

Entangling Schrödinger's cat states by seeding a Bell state or swapping the cats

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In quantum information processing, two primary research directions have emerged: one based on discrete variables (DV) and the other on the structure of quantum states in a continuous-variable (CV) space. It is increasingly recognized that integrating these two approaches could unlock new potentials, overcoming the inherent limitations of each. Here, we show that such a DV–CV hybrid approach, applied to superconducting Kerr parametric oscillators (KPOs), enables us to entangle a pair of Schrödinger's cat states by two straightforward methods. The first method involves the entanglement-preserving and deterministic conversion between Bell states in the Fock-state basis (DV encoding) and those in the cat-state basis (CV encoding). This method would allow us to construct quantum networks in the cat-state basis using conventional schemes originally developed for the Fock-state basis. In the second method, the $\sqrt{i}\text{SWAP}$ gate operation is implemented between two cat states following the procedure used for Fock-state encoding. This DV-like gate operation on CV encoding not only completes the demonstration of a universal quantum gate set in a KPO system but also enables faster and simpler gate operations compared to previous SWAP gate implementations on bosonic modes. Our work offers a simple yet powerful application of DV–CV hybridization while also highlighting the scalability of this planar KPO system.

I. INTRODUCTION

For nearly three decades, there have been two paradigms in quantum information processing: one involves discrete variables (DVs), such as photon number (Fock) states or spin states [1–4], whereas the other relies on the structure of quantum states in a continuous-variable (CV) space, such as Schrödinger's cat and Gottesman–Kitaev–Preskill states [5–7]. Recently, considerable efforts have focused on bridging DV and CV quantum information to overcome the limitations of each paradigm [8–15]. Parametrically driven Kerr nonlinear resonators, often referred to as Kerr parametric oscillators (KPOs) [16–20], offer a unique testbed for this task, particularly for exploring emergent quantum properties like entanglement in interacting quantum systems. This capability is enabled by simple one-to-one conversion between Fock and cat states via parametric pump control [21–27, 62].

In our previous work [29], we experimentally demonstrated that such conversion in a superconducting KPO preserves the quantum coherence of the system, with the underlying physics being quantum tunnelling in phase space [30, 31]. Furthermore, we

showed that single-gate operations on cat states in a KPO can be implemented similarly to conventional gate operations on the Fock-state basis [32–38].

In this work, we introduce two straightforward methods to create entangled cat states—a valuable resource for fault-tolerant quantum computation and communication [39–45]—by extending our approach that bridges DV and CV domains. The first method is the entanglement-preserving conversion from Fock-state encoding to cat-state encoding. Although there have been studies on two interacting KPOs [46, 47], the entanglement between them and its preservation during the conversion have yet to be investigated. Such a conversion suggests the possibility of constructing quantum networks in the cat basis using conventional schemes originally developed for the Fock basis, thereby reducing experimental complexity. Thus, our demonstration highlights the potential of DV–CV hybridization and may lay new groundwork for constructing quantum networks in the cat basis.

The next method is to implement the $\sqrt{i}\text{SWAP}$ gate between two cat states in a manner almost identical to that for Fock-state encoding [48]. This allows us to create entangled cat states faster than previous implementations on bosonic modes [49, 50], using only a single gate pulse. Furthermore, our implementation completes the demonstration of a universal quantum gate set, alongside the single-cat gate operations from

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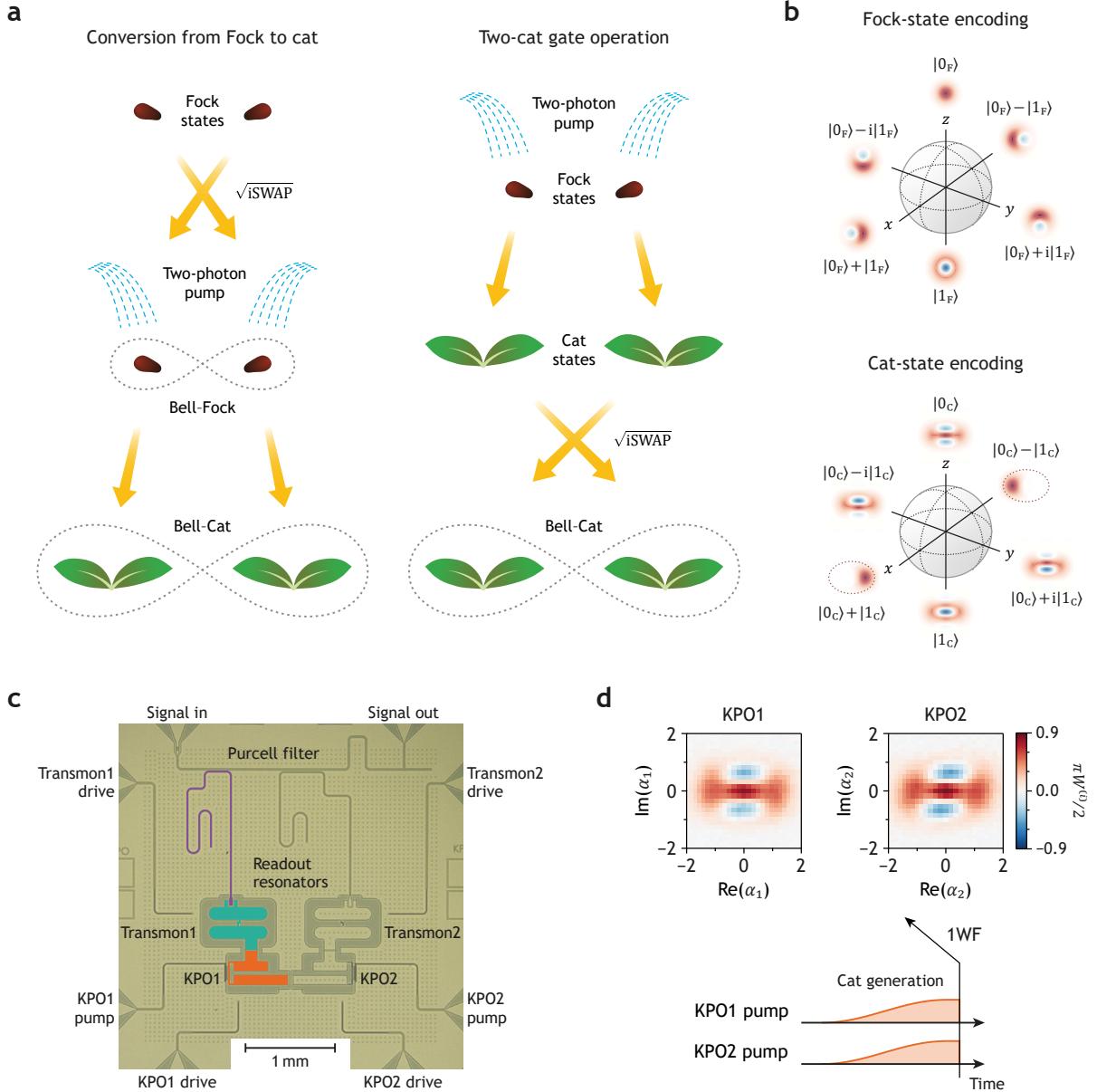


Fig. 1: Concept of the experiment. **a** Seed-sprout analogy for our methods to create Bell–Cat states. In this analogy, the seeds represent Fock state encoding, the sprouts represent cat state encoding, water sprinkles represent two-photon pumps, and gray dotted curves indicate entanglement. The normalization factor was omitted for simplicity. **b** Figure of the chip. The left side is in false colour for clarity. Each KPO is composed of 10 direct-current superconducting quantum interference devices (DC SQUIDs) with a shunting capacitor. The two KPOs are capacitively coupled. The state of each KPO is monitored by the nearby transmon (green) and its readout resonator (purple). **c** Simultaneous and independent generation of even cat states $|0_c\rangle$ from vacuum states $|0_F\rangle$ and the corresponding pulse sequence. The pulse sequence used to measure the Wigner function is omitted for simplicity (see Supplementary Fig. 2 for the full pulse sequence). The colour represents the scaled one-mode Wigner function (1WF), i.e., the number parity.

our previous work [29]. Such two-KPO gate operation for cat-state encoding, which we refer to as the two-cat gate, has not been demonstrated, despite its importance in showing the scalability of a KPO system as a promising platform for quantum information processing. Thus, our implementation supports the scalability of planar superconducting KPO systems.

For both our methods, we can make analogies to seeds (from the DV domain) sprouting (in the CV do-

main) thanks to watering (two-photon pumping), as illustrated in Fig. 1a. In this paper, we denote Fock states $|0\rangle$ and $|1\rangle$ as $|0_F\rangle$ and $|1_F\rangle$, respectively. Correspondingly, the even and odd cat states are denoted as $|0_c\rangle$ and $|1_c\rangle$ as shown in Fig. 1b. In addition, we refer to the Bell states in the Fock basis as Bell–Fock states and designate the resulting entangled cat states as Bell–Cat states.

II. SETUP

The chip used in this work is shown in Fig. 1c. It is the same chip used in our previous study [29]. The Hamiltonian of our system can be described as (see Sec. 1 of Supplementary Information for the derivation)

$$\hat{\mathcal{H}}(t) = \Delta_1 \hat{a}_1^\dagger \hat{a}_1 - \frac{K_1}{2} \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \frac{P_1(t)}{2} (\hat{a}_1^\dagger \hat{a}_1^\dagger + \hat{a}_1 \hat{a}_1) \\ + \Delta_2 \hat{a}_2^\dagger \hat{a}_2 - \frac{K_2}{2} \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 + \frac{P_2(t)}{2} (\hat{a}_2^\dagger \hat{a}_2^\dagger + \hat{a}_2 \hat{a}_2) \\ + g (\hat{a}_1^\dagger \hat{a}_2 e^{+i\Delta_p t} + \hat{a}_1 \hat{a}_2^\dagger e^{-i\Delta_p t}). \quad (1)$$

Here, we are working in units where $\hbar = 1$; \hat{a}_i and \hat{a}_i^\dagger are the ladder operators for the KPO*i* ($i = 1, 2$); Δ_i ($\equiv \omega_{K_i} - \omega_{p_i}/2$) is the KPO-pump frequency detuning, where ω_{K_i} is the transition frequency between the $|0_F\rangle$ and $|1_F\rangle$ states, and ω_{p_i} is the frequency of the two-photon pump; K_i is the self-Kerr coefficient; P_i is the amplitude of the pump; g is the coupling constant; and Δ_p [$\equiv (\omega_{p1} - \omega_{p2})/2$] is half of the detuning between the two pumps. The Hamiltonian in Eq. (1) is in the rotating frame defined by $\hat{\mathcal{H}}_0 = \sum_i (\omega_{p_i}/2) \hat{a}_i^\dagger \hat{a}_i$. See Supplementary Table 1 for the values of these system parameters.

The cat states are generated adiabatically using the pump pulse with the profile $\sin^2(\pi t/2\tau_{\text{ramp}})$, where the ramping time τ_{ramp} is 1 μs (see Methods for more details). Throughout this work, for both KPOs, the P/K ratio is chosen to be 1.0, where the Kerr coefficient is approximately 2 MHz after ramping up the pump, and the pump detuning [Δ_1 and Δ_2 in Eq. (1)] is chosen to be 1.0 MHz. Since the detuning between the two KPOs (144 MHz) is nearly 20 times larger than the coupling (8 MHz), the interaction is effectively turned off on the timescale of the measurements; thus, cat states can be generated and measured independently and simultaneously as shown in Fig. 1d.

