ORIE 5530: Modeling Under Uncertainty

Homework Assignment 4

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1. This is follow-up on the simulation question from homework 3 and you can build on the code you created then. Consider the **Continuous Time Markov chain** with the transition probability matrix

$$P = \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0.3 & 0 & 0.7 \\ 0.85 & 0.15 & 0 \end{bmatrix}$$

and with rate vector $(\lambda_1, \lambda_2, \lambda_3) = (1, 2, 3)$. That is, the chain remains in state 1 for an exponential amount of time with mean 1, in state 2 with mean 1/2 and in state 3 with mean of a third.

   a. In class, we discussed an approach to simulate a continuous time chain based on exponentials and the code we have to simulate the discrete chain.
      Simulate and plot a single realization of $X(t)$ between $t = 0$ and $t = 1000$. (Notice this is a continuous time chain, it will move not necessarily on integer time epochs.)

   b. Let us assume you collect a reward of i^2 when in state i. Compute, over this realization

$$\frac{1}{1000}\int_0^{1000} (X(s))^2 ds,$$

      Compute the stationary distribution of the CTMC and, from it, the long-run average expected reward. How does this number compare the number you obtained from the simulation.

2. A biology model of cell splitting: A cell can be in one of two states A and B. When a cell is in state A, it moves to state B after an exponential amount of time with parameter β. When in state B it splits into two state-A cells after an exponential time with parameter α. We are interested in tracking the number of cells in state A and those in state B at any given time. That is $N(t) = (N_A(t), N_B(t))$. Characterize the transition of this process in the two equivalent ways:

   – A rate vector λ—that is, specifying the parameter λ_{n_A,n_B} of the exponential amount of time the chain stays in a state (n_A, n_B)—and a transition probability matrix P.

   – A transition-rates matrix Q.

4. An inventory manager has the following policy. As long as there are 5 units in inventory, there is no need to order extra items. Demand arrives according to a Poisson process with rate μ (meaning customers come and ask for the product every $exp(\mu)$ time). As soon as the first customer takes a product (and inventory falls to 4), the inventory manager calls the supplier. The delivery time is random: the supplier will show up within an $exp(\ell)$ amount of time. When the supplier show up, the inventory will be replenished to bring it back to 5. Of course, if the inventory is empty, arriving customers leave empty handed.

   a. Build a CTMC with 6 states that models the inventory evolution.

   b. Say we start at 4 and the manager just called the supplier. What is the expected time until we either go back to 5 (the supplier arrives) or we go down to 3 (another customer takes an item).

   c. Say, now, we start at 5. What is the expected time until we run out of inventory and start disappointing customers?
      Hint: Think in terms of the first time until running out of inventory from each state and their relationships.

d. In the long-run, what is the fraction of time that arriving customers will be leaving empty-handed (no inventory).

   Part d. does not rely on parts b. and c.