# Dynamical Casimir effect under the effect of gravitational waves

How about the question of irreversibility in dynamical Casimir effect when gravitational waves are considered?.

## I. SCALAR FIELD IN CURVED SPACETIME

We consider a real massless scalar field $\Phi(x^\mu)$ in a fixed curved background whose spacetime structure is described by a $(3+1)$-dimensional manifold $\mathcal{M}$, with a Lorentzian metric $g_{\mu\nu}$. The correspondent field action is

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi. \tag{1}$$

By taking (1) to be stationary one obtains the Klein-Gordon equation in curved spacetime

$$\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\right)\Phi(x^\mu) = 0. \tag{2}$$

The field conjugated momentum in this terms is defined as

$$\pi(x^\mu) = -\sqrt{-g}g^{00}\partial_0\Phi(x^\mu).$$

## II. GRAVITATIONAL WAVES

In the linear approximation, the spacetime metric $g_{\mu\nu}$ can be expanded around a flat metric in terms of a small perturbation $h_{\mu\nu}$, such as in

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3}$$

where $|h_{\mu\nu}| \ll 1$ and its inverse satisfies $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. Using the "traceless" metric

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}, \tag{4}$$

with $h$ standing for the trace of $h_{\mu\nu}$, one can show that the Einstein's equations becomes

$$\Box\overline{h}_\mu = -16\pi T_{\mu\nu}, \tag{5}$$

as long as we impose the Lorenz Gauge $\partial_\sigma\overline{h}^{\sigma\lambda} = 0$ and the Transverse-traceless Gauge $\overline{h}^\mu_\mu = 0$ (or $\overline{h}_{\mu\nu} = h_{\mu\nu}$) and $\partial^\mu h_{\mu\nu} = 0$.

Considering vacuum solutions ($T_{\mu\nu} = 0$), we can write

$$\overline{h}_{\mu\nu} = A_{\mu\nu}e^{ik_\mu x^\mu}. \tag{6}$$

Setting the coordinate axes so that the gravitational wave is propagating in the $z$-direction, the linearized metric tensor is expressed as

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1+h_+ & h_\times & 0 \\ 0 & h_\times & 1-h_+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

and

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1-h_+ & -h_\times & 0 \\ 0 & -h_\times & 1+h_+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

where

$$h_+ = A_+\cos[\omega(z-t)] \tag{9}$$
$$h_\times = A_\times\cos[\omega(z-t)+\delta]. \tag{10}$$

Using the above metric in the dynamical equation (2), one obtain [1]

$$\Box\Phi - \left[h_+(\partial_x x - \partial_{yy}) + 2h_\times\partial_y\partial_x\right]\Phi = 0, \tag{11}$$

where we have used the determinant $g = -(1 - h_+^2 - h_\times^2) \approx -1$.

## III. INSTANTANEOUS BASIS APPROACH

Lets consider the quantized field $\hat{\Phi}(\mathbf{r}, t)$ written in the euclidean coordinates. Here we shall concentrate in the case in which the scalar field is confined in a cuboid composed by six parallel ideal mirrors with the following Dirichlet Boundary conditions

$$\Phi(0, y, z, t) = \Phi(L_x, y, z, t) = 0; \tag{12}$$
$$\Phi(x, 0, z, t) = \Phi(x, L_y, z, t) = 0; \tag{13}$$
$$\Phi(x, y, 0, t) = \Phi(x, y, L(t), t) = 0, \tag{14}$$

where $L(t)$ describe an externally prescribed trajectory. Such a system must respect the following set of differential equations for $\hat{\Phi}$ and its conjugate field $\hat{\Pi}$

$$\partial_t\hat{\Phi}(\mathbf{r}, t) = \hat{\Pi}(\mathbf{r}, t) \tag{15a}$$
$$\partial_t\hat{\Pi}(\mathbf{r}, t) = \nabla^2\hat{\Phi}(\mathbf{r}, t) + h_+(z,t)(\partial_x^2 - \partial_y^2)\Phi(\mathbf{r}, t) + 2h_\times(z,t)\partial_x\partial_y\Phi(\mathbf{r}, t). \tag{15b}$$

