ORIE 5530: Modeling Under Uncertainty

Homework Assignment 3

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1. This is a simulation question. Consider the Markov chain with the transition probability matrix

$$P = \begin{bmatrix} 0.4 & 0.38 & 0.22 \\ 0.12 & 0.7 & 0.18 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$$

   a. Simulate and plot a single realization $X_0, \ldots, X_n$ of this Markov chain (start it with $X_0 = 1$). The plot should have time as the X-axis (going from 0 to n) and the y-axis should have 1,2,3. In each point in time the Markov chain will be in one of the states. Use the time horizon $n = 100$.

   b. Increase n to a 1000 and. Generate a single realization compute, over this realization

$$\frac{1}{n}\sum_{k=0}^{n} X_k,$$

   and, for $\beta = 0.9$

$$\sum_{k=0}^{n} \beta^k (X_k)^2.$$

   As before, use $X_0 = 1$ for your initial condition.

   c. Now repeat this over many realization and average to get approximations for the expectations

$$E[\frac{1}{n}\sum_{k=0}^{n} X_k | X_0 = 1] \text{ and } E[\sum_{k=0}^{n} \beta^k (X_k)^2 | X_0 = 1].$$

   d. Compute analytically $E[\sum_{k=0}^{n} \beta^k (X_k)^2 | X_0 = 1]$ and compare to the simulation result.

2. Conditional Expectation.

   Two players A and B play a series of games with A winning each game with probability p. The overall winner is the first player to have **won two more games than the other**.

   (a) Find the probability that A is the overall winner.

   (b) Find the expected number of games played.

   (Hint: What if you know the outcomes of the first several games?)

3. Conditional expectation.

   A particle moves along the following graph so that at each step it is equally likely to move to any of its neighbours. Starting at 0 show that the expected number of steps it takes to reach n is n^2.

   (Hint: Let T_i be the number of steps it takes to go from vertex $i - 1$ to i, where $i = 1, \ldots, n$. Try to find $E[T_i]$ for all i recursively.)