For multi-degree freedom systems, the equation of motion in local coordinate system is given as:



or in matrix from:



These equations are coupled and are cumbersome to solve for more than two degree of freedom systems, thus they are converted to modal coordinates as:



Where the modal mass, modal damping and model stiffness are separately:







And  is the Eigen vector (mode shapes)

The Frequency Response Function (FRF) of the system can be write as:



The element in row *i* and column *l* of  matrix may be given as residues form:



Where *k* – mode number. So the complete transfer function can be represented in the following matrix form:



Where *n* – number of modes.

The mode shapes are found from the estimated residues. Displacement vector can be expressed by its mode shapes and modal transfer functions.



Thus,



Note that the modal mass for mode *k* using the unscaled modal matrix is:



Thus  represents the normalization of each eigenvector with the square root of the modal mass.





This is a convenient way to identify the modal parameters, i.e. mode shapes, modal stiffness and modal damping of the structure.

 - the mass normalized mode shape



Where,



In our case, the residue matrix for specific modes will be of the form:



Where, *k*=1,2,…, *n* for *n* number of modes. When we choose move the hammer to impact all the point on the tool-holder combination and measure the vibration at point 1 where the accelerometer is mounted. The matrix can be written as:



The transfer function  is measured by hitting the structure at point 1 and measuring at point 1, i.e. where the accelerometer is mounted. This is known as the direct transfer function. The transfer function  is measured by hitting the structure at point 2 and measuring at point 1. This is a cross transfer function.





**Where, the *u*11*u*11 means the direct displacement response residues of point 1 contributed by the first mode. E.g. for *u*11, First subscript denote the measured or impact point, and Second subscript denote the modal number.**

If let 



When *ω*=*ωn*1 , leads to a negligible contribution from *ωn*2 and *ωn*3, and the first part become equal to:



So we have:





Similarly, when *ω*=*ωn*2 and *ω*=*ωn*3,









Similarly,



Again, substituting  and then *ω*=*ωn*1, *ω*=*ωn*2 and *ω*=*ωn*3 subsequently, gives:

; ; 

Similarly,

; ; 

; ; 

; ; 

The real and imaginary parts of the FRFs are used to identify the modal parameters. In the imaginary part of the FRF - the peaks correspond to the natural frequencies *ωnk*. The difference between the frequencies of local maximum (*ω*1*k*) and minimum (*ω*2*k*) values on the real part of the FRF is used to determine the modal damping ratio as follows:



The negative peak value from the imaginary part (*Hk*) of the FRF is also used to find the modal stiffness value as follows:



Having obtained the modal stiffness, the modal mass and the modal damping can also be evaluated as:



And



The value of the measurement points are given in the table.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **φ(μm/N)** | R(1031) | R(1104) | I(1074) | R(1914) | R(2092) | I(1961) | R(2546) | R(2783) | I(2617) |
| H11 | 0.4049 | -0.2891 | -0.6433 | 0.1609 | -0.0588 | -0.2138 | 0.1084 | -0.1173 | -0.2180 |
| H12 | 0.2244 | -0.2384 | -0.4380 | 0.0322 | -0.0355 | -0.0879 | 0.0009 | -0.0451 | -0.0491 |
| H13 | 0.1470 | -0.1814 | -0.3077 | -0.0108 | -0.0274 | -0.0228 | -0.0240 | 0.0014 | 0.0154 |
| H14 | 0.0994 | -0.1344 | -0.2158 | -0.0252 | -0.0016 | 0.0166 | -0.0205 | 0.0091 | 0.0300 |
| H15 | 0.0730 | -0.0947 | -0.1566 | -0.0282 | 0.0037 | 0.0266 | -0.0163 | 0.0143 | 0.0328 |