III. RESULTS

A. Conversion from Fock to cat

We first prepare all four types of Bell–Fock state, $|0_F 0_F\rangle \pm |1_F 1_F\rangle$ and $|0_F 1_F\rangle \pm |1_F 0_F\rangle$. Subsequent two-photon pumping to each KPO converts the Bell–Fock state into the same type of Bell–Cat state; for instance, from $|0_F 0_F\rangle + |1_F 1_F\rangle$ to $|0_C 0_C\rangle + |1_C 1_C\rangle$ (see Fig. 2c for the pulse sequence). This approach relies on the fundamental property of entanglement, namely, that “entanglement is preserved under local unitary operations” [51].

The Bell–Fock state is prepared by activating the interaction between the KPOs by applying a parametric pulse with either the frequency $\omega_{K1} + \omega_{K2}$ or $\omega_{K1} - \omega_{K2}$ to the pump ports [3]. A parametric pulse with each frequency induces the transitions

between $|0_F 0_F\rangle$ and $|1_F 1_F\rangle$, and between $|0_F 1_F\rangle$ and $|1_F 0_F\rangle$ based on the three-wave mixing capability of our KPOs. Using such transitions, we can create states $|0_F 0_F\rangle + e^{i\phi_s} |1_F 1_F\rangle$ and $|0_F 1_F\rangle + e^{i\phi_d} |1_F 0_F\rangle$, where the phases ϕ_s and ϕ_d are determined by the phase of the parametric pulse (virtual Z gate). We refer to this pulse as the Bell-preparation pulse. Rabi oscillations associated with the Bell-preparation pulse are shown in Fig. 2a.

For the full characterization of such entangled quantum states, we measured the two-mode Wigner functions (2WFs) (Fig. 2d,e) [42, 52] because the one-mode Wigner functions (1WFs) cannot provide information on entanglement—all Bell states show the same 1WF, which is identical to that of the fully mixed state (Fig. 2b). The 2WFs of the target Bell–Fock and Bell–Cat states are shown in Supplementary Fig. 5.

We observe all essential features in the 2WF of Bell–Cat states (Fig. 2e). Firstly, in the Re–Re plots with $\text{Im}(\alpha_i) = 0$ ($i = 1, 2$), two red circles aligned diagonally indicate the correlation between the two KPOs, similar to the results in Ref. [47]. The alignment direction of the red circles represents the sign of the superposition. The colour of the centre circle, which represents the joint number parity, indicates the type of Bell state; for instance, $|01\rangle + e^{i\phi} |10\rangle$ shows a blue centre regardless of whether the basis is Fock or cat. Secondly, the interference pattern in the Im–Im plot with $\text{Re}(\alpha_i) = 0$ demonstrates that the correlation is of quantum nature.

Note that the patterns in Fig. 2d,e illustrate how the 2WFs of Bell–Fock states evolve to those of Bell–Cat states: As the pump amplitude increases, the pattern in Fig. 2d elongates along the diagonal axis, eventually resembling the Re–Re plots in Fig. 2e. Regarding the Im–Im plots of Bell–Fock states, those of $|0_F 1_F\rangle \pm |1_F 0_F\rangle$ are identical to the Re–Re plots, whereas the Im–Im plot of $|0_F 0_F\rangle \pm |1_F 1_F\rangle$ matches the Re–Re plot of $|0_F 0_F\rangle \mp |1_F 1_F\rangle$, as confirmed by our measurements (not shown). The Im–Im plots in Fig. 2e can be interpreted as a compressed version of the plots in Fig. 2d along the diagonal axis. These 2WF patterns show the profound connection between quantum correlations in the Bell–Fock and Bell–Cat states.

The fidelity between the experimentally created Bell–Fock states and the target Bell–Fock states is 0.81 ± 0.01 (the error represents the standard deviation). This fidelity was obtained by reconstructing the density matrix from the measured 2WFs (see Methods). By simulating our Bell–Fock preparation process via the Lindblad master equation with the system parameters in Supplementary Table 1, we find that approximately half of the infidelity is caused by thermal excitation and the other half by relaxations such as single-photon loss and dephasing. See Sec. 2 of Supplementary Information for more details on the simulation.

The fidelity between the experimentally created Bell–Cat states and the target states is 0.61 ± 0.04 . For completely mixed cat states, this value would be 0.25.

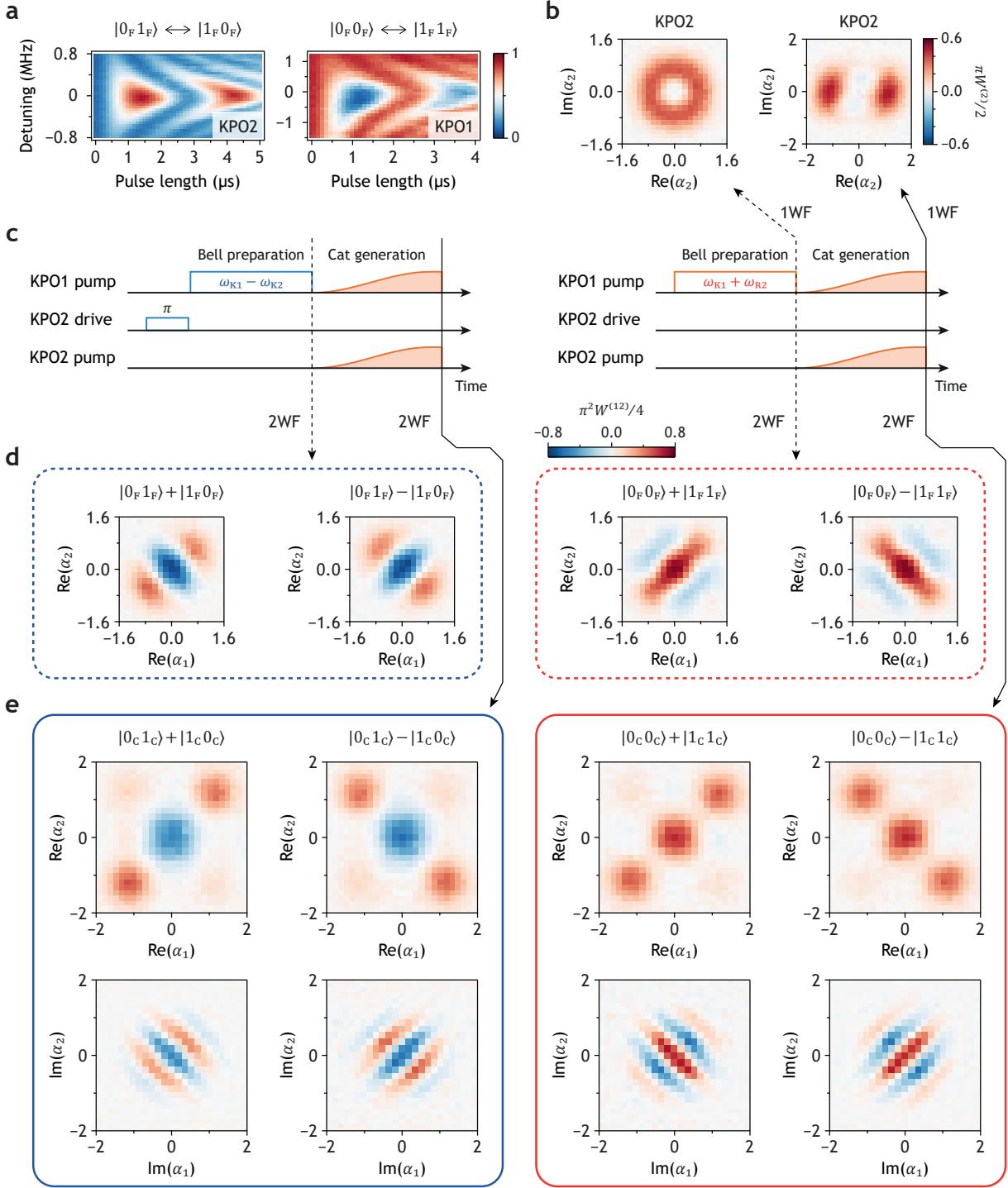


Fig. 2: Converting Bell-Fock states to Bell-Cat states. **a** Rabi oscillations for Bell-preparation pulse. Regarding Rabi oscillations associated with $|0_F 1_F\rangle \leftrightarrow |1_F 0_F\rangle$ transitions, the colour represents the population of the $|0_F\rangle$ state of KPO2 and zero detuning corresponds to the frequency $\omega_{K1} - \omega_{K2}$. As for Rabi oscillations associated with $|0_F 0_F\rangle \leftrightarrow |1_F 1_F\rangle$ transitions, the colour represents the population of the $|0_F\rangle$ state of KPO1 and zero detuning corresponds to the frequency $\omega_{K1} + \omega_{K2} - \Delta_{AC}$, where Δ_{AC} is an AC Stark-like frequency shift whose value is 21 MHz in this measurement. **b** Measured one-mode Wigner function (1WF) of Bell-Fock and Bell-Cat states. **c** Pulse sequences for Bell-Fock state preparation and Bell-Cat state generation. The amplitude and length of pulses are not to scale. **d,e** Measured two-mode Wigner function (2WF) for Bell-Fock (d) and Bell-Cat (e) states. In Re-Re plots, $\text{Im}(\alpha_1) = \text{Im}(\alpha_2) = 0$, whereas in Im-Im plots, $\text{Re}(\alpha_1) = \text{Re}(\alpha_2) = 0$. The colour represents the joint number parity.

Further suppression of the fidelity during the conversion process is primarily caused by single-photon loss. The most notable symptom in the 2WF caused by single-photon loss is that the colour of the centre circle in the Re–Re plots with $\text{Im}(\alpha_i) = 0$ ($i = 1, 2$) and the interference pattern in the Im–Im plots with $\text{Re}(\alpha_i) = 0$ decay with time [42, 53], resulting in a weaker contrast than the 2WF plots of the target Bell–Cat states (shown in Supplementary Fig. 5), as shown in Fig. 2e. The simulation with the Lindblad master equation gives a fidelity of about 0.71, which is reasonably close to our experimental result.

Dephasing caused by low-frequency noise does not affect the fidelity during and after the ramping of the pump because the cat states in KPOs are protected by the energy gap [24, 54]. Experimental evidence indicates that the primary source of relaxation for cat states in a KPO is single-photon loss [29, 34, 55]. However, dephasing during the Bell–Fock state preparation, specifically fluctuations in the phase of the Bell-preparation pulse, causes another notable symptom as shown in Fig. 2e: Two corners in the Re–Re plots are slightly pink, whereas those of the target Bell–Cat states should be completely white (see Supplementary Fig. 5) [47]. Thus, the fidelity can be improved by suppressing single-photon loss, thermal excitation, and dephasing.

If we assume $T_1 = T_2 = 100 \mu\text{s}$ [56–59] and a thermal photon number of 0.01, the fidelity of the Bell–Fock states is approximately 0.96 using the same Bell-preparation pulse, as simulated by the Lindblad master equation. The primary source of remaining infidelity arises from unwanted higher-state excitations due to the small Kerr coefficient. This issue can be mitigated by employing DRAG-like pulses [60]. Importantly, since our KPO is frequency-tunable, techniques such as spin echo are necessary to achieve a long T_2 .

