Locally Tomographic Shadows

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Overview

Local Tomography (LT) posits that the state of a composite system AB, is determined by the joint probabilities it assigns to separate, "local" measurements on A and B.

Classical probability theory and *complex* QM satisfy LT, but Real QM ($\mathbb{R}\mathbf{QM}$) does not.

This is clear on dimensional grounds, but let's look a bit deeper.

Let $\mathbf{H}$, $\mathbf{K}$ be (here, f.d.) real Hilbert spaces. Write $\mathcal{L}_s(\mathbf{H})$, $\mathcal{L}_a(\mathbf{H})$ for the spaces of symmetric, resp. anti-symmetric operators on $\mathbf{H}$, and similarly for $\mathbf{K}$. Let

$$\mathcal{L}_{ss} := \mathcal{L}_s(\mathbf{H}) \otimes \mathcal{L}_s(\mathbf{K}) \text{ and } \mathcal{L}_{aa} := \mathcal{L}_a(\mathbf{H}) \otimes \mathcal{L}_a(\mathbf{K})$$

Then

$$\mathcal{L}_s(\mathbf{H} \otimes \mathbf{K}) = \mathcal{L}_{ss} \oplus \mathcal{L}_{aa}$$

NB: and *orthogonal* decomposition w.r.t. trace inner product.

So if ρ's a density operator on $\mathbf{H} \otimes \mathbf{K}$,

$$\rho = \rho_{ss} + \rho_{aa}$$

with $\rho_{ss} \in \mathcal{L}_{ss}$ and $\rho_{aa} \in \mathcal{L}_{aa}$.

Given effects $a \in \mathcal{L}_s(\mathbf{H})$ and $b \in \mathcal{L}_s(\mathbf{K})$, $a \otimes b \in \mathcal{L}_{ss}$, so $\mathrm{Tr}((a \otimes b)\rho_{aa}) = 0$. Hence,

$$\mathrm{Tr}((a \otimes b)\rho) = \mathrm{Tr}((a \otimes b)\rho_{ss}).$$

States with the same $\mathcal{L}_{ss}$ component are *locally indistinguishable* in real QM.

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First, we need to say what we mean by a probabilistic theory.

Plan

1. Probabilistic Theories revisited
2. Construction of the LT shadow
3. The shadow of RQM
4. Some questions

I. Probabilistic Theories Revisited

For our purposes, a **probabilistic model** is pair $(\mathbb{V}, u)$ where

- $\mathbb{V}$ is an ordered real vector space, with positive cone $\mathbb{V}_+$;
- u is a strictly positive linear functional on $\mathbb{V}$, referred to as the *unit effect* of the model.

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States are elements $\alpha \in \mathbb{V}_+$ with $u(\alpha) = 1$.

Effects (measurement outcomes) are elements $a \in \mathbb{V}^*$ with $0 \leq a \leq u$: $a(\alpha)$ is the probability of a's of occurring in state α.

Processes from $(\mathbb{V}_1, u_1)$ to $(\mathbb{V}_2, u_2)$ are positive bilinear mappings $\phi : \mathbb{V}_1 \to \mathbb{V}_2$ with $u_2(\phi(\alpha)) \leq u_1(\alpha)$ for all $\alpha \in \mathbb{V}_{1+}$.

Standing assumptions and conventions:

- All models are finite-dimensional,
- We identify $\mathbb{V}$ with $\mathbb{V}^{**}$;

  $L^k(\mathbb{V}_1, ..., \mathbb{V}_k)$ is the space of k-linear forms on $\mathbb{V}_1 \times \cdots \times \mathbb{V}_k$;
- $\otimes$ is always the tensor product of vector spaces/linear maps,

  where
- we take $\mathbb{V} \otimes \mathbb{W} := \mathcal{L}^2(\mathbb{V}^*, \mathbb{W}^*)$ i.e., $(\alpha \otimes \beta)(a, b) := \alpha(a)\beta(b)$

A non-signaling composite of $(\mathbb{V}_1, u_1)$ and $(\mathbb{V}_2, u_2)$ is a model $(\mathbb{V}, u)$ plus positive linear mappings

$$m : \mathbb{V}_1 \times \mathbb{V}_2 \to \mathbb{V}$$

$$\pi : \mathbb{V}_1^* \times \mathbb{V}_2^* \to \mathbb{V}^*$$

such that

(i) $\pi(a, b)m(\alpha, \beta) = a(\alpha)b(\beta)$

(ii) $\pi(u_1, u_2) = u$.

