Solving the SIR-Macro Model in "The Macroeconomics of Epidemics"

Bence Bardóczy[*]

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This note describes how to compute transition dynamics in the SIR-Macro model developed by Eichenbaum, Rebelo, and Trabandt (2020). An implementation in Python is available on GitHub https://github.com/bbardoczy/sir-macro.

1 General model

Let's start by summarizing the equilibrium conditions of the SIR-Macro model in its most general form. For a detailed description, please consult the paper.

- **Vaccine** that protects susceptible people from getting infected is discovered with probability ν_t.
- **Treatment** that cures infected people is discovered with probability ξ_t.
- **Susceptible people**:

  The Bellman equation is

$$U_t^s = \max_{c_t^s, n_t^s, \tau_t} \left\{ \log c_t^s - \frac{\theta}{2}(n_t^s)^2 + (1-\nu_t)\beta \left[(1-\tau_t)U_{t+1}^s + \tau_t U_{t+1}^i\right] + \nu_t \beta U_{t+1}^r \right\}$$

$$\text{s.t.} \ \ (1+\mu_{ct})c_t^s = A n_t^s + \Gamma_t \qquad (\lambda_t^s)$$

$$\tau_t = \pi_1 c_t^s (I_t C_t^I) + \pi_2 n_t^s (I_t N_t^I) + \pi_3 I_t \qquad (\mu_t^s)$$

  The FOCs are

$$0 = \frac{1}{c_t^s} - \lambda_t^s (1+\mu_{ct}) - \mu_t^s \pi_1 (I_t C_t^I) \tag{1.1}$$

$$0 = -\theta n_t^s + \lambda_t^s A - \mu_t^s \pi_2 (I_t N_t^I) \tag{1.2}$$

$$0 = \beta(1-\nu_t)\left(U_{t+1}^s - U_{t+1}^i\right) - \mu_t^s \tag{1.3}$$

- **Infected people**:

  The Bellman equation is

$$U_t^i = \max_{c_t^i, n_t^i} \left\{ \log c_t^i - \frac{\theta}{2}(n_t^i)^2 + (1-\xi_t)\beta \left[(1-\pi_r-\pi_{dt})U_{t+1}^i + \pi_r U_{t+1}^r\right] + \xi_t \beta U_{t+1}^r \right\}$$

$$\text{s.t.} \ \ (1+\mu_{ct})c_t^i = A\phi n_t^i + \Gamma_t \qquad (\lambda_t^i)$$

  The FOCs are

$$0 = \frac{1}{c_t^i} - \lambda_t^i (1+\mu_{ct}) \tag{1.4}$$

$$0 = -\theta n_t^i + \lambda_t^i \phi A \tag{1.5}$$

[*]Northwestern University. Email: bardoczy@u.northwestern.edu.

- **Recovered people**:

  The Bellman equation is

$$\begin{aligned} U_t^r &= \max_{c_t^r, n_t^r} \left\{ \log c_t^r - \frac{\theta}{2} (n_t^r)^2 + \beta U_{t+1}^r \right\} \\ \text{s.t. } (1+\mu_{ct}) c_t^r &= A n_t^r + \Gamma_t \qquad (\lambda_t^r) \end{aligned}$$

  The FOCs are

$$0 = \frac{1}{c_t^r} - \lambda_t^r (1+\mu_{ct}) \tag{1.6}$$

$$0 = -\theta n_t^r + \lambda_t^r A \tag{1.7}$$

- **Epidemic laws of motion**:

$$\pi_{dt-1} = \pi_d + \kappa I_{t-1}^2 \tag{1.8}$$

$$T_{t-1} = \pi_1 (S_{t-1} c_{t-1}^s)(I_{t-1} c_{t-1}^i) + \pi_2 (S_{t-1} n_{t-1}^s)(I_{t-1} n_{t-1}^i) + \pi_3 S_{t-1} I_{t-1} \tag{1.9}$$

$$S_t = S_{t-1} - T_{t-1} \tag{1.10}$$

$$I_t = I_{t-1} + T_{t-1} - (\pi_r + \pi_{dt-1}) I_{t-1} \tag{1.11}$$

$$R_t = R_{t-1} + \pi_r I_{t-1} \tag{1.12}$$

$$D_t = D_{t-1} + \pi_{dt-1} I_{t-1} \tag{1.13}$$

  starting from $S_0 = 1 - \varepsilon, I_0 = \varepsilon, R_0 = 0, D_0 = 0$.

- **Medical preparedness**:

$$\pi_{dt} = \pi_d + \kappa I_t^2 \tag{1.14}$$

- **Government budget**:

$$\mu_{ct} C_t = \Gamma_t (S_t + I_t + R_t) \tag{1.15}$$

- **Aggregation**:

$$C_t = S_t c_t^s + I_t c_t^i + R_t c_t^r \tag{1.16}$$

$$N_t = S_t n_t^s + I_t n_t^i + R_t n_t^r \tag{1.17}$$

- **Production**:

$$C_t = A \left(S_t n_t^s + \phi I_t n_t^i + R_t n_t^r \right) \tag{1.18}$$

2 Algorithm

We're looking for H-period long sequences of endogenous variables. H has to be long enough so that the model gets close to their post-epidemic steady state. We found that $H = 250$ suffices for all cases in the paper. The problem boils down to finding the root of a function. We'll use a quasi-Newton method to do so.

