HW4_Q1

November 23, 2019

[29]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
```

[37]:
```
markov_chain = np.array([[0.0,0.6,0.4],[0.3,0.0,0.7],[0.85,0.15,0.0]])
beta = np.array([1,1/2,1/3])
```

(a) Simulate and plot a single realization of X(t) between t = 0 and t = 1000 . (Notice this is a continuous time chain, it will move not necessarily on integer time epochs.)

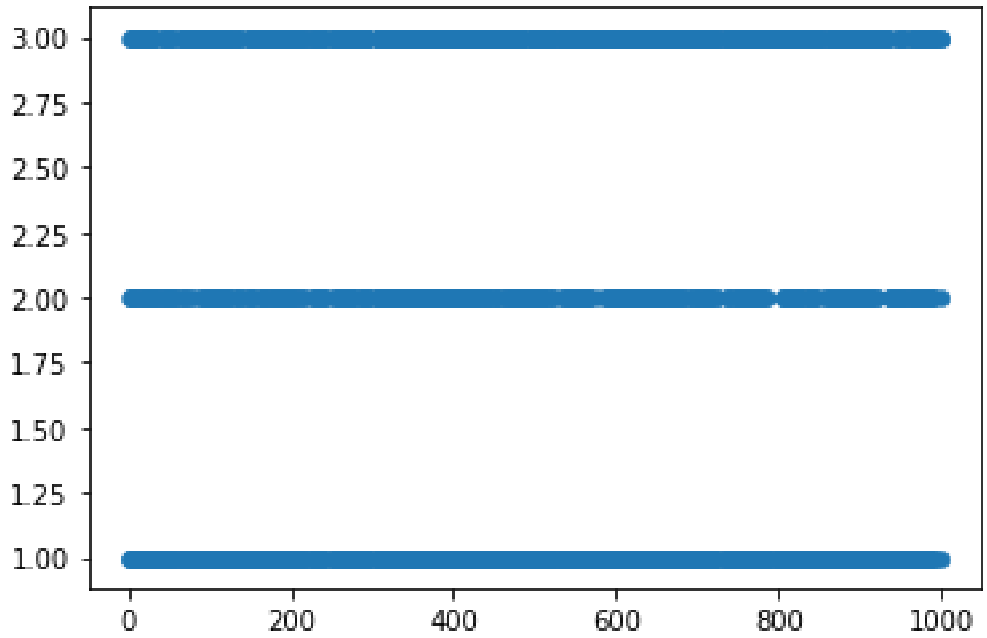
[41]:
```
def markov_chain_simulation(markov_chain, beta, T = 1000):
    results = []
    time = []
    length = []
    xi = 0
    t = np.random.exponential(beta[xi])
    results.append(1)
    time.append(t)
    length.append(t)

    while(t <= T):
        xi = np.random.choice([0,1,2], p = markov_chain[xi])
        l = np.random.exponential(beta[xi])
        t += l
        time.append(t)
        length.append(l)
        results.append(xi + 1)

    return time, results, length

X,Y,I = markov_chain_simulation(markov_chain, beta)
plt.scatter(X,Y, linewidth = 0.5)
```

[41]:
```
<matplotlib.collections.PathCollection at 0x11689d5c0>
```



(b)Compute the stationary distribution of the CTMC and, from it, the long-run average expected reward. How does this number compare the number you obtained from the simulation.

[42]:
```
print(markov_chain)
```

```
[[0.   0.6  0.4 ]
 [0.3  0.   0.7 ]
 [0.85 0.15 0.  ]]
```

[43]:
```
beta2 = np.array([1,2,3])
Q = (markov_chain.T * beta2).T
print(Q)
```

```
[[0.   0.6  0.4 ]
 [0.6  0.   1.4 ]
 [2.55 0.45 0.  ]]
```

[44]:
```
Q[0][0] = -1
Q[1][1] =-2
Q[2][2] = -3
print(Q)
```

```
[[-1.    0.6   0.4 ]
 [ 0.6  -2.    1.4 ]
 [ 2.55  0.45 -3.  ]]
```

[46]:
```
s = beta2 **2@[537/899, 198/899, 164/899]
print(s)
```

