The Qudit ZH-Calculus: Generalised Toffoli+Hadamard and Universality[1]

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[1]arXiv:2307.10095

Motivation

Previous Work

- In qubits, there are three possible graphical calculi (ZX, ZW and ZH)[2]
- ZX and ZW have proposal for generalizing to qudit, ZH does not
- Phase-free ZH-Calculus is equivalent to Toffoli+H circuits[3]

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This Paper

- First generalization of ZH to qudits and universality for linear maps
- A generalization of the Toffoli+H gateset to qudits and computational universality

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Overview

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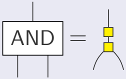
The H-box

We want an *H*-box that...

1. ...generalizes the Discrete Fourier Transform
   $H|k\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \omega^{ik} |i\rangle$ for $\omega = e^{2\pi i/d}$
2. ...generalizes the qubit AND-gate construction 

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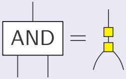
2. ...generalizes the qubit AND-gate construction 

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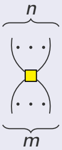
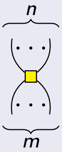


$$:= \frac{1}{\sqrt{d}} \sum_{\substack{i_1,\ldots,i_m \in \{0,1\} \\ j_1,\ldots,j_n \in \{0,1\}}} \omega^{i_1 \cdot \ldots \cdot i_m \cdot j_1 \cdot \ldots \cdot j_n} |j_1 \ldots j_n\rangle\langle i_1 \ldots i_m|$$

The Generators 1/2

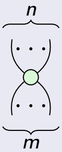
H-Box



$$:= \frac{1}{\sqrt{d}} \sum_{\substack{i_1,\dots,i_m \in \{0,1\} \\ j_1,\dots,j_n \in \{0,1\}}} \omega^{i_1 \cdot \ldots \cdot i_m \cdot j_1 \cdot \ldots \cdot j_n} |j_1 \dots j_n\rangle\langle i_1 \dots i_m|$$

Can replace ω with some r to get the "r-labelled" H-box $H(r)$.

Z-Spider

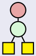


$$:= \sum_{i=0}^{d-1} |i\rangle^{\otimes n} \langle i|^{\otimes m}$$

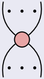
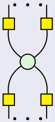
The Generators 2/2

$\sqrt{d}$ and $1/\sqrt{d}$

$\blacktriangle :=$ 

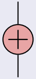
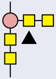
$\bigstar :=$ 

X-Spider

 $:=$ 

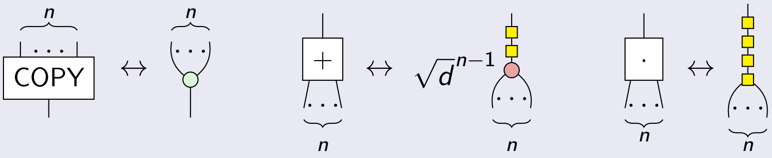
Pauli-X

Qudit Pauli-X: $|i\rangle \mapsto |i +_d 1\rangle$

 $:=$ 

An appeal to arithmetic modulo d

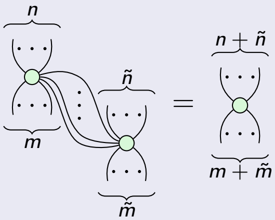
Spider Math



Generalizes qubit relationship of H-box and AND-gate - AND is multiplication modulo 2!

