Introduction to spatio-temporal models

FW 891

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Purpose

- Goal
- Refresher on spatial random field models
- Extensions in space-time
  - IID spatiotemporal random field
  - random walk spatiotemporal random field
  - AR(1) spatiotemporal random field
- Code demonstration for the AR(1) st model

A quick refresher

We have now seen temporal models

$$z_t = \alpha + z_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{N}\left(0, \sigma_p^2\right), \qquad \text{[process equation]}$$

$$y_t = z_t + \eta_t, \quad \eta_t \sim \mathrm{N}\left(0, \sigma_o^2\right). \qquad \text{[observation equation]}$$

- The time-series follows a random walk with drift or trend term α
- σ_p^2 represents process variance
- σ_o^2 represents obsrvation variance
- ε_t represent process errors
- η_t represent observation errors

Auger-Methe et al. 2021

And we have also seen spatial models

$$\epsilon_s \sim \mathrm{MVN}(0, \Sigma)$$

$$\text{where } \Sigma \text{ is a covariance matrix with:}$$

$$\Sigma_{i,j} = \sigma_{gp}^2 \exp(-\mathrm{distances}_{i,j}/\theta_{gp}),\ \text{if } i \neq j$$

$$\Sigma_{i,j} = \sigma_{gp}^2,\ \text{if } i = j$$

- ϵ_s is a spatially explicit random effect
- θ_{gp} controls how quickly the correlation decays between locations
- σ_{gp}^2 is the marginal variability of the spatial function at all locations
- $\mathrm{distances}_{i,j}$ is a matrix describing Euclidean distance between sample locations

Cressie and Wikle 2011; Anderson and Ward 2019

The problem

- What if our data are indexed not only in space, but also in time?

Cressie and Wikle 2011; Anderson and Ward 2019

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- We might need to account for this additional dependency in our data

Cressie and Wikle 2011; Anderson and Ward 2019

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- What if our data are indexed not only in space, but also in time?
- We might need to account for this additional dependency in our data
  - Why should we care about dependency?

Hurlburt 1984; Cressie and Wikle 2011; Anderson and Ward 2019

The problem

- What if our data are indexed not only in space, but also in time?
- We might need to account for this additional dependency in our data
  - Why should we care about dependency?
- More generally, why go to all of this effort with spatiotemporal data?

Hurlburt 1984; Cressie and Wikle 2011; Anderson and Ward 2019

Spatiotemporal random fields

see also this helpful link from the sdmTMB documentation

IID spatiotemporal random fields

- This is the simplest spatio-temporal random field

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IID spatiotemporal random fields

- This is the simplest spatio-temporal random field

$$\boldsymbol{\varepsilon}_{s,t} \sim \mathrm{MVN}(\mathbf{0}, \Sigma)$$

- Think of $\boldsymbol{\varepsilon}_{s,t}$ as a matrix of sites by years
- The spatially correlated random effects are independent across time-steps

How do we get Σ?

Σ is a covariance matrix with:

$$\Sigma_{i,j} = \sigma_{gp}^2 \exp(-\text{distances}_{i,j}/\theta_{gp}), \text{ if } i \neq j$$

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- We could also get Σ via other correlation kernels (Gaussian, Matern, etc.)

Let's simulate some spatiotemporal fields in Stan

- simulate an iid spatio-temporal random field with an exponential kernel

```r
library(tidyverse)
library(cmdstanr)
library(tidybayes)

set.seed(13)
n_site <- 500 # number of sampling locations for each year
# simulate random x,y site locations:
g <- data.frame(
  easting = runif(n_site, 0, 10),
  northing = runif(n_site, 0, 10)
)

locs <- unique(g)
dist_sites <- as.matrix(dist(locs)) # distances among sites
n_year <- 8 # number of years
```

Let's simulate some spatiotemporal fields in Stan

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```r
# model parameters to simulate
gp_theta <- 1 # Gaussian process scale parameter
gp_sigma <- 0.15 # Gaussian process variance / spatial noise parameter
sim_data <-
  list(
    n_sites = nrow(locs),
    n_year = n_year,
    dist_sites = dist_sites,
    gp_theta = gp_theta,
    gp_sigma = gp_sigma
  )

