Federal University of Goiás

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Quantum Pequi Group

Effective Hamiltonian aproach for the Dynamical Casimir Effect

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Calculations notes

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Contents

Chapter 1

Effective Hamiltonian

1.1 The theory

In order to build the effective Hamiltonian for the DCE, let us consider a scalar field $\phi(x,t)$ with a Lagrangian density

$$\mathcal{L}(x,t) = \frac{1}{2}\left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right]. \tag{1.1}$$

We now confine such field inside a one-dimensional cavity whose size is time-dependent. We can implement such system by considering that one of the mirror of the cavity moves accordingly with a given trajectory, in such a way its position is described by $x = q(t)$, while the other one is fixed at some position arbitrarily chosen as $x = 0$. If the mirror are perfect, the boundary conditions we impose on the field take the form $\phi(0,t) = \phi(q(t),t) = 0$.

From the Euler-Lagrange equations we obtain

$$\partial_x^2\phi = \partial_t\pi, \tag{1.2}$$

where $\pi = \partial\mathcal{L}/\partial\dot{\phi} = \partial_t\phi$ is the conjugated momenta of the field.

At every instant of time we can expand the field as

$$\phi(x,t) = \sum_k \frac{1}{\sqrt{2\omega_k(t)}}\left[a_k f_k(x,t) + a_k^* f_k^*(x,t)\right] \tag{1.3}$$

where $\{\psi(x,t)\}$ is a set of *instantaneous* mode functions. Since the field must be a solution of the wave equation (2.1), we impose that $f_k(x,t) = \varphi_k(x)e^{-i\omega_k(t)t}$, resulting in the following set of equations

$$\partial_x^2\varphi_k + \omega_k^2(t)\varphi_k = 0, \tag{1.4}$$

subjected to the boundary conditions $\varphi_k(0,t) = \varphi_k(q(t),t) = 0$ and a normalization condition $\int_0^{q(t)} \varphi_k\varphi_j\, dx = \delta_{jk}$. It is important to observe here that the time-dependence of the frequency of the mode comes from the one in the mirror's trajectory. This is an implicit time-dependence that allows us to define the set $\{f(x,t)\}$ as the instantaneous basis vectors [CITE]. Of course, the quantization procedure will not directly lead us to physically meaningful creation and annihilation operators for such modes, but this will not be problematic in the following.

In order to construct the corresponding quantum field version of our theory we follow the canonical scheme. We begin by promoting the classical scalar field and its associate momentum to quantum operators satisfying the standard equal-time commutation relations

$$\begin{aligned}\left[\hat{\phi}(x,t),\hat{\pi}(x',t)\right] &= i\delta(x-x') \\ \left[\hat{\phi}(x,t),\hat{\phi}(x',t)\right] &= [\hat{\pi}(x,t),\hat{\pi}(x',t)] = 0.\end{aligned} \tag{1.5}$$

The first consequence is that we must replace the complex amplitudes a_k and a_k^* by annihilation and creation operators $\hat{a}_k$ and $\hat{a}_k^\dagger$, which inherit the standard commutation relations $\left[\hat{a}_k,\hat{a}_j^\dagger\right] = \delta_{kj}$, $[\hat{a}_k,\hat{a}_j] = \left[\hat{a}_k^\dagger,\hat{a}_j^\dagger\right] = 0$. After introducing the algebraic structure, to complete the quantization program we must define a Hilbert space in order to give the theory a particle interpretation: first we characterise the vacuum state of the field as the state which is annihilated by all the $\hat{a}_k$:

$$\hat{a}_k|0\rangle = 0, \quad \forall k \tag{1.6}$$

Then we define the one-particle Hilbert space by applying the creation operators $\hat{a}_k^\dagger$ to the vacuum state

$$\hat{a}_k^\dagger|0\rangle = |1_k\rangle, \tag{1.7}$$

and finally we build the complete Fock space by repeated application of the particle creator:

$$\left|n_{k_1}^1, n_{k_2}^2, s\right\rangle = \prod_i \frac{1}{\sqrt{n^i!}}\left(\hat{a}_{k_i}^\dagger\right)^{n^i}|0\rangle\,. \tag{1.8}$$

We observe here that such states, in special the vacuum state, changes as time goes on, since the basis is dynamical. In other words, at every instant of time we have a different set of modes, implying a different Hilbert space and distinct creation and annihilation operators.

Given these definitions, we can write the field operator and its cojugated as

$$\hat{\phi}(x,t) = \sum_k \frac{1}{\sqrt{2\omega_k(t)}}\left(\hat{a}_k(t) + \hat{a}_k^\dagger(t)\right)\varphi_k(x,t), \tag{1.9}$$

$$\hat{\pi}(x,t) = i\epsilon(x,t)\sum_k \sqrt{\frac{\omega_k(t)}{2}}\left(\hat{a}_k^\dagger(t) - \hat{a}_k(t)\right)\varphi_k(x,t), \tag{1.10}$$

where $\hat{a}_k(t) = \hat{a}_k e^{-\omega_{k,q(t)}t}$. With these expressions it is straightforward to write down the instantaneous field Hamiltonian as the Legendre transformation of the Lagrangian, given us (see Appendix B for details)

$$\begin{aligned} \hat{H}_{\text{eff}} &= \frac{1}{2}\int_0^{q(t)} dx \left(\hat{\pi}\partial_t\hat{\phi} - \hat{\phi}\partial_t\hat{\pi}\right) \\ &= \sum_k \omega_k(t)\hat{a}_k^\dagger\hat{a}_k + i\sum_k \xi_k(t)\left(\hat{a}_k^{\dagger 2} - \hat{a}_k^2\right) \\ &+ i\sum_{\substack{j,k \\ k\neq j}} \mu_{kj}(t)\left(\hat{a}_k^\dagger\hat{a}_j^\dagger + \hat{a}_k^\dagger\hat{a}_j - \hat{a}_j\hat{a}_k - \hat{a}_j^\dagger\hat{a}_k\right), \end{aligned} \tag{1.11}$$

where we have defined

$$\begin{aligned} \mu_{k,j}(t) &= \frac{1}{2}\sqrt{\frac{\omega_k}{\omega_j}}G_{k,j}(t) \\ \xi_k(t) &= \frac{1}{4\omega_k(t)}\frac{d\omega_k(t)}{dt} + \frac{G_{k,k}(t)}{2} \end{aligned} \tag{1.12}$$

with

$$G_{k,j}(t) := -\int_0^{q(t)} \mathrm{d}x\varphi_k(x,t)\partial_t\varphi_j(x,t). \tag{1.13}$$

Once obtained the effective Hamiltonian, the time evolution of the system is governed by the Schrodinger equation

$$i\partial_t|\Psi(t)\rangle = \hat{H}_{\text{eff}}|\Psi(t)\rangle \tag{1.14}$$

As pointed out by Law [CITE], the formalism is based on a set of instantaneous basis functions $\{\varphi(x,t\}$, the vacuum states changes accordingly with the change in the time parameter, so the bosons associated with the creation and annihilation operators may not be regarded as real particles while the cavity is still in motion. Only in the static situation such operators become unique, thus acquiring the usual physical meaning.

1.2 Perturbation theory

1.2.1 Number of particles

From the Schrodinger picture (where the time dependence is presented in the quantum states) we transition to the interaction picture where the time dependence is shared with the operators. With the help of the free field $\hat{H}_0 = \sum_k \omega_k(t)\hat{a}_k^\dagger\hat{a}_k$ and its evolution operator $\hat{U}_0(t) = \exp\left\{-i\int_0^t d\tau\hat{H}_0(\tau)\right\}$, the interaction picture ket $|\psi\rangle_I$ and an arbitrary time dependent observable $\hat{A}_I(t)$ are related with their Schrodinger picture counterpart by the relations

$$\begin{aligned} |\psi(t)\rangle_I &= \hat{U}_0^\dagger(t)|\psi(t)\rangle_S, \\ \hat{A}_I(t) &= \hat{U}_0^\dagger(t)\hat{A}_S\hat{U}_0(t), \end{aligned} \tag{1.15}$$

where the correct the Schrodinger equation is given by

$$i\frac{d|\psi(t)\rangle_I}{dt} = \hat{V}_I|\psi(t)\rangle_I, \tag{1.16}$$

and $\hat{V}_S = \hat{H}_{\text{eff}} - \hat{H}_0$. From this perspective the effective Hamiltonian () can be written in the interaction picture as (see appendix B)

$$\hat{H}_I = i\sum_k \left\{ \xi_k(t)\mathcal{A}_k^{\dagger 2}(t) + \sum_{j(\neq k)} \mu_{kj}(t)\mathcal{A}_k^\dagger(t)\left(\mathcal{A}_j^\dagger(t) + \mathcal{A}_j(t)\right) - \text{h.c.} \right\}, \tag{1.17}$$

where $\hat{\mathcal{A}}_k(t) = \hat{a}_k e^{-i\tilde{\omega}_k(t)}$ with $\tilde{\omega}_k(t) = \exp\{-\int^t \omega_k(\tau)d\tau\}$.

The dynamics of the system's density operator $\rho(t)$ can be determined with the help of the dynamical equation

$$\frac{d}{dt}\rho(t) = -i\left[\hat{H}_I(t), \rho(t)\right]. \tag{1.18}$$

Considering $\dot{q}(t)$ as a perturbation parameter, a formal solution for $\rho(t)$ in the second order of $H_I(t)$ can be putted as

$$\rho(t) = \rho(0) - i\int_0^t d\tau\left[\hat{H}_I(\tau), \rho(0)\right] - \int_0^t dt'\int_0^{t'} d\tau\left[\hat{H}_I(t'), \left[\hat{H}_I(\tau), \rho(0)\right]\right] + \mathcal{O}(\hat{H}_I^3(t)), \tag{1.19}$$

allowing us to calculate the number of photons created in the cavity with the formula

$$\langle N(t)\rangle = \text{Tr}\left\{\rho(t)\hat{N}\right\} = \text{Tr}\left\{\sum_{k'}\rho(t)\hat{a}_{k'}^\dagger\hat{a}_{k'}\right\}.$$

After computing a complicated expression for the density operator in the energy basis (see appendix C) we can notice that if we investigate the case in which the system is initially prepared in the vacuum state, all the off-diagonal terms of $\rho(t)$ must vanish in the calculation of the trace of $\rho(t)\hat{N}$. With this reasoning we can simplify our considerations by paying attention only to the diagonal terms of $\rho(t)$ which reads

$$\begin{aligned} \rho_{\text{diag}}(t) = \hat{\rho}(0) - \sum_{k'}\int_0^t dt'\int_0^{t'} d\tau \Bigg\{ & \chi_{k',k'}(\tau)\chi_{k',k'}(t')\left[\hat{\mathcal{A}}_{k'}^2(t')\hat{\mathcal{A}}_{k'}^{\dagger 2}(\tau)\hat{\rho}(0) - \hat{\mathcal{A}}_{k'}^{\dagger 2}(t')\hat{\rho}(0)\hat{\mathcal{A}}_{k'}^2(\tau)\right] \\ & - \sum_{j'(\neq k')}\chi_{k',j'}(\tau)\chi_{k',j'}(t')\hat{\mathcal{A}}_{k'}^\dagger(t')\hat{\mathcal{A}}_{j'}^\dagger(t')\hat{\rho}(0)\hat{\mathcal{A}}_{j'}(\tau)\hat{\mathcal{A}}_{k'}(\tau) + \text{h.c.} \Bigg\}. \end{aligned} \tag{1.20}$$

With the help of the last expression the number of particles can be find as

$$\langle N(t)\rangle = \sum_k \left(\eta_k(t) + \sum_{j(\neq k)} \eta_{kj}(t) \right)$$

where

$$\begin{aligned}
\eta_k(t) &= \langle 2_k|\rho(t)\hat{N}|2_k\rangle = 8P\left[\xi_k(t); 2\omega_k(t)\right] \\
\eta_{kj}(t) &= \langle 1_k, 1_j|\rho(t)\hat{N}|1_k, 1_j\rangle = 4P\left[\mu_{kj}(t) + \mu_{jk}(t); \omega_k(t) + \omega_j(t)\right].
\end{aligned}$$

and

$$P\left[a(t); b(t)\right] = \int_0^t dt' \int_0^{t'} d\tau a(\tau) a(t') \cos\left[b(\tau) - b(t')\right].$$

1.2.2 Reversibility of the DCE and the diagonal entropy

The search for an adequate thermodynamical description of isolated quantum systems is still a subtle subject due to the lack of a satisfactory definition of entropy. This mainly occurs because the natural candidate, the von Neumman entropy $S(\rho) = -\operatorname{Tr}\rho\ln\rho$, is not compatible with the second law of thermodynamics for closed systems since its value is conserved for any unitary process. Moreover, for sufficiently complex systems that have reached a steady state, all knowledge about arbitrary time-averaged observables is contained in the diagonal elements of ρ in the time-independent energy eigenbasis [1], meaning that $S(\rho)$ carries superfluous information that do not appears in thermodynamic measurements.

The argument for the necessity of the thermodynamic entropy to depend only on the diagonal elements of ρ (in the energy basis) can also be put forward by considering that in the quantum realm, heat can be defined as the energy increase caused by transitions between different energy levels [2]. From the adiabatic theorem for quantum mechanics, a slow changing Hamiltonian can not induce such transitions and any explicit time-dependence must appears only at the coherences terms. This last fact means that in the adiabatic limit (quasi-static processes), the diagonal elements of ρ must remain unchanged in time, as well as any entropic quantity sensible to only those terms.

From this reasoning, in order to study the reversibility in the DCE associated with our model of moving cavity, we must consider the diagonal entropy [1]

$$S_d(\rho_{\text{diag}}) = -\sum_n \rho_{\text{diag}}^{(n)} \ln \rho_{\text{diag}}^{(n)}, \tag{1.21}$$

as the main figure of merit, where $\rho_{\text{diag}}^{(n)} = \langle n|\rho|n\rangle$ are the diagonal elements of the system's density matrix in relation to the set $\{|n\rangle\}$ of the time-independent energy eigenstates. With the help of the expression

(1.20), the diagonal entropy then takes the form of

$$S_d(t) = -\rho^{(0)}_{\text{diag}}(t)\ln\rho^{(0)}_{\text{diag}}(t) - \sum_{k'}\left(\rho^{(2_k)}_{\text{diag}}(t)\ln\rho^{(2_k)}_{\text{diag}}(t) + \sum_{j'(\neq k')}\rho^{(1_k1_j)}_{\text{diag}}(t)\ln\rho^{(1_k1_j)}_{\text{diag}}(t)\right)$$

with

$$\rho^{(0)}_{\text{diag}}(t) = \langle 0|\rho_{\text{diag}}(t)|0\rangle = 1 - \frac{1}{2}\sum_k\left(\eta_k(t) + \sum_{j(\neq k)}\eta_{kj}(t)\right)$$
$$\rho^{(2_k)}_{\text{diag}}(t) = \langle 2_k|\rho_{\text{diag}}(t)|2_k\rangle = \frac{1}{2}\eta_k(t)$$
$$\rho^{(1_k1_j)}_{\text{diag}}(t) = \langle 1_k, 1_j|\rho_{\text{diag}}(t)|1_k, 1_j\rangle = \frac{1}{2}\eta_{kj}(t).$$

1.3 Periodic motion

In order to employ perturbation theory, we must confine our attention to the weakly perturbed regime where the cavity perform a periodic motion with small amplitude. In this moving-mirror setup the field mode function can be defined with the analogous form of equation (2.4)

$$f_k(x,t) = \sqrt{\frac{2}{q(t)}}\sin\left[\omega_k(t)x\right]e^{-i\omega_k(t)t},$$

with an implicit time-dependence in the mode frequency $\omega_k(t) = \pi k/q(t)$, allowing us to directly obtain the time dependent parameters,

$$\begin{aligned}\xi_k(t) &= \frac{1}{4}\frac{d}{dt}\ln q(t),\\ \mu_{kj}(t) &= \frac{1}{2}(-1)^{j+k}\frac{kj}{j^2-k^2}\left(\frac{k}{j}\right)^{1/2}\frac{d}{dt}\ln q(t).\end{aligned} \tag{1.22}$$

A convenient mirror trajectory to be chosen is

$$q(t) = L_0\exp\{\epsilon E(t)\} \approx L_0\left[1+\epsilon E(t)\right], \tag{1.23}$$

where $E(t)$ is a periodic function and $\epsilon \ll 1$ is a small dimensionless parameter needed to keep the system with well defined frequencies $\omega_k = k\pi/L_0$ with $k = 1, 2, \ldots$ such that $\tilde{\omega}_k(t) = \omega_k t$. The no-adiabatic functions $\xi_k(t)$ and $\mu_{kj}(t)$ takes the form of

$$\xi_k(t) = \frac{\epsilon}{4}\frac{d}{dt}E(t), \qquad \mu_{kj}(t) = \frac{\epsilon}{2}(-1)^{j+k}\frac{kj}{j^2-k^2}\left(\frac{k}{j}\right)^{1/2}\frac{d}{dt}E(t), \tag{1.24}$$

We can then calculate the coefficients

$$\eta_k(t) = \frac{\epsilon^2}{2} P\left[\frac{d}{dt}E(t); \frac{2\pi k}{L}t\right]$$
$$\eta_{kj}(t) = \frac{2kj}{(j+k)^2}\epsilon^2 P\left[\frac{d}{dt}E(t); \frac{\pi}{L}(k+j)t\right]$$

with

$$P\left[\frac{d}{dt}E(t); \frac{\pi\kappa}{L}t\right] = \int_0^t dt' \int_0^{t'} d\tau \frac{d}{d\tau}E(\tau)\frac{d}{dt'}E(t')\cos\left[\frac{\kappa\pi}{L}(\tau - t')\right].$$

For the case in which the second mirror perform harmonic oscillations with small amplitude we can adequately choose $E(t) = \cos\Omega_p t$ with $\Omega_p = p\pi/L$ implying that we can calculate (see Appendix E)

$$P\left[\sin\left(\Omega_p t\right); \frac{\pi\kappa}{L}t\right] = \frac{\pi^2 p^2}{L^2}\int_0^t dt' \int_0^{t'} d\tau \sin\left(\frac{p\pi}{L}\tau\right)\sin\left(\frac{p\pi}{L}t'\right)\cos\left[\frac{\kappa\pi}{L}(\tau - t')\right] \tag{1.25}$$

$$= \begin{cases} \frac{1}{16}\left[1 + \frac{2\kappa^2\pi^2}{L^2}t^2 - \cos\left(\frac{2\kappa\pi t}{L}\right) - \frac{2\kappa\pi t}{L}\sin\left(\frac{2\kappa\pi t}{L}\right)\right] & \text{for} \quad p = \kappa \\ \frac{1}{2}\frac{p^3}{(p^2-\kappa^2)}\left[\frac{1}{2p}\cos\left(\frac{2p\pi t}{L}\right) - \frac{1}{p-\kappa}\cos(p-\kappa)\frac{\pi t}{L} - \frac{1}{p+\kappa}\cos(p+\kappa)\frac{\pi t}{L} + \frac{2p}{p^2-\kappa^2} - \frac{1}{2p}\right] & \text{for} \quad p \neq \kappa \end{cases} \tag{1.26}$$

or simply $P(t;\kappa,p)$.

To show the adequacy of the latter formalism, we compute the number of particles and the diagonal entropy produced when the cavity oscillates in resonance with some unperturbed field frequency $\Omega_p = p\pi/L$ with $p = 1, 2, \ldots$. Using a rotating wave approximation (RWA) in equation (1.25) (as can be seen in Appendix **??**), the number of particles in second order in ϵ can be obtained as

$$\langle N(t)\rangle = \frac{\pi^2\epsilon^2}{48L^2}p(p^2-1)t^2$$

which is in accordance with the literature value. In contrast, the diagonal entropy in both cases reads

$$S_d(t) = \frac{1}{2}\langle\hat{N}(t)\rangle\left(1 - \ln\frac{1}{2}\langle\hat{N}(t)\rangle\right) + \mathcal{O}(\epsilon^4).$$

1.3.1 Uniform velocity

Let us briefly analyse the particular case where the cavity moves with a small uniform velocity ϵv and a trajectory

$$q(t) = Le^{\epsilon\frac{v}{L}t} \approx L + \epsilon v t.$$

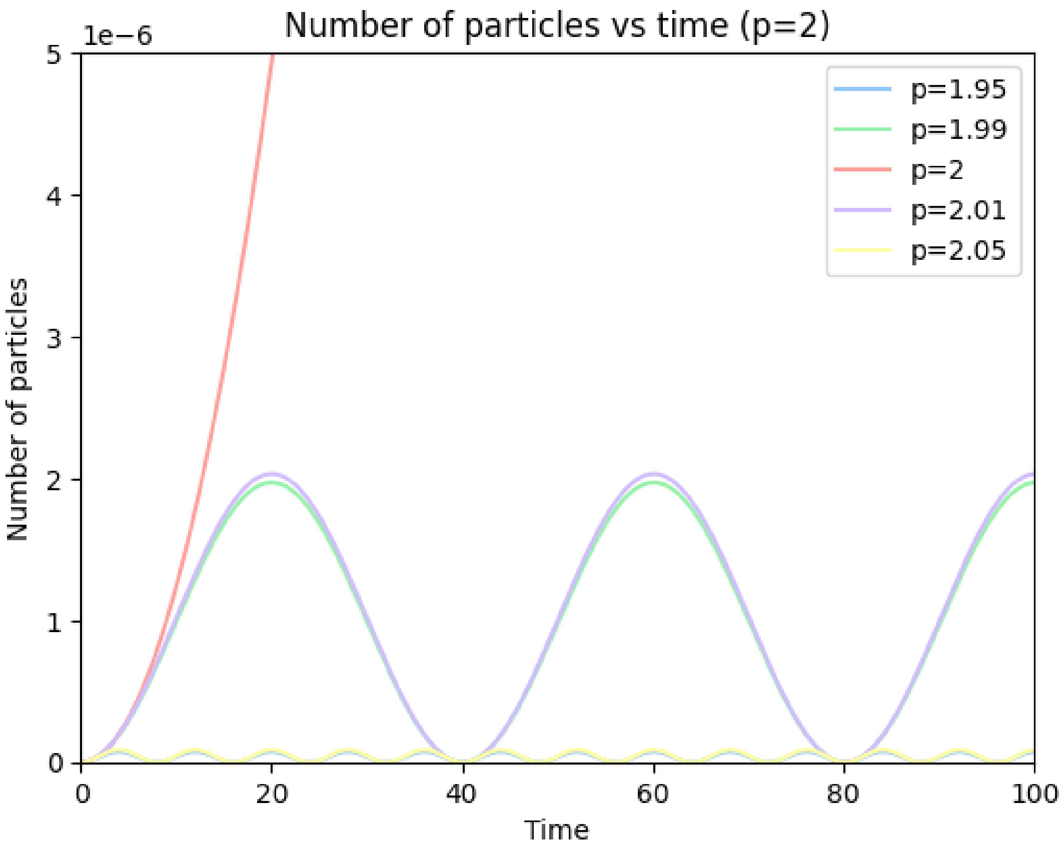

FIGURE 1.1: Number of particles for different possible cavity frequencies.

