Supplemental Material: Work distributions on quantum fields

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I. DETAILS OF THE CALCULATION OF THE STATE OF THE QUBIT

A. Field in the vacuum state

We now proceed to calculate the different terms of the perturbative expansion in (9) in the main text. Clearly, $\hat{\rho}_T^{(0)} = \mathrm{Tr}_\phi\left(\hat{\rho}_0\right) = |+\rangle\langle+|$. It is easy to see that the first order term $\hat{\rho}_T^{(1)}$ will vanish. When taking the trace over the field, all the free evolution terms will end up multiplying the vacuum state, either at their left or at their right, so they will disappear, leaving $\langle\Omega|\,\hat{U}^{(1)}\,|\Omega\rangle$ or $\langle\Omega|\,\hat{U}^{\dagger(1)}\,|\Omega\rangle$. This is zero since

$$\hat{U}^{(1)} = -\mathrm{i}\lambda\int_{-\infty}^{\infty}\mathrm{d}t\int\mathrm{d}^3\boldsymbol{x}\chi(t)F(\boldsymbol{x})\hat{\phi}(t,\boldsymbol{x}), \tag{1}$$

and $\langle\Omega|\,\hat{\phi}(t,\boldsymbol{x})\,|\Omega\rangle = 0\ \forall\ t,\,\boldsymbol{x}$.

$\hat{\rho}_T^{(2)}$ is the sum of two contributions, one involving products with $\hat{U}^{(1)}$ and $\hat{U}^{\dagger(1)}$, and the other with $\hat{U}^{(2)}$. Let us focus on the first family of terms.

As an example, we explicitly calculate the coefficient associated to the component $\frac{|0\rangle\langle 0|}{2}$ of the density matrix of the qubit.

$$\begin{aligned}
\mathrm{Tr}\left(\hat{U}^{(1)}e^{-\mathrm{i}\mu\hat{H}_0}\,|\Omega\rangle\langle\Omega|\,e^{\mathrm{i}\mu\hat{H}_0}\hat{U}^{\dagger(1)}\right) &= \mathrm{Tr}\left(\hat{U}^{(1)}\,|\Omega\rangle\langle\Omega|\,\hat{U}^{\dagger(1)}\right)\\
&= \mathrm{Tr}\left(\lambda^2\int_{-\infty}^{\infty}\mathrm{d}t\chi(t)\int_{-\infty}^{\infty}\mathrm{d}t'\chi(t')\int\mathrm{d}^3\boldsymbol{x}F(\boldsymbol{x})\int\mathrm{d}^3\boldsymbol{x}'F(\boldsymbol{x}')\hat{\phi}(t,\boldsymbol{x})\,|\Omega\rangle\langle\Omega|\,\hat{\phi}(t',\boldsymbol{x}')\right)\\
&= \mathrm{Tr}\left(\lambda^2\int\frac{\mathrm{d}^3\boldsymbol{k}\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega_{\boldsymbol{k}}}\sqrt{2\omega_{\boldsymbol{k}'}}}\int_{-\infty}^{\infty}\mathrm{d}t\chi(t)e^{i\omega_{\boldsymbol{k}}t}\int\mathrm{d}^3\boldsymbol{x}F(\boldsymbol{x})e^{-\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\right.\\
&\qquad\left.\times\int_{-\infty}^{\infty}\mathrm{d}t'\chi(t')e^{-\mathrm{i}\omega_{\boldsymbol{k}'}t'}\int\mathrm{d}^3\boldsymbol{x}'F(\boldsymbol{x}')e^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{x}'}|\boldsymbol{k}\rangle\langle\boldsymbol{k}'|\right)\\
&= \lambda^2\,\mathrm{Tr}\left(\int\frac{\mathrm{d}^3\boldsymbol{k}\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3\sqrt{2\omega_{\boldsymbol{k}}}\sqrt{2\omega_{\boldsymbol{k}'}}}\widetilde{\chi}(\omega_{\boldsymbol{k}})\widetilde{\chi}(-\omega_{\boldsymbol{k}'})\widetilde{F}(-\boldsymbol{k})\widetilde{F}(-\boldsymbol{k}')|\boldsymbol{k}\rangle\langle\boldsymbol{k}'|\right)\\
&= \lambda^2\int\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_{\boldsymbol{k}}}|\widetilde{\chi}(\omega_{\boldsymbol{k}})|^2|\widetilde{F}(\boldsymbol{k})|^2,
\end{aligned}\tag{2}$$

