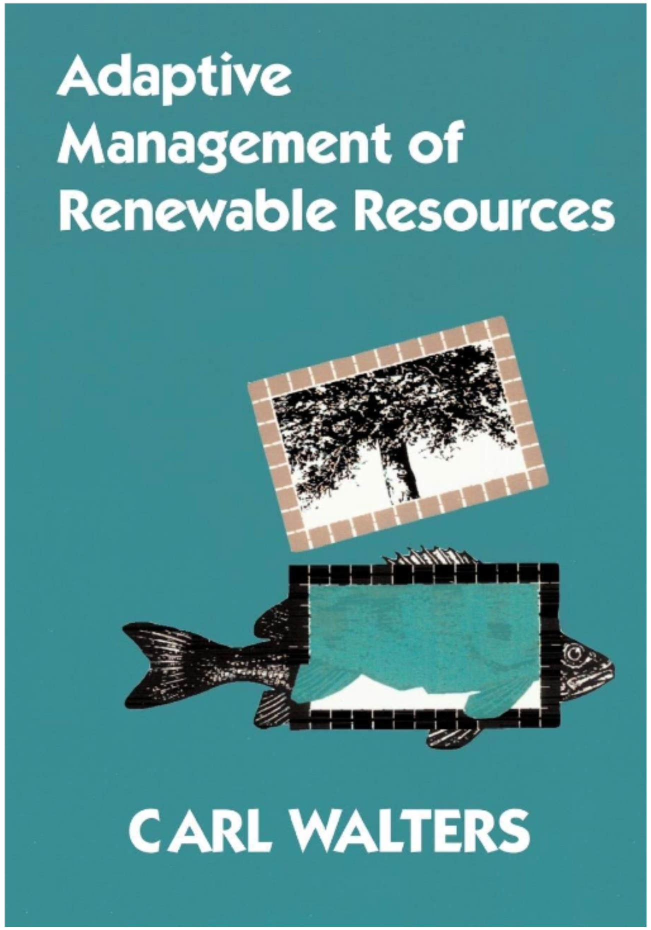
Walters 1986; Photo: Wikipedia

Other applications in ecology and resource management?



Walters 1986; Photo: Wikipedia

A toy example: normal dynamic linear model

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  - Evenly spaced points in time $t (t = 1, 2, \ldots, T)$

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The toy example in math:

$$z_t = \alpha + z_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{N}\left(0, \sigma_p^2\right),$$ [process equation]