The Rules 1/2

Z-Fusion (zs)



The Rules 1/2

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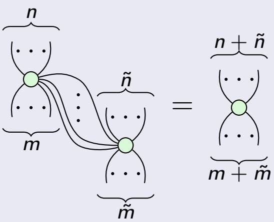
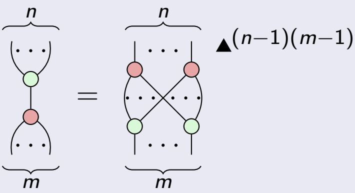


Z/X-Bialgebra (ba1)



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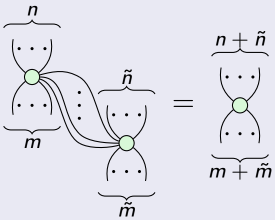

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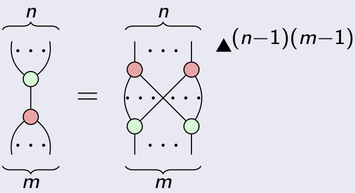
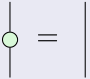
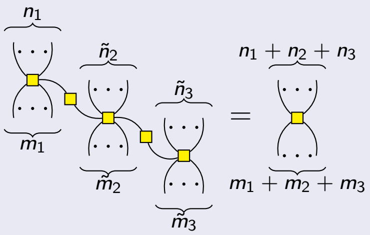


Identity (id)



The Rules 2/2

H-Contraction (hs)



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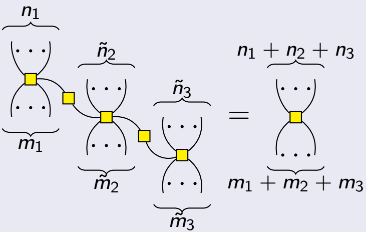

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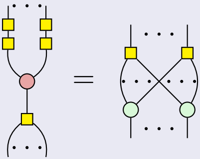

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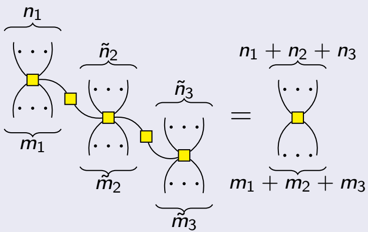

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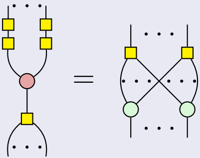
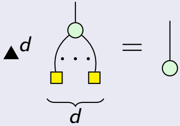
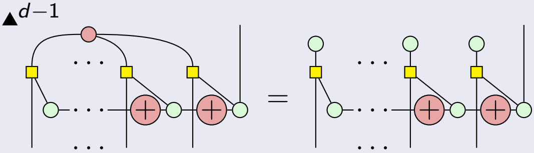


Cyclic (c)



Bonus Rule

Ortho (o)



$$\forall x_0, \ldots, x_{d-1}, y: \; x_0 y = \ldots = x_{d-1}(y + d - 1)$$

$$\iff$$

$$\forall i \in \{0, \ldots, d-1\}: \; x_i(y+i) = 0$$

Because $\{y, y+1, \ldots, y+d-1\} = \mathbb{Z}/d\mathbb{Z} \ni 0$

Overview

Many ways to describe linear maps

1. Mapping of basis states:

$$|i\rangle \mapsto |i\rangle + |i +_d 1\rangle$$

Computes rows of Pascal's triangle as column vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \overset{R}{\rightsquigarrow} \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \overset{R}{\rightsquigarrow} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \overset{R}{\rightsquigarrow} \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \overset{R}{\rightsquigarrow} \dots$$

Many ways to describe linear maps

1. Mapping of basis states:

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2. Matrix:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 1 & 1 \end{pmatrix}$$

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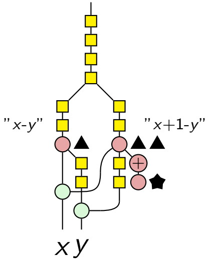
5. ... ZH-Diagram!



Post-select with 0-labelled H-box and bend y-wire to get $|i\rangle \mapsto |i\rangle + |i +_d 1\rangle$.

Universality with labeled H-boxes

Generalizes to matrices that are $r, 1$-valued instead of $0, 1$-valued

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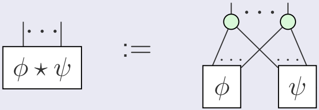
Can decompose any matrix as entry-wise product of such matrices, e.g.

