Proving bounds on gap universal AQC

lundi 15 juin 2020 08:37

$$H(s)\Big|_{\Omega_0} = (1-s)\begin{pmatrix} 0 & \cdots & & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} + s \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & & & \\ \vdots & & 1 & \ddots & \\ & & \ddots & \ddots & -\frac{1}{2} \\ 0 & & & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Bound for $s < 1/3$

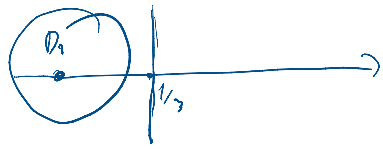
Gaschgorin circle theorem: Let A be any matrix with entries a_{ij}. Consider the disk D_i (for $1 \leq i \leq n$) in the complex plane defined as $D_i = \{ z \mid |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \}$. Then the eigenvalues of A are contained in $\cup_i D_i$ and any connected component of $\cup_i D_i$ contains as many eigenvalues of A as the number of disks in that component.

- $i = 1$

  $\rightarrow a_{11} = \frac{1}{2} s < 1/6$ for $s < 1/3$

  $\rightarrow \sum_{j \neq 1} |a_{1j}| = \frac{1}{2} s < 1/6$

  $\Rightarrow D_1 = \{ z \mid |z - \underbrace{c_1}_{\leq 1/6}| \leq \underbrace{r_1}_{\leq 1/6} \}$

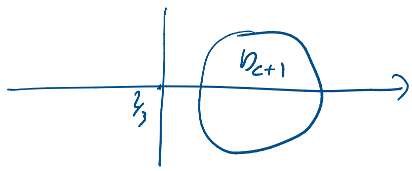


- $i = L+1$

  $\rightarrow a_{L+1,L+1} = 1 - \frac{1}{2} s > 5/6$

  $\rightarrow \sum_{j \neq L+1} |a_{L+1\,j}| = \frac{s}{2} < 1/6$ (error in RMP paper? eq. (5b))

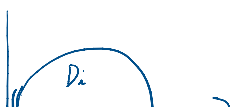
$$D_{L+1} = \{ z \mid |z - \underbrace{c_{L+1}}_{> 5/6}| \leq \underbrace{r_1}_{< 1/6} \}$$



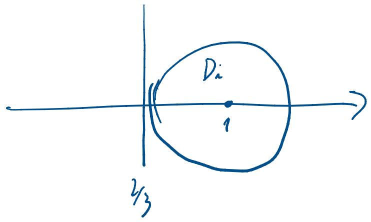
- $i \neq 1, L+1$

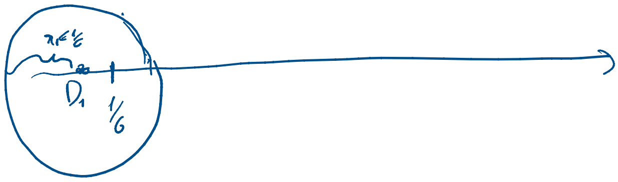
  $\rightarrow a_{ii} = 1$

  $\rightarrow \sum_{j \neq i} |a_{ij}| = s < \frac{1}{3}$



$\to \sum_{j\neq i} |a_{ij}| = s < \frac{1}{3}$





Conclusions: for $s < 1/3$ gap $\geq 1/3 \Rightarrow$ minimum separation between D_1 and other disks.

Bound for $s \geq 1/3$

Consider $G(s) \equiv \mathbb{1} - H_{s_0}(s) = \begin{pmatrix} 1-\frac{1}{2}s & \frac{1}{2}s & 0 & \cdots \\ \frac{1}{2}s & 0 & \frac{1}{2}s & \\ & \ddots & \ddots & \ddots \\ & & 0 & \frac{1}{2}s \end{pmatrix}$

$\to$ Matrix is hermitian and has all non-negative entries

$\to G(s)^{L+1}$ has all positive entries. for $0 < s \leq 1$

Perron's theorem: Let G be an Hermitian matrix with real non-negative entries. If $\exists$ a finite k s.t. all entries of G^k are positive, then G's largest eigenvalue is positive and unique. The eigenvector corresponding to this eigenvalue has only strictly positive entries.

$\to$ G's largest eigenvalue $\mu > 0$

$\to$ eigenvector $\vec{\alpha} = (\alpha_1, \ldots, \alpha_{L+1})$, with $\alpha_i > 0$.