Note m defines a linear mapping

$$m : \mathbb{V}_1 \otimes \mathbb{V}_2 \to \mathbb{V},$$

and π dualizes to give another,

$$\pi^* : \mathbb{V} \to \mathcal{L}^2(\mathbb{V}_1^*, \mathbb{V}_2^*)^* = \mathbb{V}_1 \otimes \mathbb{V}_2.$$

$(\mathbb{V}, u)$ is locally tomographic (LT) iff product effects separate states — equivalently, $\mathbb{V} \simeq \mathbb{V}_1 \otimes \mathbb{V}_2$.

Two extremal cases:

- The minimal tensor product $\mathbb{V}_1 \otimes_{\text{min}} \mathbb{V}_2$: cone generated by separable states.
- The maximal tensor product $\mathbb{V}_1 \otimes_{\text{max}} \mathbb{V}_2$: cone generated by tensors positive on product effects.

The definition of a composite just says we have positive linear mappings

$$\mathbb{V}_1 \otimes_{\text{min}} \mathbb{V}_2 \xrightarrow{m} \mathbb{V} \xrightarrow{\pi^*} \mathbb{V}_1 \otimes_{\text{max}} \mathbb{V}_2$$

composing to the identity.

Write **Prob** for the category of probabilistic models and processes. A probabilistic theory is a functor

$$\mathbb{V} : \mathcal{C} \to \textbf{Prob}$$

, where

- $\mathcal{C}$ is a symmetric monoidal category ("actual" physical systems and processes, or mathematical proxies for these)
- $\mathbb{V}(AB)$ is a non-signaling composite of $\mathbb{V}(A)$ and $\mathbb{V}(B)$
- $\mathbb{V}(I) = \mathbb{R}$.

We assume $\mathbb{V}$ is *injective on objects*, which makes $\mathbb{V}(\mathcal{C})$ a subcategory of **Prob**, with a well-defined monoidal structure given (on objects) by

$$\mathbb{V}(A), \mathbb{V}(B) \mapsto \mathbb{V}(AB).$$

II. Locally Tomographic Shadows

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- objects are finite lists $\vec{A} = (A_1, ..., A_n)$ of objects $A_i \in \mathcal{C}$ $(A_i \neq I)$ standing for $\Pi\vec{A} := \Pi_{i=1}^{n} A_i$ in $\mathcal{C}$ with the indicated decomposition.

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A strict symmetric monoidal category under concatenation; empty string is the unit.

Suppose that $\vec{A} := (A_1, ..., A_n) \in \mathcal{C}^*$ with composite $\Pi\vec{A} := A \in \mathcal{C}$: there's a positive linear mapping

$$\mathsf{LT}_{\vec{A}} := \pi^*_{\vec{A}} : \mathbb{V}(A) \longrightarrow \mathcal{L}^n(\mathbb{V}^*(A_1), ..., \mathbb{V}^*(A_n))$$

restricting $\omega \in \mathbb{V}(A)$ to product effects:

$$\widetilde{\omega}(a_1, ..., a_n) := \pi^*_{\vec{A}}(\omega)(a_1, ..., a_n) = (a_1 \otimes \cdots \otimes a_n)(\omega)$$

for all $(a_1, ..., a_n) \in \mathbb{V}^*(A_1) \times \cdots \times \mathbb{V}^*(A_n)$.

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for all $(a_1, ..., a_n) \in \mathbb{V}^*(A_1) \times \cdots \times \mathbb{V}^*(A_n)$.

We call $\widetilde{\omega}$ the *local shadow* of ω

Let $\widetilde{\mathbb{V}}(\vec{A})$ be the space $\bigotimes_i \mathbb{V}(A_i)$, ordered by the cone

$$\widetilde{\mathbb{V}}(\vec{A})_+ := \mathsf{LT}_{\vec{A}}(\mathbb{V}(\Pi A)_+)$$

of local shadows of elements of $\mathbb{V}(\Pi\vec{A})_+$.

With $\widetilde{u}_{\vec{A}} = u_{A_1} \otimes \cdots \otimes u_{A_n}$, $(\widetilde{\mathbb{V}}(\vec{A}), \widetilde{u}_{\vec{A}})$ is a model, the *locally tomographic shadow* of $(\mathbb{V}(A), u_A)$ with respect to the given decomposition.

Notation: Write

$$\widetilde{\mathbb{V}}(\vec{A}) \boxtimes \widetilde{\mathbb{V}}(\vec{B}) \; := \; \widetilde{\mathbb{V}}(\vec{A}\vec{B}).$$

In particular, for $A, B \in \mathcal{C}$,

$$\mathbb{V}(A) \boxtimes \mathbb{V}(B) \; = \; \widetilde{\mathbb{V}}(A, B) \; = \; \mathbb{V}(A) \otimes \mathbb{V}(B),$$

but ordered by the cone $\widetilde{\mathbb{V}}(A, B)$ generated by local shadows $\widetilde{\omega}$ of states $\omega \in \mathbb{V}(AB)$.