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \star \begin{pmatrix} 1 & b \\ 1 & 1 \end{pmatrix} \star \begin{pmatrix} 1 & 1 \\ c & 1 \end{pmatrix}$$



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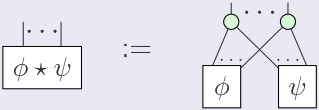

Technically, suffices to construct diagrams for matrices that only have single non-1 entry, but ideas from previous slides leads to significantly smaller diagrams

Universality for $\mathbb{Z}[\omega]$

Want universality without adjoining labelled H-boxes as new generators

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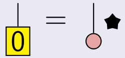
Idea

- We know how to construct $H(0)$:



- Find diagram for map S which increments H-box label:

$$H(n+1) = SH(n)$$

- Find diagram for $H(-1)$ and use $H(-1) \star H(n) = H(-n)$

Successor Map

Needs to satisfy

$$\begin{array}{rcccccccccccc}
1 & = & s_{00} & + & s_{01}a & + & s_{02}a^2 & + & \cdots & + & s_{0(d-1)}a^{d-1} \\
a+1 & = & s_{10} & + & s_{11}a & + & s_{12}a^2 & + & \cdots & + & s_{1(d-1)}a^{d-1} \\
\vdots & & & \vdots & & \vdots & & \vdots & & \vdots & & & . \\
(a+1)^{d-1} & = & s_{(d-1)0} & + & s_{(d-1)1}a & + & s_{12}a^2 & + & \cdots & + & s_{(d-1)(d-1)}a^{d-1}
\end{array}$$

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Binomial Theorem

$$(a+1)^j = \sum_{i=0}^{j} \binom{j}{i} a^i.$$

$\Rightarrow$ S encodes Pascal's triangle, e.g. $S^T|c\rangle = R^c|0\rangle$

Multiplexer

Insight

$$S^T|c\rangle = R^c|0\rangle$$

$$\iff$$

S^T is multiplexer for $R^0|0\rangle, ..., R^{d-1}|0\rangle$ with control $|c\rangle$

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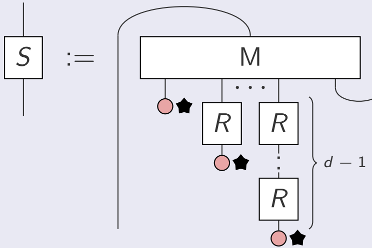
S^T is multiplexer for $R^0|0\rangle, ..., R^{d-1}|0\rangle$ with control $|c\rangle$

$$M : |x_0 ... x_{d-1}\rangle \otimes |c\rangle \;\mapsto\; \begin{cases} |x_c\rangle & x_j = 0 \text{ for all } j \neq c \\ 0 & \text{otherwise.} \end{cases}$$

Successor



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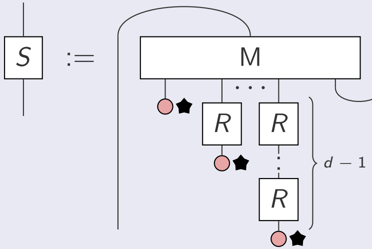
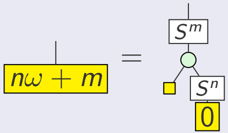


So far: All non-negative integers through successive application of S to $H(0)$

Negative Integers

Unlabeled H-box $=$ ω-labeled H-box



Negative Integers

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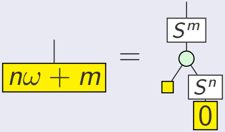


Elements $f \in \mathbb{Z}[\omega]$ have the form

$$f = \sum_{i=0}^{d-1} n_i \omega^i = n_0 + \omega(n_1 + \omega(\ldots + \omega n_{d-1})\ldots)$$

for $n_0, \ldots, n_{d-1} \in \mathbb{Z}$

-1

Theorem

$$\omega + \omega^2 + \ldots + \omega^{d-1} = -1$$

-1

Theorem

$$\omega + \omega^2 + \ldots + \omega^{d-1} = -1$$

Final pieces:

