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Note

On the space chromatic number ☆

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Abstract

The chromatic number of the space is the minimum number of colors needed to color all points of the Euclidean space so that no two points of the same color are at unit distance. We show that this number is at least 6, improving the best-known previous bound of 5.

1. Introduction

The chromatic number of the plane is the minimum number of colors needed to color all points of the plane so that no two points of the same color are at unit distance. (We refer to colorings that satisfy this property as *proper*.)

Alternatively, a *unit distance graph* in the d-dimensional Euclidean space ($\mathbb{E}^d$) is a graph that can be embedded in $\mathbb{E}^d$, so that the distance between each two adjacent vertices is 1.

Assuming the Axiom of Choice (or the weaker Erdős–De Bruijn Theorem [5]), the chromatic number of the plane is the maximal chromatic number of a finite unit distance graph in $\mathbb{E}^2$.

The problem of finding the chromatic number of the plane is also known as the 'Hadwiger–Nelson problem' or 'a second four-color problem'. It was originated around 1950 by Edward Nelson, first published in [13] and remains open till today! For a historical survey on this problem see [30,3,18], cf. also [21,36].

The best-known lower and upper bounds on the chromatic number of the plane are 4 and 7, where the lower bound is due to Nelson and follows from the fact that the Spindle graph in Fig. 1 is a unit distance graph in the plane with chromatic

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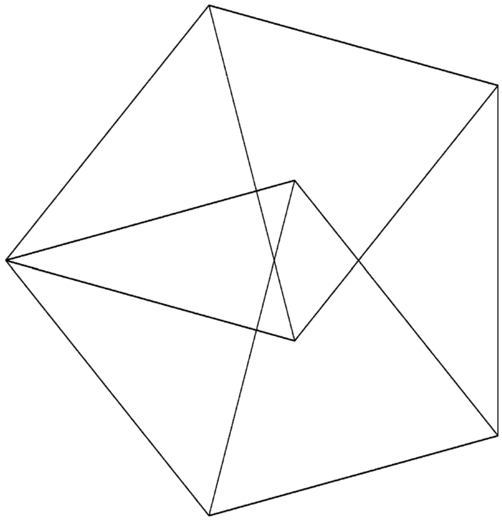

Fig. 1. The Spindle(Moser) graph, a minimum unit distance graph in $\mathbb{E}^2$ with chromatic number 4.

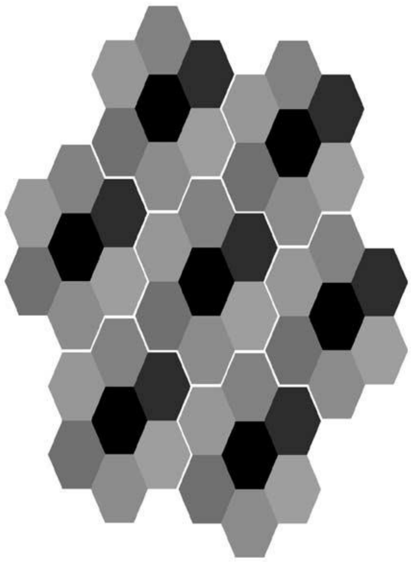

Fig. 2. A proper 7-coloring of the plane.

number 4. The upper bound is due to John Isbell and follows from the 7-coloring of the plane that is seen in Fig. 2, which is based on a regular hexagon packing of the plane with each hexagon of diameter slightly less than 1.

Hadwiger [14] proved, as a special case of a result for higher-dimensional Euclidean spaces, that if the points of the plane are partitioned into five congruent closed sets, then at least one of the sets contains, for every positive real number d, a pair of points of distance d. The same hexagon packing construction mentioned above appears there as an example that this does not hold, in general, for seven closed sets, cf. [15, p. 24].

Following a suggestion of Erdős, Heppes investigated the problem of finding the polychromatic number of the plane, that is, the minimum number of colors needed to color all points of the plane so that no color class realizes all distances. The best known lower and upper bounds here are 4 and 6, where the lower bound is due to [25] and the upper bound to S.B. Stechkin. Woodall [36] gave another proof of the upper bound, where the covering of the plane uses six closed sets. Soifer [33] showed that it is possible to find a 6-coloring such that all the forbidden distances but one are 1, cf. also [16,17,32,34]. For a historical survey see [31].

