A Living Review of Quantum Computing for Plasma Physics

Abstract

A recent report of the United States Department of Energy "Quantum for Fusion, Fusion for Quantum" has highlighted several opportunities for scientific discovery and technology advances at the interface of quantum computing and quantum technologies with fusion and plasma physics.

Quantum Computing promises accelerated simulation of certain classes of problems, in particular in plasma physics. The goal of this document is to provide a comprehensive list of citations for those developing and applying these approaches to experimental or theoretical analyses. As a living document, it will be updated as often as possible to incorporate the latest developments. Suggestions are most welcome.

The purpose of this note is to collect references for quantum algorithms already relevant to plasma physics. A minimal number of categories is chosen in order to be as useful as possible. Note that papers may be referenced in more than one category.

To facilitate search, the tags `FTol` (fault-tolerant quantum computing), `NISQ` (noisy-intermediate scale quantum computing), `QAnn` (quantum annealing), `QIns` (quantum-inspired), and `Tool` (generally useful tool) are applied if clearly appropriate.

Color-filled tags indicate the type of content. Since most papers contain some form of theoretical analysis, we use the theoretical tag `Theo` solely for the papers with analytical results, and no considerable numerical or experimental results. The tag `Num` marks papers with numerical simulations, but no experimental results run on quantum devices. Finally, `Exp` marks papers with displayed experimental results. We may ommit this tag if the paper is referenced and tagged in a subsequent subsection.

The fact that a paper is listed in this document does not endorse or validate its content - that is for the community (and for peer-review) to decide. Furthermore, the classification here is a best attempt and may have flaws - please let us know if (a) we have missed a paper you think should be included, (b) a paper has been misclassified or wrongly tagged, or (c) a citation for a paper is not correct or if the journal information is now available.

In order to be as useful as possible, this document will continue to evolve so please check back[1] before you write your next paper. You can simply download the .bib file to get all of the latest references. Please consider citing Ref. [AC22] when referring to this living review.

- **Reviews**
  Below are links to (static) general and specialized reviews.
  - **Modern review** [DS20]
    Demonstration that many plasma-wave problems are naturally representable in a quantumlike form and thus are naturally fit for quantum computers. General plasma problems that include non-Hermitian dynamics (instabilities, irreversible dissipation) and nonlinearities. Potential applications of hybrid quantum-classical computers.
- **Linear**
  Linear methods are best suited for quantum computers.
  - **Heat equation** [LMS20].
    Classical and quantum algorithms for the heat equation.
  - **Vlasov** [ESP19].
    Casestudy of linear Landau damping.
  - **Quantum signal processing for plasma wave simulation** [NSD21].
    Numerical modeling of radio-frequency waves in plasma emulated on a classical computer.

[1] See http://epp.ist.utl.pt/qppq/.

- **Nonlinear**
  Nonlinear dynamics are not directly suited for quantum computers. However, several techniques can be used to embed the problem within a linear framework.
  - **Carleman linearization** [Liu+20].
    Using Carleman linearization [KS91] for the Burgers equation
  - **Bosonic** [Llo+20].
    Description on how to model nonlinear problems as Bose-Einstein condensates.
  - **Nonlinear maps** [DS21].
    Computing a general differentiable invertible nonlinear map on a quantum computer using only linear unitary operations. The price of this universality is that the original map is represented adequately only on a finite number of iterations.
  - **Linear embeddings** [ESP21].
    Linear embedding of nonlinear dynamical systems and prospects for efficient quantum algorithms
  - **Cubic couplings** [Shi+21].
    The 3-wave mixing problem appears in areas such as nonlinear optics, gauge theories, as well as plasma and fluid dynamics.
  - **Koopman-von Neumann** [Jos20].
    Koopman-von Neumann approach to Quantum Mechanics in the context of plasma physics..
- **Variational/differentiable linear/nonlinear**
  Nonlinear variational methods put more of the computational effort on calculating a cost function than in a linear embedding.
  - **Variational quantum algorithms for nonlinear problems** [Lub+20a].
    The authors show that nonlinear partial differential equations can be efficiently solved by variational quantum computing by using multiple copies of variational quantum states to treat nonlinearities efficiently and by introducing tensor networks as a programming paradigm. Example of the nonlinear Schrödinger equation. Demonstration that the variational quantum ansatz can be exponentially more efficient than matrix product states. Results obtained on an IBM Q device.
  - **Toward Cosmological Simulations of Dark Matter on Quantum Computers** [MS21].
    Schrödinger–Poisson equations for the evolution of self-gravitating dark matter, based on a hybrid quantum–classical variational algorithm framework (see [Lub+20a]).