With the same relaxation times and thermal population, the simulation shows that the fidelity of the Bell–Cat state can reach approximately 0.93 using the same cat generation pulse. The primary source of reduced fidelity in this case is population leakage out of the computational subspace caused by non-adiabatic transitions during cat generation. This leakage can be suppressed by employing a counterdiabatic or numerically optimized pulse [61, 62]. (In this work, a counterdiabatic pulse was not used, unlike in our previous work [29]. See Methods for more information.)

B. Two-cat gate operation

One interesting and useful property of this KPO system is that we can use the same type of parametric pulse for two-qubit gate operation both in the Fock and cat state encoding [48]. In this work, the parametric pulse with the frequency $\omega_{K1} - \omega_{K2}$, which we used to prepare the Bell–Fock state, was also used for the gate operation.

We observe the Rabi-like oscillations in the parity

of each KPO, which we call the two-cat Rabi, as a function of the phase and the detuning of the parametric pulse, which we call the gate pulse (Fig. 3a,b). Here, the gate phase ϕ_g is the phase relative to the pumps, and the gate detuning Δ_g is the detuning from $(\omega_{p1} - \omega_{p2})/2$. For this measurement, we first prepare $|0_{F1F}\rangle$ and convert it to $|0_{C1C}\rangle$ by applying the pumps. Then, we apply the gate pulse, in addition to the pumps, as shown in Fig. 3c.

Note that the two KPOs exhibit the same two-cat Rabi oscillations but with opposite parities. From the simulation, we determined the gate amplitude to be 2.96 MHz (see Supplementary Fig. 4a and its caption for details). One-mode Wigner functions show that during the Rabi oscillations, the state evolves from $|0_{C1C}\rangle$ (no gate) to $|1_{C0C}\rangle$ (iSWAP). To determine the intermediate quantum state between these two points, a 2WF measurement with an additional offset in displacement is required. This is because 1WF and the Re–Re (Im–Im) plot in 2WF without an additional displacement along the imaginary (real) axes cannot distinguish the following three states: $|0_{C1C}\rangle \pm i|1_{C0C}\rangle$, which are the states after the $\sqrt{i\text{SWAP}}$ gate, and the mixture of $|0_{C1C}\rangle$ and $|1_{C0C}\rangle$. The Re–Re plot with an additional offset shows that the state is $|0_{C1C}\rangle - i|1_{C0C}\rangle$ (the plot at the bottom of Fig. 3d), confirming that the two-cat gate operation is the $\sqrt{i\text{SWAP}}$ gate (see Supplementary Fig. 5).

The $\sqrt{i\text{SWAP}}$ gate time, 275 ns, is significantly faster than recent implementations of similar SWAP gate operations on bosonic modes [49, 50]. This short gate time is possible because the beam-splitter interaction is inherently built into the Hamiltonian [Eq. (1)], and the KPO system enables us to adopt schemes for gate operations in Fock-state encoding. The primary limitations on our gate time are the AC Stark-like frequency shift induced by the gate pulse above a certain amplitude threshold, which would introduce unwanted Z-gate operations, and the small cat size. Additionally, Ref. [48] suggested performing a similar gate operation using the frequency $(\omega_{p1} + \omega_{p2})/2$, as we demonstrated in Fig. 2 for the preparation of Bell–Fock states. We did not pursue this approach because the amplitude threshold for the AC Stark-like frequency shift is almost zero at $(\omega_{p1} + \omega_{p2})/2$. Therefore, suppressing the AC Stark-like frequency shift at the circuit design level and increasing the cat size will enable faster gate operations and enhance functionalities.

Similarly to the Bell–Fock state preparation, the sign of the superposition can be flipped by adding π in the phase of the two-cat gate pulse. Unlike the conversion from the Bell–Fock to Bell–Cat states, however, we cannot create a Bell–Cat state with an arbitrary phase. The reason is that once the pumps are turned on, the pump phase becomes the reference phase; consequently, we can no longer use the virtual Z gate as implied in Fig. 3a. Thus, the Bell–Cat state we create in this work by the two-cat gate is limited to $|0_{C1C}\rangle \pm i|1_{C0C}\rangle$.

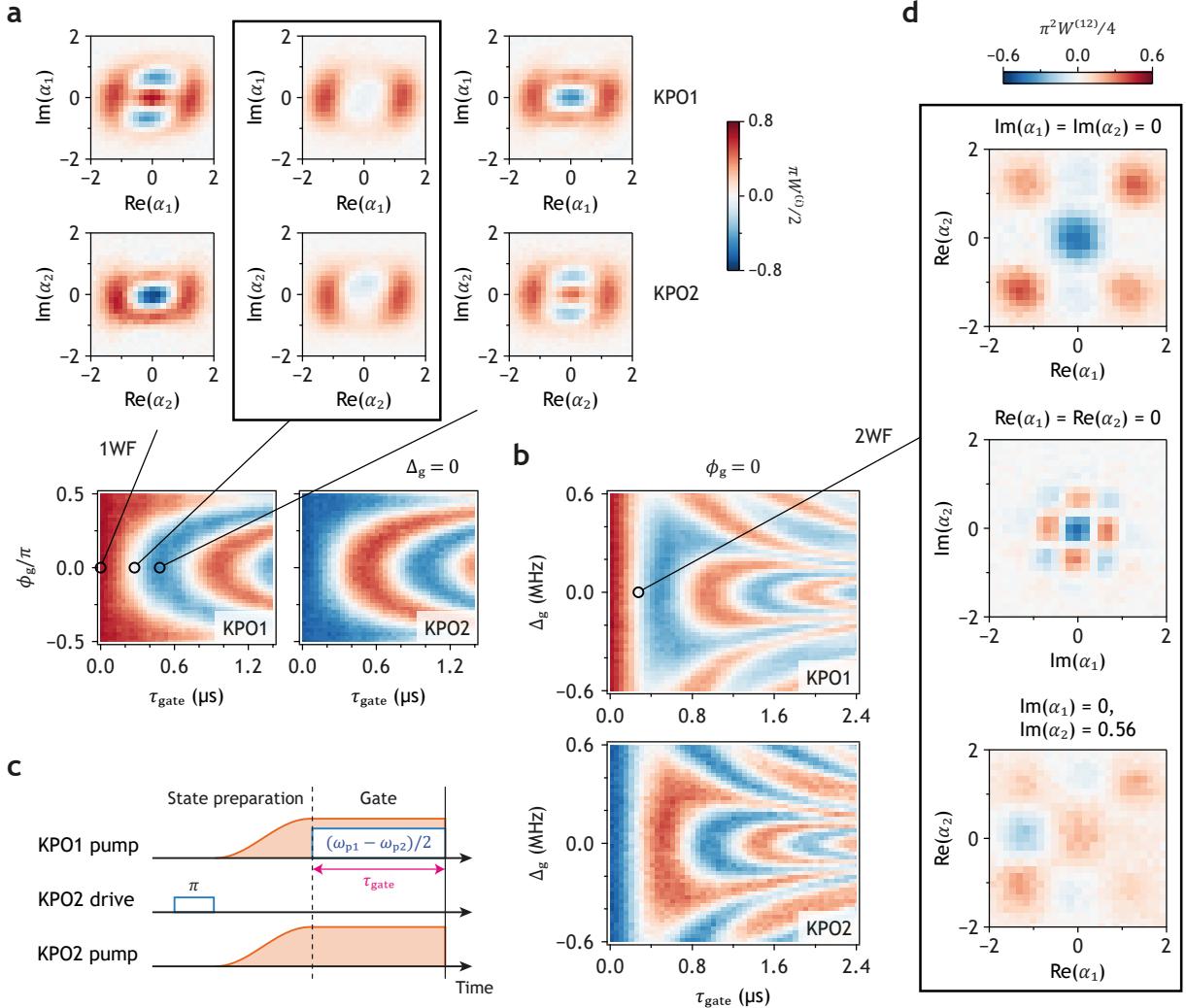


Fig. 3: Two-cat gate operation. **a,b** Two-cat Rabi oscillations between $|0_C1_C\rangle$ and $|1_C0_C\rangle$. The colours represent the number parity of each KPO. ϕ_g and Δ_g represent the phase and detuning of the gate pulse, respectively. Zero detuning ($\Delta_g = 0$) means that the frequency of the gate pulse is equal to $(\omega_{p1} - \omega_{p2})/2$. One-mode Wigner functions at times corresponding to no gate (0 ns), $\sqrt{i\text{SWAP}}$ gate (275 ns), and $i\text{SWAP}$ gate (480 ns) are shown above. **c** Pulse sequence for the two-cat Rabi. **d** Two-mode Wigner functions of the KPO state after the $\sqrt{i\text{SWAP}}$ gate. All Wigner functions showing the results of the $\sqrt{i\text{SWAP}}$ gate are enclosed in black frames.

The gate-detuning dependence of the two-cat Rabi exhibits the characteristic pattern observed in cat Rabi oscillations for the X gate [29]. This suggests that, as pointed out in Ref. [29], when mapping the dynamics of cat states to that of interacting two-level qubits, two tones with opposite gate detuning are required. In such a two-level qubit system, the same pattern can be reproduced by modulating the coupling constant with two frequencies, ω_g and $2(\omega_{q1} - \omega_{q2}) - \omega_g$, where ω_{qi} is the transition frequency of the two-level qubit i ($i = 1, 2$). In this case, zero detuning corresponds to $\omega_g = \omega_{q1} - \omega_{q2}$. For further discussion and simulation results, see Sec. 6 of Supplementary Information.

The fidelity of the $|0_C1_C\rangle \pm i|1_C0_C\rangle$ states is 0.60 ± 0.04 , which is almost identical to that achieved by the conversion from Bell–Fock to Bell–Cat states. This result is not surprising because, although the pulse

length of the $\sqrt{i\text{SWAP}}$ gate on the cat states (275 ns) is less than half that of the Bell-preparation pulse (730 ns), the contrast of the two-cat Rabi oscillations attenuates faster than that of the Rabi oscillations used for the Bell–Fock state preparation (compare Fig. 3b and the left plot of Fig. 2a). More quantitatively, the decay time of the two-cat Rabi oscillation is 3 μs for both KPOs, whereas that of the Rabi oscillations used in Bell–Fock state preparation is longer than 10 μs . The simulation using the Lindblad master equation suggests that the photon lifetime of both KPOs to reproduce the data in Fig. 3b is 10 μs (see Supplementary Fig. 4a). This photon lifetime falls within the observed range (Supplementary Table 1). Thus, the main sources of infidelity in this case are also single-photon loss and thermal excitation.

Lastly, we point out that the same $\sqrt{i\text{SWAP}}$ gate operation can be performed between cat states with

different mean photon numbers. This property may provide significant flexibility when constructing a KPO-based quantum network, particularly for the scheme developed in Refs. [63, 64]. The simulation results can be found in Supplementary Fig. 4c.

IV. DISCUSSION

To summarize, we demonstrate two intuitive methods for entangling cat states by adopting a DV–CV hybrid approach. This hybridization is achieved through Hamiltonian engineering, combining moderate Kerr nonlinearity and two-photon pumping. It enables coherent treatment of Bell–Fock and Bell–Cat states, facilitating gate operations directly on the cat basis without the need for ancilla qubits or individual Fock state control. One consequence is the entanglement-preserving conversion from Bell–Fock to Bell–Cat states. The other is the fast and simple $\sqrt{i}\text{SWAP}$ gate operation on the cat states, thereby completing the demonstration of a universal quantum gate set. Therefore, our superconducting planar KPO system is not only a potentially scalable quantum information processing unit but also a potent platform for DV–CV hybridization.