Since the amplitudes $A_+$ and $A_\times$ are very small, we are allowed to decompose the quantized field in terms of solutions for the free scalar field (Eq. (**??**) when $A_+ = A_\times = 0$). For this matter, let us consider a set of *instantaneous basis function*

$$\varphi_k(\mathbf{r}; t) = \sqrt{\frac{8}{L_x L_y L(t)}}\sin\left(\frac{k_x\pi}{L_x}x\right)\sin\left(\frac{k_y\pi}{L_y}y\right) \tag{16}$$
$$\times\sin\left(\frac{k_z\pi}{L(t)}z\right), \tag{17}$$

which are generally used as a basis function to the free field solution. Then we expand $\hat{\Phi}$ and its field conjugate $\hat{\Pi} = \partial_t \hat{\Phi}$ in terms of the *instantaneous* creation and annihilation operators $\hat{a}_{\mathbf{k}}(t)$ and $\hat{a}^{\dagger}_{\mathbf{k}}(t)$, as in

$$\hat{\Phi} = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}(t)}} \left[ \hat{a}_{\mathbf{k}}(t) e^{-i\Omega_{\mathbf{k}}(t)} + \text{H.c.} \right] \varphi_{\mathbf{k}}(\mathbf{r};t), \quad \text{(18a)}$$

$$\hat{\Pi} = i \sum_{\mathbf{k}} \sqrt{\frac{\omega_{\mathbf{k}}(t)}{2}} \left[ \hat{a}^{\dagger}_{\mathbf{k}}(t) e^{i\Omega_{\mathbf{k}}(t)} - \text{H.c.} \right] \varphi_{\mathbf{k}}(\mathbf{r};t), \quad \text{(18b)}$$

where H.c. stands for Hermitian conjugated and $\Omega_{\mathbf{k}}(t) = \int_0^t dt' \omega_{\mathbf{k}}(t')$, with eigenfrequencies

$$\omega_{\mathbf{k}}(t) = \pi \sqrt{\left(\frac{k_x}{L_x}\right)^2 + \left(\frac{k_y}{L_y}\right)^2 + \left(\frac{k_z}{L(t)}\right)^2} \quad (19)$$

## IV. OBTAINING THE DYNAMICAL EQUATIONS FOR $\hat{a}_k(t)$ AND $\hat{a}^{\dagger}_k(t)$

Considering $\hat{\mathscr{A}}_{\mathbf{k}} = \hat{a}_{\mathbf{k}}(t) e^{-i\Omega_{\mathbf{k}}(t)}$, one take the time derivative for the fields as

$$\begin{aligned} \dot{\hat{\Phi}} &= \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( \hat{\mathscr{A}}_{\mathbf{k}} + \hat{\mathscr{A}}^{\dagger}_{\mathbf{k}} \right) \left( \dot{\varphi}_{\mathbf{k}} - \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \varphi_{\mathbf{k}} \right) \quad \text{(20a)} \\ &+ \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( \dot{\hat{a}}_{\mathbf{k}} e^{-i\Omega_{\mathbf{k}}} + \dot{\hat{a}}^{\dagger}_{\mathbf{k}} e^{i\Omega_{\mathbf{k}}} \right) \varphi_{\mathbf{k}} + \hat{\Pi} \\ \dot{\hat{\Pi}} &= i \sum_{\mathbf{k}} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left( \hat{\mathscr{A}}^{\dagger}_{\mathbf{k}} - \hat{\mathscr{A}}_{\mathbf{k}} \right) \left( \dot{\varphi}_{\mathbf{k}} + \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \varphi_{\mathbf{k}} \right) \quad \text{(20b)} \\ &+ i \sum_{\mathbf{k}} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left( \dot{\hat{a}}^{\dagger}_{\mathbf{k}} e^{i\Omega_{\mathbf{k}}} - \dot{\hat{a}}_{\mathbf{k}} e^{-i\Omega_{\mathbf{k}}} \right) \varphi_{\mathbf{k}} + \nabla^2 \hat{\Phi}, \end{aligned}$$

where for conciseness we have suppressed the notation of time and spatial dependence in all terms in (20) and the upper dot convey time derivative. Comparing (15) with (20a) and (20b), we can isolate the time derivative of the ladder operators by computing

$$\begin{aligned} \int_{\Sigma(t)} \mathrm{d}^3 r \varphi_{\mathbf{j}} \left( \dot{\hat{\Phi}} - \hat{\Pi} \right) &= \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( \dot{\hat{a}}^{\dagger}_{\mathbf{k}} e^{i\Omega_{\mathbf{k}}} + \dot{\hat{a}}_{\mathbf{k}} e^{-i\Omega_{\mathbf{k}}} \right) \delta_{\mathbf{kj}} \\ &+ \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( \hat{\mathscr{A}}_{\mathbf{k}} + \hat{\mathscr{A}}^{\dagger}_{\mathbf{k}} \right) \left( G_{\mathbf{kj}} - \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \delta_{\mathbf{kj}} \right) = 0 \end{aligned}$$