where in the last step we have used that $FT[f](x) = (FT[f](-x))^*$, for a real function f. We are assuming that both the switching and the smearing are real functions. The calculation for the $\frac{|1\rangle\langle 0|}{2}$ coefficient is analogous, the only difference being the presence of a factor $e^{-\mathrm{i}\mu\hat{H}_0}$ multiplying the ket vectors $|\boldsymbol{k}\rangle$. Since $e^{-\mathrm{i}\mu\hat{H}_0}|\boldsymbol{k}\rangle = e^{-\mathrm{i}\mu\omega_{\boldsymbol{k}}}|\boldsymbol{k}\rangle$, we obtain that

$$\mathrm{Tr}\left(e^{-\mathrm{i}\mu\hat{H}_0}\hat{U}^{(1)}\,|\Omega\rangle\langle\Omega|\,e^{\mathrm{i}\mu\hat{H}_0}\hat{U}^{\dagger(1)}\right) = \lambda^2\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega_{\boldsymbol{k}}}|\widetilde{\chi}(\omega_{\boldsymbol{k}})|^2|\widetilde{F}(\boldsymbol{k})|^2 e^{-\mathrm{i}\mu\omega_{\boldsymbol{k}}}. \tag{3}$$

The rest of the components are the Hermitian conjugates of these.

We now calculate the remaining terms. That is, the terms that involve products with $\hat{U}^{(2)}$. Let us start by obtaining $\mathrm{Tr}_\phi\left(\hat{U}^{(2)}e^{-\mathrm{i}\mu\hat{H}_0}\,|\Omega\rangle\langle\Omega|\,e^{\mathrm{i}\mu\hat{H}_0}\otimes\frac{|0\rangle\langle 0|}{2} + H.c\right) = \langle\Omega|\,\hat{U}^{(2)}\,|\Omega\rangle\,\frac{|0\rangle\langle 0|}{2} + H.c.$ This is simply

$$\begin{aligned}
&-\lambda^2\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{t}\mathrm{d}t'\int\mathrm{d}^3\boldsymbol{x}\int\mathrm{d}^3\boldsymbol{x}'\chi(t)\chi(t')F(\boldsymbol{x})F(\boldsymbol{x}')\,\langle\Omega|\,\hat{\phi}(t,\boldsymbol{x})\hat{\phi}(t',\boldsymbol{x}')\,|\Omega\rangle\,\frac{|0\rangle\langle 0|}{2} + H.c \qquad (4)\\
&\qquad= -\lambda^2\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{t}\mathrm{d}t'\int\mathrm{d}^3\boldsymbol{x}\int\mathrm{d}^3\boldsymbol{x}'\chi(t)\chi(t')F(\boldsymbol{x})F(\boldsymbol{x}')2\,\mathrm{Re}[\mathcal{W}(t,\boldsymbol{x},t',\boldsymbol{x}')]\frac{|0\rangle\langle 0|}{2},
\end{aligned}$$

where $\mathcal{W}(t,\boldsymbol{x},t',\boldsymbol{x}') = \langle\Omega|\,\hat{\phi}(t,\boldsymbol{x})\hat{\phi}(t',\boldsymbol{x}')\,|\Omega\rangle$ is the Wightman function. The same is obtained for the other cases. This is because all the $e^{\pm i\mu\hat{H}_0}$ end up multiplying the vacuum state when taking the trace, so they disappear leaving simply $\langle\Omega|\,\hat{U}^{(2)}\,|\Omega\rangle + H.c.$ Therefore, the contribution of these terms to the reduced state of the qubit is $A\,|+\rangle\langle+|$, where