We suggest several future research directions extending this work. First, we can construct quantum networks in the cat basis. Note that our methods are compatible with previously demonstrated quantum network constructions in the Fock basis [63–65]. This means we can create more complex entangled states, such as Greenberger–Horne–Zeilinger or cluster states, in the cat basis simply by replacing transmon/Xmon qubits with KPOs and converting the basis from Fock to cat states. This approach will significantly reduce the complexity of constructing quantum networks using bosonic modes. We can also create travelling entangled-cat states by coupling our system to transmission lines [61]. Combining Hamiltonian engineering with dissipation engineering may enable us to create highly coherent cat states [66–69]. Finally, employing other multiphoton pumps may open new possibilities [70–82], such as exploring condensed matter physics in time crystals [83–85] and autonomous quantum error correction [86].

V. METHODS

A. Cat state generation

As mentioned in the main text, the ramping time of the pump for cat-state generation is 1 μs . This ramping time is much longer than that in our previous work (300 ns) [29] because the counterdiabatic pulse did not work. We believe that the reason is the reduction in Kerr coefficient from about 3 MHz to 2 MHz after ramping up the pump (see Supplementary Table 1), whereas the Kerr coefficient in Ref. [29] in-

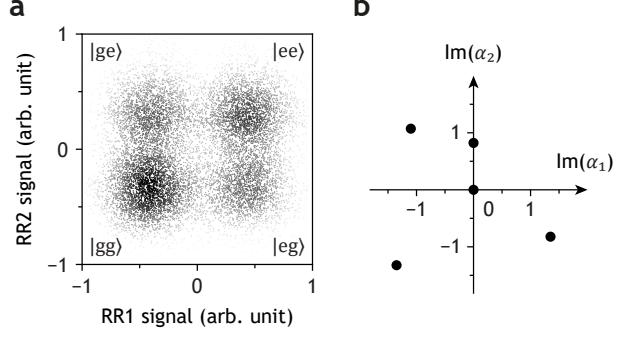


Fig. 4: Configurations for two-mode Wigner tomography. **a** Typical single-shot results. RR stands for “readout resonator”. **b** Coordinates of offset displacement for Re-Re plots of two-mode Wigner tomography.

creases slightly from 2.86 MHz to 3.13 MHz.

During the ramping, we change the pump frequency, i.e., chirp the pump pulse, for two reasons: One is to compensate for unwanted AC Stark-like frequency shifts in $2\omega_{K1}$ and $2\omega_{K2}$, which are approximately -10 MHz at the target pump amplitude [29]. The other reason is that the pump detuning must start from zero and then approach the target value adiabatically to create high-fidelity cat states.

B. Wigner-function measurements

The one-mode Wigner function of the KPO is given by [87]

$$W^{(i)}(\alpha_i) = \frac{2}{\pi} \text{Tr} \left[\hat{D}^\dagger(\alpha_i) \rho^{(i)} \hat{D}(\alpha_i) \hat{\Pi}^{(i)} \right], \quad (2)$$

where $\hat{D}(\alpha_i) = \exp(\alpha_i \hat{a}_i^\dagger - \alpha_i^* \hat{a}_i)$ is the displacement operator, $\hat{\Pi}^{(i)} = \exp(i\pi \hat{a}_i^\dagger \hat{a}_i)$ is the photon-number parity operator, and $\rho^{(i)}$ is the density matrix of KPO*i* ($i = 1, 2$). Similarly, the two-mode Wigner function is given by [42]

$$\begin{aligned} W^{(12)}(\alpha_1, \alpha_2) &= \frac{4}{\pi^2} \text{Tr} \left[\hat{D}^\dagger(\alpha_2) \hat{D}^\dagger(\alpha_1) \rho^{(12)} \hat{D}(\alpha_1) \hat{D}(\alpha_2) \hat{\Pi}^{(12)} \right] \quad (3) \\ &= \frac{4}{\pi^2} \langle \hat{\Pi}^{(12)}(\alpha_1, \alpha_2) \rangle, \end{aligned} \quad (4)$$

where $\rho^{(12)}$ is the density matrix of the two-KPO system and $\hat{\Pi}^{(12)} = \hat{\Pi}^{(1)} \hat{\Pi}^{(2)}$ is the joint parity of the two KPOs. This operator can be measured by the joint probabilities of the transmons being in their ground/excited state P_{jk} ($j, k \in \{g, e\}$) [52]:

$$\langle \hat{\Pi}^{(12)}(\alpha_1, \alpha_2) \rangle = P_{ee} + P_{gg} - P_{eg} - P_{ge}. \quad (5)$$

In the experiment, this was accomplished by fitting the single-shot readout data (Fig. 4a) with a two-dimensional Gaussian function for all pixels of the Wigner function plots.

The two-mode Wigner functions of the target states in Supplementary Fig. 5 are obtained using the Cahill–Glauber formula [88–90]:

$$W^{(12)}(\alpha_1, \alpha_2) = \frac{4}{\pi^2} \text{Tr} [\rho^{(12)} \hat{T}(\alpha_1) \hat{T}(\alpha_2)] \quad (6)$$

$$= \frac{4}{\pi^2} \sum_{\{\bar{n}_i\}=0}^{N_i} \sum_{\{m_i\}=0}^{N_i} \prod_{i=1}^2 \langle n_i | \hat{T}(\alpha_i) | m_i \rangle \times \langle \{m_i\} | \rho^{(12)} | \{n_i\} \rangle, \quad (7)$$

where \hat{T} is the complex Fourier transform of the displacement operator, and N_i is the dimension of the Hilbert space of KPO*i*. For $m_i \geq n_i$,

$$\langle n_i | \hat{T}(\alpha_i) | m_i \rangle = \sqrt{\frac{n_i!}{m_i!}} (-1)^{n_i} (2\alpha_i^*)^{\delta_i} \times L_{n_i}^{(\delta_i)}(4|\alpha_i|^2) \exp(-2|\alpha_i|^2), \quad (8)$$

where $\delta_i \equiv m_i - n_i$ and $L_{n_i}^{(\delta_i)}(x)$ are the associated Laguerre polynomials. For $m_i < n_i$, we can use the following property:

$$\langle n_i | \hat{T}(\alpha_i) | m_i \rangle = \langle m_i | \hat{T}(\alpha_i^*) | n_i \rangle. \quad (9)$$

C. Density-matrix reconstruction

A two-mode Wigner function is a four-dimensional function. Since our signal-to-noise ratio is marginal, as shown in Fig. 4a, collecting such a large data set— $13 \times 13 \times 13 \times 13$ pixels, for example—is impractical. Instead, we measured 10 two-dimensional plots, each of which has 17×17 pixels in the range $-1.6 \leq \alpha_i \leq 1.6$ ($i = 1, 2$). Among these 10 plots, half are Re–Re plots with imaginary offset displacements and the other half are Im–Im plots with real offset displacements. For Re–Re plots, the imaginary offset displacements are given as follows (Fig. 4b): $\{(\text{Im}(\alpha_1), \text{Im}(\alpha_2))\} = \{(0, 0), (0, +0.82), (-1.10, +1.07), (-1.35, -1.32), (+1.35, -0.82)\}$. The same values are used for the real offset displacements for Im–Im plots.

We found that, with the data set simulated from the target Bell–Cat states, the reconstruction fidelity is >0.99 . We also checked the reconstruction fidelity of non-ideal data sets. For example, we prepared low-quality Bell–Cat states by simulating the Lindblad master equation with $T_1 = 10$ μ s for both KPOs after 2 μ s waiting; the resulting fidelity between this state and the initial state was 0.57, which is similar to our results. The reconstruction fidelity from 2WFs of these low-quality states is still >0.93 .

For Bell–Fock states, the dimensions of the Hilbert space are set to 3×3 . For Bell–Cat states, the dimensions of the Hilbert space are set to 8×8 because, for the ideal Bell–Cat states with $P/K = 1$ and 1 MHz of pump detuning, the occupation probability at $| \geq 8 \rangle$ is less than 10^{-4} .

The algorithm for reconstruction followed the idea from Refs. [91, 92], which use gradient descent to reconstruct a density matrix with a projection step. A loss function between the measured data and that obtained from an estimated density matrix is minimized to obtain the reconstructed density matrix starting from a random initialization. We simplified the method to directly apply gradient descent (Adam [93]) on a matrix T , that is projected to construct an estimate of the physical density matrix using the Cholesky decomposition. At each gradient-descent step, the loss function is minimized followed by a projection step where the matrix T is converted to a lower triangular matrix with real-valued diagonal elements by discarding the upper-triangular part and making the diagonal real. This step allows us to obtain a density matrix $\rho = \frac{T^\dagger T}{\text{Tr}(T^\dagger T)}$ that is guaranteed to be physical. The Python libraries used were QuTiP [94, 95], NumPy [96], and JAX [97].

Data availability: All data are available in the main text or in the supplementary materials.

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Supplementary Information for “Entangling Schrödinger’s cat states by seeding a Bell state or swapping the cats”

1. DERIVATION OF THE HAMILTONIAN

In this section, we derive the Hamiltonian of two interacting Kerr parametric oscillators (KPOs). The circuit diagram is shown in Supplementary Fig. 1. The Lagrangian of the circuit is given by

$$\mathcal{L} = \mathcal{T} - \mathcal{U},$$

where

$$\begin{aligned}\mathcal{T} &= \left(\frac{\Phi_0}{2\pi}\right)^2 \left[\frac{C_{s1} + (C_{Ja1} + C_{Jb1})/N_1}{2} \dot{\phi}_1^2 + \frac{C_{s2} + (C_{Ja2} + C_{Jb2})/N_2}{2} \dot{\phi}_2^2 + \frac{C_{int}}{2} (\dot{\phi}_1 - \dot{\phi}_2)^2 \right] \\ &= \frac{C_{K1}}{2} \dot{\phi}_1^2 + \frac{C_{K2}}{2} \dot{\phi}_2^2 - C_{int} \dot{\phi}_1 \dot{\phi}_2, \\ \mathcal{U} &= -N_1 E_{Ja1} \cos\left(\frac{\phi_1}{N_1} - r_{a1} \varphi_{ex1}\right) - N_1 E_{Jb1} \cos\left(\frac{\phi_1}{N_1} + r_{b1} \varphi_{ex1}\right) \\ &\quad - N_2 E_{Ja2} \cos\left(\frac{\phi_2}{N_2} - r_{a2} \varphi_{ex2}\right) - N_2 E_{Jb2} \cos\left(\frac{\phi_2}{N_2} + r_{b2} \varphi_{ex2}\right).\end{aligned}$$