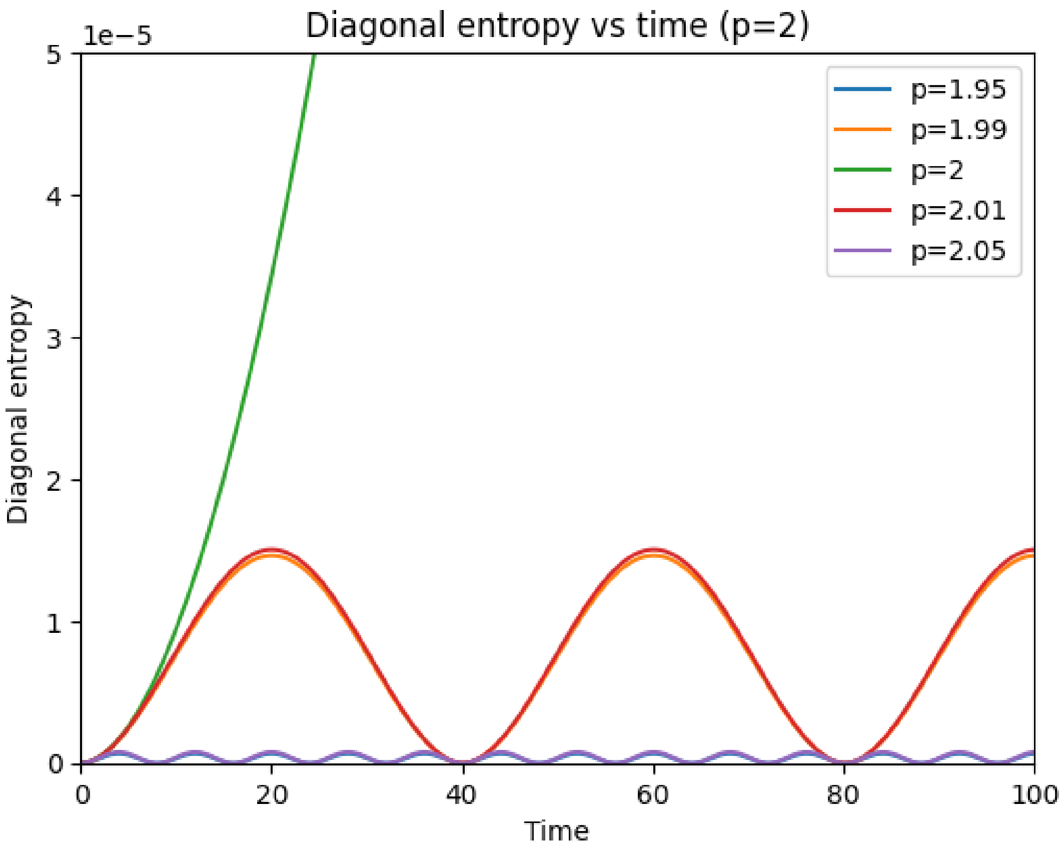

FIGURE 1.2: Diagonal entropy production for different possible cavity frequencies.

If we attain to the cavity description in instances of time such that $\epsilon vt \ll L$, we are allowed to use the last formalism and consider the motion to be characterized by the function $E(t) = vt/L$ as defined in (1.23). We can then calculate the coefficients

$$\eta_k(t) = \epsilon^2 P\left[\frac{v}{L};\frac{2\pi k}{L}t\right]$$
$$\eta_{kj}(t) = \frac{2kj}{(j+k)^2} P\left[\frac{v}{L};\frac{\pi}{L}(k+j)t\right]$$

where

$$P\left[\frac{v}{L};\frac{\kappa\pi t}{L}\right] = \frac{v^2}{L^2}\int_0^t dt' \int_0^{t'} d\tau \cos\kappa\,(\tau - t') = \frac{v^2}{\kappa^2 L^2}\left[1-\cos\left(\frac{\kappa\pi}{L}t\right)\right]$$

Therefore,

$$\sum_k \eta_k = \frac{\epsilon^2 v^2}{4L^2}\left[\sum_k \frac{1}{k^2} - \sum_k \frac{\cos 2k\pi\frac{t}{L}}{\kappa^2}\right] = \frac{\epsilon^2 v^2 \pi^2}{24L^2}\left[1 - 6\,\mathrm{B}_2\left(\left\{\frac{t}{L}\right\}\right)\right]$$

where $\mathrm{B}_2(x) = x^2 - x + 1/6$ is the Bernoulli polynomial of order 2 and $\{x\}$ symbolizes the fractional part of x. Therefore, the total number of particles becomes

$$\langle N(t)\rangle = \frac{\epsilon^2 v^2 \pi^2}{24L^2}\left[1 - 6\,\mathrm{B}_2\left(\left\{\frac{t}{L}\right\}\right)\right] + \frac{2\epsilon^2 v^2}{L^2}\sum_k \sum_{j(\neq k)} \frac{1}{(k+j)^4}\left[1-\cos\left(\frac{(k+j)\pi}{L}t\right)\right]$$
$$\frac{\epsilon^2 v^2 \pi^2}{24L^2}\left[1 - 6\,\mathrm{B}_2\left(\left\{\frac{t}{L}\right\}\right)\right] + \frac{2\epsilon^2 v^2}{L^2}\sum_k \sum_{j(\neq k)} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!(k+j)^4}\left(\frac{(k+j)\pi}{L}t\right)^{2n}$$

the coefficients for the density operator can be written in terms of

$$\rho^{(0)}_{\text{diag}}(t) = 1 - \frac{\epsilon^2 v^2}{96}\left[1 - 6\,\mathrm{B}_2(t/L)\right]$$
$$\rho^{(2_k)}_{\text{diag}}(t) = \frac{1}{16}\frac{\epsilon^2 v^2}{k^2\pi^2}\left[1-\cos\left(\frac{2k\pi}{L}t\right)\right] = \left(\frac{\sqrt{2}\epsilon v}{4k\pi}\sin\frac{k\pi t}{L}\right)^2$$
$$\rho^{(1_k 1_j)}_{\text{diag}}(t) = \left[\frac{\epsilon v}{4\pi}\frac{\sqrt{2kj}k}{(j^2-k^2)(k+j)}\sin\left(\frac{(k+j)\pi}{2L}t\right)\right]^2.$$

The correspondent diagonal entropy can be calculated as

$$S_d(t) = \frac{\epsilon^2 v^2}{96}\left[1 - 6\,\mathrm{B}_2(\{t/L\})\right] - \sum_k \left(\frac{\epsilon v}{2k\pi}\sin\frac{k\pi t}{L}\right)^2 \ln\frac{\sqrt{2}\epsilon v}{4k\pi}\sin\frac{k\pi t}{L}$$
$$- s\sum_k \sum_{j(\neq k)} \left[\frac{\epsilon v}{2\pi}\frac{\sqrt{kj}k}{(j^2-k^2)(k+j)}\sin\left(\frac{(k+j)\pi}{2L}t\right)\right]^2 \ln\left[\frac{\epsilon v}{4\pi}\frac{\sqrt{2kj}k}{(j^2-k^2)(k+j)}\sin\left(\frac{(k+j)\pi}{2L}t\right)\right]$$

$$G_{k,j}(t) := -\int_0^{q(t)} \mathrm{d}x \varphi_k(x,t)\partial_t\varphi_j(x,t) =$$

with

$$\begin{aligned}
\varphi_k(x,t) &= \sqrt{\frac{2}{q(t)}} \sin\left[\frac{k\pi}{q(t)}x\right] \\
\partial_t\varphi_k(x,t) &= \sqrt{\frac{2}{q(t)}} \cos\left[\frac{k\pi}{q(t)}x\right]\left(-k\pi x\frac{\dot{q}}{q^2}\right) + \frac{1}{2}\sqrt{2q(t)}\left(-\frac{\dot{q}}{q^2}\right)\sin\left[\frac{k\pi}{q(t)}x\right] \\
&= -k\pi x\frac{\dot{q}}{q^2}\sqrt{\frac{2}{q(t)}}\cos\left[\frac{k\pi}{q(t)}x\right] - \frac{\dot{q}}{q^2}\sqrt{\frac{q(t)}{2}}\sin\left[\frac{k\pi}{q(t)}x\right] \\
&= -\frac{\dot{q}}{q^2}\left\{k\pi x\sqrt{\frac{2}{q(t)}}\cos\left[\frac{k\pi}{q(t)}x\right] + \sqrt{\frac{q(t)}{2}}\sin\left[\frac{k\pi}{q(t)}x\right]\right\}
\end{aligned}$$

therefore

$$\begin{aligned}
&\varphi_k(x,t)\partial_t\varphi_k(x,t) \\
&\qquad = -\frac{\dot{q}}{q^2}\left\{\frac{2}{q}j\pi x\sin\left[\frac{k\pi}{q(t)}x\right]\cos\left[\frac{j\pi}{q(t)}x\right] + \sin\left[\frac{k\pi}{q(t)}x\right]\sin\left[\frac{j\pi}{q(t)}x\right]\right\}
\end{aligned}$$

and

$$\begin{aligned}
G_{j,k} &= -\int_0^{q(t)} dx\ \varphi_k(x,t)\partial_t\varphi_k(x,t) \\
&= \frac{\dot{q}}{q^2}\left\{j\pi\frac{2}{q(t)}\int_0^{q(t)} dx\ x\sin\left[\frac{k\pi}{q(t)}x\right]\cos\left[\frac{j\pi}{q(t)}x\right] + \int_0^{q(t)} dx\ \sin\left[\frac{k\pi}{q(t)}x\right]\sin\left[\frac{j\pi}{q(t)}x\right]\right\}
\end{aligned}$$

for $k \neq j$

$$\begin{aligned}
&j\pi\frac{\dot{q}}{q^2}\frac{2}{q(t)}\int_0^{q(t)} dx\ x\sin\left[\frac{k\pi}{q(t)}x\right]\cos\left[\frac{j\pi}{q(t)}x\right] \\
&j\pi\frac{\dot{q}}{q^2}\frac{2}{q(t)}\left[\frac{q^2}{2\pi}\left(\frac{(-1)^{j-k}}{j-k} - \frac{(-1)^{j+k}}{j+k}\right)\right] = j\frac{\dot{q}}{q}(-1)^{j+k}\frac{j+k-j+k}{j^2-k^2} \\
&= (-1)^{j+k}\frac{2jk}{j^2-k^2}\frac{\dot{q}}{q}
\end{aligned}$$

for $j = k$

$$k\pi\frac{\dot{q}}{q^2}\frac{2}{q(t)}\int_0^{q(t)} dx\ x\sin\left[\frac{k\pi}{q(t)}x\right]\cos\left[\frac{k\pi}{q(t)}x\right] = (-\frac{q^2}{4k\pi})\frac{\dot{q}}{q^2}\frac{2}{q(t)} - \frac{1}{2}\frac{\dot{q}}{q}$$

$$\int_0^{q(t)} dx\ \sin\left[\frac{k\pi}{q(t)}x\right] \sin\left[\frac{j\pi}{q(t)}x\right] = \frac{q}{2}\delta_{j,k}$$

we have then

$$G_{k,j}(t) = (-1)^{j+k}\frac{2jk}{j^2-k^2}\frac{\dot{q}}{q}$$
$$G_{k,k}(t) = 0$$

for

$$\begin{aligned} \mu_{k,j}(t) &= \frac{1}{2}\sqrt{\frac{\omega_k}{\omega_j}}G_{k,j}(t) = (-1)^{j+k}\frac{jk}{j^2-k^2}\sqrt{\frac{k}{j}}\frac{\dot{q}}{q} \\ \xi_k(t) &= \frac{1}{4\omega_k(t)}\frac{d\omega_k(t)}{dt} + \frac{G_{k,k}(t)}{2} = \frac{1}{4}\frac{\dot{q}}{q} \end{aligned} \tag{1.27}$$

we have therefore

$$\begin{aligned} \mu_{k,j} + \mu_{j,k} &= (-1)^{j+k} jk \left[\frac{\sqrt{\frac{k}{j}}}{j^2-k^2} + \frac{\sqrt{\frac{j}{k}}}{k^2-j^2}\right] = (-1)^{j+k}\frac{jk}{j^2-k^2}\left[\frac{k-j}{\sqrt{jk}}\right] \\ &= (-1)^{j+k}\frac{\sqrt{jk}}{j+k} \end{aligned}$$

Chapter 2

Effective Hamiltonian - 3D

2.1 The dynamical Casimir effect

Let us consider a three-dimensional cavity composed by three pairs of mirrors in the configuration of a rectangular cuboid with dimensions L_x, L_y and L_z. The mirrors placed in the x and y-axis are chosen to remain at rest, while one of the z-axis mirrors is allowed to follow a prescribed trajectory $z = L_z(t)$. Confined inside the cavity we consider a quantum scalar field $\hat{\phi}(\mathbf{x}, t)$ satisfying the wave equation

$$\nabla^2 \hat{\phi}(\mathbf{x}, t) = \partial_t \hat{\pi}(\mathbf{x}, t) \tag{2.1}$$

and the standard equal-time commutation relations

$$\begin{aligned} \left[\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)\right] &= i\delta(\mathbf{x} - \mathbf{x}'), \\ \left[\hat{\phi}(\mathbf{x}, t), \hat{\phi}(\mathbf{x}', t)\right] &= [\hat{\pi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)] = 0, \end{aligned} \tag{2.2}$$

where $\hat{\pi} = \partial_t \hat{\phi}$ is the conjugated momenta of the field. If the mirrors are ideal, the boundary conditions we impose on the field take the form $\hat{\phi}(\mathbf{r}, t)|_{\text{mirrors}} = 0$.

Taking the mirrors to be initially at rest (with static boundary conditions $L_z(t < 0) = L_0$), for $t < 0$ the field can be decomposed as

$$\hat{\phi}^{\text{in}}(\mathbf{x}, t) = \sum_{\mathbf{n}} \frac{1}{\sqrt{2\omega_{\mathbf{n}}}} \left[f_{\mathbf{n}}^{\text{in}}(\mathbf{x}, t)\hat{b}_{\mathbf{n}} + f_{\mathbf{n}}^{\text{in}*}(\mathbf{x}, t)\hat{b}_{\mathbf{n}}^{\dagger} \right], \tag{2.3}$$

where the steady-state mode functions

$$f_{\mathbf{n}}^{\text{in}}(\mathbf{x}, t < 0) = \sqrt{\frac{8}{L_x L_y L_0}} \sin\left(\frac{n_x \pi}{L_x} x\right) \sin\left(\frac{n_y \pi}{L_y} y\right) \sin\left(\frac{n_z \pi}{L_0} z\right) e^{-i\omega_{\mathbf{n}} t} \tag{2.4}$$

are complex valued solutions of the wave equation with frequencies

$$\omega_{\mathbf{n}} = \pi \sqrt{\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2 + \left(\frac{n_z}{L_0}\right)^2}, \tag{2.5}$$

where n_x, n_y and n_z are positive integers and $\mathbf{n} = (n_x, n_y, n_z)$. The annihilation (creation) operator $\hat{b}_{\mathbf{n}}$ ($\hat{b}^\dagger_{\mathbf{n}}$) satisfies the standard commutation relations

$$\left[\hat{b}_{\mathbf{n}}, \hat{b}^\dagger_{\mathbf{m}}\right] = \delta_{\mathbf{nm}}, \quad \left[\hat{b}_{\mathbf{n}}, \hat{b}_{\mathbf{m}}\right] = \left[\hat{b}^\dagger_{\mathbf{n}}, \hat{b}^\dagger_{\mathbf{m}}\right] = 0.$$

The initial vacuum state $|0;\text{in}\rangle$ is defined as the state annihilated by all $\hat{b}_{\mathbf{n}}$.

For $t > 0$, one of the z-mirrors starts moving and the quantum field is subjected to non-stationary boundary conditions. The changes in the vacuum mode structure then translate, therefore into the existence of new set of mode functions $f^{\text{out}}_{\mathbf{n}}(\mathbf{x}, t)$ such that the decomposition of the field reads

$$\hat{\phi}(\mathbf{x}, t) = \sum_{\mathbf{n}} \frac{1}{\sqrt{2\omega_{\mathbf{n}}}} \left[f^{\text{out}}_{\mathbf{n}}(\mathbf{x}, t)\hat{b}_{\mathbf{n}} + f^{\text{out}*}_{\mathbf{n}}(\mathbf{x}, t)\hat{b}^\dagger_{\mathbf{n}}\right], \tag{2.6}$$

with $f^{\text{out}}_{\mathbf{n}}$ standing for the instantaneous modes to be defined in the next section. Supposing that after some interval of time T, the moving mirror returns to the the static configuration in its initial position L_0. For $t > T$ we have a new set of physical ladder operators, $\hat{a}_{\mathbf{m}}$ and $\hat{a}^\dagger_{\mathbf{m}}$, with the following expansion for the field

$$\hat{\phi}^{\text{out}}(\mathbf{x}, t) = \sum_{\mathbf{n}} \frac{1}{\sqrt{2\omega_{\mathbf{n}}}} \left[f^{\text{in}}_{\mathbf{n}}(\mathbf{x}, t)\hat{a}_{\mathbf{n}} + f^{\text{in}*}_{\mathbf{n}}(\mathbf{x}, t)\hat{a}^\dagger_{\mathbf{n}}\right].$$

According with the Bogoliubov transformations both set of creation and annihilation operators (2.3) and (2.1) are related by

$$\hat{a}_{\mathbf{m}} = \sum_{\mathbf{n}} \left(\alpha_{\mathbf{nm}}\hat{b}_{\mathbf{n}} + \beta^*_{\mathbf{nm}}\hat{b}^\dagger_{\mathbf{n}}\right). \tag{2.7}$$

From this is immediate that the initial vacuum state (as seen after the motion of the mirror) now contains a non-vanishing number of particles

$$\langle \hat{N} \rangle = \sum_{\mathbf{m}} \langle 0;\text{in}|\hat{a}^\dagger_{\mathbf{m}}\hat{a}_{\mathbf{m}}|0;\text{in}\rangle = \sum_{\mathbf{n},\mathbf{m}} |\beta_{\mathbf{nm}}|^2, \tag{2.8}$$

which characterizes the DCE.

In the next section we describe and effective Hamiltonian approach for the DCE. This formalism, introduced by C. K. Law formalism [**law**] is suitable for the study of entropy production due to the perturbations imposed on the field.

2.2 Effective Hamiltonian

We start by reviewing the effective Hamiltonian formalism developed in Ref. [**law**]. This will be the main tool we employ in order to investigate the entropy production due to the dynamical Casimir effect.

Inside the cavity, the field can be described in terms of instantaneous mode functions.

$$f_{\mathbf{n}}^{\text{out}}(\mathbf{x},t) = \varphi_{\mathbf{n},L_z(t)}(\mathbf{x})e^{-i\omega_{\mathbf{n},L_z(t)}t},$$

which satisfies the differential equation

$$\nabla^2\varphi_{\mathbf{n},L_z(t)}(\mathbf{x}) + \omega^2_{\mathbf{n},L_z(t)}\varphi_{\mathbf{n},L_z(t)}(\mathbf{x}) = 0, \tag{2.9}$$

subjected to the boundary conditions $\varphi_{\mathbf{n},L_z(t)}(\mathbf{x})|_{\text{mirrors}} = 0$ and the normalization $\int_{\mathcal{V}(t)} d^3\mathbf{x}\, \varphi_{\mathbf{n},L_z(t)}(\mathbf{x})\varphi_{\mathbf{m},L_z(t)}(\mathbf{x}) = \delta_{\mathbf{nm}}$. The subscript notation means that the frequency $\omega_{\mathbf{n}}$ and the mode function φ_k depend implicitly on time through the mirror motion $L_z(t)$.