$$\begin{aligned} \int_{\Sigma(t)} \mathrm{d}^3 r \varphi_{\mathbf{j}} \left( \dot{\hat{\Pi}} - \nabla^2 \hat{\Phi} \right) &= i \sum_{\mathbf{k}} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left( \dot{\hat{a}}^{\dagger}_{\mathbf{k}} e^{i\Omega_{\mathbf{k}}} - \dot{\hat{a}}_{\mathbf{k}} e^{-i\Omega_{\mathbf{k}}} \right) \delta_{\mathbf{kj}} \\ &+ i \sum_{\mathbf{k}} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left( \hat{\mathscr{A}}_{\mathbf{k}} - \hat{\mathscr{A}}^{\dagger}_{\mathbf{k}} \right) \left( G_{\mathbf{jk}} - \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \delta_{\mathbf{kj}} \right) \\ &= \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( \hat{\mathscr{A}}_{\mathbf{k}} + \hat{\mathscr{A}}^{\dagger}_{\mathbf{k}} \right) \left[ H^{+}_{\mathbf{kj}} + 2H^{\times}_{\mathbf{kj}} \right] \end{aligned}$$

where it was used

$$\delta_{\mathbf{kj}} = \int_{\Sigma(t)} \mathrm{d}^3 r \; \varphi_{\mathbf{k}} \varphi_{\mathbf{j}} \quad (21)$$

$$G_{\mathbf{kj}} := \int_{\Sigma(t)} \mathrm{d}^3 r \; \varphi_{\mathbf{j}} \dot{\varphi}_{\mathbf{k}} \quad (22)$$

$$H^{+}_{\mathbf{kj}} := \int_{\Sigma(t)} \mathrm{d}^3 r \; h_{+}(z,t) \varphi_{\mathbf{j}} (\partial_x^2 - \partial_y^2) \varphi_{\mathbf{k}} \quad (23)$$

$$H^{\times}_{\mathbf{kj}} := \int_{\Sigma(t)} \mathrm{d}^3 r \; h_{\times}(z,t) \varphi_{\mathbf{j}} \partial_x \partial_y \varphi_{\mathbf{k}} \quad (24)$$

By defining

$$\mu_{\mathbf{kj}} = \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{j}}}} G_{\mathbf{jk}} - \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \delta_{\mathbf{kj}} \quad (25)$$

$$\nu_{\mathbf{kj}} = \frac{1}{\sqrt{2\omega_{\mathbf{k}}\omega_{\mathbf{j}}}} \left[ H^{+}_{\mathbf{kj}} + 2H^{\times}_{\mathbf{kj}} \right] \quad (26)$$

we obtain from (21) and (??)

$$\begin{aligned} \dot{\hat{a}}^{\dagger}_{\mathbf{j}} e^{i\Omega_{\mathbf{j}}} + \dot{\hat{a}}_{\mathbf{j}} e^{-i\Omega_{\mathbf{j}}} &= \sum_{\mathbf{k}} \mu_{\mathbf{kj}} \left( \hat{\mathscr{A}}_{\mathbf{k}} + \hat{\mathscr{A}}^{\dagger}_{\mathbf{k}} \right), \\ \dot{\hat{a}}^{\dagger}_{\mathbf{j}} e^{i\Omega_{\mathbf{j}}} - \dot{\hat{a}}_{\mathbf{j}} e^{-i\Omega_{\mathbf{j}}} &= \sum_{\mathbf{k}} \mu_{\mathbf{jk}} \left( \hat{\mathscr{A}}_{\mathbf{k}} - \hat{\mathscr{A}}^{\dagger}_{\mathbf{k}} \right) \\ &\quad - i \sum_{\mathbf{k}} \nu_{\mathbf{kj}} \left( \hat{\mathscr{A}}_{\mathbf{k}} + \hat{\mathscr{A}}^{\dagger}_{\mathbf{k}} \right), \end{aligned}$$