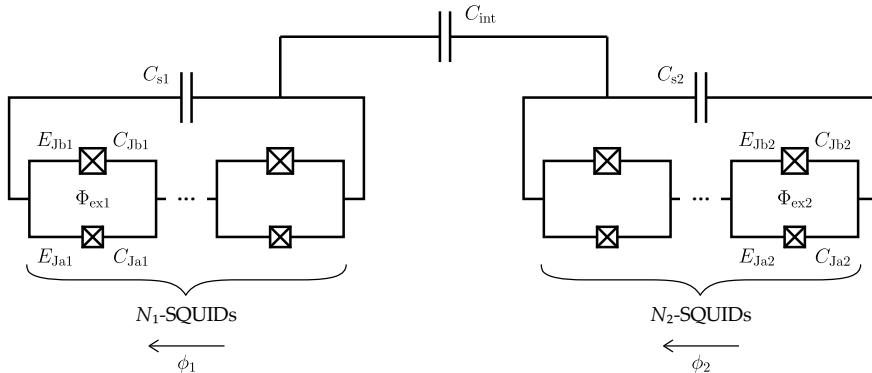
Here, Φ_0 is the magnetic flux quantum, $C_{Ki} \equiv (\Phi_0/2\pi)^2 [C_{si} + (C_{Ja_i} + C_{Jb_i})/N_i + C_{int}]$, $C_{int} \equiv (\Phi_0/2\pi)^2 C_{int}$, $\varphi_{exi} \equiv 2\pi\Phi_{exi}/\Phi_0$, and $r_{ai} + r_{bi} = 1$ ($i = 1, 2$), where these scaling parameters are determined by the irrotational constraint [1]. Note that all capacitances are renormalized by a factor of $(\Phi_0/2\pi)^2$. Using the formula $A \cos(x) + B \sin(x) = R \cos(x - \lambda)$, where $R = \sqrt{A^2 + B^2}$ and $\tan(\lambda) = B/A$, we obtain

$$\mathcal{U} = -N_1 E_{J1} \cos\left(\frac{\phi_1}{N_1} - \lambda_1\right) - N_2 E_{J2} \cos\left(\frac{\phi_2}{N_2} - \lambda_2\right),$$

where

$$\begin{aligned}E_{Ji} &= \sqrt{E_{Ja_i}^2 + E_{Jb_i}^2 + 2E_{Ja_i}E_{Jb_i} \cos(\varphi_{exi})}, \\ \lambda_i &= \arctan\left(\frac{E_{Ja_i} \sin(r_{ai} \varphi_{exi}) - E_{Jb_i} \sin(r_{bi} \varphi_{exi})}{E_{Ja_i} \cos(r_{ai} \varphi_{exi}) + E_{Jb_i} \cos(r_{bi} \varphi_{exi})}\right).\end{aligned}$$

To account for the flux-bias modulation, we decompose φ_{exi} into static and oscillating parts, i.e., $\varphi_{dc_i} + \varphi_{aci}(t)$. The oscillating part is the parametric pump given by $\varphi_{aci}(t) = 2\varphi_{aci}^{(0)} \cos(\omega_{pi} t + \theta_{pi})$, where ω_{pi} and θ_{pi} are



Supplementary Fig. 1: Circuit diagram of a superconducting KPO. The KPO consists of N_i direct-current superconducting quantum interference devices (DC SQUIDs), where $i = 1, 2$. E_{Ja_i} and C_{Ja_i} (E_{Jb_i} and C_{Jb_i}) represent the Josephson energy and the capacitance of the smaller (larger) junction in each DC SQUID, respectively. C_{si} denotes the shunting capacitance of the KPO, C_{int} is the coupling capacitance between the two KPOs, and Φ_{exi} is the external magnetic flux threaded through one DC SQUID.

the frequency and phase of the pump applied to KPO*i*, respectively. Then, we introduce a variable change, $\phi_{\text{dci}} \rightarrow \phi_{\text{dci}} + N_i \lambda_i$. The oscillating part of λ_i is ignored here because the terms associated with it disappear after the rotating-wave approximation [2]. Since $\varphi_{\text{aci}}^{(0)} \ll 2\pi$, we take the Taylor expansion at $\varphi_{\text{exi}} = \varphi_{\text{dci}}$:

$$\mathcal{U} \approx -N_1 \left(E_{J1}^{(0)} + E_{J1}^{(1)} \varphi_{\text{aci}} + E_{J1}^{(2)} \varphi_{\text{aci}}^2 \right) \cos\left(\frac{\phi_1}{N_1}\right) - N_2 \left(E_{J2}^{(0)} + E_{J2}^{(1)} \varphi_{\text{aci}} + E_{J2}^{(2)} \varphi_{\text{aci}}^2 \right) \cos\left(\frac{\phi_2}{N_2}\right),$$

where

$$\begin{aligned} E_{Ji}^{(0)} &= \sqrt{E_{\text{Jai}}^2 + E_{\text{Jbi}}^2 + 2E_{\text{Jai}}E_{\text{Jbi}} \cos(\varphi_{\text{dci}})}, \\ E_{Ji}^{(1)} &= -\frac{1}{E_{Ji}^{(0)}} E_{\text{Jai}}E_{\text{Jbi}} \sin(\varphi_{\text{dci}}), \\ E_{Ji}^{(2)} &= -\frac{1}{2(E_{Ji}^{(0)})^3} [E_{\text{Jai}}E_{\text{Jbi}}(E_{\text{Jai}}^2 + E_{\text{Jbi}}^2) \cos(\varphi_{\text{dci}}) + (E_{\text{Jai}}E_{\text{Jbi}})^2 \{\cos^2(\varphi_{\text{dci}}) + 1\}]. \end{aligned}$$

Here, the φ_{aci} terms give the parametric pump, whereas the φ_{aci}^2 terms induce a small frequency shift, which we refer to as an AC Stark-like frequency shift for convenience.

The conjugate number operators are defined by

$$\hbar N_{\phi 1} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = C_{K1} \dot{\phi}_1 - C_{\text{int}} \dot{\phi}_2, \quad \hbar N_{\phi 2} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} = -C_{\text{int}} \dot{\phi}_1 + C_{K2} \dot{\phi}_2,$$

where the reduced Planck constants are inserted to make $N_{\phi 1}$ and $N_{\phi 2}$ dimensionless. In matrix form,

$$\hbar \begin{pmatrix} N_{\phi 1} \\ N_{\phi 2} \end{pmatrix} = \begin{pmatrix} C_{K1} & -C_{\text{int}} \\ -C_{\text{int}} & C_{K2} \end{pmatrix} \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}.$$

Using the inverse capacitance matrix, we obtain

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = \frac{\hbar}{C_{K1}C_{K2} - C_{\text{int}}^2} \begin{pmatrix} C_{K2} & C_{\text{int}} \\ C_{\text{int}} & C_{K1} \end{pmatrix} \begin{pmatrix} N_{\phi 1} \\ N_{\phi 2} \end{pmatrix}.$$

The resulting Hamiltonian is (hereafter, we add a hat to an operator for clarity)

$$\begin{aligned} \hat{\mathcal{H}} = & 4E_{C1}\hat{N}_{\phi 1}^2 + 4E_{C2}\hat{N}_{\phi 2}^2 + 8E_{\text{int}}\hat{N}_{\phi 1}\hat{N}_{\phi 2} \\ & - N_1 E_{K1} \cos\left(\frac{\hat{\phi}_1}{N_1}\right) - 2\gamma_1 \cos(\omega_{p1}t + \theta_{p1}) \cos\left(\frac{\hat{\phi}_1}{N_1}\right) - N_2 E_{K2} \cos\left(\frac{\hat{\phi}_2}{N_2}\right) - 2\gamma_2 \cos(\omega_{p2}t + \theta_{p2}) \cos\left(\frac{\hat{\phi}_2}{N_2}\right), \end{aligned}$$

where

$$E_{C1} \equiv \frac{\hbar^2 C_{K2}}{8(C_{K1}C_{K2} - C_{\text{int}}^2)}, \quad E_{C2} \equiv \frac{\hbar^2 C_{K1}}{8(C_{K1}C_{K2} - C_{\text{int}}^2)}, \quad E_{\text{int}} \equiv \frac{\hbar^2 C_{\text{int}}}{8(C_{K1}C_{K2} - C_{\text{int}}^2)}.$$

Here, we define $E_{Ki} \equiv E_{Ji}^{(0)} + 2E_{Ji}^{(2)}(\varphi_{\text{aci}}^{(0)})^2$ and $\gamma_i \equiv N_i E_{Ji}^{(1)} \varphi_{\text{aci}}^{(0)}$. The φ_{aci}^2 term is averaged out by using $\langle \varphi_{\text{aci}}^2 \rangle = 2(\varphi_{\text{aci}}^{(0)})^2$ and absorbed into E_{Ki} .

We move to the occupation-number representation by defining

$$\hat{N}_{\phi i} = iN_{\phi i}^{(0)}(\hat{a}_i^\dagger - \hat{a}_i) \quad \text{and} \quad \hat{\phi}_i = \phi_i^{(0)}(\hat{a}_i^\dagger + \hat{a}_i),$$

where $N_{\phi i}^{(0)} = \sqrt[4]{E_{Ki}/32N_iE_{Ci}}$ and $\phi_i^{(0)} = \sqrt[4]{2N_iE_{Ci}/E_{Ki}}$ are the zero-point fluctuations. Then we have

$$\begin{aligned} \hat{\mathcal{H}} = & \hbar\omega_{K1}^{(0)}\hat{a}_1^\dagger\hat{a}_1 + \hbar\omega_{K2}^{(0)}\hat{a}_2^\dagger\hat{a}_2 + \hbar g(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_1\hat{a}_2^\dagger - \hat{a}_1\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2^\dagger) \\ & - N_1 E_{K1} \left[1 + \frac{1}{24} \left(\frac{\hat{\phi}_1}{N_1} \right)^4 - \dots \right] - 2\gamma_1 \cos(\omega_{p1}t + \theta_{p1}) \left[1 - \frac{1}{2} \left(\frac{\hat{\phi}_1}{N_1} \right)^2 + \frac{1}{24} \left(\frac{\hat{\phi}_1}{N_1} \right)^4 - \dots \right] \\ & - N_2 E_{K2} \left[1 + \frac{1}{24} \left(\frac{\hat{\phi}_2}{N_2} \right)^4 - \dots \right] - 2\gamma_2 \cos(\omega_{p2}t + \theta_{p2}) \left[1 - \frac{1}{2} \left(\frac{\hat{\phi}_2}{N_2} \right)^2 + \frac{1}{24} \left(\frac{\hat{\phi}_2}{N_2} \right)^4 - \dots \right], \end{aligned}$$

where

$$\hbar\omega_{Ki}^{(0)} \equiv \sqrt{\frac{8E_{Ci}E_{Ki}}{N_i}}, \quad \hbar g \equiv E_{\text{int}} \sqrt[4]{\frac{4E_{K1}E_{K2}}{N_1N_2E_{C1}E_{C2}}} = \frac{1}{2} \frac{E_{\text{int}}}{\sqrt{E_{C1}E_{C2}}} \sqrt{\omega_{K1}^{(0)}\omega_{K2}^{(0)}}. \quad (1)$$

By using the formula (derived on the basis of Ref. [3])

$$(\hat{a}^\dagger \pm \hat{a})^n = \sum_{k=0}^n \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(\pm 1)^{m-k} n!}{(n-k)!(k-2m)!m!2^m} (\hat{a}^\dagger)^{k-2m} \hat{a}^{n-k},$$

we obtain

$$\begin{aligned} -\frac{E_{Ki}}{24N_i^3} \hat{\phi}_i^4 &\Rightarrow -\hbar K_i \hat{a}_i^\dagger \hat{a}_i - \frac{\hbar K_i}{2} \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i \\ &\quad - \frac{\hbar K_i}{12} (6\hat{a}_i^\dagger \hat{a}_i^\dagger + 6\hat{a}_i \hat{a}_i + 4\hat{a}_i^\dagger \hat{a}_i \hat{a}_i \hat{a}_i + 4\hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i + \hat{a}_i \hat{a}_i \hat{a}_i \hat{a}_i + \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i^\dagger), \\ \frac{\gamma_i}{N_i^2} \cos(\omega_{pi}t + \theta_{pi}) \hat{\phi}_i^2 &\Rightarrow \hbar P_i \cos(\omega_{pi}t + \theta_{pi}) (2\hat{a}_i^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_i^\dagger + \hat{a}_i \hat{a}_i), \end{aligned} \quad (2)$$

where $\hbar K_i \equiv E_{Ci}/N_i^2$ and $\hbar P_i \equiv \varphi_{aci}^{(0)} E_{Ji}^{(1)} \sqrt{2E_{Ci}/N_i E_{Ki}}$.