p1 = [1.4111 0.9608 0.6749 0.4734 0.3435]*T*

p2 = [1.7165 0.7057 0.1830 -0.1333 -0.2136]*T*

p3 = [2.0023 0.4510 -0.1414 -0.2755 -0.3013]*T*





1. clc
2. clear
3. close all
5. %% Data Load
6. % load Point1\_Channel1\_CH1.txt
8. a=dir(fullfile('E:\Work\UBC\Courses\MECH592\_Vibration\Project\_III\\*.txt'));
9. **for** ii=1:length(a)
10. load(fullfile('E:\Work\UBC\Courses\MECH592\_Vibration\Project\_III',a(ii).name));
11. end
13. % first mode real peak, valley and image valley
14. % second mode real peak, valley and image valley
15. G11=[0.4049 -0.2891 -0.6433 0.1609  -0.0588 -0.2138 0.1084  -0.1173 -0.2180]\*1e-6;
16. G12=[0.2244 -0.2384 -0.4380 0.0322  -0.0355 -0.0879 0.0009  -0.0451 -0.0491]\*1e-6;
17. G13=[0.1470 -0.1814 -0.3077 -0.0108 -0.0274 -0.0228 -0.0240 0.0014  0.0154]\*1e-6;
18. G14=[0.0994 -0.1344 -0.2158 -0.0252 -0.0016 0.0166  -0.0205 0.0091  0.0300]\*1e-6;
19. G15=[0.0730 -0.0947 -0.1566 -0.0282 0.0037  0.0266  -0.0163 0.0143  0.0328]\*1e-6;
21. w=(101:3500)\*2\*pi;
22. %% First modal
23. f1=1074;
24. w1=2\*pi\*f1;
25. zeta1=(1104-1031)/(2\*f1);
27. k11=-1/(2\*zeta1\*G11(3));
28. m11=k11/(w1^2);
29. frf111=(w1^2/k11)./(w1^2 - w.^2 + 1i\*2\*zeta1\*w1.\*w);
31. k12=-1/(2\*zeta1\*G12(3));
32. m12=k12/(w1^2);
33. frf112=(w1^2/k12)./(w1^2 - w.^2 + 1i\*2\*zeta1\*w1.\*w);
35. k13=-1/(2\*zeta1\*G13(3));
36. m13=k13/(w1^2);
37. frf113=(w1^2/k13)./(w1^2 - w.^2 + 1i\*2\*zeta1\*w1.\*w);
39. k14=-1/(2\*zeta1\*G14(3));
40. m14=k14/(w1^2);
41. frf114=(w1^2/k14)./(w1^2 - w.^2 + 1i\*2\*zeta1\*w1.\*w);
43. k15=-1/(2\*zeta1\*G15(3));
44. m15=k15/(w1^2);
45. frf115=(w1^2/k15)./(w1^2 - w.^2 + 1i\*2\*zeta1\*w1.\*w);
47. %% Second modal
48. f2=1961;
49. w2=2\*pi\*f2;
50. zeta2=(2092-1914)/(2\*f2);
52. k21=-1/(2\*zeta2\*G11(6));
53. m21=k21/(w2^2);
54. frf211=(w2^2/k21)./(w2^2 - w.^2 + 1i\*2\*zeta2\*w2.\*w);
56. k22=-1/(2\*zeta2\*G12(6));
57. m22=k22/(w2^2);
58. frf212=(w2^2/k22)./(w2^2 - w.^2 + 1i\*2\*zeta2\*w2.\*w);
60. k23=-1/(2\*zeta2\*G13(6));
61. m23=k23/(w2^2);
62. frf213=(w2^2/k23)./(w2^2 - w.^2 + 1i\*2\*zeta2\*w2.\*w);
64. k24=-1/(2\*zeta2\*G14(6));
65. m24=k24/(w2^2);
66. frf214=(w2^2/k24)./(w2^2 - w.^2 + 1i\*2\*zeta2\*w2.\*w);
68. k25=-1/(2\*zeta2\*G15(6));
69. m25=k25/(w2^2);
70. frf215=(w2^2/k25)./(w2^2 - w.^2 + 1i\*2\*zeta2\*w2.\*w);
71. %% Third modal
72. f3=2617;
73. w3=2\*pi\*f3;
74. zeta3=(2092-1914)/(2\*f3);
76. k31=-1/(2\*zeta3\*G11(9));
77. m31=k31/(w3^2);
78. frf311=(w3^2/k31)./(w3^2 - w.^2 + 1i\*2\*zeta3\*w3.\*w);
80. k32=-1/(2\*zeta3\*G12(9));
81. m32=k32/(w3^2);
82. frf312=(w3^2/k32)./(w3^2 - w.^2 + 1i\*2\*zeta3\*w3.\*w);