Consider stochastic matrix:

$$P_{ij} = \frac{\alpha_j}{\mu\, \alpha_i} G_{ij} \geq 0 \quad ; \quad \sum_j P_{ij} = \frac{1}{\mu \alpha_i} \sum_j G_{ij} \alpha_j = 1$$

If $(\alpha_1 v_1, \ldots, \alpha_{L+1} v_{L+1})$ is a left eigenvector of P with

If $(\alpha_1 v_1, \ldots, \alpha_{L+1} v_{L+1})$ is a left eigenvector of P with eigenvalue $\frac{\nu}{\mu}$

$$\sum_i \alpha_i v_i P_{ij} = \frac{\nu}{\mu} \alpha_j v_j \iff \sum_i \cancel{\alpha_i} \frac{v_i \, \alpha_j \, G_{ij}}{\mu \, \cancel{\alpha_i}} = \frac{\nu}{\cancel{\mu}} \cancel{\alpha_j} v_j$$

$$\iff \sum_i v_i G_{ij} = \nu v_j$$

$\vec{v} = (v_1, \ldots, v_{L+1})$ is a eigenvector of G with eigenvalue ν

and vice-versa.

$\Rightarrow$ Taking $\vec{v} = \vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$,

$$\vec{\Pi} = \frac{1}{Z} (\alpha_1^2, \ldots, \alpha_{L+1}^2), \quad Z = \sum_i \alpha_i^2$$

is a left eigenvector of P with the maximum eigenvalue 1.

$P \vec{\Pi} = \vec{\Pi} \rightarrow$ limiting distribution of P

Def. Gap of P: $\frac{\delta}{\mu} \Rightarrow$ 2nd largest EV of P is $1 - \frac{\delta}{\mu} = \frac{\mu - \delta}{\mu}$

$\Rightarrow$ Gap of $G = \mu - \delta \Rightarrow$ Gap of $H = \mathbb{1} - G$

To compute gap of P we use the conductance bound (Sinclair, Jerrum 1989)

To define conductance of P, $\varphi(P)$, let us define a non-empty set $B \in \{1, 2, \ldots, L+1\}$ satisfying

$$\sum_{i \in B} \Pi_i \leq 1/2 \quad ; \quad \vec{\Pi} = \frac{1}{Z} (\alpha_1^2, \ldots, (\alpha_{L+1})^2)$$

$\vec{\alpha}$ principal eigenvector of G

<u>Conductance $\varphi(P)$</u>:

$$\varphi(P) = \min_B \frac{F(B)}{\ldots(B)} \quad \text{where} \quad F(B) = \sum_{i \in B} \sum_{j \notin B} \Pi_i P_{ij}$$

$$\varphi(P) = \min_B \frac{F(B)}{\pi(B)} \quad \text{where} \quad F(B) = \sum_{i \in B} \sum_{j \notin B} \pi_i \, r_{ij}$$

$$\pi(B) = \sum_{i \in B} \pi_i$$

Conductance bound (Jerrum, Sinclair 1989)

$$\Delta(P) \geq \frac{1}{2} \varphi(P)^2$$

To compute the conductance, we first show that $\vec{\alpha}$ is a monotone i.e.,

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n .$$

To show this, first note that if $\vec{v}$ is a monotone $G(s)\,\vec{v} = \vec{w}$ is also a monotone ($G(s)$ preserves monotonicity)

Proof:

$$w_1 = \left(1 - \tfrac{1}{2} s\right) v_1 + \frac{1}{2} s\, v_2$$

$$w_k = \frac{1}{2} s\, v_{k-1} + \tfrac{1}{2} s\, v_{k+1}$$

$$w_{\ell+1} = \frac{1}{2} s\, v_\ell + \frac{1}{2} s\, v_{\ell+1}$$

Therefore:

$$w_1 - w_2 = (1-s)\, v_1 + \tfrac{1}{2} s\,(v_2 - v_3)$$

$$w_k - w_{k+1} = \frac{1}{2} s\,(v_1 - v_2 + v_3 - v_4), \quad 2 \leq k \leq \ell - 1$$

$$w_\ell - w_{\ell+1} = \frac{1}{2} s\,(v_{\ell-1} - v_\ell)$$

which are all ≥ 0 since $\vec{v}$ is a monotone.