Now we move to the rotating frame whose Hamiltonian is defined by

$$\hat{\mathcal{H}}_0/\hbar = \frac{\omega_{p1}}{2} \hat{a}_1^\dagger \hat{a}_1 + \frac{\omega_{p2}}{2} \hat{a}_2^\dagger \hat{a}_2$$

and make the rotating-wave approximation. Then, the surviving terms are

$$\begin{aligned} \hat{\mathcal{H}}_{\text{rot}}/\hbar &= e^{i\hat{\mathcal{H}}_0 t/\hbar} (\hat{\mathcal{H}} - \hat{\mathcal{H}}_0) e^{-i\hat{\mathcal{H}}_0 t/\hbar} \times (1/\hbar) \\ &\approx \Delta_1 \hat{a}_1^\dagger \hat{a}_1 - \frac{K_1}{2} \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \frac{P_1}{2} (\hat{a}_1^\dagger \hat{a}_1^\dagger + \hat{a}_1 \hat{a}_1) + \Delta_2 \hat{a}_2^\dagger \hat{a}_2 - \frac{K_2}{2} \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 + \frac{P_2}{2} (\hat{a}_2^\dagger \hat{a}_2^\dagger + \hat{a}_2 \hat{a}_2) \\ &\quad + g [\hat{a}_1^\dagger \hat{a}_2 e^{i(\Delta_p t + \theta_p)} + \hat{a}_1 \hat{a}_2^\dagger e^{-i(\Delta_p t + \theta_p)}]. \end{aligned} \quad (3)$$

Here, $\Delta_i (\equiv \omega_{Ki} - \omega_{pi}/2)$ is the KPO-pump frequency detuning, where $\omega_{Ki} (\equiv \omega_{Ki}^{(0)} - K_i)$ is the transition frequency between the $|0_F\rangle$ and $|1_F\rangle$ states; $\Delta_p \equiv (\omega_{p1} - \omega_{p2})/2$ and $\theta_p \equiv (\theta_{p1} - \theta_{p2})/2$. Note that the unitary transformation $\hat{a}_i \rightarrow \hat{a}_i e^{-i\theta_{pi}/2}$ was performed. Setting $\theta_p = 0$ gives Eq. (1) in the main text.

In the same rotating frame used in Supplementary Eq. (3), the Hamiltonian describing both the Bell-preparation pulse for the $|0_F 1_F\rangle \pm |1_F 0_F\rangle$ state and the two-cat gate operation is given by [4]

$$\begin{aligned} \hat{\mathcal{H}}_g/\hbar &= 2P_g \cos(\omega_g t + \theta_g) \hat{a}_1^\dagger \hat{a}_1 \\ &= 2P_g \cos[(\Delta_p + \Delta_g)t + \theta_g] \hat{a}_1^\dagger \hat{a}_1, \end{aligned} \quad (4)$$

where P_g and θ_g are the amplitude and phase of the pulse, respectively, and $\Delta_g \equiv \omega_g - \Delta_p$. The factor of 2 on the right-hand side of the equation originated from Supplementary Eq. (2). For simulation, θ_g must be 0 or π to obtain $|0_F 1_F\rangle \pm |1_F 0_F\rangle$. The Bell-preparation pulse for the $|0_F 0_F\rangle \pm |1_F 1_F\rangle$ state can be described as [4]

$$\begin{aligned} \hat{\mathcal{H}}_g/\hbar &= \frac{P_g}{2} [\hat{a}_1^\dagger \hat{a}_1^\dagger e^{i\{(\omega_{p1}-\omega_g)t-\theta_g\}} + \hat{a}_1 \hat{a}_1 e^{-i\{(\omega_{p1}-\omega_g)t-\theta_g\}}] \\ &= \frac{P_g}{2} [\hat{a}_1^\dagger \hat{a}_1^\dagger e^{+i\{(\Delta_p-\Delta_g)t-\theta_g\}} + \hat{a}_1 \hat{a}_1 e^{-i\{(\Delta_p-\Delta_g)t-\theta_g\}}], \end{aligned}$$

where $\Delta_g \equiv \omega_g - (\omega_{p1} + \omega_{p2})/2$ and θ_{p1} is assumed to be zero for simplicity. Note that, for simulation, θ_g must be $\pm\pi/2$ to generate $|0_F 0_F\rangle \pm |1_F 1_F\rangle$; $\theta_g = 0$ gives $|0_F 0_F\rangle \pm i|1_F 1_F\rangle$.

2. SIMULATION OF BELL-CAT STATE GENERATION

We simulate our Bell-Cat generation scheme via conversion from Fock states to cat states by solving the following Lindblad master equation with QuTiP [5, 6]:

$$\frac{\partial \rho(t)}{\partial t} = -\frac{i}{\hbar} [\hat{\mathcal{H}}(t), \rho(t)] + \{\gamma_{K1} \mathcal{D}[\hat{a}_1] + \gamma_{K2} \mathcal{D}[\hat{a}_2] + \gamma_{\phi1} \mathcal{D}[\hat{a}_1^\dagger \hat{a}_1] + \gamma_{\phi2} \mathcal{D}[\hat{a}_2^\dagger \hat{a}_2]\} \rho(t), \quad (5)$$

Supplementary Table 1: Measured system parameters. Φ_0 is the magnetic flux quantum. “RR” stands for readout resonator. For the self-Kerr coefficient of the KPOs, the value in parentheses is the coefficient when the pump is on. For more rigorous definitions, see Supplementary Eqs. (3) and (7).

Physical quantity	Symbol	System 1	System 2
Number of DC SQUIDs in KPO	N_i	10	10
External magnetic flux threaded through one DC SQUID	Φ_{exi}	$0.229\Phi_0$	$0.191\Phi_0$
Transition frequency between the $ 0\rangle$ and $ 1\rangle$ states of KPO	$\omega_{\text{K}i}/2\pi$	2.5635 GHz	2.4199 GHz
Self-Kerr coefficient of KPO	$K_i/2\pi$	2.86 (2.05) MHz	2.91 (1.90) MHz
Longitudinal relaxation time of KPO	$T_1^{\text{K}i}$	13 ± 3 μ s	20 ± 6 μ s
Transverse relaxation time of KPO	$T_2^{\text{K}i}$	4.8 ± 0.4 μ s	4.3 ± 0.8 μ s
Coupling constant between two KPOs	$g/2\pi$	8.0 MHz	
Transition frequency between $ g\rangle$ and $ e\rangle$ states of transmon	$\omega_{\text{T}i}/2\pi$	3.8076 GHz	3.6216 GHz
Anharmonicity of transmon	$K_{\text{T}i}/2\pi$	221.3 MHz	225.9 MHz
Longitudinal relaxation time of transmon	$T_1^{\text{T}i}$	49 ± 7 μ s	37 ± 4 μ s
Transverse relaxation time of transmon	$T_2^{\text{T}i}$	25 ± 2 μ s	13 ± 1 μ s
Cross-Kerr coefficient between KPO and transmon	$\chi_{\text{KT}i}/2\pi$	1.69 MHz	1.72 MHz
Higher-order cross-Kerr coefficient	$\chi_{\text{KKT}i}/2\pi$	-7 kHz	-7 kHz
Number of thermally excited photons in KPO	n_{thi}	0.09	0.08
Resonance frequency of RR	$\omega_{\text{R}i}/2\pi$	7.6079 GHz	7.4641 GHz
Cross-Kerr coefficient between transmon and RR	$\chi_{\text{TR}i}/2\pi$	0.37 MHz	0.39 MHz

where ρ is the density matrix of the two-KPO system, $\mathcal{D}[\hat{O}]\rho = \hat{O}\rho\hat{O}^\dagger - \frac{1}{2}\hat{O}^\dagger\hat{O}\rho - \frac{1}{2}\rho\hat{O}^\dagger\hat{O}$, γ_{Ki} is the single-photon loss rate of KPO*i*, and $\gamma_{\phi i}$ is the dephasing rate of KPO*i* ($i = 1, 2$).

We first simulate the Bell–Fock state preparation. In a frame rotating with the frequency $(\omega_{\text{K}1} + \omega_{\text{K}2})/2$, the Hamiltonian describing the Bell-preparation pulse for the $|0_{\text{F}}0_{\text{F}}\rangle \pm |1_{\text{F}}1_{\text{F}}\rangle$ state is given by

$$\hat{\mathcal{H}}_{\text{BFS}}/\hbar = \left(\frac{\omega_{\text{K}1} - \omega_{\text{K}2}}{2} + \delta_g \right) \hat{a}_1^\dagger \hat{a}_1 - \frac{K_1}{2} \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \left(\frac{\omega_{\text{K}2} - \omega_{\text{K}1}}{2} + \delta_g \right) \hat{a}_2^\dagger \hat{a}_2 - \frac{K_2}{2} \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 + g \left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger \right) \pm i \frac{P_g}{2} \left(\hat{a}_1^\dagger \hat{a}_1^\dagger - \hat{a}_1 \hat{a}_1 \right), \quad (6)$$

where δ_g is the small detuning of -20 kHz at which the centre of the Rabi pattern is. We solve Supplementary Eq. (5) with Supplementary Eq. (6), using the thermal state as the initial state. In this simulation, both single-photon loss and dephasing are considered.

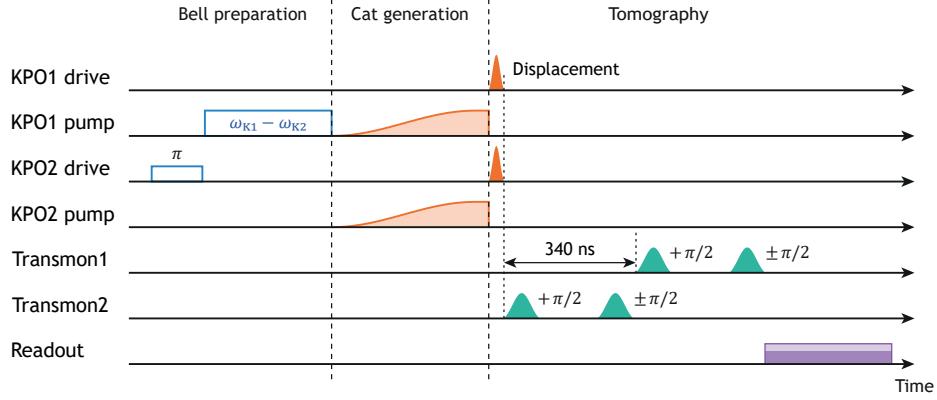
Next, we simulate cat-state generation by solving Supplementary Eq. (5) with Supplementary Eq. (3), using the solution of the Bell–Fock preparation simulation as the initial state. Here, only single-photon loss is considered because dephasing caused by low-frequency noise does not affect the fidelity, as pointed out in the main text. The coupling term in Supplementary Eq. (3) was omitted at this step because, without a Bell-preparation or gate pulse, its contribution would be averaged out.