- **Solving nonlinear differential equations with differentiable quantum circuits** [KPE21].
  Chebyshev quantum feature map as a basis set of fitting polynomials. As an example of application, the authors solve Navier-Stokes equations and compute density, temperature, and velocity profiles for the fluid flow in a convergent-divergent nozzle.
- **Variational quantum simulations of stochastic differential equations** [Kub+21].
  Approximating the target SDE by a trinomial tree structure with discretization. The embedding of the probability distribution is done directly in the amplitudes of the quantum state.
- **Variational quantum algorithm for the Feynman-Kac formula** [Alg+22].
  Variational quantum imaginary time evolution based on the correspondence between the Feynman-Kac partial differential equation (PDE) and the Wick-rotated Schrödinger equation. Generalizes to multidimensional system of stochastic differential equations. Since classical PDEs preserve probability distributions instead of preserving the $\ell - 2$ norm, a proxy norm is introduced to keep the numerical solution approximately normalized throughout the evolution.

- **General variational**
  Variational approaches to solve general or specific classes of partial differential equations.
  - **Evolution equations** [LEK22].
    The proposed Variational Quantum Evolution Equation Solver solves general evolution equations. An analysis of the heat equation is presented, and a scheme for solving nonlinear equations such as the reaction-diffusion and the incompressible Navier-Stokes equations is demonstrated.
  - **Advection-Diffusion equation** [Dem+22].
    A resource-friendly version of the Variational Quantum Linear Solver is experimentally tested for the advection-diffusion equation.

- **Lattice**
  Lattice methods have been successfully applied in QED and QCD models.
  - **Real-time lattice scalar QED** [Shi+18].
    Real-time lattice quantum electrodynamics provides a unique tool for simulating plasmas in the strong-field regime, where collective plasma scales are not well separated from relativistic-quantum scales. Scalar QED is studied as a toy model.
  - **Maxwell's equations** [Vah+20].
    Propagation of wavepackets in dielectric media. A second-order accurate 4-spinor scheme is developed and tested successfully for two-dimensional (2-D)

propagation of a Gaussian pulse in a uniform medium whereas for normal (1-D) incidence of an electromagnetic Gaussian wave packet onto a dielectric interface requires 8-component spinors because of the coupling between the two electromagnetic polarizations.

- System of linear equations `NISQ` [Bra+20; HBR19; Xu+21], `QAnn` Exp [BL22], `QIns` Theo [SM21].
- System of nonlinear equations `NISQ` [Lub+20b].
  - System of polynomial equations `QAnn` Exp [Cha+19].
- Ordinary differential equations `QAnn` [Zan+21].
  - Linear `QAnn` Exp [Zan+21].
    * Second-order `QAnn` Exp [SS19].
  - Nonlinear `QAnn` Num [Zan+21].
- Partial differential equations `FTol` [CLO21], `QIns` [Gar21] (check refs).
  - Linear `QAnn` Num [CS22].
    * Laguerre `QAnn` Num [CS22].
    * Wave `QAnn` Num [CS22].
    * Non-homogeneous [Bra+20].
      · Poisson `NISQ` [Bra+20; Liu+21; Sat+21], `FTol` [Cao+13].
    * Vlasov-Poisson `QIns` Num [YL22] (check refs).
    * Fokker-Planck `QIns` Num [Gar21].
  - Parabolic `QIns` [Pat+22].
    * Black-Scholes-Barenblatt `QIns` Num[Pat+22].
    * Hamilton-Jacobi-Bellman `QIns` Num[Pat+22].
  - Black-Scholes [FJO21; MK22].
  - Helmholtz [Ewe+22].
  - Heat `NISQ` [FJO21; MK22; LEK22], `FTol` [LMS22].
  - Evolution `NISQ` [LEK22].
    * Reaction-diffusion `NISQ` [LEK22].
    * Incompressible Navier-Stokes `NISQ` [LEK22].
- Quantum simulation
  - Imaginary time evolution `NISQ` [McA+19].

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