With these expressions it is straightforward to write down an effective Hamiltonian as [**law**]

$$\begin{aligned}\hat{H}_{\text{eff}} &= \frac{1}{2}\int_{\mathcal{V}(t)} d^3\mathbf{x}\left(\hat{\pi}\partial_t\hat{\phi} - \hat{\phi}\partial_t\hat{\pi}\right) = \sum_{\mathbf{n}} \omega_{\mathbf{n},L_z(t)}\hat{a}^\dagger_{\mathbf{n}}\hat{a}_{\mathbf{n}} &(2.10)\\ &+ \frac{i}{2}\sum_{\mathbf{m},\mathbf{n}}\left[\chi^{(+)}_{\mathbf{n},\mathbf{m}}(t)\left(\hat{a}^\dagger_{\mathbf{n}}\hat{a}^\dagger_{\mathbf{m}} - \hat{a}_{\mathbf{n}}a_{\mathbf{m}}\right) + \chi^{(-)}_{\mathbf{n},\mathbf{m}}(t)\left(\hat{a}^\dagger_{\mathbf{n}}a_{\mathbf{m}} - \hat{a}^\dagger_{\mathbf{m}}\hat{a}_{\mathbf{n}}\right)\right], &(2.11)\end{aligned}$$

where beyond the field mode structure with instantaneous frequencies $\omega_{\mathbf{n},L(t)}$, there are additional terms representing non-adiabatic processes such as the creation of pairs of particles (even from the vacuum state) as well as their scattering to different modes. The coefficients $\chi^{(+)}_{\mathbf{n},\mathbf{m}}(t)$ and $\chi^{(-)}_{\mathbf{n},\mathbf{m}}(t)$ which govern the time-scales of these processes are defined as

$$\chi^{(\pm)}_{\mathbf{n},\mathbf{m}}(t) = \frac{v^{(\pm)}_{\mathbf{n},\mathbf{m}}}{\omega_{\mathbf{n},L(t)} \pm \omega_{\mathbf{m},L(t)}}\frac{\dot{L}_z(t)}{L_z(t)}$$

where

$$v^{(\pm)}_{\mathbf{n},\mathbf{m}} = \frac{1}{2}\left[\frac{\omega^2_{\mathbf{m},L(t)} - \omega^2_{\mathbf{n},L(t)}}{\sqrt{\omega_{\mathbf{n},L(t)}\omega_{\mathbf{m},L(t)}}}g_{\mathbf{n},\mathbf{m}} - L_z(t)\frac{\omega_{\mathbf{n},L(t)} \pm \omega_{\mathbf{m},L(t)}}{\omega_{\mathbf{n},L_z(t)}}\frac{\partial\omega_{\mathbf{n},L_z(t)}}{\partial L_z}\delta_{\mathbf{n},\mathbf{m}}\right],$$

and

$$g_{\mathbf{n},\mathbf{m}} = -g_{\mathbf{m},\mathbf{n}} = -L_z(t)\int_0^{L_z(t)} dz\, \varphi_{\mathbf{n},L(t)}\frac{\partial\varphi_{\mathbf{m},L(t)}}{\partial L_z}. \tag{2.12}$$

The time evolution of the system is governed by the Schrodinger equation

$$i\partial_t|\Psi(t)\rangle = \hat{H}_{\text{eff}}|\Psi(t)\rangle. \tag{2.13}$$

As pointed out by Law [**law**], as the formalism is based on a set of instantaneous basis functions $\{\varphi_{\mathbf{n},L(t)}(\mathbf{x})\}$, the vacuum states changes accordingly with the change in the time parameter, so the bosons associated with the creation and annihilation operators may not be regarded as real particles while the cavity is still in motion. Only in the static situation such operators become unique, thus acquiring the usual physical meaning.

Moving now to the interaction picture, the effective Hamiltonian reads

$$\hat{H}_I = \frac{i}{2}\sum_{\mathbf{m},\mathbf{n}}\left[\chi^{(+)}_{\mathbf{n},\mathbf{m}}(t)\left(\hat{\mathcal{A}}^\dagger_{\mathbf{n}}\hat{\mathcal{A}}^\dagger_{\mathbf{m}} - \hat{\mathcal{A}}_{\mathbf{n}}\hat{\mathcal{A}}_{\mathbf{m}}\right) + \chi^{(-)}_{\mathbf{n},\mathbf{m}}(t)\left(\hat{\mathcal{A}}^\dagger_{\mathbf{n}}\hat{\mathcal{A}}_{\mathbf{m}} - \hat{\mathcal{A}}^\dagger_{\mathbf{m}}\hat{\mathcal{A}}_{\mathbf{n}}\right)\right], \tag{2.14}$$

where $\hat{\mathcal{A}}_{\mathbf{n}}(t) = \hat{a}_{\mathbf{n}}\exp\{-i\tilde{\omega}_{\mathbf{n}}(t)\}$ and $\tilde{\omega}_{\mathbf{n},L_z(t)}(t) = \int^t dt'\omega_{\mathbf{n},L_z(t')}$. The time evolution of the system's density operator $\hat{\rho}(t)$ can be determined with the help of the dynamical equation $\dot{\hat{\rho}}(t) = -i\left[\hat{H}_I, \hat{\rho}(t)\right]$. Considering $\dot{L}_z(t)$ as a perturbation parameter, a formal solution for $\hat{\rho}(t)$ in the second order of $\hat{H}_I(t)$ can be putted as

$$\begin{aligned}\hat{\rho}(t) = \hat{\rho}(0) - i\int_0^t dt'\left[\hat{H}_I(t'), \hat{\rho}(0)\right] \\ - \int_0^t dt'\int_0^{t'} dt''\left[\hat{H}_I(t'), \left[\hat{H}_I(t''), \hat{\rho}(0)\right]\right].\end{aligned}$$

We are interested in the case where the cavity is initially prepared in the vacuum state $\hat{\rho}(0) = |0\rangle\langle 0|$ (from here on we use the shorthand notation $|0\rangle \equiv |0;\text{in}\rangle$). Under such initial condition, the number of particles created inside the cavity due do the DCE can be written as

$$\langle\hat{N}(t)\rangle = \operatorname{Tr}\left\{\sum_{\mathbf{n}}\hat{\rho}(t)\hat{a}^\dagger_{\mathbf{n}}\hat{a}_{\mathbf{n}}\right\} = \sum_{\mathbf{n},\mathbf{m}}\mathcal{N}_{\mathbf{n},\mathbf{m}}, \tag{2.15}$$

with

$$\mathcal{N}_{\mathbf{n},\mathbf{m}}(t) := 2\,\mathrm{Re}\int_0^t dt'\int_0^{t'} dt''\,\chi^{(+)}_{\mathbf{n},\mathbf{m}}(t'')\chi^{(+)}_{\mathbf{n},\mathbf{m}}(t')e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}},$$

which can be identified, from equation (2.8), as the Bogoliubov coefficient $|\beta_{\mathbf{n},\mathbf{m}}|^2$.

Será que esta transição não está brusca? We are now ready to study irreversibility induced by the DCE. In order to quantity irreversibility, we consider the diagonal entropy [1]

$$S_d(\rho_{\text{diag}}) = -\sum_{\eta}\rho^{(\eta)}_{\text{diag}}\ln\rho^{(\eta)}_{\text{diag}}, \tag{2.16}$$

as the main figure of merit, where $\rho^{(\eta)}_{\text{diag}} = \langle\boldsymbol{\eta}|\rho|\boldsymbol{\eta}\rangle$ are the diagonal elements of the system's density operator in the time-independent energy eigenstates, defined as

$$|\boldsymbol{\eta}\rangle = |\eta_{\mathbf{n}_1}, \eta_{\mathbf{n}_2}, \ldots\rangle = \prod_{\eta_{\mathbf{n}_i}}\frac{1}{\sqrt{\eta_{\mathbf{n}_i}!}}\left(a^\dagger_{\mathbf{n}_i}\right)^{\eta_{\mathbf{n}_i}}|0\rangle,$$

for the number state populated with $\eta_{\mathbf{n}_i}$ photons in the $\mathbf{n}_i$-th mode..

From Eq. (2.16), the diagonal entropy can be directly computed, resulting in

$$S_d(t) = -\rho^{(0)}_{\text{diag}}(t)\ln\rho^{(0)}_{\text{diag}}(t) - \sum_{\mathbf{n},\mathbf{m}}\rho^{(\mathbf{n},\mathbf{m})}_{\text{diag}}(t)\ln\rho^{(\mathbf{n},\mathbf{m})}_{\text{diag}}(t)$$

with

$$\begin{aligned}
\rho^{(0)}_{\text{diag}}(t) &:= \langle 0|\hat{\rho}(t)|0\rangle = 1 - \frac{1}{2}\langle\hat{N}(t)\rangle \\
\rho^{(\mathbf{n},\mathbf{m})}_{\text{diag}}(t) &:= \begin{cases} \langle 2_{\mathbf{n}}|\hat{\rho}(t)|2_{\mathbf{n}}\rangle, & \text{if } \mathbf{n}=\mathbf{m} \\ \langle 1_{\mathbf{n}},1_{\mathbf{m}}|\hat{\rho}(t)|1_{\mathbf{n}},1_{\mathbf{m}}\rangle, & \text{if } \mathbf{n}\neq\mathbf{m} \end{cases} \\
&= \frac{1}{2}\mathcal{N}_{\mathbf{n},\mathbf{n}}(t).
\end{aligned}$$

In order to illustrate the behavior of the entropy production, we concentrate in the case when the field inside the cavity is weakly perturbed by the mirrors in a periodically manner. In this moving-mirror setup the field mode function can be defined with the analogous form of Eq. (2.4)

$$f^{\text{out}}_{\mathbf{n}}(\mathbf{x},t) = \sqrt{\frac{8}{L_xL_yL_z(t)}}\sin\left(\frac{n_x\pi}{L_x}x\right)\sin\left(\frac{n_y\pi}{L_y}y\right)\sin\left(\frac{n_z\pi}{L_z(t)}z\right)e^{-i\omega_{\mathbf{n},L(t)}t},$$

with an implicit time-dependence in the mode frequency $\omega_{\mathbf{n},L(t)}$, allowing us to directly obtain the parameter,

$$g_{\mathbf{n},\mathbf{m}} \quad = \quad \begin{cases} (-1)^{j_z+k_z}\frac{2n_zm_z}{m_z^2-n_z^2}\delta_{n_x,m_x}\delta_{n_y,m_y}, & \text{if } \mathbf{n}\neq\mathbf{m} \\ 0, & \text{if } \mathbf{n}=\mathbf{m} \end{cases} \tag{2.17}$$

A convenient class of mirror trajectory to be chosen is when the second mirror

$$L_z(t) = L_0\left[1+\epsilon\xi(t)\right],$$

where $\xi(t)$ is a periodic function and $\epsilon \ll 1$ is a small dimensionless number, needed to keep the field with well defined frequencies $\tilde{\omega}_{\mathbf{n},L(t)} = \omega_{\mathbf{n}}t$ as defined in (2.5).

In this regime, we directly obtain

$$\mathcal{N}_{\mathbf{n},\mathbf{m}}(t) = \epsilon^2 v^2_{\mathbf{n},\mathbf{m}}\,\text{Re}\int_0^T dt'\int_0^T dt''\,\xi(t'')\xi(t')e^{-i(\omega_{\mathbf{n}}+\omega_{\mathbf{m}})(t''-t'')},$$

We calculate the last expression for the in which the mirror performs harmonic oscillations

$$\xi(t) = \sin\Omega^{\mu}_{\mathbf{k}}t,$$

with an arbitrary frequency $\Omega_{\mathbf{k}}^{(\mu)} = \mu\omega_{\mathbf{k}}$, which is a real multiple μ of some unperturbed field frequency $\omega_{\mathbf{k}}$. By defining $\Omega \equiv \Omega_{\mathbf{k}}^{\mu}$ and $\omega \equiv \omega_{\mathbf{n}} + \omega_{\mathbf{m}}$

$$\mathcal{N}_{\mathbf{n},\mathbf{m}}(t) = \begin{cases} \frac{1}{2}\frac{\epsilon^2 v^2}{\omega^2}\left[\omega^2 t^2 + 2\omega t \sin(2\omega t) + \cos(2\omega t) - 1\right] & \text{for} \quad \Omega = \omega \\ \frac{\Omega^2}{2\omega^2}\frac{\epsilon^2 v^2}{\Omega^2-\omega^2}\left[1 + \frac{4\omega^2}{\Omega^2-\omega^2} + \frac{2\omega}{\Omega+\omega}\cos(\Omega+\omega)t - \frac{2\omega}{\Omega-\omega}\cos(\Omega-\omega)t - \cos 2\Omega t\right] & \text{for} \quad \Omega \neq \omega. \end{cases} \tag{2.18}$$

As the diagonal terms of ρ are proportional to the last expression, we can interpret them as quantifying the different probabilities of finding the system in one of its energy eigenstates for different cavity frequencies Ω_μ. This makes clear that only at resonance, when the cavity oscillates at some unperturbed cavity field frequency, is when the amplification of the field energy happens. Out of the resonance frequencies, the probability of finding the system in different energy excitation oscillates very weakly in magnitude.

To show the adequacy of the latter formalism, we compute the number of particles and the diagonal entropy produced when the cavity oscillates in resonance with some unperturbed field frequency $\Omega_p = p\omega_1$ for $p = 1, 2, \ldots,$. Under the rotating wave approximation and up to second order, Eq. (1.25) becomes

$$\langle \hat{N}(t) \rangle = \frac{1}{6}p(p^2-1)\tau^2, \tag{2.19}$$

which is in accordance with literature [3]. Note that the above expression is valid under perturbation theory involving time and, therefore, it is a good approximation only when $\tau \ll 1$.

The diagonal entropy, our focus of interest here, takes the form

$$S_d(t) = \frac{1}{2}\hat{N}(t)\left[1 - \ln\frac{1}{2}\hat{N}(t) + \ln\frac{p(p^2-1)}{6} - \frac{6f(p)}{p(p^2-1)}\right].$$

Chapter 3

Dodonov method of Gaussian States

3.1 Introduction

Direct experimental detection is difficulty because the mirror would need to move at velocities near of the light but real mirror would not support such there is a maximum velocity achievable

[4]

3.1.1 Instantaneous basis method (IBM)

Another approach to study the DCE in the one-dimensional case by expanding the modes defined in equation (2.6) in a series with respect to the instantaneous basis

$$f_n^{\text{out}}(x, t > 0) = \sum_{k=1}^{\infty} \varphi_{k,L(t)}(x) Q_k^{(n)}(t),$$

with

$$\varphi_{k,L(t)}(x) = \sqrt{\frac{2}{L(t)}} \sin\left(\frac{k\pi}{L(t)} x\right)$$

and the initial conditions

$$Q_k^{(n)}(0) = \delta_{kn}, \qquad \dot{Q}_k^{(n)}(t) = -i\omega_n \delta_{kn}, \qquad k, n = 1, 2, \ldots.$$

From the restrictions imposed by the wave equation on the instantaneous coefficient we obtain the set of coupled differential equations

$$\ddot{Q}_k^{(n)} + \omega_k^2(t) Q_k^{(n)} = 2 \sum_{j=1}^{\infty} G_{kj}(t) \dot{Q}_j^{(n)} + \sum_{j=1}^{\infty} \dot{G}_{kj} Q_k^{(n)} + \mathcal{O}(G_{kj}^2) \tag{3.1}$$

where $\omega_k(t) = k\pi/L(t)$ and the coefficients $G_{jk}(t)$ (for $j \neq k$)

$$G_{jk} = -G_{kj} = (-1)^{k-j} \frac{2kj}{(j^2 - k^2)} \frac{\dot{L}(t)}{L(t)}.$$

If the wall returns to its initial position $x = L_0$ after some interval of time T then

$$Q_k^{(n)}(t) = \alpha_{nk} e^{-\omega_k t} + \beta_{nk} e^{\omega_k t}, \qquad k, n = 1, 2, \dots . \tag{3.2}$$

A simplification for the problem occurs for the case in which the mirror perform small oscillations at the frequency of some unperturbed field eigenfrequency $\omega_p = p\omega_1$

$$L(t) = L_0 \left[1 + \epsilon \cos\left(p\omega_1 t\right)\right], \qquad (p = 1, 2, \dots).$$

Inserting expression (3.2) into (3.1), we obtain a set of differential equations from which is possible to extract the Bogoluibov coefficients α_{nm} and β_{nm} by assuming that they vary slowly in time.

Parametric Oscillations

Assuming the the Bogoluibov coefficients α_{nm} and β_{nm} having derivatives proportional to ϵ and second-order derivatives to ϵ^2 (slowly varying in time) we can obtain expressions putting expression (3.2) into (3.1). As a result we obtain a set of differential equations

$$\begin{aligned}
\frac{d}{d\tau}\alpha_{nm} &= (-1)^p \left[(m+p)\alpha_{n(m+p)} + (m-p)\alpha_{n(m-p)}\right] \\
\frac{d}{d\tau}\beta_{nm} &= (-1)^p \left[(m+p)\alpha_{n(m+p)} + (m-p)\beta_{n(m-p)}\right]
\end{aligned}$$

with the initial conditions $\alpha_{nm}(0) = \delta_{nm}$ and $\beta_{nm}(0) = 0$ where $\tau = \frac{1}{2}\epsilon\omega_1 t = 2\alpha t$. Due to the initial conditions the solutions must automatically respect the relation $\alpha_{(k+np)(j+mp)} \equiv 0$ if $j \neq k$, therefore, the non-zero coefficients for the principal modes $\mu = p(k + 1/2)$ for $k = 0, 1, 2, \dots$ and $p = 0, 2, 4 \dots$ must read

$$\begin{aligned}
\alpha_{(p/2+np)(p/2+mp)} &= \frac{\Gamma(n+3/2)\kappa^{n-m}}{\Gamma(m+3/2)\Gamma(1+n-m)} \\
&\quad \times \mathrm{F}(n+1/2, -m-1/2; 1+n-m; \kappa^2) \\
\beta_{(p/2+np)(p/2+mp)} &= \frac{(-1)^m \Gamma(m+1/2)\Gamma(n+3/2)\kappa^{n+m+1}}{\pi\Gamma(2+n+m)} \\
&\quad \times \mathrm{F}(n+1/2, m+1/2; 2+n+m; \kappa^2)
\end{aligned}$$

where F is the hypergeometrica l function. Using the elliptic integrals

$$\begin{aligned}
\mathbf{K}(\kappa) &= \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-\kappa^2\sin^2\alpha}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; \kappa^2\right) \\
\mathbf{E}(\kappa) &= \int_0^{\pi/2} d\alpha \sqrt{1-\kappa^2\sin^2\alpha} = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; \kappa^2\right)
\end{aligned}$$

we can write the first Bogoliubov coefficients as

$$\begin{aligned}
\alpha_{11} &= \frac{2}{\pi}\mathbf{E}(\kappa) \\
\beta_{11} &= \frac{2}{\pi\kappa}\left[\tilde{\kappa}^2\mathbf{K}(\kappa) - \mathbf{E}(\kappa)\right], \\
\alpha_{13} &= \frac{2\sqrt{3}}{9\pi\kappa}\left[(1-2\kappa^2)\mathbf{E}(\kappa) - \tilde{\kappa}^2\mathbf{K}(\kappa)\right] \\
\beta_{13} &= \frac{2\sqrt{3}}{9\kappa^2}\left[(2-\kappa^2)\mathbf{E}(\kappa) - 2\tilde{\kappa}^2\mathbf{K}(\kappa)\right],
\end{aligned}$$

where the principal mode index p appears implicitly in $\kappa = S(p\tau)/\sqrt{1-S^2(p\tau)}$ and $\tilde{\kappa} \equiv \sqrt{1-\kappa^2}$ with $S(\tau) = \sinh(\tau)$.

3.1.2 Diagonal entropy for the first mode $(k = 1)$ of a (1+1)-dimensional cavity

In order to compute the statistical properties of the particles created in the cavity we can attend to two key points. The first refers to the fact that, as seen the first part, the time evolution of moving cavity systems can be described in the Schrodinger picture with the help of a quadratic multidimensional time-dependent Hamiltonian. The second key point is the fact that the evolution governed by quadratic Hamiltonian's transforms any Gaussian state to another Gaussian state which by the way is completely determined by the average values of the field quadrature and by their variances

As the vacuum state belongs to this wider class of Gaussian states, it is in fact possible to describe our initial state in terms of the Wigner-Weyl formulation of quantum mechanics, with a simple distribution function for the m-th mode

$$W_m(\mathbf{q}) = \frac{1}{\sqrt{2\pi \det \mathbf{\Sigma}}} e^{-\frac{1}{2}(\mathbf{q}-\langle\mathbf{q}\rangle)\mathbf{\Sigma}^{-1}(\mathbf{q}-\langle\mathbf{q}\rangle)},$$

which depends only on the average values of the quadrature operators $\mathbf{q} = (\hat{q}_m, \hat{p}_m)$ and their variances $\Sigma_{ij} = \langle q_m^i q_m^j\rangle - \langle q_m^i\rangle\langle q_m^j\rangle$ with $i, j = 1, 2$. The quadrature variances defines a symmetric matrix

$$\mathbf{\Sigma} \equiv \begin{pmatrix} \sigma_m^q & \sigma_m^{qp} \\ \sigma_m^{qp} & \sigma_m^p \end{pmatrix}$$

with elements

$$\sigma_m^q = \langle \hat{q}_m^2\rangle - \langle\hat{q}_m\rangle^2, \qquad \sigma_m^p = \langle \hat{p}_m^2\rangle - \langle\hat{p}_m\rangle^2,$$
$$\sigma_m^{qp} = \frac{1}{2}\langle \hat{p}_m\hat{q}_m + \hat{q}_m\hat{p}_m\rangle - \langle\hat{q}_m\hat{p}_m\rangle.$$

Average values can be computed as

$$\langle f(\mathbf{q})\rangle = \mathrm{Tr}\,\hat{\rho} f(\mathbf{q}) = \int f(\mathbf{q}) W_m(\mathbf{q}) d^N\mathbf{q}$$

For the initial vacuum state where all average values $\langle \hat{q}_m \rangle = \langle \hat{p}_m \rangle = 0$, the diagonal density matrix components can be calculated in terms of the quadrature variance with the help of the formula [3]

$$\rho_{nn}^{(m)} = \frac{2\left[\left(2\sigma_m^q - 1\right)\left(2\sigma_m^p - 1\right)\right]^{n/2}}{\left[\left(2\sigma_m^q + 1\right)\left(2\sigma_m^p + 1\right)\right]^{(n+1)/2}} \times \mathrm{P}_n\left(\frac{4\sigma_m^q\sigma_m^p - 1}{\sqrt{(4(\sigma_m^q)^2 - 1)(4(\sigma_m^p)^2 - 1)}}\right),$$

where P_n is the Legendre polynomial of order n. For an initial vacuum state ($\langle \hat{q}_m \rangle = \langle \hat{p}_m \rangle = 0$), using expression (2.7) its possible to show that

$$\sigma_{2m+1}^q = \frac{1}{2}\sum_{n=1}^{\infty} \left|\alpha_{nm} + \beta_{nm}\right|^2$$
$$\sigma_{2m+1}^p = \frac{1}{2}\sum_{n=1}^{\infty} \left|\alpha_{nm} - \beta_{nm}\right|^2.$$

In the special case of parametric oscillation, from the recursive relations that arises from the Bogoliubov coefficient symmetries, it turn out to be possible to simplify the last expression to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_{m=2\mu+1}^q = -m\left[\alpha_{1m} + \beta_{1m}\right]^2$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_{m=2\mu+1}^p = +m\left[\alpha_{1m} - \beta_{1m}\right]^2.$$

with $\mu = 1, 2, \ldots$. From the above expression is possible to find their Taylor expansions

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_{2m+1}^q\,|_{\tau\to\infty} = 0; \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_{2m+1}^p\,|_{\tau\to\infty} = \frac{16}{\pi^2(2m+1)}$$

2

$$\left.\begin{array}{r}\sigma_{2m+1}^q\,|_{\tau\to 0}\\ \sigma_{2m+1}^p\,|_{\tau\to 0}\end{array}\right\} = \frac{1}{2} \mp \tau^{2m+1}\left[\frac{(2m-1)!!}{m!}\right]^2 \times \left[1 \mp \frac{2m+1}{(m+1)^2}\tau + \mathcal{O}(\tau^2)\right]$$

For the first mode of the field ($m = 1$), in the short time limit $\tau \ll 1$ ($\tau = 2\alpha t$), the expression for the variances are $\sigma_1^q = \frac{1}{2}\left(1 - 2\tau + 2\tau^2 + \mathcal{O}(\tau^3)\right)$ and $\sigma_1^p = \frac{1}{2}\left(1 + 2\tau + 2\tau^2 + \mathcal{O}(\tau^3)\right)$, so the density matrix takes the form of

$$\rho_{nn}^{(1)}(\tau \ll 1) = (-1)^{n/2}\tau^n \left(1 - \frac{2n+2}{2}\tau^2 + \mathcal{O}(\tau^3)\right) P_n(0)$$
$$= \frac{(2k)!}{(2^k k!)^2}\tau^{2k}\left(1 - \frac{k+1}{2}\tau^2\right)$$

where $P_{2k+1}(0) = 0$ and $P_{2k} = (-1)^k(2k)!/2^{2k}(k!)^2$ for $k = 0, 1, 2, \ldots$.