# compile the model
sim_mod <- cmdstan_model("week10/src/sim_st_iid.stan")
```

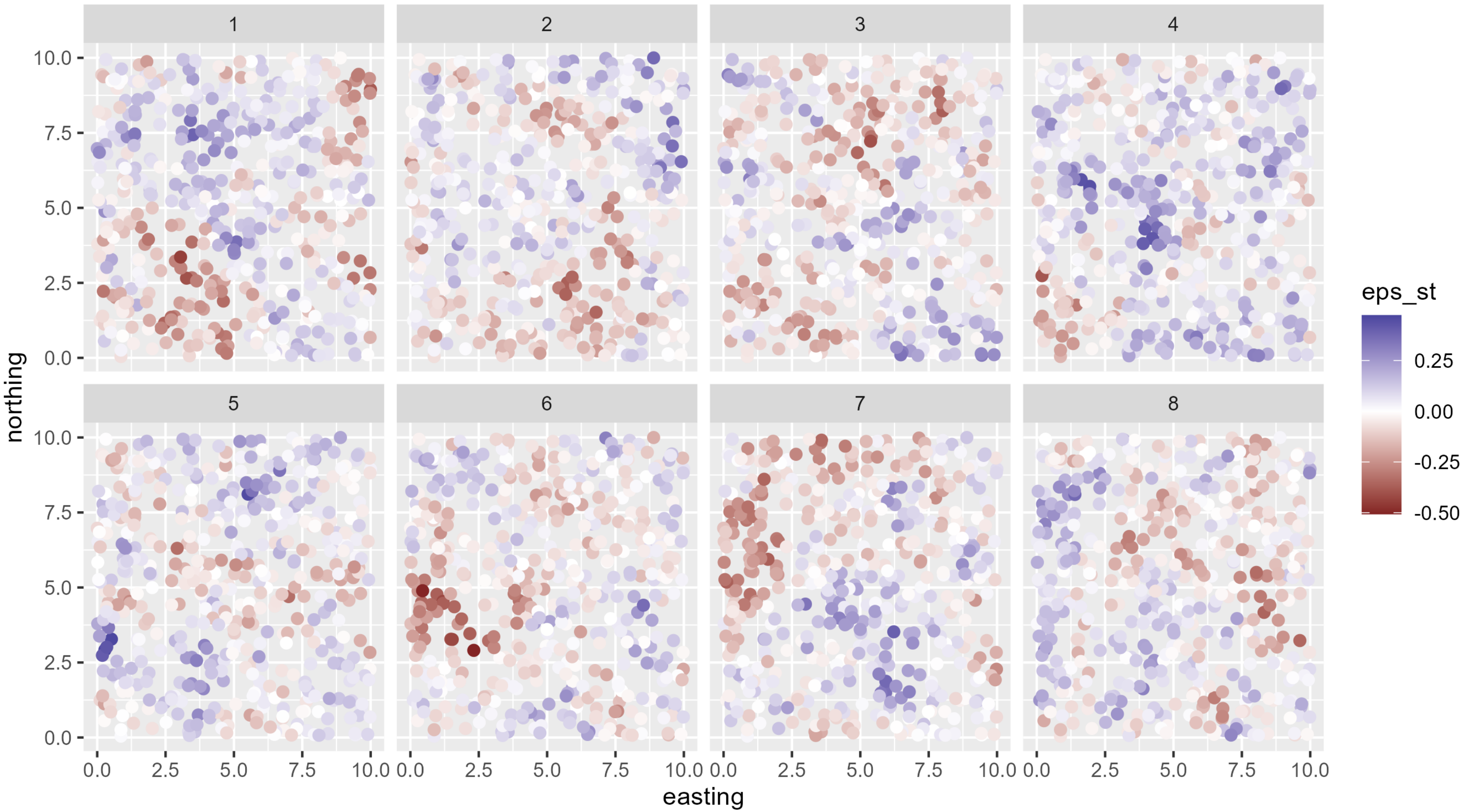
Let's simulate a spatiotemporal field

- simulate an iid spatio-temporal random field with an exponential kernel

```
# simulate data
sim_s <- sim_mod$sample(
  data = sim_data,
  fixed_param = TRUE, iter_warmup = 0, iter_sampling = 1,
  chains = 1, seed = 1
)