The $|0_{\text{F}}1_{\text{F}}\rangle \pm |1_{\text{F}}0_{\text{F}}\rangle$ state preparation can also be simulated similarly by using the following Hamiltonian in a frame rotating with the frequency $\omega_{\text{K}1}$:

$$\hat{\mathcal{H}}_{\text{BFD}}/\hbar = -\frac{K_1}{2} \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + (\omega_{\text{K}1} - \omega_{\text{K}2}) \hat{a}_2^\dagger \hat{a}_2 - \frac{K_2}{2} \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 + g \left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger \right) \pm 2P_g \cos[(\omega_{\text{K}1} - \omega_{\text{K}2})t] \hat{a}_1^\dagger \hat{a}_1.$$

3. SYSTEM PARAMETERS

The measured system parameters are presented in Supplementary Table 1. The Hamiltonian was characterized by analysing Rabi oscillations in the $|0_{\text{F}}\rangle$ state population, following the methodology outlined in Ref. [8], except for the coupling constant g . The value of g was determined by tuning the global flux bias to the point where the transition frequency of both KPOs becomes equal and then observing the anti-crossing in the KPO signal. At this flux bias, where $\omega_{\text{K}1}/2\pi = \omega_{\text{K}2}/2\pi = 2.237$ GHz, $g/2\pi$ was found to be 7.2 MHz. Consequently, the coupling constant at our working flux bias, where $\omega_{\text{K}1}/2\pi = 2.564$ GHz and $\omega_{\text{K}2}/2\pi = 2.420$ GHz, would be 8.0 MHz, according to Supplementary Eq. (1).



Supplementary Fig. 2: Full pulse sequence of Fig. 2c. In the “tomography” part, the displacement pulses applied to both KPOs implement the displacement operators described in Eqs. (2) and (3) in the main text. The subsequent $\pi/2$ pulses applied to the transmons map the number parity of the KPOs to the transmon states. Then, the states of the two transmons are measured simultaneously. For this measurement, two microwave pulses with the frequencies of the readout resonators are combined and applied to the chip. Regarding the 340 ns delay in the transmon1 pulse sequence, see the supplementary text.

Supplementary Table 2: Parameters for the pulse shape. “ σ ” represents the standard deviation of Gaussian pulses. For flap-top Gaussian pulses, σ indicates the standard deviation of the pulse edges. The time intervals between two $\pi/2$ transmon pulses for the parity measurement are 278 ns for KPO1 and 280 ns for KPO2.

Operation	Data	Element	Pulse shape	Pulse length (ns)	σ (ns)
Cat generation	Figs. 1–3	KPO	$\sin^2(\pi t/2\tau_{\text{ramp}})$	1000	
Bell preparation	Fig. 2 ($ 0_F1_F\rangle$)	KPO2	Flat-top Gaussian	955	1
	Fig. 2 ($ 0_F1_F\rangle \pm 1_F0_F\rangle$)	KPO1	Flat-top Gaussian	730	1
	Fig. 2 ($ 0_F0_F\rangle \pm 1_F1_F\rangle$)	KPO1	Flat-top Gaussian	852	1
Two-cat Rabi \sqrt{iSWAP} gate	Fig. 3	KPO1	Flat-top Gaussian	Variable	1
				275	1
Parity measurement	All Wigner functions	KPO	Gaussian	10	2.5
		Transmon	Gaussian ($\pi/2$)	25	6.25
Rabi	Fig. 4	KPO	Flat-top Gaussian	Variable	1
		Transmon	Gaussian (π)	2000	500
Transmon readout	All data	Readout resonator	Flat-top Gaussian	2500	4

The calibration between the AWG setting value and the displacement in phase space was achieved by fitting the parity measurement of the vacuum state with the following Hamiltonian [7, 8]:

$$\hat{\mathcal{H}}_{\text{fit}i}/\hbar = \Delta_{Ki}\hat{a}_i^\dagger\hat{a}_i - \frac{K_i}{2}\hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i + \beta_i(\hat{a}_i^\dagger e^{-i\Delta_{dT_i}t} + \hat{a}_i e^{+i\Delta_{dT_i}t}) + \hbar\Delta_{Ti}\hat{b}_i^\dagger\hat{b}_i - \frac{K_{Ti}}{2}\hat{b}_i^\dagger\hat{b}_i^\dagger\hat{b}_i\hat{b}_i + \beta_{Ti}(\hat{b}_i^\dagger + \hat{b}_i) + \chi_{KTi}\hat{a}_i^\dagger\hat{a}_i\hat{b}_i^\dagger\hat{b}_i + \chi_{KKTi}\hat{a}_i^\dagger\hat{a}_i(\hat{a}_i^\dagger\hat{a}_i - 1)\hat{b}_i^\dagger\hat{b}_i. \quad (7)$$

Here, $\Delta_{Ki} \equiv \omega_{Ki} - \omega_{Tdi}$, where ω_{Tdi} is the frequency of the drive for transmon i ($i = 1, 2$); $\Delta_{dT_i} \equiv \omega_{di} - \omega_{Tdi}$; and $\Delta_{Ti} \equiv \omega_{Ti} - \omega_{Tdi}$. For more details, see Ref. [8].

4. FULL PULSE SEQUENCE

The full pulse sequence, including that for Wigner-function measurement via the Lutterbach–Davidovich method [9–11], is depicted in Supplementary Fig. 2. Specific details regarding pulse shapes are outlined in Supplementary Table 2. Notably, a 340 ns delay is introduced between the displacement pulse for KPO1 and

the first $\pi/2$ pulse for transmon1 to prevent the unintended excitation of transmon1 triggered by the $\pi/2$ pulses for transmon2. This precaution is necessary owing to the small frequency difference (only 35 MHz) between the $|e\rangle$ - $|f\rangle$ transition of transmon1 and the $|g\rangle$ - $|e\rangle$ transition of transmon2, falling within the bandwidth of the $\pi/2$ pulses for Wigner function measurement. By comparing two one-mode Wigner function measurements of KPO1—one obtained from the simultaneous measurement of both transmons with the delay, and the other from the transmon1 measurement only without the delay—we found that this delay does not severely degrade the quality of the Wigner function measurement, as the Kerr evolution preserves the number parity.

5. ELECTRONICS AND CABLING

The pulses for the transmon and the readout resonator were generated by arbitrary waveform generators (AWGs), local oscillators, and IQ mixers. For the KPO, pulses were generated using Keysight M8195A (labelled “Fast AWG” in Supplementary Fig. 3a). This instrument has four channels, with each channel having two outputs, such as Ch. 1 and Ch. $\bar{1}$ (see the leftmost part of Supplementary Fig. 3a). The pulse shapes from these two outputs are identical, with only a 180° phase difference. In Ch. 1 (and Ch. $\bar{1}$), both the KPO1 drive and low-frequency pulses near the frequency $(\omega_{K1} - \omega_{K2})/2\pi = 144$ MHz are generated. The pulses from Ch. 1 pass through a bandpass filter to suppress the low-frequency component, while the pulses from Ch. $\bar{1}$ pass through multiple low-pass filters to cut out the gigahertz components. The KPO1 pump and the low-frequency pulses are combined in the mixing chamber and inserted into the KPO1 pump port of the chip (Fig. 1c).

6. TWO-CAT RABI OSCILLATIONS IN INTERACTING TWO-LEVEL SYSTEM QUBITS

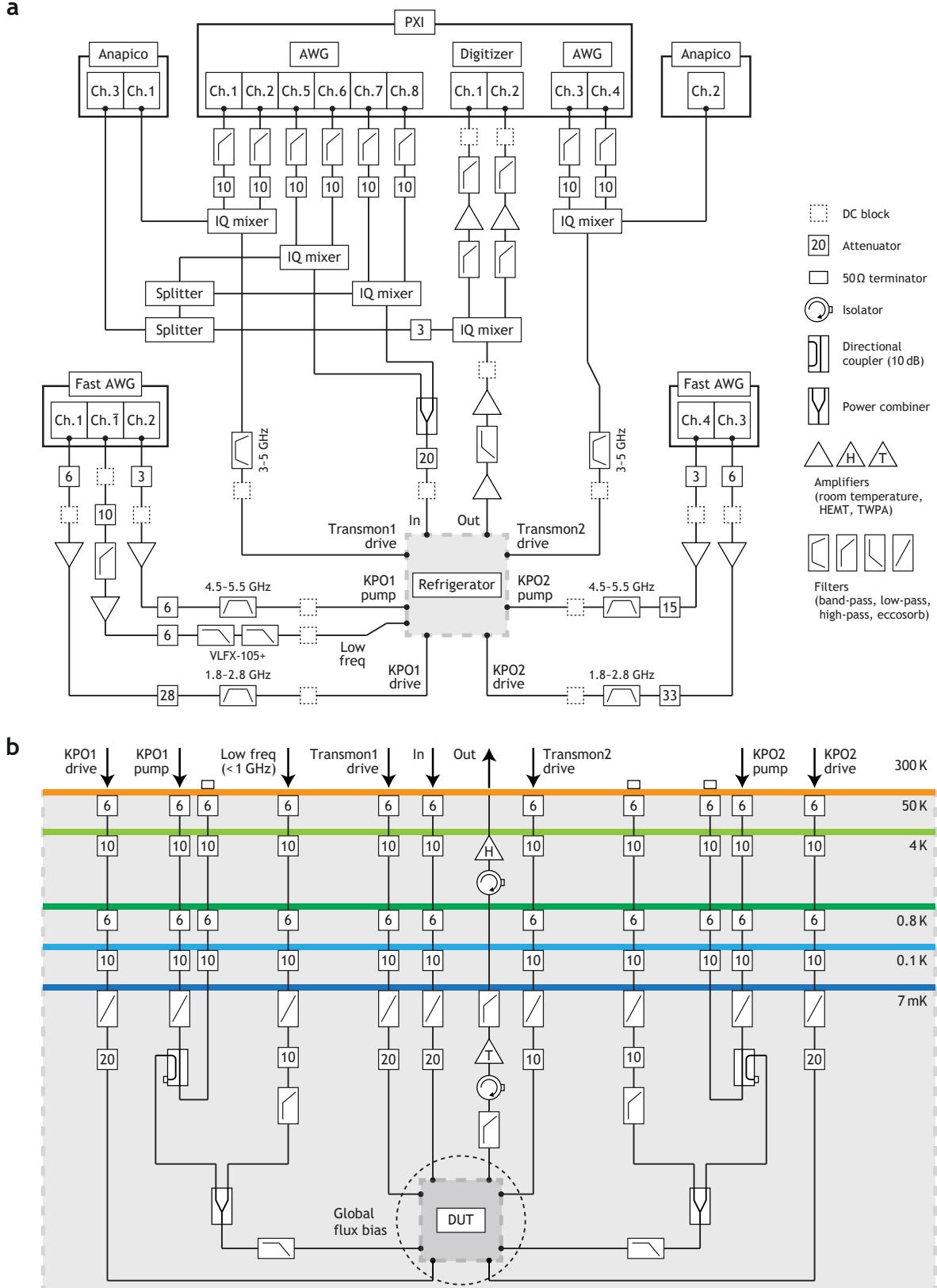
As mentioned in the main text, the two-cat Rabi oscillations can be reproduced in interacting two-level qubits, whose resonance frequencies are ω_{q1} and ω_{q2} . We first consider $|0_F 1_F\rangle \leftrightarrow |1_F 0_F\rangle$ transitions induced by the parametric modulation of the coupling constant with the frequency ω_g . The Hamiltonian describing this process is given by [12]

$$\hat{\mathcal{H}}_{Rs}/\hbar = J_g \cos[(\Delta_q + \Delta_g)t] (\hat{\sigma}_+^{(1)} \hat{\sigma}_-^{(2)} e^{+i\Delta_q t} + \hat{\sigma}_-^{(1)} \hat{\sigma}_+^{(2)} e^{-i\Delta_q t}), \quad (8)$$