The same problems can be asked for any metric space. The chromatic number of the d-dimensional sphere of radius r is the minimum number of colors possible in a coloring of the points of the surface of S_r^d in which any two points at unit (chordal) distance, apart are colored differently. It is easily seen that $\chi(S_r^d)$ is 1 for $r < \frac{1}{2}$ and 2 when $r = \frac{1}{2}$. Simmons [27–29] showed that $3 \leqslant \chi(S_r^2) \leqslant 4$ where

$$\tfrac{1}{2} < r \leqslant \sqrt{(3-\sqrt{3})/2};$$

that $3 \leqslant \chi(S_r^2) \leqslant 5$ where

$$\sqrt{(3-\sqrt{3})/2} < r \leqslant \tfrac{1}{\sqrt{3}}$$

(the upper bound is due to E. Straus); that $4 \leqslant \chi(S_r^2)$ for $r \geqslant \frac{1}{\sqrt{3}}$, disproving a conjecture of Erdős for $r = \frac{1}{\sqrt{3}}$ and that $4 = \chi(S^2_{1/\sqrt{2}})$. Lovász [23] showed that $n \leqslant \chi(S_r^{n-1})$ for all $r > \frac{1}{2}$, answering a question of Erdős and R.L. Graham posed at the Hungarian Combinatorial Conference in Eger, Hungary, in 1981. For a survey of results cf. [18,24].

The best known lower and upper bounds for the chromatic number of the space are 5 and 21 [25,1], cf. also [24]. These bounds are easy generalizations of the bounds in the plane (see [35]). The best known general bounds for the chromatic number of $\mathbb{E}^d$ are

$$(1+\mathrm{o}(1))\left(\frac{6}{5}\right)^n \leqslant \chi(\mathbb{E}^d) \leqslant (3+\mathrm{o}(1))^n.$$

The upper bound was obtained by [22], and the lower bound by [12], showing that $\chi(\mathbb{E}^d)$ grows exponentially with n (see also [6,7,35]).

The analogue problems where $\mathbb{E}^d$ is replaced by $\mathbb{Q}^d$ were extensively studied as well. Woodall [36] had proved that $\chi(\mathbb{Q}^2) = 2$, Benda and Perles proved in 1977 that $\chi(\mathbb{Q}^3) = 2$ and that $\chi(\mathbb{Q}^4) = 4$ though the manuscript was never published. A copy can be found in [19]. Chilakamarri [2] proved that $\chi(\mathbb{Q}^5) \geqslant 6$. For detailed results on the subject see [37].

Scheinerman and Ullman [26] improves the bounds on the fractional chromatic number of the plane showing that

$$3.555 \leqslant \chi_f(\mathbb{E}^2) \leqslant 4.36.$$

Falconer [11] showed that if we further require the coloring to be measurable (as a function from $\mathbb{E}^3$ to the integers) then there is no proper $d+2$ coloring of $\mathbb{E}^d$ where $d \geqslant 2$. In particular, there are no proper measurable 4 colorings of the plane and no proper measurable 5 colorings of the space($\mathbb{E}^3$).

Juhász [20] showed that for an arbitrary size 4 configuration of points K in the Euclidean plane and any {red,blue} coloring of the plane there exists either an all-red configuration congruent to K or two blue points unit distance, apart. This settled a question raised in [9]. In the opposite direction a configuration of size 12 and an appropriate 2-coloring were given for which this property does not hold. Csizmadia and Tóth [4] later improved this to a size 8 configuration.

Here we improve the lower bound for the chromatic number of the space and give a new simple proof for Falconer's result in the space. As a corollary, we prove a simple 2-coloring Ramsey-type result for the space.

The connection between the chromatic number and such Ramsey-type results was first formalized in [8–10].

2. The main theorem

Theorem 1. *The chromatic number of the space is at least* 6.

Note that this result would follow immediately from an improvement of the lower bound for the chromatic number of the unit sphere, but this remains open.

Let s,t be two arbitrary points unit distanced apart in $\mathbb{E}^3$ and let $C = C_{\{s,t\}}$ be the circle of points that are unit distance from both s and t. Fix a sequence of distinct points (p, p_1, p_2, q) on C that satisfy $|p - p_1| = |p_1 - p_2| = |p_2 - q| = 1$. Now let τ be a rotation of the space around the line $l = l(p,q)$ defined by the condition $|\bar{p}_i - p_i| = 1$ for $i = 1,2$, where we denote $\tau(x)$ by $\bar{x}$. Let G be the unit distance graph over $\{s,t,p,p_1,p_2,q,\bar{s},\bar{t},\bar{p}_1,\bar{p}_2\}$ as seen in Fig. 3. Given a vertex coloring c of G, we call an edge (p,q) of G monochromatic if $c(p)$ and $c(q)$ are the same.