# extract the simulated data
eps_st <- matrix(sim_s$draws("eps_st", format = "draws_matrix"),
  nrow = n_site, ncol = n_year
)
```

IID spatiotemporal random field



AR(1) spatiotemporal random field

AR(1) spatiotemporal random field

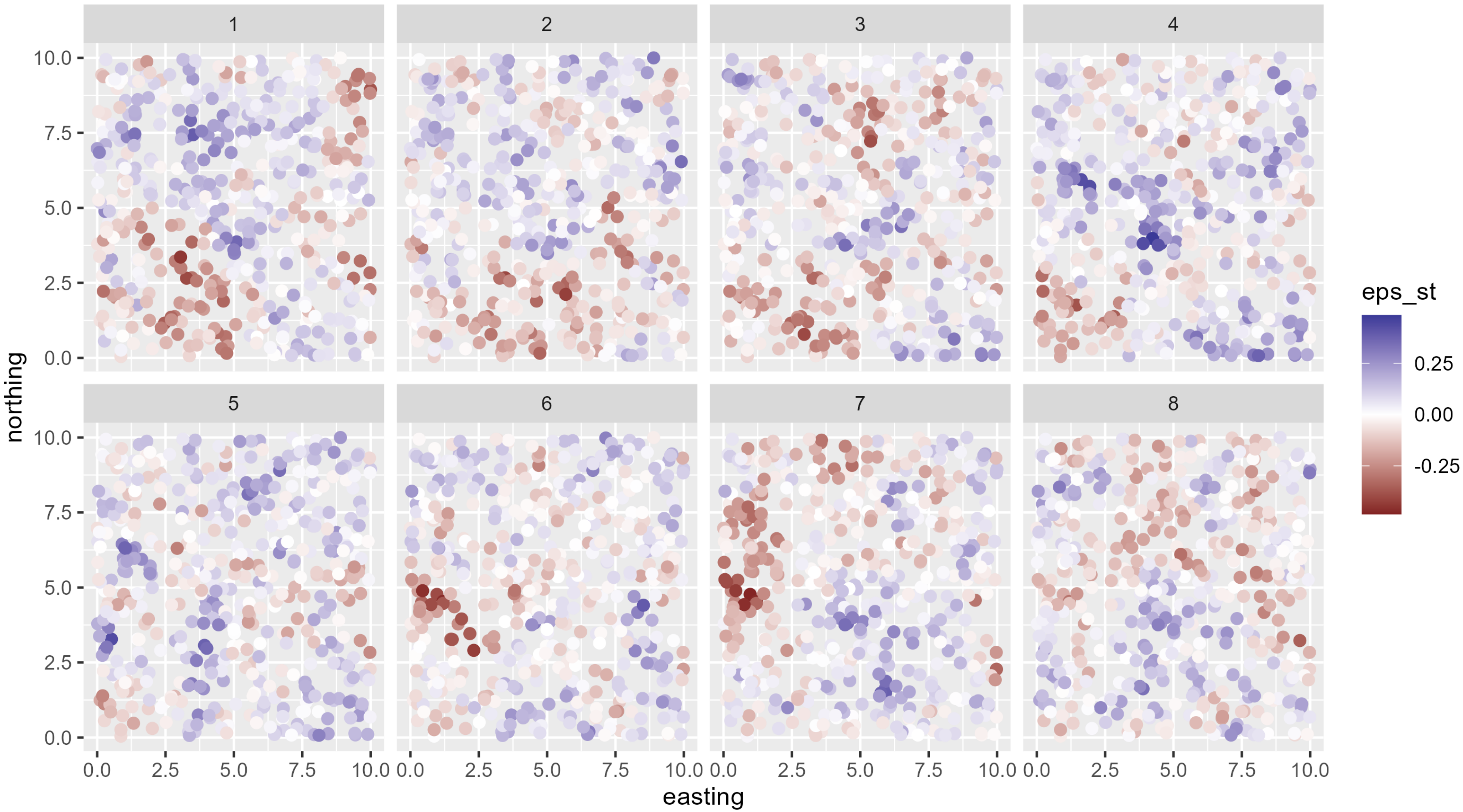
- Autoregressive lag-1 spatiotemporal field

$$\boldsymbol{\delta}_{t=1} \sim \mathrm{MVN}(\mathbf{0}, \Sigma)$$
$$\boldsymbol{\delta}_{t>1} = \rho \boldsymbol{\delta}_{t-1} + \sqrt{1-\rho^2}\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_t \sim \mathrm{MVN}(\mathbf{0}, \Sigma),$$

