Using Fisher Information to diagnose DPT

R. J. Lewis-Swan *et. al.* propose the quantum Fisher information given by

$$F(\lambda, t) = -4 \left. \frac{\partial^2 \mathcal{F}(\lambda, \delta\lambda, t)}{\partial(\delta\lambda)^2} \right|_{\delta\lambda \to 0} \tag{1}$$

to diagnose dynamical phase transitions, where $\mathcal{F}(\lambda, \delta\lambda, t) = |\langle \psi(\lambda, t) | \psi(\lambda + \delta\lambda, t) \rangle| = |\langle \psi_0 | e^{+iH(\lambda)t} e^{-iH(\lambda+\delta\lambda)t} | \psi_0 \rangle|$ and $H(\lambda) = H_0 + \lambda H_1$ with $|\psi_0\rangle$ being an eigenstate of H_0. The overlap can be rewritten as

$$\begin{aligned} \mathcal{F}(\lambda, \delta\lambda, t) &= |\langle \psi_0 | e^{+i(H_0 + \lambda H_1)t} e^{-i(H_0 + (\lambda+\delta\lambda)H_1)t} | \psi_0 \rangle| \\ &= |\langle \psi_0 | e^{-i\delta\lambda H_1 t} | \psi_0 \rangle| \\ &= |\langle \psi_0 | \psi(\delta\lambda, t) \rangle| \\ &= \mathcal{F}(\delta\lambda, t). \end{aligned} \tag{2}$$

Since we take the limit $\delta\lambda \to 0$ afterwards, the Fisher information will be a function of the time only. In our analysis, we interpret the overlap $\mathcal{F}$ as the LE from the context of DQPT, thus we denote it by $\mathcal{L}(\delta\lambda, t)$ henceforth. Let us mention that the overlap $\mathcal{F}$ can be written in terms of the fidelity

$$f(\rho, \sigma) = \left[\mathrm{Tr} \left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} \right) \right]^2 \tag{3}$$

as

$$\mathcal{F}(\delta\lambda, t) = \mathcal{L}(\delta\lambda, t) = \sqrt{f\Big(|\psi_0\rangle\langle\psi_0|, |\psi(\delta\lambda, t)\rangle\langle\psi(\delta\lambda, t)| \Big)}. \tag{4}$$

Moving to Céleri and Serra's paper context, we identify the density matrix $\rho = |\psi_0\rangle\langle\psi_0|$ as the state of the system which has undergone an ideal unitary process, since

$$|\psi_0\rangle\langle\psi_0| = U(\lambda = 0, t) |\psi_0\rangle\langle\psi_0| U^\dagger(\lambda = 0, t) \tag{5}$$

and $\sigma = |\psi(\delta\lambda, t)\rangle\langle\psi(\delta\lambda, t)|$ as the state of the system which has undergone an irreversible unitary process, since

$$|\psi(\delta\lambda, t)\rangle\langle\psi(\delta\lambda, t)| = U(\lambda = \delta\lambda, t) |\psi_0\rangle\langle\psi_0| U^\dagger(\lambda = \delta\lambda, t). \tag{6}$$

Their paper gives an approximation for the Fisher information in terms of the relative entropy envolving two close probability distributions

$$S(p_{\lambda+\delta\lambda} || p_\lambda) \approx \frac{\delta\lambda^2}{2} F(p_\lambda). \tag{7}$$

Landi's paper argues that an analogous relation also holds for density matrices depending on some set of experimentally controllable parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ...)^T$ differing by a small amount $\delta\boldsymbol{\lambda}$.

$$S(\rho(\boldsymbol{\lambda} + \delta\boldsymbol{\lambda})||\rho(\boldsymbol{\lambda})) \approx \frac{1}{2}\delta\boldsymbol{\lambda}^T F(\rho(\boldsymbol{\lambda}))d\boldsymbol{\lambda}. \tag{8}$$

For our purpose, we consider the set $\boldsymbol{\lambda} = (\lambda)$ containing only one parameter, thereby we obtain the same relation in (7) with the probability distributions replaced by the density matrices (5) and (6). Then, using (1), (4) and (8), we'll have

$$S\Big(|\psi(\delta\lambda, t)\rangle\langle\psi(\delta\lambda, t)| \;\Big\|\; |\psi_0\rangle\langle\psi_0|\Big) \approx \frac{1}{2}\delta\lambda^2 \frac{\partial^2}{\partial(\delta\lambda)^2}\mathcal{L}(\delta\lambda, t)\Big|_{\delta\lambda\to 0} \tag{9}$$