To show $\vec{\alpha}$ is a monotone we will show that

$$\lim_{k \to \infty} \left(\frac{1}{\mu_1} G(s) \right)^k \vec{1} = c_1 \vec{\alpha}$$

$\vec{1} = (1, 1 \dots, 1)$ is a monotone

and $c_1 > 0$

Proof: Let μ_i and $|v_i\rangle$ be the eigenvalues and eigenvectors of $G(s)$

Since $\{|v_i\rangle\}_i$ form a basis we can write

$$\sum c_i |v_i\rangle = (1, \dots, 1) \quad . \quad \text{Here } \mu_1 = \mu \text{ and } |v_1\rangle = \vec{\alpha}$$

Since ...

$$\sum_i c_i |v_i\rangle = (1, \dots, 1)$$. Here $\mu_1 = \mu$ and $|v_1\rangle = \vec{\alpha}$

Then

$$\left(\frac{1}{\mu_1} G(s)\right)^k \sum_i c_i |v_i\rangle = \sum_i \left(\frac{\mu_i}{\mu_1}\right)^k c_i |v_i\rangle \quad , \text{ with } \mu_i < \mu_1 \text{ from Perron's theorem}$$

Hence

$$\lim_{k\to\infty} \left(\frac{1}{\mu_1} G(s)\right)^k \vec{1} = c_1 |v_1\rangle \equiv c_1 \vec{\alpha}$$

$$c_1 = \vec{1} \cdot \vec{\alpha} = \sum_i \alpha_i > 0 \quad \text{since } \alpha_i > 0$$

Hence $\vec{\alpha}$ is a monotone.

Calculation of the conductance

1) First, let us consider the case where index $1 \in B$.
Let k be the smallest index for which $k \in B$ but $k+1 \notin B$.

$$F(B) = \sum_{i\in B} \sum_{j\notin B} \Pi_i P_{ij} \geq \Pi_k P_{k,k+1}$$

$$= \Pi_k \frac{\sqrt{\Pi_{k+1}}}{\mu \sqrt{\Pi_k}} [G(s)]_{k,k+1} = \frac{\sqrt{\Pi_k \Pi_{k+1}}}{\mu} [G(s)]_{k,k+1}$$

$$\geq \Pi_{k+1} [G(s)]_{k,k+1} \qquad = \Pi_{k+1} \frac{s}{2} \geq \frac{1}{6} \Pi_{k+1}$$

because $\vec{\Pi}$ is a monotone so $\Pi_k \geq \Pi_{k+1}$ $\qquad$ $s \geq 1/3$

$$\Rightarrow \quad F(B = \{1, k, \dots\}) \geq \frac{\Pi_{k+1}}{6}$$

Now since by definition $\Pi(B) \leq 1/2$ then $\Pi(\overline{B}) = \sum_{i \notin B} \Pi_i \geq 1/2 \quad (*_1)$

complement of B

By definition of k, $k+1$ is the lowest index that belongs to $\overline{B}$.

$$\Pi(\overline{B}) = \sum \Pi_i \leq \Pi_{k+1} |\overline{B}| \leq \Pi_{k+1} L \qquad (*_2)$$

By definition of ...

$$\pi(\overline{B}) = \sum_{i\in\overline{B}} \pi_i \underset{\substack{\uparrow\\ \vec{\pi} \text{ is monotone}}}{\leq} \pi_{k+1}\,|\overline{B}| \leq \pi_{k+1} L \qquad (\mathcal{X}_2)$$

$$(\mathcal{X}_1) + (\mathcal{X}_2) \implies \pi_{k+1} \geq \frac{1}{2L}$$

Finally
$$\frac{F(B=\{1,k,\dots\})}{\pi(B)} \geq \frac{\pi_{k+1}}{6\,\pi(B)} \geq \frac{1}{12L\,\pi(B)} \underset{\substack{\downarrow\\ \pi(B)\leq 1/2}}{\geq} \frac{1}{6L}$$

2) Now, we consider the case $1 \notin B$.

Let k be the smallest index s.t. $k \notin B$ but $k+1 \in B$.

$$F(B) = \sum_{i\in B}\sum_{j\notin B} \pi_i P_{ij} \geq \pi_{k+1} P_{k+1,k} \geq \frac{\pi_{k+1}}{6}$$
$$(\;)\,:\,\frac{s}{2}$$

$$\pi(B) = \sum_{i\in B} \pi_i \leq L\,\pi_{k+1}$$

$$\varphi(B) = \frac{F(B)}{\pi(B)} \geq \frac{1}{6L}$$

$\underset{\text{conductance bound}}{\implies}$ $\Delta(P) \geq \frac{1}{2}\left(\frac{1}{6L}\right)^2$; $\Delta(G) = \mu\,\Delta(P) = (1-\lambda)\,\Delta(H)$