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The toy example in pictures

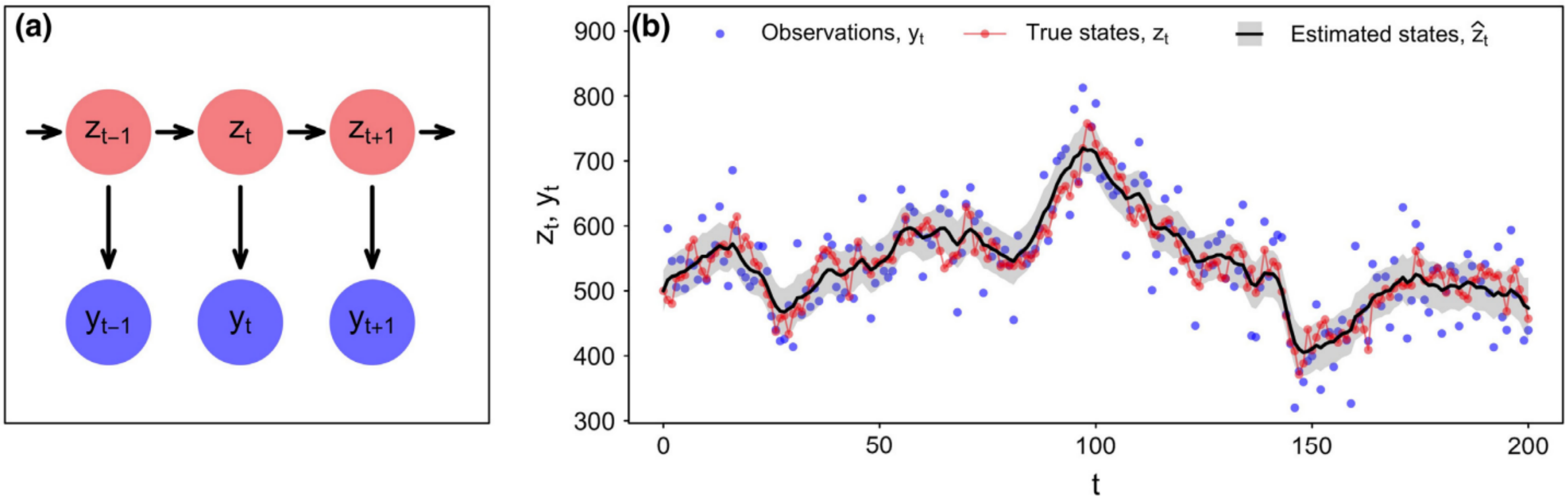

FIG. 1. The dependence structure and evolution of the two time series comprising a simple univariate state–space model. (a) Dependence relationships with arrows, demonstrating that once the dependence of the observations y_t on the states z_t is accounted for, the observations are assumed independent. (b) Our toy model (Eqs. 1, 2). The blue and red dots are the simulated observations and states, respectively. The black line and gray band are the estimated states and associated 95% confidence intervals. The true states, but not the observations, usually fall in the 95% confidence intervals. This demonstrates that the state estimates can be a closer approximation of the truth than the observations.

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Two main assumptions

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Aeberhard et al. 2018; Auger-Methe et al. 2021

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- In a pop dy context, this could be interpretted to mean that y_t values are autocorrelated because the underlying process driving them is autocorrelated through time

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The toy example needs one more thing:

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  - Typically referred to as the initialization equation
- We'll set $z_{t=0} = z_0$, where z_0 is an estimated parameter

Auger-Methe et al. 2021

Handling nonlinearity

- Our toy example is useful for teaching but rarely helpful in practice

Jamieson and Brooks 2004; Auger-Methe et al. 2021

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\begin{aligned}
z_t &= z_{t-1}\exp(\beta_0 + \beta_1 z_{t-1} + \varepsilon_t), \quad \varepsilon_t \sim \mathrm{N}\left(0, \sigma_p^2\right) && \text{[process equation]} \\
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- stochastic logistic model

Jamieson and Brooks 2004; Auger-Methe et al. 2021

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$$w_t = w_{t-1} + \beta_0 + \beta_1 \exp(w_{t-1}) + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{N}\left(0, \sigma_p^2\right) \qquad \text{[process eq.]}$$

$$y_t = \exp(w_t) + \eta_t, \quad \eta_t \sim \mathrm{N}\left(0, \sigma_o^2\right) \qquad \text{[observation eq.]}$$

$$\text{where } w_t = log(z_t)$$

- playing with math and modeling population on logarithmic scale

Jamieson and Brooks 2004; Auger-Methe et al. 2021

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$$z_t = z_{t-1} \exp(\beta_0 + \beta_1 \log(z_{t-1}) + \varepsilon_t), \quad \varepsilon_t \sim \mathrm{N}\left(0, \sigma_p^2\right) \quad \text{[process eq.]}$$

- Gompertz model, which assumes that per-unit-abundance growth rate depends on log abundance

Jamieson and Brooks 2004; Auger-Methe et al. 2021

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$$w_t = \beta_0 + (1 + \beta_1)\, w_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{N}\left(0, \sigma_p^2\right), \qquad \text{[process equation]}$$