The effect cone of $\widetilde{\mathbb{V}}(A_1, ..., A_n)$ has a nice characterization:

Lemma: $\widetilde{\mathbb{V}}(A_1, ..., A_n)^*_+ \simeq \mathbb{V}^*(\Pi_i A_i)_+ \cap (\bigotimes_i \mathbb{V}^*(A_i))$.

In the bipartite case:

$$(\mathbb{V}(A) \boxtimes \mathbb{V}(B))^*_+ \simeq \mathbb{V}(AB)^*_+ \cap (\mathbb{V}(A)^* \otimes \mathbb{V}(B)^*).$$

What about processes?

Let $A = \Pi\vec{A}$ and $B = \Pi\vec{B}$ where $\vec{A} = (A_1, ..., A_n)$ and $\vec{B} = (B_1, ..., B_k)$. The following is routine:

Lemma: *Let $\Phi : \mathbb{V}(\Pi\vec{A}) \to \mathbb{V}(\Pi\vec{B})$ be a positive linear mapping. The following are equivalent:*

(a) Φ maps $Ker(LT_{\vec{A}})$ into $Ker(LT_{\vec{B}})$.

(b) If $\omega, \omega' \in \mathbb{V}(A)$ are locally indistinguishable, so are $\Phi(\omega), \Phi(\omega')$ in $\mathbb{V}(B)$.

(c) There exists a linear mapping $\phi : \bigotimes_i \mathbb{V}(A_i) \to \bigotimes_j \mathbb{V}(B_j)$ such that $LT_{\vec{B}} \circ \Phi = \phi \circ LT_{\vec{A}}$

A positive linear mapping $\Phi : \mathbb{V}(A) \to \mathbb{V}(B)$ satisfying these conditions is *locally positive* (with respect to the specified decompositions).

The linear mapping ϕ in part (c) is then uniquely determined. We call it the *shadow* of Φ, writing $\phi = \mathrm{LT}(\Phi)$.

Lemma: *If $\Phi : \mathbb{V}(A) \to \mathbb{V}(B)$ is locally positive, then $\phi = LT(\Phi)$ is positive as a mapping $\widetilde{\mathbb{V}}(A_1, ..., A_m) \to \widetilde{\mathbb{V}}(B_1, ..., B_n)$.*

Locally positive maps are reasonably abundant, but do exclude some important morphisms in $\mathbb{R}\mathbf{QM}$.

Examples:

(a) If σ and α are swap and associator morphisms in $\mathcal{C}$, $\mathbb{V}(\sigma)$ is locally positive, but $\mathbb{V}(\alpha)$ need not be.

(b) if α is a state on $A = A_1 \otimes \cdots \otimes A_n$, then the corresponding mapping $\alpha : \mathbb{R} = \mathbb{V}(I) \to \mathbb{V}(A)$ given by $\alpha(1) = \alpha$ is trivially locally positive (the kernel of LT_I is trivial). But an effect $a : \mathbb{V}(A) \to \mathbb{R}$ need not be locally positive.

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Call a morphism $\Pi\vec{A} \xrightarrow{\phi} \Pi\vec{B}$ *local* iff $\mathbb{V}(\phi) : \mathbb{V}(\Pi\vec{A}) \to \mathbb{V}(\Pi\vec{B})$ is locally positive (relative to the preferred factorizations of A and B).

Write $\text{Loc}(\mathcal{C}, \mathbb{V})$ for the monoidal subcategory (it is one) of $\mathcal{C}^*$ having the same objects but only local morphisms.

Lemma: $\widetilde{\mathbb{V}} : Loc(\mathcal{C}, \mathbb{V}) \to$ **Prob** *is a locally tomographic probabilistic theory — the* locally tomographic shadow *of* $\mathbb{V}$*.*

IV. The Shadow of Real Quantum Theory

For simplicity, let $\mathbf{H} = \mathbf{K}$, writing $\mathcal{L}_s$ for $\mathcal{L}_s(\mathbf{H})$. Recall

$$\mathcal{L}_s(\mathbf{H} \otimes \mathbf{H}) = \mathcal{L}_{ss} \oplus \mathcal{L}_{aa},$$

where

$$\mathcal{L}_{ss} = \mathcal{L}_s(\mathbf{H}) \otimes \mathcal{L}_s(\mathbf{K}) \text{ and } \mathcal{L}_{aa} = \mathcal{L}_a(\mathbf{H}) \otimes \mathcal{L}_a(\mathbf{K}).$$

Then LT is just the projection onto $\mathcal{L}_{ss}$. This is just $\mathrm{Sym} \otimes \mathrm{Sym}$, where

$$\mathrm{Sym}(a) := \frac{1}{2}(a + a^t).$$

But $\mathrm{Sym} \otimes \mathrm{Sym}$ is not positive!