1. Specify containment policies and discovery probabilities $\{\mu_{ct}, \xi_t, \nu_t\}_{t=0}^{H-1}$.

2. Guess employment $\{n_t^s, n_t^i, n_t^r\}_{t=0}^{H-1}$.

   In practice, we start from the steady state.

3. Solve for consumption:

$$c_t^r = \frac{A}{(1+\mu_{ct})\theta n_t^r} \tag{2.1}$$

$$\Gamma_t = (1+\mu_{ct})c_t^r - An_t^r \tag{2.2}$$

$$c_t^i = \frac{A\phi n_t^i + \Gamma_t}{1+\mu_{ct}} \tag{2.3}$$

$$c_t^s = \frac{An_t^s + \Gamma_t}{1+\mu_{ct}} \tag{2.4}$$

4. Iterate forward the epidemic laws of motion.

   Initial conditions:

$$S_0 = 1 - \varepsilon \tag{2.5}$$

$$I_0 = \varepsilon \tag{2.6}$$

$$R_0 = 0 \tag{2.7}$$

$$D_0 = 0 \tag{2.8}$$

   Iteration for $t = 1, \ldots, H-1$:

$$\pi_{dt-1} = \pi_d + \kappa I_{t-1}^2 \tag{2.9}$$

$$T_{t-1} = \pi_1(S_{t-1}c_{t-1}^s)(I_{t-1}c_{t-1}^i) + \pi_2(S_{t-1}n_{t-1}^s)(I_{t-1}n_{t-1}^i) + \pi_3 S_{t-1}I_{t-1} \tag{2.10}$$

$$S_t = S_{t-1} - T_{t-1} \tag{2.11}$$

$$I_t = I_{t-1} + T_{t-1} - (\pi_r + \pi_{dt-1})I_{t-1} \tag{2.12}$$

$$R_t = R_{t-1} + \pi_r I_{t-1} \tag{2.13}$$

$$D_t = D_{t-1} + \pi_{dt-1}I_{t-1} \tag{2.14}$$

5. Solve for value functions by backward iteration.

   Transmission probability:

$$\tau_t = \pi_1 c_t^s(I_t C_t^I) + \pi_2 n_t^s(I_t N_t^I) + \pi_3 I_t \tag{2.15}$$

   Terminal conditions:

$$U_H^r = U_{ss} \tag{2.16}$$

$$U_H^i = \frac{\log(\phi c_{ss}^i) - \frac{\theta}{2}(n_{ss}^i)^2 + \beta(1-\xi_{H-1})\pi_r U_H^r + \beta\xi_{H-1}U_H^r}{1 - \beta(1-\pi_r-\pi_{dH-1})(1-\xi_{H-1})} \tag{2.17}$$

$$U_H^s = U_H^r \tag{2.18}$$

Iterate backwards:

$$U_t^r = \log c_H^r - \frac{\theta}{2}(n_H^r)^2 + \beta U_{t+1}^r \tag{2.19}$$

$$U_t^i = \log c_H^i - \frac{\theta}{2}(n_H^i)^2 + (1-\xi_t)\beta\left[(1-\pi_r-\pi_{dt})U_{t+1}^i + \pi_r U_{t+1}^r\right] + \xi_t\beta U_{t+1}^r \tag{2.20}$$

$$U_t^s = \log c_t^s - \frac{\theta}{2}(n_t^s)^2 + (1-\nu_t)\beta\left[(1-\tau_t)U_{t+1}^s + \tau_t U_{t+1}^i\right] + \nu_t\beta U_{t+1}^r \tag{2.21}$$

6. Multipliers and aggregates:

$$\mu_t^s = \beta(1-\nu_t)(U_{t+1}^s - U_{t+1}^i) \tag{2.22}$$

$$\lambda_t^s = \frac{\theta n_t^s + \mu_t^s \pi_2 I_t n_t^i}{A} \tag{2.23}$$

$$\lambda_t^i = \frac{\theta n_t^i}{\phi A} \tag{2.24}$$

$$C_t = S_t c_t^s + I_t c_t^i + R_t c_t^r \tag{2.25}$$

$$N_t = S_t n_t^s + I_t n_t^i + R_t n_t^r \qquad \text{(these are raw hours!)} \tag{2.26}$$

7. Residuals:

$$0 = \mu_{ct} C_t - \Gamma_t(S_t + I_t + R_t) \tag{2.27}$$

$$0 = \lambda_t^i(1+\mu_{ct}) - \frac{1}{c_t^i} \tag{2.28}$$

$$0 = \lambda_t^s(1+\mu_{ct}) + \mu_t^s \pi_1 (I_t C_t^I) - \frac{1}{c_t^s} \tag{2.29}$$

8. Update guesses using a Newton step.

   Let x^j be the current guess and f be the function that maps guesses into residuals. The next guess should be

$$x^{j+1} = x^j - [f'(x_{ss})]^{-1} f(x^j) \tag{2.30}$$

   Note that there's no need to recompute the Jacobian f' at every step. The Jacobian at the initial guess is a sufficiently good updating rule to make the algorithm converge in 8-10 steps to a tolerance level of 10^{-8}. This trick provides large efficiency gains whenever it's costly to compute the Jacobian.

References

Eichenbaum, M. S., S. Rebelo, and M. Trabandt (2020): "The Macroeconomics of Epidemics," NBER Working Paper 26882.