84. k33=-1/(2\*zeta3\*G13(9));
85. m33=k33/(w3^2);
86. frf313=(w3^2/k33)./(w3^2 - w.^2 + 1i\*2\*zeta3\*w3.\*w);
88. k34=-1/(2\*zeta3\*G14(9));
89. m34=k34/(w3^2);
90. frf314=(w3^2/k34)./(w3^2 - w.^2 + 1i\*2\*zeta3\*w3.\*w);
92. k35=-1/(2\*zeta3\*G15(9));
93. m35=k35/(w3^2);
94. frf315=(w3^2/k35)./(w3^2 - w.^2 + 1i\*2\*zeta3\*w3.\*w);
95. %%
96. figure(1)
97. subplot(211)
98. plot(Point\_1\_Channel\_1\_CH1(101:3500,1),Point\_1\_Channel\_1\_CH1(101:3500,2),'b-','linewidth',2);
100. hold on
101. plot(w/2/pi,real(frf111),'r-','linewidth',2);
102. plot(w/2/pi,real(frf211),'r--','linewidth',2);
103. plot(w/2/pi,real(frf311),'r-.','linewidth',2);
105. legend('\fontsize{10}\fontname{Times New Roman}\itReal','location','northwest')
106. set(gca,'FontSize', 10 ,'FontName', 'Times New Roman')
107. title('\fontsize{10}FRF G11')
108. xlabel('\fontsize{10}\fontname{Times New Roman}\it f\rm/ Hz')
109. ylabel('\fontsize{10}\fontname{Times New Roman}\it FRF\rm/ m·N^{-1}')
110. subplot(212)
111. plot(Point\_1\_Channel\_1\_CH1(101:3500,1),Point\_1\_Channel\_1\_CH1(101:3500,3),'b-','linewidth',2);
112. hold on
113. plot(w/2/pi,imag(frf111),'r-','linewidth',2);
114. plot(w/2/pi,imag(frf211),'r--','linewidth',2);
115. plot(w/2/pi,imag(frf311),'r-.','linewidth',2);
117. legend('\fontsize{10}\fontname{Times New Roman}\itImag','location','northwest')
118. set(gca,'FontSize', 10 ,'FontName', 'Times New Roman')
119. set(gcf,'unit','centimeters','position',[0 10 13.53 9.03],'color','white');%??word?13.5,9?
120. xlabel('\fontsize{10}\fontname{Times New Roman}\it f\rm/ Hz')
121. ylabel('\fontsize{10}\fontname{Times New Roman}\it FRF\rm/ m·N^{-1}')
123. figure(2)
124. subplot(211)
125. plot(Point\_2\_Channel\_1\_CH1(101:3500,1),Point\_2\_Channel\_1\_CH1(101:3500,2),'b-','linewidth',2);
126. hold on
127. plot(w/2/pi,real(frf112),'r-','linewidth',2);
128. plot(w/2/pi,real(frf212),'r--','linewidth',2);
129. plot(w/2/pi,real(frf312),'r-.','linewidth',2);
130. legend('\fontsize{10}\fontname{Times New Roman}\itReal','location','northwest')
131. set(gca,'FontSize', 10 ,'FontName', 'Times New Roman')
132. title('\fontsize{10}FRF G12')
133. xlabel('\fontsize{10}\fontname{Times New Roman}\it f\rm/ Hz')
134. ylabel('\fontsize{10}\fontname{Times New Roman}\it FRF\rm/ m·N^{-1}')
135. subplot(212)
136. plot(Point\_2\_Channel\_1\_CH1(101:3500,1),Point\_2\_Channel\_1\_CH1(101:3500,3),'b-','linewidth',2);
137. hold on
138. plot(w/2/pi,imag(frf112),'r-','linewidth',2);
139. plot(w/2/pi,imag(frf212),'r--','linewidth',2);
140. plot(w/2/pi,imag(frf312),'r-.','linewidth',2);