From the last system is easy to isolate $\dot{\hat{a}}_j$ and $\dot{\hat{a}}^{\dagger}_j$ as

$$\dot{\hat{a}}_{\mathbf{j}} = \sum_{\mathbf{k}} \left( A_{\mathbf{kj}} \hat{a}_{\mathbf{k}} + B^{*}_{\mathbf{kj}} \hat{a}^{\dagger}_{\mathbf{k}} \right), \quad \text{(27a)}$$

$$\dot{\hat{a}}^{\dagger}_{\mathbf{j}} = \sum_{\mathbf{k}} \left( B_{\mathbf{kj}} \hat{a}_{\mathbf{k}} + A^{*}_{\mathbf{kj}} \hat{a}^{\dagger}_{\mathbf{k}} \right), \quad \text{(27b)}$$

where

$$A_{\mathbf{kj}} = \frac{1}{2} \left[ \mu_{\mathbf{kj}} - \mu_{\mathbf{jk}} + i\nu_{\mathbf{kj}} \right] e^{-i[\Omega_{\mathbf{k}} - \Omega_{\mathbf{j}}]} \quad (28)$$

$$B_{\mathbf{kj}} = \frac{1}{2} \left[ \mu_{\mathbf{kj}} + \mu_{\mathbf{jk}} - i\nu_{\mathbf{kj}} \right] e^{-i[\Omega_{\mathbf{k}} + \Omega_{\mathbf{j}}]}. \quad (29)$$

## V. OBTAINING THE BOGOLIUBOV COEFFICIENTS

From this we can calculate the number of particles created from the vacuum state as simply

$$N(t) = \sum_{\mathbf{kj}} |\beta_{\mathbf{kj}}(t)|^2 \quad \text{with} \quad \beta_{\mathbf{kj}}(t) = \int_0^t \mathrm{d}t' B_{\mathbf{kj}}(t'). \quad (30)$$

Considering $B_{\mathbf{kj}} = B^{0}_{\mathbf{kj}} - iB^{G}_{\mathbf{kj}}$, where

$$B^{0}_{\mathbf{kj}} = \frac{1}{2} \left[ \mu_{\mathbf{kj}} + \mu_{\mathbf{jk}} \right] e^{-i[\Omega_{\mathbf{k}} + \Omega_{\mathbf{j}}]} \quad (31)$$

$$B^{G}_{\mathbf{kj}} = \frac{1}{2} \nu_{\mathbf{kj}} e^{-i[\Omega_{\mathbf{k}} + \Omega_{\mathbf{j}}]} \quad (32)$$

are the coefficients representing the free contribution and the gravity corrections. In this manner we can see, that

$$N(t) = N_0(t) - N_G(t), \tag{33}$$

where

$$N_0(t) = \sum_{\mathbf{kj}} \left| \int_0^t \mathrm{d}t' B^0_{\mathbf{kj}}(t) \right|^2 \tag{34}$$

$$N_G(t) = \sum_{\mathbf{kj}} \left| \int_0^t \mathrm{d}t' B^G_{\mathbf{kj}}(t) \right|^2 \tag{35}$$

are, correspondently, the free contribution and the gravity correction for the number of particles created from the vacuum. So, as expected, the number of particles must decrease due to the gravitational coupling.

## Appendix A: Calculating the $G_{kj}$ Coefficients

If we decompose $\varphi_{\mathbf{k}}(\mathbf{r};t) = \varphi_{\mathbf{k}_{||}}(x,y)\varphi_{k_z}(z,t)$, where $\mathbf{k}_{||} = (k_x, k_y)$ and

$$\varphi_{\mathbf{k}_{||}}(x,y) := \sqrt{\frac{4}{L_x L_y}} \sin\left(\frac{k_x \pi}{L_x} x\right) \sin\left(\frac{k_y \pi}{L_y} y\right) \tag{A1}$$

$$\varphi_{k_z}(z,t) := \sqrt{\frac{2}{L(t)}} \sin\left(\frac{k_z \pi}{L(t)} z\right), \tag{A2}$$

we must obtain

$$\begin{aligned} G_{\mathbf{kj}} &= \int_{\Sigma(t)} \mathrm{d}^3 r \ \varphi_{\mathbf{j}} \dot{\varphi}_{\mathbf{k}} \\ &= \int_0^{L_x} \mathrm{d}x \int_0^{L_y} \mathrm{d}y \varphi_{\mathbf{k}_{||}}(x,y) \varphi_{\mathbf{j}_{||}}(x,y) \\ &\times \int_0^{L(t)} \mathrm{d}z \varphi_{k_z}(z,t) \dot{\varphi}_{j_z}(z,t) \\ &= (1)^{j_z - k_z} \frac{2 k_z j_z}{j_z^2 - k_z^2} \frac{\dot{L}(t)}{L(t)} \delta_{k_x, j_x} \delta_{k_y, j_y} \end{aligned} \tag{A3}$$


[1] S. Morales and A. Dasgupta. Scalar and fermion field interactions with a gravitational wave. Class. Quantum Grav. **37**, 105001 (2020).

[2] G. de Oliveira, and L. C. Céleri. Thermodynamic entropy production in the dynamical Casimir effect. https://arxiv.org/abs/2309.07847 (2023).