

# Applications of laser-wakefield scaling laws

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## Abstract

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We apply analytical scaling laws to the parameters of the FLAME laser system, in order to estimate the maximum obtainable electron energy and total charge in a laser-wakefield acceleration scenario. We also roughly predict the expected betatron radiation spectrum, emitted by the accelerated electrons.

## 1. FLAME parameters

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The FLAME Laser System (FLAME) has a wavelength  $\lambda_L = 800$  nanometers. The operating laser pulse energy can range from 0.03 J up to 3 J on target.<sup>1</sup> Inside the laser focus, the energy is roughly 40% of that. The pulse duration can range from 28 fs up to a maximum of 300 fs. The beam waist is  $w_0 = 15 \mu\text{m}$ . We can take the pulse duration to be  $\tau_L = 30$  fs at FWHM in intensity.

When using a gas-jet target, available plasma density range is between  $1$  and  $5 \times 10^{18} \text{ cm}^{-3}$ , corresponding to an acceleration length between 5 and 1 mm.

## 2. Electron acceleration estimate

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For optimal acceleration, we impose the condition that the dephasing length should be equal to the pump depletion length  $L_{\text{dephasing}} = L_{\text{depletion}}$  [1].

$$L_{\text{dephasing}} = \frac{4}{3} \left( \frac{\omega_L}{\omega_p} \right)^2 \frac{\sqrt{a_0}}{k_p} = L_{\text{depletion}} = \left( \frac{\omega_L}{\omega_p} \right)^2 c \tau_L$$

In practical units, we have  $\lambda_p[\mu\text{m}] = 3.34 \times 10^{10} (n_p[\text{cm}^{-3}])^{-1/2}$ , therefore  $k_p[\mu\text{m}^{-1}] = 2\pi/\lambda_p[\mu\text{m}] = 1.88 \times 10^{-10} (n_p[\text{cm}^{-3}])^{1/2}$  and  $\omega_p[\text{fs}^{-1}] = 5.64 \times 10^{-11} (n_p[\text{cm}^{-3}])^{1/2}$ . This allows us to obtain the ratio  $a_0/n_p$  from above.

$$\frac{a_0}{n_p[\text{cm}^{-3}]} = 1.79 \times 10^{-21} (\tau_L[\text{fs}])^2$$

### Plasma density and normalized vector potential

If we want to use any external guiding (eg via capillaries), then we also need to impose the condition for self-guided propagation of the laser:

$$a_0 \geq \left( \frac{n_c}{n_p} \right)^{1/5}$$

with the critical density  $n_c[\text{cm}^{-3}] = 1.12 \times 10^{21} / (\lambda_L[\mu\text{m}])^2$ , which is  $1.75 \times 10^{21} \text{ cm}^{-3}$  in our case. Substituting above and taking the lower limit, we get

$$n_p[\text{cm}^{-3}] = \frac{1.12 \times 10^{21}}{a_0^5 (\lambda_L[\mu\text{m}])^2}$$

Substituting  $n_p$  in the ratio  $a_0/n_p$  from above, we finally get:

$$a_0 \approx 2^{1/6} \left( \frac{\tau_L [\text{fs}]}{\lambda_L [\mu\text{m}]} \right)^{1/3}$$

and

$$n_p [\text{cm}^{-3}] = 6.29 \times 10^{20} \frac{1}{(\lambda_L [\mu\text{m}])^{1/3} (\tau_L [\text{fs}])^{5/3}}$$

For FLAME parameters, we get  $a_0 \approx 3.7$  and  $n_p \approx 2.3 \times 10^{18} \text{cm}^{-3} = 13 \times 10^{-4} n_c$ . For  $a_0 \geq 4 - 5$  we also get self-injection from pure Helium. Helium has the ionization energies 24.59 eV ( $\text{He}^+$ ) and 54.42 ( $\text{He}^{2+}$ ), corresponding to laser intensities  $1.4 \times 10^{15}$ , respectively  $8.8 \times 10^{15} \text{W/cm}^2$  [2], and will therefore be easily ionized by the laser prepulse. Knowing the density, we get  $\omega_p = 8.63 \times 10^{-2} \text{fs}^{-1}$ , and more importantly,  $\omega_p^{-1} = 11.59 \text{fs}$ , which is the same order of magnitude as the pulse duration  $\tau_L = 30 \text{fs}$ . For  $\tau_L \gg \omega_p^{-1}$  (eg.  $\tau_L = 600 \text{fs}$ ), one would either be in the SMLWFA or DLA regime, depending on the value of  $a_0$ .