$$A = -\lambda^2 \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}^3\boldsymbol{x} \int \mathrm{d}^3\boldsymbol{x}' \chi(t)\chi(t')F(\boldsymbol{x})F(\boldsymbol{x}') 2\,\mathrm{Re}[\mathcal{W}(t,\boldsymbol{x},t',\boldsymbol{x}')]. \tag{5}$$

Adding everything and noting that (2) is equal to (5) we obtain, after applying the second Hadamard on the qubit, that the reduced state can be written as

$$\hat{\rho}_\mu = \frac{1}{2}\left(\mathbb{I} + |-\rangle\langle+|\left(1+\lambda^2\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega_{\boldsymbol{k}}}|\widetilde{\chi}(\omega_{\boldsymbol{k}})|^2|\widetilde{F}(\boldsymbol{k})|^2\left(e^{-\mathrm{i}\mu\omega_{\boldsymbol{k}}}-1\right)\right)\right. \tag{6}$$

$$\left. + |+\rangle\langle-|\left(1+\lambda^2\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega_{\boldsymbol{k}}}|\widetilde{\chi}(\omega_{\boldsymbol{k}})|^2|\widetilde{F}(\boldsymbol{k})|^2\left(e^{\mathrm{i}\mu\omega_{\boldsymbol{k}}}-1\right)\right)\right). \tag{7}$$

B. Field in a finite-temperature KMS state

Some properties that we use throughout these calculation are:

$$\mathrm{Tr}\left(\hat{\phi}\hat{\rho}_\beta\right) = 0, \tag{8}$$

$$\left[\hat{\rho}_\beta, e^{-it\hat{H}_0}\right] = 0. \tag{9}$$

As before, we calculate the reduced state of the qubit with a Dyson expansion. The first order term is again zero. As an example of why this is the case, we calculate

$$\mathrm{Tr}\left(e^{-\mathrm{i}\mu\hat{H}_0}\hat{U}^{(1)}\hat{\rho}_\beta e^{\mathrm{i}\mu\hat{H}_0}\right) + \mathrm{Tr}\left(e^{-\mathrm{i}\mu\hat{H}_0}\hat{\rho}_\beta e^{\mathrm{i}\mu\hat{H}_0}\hat{U}^{\dagger(1)}\right) = \mathrm{Tr}\left(\hat{U}^{(1)}\hat{\rho}_\beta\right) + Tr\left(\hat{\rho}_\beta\hat{U}^{\dagger(1)}\right), \tag{10}$$

where we have used the cyclic property of the trace and (9). Finally, using the linearity of the trace, the expression for $\hat{U}^{(1)}$ and (8), we obtain that both terms are zero. A similar procedure can be used to check that all the other contributions to the first order correction are zero.

We now calculate the second order terms, starting by the ones that only involve products of $\hat{U}^{(1)}$ and $\hat{U}^{\dagger(1)}$. We derive here only the coefficient of $\frac{|0\rangle\langle1|}{2}$. The other cases follow analogously.

$$\mathrm{Tr}\left(\hat{U}^{(1)}e^{-\mathrm{i}\mu\hat{H}_0}\hat{\rho}_\beta\hat{U}^{\dagger(1)}e^{\mathrm{i}\mu\hat{H}_0}\right) = \mathrm{Tr}\left(e^{\mathrm{i}\mu\hat{H}_0}\hat{U}^{(1)}e^{-\mathrm{i}\mu\hat{H}_0}\hat{\rho}_\beta\hat{U}^{\dagger(1)}\right). \tag{11}$$