In this case the diagonal entropy for the first mode can be computed as

$$S_d^{(1)}(\tau \ll 1) = -\sum_n \rho_{nn}^{(1)} \ln \rho_{nn}^{(1)} =$$
$$-\rho_{00}^{(1)} \ln \rho_{00}^{(1)} - \rho_{22}^{(1)} \ln \rho_{22}^{(1)} + \mathcal{O}(\tau^3)$$
$$= 2\alpha^2 t^2 \left(1 - \ln 2\alpha^2 t^2\right) + \mathcal{O}(\tau^3)$$

For the long-time limit $\tau \gg 1$ the system variances take the asymptotic behaviour $\sigma_1^q \to 2/\pi^2$ and $\sigma_1^p \to 16\tau/\pi^2$so our density matrix takes the form of

$$\rho_{nn}^{(1)}(\tau \gg 1) = C_n \Sigma_1^{-1/2} + \mathcal{O}(1/\tau)$$

with $\Sigma_1 \equiv \sigma_1^q \sigma_1^p$ at $\tau \gg 1$ and with

$$C_n = \frac{1}{\sqrt{1+e}}\left(\frac{1-e}{\sqrt{1-e^2}}\right)^n \mathrm{P}_n\left(\frac{1}{\sqrt{1-e^2}}\right) \text{ with } e = \frac{1}{2\sigma_1^q}.$$

The correspondent diagonal entropy is now written as

$$S_d^{(1)}(\tau \gg 1) \approx \frac{1}{2}\ln \Sigma_1 \sum_{n=0}^{\infty} C_n \Sigma_1^{-1/2}$$
$$-\sum_{n=0}^{\infty} C_n \ln(C_n)\Sigma_1^{-1/2}$$
$$=\frac{1}{2}\,\mathrm{Tr}\{\hat{\rho}\}\ln \Sigma_1 + S(C_n)\Sigma_1^{-1/2},$$

where $S(C_n) \equiv -\sum_n C_n \ln(C_n)$. Let us analyse the convergence of each one of those terms. For $n \gg 1$ we can use the asymptotic formula

$$\mathrm{P}_n\left(\frac{1}{\sqrt{1-e^2}}\right) = \frac{1}{\sqrt{2\pi n e}}\frac{(1+e)^{\frac{n+1}{2}}}{(1-e)^{n/2}} + \mathcal{O}(1/n),$$

so by taking the term C_n has the following asymptotic limit

$$C_{n\gg 1} = \frac{1}{\sqrt{\pi n}} \qquad \text{for } n \gg 1.$$

This means that for $N \to \infty$, we have $\sum_n C_n \leq \sum_n C_{n\gg 1} \leq \frac{1}{\sqrt{\pi}}\int_0^N \frac{dn}{\sqrt{n}} = \frac{2}{\sqrt{\pi}}\sqrt{N}$, which means that

$$\mathrm{Tr}\{\hat{\rho}\} = \sum_n C_n \Sigma_1^{-1/2} \leq \frac{2}{\sqrt{\pi}} \frac{\lim_{N\to\infty}\sqrt{N}}{\lim_{\Sigma\to\infty}\sqrt{\Sigma}} \approx 1.128\ldots,$$

showing that the trace of ρ converges and it is bounded to be less than $2/\sqrt{\pi}$. On the other hand,

$$\begin{aligned}
S(C_n)\Sigma_1^{-1/2} &\leq \frac{1}{2}\frac{1}{\sqrt{\Sigma}}\int_0^N \frac{\ln \pi n}{\sqrt{\pi n}} dn \\
&= \frac{1}{2}\left(\frac{2}{\sqrt{\pi}}\frac{\sqrt{N}}{\sqrt{\Sigma}}\ln \pi N - \frac{4}{\sqrt{\pi}}\frac{\sqrt{N}}{\sqrt{\Sigma}}\right) \\
&\sim \frac{1}{2}\ln \pi N - 1
\end{aligned}$$

meaning that the last series diverges logarithmically such that the diagonal entropy is of the order of

$$\begin{aligned}
S_d^{(1)}(\tau \gg 1) &= \frac{1}{2}\ln \Sigma_1 - \sum_{n=0}^{\infty} C_n \ln(C_n)\Sigma_1^{-1/2} \\
&\sim \frac{1}{2}\ln\frac{32}{\pi^4} + \frac{1}{2}\ln\tau - 1 + \frac{1}{2}\ln \pi N + \mathcal{O}(\tau^{-1}\ln\tau)
\end{aligned}$$

The relation between the Shannon entropy of the C_n coefficients and the diagonal entropy is given by

$$S(C_n) = \sqrt{\frac{32}{\pi^4}\tau}\left(S_d^{(1)} - \frac{1}{2}\ln\frac{32}{\pi^2}\tau\right).$$

3.1.3 Diagonal entropy for the first mode $(k = 1)$ of a (3+1)-dimensional cavity

For a 3th-dimensional cavity the quadrature variances are given by $\sigma_m^q = \frac{1}{2}e^{-2\tau}$ and $\sigma_m^p = \frac{1}{2}e^{2\tau}$and the diagonal component of the density matrix take the form of

$$\rho_{nn}^{(m)}(\tau) = \frac{(2k)!}{(2^k k!)^2}\frac{\tanh^{2k}(\tau)}{\cosh\tau},$$

with $n = 2k$ and $k = 0, 1, 2, \ldots$ where was used $\tau = \gamma t$. For the short-limit $\tau \ll 1$, we have $\tanh^{2k}(\tau) = \tau^{2k} + \mathcal{O}(\tau^3)$ and $1/\cosh(\tau) = 1 - \frac{1}{2}\tau^2 + \mathcal{O}(\tau^3)$ so

$$\rho_{nn}^{(m)}(\tau \ll 1) = \frac{(2k)!}{(2^k k!)^2}\tau^{2k}\left(1 - \frac{1}{2}\tau^2\right) + \mathcal{O}(\tau^3)$$

and the diagonal entropy becomes

$$
\begin{aligned}
S_d(\tau \ll 1) &= -\sum_{k=0}^{\infty} \rho_{2k}^{(m)} \ln \rho_{2k}^{(m)} = -\rho_0^{(m)} \ln \rho_0^{(m)} - \rho_1^{(m)} \ln \rho_1^{(m)} = \\
&= -\left(1 - \frac{1}{2}\tau^2\right) \ln \left(1 - \frac{1}{2}\tau^2\right) - \frac{\tau^2}{2} \ln \frac{\tau^2}{2} + \mathcal{O}(\tau^3) \\
&= \frac{\tau^2}{2}\left(1 - \ln \frac{\tau^2}{2}\right) + \mathcal{O}(\tau^4)
\end{aligned}
$$

For the long-time limit $\tau \gg 1$, we have $\sigma_m^q \to 0$, $\tanh(\tau) \to 1$ and $\cosh(\tau) \to \frac{1}{2}e^{\tau} = \sqrt{\sigma_m^p/2}$

$$
\rho_{nn}^{(m)}(\tau \gg 1) = C_n' \Sigma_1'^{-1/2} \quad \text{with} \quad C_{2k}' = \frac{(2k)!}{\sqrt{2}(2^k k!)^2},
$$

For $n \gg 1$ we have $k! \to \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$ so $C_{2k}' \to \frac{1}{\sqrt{2\pi k}} = \frac{1}{\sqrt{\pi n}}$ which again, it is compatible with the inequality $\mathrm{Tr}\{\hat{\rho}\} \leq 2/\sqrt{\pi} \approx 1.128$ when we rearrange expressions such that $\Sigma_1' = \sigma_m^p/4$.

Using the last expression we find that the diagonal entropy must be given by

$$
\begin{aligned}
S_d^{(m)}(\tau \gg 1) &= -\mathrm{Tr}\{\hat{\rho}\} \ln \Sigma_1'^{-1/2} + S(C_n')\Sigma_1'^{-1/2} \\
&= \tau - 2\ln 2 + \frac{e^{-\tau}}{\sqrt{8}} S(C_n'),
\end{aligned}
$$

which increases linearly with time.

$$
S_{\text{diag}}^{(1)} - S_{\text{em}}^{(1)} = \sqrt{\frac{\pi^4 \tau}{32}} S(C_n) - 1 \tag{3.3}
$$

where $S(C_n)$ is the Shannon entropy for the coefficients $C_n = \frac{1}{\sqrt{1+e}}\left(\frac{1-e}{\sqrt{1-e^2}}\right)^n \mathrm{P}_n\left(\frac{1}{\sqrt{1-e^2}}\right)$ with $e = \frac{\pi^4}{4}$

Appendix A

Derivation of the effective Hamiltonian

By definition, we can write the field $\hat{\phi}(\mathbf{x},t)$, its conjugated momenta $\hat{\pi}(\mathbf{x},t)$ and their time derivatives (with $\hat{a}_{\mathbf{n}}(t) = \hat{a}(0)e^{-i\omega_{\mathbf{n}}(t)t}$) as

$$\hat{\phi}(\mathbf{x},t) = \sum_{\mathbf{n}} \frac{1}{\sqrt{2\omega_{\mathbf{n}}(t)}} \left[\hat{a}_{\mathbf{n}}(t) + \hat{a}^{\dagger}_{\mathbf{n}}(t)\right] \varphi_{\mathbf{n}}(\mathbf{x},t) \tag{A.1}$$

$$\hat{\pi}(\mathbf{x},t) = i\sum_{\mathbf{n}} \sqrt{\frac{\omega_{\mathbf{n}}(t)}{2}} \left[\hat{a}^{\dagger}_{\mathbf{n}}(t) - \hat{a}_{\mathbf{n}}(t)\right] \varphi_{\mathbf{n}}(\mathbf{x},t) \tag{A.2}$$

$$\partial_t\hat{\phi}(\mathbf{x},t) = \sum_{\mathbf{n}} \frac{1}{\sqrt{2\omega_{\mathbf{n}}(t)}} \left[\hat{a}_{\mathbf{n}}(t) + \hat{a}^{\dagger}_{\mathbf{n}}(t)\right] \left[\partial_t\varphi_{\mathbf{n}}(\mathbf{x},t) - \frac{1}{2\omega_{\mathbf{n}}}\frac{d\omega_{\mathbf{n}}}{dt}\varphi_{\mathbf{n}}(\mathbf{x},t)\right] \tag{A.3}$$

$$+\, i\sum_{\mathbf{n}} \sqrt{\frac{\omega_{\mathbf{n}}(t)}{2}} \left[\hat{a}^{\dagger}_{\mathbf{n}}(t) - \hat{a}_{\mathbf{n}}(t)\right] \varphi_{\mathbf{n}}(\mathbf{x},t) \tag{A.4}$$

$$\partial_t\hat{\pi}(\mathbf{x},t) = i\sum_{\mathbf{n}} \sqrt{\frac{\omega_{\mathbf{n}}(t)}{2}} \left[\hat{a}^{\dagger}_{\mathbf{n}}(t) - \hat{a}_{\mathbf{n}}(t)\right] \left[\partial_t\varphi_{\mathbf{n}}(t) + \frac{1}{2\omega_{\mathbf{n}}}\frac{d\omega_{\mathbf{n}}}{dt}\varphi_{\mathbf{n}}(\mathbf{x},t)\right] \tag{A.5}$$

$$-\sum_{\mathbf{n}} \sqrt{\frac{\omega_{\mathbf{n}}(t)}{2}}\omega_{\mathbf{n}} \left[\hat{a}_{\mathbf{n}}(t) + \hat{a}^{\dagger}_{\mathbf{n}}(t)\right] \varphi_{\mathbf{n}}(\mathbf{x},t). \tag{A.6}$$

In order to calculate the Hamiltonian, we begin by computing the first part (ignoring the notation for the explicit time dependence of each term)

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathcal{V}(t)} d^3\mathbf{x}\;\hat{\pi}\;\partial_t\hat{\phi} = \\
&= \frac{1}{4}\sum_{\mathbf{n},\mathbf{m}}\Bigg\{ i\sqrt{\frac{\omega_\mathbf{n}}{\omega_\mathbf{m}}}\left(\hat{a}^\dagger_\mathbf{n}-\hat{a}_\mathbf{n}\right)\left(\hat{a}_\mathbf{m}+\hat{a}^\dagger_\mathbf{m}\right)\left(\int_{\mathcal{V}(t)} d^3\mathbf{x}\;\varphi_\mathbf{n}\partial_t\varphi_\mathbf{m} - \frac{1}{2\omega_\mathbf{m}}\frac{d\omega_\mathbf{m}}{dt}\int_{\mathcal{V}(t)} d^3\mathbf{x}\;\varphi_\mathbf{n}\varphi_\mathbf{m}\right) \\
&- \sqrt{\omega_\mathbf{n}\omega_\mathbf{m}}\left(\hat{a}^\dagger_\mathbf{n}-\hat{a}_\mathbf{n}\right)\left(\hat{a}^\dagger_\mathbf{m}-\hat{a}_\mathbf{m}\right)\int_{\mathcal{V}(t)} d^3\mathbf{x}\varphi_\mathbf{n}\varphi_\mathbf{m}\Bigg\} \\
&= \frac{1}{4}\sum_{\mathbf{n},\mathbf{m}}\Bigg\{ i\sqrt{\frac{\omega_\mathbf{n}}{\omega_\mathbf{m}}}\left(\hat{a}^\dagger_\mathbf{n}-\hat{a}_\mathbf{n}\right)\left(\hat{a}_\mathbf{m}+\hat{a}^\dagger_\mathbf{m}\right)\left(-G_{\mathbf{n},\mathbf{m}} - \frac{1}{2\omega_\mathbf{m}}\frac{d\omega_\mathbf{m}}{dt}\delta_{\mathbf{n},\mathbf{m}}\right) \\
&- \sqrt{\omega_\mathbf{n}\omega_\mathbf{m}}\left(\hat{a}^\dagger_\mathbf{n}-\hat{a}_\mathbf{n}\right)\left(\hat{a}^\dagger_\mathbf{m}-\hat{a}_\mathbf{m}\right)\delta_{\mathbf{n},\mathbf{m}}\Bigg\} \\
&= \frac{1}{2}\sum_{\mathbf{n},\mathbf{m}}\Bigg\{ -\frac{i}{2}\Lambda_{\mathbf{n},\mathbf{m}}\left(\hat{a}^\dagger_\mathbf{n}-\hat{a}_\mathbf{n}\right)\left(\hat{a}_\mathbf{m}+\hat{a}^\dagger_\mathbf{m}\right) - \sqrt{\omega_\mathbf{n}\omega_\mathbf{m}}\left(\hat{a}^\dagger_\mathbf{n}-\hat{a}_\mathbf{n}\right)\left(\hat{a}^\dagger_\mathbf{m}-\hat{a}_\mathbf{m}\right)\delta_{\mathbf{n},\mathbf{m}}\Bigg\} \\
&= -\frac{1}{2}\sum_{\mathbf{n}}\Bigg\{\frac{\omega_\mathbf{n}}{2}\left(\hat{a}^\dagger_\mathbf{n}-\hat{a}_\mathbf{n}\right)\left(\hat{a}^\dagger_\mathbf{n}-\hat{a}_\mathbf{n}\right) + \sum_{\mathbf{m}}\frac{i}{2}\Lambda_{\mathbf{n},\mathbf{m}}\left(\hat{a}^\dagger_\mathbf{n}-\hat{a}_\mathbf{n}\right)\left(\hat{a}_\mathbf{m}+\hat{a}^\dagger_\mathbf{m}\right)\Bigg\}
\end{aligned}
$$

,

where $\Lambda_{\mathbf{n},\mathbf{m}} = \sqrt{\frac{\omega_\mathbf{n}}{\omega_\mathbf{m}}}G_{\mathbf{n},\mathbf{m}} + \frac{1}{2\omega_\mathbf{m}}\frac{d\omega_\mathbf{m}}{dt}\delta_{\mathbf{n},\mathbf{m}}$. The other part is

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathcal{V}(t)} d^3\mathbf{x}\;\hat{\phi}\;\partial_t\hat{\pi} = \\
&= \frac{1}{4}\sum_{\mathbf{n},\mathbf{m}}\Bigg\{ i\sqrt{\frac{\omega_\mathbf{m}}{\omega_\mathbf{n}}}\left(\hat{a}^\dagger_\mathbf{n}+\hat{a}_\mathbf{n}\right)\left(\hat{a}^\dagger_\mathbf{m}-\hat{a}_\mathbf{m}\right)\left(\int_{\mathcal{V}(t)} d^3\mathbf{x}\;\varphi_\mathbf{n}\partial_t\varphi_\mathbf{m} + \frac{1}{2\omega_\mathbf{m}}\frac{d\omega_\mathbf{m}}{dt}\int_{\mathcal{V}(t)} d^3\mathbf{x}\;\varphi_\mathbf{n}\varphi_\mathbf{m}\right) \\
&- \sqrt{\frac{\omega_\mathbf{m}}{\omega_\mathbf{n}}}\omega_\mathbf{m}\left(\hat{a}^\dagger_\mathbf{n}+\hat{a}_\mathbf{n}\right)\left(\hat{a}^\dagger_\mathbf{m}+\hat{a}_\mathbf{m}\right)\int_{\mathcal{V}(t)} d^3\mathbf{x}\;\varphi_\mathbf{n}\varphi_\mathbf{m}\Bigg\} \\
&= \frac{1}{4}\sum_{\mathbf{n},\mathbf{m}}\Bigg\{ i\sqrt{\frac{\omega_\mathbf{m}}{\omega_\mathbf{n}}}\left(\hat{a}^\dagger_\mathbf{n}+\hat{a}_\mathbf{n}\right)\left(\hat{a}^\dagger_\mathbf{m}-\hat{a}_\mathbf{m}\right)\left(G_{\mathbf{m},\mathbf{n}} + \frac{1}{2\omega_\mathbf{m}}\frac{d\omega_\mathbf{m}}{dt}\delta_{\mathbf{n},\mathbf{m}}\right) \\
&- \sqrt{\frac{\omega_\mathbf{m}}{\omega_\mathbf{n}}}\omega_\mathbf{m}\left(\hat{a}^\dagger_\mathbf{n}+\hat{a}_\mathbf{n}\right)\left(\hat{a}^\dagger_\mathbf{m}+\hat{a}_\mathbf{m}\right)\delta_{\mathbf{n},\mathbf{m}}\Bigg\} \\
&= \frac{1}{4}\sum_{\mathbf{n},\mathbf{m}}\Bigg\{\frac{i}{2}\Lambda_{\mathbf{m},\mathbf{n}}\left(\hat{a}^\dagger_\mathbf{n}+\hat{a}_\mathbf{n}\right)\left(\hat{a}^\dagger_\mathbf{m}-\hat{a}_\mathbf{m}\right) - \sqrt{\frac{\omega_\mathbf{m}}{\omega_\mathbf{n}}}\omega_\mathbf{m}\left(\hat{a}^\dagger_\mathbf{n}+\hat{a}_\mathbf{n}\right)\left(\hat{a}^\dagger_\mathbf{m}+\hat{a}_\mathbf{m}\right)\delta_{\mathbf{n},\mathbf{m}}\Bigg\} \\
&= -\frac{1}{2}\sum_{\mathbf{n}}\Bigg\{\frac{\omega_\mathbf{n}}{2}\left(\hat{a}^\dagger_\mathbf{n}+\hat{a}_\mathbf{n}\right)\left(\hat{a}^\dagger_\mathbf{n}+\hat{a}_\mathbf{n}\right) - \sum_{\mathbf{m}}\frac{i}{2}\Lambda_{\mathbf{m},\mathbf{n}}\left(\hat{a}^\dagger_\mathbf{n}+\hat{a}_\mathbf{n}\right)\left(\hat{a}^\dagger_\mathbf{m}-\hat{a}_\mathbf{m}\right)\Bigg\}
\end{aligned}
$$