### Beam waist

We also need to match the beam waist  $w_0$  of the focused Gaussian laser pulse to the plasma, giving the condition  $w_0 k_p = 2\sqrt{a_0}$ , so

$$w_0 [\mu\text{m}] = 1.06 \times 10^{10} \left( \frac{a_0}{n_p [\text{cm}^{-3}]} \right)^{1/2} = 44.84 \frac{\tau_L [\text{fs}]}{100}$$

For FLAME, we get  $w_0 \approx 13 \mu\text{m}$  at  $1/e^2$  intensity, so FWHM (1/2 intensity) =  $w_0 \sqrt{2 \ln 2} \approx 18 \mu\text{m}$ .

### Laser energy, power and intensity

We now estimate the required laser energy  $\varepsilon_L$ . The peak laser intensity in the focal plane is

$$I_L [10^{18} \text{W/cm}^2] \approx 6 \times 10^4 \frac{\varepsilon_L [\text{J}]}{\tau_L [\text{fs}] (w_0 [\mu\text{m}])^2}$$

so for our parameters  $I_L \approx 2.96 \times 10^{19} \text{W/cm}^2$ . For a relative scale, the atomic Coulomb field is on the order of  $10^{14} \text{W/cm}^2$  and relativistic effects become important for laser intensities above  $10^{17} \text{W/cm}^2$  ( $a_0 \geq 1$ ), while QED effects such as radiation reaction only become important for intensities beyond  $\sim 2 \times 10^{21} \text{W/cm}^2$ . We know that  $a_0 = 0.855 \lambda_L [\mu\text{m}] (I_L [10^{18} \text{W/cm}^2])^{1/2}$ , so

$$\varepsilon_L [\text{J}] = 2.28 \times 10^{-5} a_0^2 (w_0 [\mu\text{m}])^2 \frac{\tau_L [\text{fs}]}{(\lambda_L [\mu\text{m}])^2} = 5.68 \times 10^{-6} \frac{(\tau_L [\text{fs}])^{11/3}}{(\lambda_L [\mu\text{m}])^{8/3}}$$

Therefore, for FLAME we need  $\varepsilon_L \approx 2.7 \text{J}$  on target. The laser power on target is

$$P [\text{TW}] = 2.41 \times 10^{18} \frac{a_0^3}{n_p [\text{cm}^{-3}]} \frac{1}{(\lambda_L [\mu\text{m}])^2}$$

$$P[\text{TW}] = 2 \times 10^3 \sqrt{\frac{\ln 2}{\pi}} \frac{1}{\tau_L[\text{fs}]} \varepsilon_L[\text{J}] = 5.34 \times 10^{-3} \left( \frac{\tau_L[\text{fs}]}{\lambda_L[\mu\text{m}]} \right)^{8/3}$$

so approximately 84 TW in our case.

### Acceleration distance

The optimum propagation distance is equal to  $L = L_{\text{dephasing}} (= L_{\text{depletion}})$ .

$$L[\text{mm}] = 3.34 \times 10^{17} \frac{\tau_L[\text{fs}]}{(\lambda_L[\mu\text{m}])^2} \frac{1}{n_p[\text{cm}^{-3}]} = 5.31 \times 10^{-4} \frac{(\tau_L[\text{fs}])^{8/3}}{(\lambda_L[\mu\text{m}])^{5/3}}$$

which for FLAME amounts to  $L \approx 7$  mm.