141. legend('\fontsize{10}\fontname{Times New Roman}\itImag','location','northwest')
142. set(gca,'FontSize', 10 ,'FontName', 'Times New Roman')
143. set(gcf,'unit','centimeters','position',[0 10 13.53 9.03],'color','white');%??word?13.5,9?
144. xlabel('\fontsize{10}\fontname{Times New Roman}\it f\rm/ Hz')
145. ylabel('\fontsize{10}\fontname{Times New Roman}\it FRF\rm/ m·N^{-1}')
147. figure(3)
148. subplot(211)
149. plot(Point\_3\_Channel\_1\_CH1(101:3500,1),Point\_3\_Channel\_1\_CH1(101:3500,2),'b-','linewidth',2);
150. hold on
151. plot(w/2/pi,real(frf113),'r-','linewidth',2);
152. plot(w/2/pi,real(frf213),'r--','linewidth',2);
153. plot(w/2/pi,real(frf313),'r-.','linewidth',2);
154. legend('\fontsize{10}\fontname{Times New Roman}\itReal','location','northwest')
155. set(gca,'FontSize', 10 ,'FontName', 'Times New Roman')
156. title('\fontsize{10}FRF G13')
157. xlabel('\fontsize{10}\fontname{Times New Roman}\it f\rm/ Hz')
158. ylabel('\fontsize{10}\fontname{Times New Roman}\it FRF\rm/ m·N^{-1}')
159. subplot(212)
160. plot(Point\_3\_Channel\_1\_CH1(101:3500,1),Point\_3\_Channel\_1\_CH1(101:3500,3),'b-','linewidth',2);
161. hold on
162. plot(w/2/pi,imag(frf113),'r-','linewidth',2);
163. plot(w/2/pi,imag(frf213),'r--','linewidth',2);
164. plot(w/2/pi,imag(frf313),'r-.','linewidth',2);
165. legend('\fontsize{10}\fontname{Times New Roman}\itImag','location','northwest')
166. set(gca,'FontSize', 10 ,'FontName', 'Times New Roman')
167. set(gcf,'unit','centimeters','position',[0 10 13.53 9.03],'color','white');%??word?13.5,9?
168. xlabel('\fontsize{10}\fontname{Times New Roman}\it f\rm/ Hz')
169. ylabel('\fontsize{10}\fontname{Times New Roman}\it FRF\rm/ m·N^{-1}')
171. figure(4)
172. subplot(211)
173. plot(Point\_4\_Channel\_1\_CH1(101:3500,1),Point\_4\_Channel\_1\_CH1(101:3500,2),'b-','linewidth',2);
174. hold on
175. plot(w/2/pi,real(frf114),'r-','linewidth',2);
176. plot(w/2/pi,real(frf214),'r--','linewidth',2);
177. plot(w/2/pi,real(frf314),'r-.','linewidth',2);
178. legend('\fontsize{10}\fontname{Times New Roman}\itReal','location','northwest')
179. set(gca,'FontSize', 10 ,'FontName', 'Times New Roman')
180. title('\fontsize{10}FRF G14')
181. xlabel('\fontsize{10}\fontname{Times New Roman}\it f\rm/ Hz')
182. ylabel('\fontsize{10}\fontname{Times New Roman}\it FRF\rm/ m·N^{-1}')
183. subplot(212)
184. plot(Point\_4\_Channel\_1\_CH1(101:3500,1),Point\_4\_Channel\_1\_CH1(101:3500,3),'b-','linewidth',2);
185. hold on
186. plot(w/2/pi,imag(frf114),'r-','linewidth',2);
187. plot(w/2/pi,imag(frf214),'r--','linewidth',2);
188. plot(w/2/pi,imag(frf314),'r-.','linewidth',2);