$\lambda \to$ GS of H

$\lambda \leq \langle\psi|H(s)|\psi\rangle \quad \forall|\psi\rangle$

$\lambda \leq \langle\gamma(0)|H(s)|\gamma(0)\rangle = \frac{s}{2} \leq \frac{1}{2}$

$$\implies \Delta(H) \geq \frac{1}{4}\left(\frac{1}{6L}\right)^2$$

Gap bound in the entire Hilbert space:

$$H(s) = (1-s)H_{init} + s\,H_{final} = H_{input} + H_c + (1-s)H_{c\text{-}init} + s\,H_{prop}$$

where $\sum^{L-1}$ ... at least 1 to bad clock states where

$$H_{clock} = \sum_{l=1}^{L-1} |0_l 1_{l+1}\rangle\langle 0_l 1_{l+1}|$$ → gives an energy of at least 1 to bad clock states (large gap)

$$H_{c\text{-}init} = |1\rangle\langle 1|_{c}$$

$$H_{input} = \sum_{i=1}^{n} |1\rangle\langle 1|_i \otimes |0\rangle\langle 0|_{1c}$$

It is hence enough to consider the subspace of "good" clock states.

- $\dim(\mathcal{H}_{comp.} \otimes \mathcal{H}_{g.clock}) = 2^n (L+1)$

Consider the invariant subspace $\Omega_y = \mathrm{span}\{|\gamma_y(l)\rangle\}$, with $|\gamma_y(0)\rangle = |y\rangle|0\rangle_c$, $|\gamma_y(l)\rangle = V_l \cdots V_1 |y\rangle|l\rangle_c$ for $l = 1, \dots, L$

In this subspace, H_{input} acts as

$$H_{input}\big|_{\Omega_y} = \begin{pmatrix} k(y) & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

where $k(y)$ is the number of bits "1" contained in y.

To show there is a gap in the full Hilbert space, we need to compute lower bound the GS energy of $H(s)|_{\Omega_y}$ when $y \neq 00..0$

For $y \neq 0..0$

$$H(s)\big|_{\Omega_y} = \underbrace{\begin{pmatrix} k(y) & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}}_{\substack{\text{Ground states} \\ \sum_{l=1}^{L} c_l |\gamma_y(l)\rangle \\ \text{gap} = k(y) \geq 1 \\ \text{GS energy} = 0}} + \underbrace{(1-s)\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} + s\begin{pmatrix} 1/2 & -1/2 & 0 & \cdots & 0 \\ -1/2 & 1 & -\ & & \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & & & 1 & -1/2 \\ 0 & & & -1/2 & 1/2 \end{pmatrix}}_{\substack{\text{Ground state} \\ \sum_{l=0}^{L} \alpha_l |\gamma_y(l)\rangle,\ \text{unique} \\ \text{gap}:\ \Omega\left(\frac{1}{L^2}\right) \\ \text{GS energy } g_0(s)}}$$

Geometrical Lemma:

	H_1	H_2
GS energy:	g_1	g_2
gap:	Δ_1	Δ_2

$H = H_1 + H_2$

$$g \geq g_1 + g_2 + \min(\Delta_1, \Delta_2)(1 - \cos\theta)$$

where $\cos\theta$ is the max overlap between a GS of H_1 with a GS of H_2.

between a GS of H_1 -
GS of H_2.

In our case $g_1 = g(s)$

$g_2 = 0$

$\Delta_1 = \mathcal{O}(1/L^2)$

$\Delta_2 \geq 1$

$$\cos\theta = \max_{\{e_\ell\}} \left| \sum_{\ell=1}^{L} e_\ell^* \alpha_\ell \right| = \sqrt{\sum_{\ell=1}^{L} |\alpha_i|^2} \; \underbrace{\max_{\{e_\ell\}} \left| \sum_{\ell=1}^{L} e_\ell^* \frac{\alpha_\ell}{\sqrt{\sum_{i=1}^{L} |\alpha_i|^2}} \right|}_{=1 \text{ if } e_\ell^* = \frac{\alpha_\ell^*}{\sqrt{\sum_{i=1}^{L}|\alpha_i|^2}}}$$

$$= \sqrt{1 - \underbrace{|\alpha_0|^2}_{\geq \frac{1}{L+1} \text{ because } \vec{\alpha} \text{ is monotone}}}$$

$$\leq \sqrt{1 - \frac{1}{L+1}} = \sqrt{\frac{L}{L+1}} \leq 1 - \frac{1}{2L}$$

From geometric lemma:

$$g_Y(s) \geq g_0(s) + \underbrace{(1 - \cos\theta)}_{\Omega(1/L)} \underbrace{\min(\Delta_1, \Delta_2)}_{\Omega(1/L^2)}$$

$$\geq g_0(s) + \Omega\left(1/L^3\right)$$

$\Rightarrow$ Comment: same technique used in Kitaev's QMA-completeness proof to show that for "NO" instances GS energy is $\Omega\left(\frac{1}{L^3}\right)$.