### Maximum electron energy and charge

The maximum electron energy that can be reached under these circumstances is

$$\gamma = \frac{\Delta\varepsilon}{mc^2} = \frac{2}{3} a_0 \frac{\omega_L^2}{\omega_p^2}$$

$$\gamma = 7.44 \times 10^{20} \frac{a_0}{n_p[\text{cm}^{-3}]} \frac{1}{(\lambda_L[\mu\text{m}])^2} = 1.33 \left( \frac{\tau_L[\text{fs}]}{\lambda_L[\mu\text{m}]} \right)^2$$

$$\Delta\varepsilon[\text{GeV}] = 5.11 \times 10^{-4} \gamma = 6.8 \times 10^{-4} \left( \frac{\tau_L[\text{fs}]}{\lambda_L[\mu\text{m}]} \right)^2$$

For FLAME, we get  $\Delta\varepsilon = 0.95$  GeV and a Lorentz factor  $\gamma = 1870$ .

We assume the blowout radius  $R = w_0 = 44.84 \tau_L[\text{fs}]/100 = 18 \mu\text{m}$ . The number of accelerated electrons  $N_e \propto R^3 n_p$  is equal to the ionic charge of the bubble, so [1]

$$N_e \approx 3.13 \times 10^8 \lambda_L[\mu\text{m}] (P[\text{TW}])^{1/2} = 2.28 \times 10^7 \frac{(\tau_L[\text{fs}])^{4/3}}{(\lambda_L[\mu\text{m}])^{1/3}}$$

$$N_e \approx 4.85 \times 10^{17} \frac{a_0^{3/2}}{(n_p[\text{cm}^{-3}])^{1/2}}$$

giving a charge

$$Q_e[\text{pC}] = N_e \times e[\text{pC}] \approx 3.65 \frac{(\tau_L[\text{fs}])^{4/3}}{(\lambda_L[\mu\text{m}])^{1/3}}$$

For FLAME we get  $Q_e \approx 366$  pC  $\approx 0.36$  nC and  $N_e = 2.29 \times 10^9$  electrons. As a last note, with PW lasers, the higher laser energy can be focused to a larger focal spot matched by a lower plasma density.

## External guiding

3J,  $a_0 = 2$ ,  $n_p = 5.1 \times 10^{17} \text{ cm}^{-3}$ ,  $w_0 = 21 \mu\text{m}$ ,  $\tau_L = 47 \text{ fs}$

- needs external guiding (capillaries)
- needs external injection
- can reach  $\Delta\varepsilon = 2.4 \text{ GeV}$  after 52 mm of plasma

## 3. Betatron emission

After matching the laser spot size, and pulse duration to the plasma density, we can look at betatron emission [3]. The betatron oscillation radius  $r_\beta = 3 \mu\text{m}$  for a blowout radius  $R = 21 \mu\text{m}$ . Since the betatron oscillations are limited by the bubble size, it is reasonable to assume a linear dependence on  $R$  and therefore take  $r_\beta = 2.6 \mu\text{m}$  in our case.

We now need to work out the five relevant parameters for the wiggler: the relativistic electron factor  $\gamma$ , the number of electrons  $N_e$ , the strength parameter  $K$ , the period of the wiggler  $\lambda_u$  and the number of (betatron) periods  $N$ . We already know the first two parameters. Once we have the parameters, we can calculate the critical radiation energy  $\hbar\omega_c$ , the opening angle of the radiation  $\theta_r$  and the number of emitted photons per electron and per betatron period,  $N_\gamma$ . We can then get the total number of emitted photons per laser shot,  $N_\gamma \times N_e \times N$ .

$$\begin{aligned}\theta_r[\text{mrad}] &= 10^3 \times \frac{K}{\gamma} \\ N_\gamma &= 3.31 \times 10^{-2} K \\ \hbar\omega_c[\text{keV}] &= 1.86 \times 10^{-3} \frac{K\gamma^2}{\lambda_u[\mu\text{m}]}\end{aligned}$$