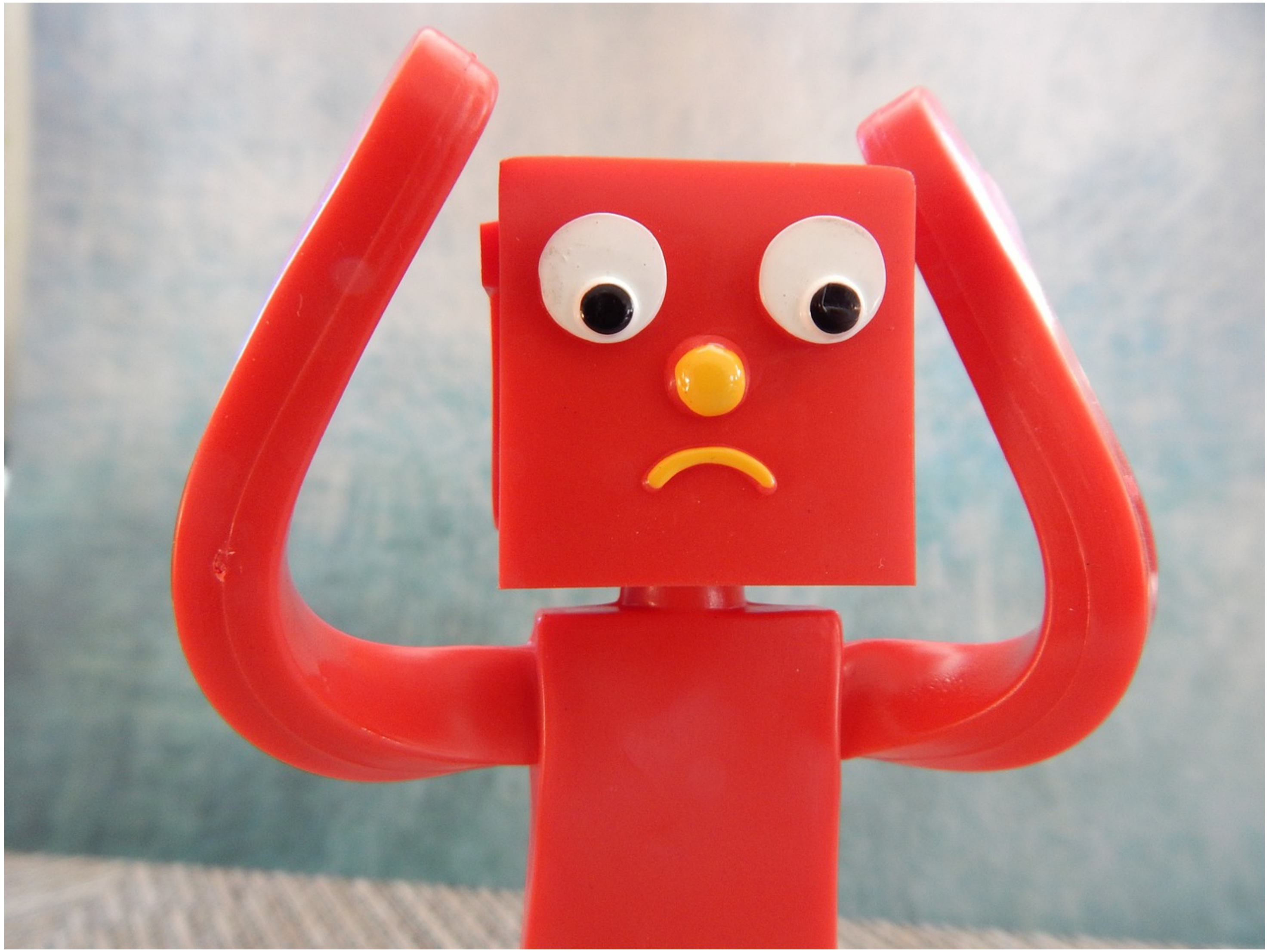
$$g_t = w_t + \eta_t, \quad \eta_t \sim \mathrm{N}\left(0, \sigma_o^2\right), \qquad \text{[observation equation]}$$

$$\text{where } w_t = log(z_t) \text{ and } g_t = log(y_t)$$

- Linearized Gompertz model

Jamieson and Brooks 2004; Auger-Methe et al. 2021

What's the point?