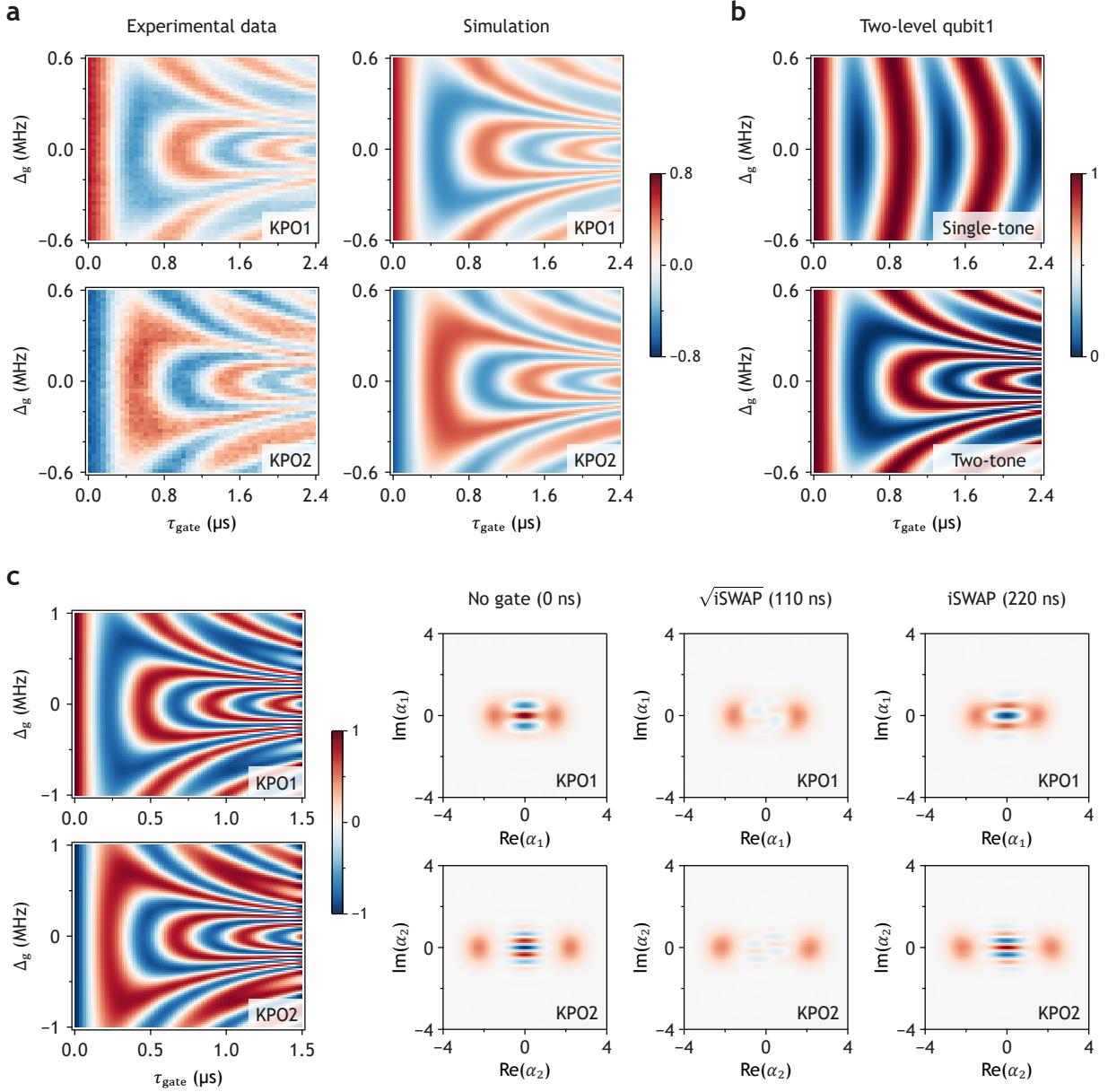
where J_g is the modulation amplitude of the coupling constant, $\Delta_q \equiv \omega_{q1} - \omega_{q2}$, $\Delta_g \equiv \omega_g - \Delta_q$, and $\hat{\sigma}_{\pm}^{(k)} \equiv (\hat{\sigma}_x^{(k)} \pm i\hat{\sigma}_y^{(k)})/2$, where $\hat{\sigma}_x^{(k)}$ and $\hat{\sigma}_y^{(k)}$ are the Pauli operators of qubit k ($k = 1, 2$). To reproduce our data for $|0_C 1_C\rangle \leftrightarrow |1_C 0_C\rangle$ transitions, we need two modulation tones with opposite detuning:

$$\hat{\mathcal{H}}_{Rd}/\hbar = \frac{J_g}{2} [\cos\{(\Delta_q + \Delta_g)t\} + \cos\{(\Delta_q - \Delta_g)t\}] (\hat{\sigma}_+^{(1)} \hat{\sigma}_-^{(2)} e^{+i\Delta_q t} + \hat{\sigma}_-^{(1)} \hat{\sigma}_+^{(2)} e^{-i\Delta_q t}). \quad (9)$$

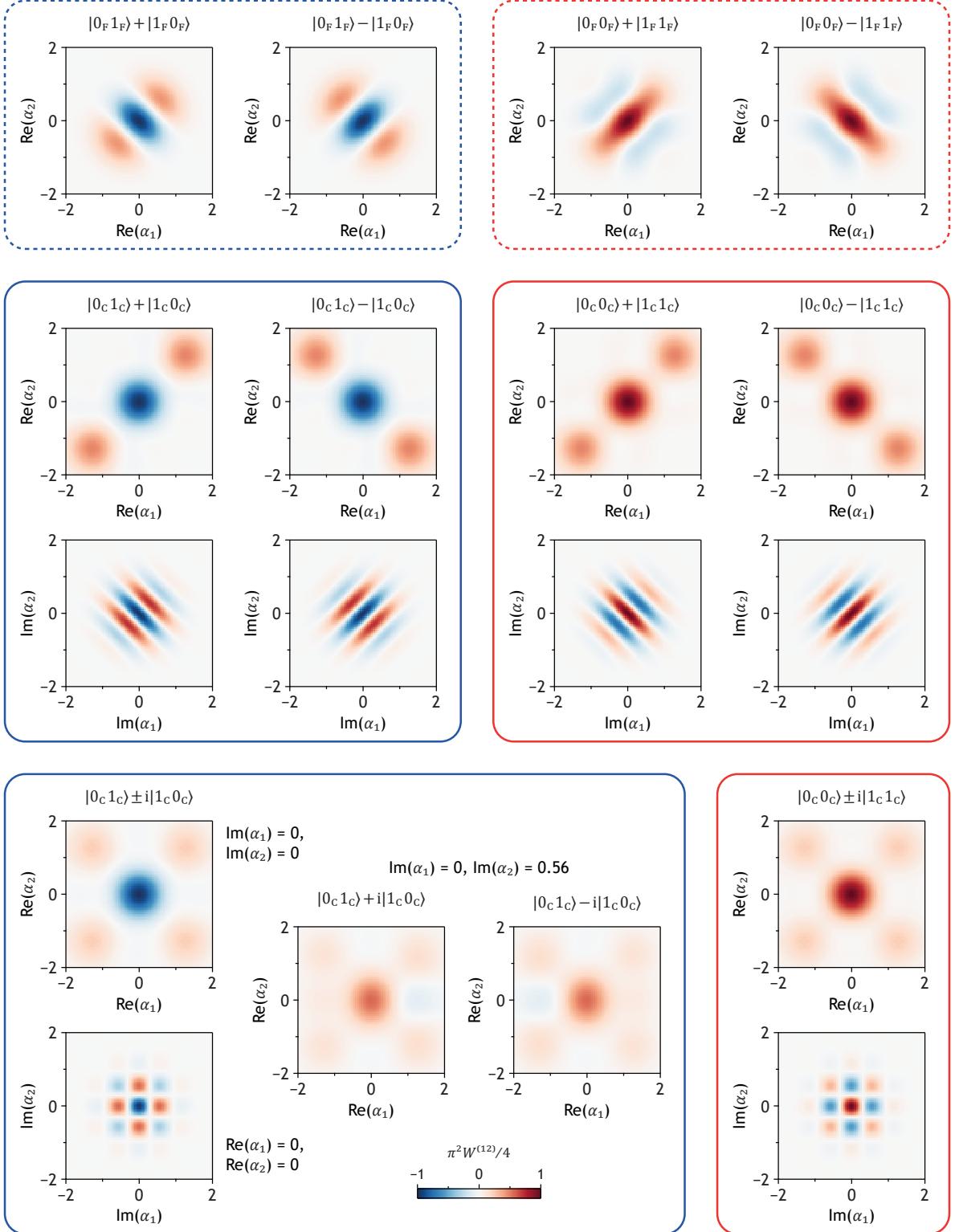
Here, the frequency $\Delta_q + \Delta_g$ corresponds to ω_g and the frequency $\Delta_q - \Delta_g$ corresponds to $2(\omega_{q1} - \omega_{q2}) - \omega_g$. The simulations using Supplementary Eqs. (8) and (9) are shown in Supplementary Fig. 4b.



Supplementary Fig. 3: a,b Experimental configuration of room-temperature electronics (a) and cryogenic components (b). The 50- Ω -terminated port next to the pump line serves to dissipate unused pump power. The isolator at the mixing chamber is a double-junction isolator. The cables for the traveling-wave parametric amplifier (TWPA) pump are not shown for simplicity. A single instrument labelled “Fast AWG” is represented as two parts for ease of recognition; similarly, “Anapico”, which provides local oscillators, is shown in two parts.



Supplementary Fig. 4: **a** Two-cat Rabi oscillations. The colours represent the number parity of each KPO. The gate amplitude from the fitting with the simulation is 2.96 MHz. For the simulation, Supplementary Eqs. (3), (4), and (5) were used with $T_1^{\text{K1}} = T_1^{\text{K2}} = 10 \mu\text{s}$; dephasing was not considered. **b** Simulation of Rabi oscillations in two interacting two-level qubits. The colours represent the population of the $|0\rangle$ state of qubit1. For this simulation, Supplementary Eqs. (8) (single tone) and (9) (two tones) with $J_m/2\pi = 1.06$ MHz were used. **c** Two-cat Rabi oscillations between two KPOs with different mean photon numbers. The mean photon number of KPO1 is 2.1, whereas that of KPO2 is 5.1. For the simulation, Supplementary Eqs. (3), (4), and (5) were used with the following conditions: The Kerr coefficient for both KPOs is 2 MHz; the pump amplitude and detuning for KPO1 are 3 MHz and 1 MHz, respectively; for KPO2, 8 MHz and 2 MHz, respectively. Relaxations are not considered in (b) and (c).



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