## Wiggler parameters

In the case of betatron oscillations, the strength parameter is given by

$$K = \sqrt{\frac{\gamma}{2}} (k_p[\mu\text{m}^{-1}]) (r_\beta[\mu\text{m}]) = 1.33 \times 10^{-10} r_\beta[\mu\text{m}] (\gamma n_p[\text{cm}^{-3}])^{1/2}$$

so  $K \approx 26$  in our case, which puts us in the wiggler regime  $K \gg 1$ .

$$\lambda_u[\mu\text{m}] = \sqrt{2} \gamma^{1/2} (\lambda_p[\mu\text{m}]) = 4.72 \times 10^{10} \left( \frac{\gamma}{n_p[\text{cm}^{-3}]} \right)^{1/2}$$

so  $\lambda_u = 1334 \mu\text{m} = 1.33 \text{ mm}$ .

## Radiation properties

We now easily obtain  $\theta_r = 14 \text{ mrad}$  and  $N_\gamma = 0.87 \text{ photons / (electron x betatron period)}$ . While the electrons radiate throughout the whole acceleration process, the main contribution comes from the part of the trajectory where their energy is maximum, ie, after the distance  $L$ . If the electron is around its maximum energy for about  $N \sim 3$  betatron periods, the total number of emitted photons per

shot will be  $N_{\gamma}^{\text{shot}} = N_{\gamma} \times N_e \times N = 0.87 \times 2.28 \times 10^9 \times 3 \approx 6 \times 10^9$  photons. Finally, the critical energy is in our case  $\hbar\omega_c = 128$  keV.

**Note:** The betatron amplitude  $r_{\beta} \propto \sqrt{a_0/n_p}$  and the number of betatron oscillations  $N \propto 1/\sqrt{n_p}$ .

## 4. Quantum effects

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We are now ready to estimate the size of the quantum corrections. Quantum effects become important when the electron energy loss due to photon emission becomes comparable to the electron energy.

### Nonlinear quantum parameter

We first look at the so-called nonlinear quantum parameter  $\chi_0$  [4] for the case of an ultrarelativistic electron counter-propagating with respect to a plane wave:

$$\chi_0 = 5.9 \times 10^{-3} \Delta\epsilon [\text{GeV}] (I_L [10^{18} \text{W/cm}^2])^{1/2}$$

giving  $\chi_0 \approx 3 \times 10^{-2}$ .  $\chi_0$  can be interpreted as the amplitude of the plane wave's electric field in the electron rest frame (in units of the critical QED field  $m^2/e = 1.3 \times 10^{16} \text{V/cm} = 1.3 \times 10^6 \text{TV/m}$ ) and control quantum effects such as photon recoil and spin. Since  $\chi_0 \ll 1$ , these effects play a minor role here. In fact our electric field is just  $E_L [\text{TV/m}] = 3.21 a_0 / \lambda_L [\mu\text{m}] = 0.151 \text{TV/cm}$ . Inside the plasma, the maximum accelerating field achievable at a given plasma density is  $\approx a_0^{1/2} E_0$  (for  $a_0 \geq 2$ ), where  $E_0$  is the cold non-relativistic wave-breaking field,  $E_0 [\text{V/cm}] \approx 0.96 (n_p [\text{cm}^{-3}])^{1/2}$ . In our case  $a_0^{1/2} E_0 = 2.84 \times 10^9 \text{V/cm}$ .

### Interaction duration

The other way to estimate quantum effects is [3]: radiation reaction can be neglected when the electron - laser interaction duration is much smaller than the electron energy loss rate, or equivalently, the number of oscillations  $N \ll N_{\text{RR}}$ , with

$$N_{\text{RR}} = 2.7 \times 10^7 \frac{\lambda_u [\mu\text{m}]}{\gamma K^2}$$

so for FLAME parameters  $N_{\text{RR}} \sim 2.8 \times 10^4 \gg N$ .

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1. In the latest experimental campaign, the aim is to use 3 J.↵