- This t's random effect is a function of previous random effect
- ρ is correlation in time, must be $[-1, 1]$
- The term $\rho\delta_{t-1} + \sqrt{1-\rho^2}$ scales the spatiotemporal to ensure it represents the steady-state marginal variance

see also this helpful link from the sdmTMB documentation

AR(1) spatiotemporal random field



A random-walk (RW) spatiotemporal field

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$$\boldsymbol{\varepsilon}_{s,t=1} \sim \mathrm{MVN}(\mathbf{0}, \Sigma)$$
$$\boldsymbol{\varepsilon}_{s,t>1} \sim \mathrm{MVN}(\boldsymbol{\varepsilon}_{\boldsymbol{s},\boldsymbol{t-1}}, \Sigma),$$

see also this helpful link from the sdmTMB documentation

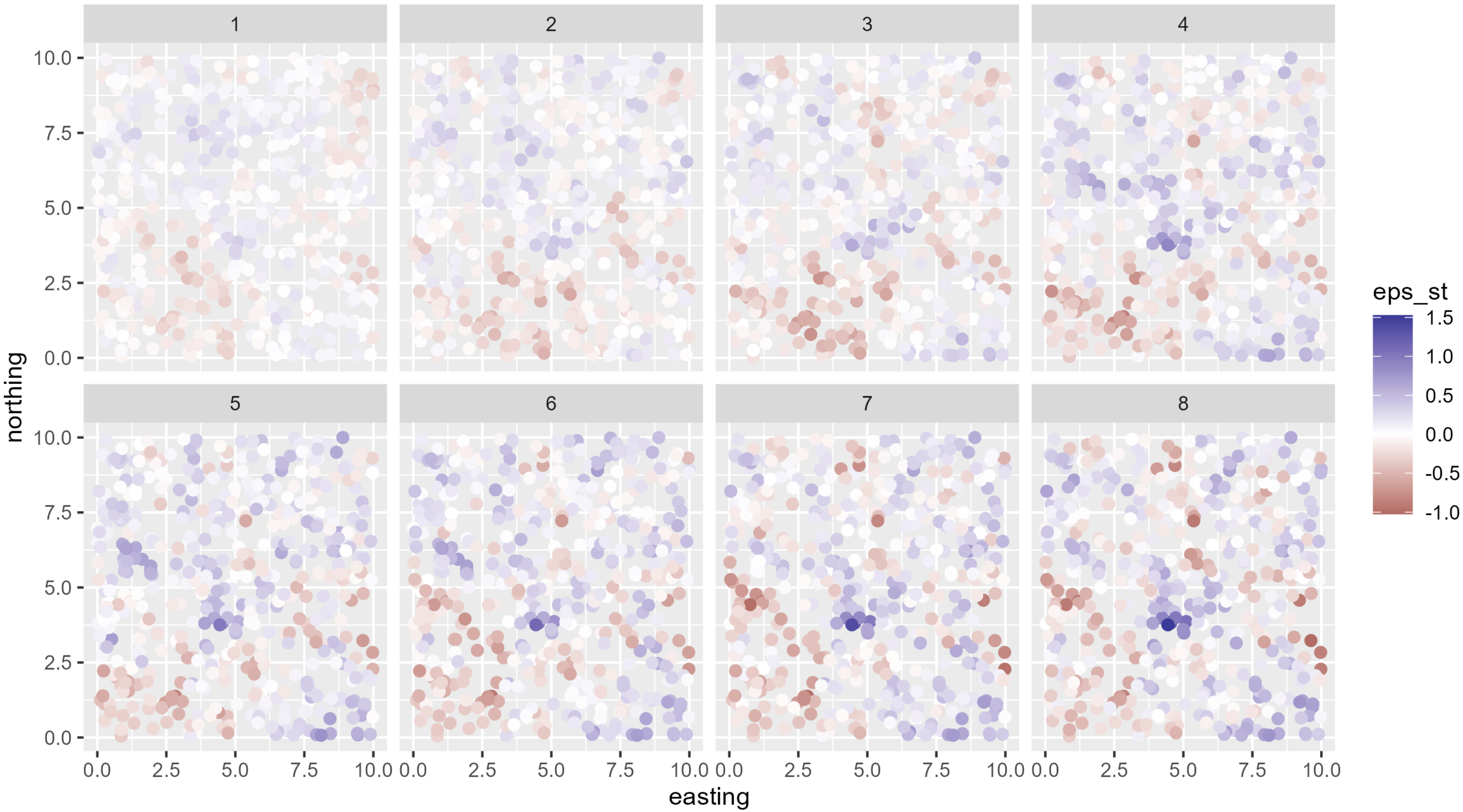
A random-walk (RW) spatiotemporal field

$$\boldsymbol{\varepsilon}_{s,t=1} \sim \mathrm{MVN}(\mathbf{0}, \Sigma)$$
$$\boldsymbol{\varepsilon}_{s,t>1} \sim \mathrm{MVN}(\boldsymbol{\varepsilon}_{\boldsymbol{s},\boldsymbol{t-1}}, \Sigma),$$