Remembering the identities $[a_{\mathbf{n}}, a^\dagger_{\mathbf{n}}] = 1$ and $[a_{\mathbf{n}}, a_{\mathbf{n}}] = [a^\dagger_{\mathbf{n}}, a^\dagger_{\mathbf{n}}] = 0$, as well as, $G_{\mathbf{m},\mathbf{n}} = -G_{\mathbf{n},\mathbf{m}}$, we can easily show that,

$$
\begin{aligned}
\hat{H} &= \frac{1}{2}\int_{\mathcal{V}(t)} d^3\mathbf{x}\left(\hat{\pi}\partial_t\hat{\phi} - \hat{\phi}\partial_t\hat{\pi}\right) \\
&= \frac{1}{2}\sum_{\mathbf{n}}\left\{\frac{\omega_{\mathbf{n}}}{2}\left(\hat{a}^\dagger_{\mathbf{n}} + \hat{a}_{\mathbf{n}}\right)\left(\hat{a}^\dagger_{\mathbf{n}} + \hat{a}_{\mathbf{n}}\right) - \sum_{\mathbf{m}}\frac{i}{2}\Lambda_{\mathbf{m},\mathbf{n}}\left(\hat{a}^\dagger_{\mathbf{n}} + \hat{a}_{\mathbf{n}}\right)\left(\hat{a}^\dagger_{\mathbf{m}} - \hat{a}_{\mathbf{m}}\right)\right\} \\
&- \frac{1}{2}\sum_{\mathbf{n}}\left\{\frac{\omega_{\mathbf{n}}}{2}\left(\hat{a}^\dagger_{\mathbf{n}} - \hat{a}_{\mathbf{n}}\right)\left(\hat{a}^\dagger_{\mathbf{n}} - \hat{a}_{\mathbf{n}}\right) + \sum_{\mathbf{m}}\frac{i}{2}\Lambda_{\mathbf{n},\mathbf{m}}\left(\hat{a}^\dagger_{\mathbf{n}} - \hat{a}_{\mathbf{n}}\right)\left(\hat{a}_{\mathbf{m}} + \hat{a}^\dagger_{\mathbf{m}}\right)\right\} \\
&= \frac{1}{2}\sum_{\mathbf{n}}\left\{\frac{\omega_{\mathbf{n}}}{2}\left[\left(a^{\dagger 2}_{\mathbf{n}} + a^\dagger_{\mathbf{n}}a_{\mathbf{n}} + a_{\mathbf{n}}a^\dagger_{\mathbf{n}} + a_{\mathbf{n}}a_{\mathbf{n}}\right) - \left(a^{\dagger 2}_{\mathbf{n}} - a^\dagger_{\mathbf{n}}a_{\mathbf{n}} - a_{\mathbf{n}}a^\dagger_{\mathbf{n}} + a_{\mathbf{n}}a_{\mathbf{n}}\right)\right]\right. \\
&\left.- \sum_{\mathbf{m}}\frac{i}{2}\left[\Lambda_{\mathbf{m},\mathbf{n}}\left(a^\dagger_{\mathbf{n}}a^\dagger_{\mathbf{m}} - a^\dagger_{\mathbf{n}}a_{\mathbf{m}} + a_{\mathbf{n}}a^\dagger_{\mathbf{m}} - a_{\mathbf{n}}a_{\mathbf{m}}\right) + \Lambda_{\mathbf{n},\mathbf{m}}\left(a^\dagger_{\mathbf{n}}a^\dagger_{\mathbf{m}} + a^\dagger_{\mathbf{n}}a_{\mathbf{m}} - a_{\mathbf{n}}a^\dagger_{\mathbf{m}} - a_{\mathbf{n}}a_{\mathbf{m}}\right)\right]\right\} \\
&= \frac{1}{2}\sum_{\mathbf{n}}\left\{\omega_{\mathbf{n}}\left(a^\dagger_{\mathbf{n}}a_{\mathbf{n}} + a_{\mathbf{n}}a^\dagger_{\mathbf{n}}\right)\right. \\
&\left.- \sum_{\mathbf{m}}\frac{i}{4}\left[\left(\Lambda_{\mathbf{n},\mathbf{m}} + \Lambda_{\mathbf{m},\mathbf{n}}\right)a^\dagger_{\mathbf{n}}a^\dagger_{\mathbf{m}} + \left(\Lambda_{\mathbf{n},\mathbf{m}} - \Lambda_{\mathbf{m},\mathbf{n}}\right)a^\dagger_{\mathbf{n}}a_{\mathbf{m}} - \left(\Lambda_{\mathbf{n},\mathbf{m}} - \Lambda_{\mathbf{m},\mathbf{n}}\right)a_{\mathbf{n}}a^\dagger_{\mathbf{m}} - \left(\Lambda_{\mathbf{n},\mathbf{m}} + \Lambda_{\mathbf{m},\mathbf{n}}\right)a_{\mathbf{n}}a_{\mathbf{m}}\right)\right\} \\
&= \frac{1}{2}\sum_{\mathbf{n}}\left\{\omega_{\mathbf{n}}\left(a^\dagger_{\mathbf{n}}a_{\mathbf{n}} + a_{\mathbf{n}}a^\dagger_{\mathbf{n}}\right)\right. \\
&\left.- \sum_{\mathbf{m}}\frac{i}{4}\left[\left(\Lambda_{\mathbf{n},\mathbf{m}} + \Lambda_{\mathbf{m},\mathbf{n}}\right)\left(a^\dagger_{\mathbf{n}}a^\dagger_{\mathbf{m}} - a_{\mathbf{n}}a_{\mathbf{m}}\right) + \left(\Lambda_{\mathbf{n},\mathbf{m}} - \Lambda_{\mathbf{m},\mathbf{n}}\right)\left(a^\dagger_{\mathbf{n}}a_{\mathbf{m}} - a_{\mathbf{n}}a^\dagger_{\mathbf{m}}\right)\right]\right\}
\end{aligned}
$$

defining $\chi^{(+)}_{\mathbf{m},\mathbf{n}} = \frac{1}{2}\left(\Lambda_{\mathbf{n},\mathbf{m}} + \Lambda_{\mathbf{m},\mathbf{n}}\right)$ and $\chi^{(-)}_{\mathbf{m},\mathbf{n}} = \frac{1}{2}\left(\Lambda_{\mathbf{n},\mathbf{m}} - \Lambda_{\mathbf{m},\mathbf{n}}\right)$, the effective Hamiltonian can be written as

$$
\hat{H}_{eff} = \sum_{\mathbf{n}}\omega_{\mathbf{n}}\left(a^\dagger_{\mathbf{n}}a_{\mathbf{n}} + \frac{1}{2}\right) - \frac{i}{2}\sum_{\mathbf{m},\mathbf{n}}\left[\chi^{(+)}_{\mathbf{n},\mathbf{m}}(t)\left(a^\dagger_{\mathbf{n}}a^\dagger_{\mathbf{m}} - a_{\mathbf{n}}a_{\mathbf{m}}\right) + \chi^{(-)}_{\mathbf{n},\mathbf{m}}(t)\left(a^\dagger_{\mathbf{n}}a_{\mathbf{m}} - a_{\mathbf{n}}a^\dagger_{\mathbf{m}}\right)\right]
$$

Considering $\Lambda_{\mathbf{n},\mathbf{m}} = \sqrt{\frac{\omega_{\mathbf{n}}}{\omega_{\mathbf{m}}}}G_{\mathbf{n},\mathbf{m}} + \frac{1}{2\omega_{\mathbf{m}}}\frac{d\omega_{\mathbf{m}}}{dt}\delta_{\mathbf{n},\mathbf{m}}$, we can write the coefficients as

$$
\begin{aligned}
\chi^{(+)}_{\mathbf{n},\mathbf{m}}(t) &= \frac{1}{2}\left(\sqrt{\frac{\omega_{\mathbf{n}}}{\omega_{\mathbf{m}}}} - \sqrt{\frac{\omega_{\mathbf{m}}}{\omega_{\mathbf{n}}}}\right)G_{\mathbf{n},\mathbf{m}} + \frac{1}{2\omega_{\mathbf{m}}}\frac{d\omega_{\mathbf{m}}}{dt}\delta_{\mathbf{n},\mathbf{m}} \\
\chi^{(-)}_{\mathbf{n},\mathbf{m}}(t) &= \frac{1}{2}\left(\sqrt{\frac{\omega_{\mathbf{n}}}{\omega_{\mathbf{m}}}} + \sqrt{\frac{\omega_{\mathbf{m}}}{\omega_{\mathbf{n}}}}\right)G_{\mathbf{n},\mathbf{m}}.
\end{aligned}
$$

We can simplify the last coefficients by writing as

$$\chi^{(\pm)}_{\mathbf{n},\mathbf{m}}(t) \quad = \quad \frac{v^{(\pm)}_{\mathbf{n},\mathbf{m}}}{\omega_{\mathbf{n},L(t)} \pm \omega_{\mathbf{m},L(t)}} \frac{\dot{L}_z(t)}{L_z(t)}$$

where

$$v^{(\pm)}_{\mathbf{n},\mathbf{m}} = \frac{1}{2}\left[\frac{\omega^2_{\mathbf{m},L(t)} - \omega^2_{\mathbf{n},L(t)}}{\sqrt{\omega_{\mathbf{n},L(t)}\omega_{\mathbf{m},L(t)}}} g_{\mathbf{n},\mathbf{m}} - L_z(t) \frac{\omega_{\mathbf{n},L(t)} \pm \omega_{\mathbf{m},L(t)}}{\omega_{\mathbf{n},L_z(t)}} \frac{\partial \omega_{\mathbf{n},L_z(t)}}{\partial L_z} \delta_{\mathbf{n},\mathbf{m}}\right],$$

and

$$g_{\mathbf{n},\mathbf{m}} = -g_{\mathbf{m},\mathbf{n}} = -L_z(t)\int_0^{L_z(t)} dz\, \varphi_{\mathbf{n},L(t)} \frac{\partial \varphi_{\mathbf{m},L(t)}}{\partial L_z}. \tag{A.7}$$

A.1 Bogoliubov coefficients

Let the equations of motion

$$\begin{aligned}
\partial_t \hat{\phi}(\mathbf{x}, t) &= \hat{\pi}(\mathbf{x}, t) \\
\partial_t \hat{\pi}(\mathbf{x}, t) &= -\omega^2_{\mathbf{n}}(t)\hat{\phi}(\mathbf{x}, t),
\end{aligned}$$

with

$$\hat{\phi}(\mathbf{x}, t) = \sum_{\mathbf{n}} \frac{1}{\sqrt{2\omega_{\mathbf{n}}(t)}} \left[\hat{a}_{\mathbf{n}}(t) + \hat{a}^\dagger_{\mathbf{n}}(t)\right] \varphi_{\mathbf{n}}(\mathbf{x}, t) \tag{A.8}$$

$$\hat{\pi}(\mathbf{x}, t) = i\sum_{\mathbf{n}} \sqrt{\frac{\omega_{\mathbf{n}}(t)}{2}} \left[\hat{a}^\dagger_{\mathbf{n}}(t) - \hat{a}_{\mathbf{n}}(t)\right] \varphi_{\mathbf{n}}(\mathbf{x}, t) \tag{A.9}$$

$$\partial_t \hat{\phi}(\mathbf{x}, t) = \sum_{\mathbf{n}} \frac{1}{\sqrt{2\omega_{\mathbf{n}}(t)}} \left[\hat{a}_{\mathbf{n}}(t) + \hat{a}^\dagger_{\mathbf{n}}(t)\right] \left[\partial_t \varphi_{\mathbf{n}}(\mathbf{x}, t) - \frac{1}{2\omega_{\mathbf{n}}} \frac{d\omega_{\mathbf{n}}}{dt} \varphi_{\mathbf{n}}(\mathbf{x}, t)\right] \tag{A.10}$$

$$+ \sum_{\mathbf{n}} \frac{1}{\sqrt{2\omega_{\mathbf{n}}(t)}} \left[\partial_t \hat{a}_{\mathbf{n}}(t) + \partial_t \hat{a}^\dagger_{\mathbf{n}}(t)\right] \varphi_{\mathbf{n}}(\mathbf{x}, t) \tag{A.11}$$

$$\partial_t \hat{\pi}(\mathbf{x}, t) = i\sum_{\mathbf{n}} \sqrt{\frac{\omega_{\mathbf{n}}(t)}{2}} \left[\hat{a}^\dagger_{\mathbf{n}}(t) - \hat{a}_{\mathbf{n}}(t)\right] \left[\partial_t \varphi_{\mathbf{n}}(t) + \frac{1}{2\omega_{\mathbf{n}}} \frac{d\omega_{\mathbf{n}}}{dt} \varphi_{\mathbf{n}}(\mathbf{x}, t)\right] \tag{A.12}$$

$$+ i\sum_{\mathbf{n}} \sqrt{\frac{\omega_{\mathbf{n}}(t)}{2}} \left[\partial_t \hat{a}^\dagger_{\mathbf{n}}(t) - \partial_t \hat{a}_{\mathbf{n}}(t)\right] \varphi_{\mathbf{n}}(\mathbf{x}, t). \tag{A.13}$$

therefore

$$\int d^3\mathbf{x}\,\varphi_{\mathbf{m}}\partial_t\hat{\phi} = \sum_{\mathbf{n}} \frac{1}{\sqrt{2\omega_{\mathbf{n}}}}\left[\hat{a}_{\mathbf{n}} + \hat{a}^\dagger_{\mathbf{n}}\right]\left[-G_{\mathbf{m},\mathbf{n}} - \frac{1}{2\omega_{\mathbf{n}}}\frac{d\omega_{\mathbf{n}}}{dt}\delta_{\mathbf{n},\mathbf{m}}\right] + \sum_{\mathbf{n}} \frac{1}{\sqrt{2\omega_{\mathbf{n}}}}\left[\partial_t\hat{a}_{\mathbf{n}} + \partial_t\hat{a}^\dagger_{\mathbf{n}}\right]\delta_{\mathbf{n},\mathbf{m}}$$

$$\int d^3\mathbf{x}\,\varphi_{\mathbf{m}}\hat{\pi} = i\sum_{\mathbf{n}}\sqrt{\frac{\omega_{\mathbf{n}}}{2}}\left[\hat{a}^\dagger_{\mathbf{n}} - \hat{a}_{\mathbf{n}}\right]\delta_{\mathbf{n},\mathbf{m}},$$

meaning

$$\partial_t\hat{a}^\dagger_{\mathbf{m}} + \partial_t\hat{a}_{\mathbf{m}} = i\omega_{\mathbf{n}}\left[\hat{a}^\dagger_{\mathbf{n}} - \hat{a}_{\mathbf{n}}\right] + \sum_{\mathbf{n}}\mu_{\mathbf{m},\mathbf{n}}\left[\hat{a}_{\mathbf{n}} + \hat{a}^\dagger_{\mathbf{n}}\right]$$

with $\mu_{\mathbf{m},\mathbf{n}} = \sqrt{\frac{\omega_{\mathbf{m}}}{\omega_{\mathbf{n}}}}\left[G_{\mathbf{m},\mathbf{n}} + \frac{1}{2\omega_{\mathbf{n}}}\frac{d\omega_{\mathbf{n}}}{dt}\delta_{\mathbf{n},\mathbf{m}}\right]$.

$$\int d^3\mathbf{x}\,\varphi_{\mathbf{m}}\partial_t\hat{\pi} = i\sum_{\mathbf{n}}\sqrt{\frac{\omega_{\mathbf{n}}}{2}}\left[\hat{a}^\dagger_{\mathbf{n}} - \hat{a}_{\mathbf{n}}\right]\left[G_{\mathbf{n},\mathbf{m}} + \frac{1}{2\omega_{\mathbf{n}}}\frac{d\omega_{\mathbf{n}}}{dt}\delta_{\mathbf{n},\mathbf{m}}\right] + i\sum_{\mathbf{n}}\sqrt{\frac{\omega_{\mathbf{n}}}{2}}\left[\partial_t\hat{a}^\dagger_{\mathbf{n}} - \partial_t\hat{a}_{\mathbf{n}}\right]\delta_{\mathbf{n},\mathbf{m}}$$

$$-\omega^2_{\mathbf{n}}\int d^3\mathbf{x}\,\varphi_{\mathbf{m}}\hat{\phi} = -\sum_{\mathbf{n}}\omega_{\mathbf{n}}\sqrt{\frac{\omega_{\mathbf{n}}}{2}}\left[\hat{a}^\dagger_{\mathbf{n}} + \hat{a}_{\mathbf{n}}\right]\delta_{\mathbf{n},\mathbf{m}},$$

meaning

$$\partial_t\hat{a}^\dagger_{\mathbf{m}} - \partial_t\hat{a}_{\mathbf{m}} = -i\omega_{\mathbf{n}}\left[\hat{a}^\dagger_{\mathbf{n}} + \hat{a}_{\mathbf{n}}\right] - \sum_{\mathbf{n}}\mu_{\mathbf{n},\mathbf{m}}\left[\hat{a}^\dagger_{\mathbf{n}} - \hat{a}_{\mathbf{n}}\right],$$

therefore

$$\partial_t\hat{a}_{\mathbf{m}} = -i\omega_{\mathbf{n}}\hat{a}^\dagger_{\mathbf{n}} + \sum_{\mathbf{n}}\frac{1}{2}\left(\mu_{\mathbf{m},\mathbf{n}} + \mu_{\mathbf{n},\mathbf{m}}\right)\hat{a}^\dagger_{\mathbf{n}} + \sum_{\mathbf{n}}\frac{1}{2}\left(\mu_{\mathbf{m},\mathbf{n}} - \mu_{\mathbf{n},\mathbf{m}}\right)\hat{a}_{\mathbf{n}},$$

$$\partial_t\hat{a}^\dagger_{\mathbf{m}} = i\omega_{\mathbf{n}}\hat{a}_{\mathbf{n}} + \sum_{\mathbf{n}}\frac{1}{2}\left(\mu_{\mathbf{m},\mathbf{n}} + \mu_{\mathbf{n},\mathbf{m}}\right)\hat{a}_{\mathbf{n}} + \sum_{\mathbf{n}}\frac{1}{2}\left(\mu_{\mathbf{m},\mathbf{n}} - \mu_{\mathbf{n},\mathbf{m}}\right)\hat{a}^\dagger_{\mathbf{n}},$$

Appendix B

Interaction picture

Let's consider a Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$ where H_0 has a known dynamics and $\hat{V}$ is a time dependent perturbation as in the form

$$\begin{aligned} \hat{H}_0 &= \sum_k \omega_k(t) a_k^\dagger a_k \\ \hat{V}(t) &= i \sum_k \left[\xi_k(t) \left(a_k^{\dagger 2} - a_k^2 \right) + \sum_{j(\neq k)} \mu_{kj} \left(a_k^\dagger a_j^\dagger + a_k^\dagger a_j - a_j a_k - a_j^\dagger a_k \right) \right]. \end{aligned} \tag{B.1}$$

With $\xi_k(t) = \frac{1}{2} G_{k,k}(t) + \frac{1}{4\omega_k(t)} \frac{d\omega_k(t)}{dt}$ and $\mu_{kj}(t) = \frac{1}{2} \left[\frac{\omega_k(t)}{\omega_j(t)} \right]^{1/2} G_{k,j}(t)$. We can then define an interaction picture with ket $|\psi\rangle_I$ and an arbitrary time dependent observable $\hat{A}(t)$ such that

$$\begin{aligned} |\psi(t)\rangle_I &= \hat{U}_0^\dagger |\psi(t)\rangle_S \\ \hat{A}_I(t) &= \hat{U}_0^\dagger(t) \hat{A}_S \hat{U}_0(t) \end{aligned} \tag{B.2}$$

where $\hat{A}$ is an arbitrary observable and $\hat{U}_0$ is an unitary operator for the time evolution of the Hamiltonian $\hat{H}_0$ that has the form

$$\hat{U}_0(t) = \exp\left\{ -i \int_0^t d\tau \hat{H}_0(\tau) \right\}. \tag{B.3}$$

In this context the correct the Schrodinger equation is given by

$$i\frac{d|\psi(t)\rangle_S}{dt} = i\frac{d}{dt}\left(\hat{U}_0 |\psi(t)\rangle_I \right) = i\left(\frac{d\hat{U}_0}{dt} |\psi(t)\rangle_I + \hat{U}_0 \frac{d|\psi(t)\rangle_I}{dt} \right) = \hat{H}|\psi(t)\rangle_S = \hat{H}\left(\hat{U}_0 |\psi(t)\rangle_I \right). \tag{B.4}$$

isolating the term $\hat{U}_0 \frac{d}{dt} |\psi(t)\rangle_I$ in the last equation (B.4) and multiplying both sides from the left by the expression $U_0^\dagger$ we have

$$\begin{aligned} i\hat{U}_0^\dagger \hat{U}_0 \frac{d|\psi(t)\rangle_I}{dt} &= \left(\hat{U}_0^\dagger \hat{H} \hat{U}_0 - U_0^\dagger \frac{dU_0}{dt} \right) |\psi(t)\rangle_I \\ i\frac{d|\psi(t)\rangle_I}{dt} &= \left[\hat{U}_0^\dagger \left(\hat{H}_0 + \hat{V} \right) \hat{U}_0 - iU_0^\dagger \frac{dU_0}{dt} \right] |\psi(t)\rangle_I \end{aligned} \tag{B.5}$$

using the fact that $\left[\hat{U}_0, \hat{H}_0\right] = 0$ we have that $\hat{U}_0^\dagger \hat{H}_0 \hat{U}_0 = \hat{H}_0$ and $U_0^\dagger \frac{dU_0}{dt} = U_0^\dagger \left(-i\hat{H}_0 U_0\right) = -i\hat{H}_0$ what implies in

$$i\frac{d|\psi(t)\rangle_I}{dt} = \left[\hat{H}_0 + \hat{U}_0^\dagger \hat{V} \hat{U}_0 - \hat{H}_0\right] |\psi(t)\rangle_I = \hat{V}_I |\psi(t)\rangle_I, \tag{B.6}$$

which is the correct Schrodinger equation for the interaction picture. This is also described by the unitary evolution operator

$$\hat{U}(t) = \mathcal{T}\left\{\exp\{i\int d\tau \hat{V}_I(\tau)\}\right\}. \tag{B.7}$$