We have that

$$\begin{aligned} e^{\mathrm{i}\mu\hat{H}_0}\hat{U}^{(1)}e^{-\mathrm{i}\mu\hat{H}_0} &= -\mathrm{i}\lambda\int_{-\infty}^{\infty}\mathrm{d}t\int\mathrm{d}^3\boldsymbol{x}\chi(t)F(\boldsymbol{x})e^{\mathrm{i}\mu\hat{H}_0}\hat{\phi}(\boldsymbol{x},t)e^{-\mathrm{i}\mu\hat{H}_0} \\ &= -\mathrm{i}\lambda\int_{-\infty}^{\infty}\mathrm{d}t\int\mathrm{d}^3\boldsymbol{x}\chi(t)F(\boldsymbol{x})\hat{\phi}(\boldsymbol{x},t+\mu). \end{aligned} \tag{12}$$

So (11) equals

$$\begin{aligned} &\lambda^2\int\mathrm{d}t\int\mathrm{d}t'\int\mathrm{d}^3\boldsymbol{x}\int\mathrm{d}^3\boldsymbol{x}'\chi(t)\chi(t')F(\boldsymbol{x})F(\boldsymbol{x}')\,\mathrm{Tr}\left(\hat{\phi}(\boldsymbol{x},t+\mu)\hat{\rho}_\beta\hat{\phi}(\boldsymbol{x}',t'+\mu)\right) \\ &\quad= \lambda^2\int\mathrm{d}t\int\mathrm{d}t'\int\mathrm{d}^3\boldsymbol{x}\int\mathrm{d}^3\boldsymbol{x}'\chi(t)\chi(t')F(\boldsymbol{x})F(\boldsymbol{x}')\mathcal{W}_\beta(\boldsymbol{x}',t',\boldsymbol{x},t+\mu). \end{aligned} \tag{13}$$

The other terms have the same structure, with the only change being in the thermal Wightman function, which is $\mathcal{W}(x',t',x,t)$ for the diagonal terms, and $\mathcal{W}(x',t',x,t-\mu)$ for the $\frac{|1\rangle\langle0|}{2}$ term.

Let us obtain now the second order terms coming from products with $\hat{U}^{(2)}$ and $\hat{U}^{\dagger(2)}$. It is easy to see, using (9) that all the terms are equal to

$$\mathrm{Tr}\left(\hat{U}^{(2)}\hat{\rho}_\beta\right) + \mathrm{Tr}\left(\hat{\rho}_\beta\hat{U}^{\dagger(2)}\right) = 2\,\mathrm{Re}\,\mathrm{Tr}\left(\hat{U}^{(2)}\hat{\rho}_\beta\right). \tag{14}$$

This finishes the proof. Since the Dyson expansion preserves the trace of the density matrix, Eq. (13) has to be equal to Eq. (14), so as to cancel the diagonals added by the perturbation terms. This is useful because it gives a much more compact expression for the reduced state of the qubit. Therefore, at the end of the Ramsey scheme, the density matrix of the qubit is:

$$\frac{1}{2}\Bigg(\mathbb{I}+|+\rangle\langle-|\left(1+\lambda^2\int \mathrm{d}t\int \mathrm{d}t'\int \mathrm{d}^3\boldsymbol{x}\int \mathrm{d}^3\boldsymbol{x}'\chi(t)\chi(t')F(\boldsymbol{x})F(\boldsymbol{x}')\left(\mathcal{W}_\beta(\boldsymbol{x}',t',\boldsymbol{x},t+\mu)-\mathcal{W}_\beta(\boldsymbol{x}',t',\boldsymbol{x},t)\right)\right) \tag{15}$$
$$+|-\rangle\langle+|\left(1+\lambda^2\left(\int \mathrm{d}t\int \mathrm{d}t'\int \mathrm{d}^3\boldsymbol{x}\int \mathrm{d}^3\boldsymbol{x}'\chi(t)\chi(t')F(\boldsymbol{x})F(\boldsymbol{x}')\left(\mathcal{W}_\beta(\boldsymbol{x}',t',\boldsymbol{x},t-\mu)-\mathcal{W}_\beta(\boldsymbol{x}',t',\boldsymbol{x},t)\right)\right)\right)\Bigg).$$