Example: Let $\{x, y\}$ be an ONB for $\mathbb{R}^2$, and let

$$z \;=\; \frac{1}{\sqrt{2}}(x \otimes y + y \otimes x)$$

and set

$$W := (\mathrm{Sym} \otimes \mathrm{Sym})(P_z) \;=\; \frac{1}{2}(P_x \otimes P_y + P_y \otimes P_x) + S \otimes S$$

where $Sx = \frac{1}{2}y$ and $Sy = \frac{1}{2}x$.

This is not a positive operator. For instance, If $v = x \otimes x - y \otimes y$, then $Wv = -\frac{1}{4}v$.

So $(\mathcal{L}_s \boxtimes \mathcal{L}_s)_+$ is strictly larger than $\mathcal{L}_{ss} \cap \mathcal{L}_+$.

A priori we have now have

$$(\mathcal{L}_s \otimes_{\mathrm{min}} \mathcal{L}_s)_+ \leq \mathcal{L}_{ss} \cap \mathcal{L}_+ < (\mathcal{L}_s \boxtimes \mathcal{L}_s)_+ \leq (\mathcal{L}_s \otimes_{\mathrm{max}} \mathcal{L}_s)_+.$$

In fact,

Theorem: All *of these embeddings are strict.*

(The hard one is the last. Uses the existence of unextendable product bases.)

The geometry of the state space in LT($\mathbb{R}$**QM**) is nontrivial. The following restates a result of Chiribella, D'Ariano and Perinotti (2009):

Theorem: *If α, β are states density operators on* $\mathbf{H} \otimes \mathbf{K}$ *with α pure, then*

$$LT(\alpha) = LT(\beta) \ \Rightarrow \ \alpha = \beta.$$

So the the LT map never identifies a pure state with any other state. Only nontrivially mixed states get "pasted togther".

Processes

Let $\Phi : \mathcal{L}_s(\mathbf{H} \otimes \mathbf{H}) \to \mathcal{L}_s(\mathbf{H} \otimes \mathbf{H})$ be a positive linear mapping. With respect to the decomposition

$$\mathcal{L}_s(\mathbf{H} \otimes \mathbf{K}) = \mathcal{L}_{s,s} \oplus \mathcal{L}_{s,a} \oplus \mathcal{L}_{a,s} \oplus \mathcal{L}_{a,a},$$

Φ has an operator matrix $\begin{bmatrix} \Phi_{ss} & \Phi_{sa} \\ \Phi_{as} & \Phi_{aa} \end{bmatrix}$ We have

Lemma: *Let Φ be as above. Then Φ is locally positive iff $\Phi_{sa} = 0$, and in this case, $LT(\Phi) = \Phi_{s,s}$.*

IV. Conclusions and questions

Compact Closure The effect $\epsilon : a \otimes b \mapsto \mathrm{Tr}(ab^t)$ is not local. Hence, LT($\mathbb{R}$**QM**) does not inherit the compact structure of $\mathbb{R}$**QM**. If $\mathcal{C}$ is compact closed, when *is* LT($\mathcal{C}, \mathbb{V}$) compact closed?

LT and Complex QM How does LT interact with the restriction-of-scalars and complexification functors $(-)_{\mathbb{R}} : \mathbb{C}\mathrm{QM} \to \mathbb{R}\mathbf{QM}$, $(-)^{\mathbb{C}} : \mathbb{R}\mathbf{QM} \to \mathbb{C}\mathrm{QM}$?

The Shadow of InvQM In (BGW, Quantum 2020), we constructed a non-LT theory **InvQM**, containing finite-dimensional real and quaternionic QM and also a relative of complex QM. What is LT(**InvQM**)?

Non-deterministic shadows Not all processes in $\mathbb{R}$**QM** are local. Suppose Alice and Bob agree that their joint state is ω. This is consistent with the true global state being any $\mu \in \mathsf{LT}_{A,B}^{-1}(\omega)$. If μ evolves under a (global) process $\phi : \mathbb{V}(AB) \to \mathbb{V}(CD)$, the result will be one of the states in $\phi(\mathsf{LT}_{A,B}^{-1}(\omega))$. If ϕ is not local, these needn't lie in a single fibre of $\mathsf{LT}_{C,D}$: parties C and D might observe any of the different states in $\mathsf{LT}_{C,D}(\phi(\mathsf{LT}_{A,B}^{-1}(\omega)))$, giving the impression that ϕ acted indeterministically. How should one quantify this extra layer of uncertainty?