Theorem 1. *The effective Hamiltonian from Theorem* **??** *can be written in the interaction picture as*

$$H_I = i\sum_k \left\{e^{i\tilde{\omega}_k(t)}\left[\xi_k(t) a_k^{\dagger 2} e^{i\tilde{\omega}_k(t)} + \sum_{j(\neq k)} \mu_{kj}(t) a_k^\dagger \left(a_j^\dagger e^{i\tilde{\omega}_j(t)} + a_j e^{-i\tilde{\omega}_j(t)}\right)\right] - h.c.\right\}, \tag{B.8}$$

Proof. In the interaction picture the effective hamiltonian (**??**) becomes

$$H_I(t) = \hat{U}_0^\dagger \hat{V} \hat{U}_0 = \exp\left\{i\sum_k \tilde{\omega}_k(t) a_k^\dagger a_k\right\} \hat{V} \exp\left\{-i\sum_k \tilde{\omega}_k(t) a_k^\dagger a_k\right\} \tag{B.9}$$

where $\tilde{\omega}_k(t) = \int_0^t d\tau \omega_k(\tau)$. Using the relation

$$e^{\alpha\hat{A}} \hat{B} e^{-\alpha\hat{A}} = \hat{B} + \alpha\left[\hat{A}, \hat{B}\right] + \frac{\alpha^2}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \dots \tag{B.10}$$

with the operators

$$\begin{aligned}
\hat{A} &= \sum_{k'} \tilde{\omega}_{k'}(t) a_{k'}^\dagger a_{k'} \\
\hat{B}_1 &= i\sum_k \xi_k(t) a_k^{\dagger 2} \\
\hat{B}_2 &= i\sum_k \sum_{j(\neq k)} \mu_{jk}(t) a_k^\dagger a_j^\dagger \\
\hat{B}_3 &= i\sum_k \sum_{j(\neq k)} \mu_{jk}(t) a_k^\dagger a_j
\end{aligned} \tag{B.11}$$

where $\hat{V} = \hat{B}_1 + \hat{B}_2 + \hat{B}_3 + \text{h.c.}$ and the commutation relations

$$\begin{aligned}
\left[\hat{A}, \hat{B}_1\right] &= i\sum_k \sum_{k'} \tilde{\omega}_{k'}(t)\xi_k(t)\left[a_{k'}^\dagger a_{k'}, a_k^{\dagger 2}\right] = i\sum_k 2\tilde{\omega}_k(t)\xi_k(t) a_k^{\dagger 2} \\
\left[\hat{A}, \hat{B}_2\right] &= i\sum_k \sum_{k'} \sum_{j(\neq k)} \tilde{\omega}_{k'}(t)\mu_{jk}(t)\left[a_{k'}^\dagger a_{k'}, a_k^\dagger a_j^\dagger\right] = i\sum_k \sum_{j(\neq k)} \mu_{jk}(t)\left(\tilde{\omega}_k(t) a_k^\dagger a_j^\dagger + a_k^\dagger a_j^\dagger \tilde{\omega}_j(t)\right) \\
\left[\hat{A}, \hat{B}_3\right] &= i\sum_k \sum_{k'} \sum_{j(\neq k)} \tilde{\omega}_{k'}(t)\mu_{jk}(t)\left[a_{k'}^\dagger a_{k'}, a_k^\dagger a_j\right] = i\sum_k \sum_{j(\neq k)} \mu_{jk}(t)\left(\tilde{\omega}_k(t) a_k^\dagger a_j - a_k^\dagger a_j \tilde{\omega}_j(t)\right)
\end{aligned} \tag{B.12}$$

$$\begin{aligned}
\left[\hat{A},\left[\hat{A},\hat{B}_1\right]\right] &= i\sum_{k,k'} 2\tilde{\omega}_{k'}(t)\tilde{\omega}_k(t)\xi_k\left[a^\dagger_{k'}a_{k'},a^{\dagger 2}_k\right] = i\sum_k \left(2\tilde{\omega}_k(t)\right)^2\xi_k(t)a^{\dagger 2}_k \\
\left[\hat{A},\left[\hat{A},\hat{B}_2\right]\right] &= i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)\tilde{\omega}_{k'}(t)\left\{\tilde{\omega}_k(t)\left[a^\dagger_{k'}a_{k'},a^\dagger_k a^\dagger_j\right]+\left[a^\dagger_{k'}a_{k'},a^\dagger_k a^\dagger_j\right]\tilde{\omega}_j(t)\right\} \\
&= i\sum_k\sum_{j(\neq k)}\mu_{jk}(t)\left(\tilde{\omega}^2_k(t)a^\dagger_k a^\dagger_j+\tilde{\omega}_k(t)a^\dagger_k a^\dagger_j\tilde{\omega}_j(t)+\tilde{\omega}_k(t)a^\dagger_k a^\dagger_j\tilde{\omega}_j(t)+\tilde{\omega}^2_j a^\dagger_k a^\dagger_j\right) \\
\left[\hat{A},\left[\hat{A},\hat{B}_3\right]\right] &= i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)\tilde{\omega}_{k'}(t)\left\{\tilde{\omega}_k(t)\left[a^\dagger_{k'}a_{k'},a^\dagger_k a_j\right]-\left[a^\dagger_{k'}a_{k'},a^\dagger_k a_j\right]\tilde{\omega}_j(t)\right\} \\
&= i\sum_k\sum_{j(\neq k)}\mu_{jk}(t)\left(\tilde{\omega}^2_k(t)a^\dagger_k a_j-\tilde{\omega}_k(t)a^\dagger_k a_j\tilde{\omega}_j(t)-\tilde{\omega}_k(t)a^\dagger_k a_j\tilde{\omega}_j(t)+\tilde{\omega}^2_j a^\dagger_k a_j\right)
\end{aligned} \tag{B.13}$$

Using the relation above we find that

$$\begin{aligned}
e^{i\hat{A}}\hat{B}_1e^{-i\hat{A}} &= \hat{B}_1+i\left[\hat{A},\hat{B}_1\right]+\frac{i^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}_1\right]\right]+\dots \\
&= i\sum_k\xi_k(t)a^{\dagger 2}_k+i\sum_k 2\tilde{\omega}_k(t)\xi_k(t)a^{\dagger 2}_k+i\sum_k\left(2\tilde{\omega}_k(t)\right)^2\xi_k(t)a^{\dagger 2}_k \\
&= i\sum_k\xi_k(t)a^{\dagger 2}_k\left\{1+i2\tilde{\omega}_k(t)+\frac{i^2}{2!}\left(2\tilde{\omega}_k(t)\right)^2+\dots\right\} = i\sum_k\xi_k(t)a^{\dagger 2}_k e^{2i\tilde{\omega}_k(t)} \\
e^{i\hat{A}}\hat{B}_2e^{-i\hat{A}} &= \hat{B}_2+i\left[\hat{A},\hat{B}_2\right]+\frac{i^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}_2\right]\right]+\dots = i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)\left\{a^\dagger_k a^\dagger_j+i\left(\tilde{\omega}_k(t)a^\dagger_k a^\dagger_j+a^\dagger_k a^\dagger_j\tilde{\omega}_j(t)\right)\right. \\
&\left.+\frac{i^2}{2!}\left(\tilde{\omega}^2_k(t)a^\dagger_k a^\dagger_j+\tilde{\omega}_k(t)a^\dagger_k a^\dagger_j\tilde{\omega}_j(t)+\tilde{\omega}_k(t)a^\dagger_k a^\dagger_j\tilde{\omega}_j(t)+\tilde{\omega}^2_j a^\dagger_k a^\dagger_j\right)\right\} \\
&= i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)\left(1+i\tilde{\omega}_k(t)+\frac{i^2}{2!}\tilde{\omega}^2_k(t)+\dots\right)a^\dagger_k a^\dagger_j\left(1+i\tilde{\omega}_j(t)+\frac{i^2}{2!}\tilde{\omega}^2_j(t)+\dots\right) \\
&= i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)e^{i\tilde{\omega}_k(t)}a^\dagger_k a^\dagger_j e^{i\tilde{\omega}_j(t)} \\
e^{i\hat{A}}\hat{B}_3e^{-i\hat{A}} &= \hat{B}_3+i\left[\hat{A},\hat{B}_3\right]+\frac{i^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}_3\right]\right]+\dots = i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)\left\{a^\dagger_k a_j+i\left(\tilde{\omega}_k(t)a^\dagger_k a_j-a^\dagger_k a_j\tilde{\omega}_j(t)\right)\right. \\
&\left.+\frac{i^2}{2!}\left(\tilde{\omega}^2_k(t)a^\dagger_k a_j-\tilde{\omega}_k(t)a^\dagger_k a_j\tilde{\omega}_j(t)-\tilde{\omega}_k(t)a^\dagger_k a_j\tilde{\omega}_j(t)+\tilde{\omega}^2_j a^\dagger_k a_j\right)\right\} \\
&= i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)\left(1+i\tilde{\omega}_k(t)+\frac{i^2}{2!}\tilde{\omega}^2_k(t)+\dots\right)a^\dagger_k a_j\left(1-i\tilde{\omega}_j(t)+\frac{i^2}{2!}\tilde{\omega}^2_j(t)+\dots\right) \\
&= i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)e^{i\tilde{\omega}_k(t)}a^\dagger_k a_j e^{-i\tilde{\omega}_j(t)}
\end{aligned} \tag{B.14}$$

Allowing us to obtain our effective Hamiltonian in the interaction picture as

$$
\begin{aligned}
\hat{H}_I &= \hat{U}_0^\dagger \hat{V} \hat{U}_0 = e^{i\hat{A}} \left(\hat{B}_1 + \hat{B}_2 + \hat{B}_3 + \text{h.c.} \right) e^{-i\hat{A}} \\
&= i \sum_k \left\{ \xi_k(t) e^{i\tilde{\omega}_k(t)} a_k^{\dagger 2} e^{i\tilde{\omega}_k(t)} + \sum_{j(\neq k)} \mu_{kj}(t) \left(e^{i\tilde{\omega}_k(t)} a_k^\dagger a_j^\dagger e^{i\tilde{\omega}_j(t)} + e^{i\tilde{\omega}_k(t)} a_k^\dagger a_j e^{-i\tilde{\omega}_j(t)} \right) \right\} + \text{h.c.} \\
&= i \sum_k \left\{ e^{i\tilde{\omega}_k(t)} \left[\xi_k(t) a_k^{\dagger 2} e^{i\tilde{\omega}_k(t)} + \sum_{j(\neq k)} \mu_{kj}(t) a_k^\dagger \left(a_j^\dagger e^{i\tilde{\omega}_j(t)} + a_j e^{-i\tilde{\omega}_j(t)} \right) \right] - \text{h.c.} \right\}
\end{aligned}
\tag{B.15}
$$

proving the last theorem. □

Appendix C

Density operator

C.1 Density operator

In this section we calculate the expression for the diagonal terms of the density operator after the following expansion

$$\hat{\rho}(t) = \hat{\rho}(0) - i\int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right] - \int_0^t dt'' \int_0^{t''} dt' \left[\hat{H}_I(t''), \left[\hat{H}_I(t'), \hat{\rho}(0)\right]\right] + \mathcal{O}(\hat{H}_I^3(t)).$$

Considering the vacuum state $\hat{\rho}(0) = |0\rangle\langle 0|$ as the initial state and the correspondent relations $\hat{\mathcal{A}}_{\mathbf{n}}\hat{\rho}(0) = \hat{\rho}(0)\mathcal{A}_{\mathbf{n}}^{\dagger} = 0$, we can obtain

$$\begin{aligned}\left[\hat{H}_I(t'), \hat{\rho}(0)\right] &= \hat{H}_I(t')|0\rangle\langle 0| - |0\rangle\langle 0|\hat{H}_I(t') \\ &= \frac{i}{2}\sum_{\mathbf{n}',\mathbf{m}'} \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \left\{\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) + \hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right\},\end{aligned} \tag{C.1}$$

for the first order term. For the second order terms

$$\begin{aligned}&\left[H(t''), \left[\hat{H}_I(t'), \rho(0)\right]\right] = H(t'')\left[\hat{H}_I(t'), \rho(0)\right] - \left[\hat{H}_I(t'), \rho(0)\right]H(t'') \\ &= \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\left[\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} - \hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\right) + \chi^{(-)}_{\mathbf{n}'',\mathbf{m}''}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''}\right)\right]\left[\hat{H}_I(t'), \hat{\rho}(0)\right] \\ &- \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\left[\hat{H}_I(t'), \hat{\rho}(0)\right]\left[\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} - \hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\right) + \chi^{(-)}_{\mathbf{n}'',\mathbf{m}''}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\left[\hat{H}_I(t'),\rho(0)\right] - \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}\left[\hat{H}_I(t'),\rho(0)\right]\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \\
&- \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\chi^{(-)}_{\mathbf{n}'',\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\left[\hat{H}_I(t'),\rho(0)\right] + \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\chi^{(-)}_{\mathbf{n}'',\mathbf{m}''}\left[\hat{H}_I(t'),\rho(0)\right]\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} \\
&+ \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\chi^{(-)}_{\mathbf{n}'',\mathbf{m}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\left[\hat{H}_I(t'),\rho(0)\right] - \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\chi^{(-)}_{\mathbf{n}'',\mathbf{m}''}\left[\hat{H}_I(t'),\rho(0)\right]\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} \\
&- \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''}\left[\hat{H}_I(t'),\rho(0)\right] + \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}\left[\hat{H}_I(t'),\rho(0)\right]\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''}.
\end{aligned}
$$

Substituting the first order commutator (C.1) in the last expression, we obtain

$$
\begin{aligned}
&\left[H(t''),\left[\hat{H}_I(t'),\rho(0)\right]\right] = \\
&-\frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}\sum_{\mathbf{n}'',\mathbf{m}''}\chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}\left\{\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}''}\hat{\rho}(0) + \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right\} \\
&+\frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}\sum_{\mathbf{n}'',\mathbf{m}''}\chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}\left\{\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} + \hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\right\} \\
&+\frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}\sum_{\mathbf{n}'',\mathbf{m}''}\chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}\left\{\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) + \hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right\} \\
&-\frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}\sum_{\mathbf{n}'',\mathbf{m}''}\chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}\left\{\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} + \hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\right\} \\
&-\frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}\sum_{\mathbf{n}'',\mathbf{m}''}\chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(-)}_{\mathbf{n}'',\mathbf{m}''}\left\{\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) + \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right\} \\
&+\frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}\sum_{\mathbf{n}'',\mathbf{m}''}\chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(-)}_{\mathbf{n}'',\mathbf{m}''}\left\{\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} + \hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\right\} \\
&+\frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}\sum_{\mathbf{n}'',\mathbf{m}''}\chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(-)}_{\mathbf{n}'',\mathbf{m}''}\left\{\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) + \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right\} \\
&-\frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}\sum_{\mathbf{n}'',\mathbf{m}''}\chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(-)}_{\mathbf{n}'',\mathbf{m}''}\left\{\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''} + \hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''}\right\}.
\end{aligned}
$$

From those last expressions, the only candidates for the diagonal terms of $\hat{\rho}(t)$ can be identified as

$$\hat{\rho}(t)|_{\text{diag}} = \hat{\rho}(0) - \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}\sum_{\mathbf{n}'',\mathbf{m}''}\int_0^t dt''\int_0^{t''} dt'\chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}$$
$$\times\left\{\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'} - \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} + \hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\right\}.$$

The above terms are only diagonal if

$$\hat{\rho}(t)|_{\text{diag}} = \hat{\rho}(0) - \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\int_0^t dt''\int_0^{t''} dt'\chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}$$
$$\times\left\{\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'} - \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'} + \hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\right\},$$

where the factor $2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}$ comes from the fact that we have twice more diagonal terms when $\mathbf{n}' \neq \mathbf{m}'$.

C.2 Number of particles

In this section we use the last expression for the density operator to compute the number of particles created inside the cavity. But before we shall compute the expression for the quantity $\hat{\rho}(t)\hat{N}$ using only the diagonal contribution of $\hat{\rho}$ (since we will take the trace of the expression anyway). Consider

$$\hat{\rho}(t)|_{\text{diag}}\hat{N} = \sum_{\mathbf{k}}\hat{\rho}(t)|_{\text{diag}}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}$$
$$= \frac{1}{4}\sum_{\mathbf{k},\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\int_0^t dt''\int_0^{t''} dt'\chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'}\left\{\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\right\}$$
$$= \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 4^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\int_0^t dt''\int_0^{t''} dt'\chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'}\left\{\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}_{\mathbf{n}'} + \text{h.c.}\right\}.$$

Again, there will be twice more terms when $\mathbf{n}' \neq \mathbf{m}'$.

Term $2_{\mathbf{n}}$

The first diagonal coefficient of $\hat{\rho}_{diag}(t)\hat{N}$ can be computed as

$$
\begin{aligned}
&\mathcal{N}_{\mathbf{n},\mathbf{n}}(t) = \langle 2_{\mathbf{n}} | \hat{\rho}(t)|_{\text{diag}} \hat{N} | 2_{\mathbf{n}} \rangle \\
&\frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} 4^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} \left\{ \langle 2_{\mathbf{n}} | \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{a}^{\dagger}_{\mathbf{n}'} \hat{a}_{\mathbf{n}'} | 2_{\mathbf{n}} \rangle + \text{h.c.} \right\}.
\end{aligned}
$$

As the coefficient $\langle 2_{\mathbf{n}} | \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{a}^{\dagger}_{\mathbf{n}'} \hat{a}_{\mathbf{n}'} | 2_{\mathbf{n}} \rangle$ are zero for $\mathbf{m}', \mathbf{n}' \neq \mathbf{n}$, the only surviving terms are

$$
\begin{aligned}
\mathcal{N}_{\mathbf{n},\mathbf{n}}(t) &= \frac{1}{4} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{n}} \chi''^{(+)}_{\mathbf{n},\mathbf{n}} \left\{ \langle 2_{\mathbf{n}} | \hat{\mathcal{A}}''^{\dagger 2}_{\mathbf{n}} \hat{\rho}(0) \hat{\mathcal{A}}'^{2}_{\mathbf{n}} \hat{a}^{\dagger}_{\mathbf{n}} \hat{a}_{\mathbf{n}} | 2_{\mathbf{n}} \rangle + \text{h.c.} \right\} \\
&= \frac{1}{4} \langle 2_{\mathbf{n}} | \hat{a}''^{\dagger 2}_{\mathbf{n}} \hat{\rho}(0) \hat{a}'^{2}_{\mathbf{n}} \hat{a}^{\dagger}_{\mathbf{n}} \hat{a}_{\mathbf{n}} | 2_{\mathbf{n}} \rangle \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{n}} \chi''^{(+)}_{\mathbf{n},\mathbf{n}} \left\{ e^{-2i[\tilde{\omega}_{\mathbf{n}}(t'') - \tilde{\omega}_{\mathbf{n}}(t')]} + \text{h.c.} \right\} \\
&= 2\,\text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{n}} \chi''^{(+)}_{\mathbf{n},\mathbf{n}} e^{-2i[\tilde{\omega}_{\mathbf{n}}(t'') - \tilde{\omega}_{\mathbf{n}}(t')]}.
\end{aligned}
$$

Term $1_{\mathbf{n}}, 1_{\mathbf{m}}$

The second type of diagonal coefficients of $\hat{\rho}_{diag}(t)\hat{N}$ can be computed as

$$
\begin{aligned}
&\mathcal{N}_{\mathbf{n},\mathbf{m}}(t)|_{\mathbf{n}\neq\mathbf{m}} = \langle 1_{\mathbf{n}}, 1_{\mathbf{m}} | \rho_{\text{diag}}(t) N | 1_{\mathbf{n}}, 1_{\mathbf{m}} \rangle \\
&= \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} 4^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} \left\{ \langle 1_{\mathbf{n}}, 1_{\mathbf{m}} | \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{a}^{\dagger}_{\mathbf{n}'} \hat{a}_{\mathbf{n}'} | 1_{\mathbf{n}}, 1_{\mathbf{m}} \rangle + \text{h.c.} \right\} \\
&= \langle 1_{\mathbf{n}}, 1_{\mathbf{m}} | \hat{a}''^{\dagger}_{\mathbf{n}} \hat{a}^{\dagger}_{\mathbf{m}} \hat{\rho}(0) \hat{a}_{\mathbf{m}} \hat{a}'_{\mathbf{n}} \hat{a}^{\dagger}_{\mathbf{n}} \hat{a}_{\mathbf{n}} | 1_{\mathbf{n}}, 1_{\mathbf{m}} \rangle \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} \\
&\times \left\{ e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'') + \tilde{\omega}_{\mathbf{m}}(t'')] - [\tilde{\omega}_{\mathbf{n}}(t') + \tilde{\omega}_{\mathbf{m}}(t')]\}} + \text{h.c.} \right\} \\
&= 2\,\text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{m}} \chi''^{(+)}_{\mathbf{n},\mathbf{m}} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'') + \tilde{\omega}_{\mathbf{m}}(t'')] - [\tilde{\omega}_{\mathbf{n}}(t') + \tilde{\omega}_{\mathbf{m}}(t')]\}}.
\end{aligned}
$$

Therefore, the total number of particles can be computed as

$$
\begin{aligned}
\langle \hat{N}(t) \rangle &= \text{Tr} \left\{ \hat{\rho}(t)|_{\text{diag}} \hat{N} \right\} = \sum_{\mathbf{n}} \mathcal{N}_{\mathbf{n},\mathbf{n}} + \sum_{\mathbf{n}} \sum_{\mathbf{m}(\neq\mathbf{n})} \mathcal{N}_{\mathbf{n},\mathbf{m}} \\
&= \sum_{\mathbf{n},\mathbf{m}} \mathcal{N}_{\mathbf{n},\mathbf{m}} = 2\,\text{Re} \sum_{\mathbf{n},\mathbf{m}} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{m}} \chi''^{(+)}_{\mathbf{n},\mathbf{m}} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'') + \tilde{\omega}_{\mathbf{m}}(t'')] - [\tilde{\omega}_{\mathbf{n}}(t') + \tilde{\omega}_{\mathbf{m}}(t')]\}}
\end{aligned}
$$

C.3 Diagonal Entropy

Let the expression for the diagonal terms of the density operator

$$\hat{\rho}(t)|_{\text{diag}} = \hat{\rho}(0) - \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}$$
$$\times \left\{ \hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'} + \text{h.c.} \right\},$$

We can calculate the diagonal coefficients as

Term 0

$$\begin{aligned}
\rho^{(0)}(t)|_{\text{diag}} &= \langle 0|\,\hat{\rho}(t)|_{\text{diag}}\,|0\rangle \\
&= 1 - \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \left\{ \langle 0|\,\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\,|0\rangle + \text{h.c.} \right\} \\
&= 1 - \frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'} \langle 0|\,\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\,|0\rangle\, 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\, \text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} \\
&= 1 - \sum_{\mathbf{n}',\mathbf{m}'} \text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} \\
&= 1 - \frac{1}{2}\langle \hat{N}(t)\rangle.
\end{aligned}$$

Since $\langle 0|\,\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\,|0\rangle = 2^{\delta_{\mathbf{n}',\mathbf{m}'}}$.