Using the expression of the Wightman function for a thermal state of inverse temperature β [1]

$$\mathcal{W}_\beta(\boldsymbol{x}',t',\boldsymbol{x},t)=\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega_{\boldsymbol{k}}\left(e^{\beta\omega_{\boldsymbol{k}}}-1\right)}\left(e^{\beta\omega_{\boldsymbol{k}}}e^{\mathrm{i}k(\boldsymbol{x}-\boldsymbol{x}')}+e^{\mathrm{i}k(\boldsymbol{x}'-\boldsymbol{x})}\right), \tag{16}$$

we can calculate the characteristic function of the work distribution

$$\widetilde{P}(\mu)=1+\lambda^2\int\frac{d^3\boldsymbol{k}}{(2\pi)^3 2\omega_{\boldsymbol{k}}\left(e^{\beta\omega_{\boldsymbol{k}}}-1\right)}|\widetilde{\chi}(\omega_{\boldsymbol{k}})|^2|\widetilde{F}(\boldsymbol{k})|^2\left(e^{\beta\omega_{\boldsymbol{k}}}+1\right)\left(\cos(\mu\omega_{\boldsymbol{k}})-1\right) \tag{17}$$
$$+\mathrm{i}\lambda^2\int\frac{d^3\boldsymbol{k}}{(2\pi)^3 2\omega_{\boldsymbol{k}}}|\widetilde{\chi}(\omega_{\boldsymbol{k}})|^2|\widetilde{F}(\boldsymbol{k})|^2\sin(\mu\omega_{\boldsymbol{k}}).$$

C. Delta-coupling to the vacuum

Let us start by defining a general coherent state of the field $|\alpha(\mathbf{k})\rangle$:

$$|\alpha(\mathbf{k})\rangle=\hat{D}_{\alpha_{(\mathbf{k})}}|\Omega\rangle=\exp\left(\int d^3\mathbf{k}[\alpha(\mathbf{k})\hat{a}^\dagger_{\mathbf{k}}-\alpha^*(\mathbf{k})\hat{a}_{\mathbf{k}}]\right)|\Omega\rangle. \tag{18}$$

When the interaction Hamiltonian couples to the field through a delta switching function, the resulting unitary is

$$\hat{U}_\phi(T)=\exp\left(-i\lambda\int d^3\mathbf{x}F(\mathbf{x})\hat{\phi}(\mathbf{x})\right)=\hat{D}_{\alpha_{(\mathbf{k})}}, \tag{19}$$

with $\alpha_{(\mathbf{k})}=\frac{-i\lambda\widetilde{F}^*(\mathbf{k})}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}}$. This can be seen writing the field operator in its mode decomposition. We now calculate the reduced state of the qubit before applying the last Hadamard gate of the Ramsey scheme. The diagonal elements of the density matrix of the qubit are both $\frac{1}{2}$. The off-diagonal terms are $\frac{1}{2}\langle\Omega|\hat{U}^\dagger_\phi(T)e^{-i\mu\hat{H}_0}\hat{U}_\phi(T)|\Omega\rangle$ and the Hermitian conjugate. Using that $e^{-i\mu\hat{H}_0}|\Omega\rangle=|\Omega\rangle$, we can rewrite the previous expression as

$$\langle\Omega|\hat{U}^\dagger_\phi(T)e^{-i\mu\hat{H}_0}\hat{U}_\phi(T)e^{i\mu\hat{H}_0}|\Omega\rangle=\langle\alpha(\mathbf{k})|\beta(\mathbf{k})\rangle, \tag{20}$$