Lemma 1. *Every proper* 5 *coloring* c *of* G *satisfies*:

- *Whenever* (p,q) *is monochromatic then neither* (p,p_2) *nor* (p_1,q) *as well as neither* $(\bar{p},\bar{p}_2)$ *nor* $(\bar{p}_1,\bar{q})$ *are monochromatic.*
- *Exactly one among* $\{(p,q),(p,p_2),(p_1,q)\}$ *as well as one among* $\{(\bar{p},\bar{q}),(\bar{p},\bar{p}_2),(\bar{p}_1,\bar{q})\}$ *is monochromatic.*

Proof. Assume G is properly 5 colored. Since p,p_1,p_2,q are all unit distance apart from two fixed points which themselves are unit distance apart, we know that some points in p,p_1,p_2,q share their color. Since $\{p,p_1\},\{p_1,p_2\},\{p_2,q\}$ are all unit distance apart, we conclude that at least one of the three equalities $c(p)=c(q)$, $c(p)=c(p_2)$, $c(p_1)=c(q)$ is satisfied. Similar considerations give that at least one of

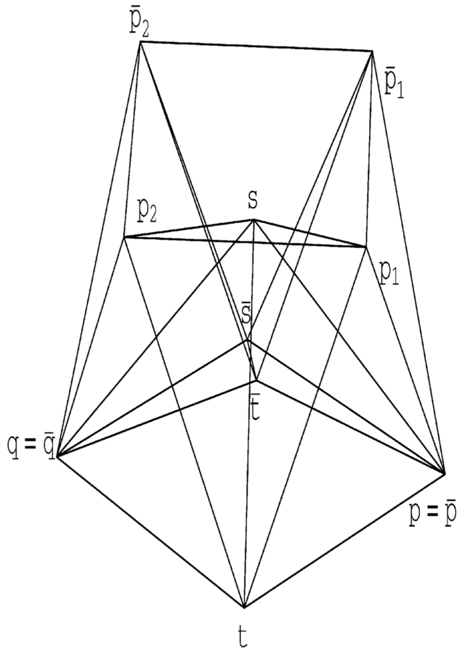

Fig. 3. The basic construction.

the edges $(\bar{p},\bar{q}),(\bar{p},\bar{p}_2),(\bar{p}_1,\bar{q})$ is monochromatic. If (p,q) is monochromatic then none of $(p,p_2),(p_1,q),(\bar{p},\bar{p}_2),(\bar{p}_1,\bar{q})$ is monochromatic, for c was proper and the lemma follows. Otherwise at least two of $(p,p_2),(p_1,q),(\bar{p},\bar{p}_2),(\bar{p}_1,\bar{q})$ are monochromatic but surely neither $(c(p)=c(p_2)$ and $c(\bar{p})=c(\bar{p}_2))$ nor $(c(p_1)=c(q)$ and $c(\bar{p}_1)=c(\bar{q}))$ can be satisfied. The lemma follows. □

Lemma 2. *If $\mathbb{E}^3$ is properly 5 colored and $c(p)\neq c(q)$ then the rotation of the circle $C_{\{p_1,\bar{p}_1\}}$ around the line $l(p,q)$, as shown in* Fig. 4, *avoids the color $c(q)$.*

Proof. Assume $\mathbb{E}^3$ is indeed properly 5 colored and $c(p)\neq c(q)$. Then using Lemma 1, either $c(q)=c(p_1)$ or $c(q)=c(\bar{p}_1)$ is satisfied. It follows that the circle $C_{\{p_1,\bar{p}_1\}}$ avoids the color $c(q)$. Applying the argument to rotations of $C_{\{p_1,\bar{p}_1\}}$ around the line $l(p,q)$ gives the required result. □

It is easy to check that $|p-q|=\frac{5}{3}$ and $|p-p_2|=|p_1-q|=\sqrt{\frac{8}{3}}$.