Term $2_{\mathbf{n}}$

$$\begin{aligned}
\rho^{(\mathbf{n},\mathbf{n})}(t)|_{\text{diag}} &= \langle 2_{\mathbf{n}}|\,\hat{\rho}_{\text{diag}}\,|2_{\mathbf{n}}\rangle = \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \\
&\quad \times \left\{ \langle 2_{\mathbf{n}}|\,\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\,|2_{\mathbf{n}}\rangle + \text{h.c.} \right\} \\
&= \frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'} \langle 2_{\mathbf{n}}|\,\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\,|2_{\mathbf{n}}\rangle\, 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\, \text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} \\
&= \text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{n}}\chi''^{(+)}_{\mathbf{n},\mathbf{n}} e^{-2i[\tilde{\omega}_{\mathbf{n}}(t'')-\tilde{\omega}_{\mathbf{n}}(t')]} \\
&= \frac{1}{2}\mathcal{N}_{\mathbf{n},\mathbf{n}}.
\end{aligned}$$

since $\langle 2_{\mathbf{n}}|\,\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\,|2_{\mathbf{n}}\rangle = 2\delta_{\mathbf{n}',\mathbf{n}}\delta_{\mathbf{m}',\mathbf{n}}$.

Term $1_{\mathbf{n}}, 1_{\mathbf{m}}$

For the diagonal terms associated with the state $|1_{\mathbf{n}}, 1_{\mathbf{m}}\rangle$ we have

$$
\begin{aligned}
\rho^{(\mathbf{n},\mathbf{m})}(t)|_{\text{diag}} = \langle 1_{\mathbf{n}}, 1_{\mathbf{m}}| \hat{\rho}_{\text{diag}} |1_{\mathbf{n}}, 1_{\mathbf{m}}\rangle = \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \\
\times \left\{ \langle 1_{\mathbf{n}}, 1_{\mathbf{m}}| \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} |1_{\mathbf{n}}, 1_{\mathbf{m}}\rangle + \text{h.c.} \right\} \\
= \frac{1}{2} \sum_{\mathbf{n}',\mathbf{m}'} \langle 1_{\mathbf{n}}, 1_{\mathbf{m}}| \hat{a}^{\dagger}_{\mathbf{n}'} \hat{a}^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{n}'} \hat{a}_{\mathbf{m}'} |1_{\mathbf{n}}, 1_{\mathbf{m}}\rangle\, 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \,\text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]} \\
= \text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{m}} \chi''^{(+)}_{\mathbf{n},\mathbf{m}} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} \\
= \frac{1}{2} \mathcal{N}_{\mathbf{n},\mathbf{m}}.
\end{aligned}
$$

C.4 In the m-th mode

C.4.1 Density operator in the m-th mode

To compute the number of particles in the m-th mode, we begin computing the expression for the diagonal density operator after tracing it over the fields modes $\mathbf{m}, \mathbf{n} \neq \mathbf{k}$, such as

$$
\begin{aligned}
\hat{\rho}_{\mathbf{k}}(t)|_{\text{diag}} = \text{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}} \left(\hat{\rho}(0) \right) - \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \\
\times \left\{ \text{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}} \left(\hat{\mathcal{A}}''_{\mathbf{n}'} \hat{\mathcal{A}}''_{\mathbf{m}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \right) - \text{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}} \left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \right) + \text{h.c.} \right\},
\end{aligned}
$$

To do so, we compute the trace operation for each one of the density operator.

First term

$$
\begin{aligned}
\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \text{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}} \left(\hat{\mathcal{A}}''_{\mathbf{n}'} \hat{\mathcal{A}}''_{\mathbf{m}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \right) \\
= \sum_{\mathbf{n}'} \text{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}} \left(\hat{\mathcal{A}}''^{2}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger 2}_{\mathbf{n}'} \hat{\rho}(0) \right) + 2 \sum_{\substack{\mathbf{n}',\mathbf{m}' \\ \mathbf{m}'\neq\mathbf{n}'}} \text{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}} \left(\hat{\mathcal{A}}''_{\mathbf{n}'} \hat{\mathcal{A}}''_{\mathbf{m}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \right) \\
= 2 \sum_{\mathbf{n}'} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} e^{-2i[\tilde{\omega}_{\mathbf{n}'}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]} |0_{\mathbf{k}}\rangle \langle 0_{\mathbf{k}}| \\
+ 2 \sum_{\substack{\mathbf{n}',\mathbf{m}' \\ \mathbf{m}'\neq\mathbf{n}'}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{m}'}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{m}'}(t')]\}} |0_{\mathbf{k}}\rangle \langle 0_{\mathbf{k}}|;
\end{aligned}
$$

Second term

$$
\begin{aligned}
&\sum_{\mathbf{n}',\mathbf{m}'} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'}\,\mathrm{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right)\\
&=\sum_{\mathbf{n}'} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'}\,\mathrm{Tr}_{\mathbf{n}'\neq\mathbf{k}}\left(\hat{\mathcal{A}}''^{\dagger 2}_{\mathbf{n}'}\hat{\rho}(0)\hat{\mathcal{A}}'^{2}_{\mathbf{n}'}\right)+\sum_{\substack{\mathbf{n}',\mathbf{m}'\\ \mathbf{m}'\neq\mathbf{n}'}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'}\,\mathrm{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right)\\
&=2\chi'^{(+)}_{\mathbf{k},\mathbf{k}}\chi''^{(+)}_{\mathbf{k},\mathbf{k}}e^{-2i[\tilde{\omega}_{\mathbf{k}}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]}\left|2_{\mathbf{k}}\right\rangle\left\langle 2_{\mathbf{k}}\right|+\sum_{\mathbf{n}'(\neq\mathbf{k})} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'}\,\mathrm{Tr}_{\mathbf{n}'\neq\mathbf{k}}\left(\hat{\mathcal{A}}''^{\dagger 2}_{\mathbf{n}'}\hat{\rho}(0)\hat{\mathcal{A}}'^{2}_{\mathbf{n}'}\right)\\
&+2\sum_{\mathbf{n}'\neq\mathbf{k}} \chi'^{(+)}_{\mathbf{n}',\mathbf{k}}\chi''^{(+)}_{\mathbf{n}',\mathbf{k}}\,\mathrm{Tr}_{\mathbf{n}'\neq\mathbf{k}}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{k}}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{k}}\hat{\mathcal{A}}'_{\mathbf{n}'}\right)\\
&+\sum_{\substack{\mathbf{n}'(\neq\mathbf{k},\mathbf{m}')\\ \mathbf{m}'(\neq\mathbf{k},\mathbf{n}')}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'}\,\mathrm{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right)
\end{aligned}
$$

$$
\begin{aligned}
&=2\chi'^{(+)}_{\mathbf{k},\mathbf{k}}\chi''^{(+)}_{\mathbf{k},\mathbf{k}}e^{-2i[\tilde{\omega}_{\mathbf{k}}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]}\left|2_{\mathbf{k}}\right\rangle\left\langle 2_{\mathbf{k}}\right|+2\sum_{\mathbf{n}'(\neq\mathbf{k})} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'}e^{-2i[\tilde{\omega}_{\mathbf{n}'}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]}\left|0_{\mathbf{k}}\right\rangle\left\langle 0_{\mathbf{k}}\right|\\
&+2\sum_{\mathbf{n}'\neq\mathbf{k}} \chi'^{(+)}_{\mathbf{n}',\mathbf{k}}\chi''^{(+)}_{\mathbf{n}',\mathbf{k}}e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{k}}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{k}}(t')]\}}\left|1_{\mathbf{k}}\right\rangle\left|1_{\mathbf{k}}\right\rangle\\
&+\sum_{\substack{\mathbf{n}'(\neq\mathbf{k},\mathbf{m}')\\ \mathbf{m}'(\neq\mathbf{k},\mathbf{n}')}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'}e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{m}'}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{m}'}(t')]\}}\left|0_{\mathbf{k}}\right\rangle\left|0_{\mathbf{k}}\right\rangle\\
&=2\chi'^{(+)}_{\mathbf{k},\mathbf{k}}\chi''^{(+)}_{\mathbf{k},\mathbf{k}}e^{-2i[\tilde{\omega}_{\mathbf{k}}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]}\left|2_{\mathbf{k}}\right\rangle\left\langle 2_{\mathbf{k}}\right|+2\sum_{\mathbf{n}'(\neq\mathbf{k})} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'}e^{-2i[\tilde{\omega}_{\mathbf{n}'}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]}\left|0_{\mathbf{k}}\right\rangle\left\langle 0_{\mathbf{k}}\right|\\
&+2\sum_{\mathbf{n}'\neq\mathbf{k}} \chi'^{(+)}_{\mathbf{n}',\mathbf{k}}\chi''^{(+)}_{\mathbf{n}',\mathbf{k}}e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{k}}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{k}}(t')]\}}\left|1_{\mathbf{k}}\right\rangle\left|1_{\mathbf{k}}\right\rangle\\
&+\sum_{\substack{\mathbf{n}'(\neq\mathbf{k},\mathbf{m}')\\ \mathbf{m}'(\neq\mathbf{k},\mathbf{n}')}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'}e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{m}'}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{m}'}(t')]\}}\left|0_{\mathbf{k}}\right\rangle\left|0_{\mathbf{k}}\right\rangle .
\end{aligned}
$$

If we trace out the modes different from $\mathbf{n}, \mathbf{m} = \mathbf{k}$ we must obtain

$$\begin{aligned}
\hat{\rho}_{\mathbf{k}}(t)|_{\text{diag}} = |0_{\mathbf{k}}\rangle \langle 0_{\mathbf{k}}| - \frac{1}{2}\,\mathrm{Re}\int_0^t dt'' \int_0^{t''} dt' \Bigg\{ & 2\sum_{\mathbf{n}'} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} e^{-2i[\tilde{\omega}_{\mathbf{n}}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]} |0_{\mathbf{k}}\rangle \langle 0_{\mathbf{k}}| \\
&+ \sum_{\substack{\mathbf{n}',\mathbf{m}' \\ \mathbf{m}'\neq\mathbf{n}'}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} |0_{\mathbf{k}}\rangle \langle 0_{\mathbf{k}}| - 2\chi'^{(+)}_{\mathbf{k},\mathbf{k}}\chi''^{(+)}_{\mathbf{k},\mathbf{k}} e^{-2i[\tilde{\omega}_{\mathbf{k}}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]} |2_{\mathbf{k}}\rangle \langle 2_{\mathbf{k}}| \\
&- 2\sum_{\mathbf{n}'(\neq\mathbf{k})} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} e^{-2i[\tilde{\omega}_{\mathbf{n}'}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]} |0_{\mathbf{k}}\rangle \langle 0_{\mathbf{k}}| \\
&- 4\sum_{\mathbf{n}'\neq\mathbf{k}} \chi'^{(+)}_{\mathbf{n}',\mathbf{k}}\chi''^{(+)}_{\mathbf{n}',\mathbf{k}} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{k}}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{k}}(t')]\}} |1_{\mathbf{k}}\rangle \langle 1_{\mathbf{k}}| \\
&- 2\sum_{\substack{\mathbf{n}'(\neq\mathbf{k},\mathbf{m}') \\ \mathbf{m}'(\neq\mathbf{k},\mathbf{n}')}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{m}'}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{m}'}(t')]\}} |0_{\mathbf{k}}\rangle \langle 0_{\mathbf{k}}| \Bigg\}.
\end{aligned}$$

C.4.2 Number of particles in the k-th mode

Let the expression for

$$\begin{aligned}
\hat{\rho}_{\mathbf{k}}(t)|_{\text{diag}}\hat{N} = \mathrm{Re}\int_0^t dt'' \int_0^{t''} dt' \Bigg\{ & \chi'^{(+)}_{\mathbf{k},\mathbf{k}}\chi''^{(+)}_{\mathbf{k},\mathbf{k}} |2_{\mathbf{k}}\rangle \langle 2_{\mathbf{k}}| \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \\
&+ 2\sum_{\mathbf{n}'\neq\mathbf{k}} \chi'^{(+)}_{\mathbf{n}',\mathbf{k}}\chi''^{(+)}_{\mathbf{n}',\mathbf{k}} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{k}}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{k}}(t')]\}} |1_{\mathbf{k}}\rangle \langle 1_{\mathbf{k}}| \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \Bigg\},
\end{aligned}$$

therefore

$$\begin{aligned}
\langle N_{\mathbf{k}}(t)\rangle &= \mathrm{Tr}\left\{\hat{\rho}_{\mathbf{k}}(t)|_{\text{diag}}\hat{N}\right\} \\
&= 2\,\mathrm{Re}\sum_{\mathbf{n}'}\int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n}',\mathbf{k}}\chi''^{(+)}_{\mathbf{n}',\mathbf{k}} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{k}}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{k}}(t')]\}} \\
&= \sum_{\mathbf{n}'}\mathcal{N}_{\mathbf{n}',\mathbf{k}}
\end{aligned}$$

C.4.3 Diagonal entropy in the $\mathbf{k}$-th mode

Term $0_{\mathbf{k}}$

$$\rho_{\mathbf{k}}^{(0)}(t) = \langle 0_{\mathbf{k}}| \hat{\rho}_{\mathbf{k}} |0_{\mathbf{k}}\rangle = 1 - \mathrm{Re}\int_0^t dt'' \int_0^{t''} dt' \Bigg\{ \sum_{\mathbf{n}'} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} e^{-2i[\tilde{\omega}_{\mathbf{n}}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]}$$
$$- \sum_{\mathbf{n}'(\neq\mathbf{k})} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} e^{-2i[\tilde{\omega}_{\mathbf{n}'}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]}$$
$$+ \sum_{\mathbf{n}',\mathbf{m}'} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}}$$
$$- \sum_{\substack{\mathbf{n}'(\neq\mathbf{k},\mathbf{m}') \\ \mathbf{m}'(\neq\mathbf{k},\mathbf{n}')}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{m}'}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{m}'}(t')]\}} \Bigg\}$$
$$= 1 - \mathrm{Re}\int_0^t dt'' \int_0^{t''} dt' \Bigg\{ \chi'^{(+)}_{\mathbf{k},\mathbf{k}}\chi''^{(+)}_{\mathbf{k},\mathbf{k}} e^{-2i[\tilde{\omega}_{\mathbf{k}}(t'')-\tilde{\omega}_{\mathbf{k}}(t')]}$$
$$+ \sum_{\mathbf{n}'(\neq\mathbf{k})} \chi'^{(+)}_{\mathbf{n}',\mathbf{k}}\chi''^{(+)}_{\mathbf{n}',\mathbf{k}} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{k}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{k}}(t')]\}} \Bigg\}$$
$$1 - \frac{1}{2}\langle \hat{N}_{\mathbf{k}}(t)\rangle.$$

Term $2_{\mathbf{k}}$

$$\rho_{\mathbf{k}}^{(2_{\mathbf{k}})} = \langle 2_{\mathbf{k}}| \hat{\rho}_{\mathbf{k}} |2_{\mathbf{k}}\rangle = \mathrm{Re}\int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{k},\mathbf{k}}\chi''^{(+)}_{\mathbf{k},\mathbf{k}} e^{-2i[\tilde{\omega}_{\mathbf{k}}(t'')-\tilde{\omega}_{\mathbf{k}}(t')]}$$
$$= \frac{1}{2}\mathcal{N}_{\mathbf{k},\mathbf{k}}$$

Term $1_{\mathbf{k}}$

$$\rho_{\mathbf{k}}^{(1_{\mathbf{k}})} = \langle 1_{\mathbf{k}}| \hat{\rho}_{\mathbf{k}} |1_{\mathbf{k}}\rangle = \mathrm{Re}\sum_{\mathbf{n}'\neq\mathbf{k}}\int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n}',\mathbf{k}}\chi''^{(+)}_{\mathbf{n}',\mathbf{k}} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{k}}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{k}}(t')]\}}$$
$$= \frac{1}{2}\sum_{\mathbf{n}'\neq\mathbf{k}} \mathcal{N}_{\mathbf{n}',\mathbf{k}}$$

Appendix D

Rotating Wave Approximation

D.1 Number of particles

In order to show the adequacy of our latter formalism we shall reproduce the literature results by computing our quantities when there is resonance between the cavity and some integral multiple of some unperturbed field frequency.

D.1.1 Total number of particles

To do so, we first consider the one-dimensional case in which the field frequency takes the form of

$$\omega_n = \frac{n\pi}{L},$$

such that

$$v^{(+)}_{n,m}(t) = \frac{1}{2}\frac{\omega_m^2 - \omega_n^2}{\sqrt{\omega_m \omega_n}} g_{m,n}(t) - L(t)\frac{\partial \omega_{n,L(t)}}{\partial L}\delta_{m,n}$$

for

$$g_{n,m} = -g_{m,n} = -L(t)\int_0^{L(t)} dz\ \varphi_n \frac{\partial \varphi_m}{\partial L}.$$

In the case of weak perturbed motion $L(t) = L_0\left[1 + \epsilon\xi(t)\right]$, we must obtain

$$g_{n,m}(t) = \begin{cases} (-1)^{m+n}\frac{2mn}{m^2-n^2} & n \neq m \\ 0 & n = m, \end{cases}$$

meaning

$$\chi^{(+)}_{n,m}(t) = -\epsilon\frac{v^{(+)}_{n,m}}{\omega_n + \omega_m}\dot{\xi}(t) \quad \text{with} \quad v^{(+)}_{n,m} = \frac{\pi}{L_0}\begin{cases} (-1)^{m+n}\sqrt{mn} & n \neq m \\ n & n = m \end{cases}.$$

meaning $v_{nm}^{(+)2} = \frac{\pi^2}{L_0^2}nm$. From this we can compute the total number of particles created in second order of ϵ as

$$N(t) = \sum_{n,m} \mathcal{N}_{n,m},$$

with

$$\begin{aligned}\mathcal{N}_{n,m} &= 2\,\mathrm{Re}\int_0^T dt'' \int_0^{t''} dt'\ \chi_{k',j'}(t')\chi_{k',j'}(t'')e^{-i[\bar{\omega}_n(t')+\bar{\omega}_m(t')-(\bar{\omega}_n(t'')+\bar{\omega}_m(t''))]} \\ &= \frac{2\epsilon^2 v_{m,n}^2}{(\omega_n+\omega_m)^2}\int_0^T dt'' \int_0^{t''} dt'\ \frac{d}{dt'}\xi(t')\frac{d}{dt''}\xi(t'')e^{-i(\omega_n+\omega_m)(t'-t'')}.\end{aligned} \tag{D.1}$$

To simplify this last integral, we can consider the following equality

$$\begin{aligned}&\int_0^T dt'' \int_0^{t''} dt'\ \frac{d}{dt'}\xi(t')\frac{d}{dt''}\xi(t'')e^{-i(\omega_n+\omega_m)(t'-t'')} \\ &\qquad = \frac{1}{2}\int_0^T dt'' \int_0^{T} dt'\ \frac{d}{dt'}\xi(t')\frac{d}{dt''}\xi(t'')e^{-i(\omega_n+\omega_m)(t'-t'')}\end{aligned}$$

Remembering also that as the cavity remain at rest for instants of time $t < 0$ and $t > T$, the motion function $\xi(t)$ must respects $\xi(0) = \xi(T) = 0$. Using those last considerations by integration by parts the expression (D.1), we must obtain

$$\mathcal{N}_{n,m} = \epsilon^2 v_{n,m}^{(+)2}\,\mathrm{Re}\int_0^T dt'' \int_0^T dt'\ \xi(t')\xi(t'')e^{-i(\omega_n+\omega_m)(t'-t'')}.$$

In this faction, we can interpret the coefficient as the square of the Bogoliubov coefficients

$$\mathcal{N}_{n,m} = |\beta_{n,m}|^2,$$

with

$$\beta_{n,m} = -i\epsilon v_{n,m}^{(+)}\int_0^T dt\ \xi(t)e^{-i(\omega_n+\omega_m)t}.$$

Considering the sinusoidal cavity motion $\xi(t) = \sin \Omega_p t$ with $\Omega = p\omega_1$ and $p = 1, 2, \ldots$, we can compute

$$\begin{aligned}
\mathcal{N}_{m,n} &= \epsilon^2 v_{m,n}^2 \int_0^T dt'' \sin \Omega_p t'' e^{i(\omega_n+\omega_m)t''} \int_0^T dt' \sin \Omega_p t' e^{-i(\omega_n+\omega_m)t'} \\
&= -\frac{1}{4}\epsilon^2 v_{m,n}^2 \int_0^T dt'' \left[e^{i[\Omega_p+(\omega_n+\omega_m)]t''} - e^{-i[\Omega_p-(\omega_n+\omega_m)t'']} \right] \\
&\times \int_0^T dt' \left[e^{i[\Omega_p-(\omega_n+\omega_m)]t'} - e^{-i[\Omega_p+(\omega_n+\omega_m)]t'} \right].
\end{aligned}$$

Considering the resonance condition $\Omega_p = \omega_n + \omega_m$, by applying the rotating wave approximation where we ignore exponentials with arguments greater than unity (as they oscillates so rapidly that in average its contributions can be neglected), we finally obtain

$$\begin{aligned}
\mathcal{N}_{n,m} &= \frac{1}{4}\epsilon^2 v_{m,n}^2 T^2 \delta_{n,p-m} \\
&= \frac{\epsilon^2\pi^2}{4L_0^2} T^2 nm \delta_{n,p-m}
\end{aligned}$$

Where as $m = p - n$ with $n = 1, 2, \ldots$, we have $m = 1, \ldots p - 1$. Therefore,

$$\begin{aligned}
N(t) = \sum_{n,m} \mathcal{N}_{n,m} &= \frac{\epsilon^2\pi^2}{4L_0^2} T^2 \sum_{m=1}^{p-1} m(m-p) \\
&= \frac{\epsilon^2\pi^2}{4L_0^2} T^2 \left(\frac{p(p^2-1)}{6} \right) \\
&= \frac{\epsilon^2\pi^2}{24L_0^2} p(p^2-1)T^2.
\end{aligned}$$

D.1.2 Number of particles in the k-th mode

The number of particles created in the k-mode in second order of time can be calculated through

$$\begin{aligned}
\langle \hat{N}_k(t) \rangle &= \sum_n \mathcal{N}_{n,k} \\
&= \frac{\epsilon^2\pi^2}{4L_0^2} T^2 \sum_n nk \, \delta_{n,p-k} \\
&= \frac{\epsilon^2\pi^2}{4L_0^2} k(p-k)T^2,
\end{aligned}$$

where $k = 1, \ldots, p - 1$.