- This t's random effect is a function of previous random effect
- No ρ, which means this thing can model nonstationarity
- Note the variance is no longer steady state variance

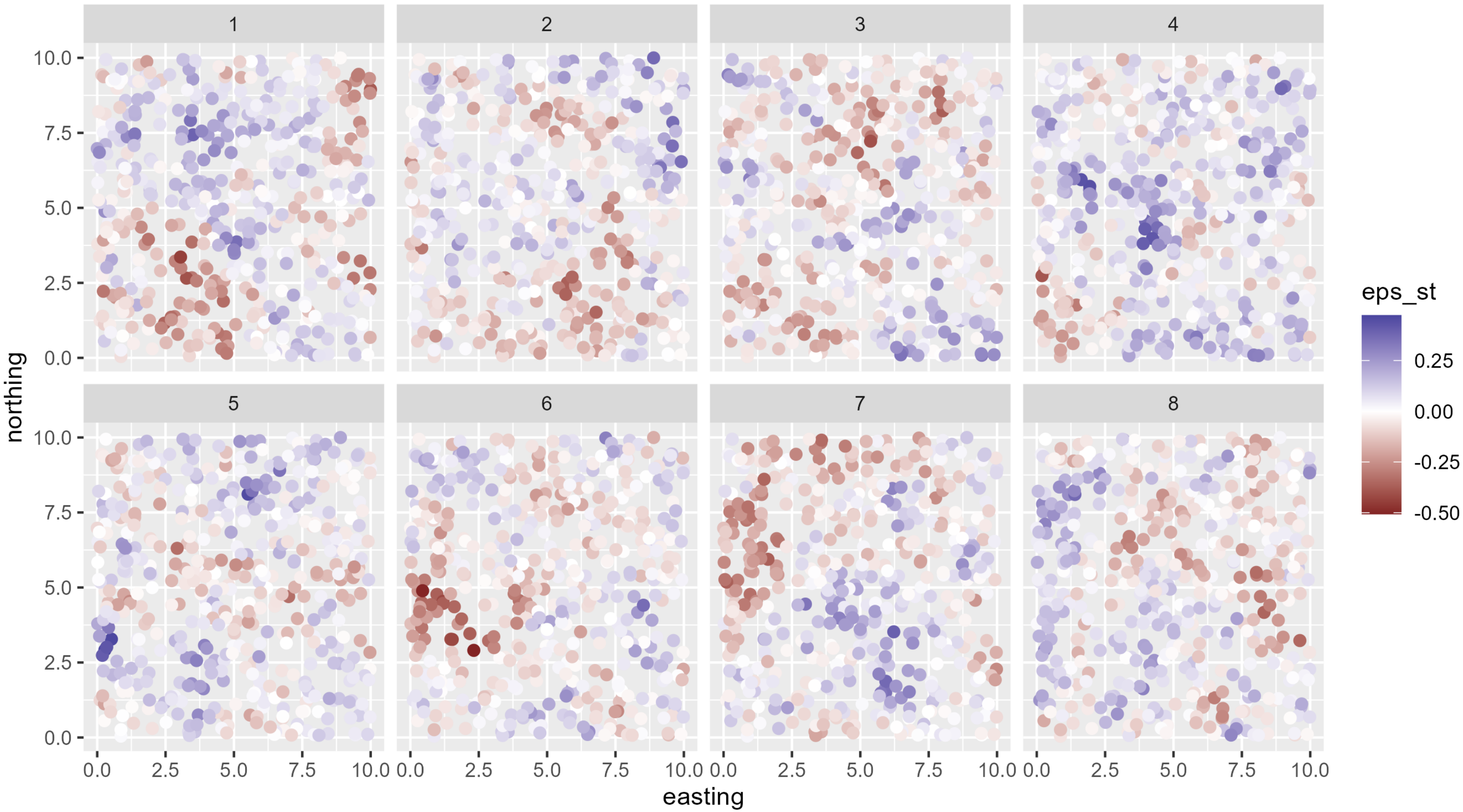
see also this helpful link from the sdmTMB documentation

RW spatiotemporal random field

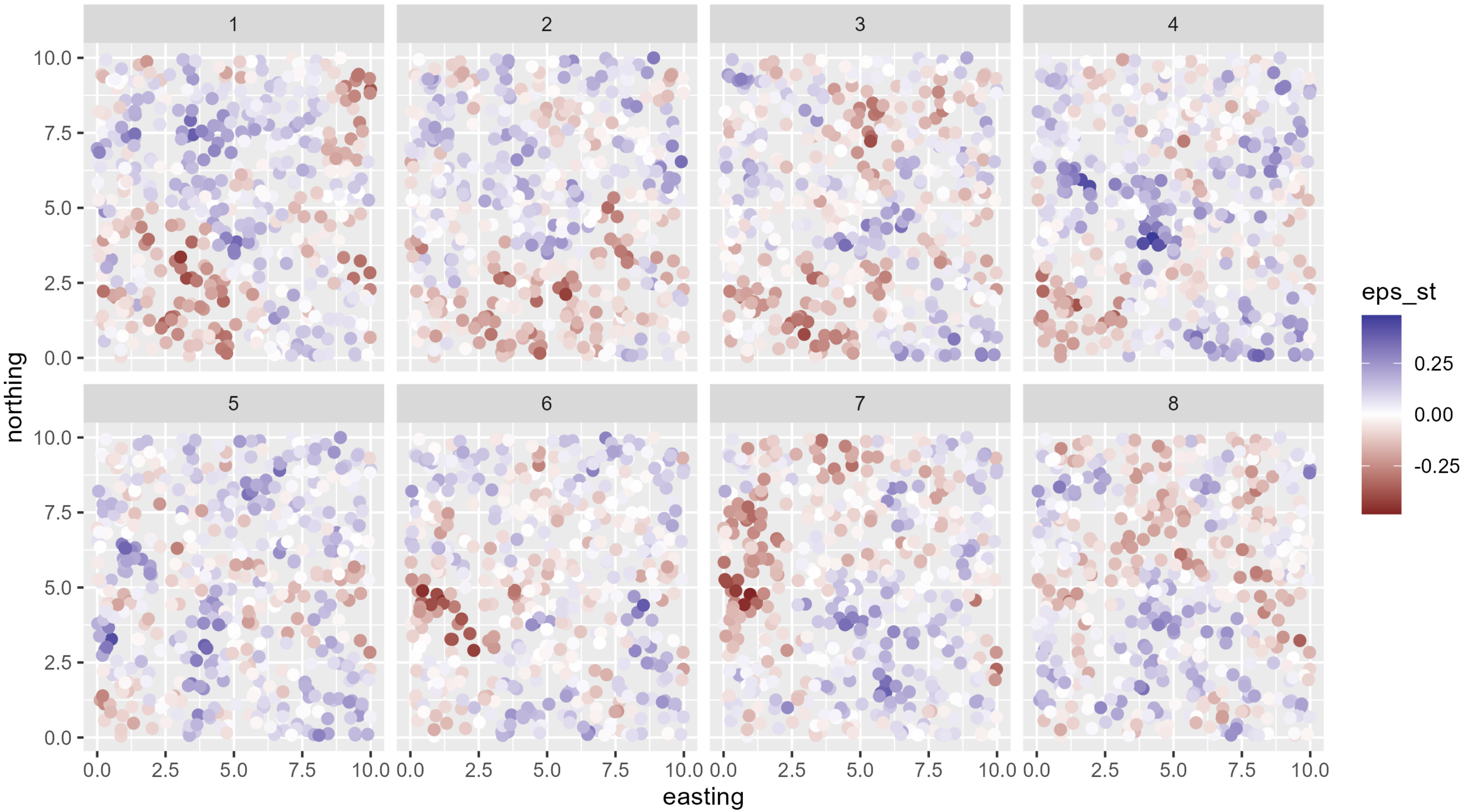


Now for a side by side of all three

IID spatiotemporal random field



AR(1) spatiotemporal random field