with $\beta(\mathbf{k})=\frac{-i\lambda\widetilde{F}^*\mathbf{k}e^{-i\omega_{\mathbf{k}}\mu}}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}}$, as can be seen by using that $e^{-i\mu\hat{H}_0}\hat{a}_{\boldsymbol{k}}e^{i\mu\hat{H}_0}=e^{i\mu\omega_{\boldsymbol{k}}}\hat{a}_{\boldsymbol{k}}$, and $e^{-i\mu\hat{H}_0}\hat{a}^\dagger_{\boldsymbol{k}}e^{i\mu\hat{H}_0}=e^{-i\mu\omega_{\boldsymbol{k}}}\hat{a}^\dagger_{\boldsymbol{k}}$. Using the expression for the inner product of two coherent states in Appendix A of [2], (20) can be simplified to

$$\langle\alpha(\mathbf{k})|\beta(\mathbf{k})\rangle=\exp\left[\lambda^2\int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}|\widetilde{F}(\mathbf{k})|^2(e^{-i\mu\omega_{\mathbf{k}}}-1)\right]. \tag{21}$$

Applying the second Hadamard to the qubit and taking the Z and Y components of the Bloch vector of the qubit finally yields

$$\widetilde{P}(\mu)=\exp\left[\lambda^2\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}|\widetilde{F}(\mathbf{k})|^2(e^{i\mu\omega_{\mathbf{k}}}-1)\right]. \tag{22}$$

Choosing a normalized spherical Gaussian centered at zero as smearing $F(r)=\frac{e^{-\frac{r^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$, and changing to polar coordinates (since the smearing is spherically symmetrical), yields for the characteristic function of a massless scalar field ($\omega_{\boldsymbol{k}}=|\boldsymbol{k}|$)

$$\widetilde{P}(\mu)=\exp\left[\frac{\lambda^2}{4\pi^2}\int_0^\infty \mathrm{d}|\boldsymbol{k}||\widetilde{F}(|\boldsymbol{k}|)|^2|\boldsymbol{k}|(e^{i\mu|\boldsymbol{k}|}-1)\right]. \tag{23}$$

Using that

$$\int_0^\infty \mathrm{d}|\boldsymbol{k}||\widetilde{F}(|\boldsymbol{k}|)|^2|\boldsymbol{k}| = \frac{1}{2\sigma^2}, \tag{24}$$

and

$$\int_0^\infty \mathrm{d}|\boldsymbol{k}||\widetilde{F}(|\boldsymbol{k})|^2|\boldsymbol{k}|e^{i\mu|\boldsymbol{k}} = \exp\left(\frac{e^{-\frac{\mu^2}{4\sigma^2}}\left(2e^{\frac{\mu^2}{4\sigma^2}}\mu\sigma\mathcal{D}\left(\frac{\mu}{2\sigma}\right) - 2e^{\frac{\mu^2}{4\sigma^2}}\sigma^2 - i\sqrt{\pi}\mu\sigma\right)}{4\sigma^4}\right), \tag{25}$$

yields for the characteristic function

$$\widetilde{P}(\mu) = \exp\left(-\lambda^2\frac{e^{-\frac{\mu^2}{4\sigma^2}}\left(2e^{\frac{\mu^2}{4\sigma^2}}\mu\sigma\mathcal{D}\left(\frac{\mu}{2\sigma}\right) - 2e^{\frac{\mu^2}{4\sigma^2}}\sigma^2 - i\sqrt{\pi}\mu\sigma\right)}{(4\pi^2)\,4\sigma^4} - \frac{\lambda^2}{8\pi^2\sigma^2}\right), \tag{26}$$

where $\mathcal{D}(x)$ is the Dawson Function, defined as $\mathcal{D}(x) = e^{-x^2}\int_0^x \mathrm{d}y e^{y^2}$.

[1] P. Simidzija and E. Martín-Martínez, Phys. Rev. D **98**, 085007 (2018).
[2] P. Simidzija and E. Martín-Martínez, Phys. Rev. D **96**, 065008 (2017).