- $H(-1) = H(\omega + \ldots + \omega^{d-1})$
- $H(-n) = H(n) \star H(-1)$

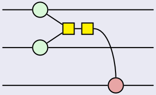
$\Rightarrow$ Diagrams for all matrices over $\mathbb{Z}[\omega]$

Overview

Toffolis and phase-free qubit ZH

Qubits

Toffoli + H approximately universal for quantum computation, and ZH allows simple reasoning about these gates:



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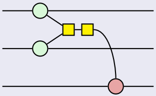

Question

What approximately universal gateset does phase-free qudit ZH easily allow us to reason about?

Toffoli generalizes to $|0\rangle$-controlled X

In odd qudit dimension d, the $|0\rangle$-controlled X suffices to realize all d-ary classical reversible function $f : \mathbb{Z}_d^n \to \mathbb{Z}_d^n$ (with ancillae)

$\rightarrow$ We derive this by explicitly constructing all possible f in $\mathcal{O}(d^n n)$ many $|0\rangle$-controlled X gates (optimal up to log-factor)

Toffoli + H generalizes to $|0\rangle$-controlled X + H

$|0\rangle$-controlled X and H are approximately universal for qudit quantum computation

$\rightarrow$ Construct Cliffords + single-qudit non-Clifford gate to get universality

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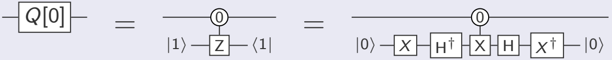
$\rightarrow$ Construct Cliffords + single-qudit non-Clifford gate to get universality

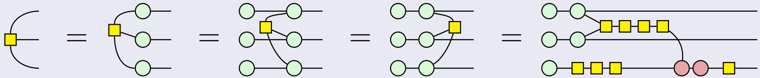
- For $d = 3$: Construct $R = \mathrm{diag}(1, 1, -1)$ gate
  - $\rightarrow$ Complicated construction, see paper...
- For $d > 3$: Construct $Q[0] = \mathrm{diag}(\omega, 1, ..., 1)$ gate



Qudit ZH is equivalent to post-selected circuits

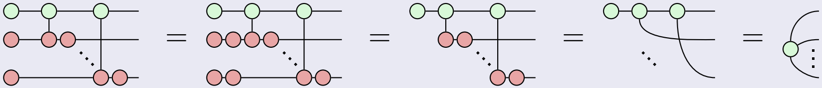
H-Box

Is just a CCZ acting on $|{+}{+}{+}\rangle$:



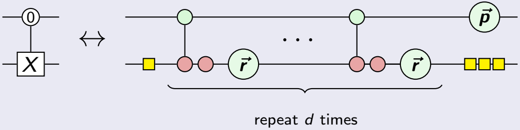
RHS is classical reversible (Toffoli-like) + H, and thus expressible via $|0\rangle$-controlled X

Z-spider



Qudit ZH can be translated to Qudit ZX

$|0\rangle$-controlled X



where $\vec{p} = \left(\omega^{\frac{-(d-1)}{2}}, \omega^{\frac{-(d-1)}{2}}, \ldots, \omega^{\frac{-(d-1)}{2}}\right)$ and
$\vec{r} = \left(\omega^{\frac{1}{d}}, \omega^{\frac{2}{d}}, \ldots, \omega^{\frac{d-1}{d}}\right)$

Result is circuit with post-selections over Clifford + $\sqrt[d]{Z}$ gateset[4].