State space models are complex

- e.g., see Thorson et al. 2016

Auger-Methe et al. 2021

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  - SSMs superior to simpler models when both the process variance and the observation error are large (e.g., de Valpine and Hastings 2002)

Auger-Methe et al. 2021

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  - Many studies have shown that SSMs provided better inference than simpler models
- Should SSMs be the default for many ecological time series?

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Nonidentifiability

The Hunt-Lenox Globe: here be dragons

Nonidentifiability

- Identifiability refers to whether there is a unique representation of the model

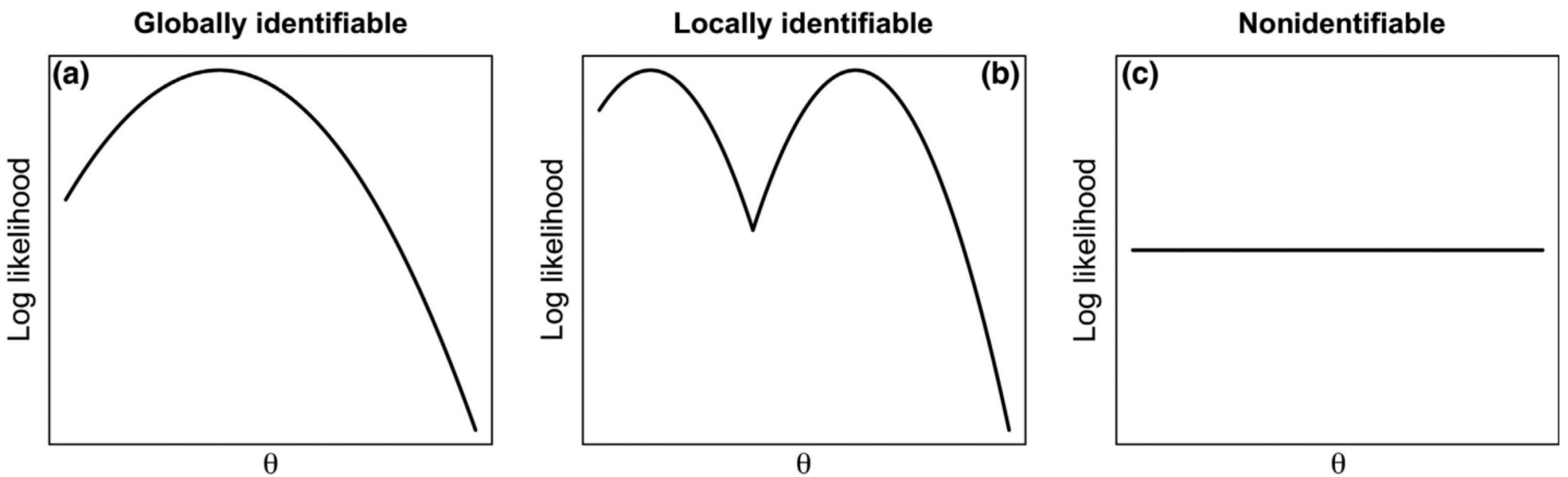

FIG. 3. Examples of log-likelihood profiles for a parameter θ under various identifiability scenarios: (a) globally identifiable, (b) locally identifiable, and (c) nonidentifiable model.

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- *One should choose priors with great care for SSMs*

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Remedies for nonidentifiability

- Reformulate the model
- Make simplifying assumptions when data are limited
- Estimate measurement errors externally
- Integrate additional data
- Try to use replicated observations
- Match temporal resolution between states and observations
    - If one has a data set with locations every 8 h, it would be challenging to estimate behavioral states lasting <16–24 h

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Wrap up

- SSMs are flexible models for time series that can be used to answer a broad range of ecological questions
- They can be used to model univariate or multivariate time series
- SSMs can be linear or nonlinear, and have discrete or continuous time steps
- Key point:

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Wrap up

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- They can be used to model univariate or multivariate time series
- SSMs can be linear or nonlinear, and have discrete or continuous time steps
- Key point:
    - Recognize that this is a crash course, and there are a great many extensions of SSMs in ecology

Auger-Methe et al. 2021

To the example

References

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