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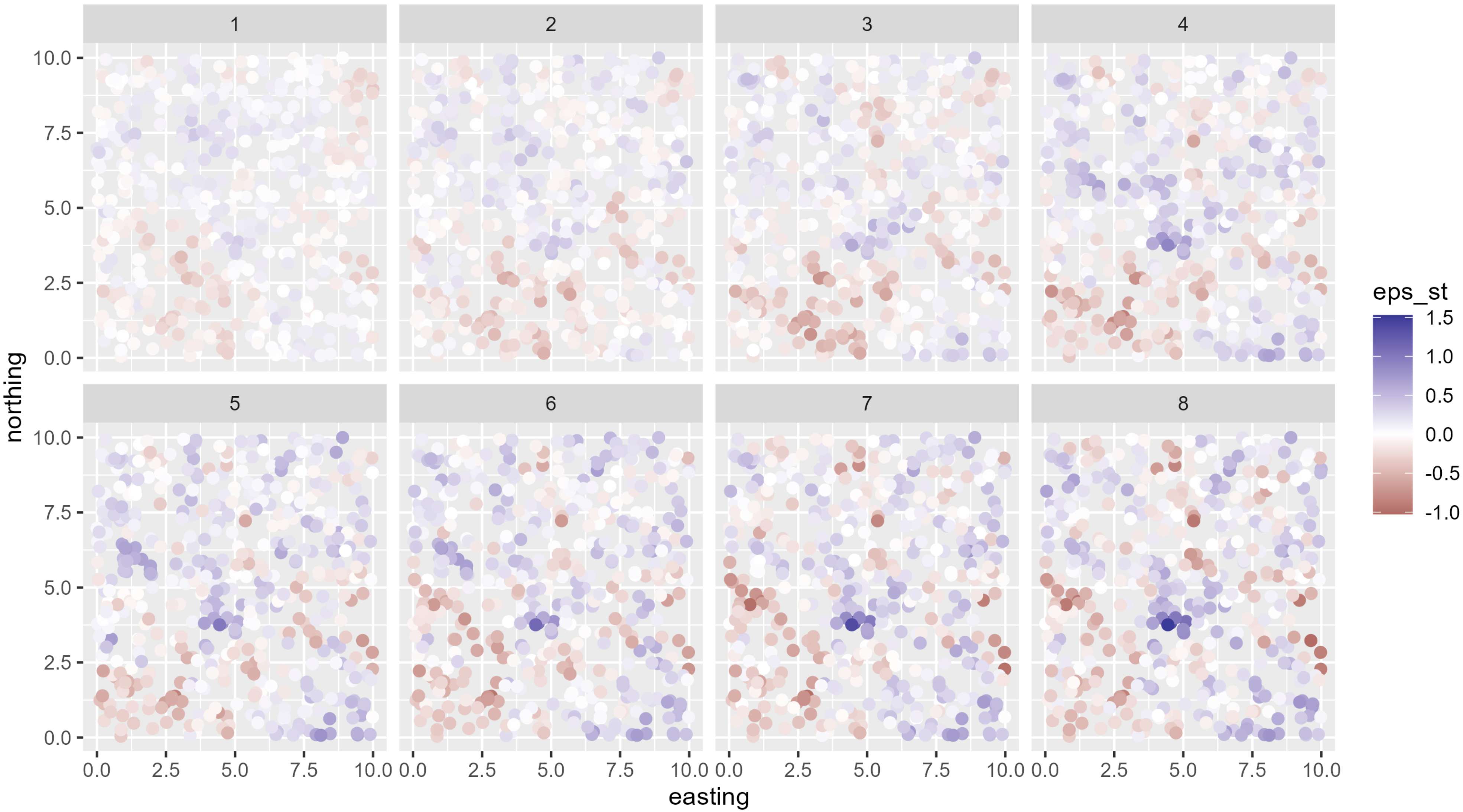

To the code

- Go to the Stan and R scripts

References

- Anderson and Ward. 2019. Black swans in space: modeling spatiotemporal processes with extremes. Ecology.
- Auger-Methe et al. 2021. A guide to state-space modeling of ecological time series. Ecological Monographs.
- Cressie and Wikle 2011. Statistics for spatio-temporal data.
- Hurlburt 1984. Pseudoreplication and the Design of Ecological Field Experiments. Ecological Monographs.