[4] Lia Yeh (2023): Scaling W states in the qudit Clifford hierarchy. In: Proceedings of the 1st International Workshop on the Art, Science, and Engineering of Quantum Programming, arXiv.2304.12504

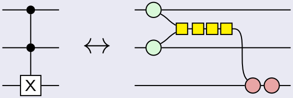
Overview

Thanks

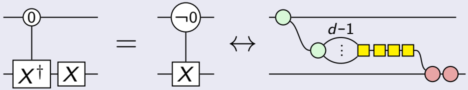
(AirBnB cat that fell asleep next to me while working on slides)

Qudit Gates

"Toffoli"

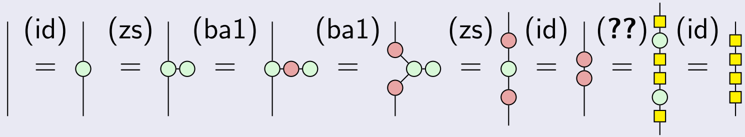


$|0\rangle$-controlled X



A Proof

Proof.



□

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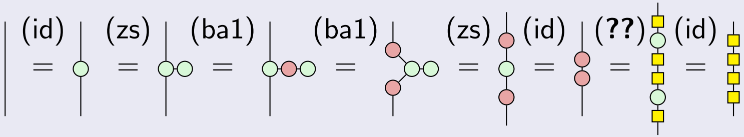
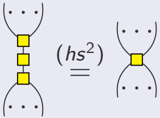


For qu*bit* ZH, this means that Hadamard self-inverseness follows from H-fusion, as



Sketch

Write a given matrix as entry-wise product of simpler matrices, e.g.

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \star \begin{pmatrix} 1 & b \\ 1 & 1 \end{pmatrix} \star \begin{pmatrix} 1 & 1 \\ c & 1 \end{pmatrix}$$

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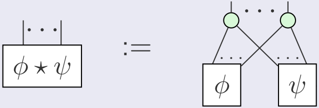


Sketch

For a matrix $L = \sum_{\vec{x},\vec{y}} \lambda_{\vec{x},\vec{y}} |\vec{y}\rangle\langle\vec{x}|$ containing only 1s and rs, describe the location of the 1s as a logical formula

$$\varphi_L(x_1, \dots, x_n, y_1, \dots, y_m) = \bigvee_{\substack{i_1,\dots i_n \\ j_1,\dots,j_m \\ \in\{0,\dots,d-1\} \\ \lambda_{i_1\dots i_n j_1\dots j_m}=1}} \bigwedge_{k=1}^{n} (x_k = i_k) \wedge \bigwedge_{\ell=1}^{m} (y_\ell = j_\ell)$$

Sketch

Inductively construct polynomial p_L such that

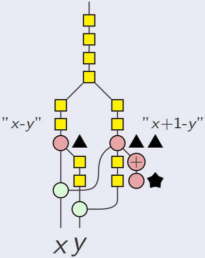
$$p_L(x_1, ..., x_n, y_1, ..., y_m) = 0 \iff \varphi_L(x_1, ..., x_n, y_1, ..., y_m)$$

Needs that $\mathbb{Z}_d$ has no zero-divisors if d prime

1. In the case of $\varphi = (p_1(x_1, ..., x_n) = p_2(x_1, ..., x_n))$ for $p_1, p_2 \in (\mathbb{Z}_d)[X_1, ..., X_n]$, set $p_\varphi = p_1 - p_2$
2. In the case of $\varphi = \neg\varphi'$, set $p_\varphi = 1 - (p_{\varphi'})^{d-1}$
3. In the case of $\varphi = \varphi_1 \vee \varphi_2$, set $p_\varphi = p_{\varphi_1} \cdot p_{\varphi_2}$

Turning Polynomial into ZH-diagram

Diagram of $|x, y\rangle \mapsto |p(x, y)\rangle$ for $p(x, y) = (x - y)(x + 1 - y)$:



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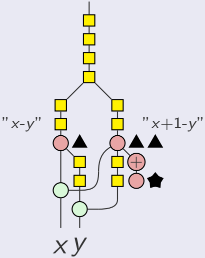

Apply $x \mapsto x^{d-1}$, post-select with $H(r) = (1, r, r^2, ..., r^{d-1}) \Rightarrow$ get state evaluating to 1 if $p(x, y) = 0$ and r otherwise