Proof of the Main Theorem. Fix a pair of points q,q_1 distanced $\frac{5}{3}$ apart in $\mathbb{E}^3$ and let C denote the circle of points p that satisfy $|q_1-p|=1,|q-p|=|q_1-q|$. If the circle C avoids the color of q, i.e. $\forall p\in C\ c(p)\neq c(q)$, then apply Lemma 2 for each pair of points (q,p), $p\in C$.

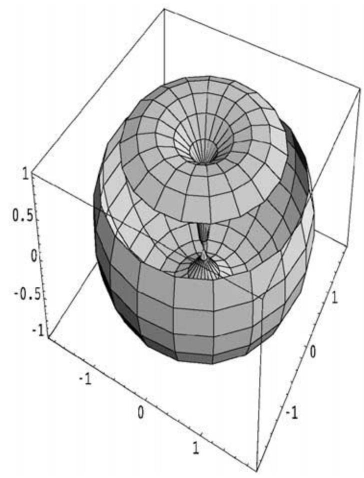

Fig. 4. A rotated circle, an inner view.

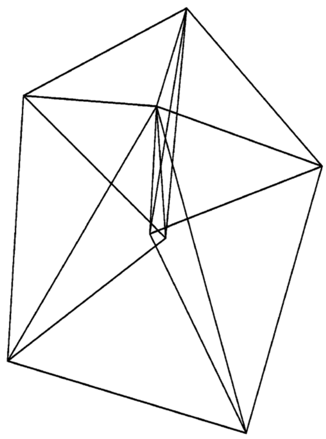

Fig. 5. The three-dimensional Spindle or Raĭskiĭ Spindle graph.

The rotation body that is obtained contains a ball of diameter $\sqrt{3}$ and, therefore, contains the three-dimensional spindle shown in Fig. 5.

It is easy to verify that this graph has no proper 4 coloring. (Actually this is a smallest unit distance graph in $\mathbb{E}^3$ with respect to this property.)

If C does not avoid the color q then we can rotate q_1 keeping q fixed so that $c(q_1) = c(q)$. This reduces the problem to the previous case. The theorem follows. $\square$

Corollary 1. *For every* $\{0,1\}$*-coloring of* $\mathbb{E}^3$ *and every subsets* A_2, A_5 *of* $\mathbb{E}^3$ *of sizes* 2 *and* 5, *respectively, either there is a translate of* A_5 *colored* 0 *or there is an isometric copy of* A_2 *colored* 1.

Proof. Assume that the statement is false. Without loss of generality there is a $\{0,1\}$-coloring f and a subset A_5 of size 5 whose translates always contain a point colored 1 and no two points unit distance apart are both colored 1. We claim then that the sets $\{\{p-a \mid f(p)=1\} \mid a \in A\}$ form the color sets of a proper 5-multi-coloring of the space. (If the point p does not belong to any of the color sets then A_5 translated by p, i.e. $\{p+a \mid a \in A\}$ is colored 0.) This is a contradiction to Theorem 1 and the corollary follows. □

Remark. The proof given above can be used to obtain an explicit unit distance graph of the space with chromatic number >5. This graph has less than 400 vertices. The author believes it is possible to reduce the size of such a graph.

3. Further remarks

The special case where the coloring is measurable admits a simpler proof.

Theorem 2. *There is no measurable proper* 5 *coloring of the space.*

Proof. Assume that such a measurable coloring exists. Fix a ball B of radius $R \geqslant 1$. Now, randomly choose a point p uniformly over B with respect to Lebesgue measure. Choose another point q uniformly over the sphere S_r centered around p with radius r where $R \gg r > 0$. Denote the probability that p and q have the same color by ρ_r. Using Lemma 1 we know that $2\rho_{\sqrt{8/3}} + \rho_{5/3}$ is at least 1. To see this, let F denote the finite point configuration $\{p, p_1, p_2, \bar{p}_1, \bar{p}_2, q\}$ as the one used in Fig. 3. Since the coloring c was measurable, the expectations of the indicator random variables for the events (p, p_2) *is monochromatic*, (p_1, q) *is monochromatic* and (p, q) *is monochromatic* is at least 1. This expected value is simply $2\rho_{\sqrt{8/3}} + \rho_{5/3}$.

On the other hand, surely for any $r \geqslant 1$, $\rho_r \leqslant \frac{2}{7}$. Simply embed the spindle graph in Fig. 1 on S_r with center q and observe that the sum of expectations of the indicator random variables for the events (p_i, q) *is monochromatic* is at most $\frac{2}{7}$.

This leads to a contradiction, and the result follows. □

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