D.2 Diagonal entropy

Considering

$$\mathcal{N}_{n,m} = \frac{6}{p(p^2-1)} N(t) nm \delta_{n,p-m},$$

the diagonal entropy can be written as

$$\begin{aligned}
S_d(t) &= -\rho^{(0)}_{\text{diag}}(t) \ln \rho^{(0)}_{\text{diag}}(t) - \sum_{n,m} \rho^{n,m}_{\text{diag}}(t) \ln \rho^{n,m}_{\text{diag}}(t) \\
&= -\left(1 - \frac{1}{2}\langle N(t)\right) \ln \left(1 - \frac{1}{2}\langle N(t)\right) - \sum_{n,m} \frac{1}{2}\mathcal{N}_{n,m} \ln \frac{1}{2}\mathcal{N}_{n,m} \\
&= \frac{1}{2}N(t) - \sum_{n,m} \frac{1}{2}\mathcal{N}_{n,m} \ln \frac{1}{2}\frac{6}{p(p^2-1)} N(t) nm\delta_{n,p-m} \\
&= \frac{1}{2}N(t) - \sum_{n,m} \frac{1}{2}\mathcal{N}_{n,m} \ln \frac{1}{2}\frac{6}{p(p^2-1)} N(t) - \sum_{n,m} \frac{1}{2}\mathcal{N}_{n,m} \ln nm\delta_{n,p-m} \\
&= \frac{1}{2}\langle N(t) - \frac{1}{2}N(t) \ln \frac{1}{2}N(t) - \frac{1}{2}N(t) \ln \frac{6}{p(p^2-1)} \\
&- \frac{1}{2}\frac{6}{p(p^2-1)} N(t) \sum_{n,m} nm\delta_{n,p-m} \ln nm\delta_{n,p-m} \\
&= \frac{1}{2}N(t)\left[1 - \ln \frac{1}{2}N(t) + \ln \frac{p(p^2-1)}{6} - \frac{6f(p)}{p(p^2-1)}\right]
\end{aligned}$$

where $f(p) = \sum_{m=1}^{p-1} (p-m)m \ln(p-m)m$

Appendix E

Computing $\mathcal{N}_{n,m}$

Let the function $\mathcal{N}_{n,m}$, with $\Omega = \Omega_\mu$ and $\omega = \omega_n + \omega_m$

$$\mathcal{N}_{n,m} = \frac{2\epsilon^2 v^2}{\omega^2} \int_0^t dt' \int_0^{t'} dt'' \frac{d}{dt''}\xi(t'') \frac{d}{dt'}\xi(t') \cos\omega(t'' - t').$$

Using $\xi(t) = \sin\Omega_\mu t$, we can develop the last integrant as

$$\begin{aligned}
&\frac{d}{dt''}\xi(t'')\frac{d}{dt'}\xi(t')\cos\omega(t''-t') = \Omega^2 \cos\Omega t'' \cos\Omega t' \cos\omega(t''-t') \\
&\quad = \frac{\Omega^2}{8}\left(e^{i\Omega t''} + e^{-i\Omega_p t''}\right)\left(e^{i\Omega t'} + e^{-i\Omega t'}\right)\left(e^{i\omega(t''-t')} + e^{-i\omega(t''-t')}\right) \\
&\quad = \frac{\Omega^2}{8}\left(e^{i\Omega t'} + e^{-i\Omega t'}\right)\left[\left(e^{i(\Omega+\omega)t''} + e^{-i(\Omega-\omega)t''}\right)e^{-i\omega t'} + \left(e^{i(\Omega-\omega)t''} + e^{-i(\Omega+\omega)t''}\right)e^{i\omega t'}\right] \\
&\quad = \frac{\Omega^2}{8}\Bigg[\left(e^{i(\Omega+\omega)t''} + e^{-i(\Omega-\omega)t''}\right)\left(e^{i(\Omega-\omega)t'} + e^{-i(\Omega+\omega)t'}\right) \\
&\qquad + \left(e^{i(\Omega-\omega)t''} + e^{-i(\Omega+\omega)t''}\right)\left(e^{i(\Omega+\omega)t'} + e^{-i(\Omega-\omega)t'}\right)\Bigg],
\end{aligned}$$

E.1 $\Omega \neq \omega$

$$\begin{aligned}
\mathcal{N}_{n,m} &= \frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \int_0^t dt' \int_0^{t'} dt'' \Bigg[\left(e^{i(\Omega+\omega)t''} + e^{-i(\Omega-\omega)t''}\right)\left(e^{i(\Omega-\omega)t'} + e^{-i(\Omega+\omega)t'}\right) \\
&\quad + \left(e^{i(\Omega-\omega)t''} + e^{-i(\Omega+\omega)t''}\right)\left(e^{i(\Omega+\omega)t'} + e^{-i(\Omega-\omega)t'}\right)\Bigg] \\
&= \frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \int_0^t dt' \Bigg[\left(e^{i(\Omega-\omega)t'} + e^{-i(\Omega+\omega)t'}\right)\int_0^{t'} dt'' \left(e^{i(\Omega+\omega)t''} + e^{-i(\Omega-\omega)t''}\right) \\
&\quad + \left(e^{i(\Omega+\omega)t'} + e^{-i(\Omega-\omega)t'}\right)\int_0^{t'} dt'' \left(e^{i(\Omega-\omega)t''} + e^{-i(\Omega+\omega)t''}\right)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Omega^2}{4i\omega^2}\epsilon^2 v^2 \int_0^t dt' \Bigg[\left(e^{i(\Omega-\omega)t'} + e^{-i(\Omega+\omega)t'}\right)\left(\frac{e^{i(\Omega+\omega)t''}}{\Omega+\omega} - \frac{e^{-i(\Omega-\omega)t''}}{\Omega-\omega}\right)\Bigg|_0^{t'} \\
&\qquad + \left(e^{i(\Omega+\omega)t'} + e^{-i(\Omega-\omega)t'}\right)\left(\frac{e^{i(\Omega-\omega)t''}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t''}}{\Omega+\omega}\right)\Bigg|_0^{t'}\Bigg] \\
&= \frac{\Omega^2}{4i\omega^2}\epsilon^2 v^2 \int_0^t dt' \Bigg[\left(e^{i(\Omega-\omega)t'} + e^{-i(\Omega+\omega)t'}\right)\left(\frac{e^{i(\Omega+\omega)t'}}{\Omega+\omega} - \frac{e^{-i(\Omega-\omega)t'}}{\Omega-\omega} - \frac{1}{\Omega+\omega} + \frac{1}{\Omega-\omega}\right) \\
&\qquad + \left(e^{i(\Omega+\omega)t'} + e^{-i(\Omega-\omega)t'}\right)\left(\frac{e^{i(\Omega-\omega)t'}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t'}}{\Omega+\omega} - \frac{1}{\Omega-\omega} + \frac{1}{\Omega+\omega}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Omega^2}{4i\omega^2}\epsilon^2 v^2 \int_0^t dt' \Bigg[e^{i(\Omega-\omega)t'}\left(\frac{e^{i(\Omega+\omega)t'}}{\Omega+\omega} - \frac{e^{-i(\Omega-\omega)t'}}{\Omega-\omega} + \frac{2\omega}{\Omega^2-\omega^2}\right) \\
&\qquad + e^{-i(\Omega+\omega)t'}\left(\frac{e^{i(\Omega+\omega)t'}}{\Omega+\omega} - \frac{e^{-i(\Omega-\omega)t'}}{\Omega-\omega} + \frac{2\omega}{\Omega^2-\omega^2}\right) \\
&\qquad + e^{i(\Omega+\omega)t'}\left(\frac{e^{i(\Omega-\omega)t'}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t'}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\right) \\
&\qquad + e^{-i(\Omega-\omega)t'}\left(\frac{e^{i(\Omega-\omega)t'}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t'}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\right)\Bigg] \\
&= \frac{\Omega^2}{4i\omega^2}\epsilon^2 v^2 \int_0^t dt' \Bigg[\left(\frac{e^{2i\Omega t'}}{\Omega+\omega} - \frac{1}{\Omega-\omega} + \frac{2\omega}{\Omega^2-\omega^2}e^{i(\Omega-\omega)t'}\right) \\
&\qquad + \left(\frac{1}{\Omega+\omega} - \frac{e^{-2i\Omega t'}}{\Omega-\omega} + \frac{2\omega}{\Omega^2-\omega^2}e^{-i(\Omega+\omega)t'}\right) \\
&\qquad + \left(\frac{e^{2i\Omega t'}}{\Omega-\omega} - \frac{1}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}e^{i(\Omega+\omega)t'}\right) \\
&\qquad + \left(\frac{1}{\Omega-\omega} - \frac{e^{-2i\Omega t'}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}e^{-i(\Omega-\omega)t'}\right)\Bigg] \\
&= \frac{\Omega^2}{4i\omega^2}\epsilon^2 v^2 \int_0^t dt' \Bigg[\frac{2\Omega}{\Omega^2-\omega^2}\left(e^{2i\Omega t'} - e^{-2i\Omega t'}\right) + \frac{2\omega}{\Omega^2-\omega^2}\left(e^{i(\Omega-\omega)t'} - e^{-i(\Omega-\omega)t'}\right) \\
&\qquad - \frac{2\omega}{\Omega^2-\omega^2}\left(e^{i(\Omega+\omega)t'} - e^{-i(\Omega+\omega)t'}\right)\Bigg] \\
&= \frac{\Omega^2}{2\omega^2}\frac{\epsilon^2 v^2}{\Omega^2-\omega^2}\int_0^t dt' \Big[2\Omega\sin 2\Omega t' + 2\omega\sin(\Omega-\omega)t' - 2\omega\sin(\Omega+\omega)t' \Big] \\
&= \frac{\Omega^2}{2\omega^2}\frac{\epsilon^2 v^2}{\Omega^2-\omega^2}\left[\frac{2\omega}{\Omega+\omega}\cos(\Omega+\omega)t' - \frac{2\omega}{\Omega-\omega}\cos(\Omega-\omega)t' - \cos 2\Omega t'\right]\Bigg|_0^t
\end{aligned}
$$

$$= \frac{\Omega^2}{2\omega^2}\frac{\epsilon^2 v^2}{\Omega^2-\omega^2}\left[1+\frac{4\omega^2}{\Omega^2-\omega^2}+\frac{2\omega}{\Omega+\omega}\cos(\Omega+\omega)t-\frac{2\omega}{\Omega-\omega}\cos(\Omega-\omega)t-\cos 2\Omega t\right]$$

E.2 $\Omega = \omega$

$$\begin{aligned}
\frac{d}{dt''}\xi(t'')\frac{d}{dt'}\xi(t')\cos\omega(t''-t') &= \omega^2\cos\omega t''\cos\omega t'\cos\omega(t''-t')\\
&= \omega^2\cos\omega t''\cos\omega t'\left(\cos\omega t''\cos\omega t'+\sin\omega t''\sin\omega t'\right)\\
&= \frac{1}{4}\omega^2\left[\sin 2\omega t''\sin 2\omega t'+\left(1+\cos 2\omega t''\right)\left(1+\cos 2\omega t'\right)\right]\\
&= \frac{1}{4}\omega^2\left[\sin 2\omega t''\sin 2\omega t'+\cos 2\omega t'\cos 2\omega t''+\cos 2\omega t''+\cos 2\omega t'+1\right]\\
&= \frac{1}{4}\omega^2\left[\cos 2\omega(t''-t')+\cos 2\omega t''+\cos 2\omega t'+1\right]
\end{aligned}$$

integrating the last expression

$$\begin{aligned}
\mathcal{N}_{n,m} &= \frac{1}{2}\epsilon^2 v^2\int_0^t dt'\int_0^{t'} dt''\left[\cos 2\omega(t''-t')+\cos 2\omega t''+\cos 2\omega t'+1\right]\\
&= \frac{1}{2}\epsilon^2 v^2\int_0^t dt'\left[\frac{1}{2\omega}\sin 2\omega(t''-t')+\frac{1}{2\omega}\sin 2\omega t''+t''\cos 2\omega t'+t''\right]\Bigg|_0^{t'}\\
&= \frac{1}{2}\epsilon^2 v^2\int_0^t dt'\left[t'\cos 2\omega t'+t'\right]\\
&= \frac{1}{8}\frac{\epsilon^2 v^2}{\omega^2}\left[2\omega^2 t^2+2\omega t\sin(2\omega t)+\cos(2\omega t)-1\right]
\end{aligned}$$

E.3 the function

$$\mathcal{N}_{\mathbf{n},\mathbf{m}}(t) =$$

$$= \begin{cases} \frac{1}{2}\frac{\epsilon^2 v^2}{\omega^2}\left[\omega^2 t^2+2\omega t\sin(2\omega t)+\cos(2\omega t)-1\right] & \text{for}\quad \Omega=\omega \\ \frac{\Omega^2}{2\omega^2}\frac{\epsilon^2 v^2}{\Omega^2-\omega^2}\left[1+\frac{4\omega^2}{\Omega^2-\omega^2}+\frac{2\omega}{\Omega+\omega}\cos(\Omega+\omega)t-\frac{2\omega}{\Omega-\omega}\cos(\Omega-\omega)t-\cos 2\Omega t\right] & \text{for}\quad \Omega\neq\omega. \end{cases} \tag{E.1}$$

E.4 Other

Now consider

$$\begin{aligned}
\mathcal{N}_{n,m} &= \frac{\epsilon^2 v^2}{\omega^2} \operatorname{Re} \int_0^t dt'' \frac{d}{dt''}\xi(t'') e^{i\omega t''} \int_0^t dt' \ \frac{d}{dt'}\xi(t') e^{-i\omega t'} \\
&= \frac{\Omega^2}{\omega^2}\epsilon^2 v^2 \operatorname{Re} \int_0^t dt'' \cos\Omega t'' e^{i\omega t''} \int_0^t dt' \ \cos\Omega t' e^{-i\omega t'} \\
&= \frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \operatorname{Re} \int_0^t dt'' \left[e^{i\Omega t''} + e^{-i\Omega t''}\right] e^{i\omega t''} \int_0^t dt' \ \left[e^{i\Omega t'} + e^{-i\Omega t'}\right] e^{-i\omega t'} \\
&= \frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \operatorname{Re} \int_0^t dt'' \left[e^{i(\Omega+\omega)t''} + e^{-i(\Omega-\omega)t''}\right] \int_0^t dt' \ \left[e^{i(\Omega-\omega)t'} + e^{-i(\Omega+\omega)t'}\right] \\
&= -\frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \operatorname{Re} \left[\frac{e^{i(\Omega+\omega)t''}}{\Omega+\omega} - \frac{e^{-i(\Omega-\omega)t''}}{\Omega-\omega}\right]\Bigg|_0^t \left[\frac{e^{i(\Omega-\omega)t'}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t'}}{\Omega+\omega}\right]\Bigg|_0^t
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \operatorname{Re} \left[\frac{e^{i(\Omega+\omega)t}}{\Omega+\omega} - \frac{e^{-i(\Omega-\omega)t}}{\Omega-\omega} - \frac{1}{\Omega+\omega} + \frac{1}{\Omega-\omega}\right] \left[\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega} - \frac{1}{\Omega-\omega} + \frac{1}{\Omega+\omega}\right] \\
&= -\frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \operatorname{Re} \left[\frac{e^{i(\Omega+\omega)t}}{\Omega+\omega} - \frac{e^{-i(\Omega-\omega)t}}{\Omega-\omega} + \frac{2\omega}{\Omega^2-\omega^2}\right] \left[\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \,\mathrm{Re}\left\{ \frac{e^{i(\Omega+\omega)t}}{\Omega+\omega}\left[\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\right]\right. \\
&\qquad\qquad - \frac{e^{-i(\Omega-\omega)t}}{\Omega-\omega}\left[\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\right] \\
&\qquad\qquad \left. + \frac{2\omega}{\Omega^2-\omega^2}\left[\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\right]\right\} \\
&= -\frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \,\mathrm{Re}\left\{ \frac{e^{2i\Omega t}}{\Omega^2-\omega^2} - \frac{1}{(\Omega+\omega)^2} - \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{i(\Omega+\omega)t}}{\Omega+\omega}\right. \\
&\qquad\qquad - \frac{1}{(\Omega-\omega)^2} + \frac{e^{-2i\Omega t}}{\Omega^2-\omega^2} + \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{-i(\Omega-\omega)t}}{\Omega-\omega} \\
&\qquad\qquad \left. + \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega} - \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega} - \frac{4\omega^2}{(\Omega^2-\omega^2)^2}\right\} \\
&= -\frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \,\mathrm{Re}\left\{ \frac{e^{2i\Omega t}}{\Omega^2-\omega^2} + \frac{e^{-2i\Omega t}}{\Omega^2-\omega^2} - \frac{1}{(\Omega+\omega)^2} - \frac{1}{(\Omega-\omega)^2}\right. \\
&\qquad\qquad + \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega} + \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{-i(\Omega-\omega)t}}{\Omega-\omega} \\
&\qquad\qquad \left. - \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{i(\Omega+\omega)t}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega} - \frac{4\omega^2}{(\Omega^2-\omega^2)^2}\right\} \\
&= -\frac{\Omega^2}{2\omega^2}\epsilon^2 v^2 \,\mathrm{Re}\left\{ \frac{\cos 2\Omega t}{\Omega^2-\omega^2} - \frac{\Omega^2+\omega^2}{(\Omega^2-\omega^2)^2} + \frac{2\omega}{\Omega^2-\omega^2}\frac{\cos(\Omega-\omega)t}{\Omega-\omega}\right. \\
&\qquad\qquad \left. - \frac{2\omega}{\Omega^2-\omega^2}\frac{\cos(\Omega+\omega)t}{\Omega+\omega} - \frac{2\omega^2}{(\Omega^2-\omega^2)^2}\right\} \\
&= \frac{1}{2}\frac{\Omega^2}{\omega^2}\frac{\epsilon^2 v^2}{\Omega^2-\omega^2}\left\{1 + \frac{4\omega^2}{\Omega^2-\omega^2} - \frac{2\omega}{\Omega-\omega}\cos(\Omega-\omega)t + \frac{2\omega}{\Omega+\omega}\cos(\Omega+\omega)t - \cos 2\Omega t\right\}
\end{aligned}
$$

Bibliography

[1] A. Polkovnikov, “Microscopic diagonal entropy and its connection to basic thermodynamic relations”, Annals of Physics **326**, 486–499 (2011).

[2] A. Polkovnikov, “Microscopic expression for heat in the adiabatic basis”, Physical Review Letters **101**, 10.1103/physrevlett.101.220402 (2008).

[3] V. V. Dodonov, “Nonstationary casimir effect and analytical solutions for quantum fields in cavities with moving boundaries”, 10.48550/ARXIV.QUANT-PH/0106081 (2001).

[4] V. V. Dodonov, “Resonance photon generation in a vibrating cavity”, **31**, 9835 (1998).