189. legend('\fontsize{10}\fontname{Times New Roman}\itImag','location','northwest')
190. set(gca,'FontSize', 10 ,'FontName', 'Times New Roman')
191. set(gcf,'unit','centimeters','position',[0 10 13.53 9.03],'color','white');%??word?13.5,9?
192. xlabel('\fontsize{10}\fontname{Times New Roman}\it f\rm/ Hz')
193. ylabel('\fontsize{10}\fontname{Times New Roman}\it FRF\rm/ m·N^{-1}')
195. figure(5)
196. subplot(211)
197. plot(Point\_5\_Channel\_1\_CH1(101:3500,1),Point\_5\_Channel\_1\_CH1(101:3500,2),'b-','linewidth',2);
198. hold on
199. plot(w/2/pi,real(frf115),'r-','linewidth',2);
200. plot(w/2/pi,real(frf215),'r--','linewidth',2);
201. plot(w/2/pi,real(frf315),'r-.','linewidth',2);
202. legend('\fontsize{10}\fontname{Times New Roman}\itReal','location','northwest')
203. set(gca,'FontSize', 10 ,'FontName', 'Times New Roman')
204. title('\fontsize{10}FRF G15')
205. xlabel('\fontsize{10}\fontname{Times New Roman}\it f\rm/ Hz')
206. ylabel('\fontsize{10}\fontname{Times New Roman}\it FRF\rm/ m·N^{-1}')
207. subplot(212)
208. plot(Point\_5\_Channel\_1\_CH1(101:3500,1),Point\_5\_Channel\_1\_CH1(101:3500,3),'b-','linewidth',2);
209. hold on
210. plot(w/2/pi,imag(frf115),'r-','linewidth',2);
211. plot(w/2/pi,imag(frf215),'r--','linewidth',2);
212. plot(w/2/pi,imag(frf315),'r-.','linewidth',2);
213. legend('\fontsize{10}\fontname{Times New Roman}\itImag','location','northwest')
214. set(gca,'FontSize', 10 ,'FontName', 'Times New Roman')
215. set(gcf,'unit','centimeters','position',[0 10 13.53 9.03],'color','white');%??word?13.5,9?
216. xlabel('\fontsize{10}\fontname{Times New Roman}\it f\rm/ Hz')
217. ylabel('\fontsize{10}\fontname{Times New Roman}\it FRF\rm/ m·N^{-1}')
219. %%
220. u11=sqrt(-2\*zeta1\*w1^2\*G11(3));
221. u12=sqrt(-2\*zeta2\*w2^2\*G11(6));
222. u13=sqrt(-2\*zeta3\*w3^2\*G11(9));
224. u21=-2\*zeta1\*w1^2\*G12(3)/u11;
225. u22=-2\*zeta2\*w2^2\*G12(6)/u12;
226. u23=-2\*zeta3\*w3^2\*G12(9)/u13;
228. u31=-2\*zeta1\*w1^2\*G13(3)/u11;
229. u32=-2\*zeta2\*w2^2\*G13(6)/u12;
230. u33=-2\*zeta3\*w3^2\*G13(9)/u13;
232. u41=-2\*zeta1\*w1^2\*G14(3)/u11;
233. u42=-2\*zeta2\*w2^2\*G14(6)/u12;
234. u43=-2\*zeta3\*w3^2\*G14(9)/u13;
236. u51=-2\*zeta1\*w1^2\*G15(3)/u11;
237. u52=-2\*zeta2\*w2^2\*G15(6)/u12;
238. u53=-2\*zeta3\*w3^2\*G15(9)/u13;
240. p1=[u11 u21 u31 u41 u51]'
241. p2=[u12 u22 u32 u42 u52]'
242. p3=[u13 u23 u33 u43 u53]'
244. figure(10)
245. y=[0 55 90 130 165];
246. plot(p1,y,'r','linewidth',2)
247. hold on
248. plot(p2,y,'g','linewidth',2)
249. plot(p3,y,'b','linewidth',2)
250. plot([0,0],[0,165],'k','linewidth',2)
251. legend('mode1','mode2','mode3')