```
3.1201334816462736
```

[52]:
```
i = np.array(I)
y = np.array(Y)
sum((y**2)*i)/1000
```

[52]:
```
3.0594543292192875
```

There is some difference but they are almost the same. They are close.

[ ]:
```

```

2. $N(t) = (N_A(t), N_B(t))$

$\lambda_{nA,nB} = n_A\beta + n_B\alpha$

$A \to B : \beta$

$B \to 2A : \alpha$

$A \to B$

$$P_{(nA,nB)(nA-1,nB-1)} = \frac{\lambda_{(nA,nB),(nA-1,nB+1)}}{\lambda_{(nA,nB),(nA-1,nB+1)} + \lambda_{(nA,nB),(nA+2,nB-1)}}$$

$$= \frac{n_A\beta}{n_A\beta + n_B\alpha}$$

$B \to 2A$

$$P_{(nA,nB)(nA+2,nB-1)} = \frac{\lambda_{(nA,nB),(nA+2,nB-1)}}{\lambda_{(nA,nB),(nA+2,nB-1)} + \lambda_{(nA,nB),(nA-1,nB+1)}} = \frac{n_B\alpha}{n_B\alpha + n_A\beta}$$

A transition probability matrix P:

$$\begin{array}{cccc} P: & (nA,nB) & (nA-1,nB+1) & (nA+2,nB-1) \\ (nA,nB) & 0 & \frac{n_A\beta}{n_A\beta + n_A\alpha} & \frac{n_B\alpha}{n_A\beta + n_B\alpha} \end{array}$$

A transition-rates matrix Q:

$$\begin{array}{cccc} Q: & (nA,nB) & (nA-1,nB+1) & (nA+2,nB-1) \\ (nA,nB) & -(n_A\beta + n_B\alpha) & n_A\beta & n_B\alpha \end{array}$$

4. (a) CTMC with 6 states of inventory $= 0, 1, 2, 3, 4, 5$

$$Q = \begin{bmatrix} -\lambda & 0 & 0 & 0 & 0 & \lambda \\ \mu & -(\mu+\lambda) & 0 & 0 & 0 & \lambda \\ 0 & \mu & -(\mu+\lambda) & 0 & 0 & \lambda \\ 0 & 0 & \mu & -(\mu+\lambda) & 0 & \lambda \\ 0 & 0 & 0 & \mu & -(\mu+\lambda) & \lambda \\ 0 & 0 & 0 & 0 & \mu & -\mu \end{bmatrix}$$

(b) Expected time is $\frac{1}{\mu+\lambda}$

(c) Let x_i be the expected time for the first time running out of inventory from state i.

$$x_0 = 0,$$

$$x_1 = \frac{1}{\mu+\lambda} + \frac{\mu}{\mu+\lambda} x_0 + \frac{\lambda}{\mu+\lambda} x_5$$

$$x_2 = \frac{1}{\mu+\lambda} + \frac{\mu}{\mu+\lambda} x_1 + \frac{\lambda}{\mu+\lambda} x_5$$

$$x_3 = \frac{1}{\mu+\lambda} + \frac{\mu}{\mu+\lambda} x_2 + \frac{\lambda}{\mu+\lambda} x_5$$

$$x_4 = \frac{1}{\mu+\lambda} + \frac{\mu}{\mu+\lambda} x_3 + \frac{\lambda}{\mu+\lambda} x_5$$

$$x_5 = \frac{\frac{1}{\mu+\lambda} + \frac{\mu}{(\mu+\lambda)^2} + \frac{\mu^2}{(\mu+\lambda)^3} + \frac{\mu^3}{(\mu+\lambda)^4} + \frac{1}{\mu}}{1 - \frac{\mu^3\lambda}{(\mu+\lambda)^4} - \frac{\mu^2\lambda}{(\mu+\lambda)^3} - \frac{\mu\lambda}{(\mu+\lambda)^2} \quad \frac{\lambda}{(\mu+\lambda)}}$$

(d) $\pi = [\pi_0 \ \pi_1 \ \pi_2 \ \pi_3 \ \pi_4 \ \pi_5]$

Since $\pi Q = 0$ $\quad \sum_{i=0}^{5} \pi_i = 1$

$$\pi_0 = \frac{\mu^5}{(\mu+\lambda)^5}$$