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SUPERSYMMETRY AND THE QUANTUM MECHANICS OF SPIN

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Bohr–Sommerfeld quantization is exact for a spin in a magnetic field. This well-known result is connected with the existence of a hidden supersymmetry. I discuss the supersymmetry and, in a broader context, the semiclassical quantum-mechanical interpretation of the Weyl character formula for compact, semisimple Lie groups.

1. Introduction

There are a number of well-known cases in which semiclassical, Bohr–Sommerfeld, quantization gives exact results for the energies of quantum-mechanical systems. These are usually systems, such as the hydrogen atom or the quantum mechanics of a particle moving on a group manifold, where there is a high degree of symmetry. The quantum mechanics of a spin precessing in a magnetic field is another example where semiclassical quantization gives the exact answer for the spectrum, and indeed the wave functions [1–3]. The purpose of this paper is to investigate this simple situation. We will discover that spin has an amusing supersymmetric extension which is connected with the exactness of the semiclassical result. The supersymmetry generalizes to an arbitrary compact, semisimple, Lie group and the result for SU(2) is seen to be a special case of the Atiyah and Bott proof [4] of the Weyl character formula.

In the course of our investigation we will take the familiar quantum mechanics of angular momentum and make it seem rather complicated, not out of perversity, but in order to illustrate some interesting mathematics. These mathematical concepts are useful in working with quantum many-body systems and the present exposition was prompted by the recent work by Wiegmann [5] and Polyakov [6]. They investigated the spin and statistics of topological excitations in the spin degrees of freedom in two-dimensional Heisenberg antiferromagnets, systems relevant for the study of high-T_c superconductors. Polyakov and Wiegmann used path-integral formulations for the quantum mechanics of spin similar to those discussed here. Indeed,

Wiegmann [5] discusses a supersymmetric extension of the coherent-state path integral for spin but it seems to be a different supersymmetry from that that I will discuss.

The outline of this paper is as follows. Sect. 2 sets the scene for the problem to be investigated and makes a connection between the exactness of the semiclassical spectrum for a spinning particle and the Weyl character formula. Sects. 3, 4 and 5 are an introduction to some basic geometric ideas needed for the rest. Sect. 3 describes the classical mechanics of a spin, including the Duistermaat–Heckman result on the classical partition function, sect. 4 discusses the fibre-bundle approach to the monopole spherical harmonics which are the wave functions of the spin, while sect. 5 reviews the conventional and coherent-state path integrals for a spin and points out the privileged position of coherent states constructed from highest weight vectors. Sect. 6 then uses the Kähler condition for the construction of $N=2$ supersymmetry and makes a connection between coherent states and the kernel of the twisted Dolbeault complex. Sect. 7 applies the G-index theorem to the Dolbeault complex and, using a method due to Witten, deforms it to yield the Weyl character formula.

2. Partition functions and characters

The energy eigenvalues of a spin-J particle in a B field are $-J|B|\ldots+J|B|$. These are exactly the energies allowed by Bohr–Sommerfeld quantization and our problem is to try and understand why this is so.

A convenient way of encoding spectral information is the partition function

$$\mathscr{Z}=\mathrm{Tr}(\mathrm{e}^{-\beta BJ_3})_{\text{spin } J}=\mathrm{e}^{\beta BJ}+\mathrm{e}^{\beta B(J-1)}+\ldots+\mathrm{e}^{-\beta BJ}. \tag{2.1}$$

Equivalently, we replace βB by $i\theta$ and obtain the character of the spin-J representation of the rotation group

$$\begin{aligned}\mathrm{e}^{i\theta J}+\mathrm{e}^{i\theta(J-1)}+\ldots+\mathrm{e}^{-i\theta J}&=\chi_J(\theta)\\&=\frac{\sin((2J+1)\theta/2)}{\sin\theta/2}\\&=\frac{\mathrm{e}^{iJ\theta}}{1-\mathrm{e}^{-i\theta}}+\frac{\mathrm{e}^{-iJ\theta}}{1-\mathrm{e}^{i\theta}}.\end{aligned} \tag{2.2}$$

The last line of eq. (2.2) is a special case of the Weyl character formula for Lie groups [4]. This formula writes the character as a sum of terms, one for each element of the Weyl group, the group of symmetries of the roots of the Lie algebra. For

SU(2) the Weyl group has two elements, the identity and the map which takes J_3 to $-J_3$. Rewriting the partition function in a similar form

$$\begin{aligned}\mathscr{Z} &= e^{\beta BJ}(1 + e^{-\beta B} + e^{-2\beta B} + \dots) \\ &\quad - e^{-\beta B(J+1)}(1 + e^{-\beta B} + e^{-2\beta B} + \dots) \\ &= \frac{e^{\beta BJ}}{1 - e^{-\beta B}} + \frac{e^{-\beta BJ}}{1 - e^{\beta B}} \end{aligned} \tag{2.3}$$

also expresses it as the sum of two terms, each associated with one of the poles of the sphere. The classical motion of the tip of the spin vector describes circles around the poles in the same way that the phase-space trajectories of a harmonic oscillator form circles about the origin. Each term is the Boltzmann weight for the spin to point to a pole, times the partition function of a harmonic oscillator with the same period as the Larmor precession. Most of the terms cancel between the sums. This suggests that the semiclassical expansion is exact because the system is somehow a set of harmonic oscillators. We will see that this is so because of the hidden supersymmetry of the problem.

3. Classical mechanics of spin

The quantum mechanics of a spin-$\frac{1}{2}$ particle is a standard topic in introductory courses – yet we hardly ever mention the *classical* mechanics of a spin*. There is a reason for omission of this apparently easy problem. The examples discussed in first courses on hamiltonian mechanics all have canonical variables which naturally fall into the category of p's or q's. Spin is the simplest example of a hamiltonian system where the elementary version taught to physicists does not apply [7, 8].

Since the only degree of freedom for a spin is its direction, the configuration space must be the two sphere S^2. There are two separate but related problems whose configuration space is S^2. The first is the motion of a rigid rod about its centre of mass with an action integral

$$S = \int \left(\tfrac{1}{2} I \dot{\boldsymbol{n}}^2 + \boldsymbol{B} \cdot \boldsymbol{n}\right) \mathrm{d}t . \tag{3.1}$$

* By "spin" I mean a system with classical variables J_i $(i = 1, 2, 3)$ whose Poisson brackets obey $\{J_i, J_j\} = \varepsilon_{ijk} J_k$. I do *not* mean the dynamics of a top. A top has six canonical variables – three configuration space q's which form the group manifold $SO(3) \cong S^3$ (locally) and three more p's which go together with these to make up a phase space which is the cotangent bundle $T^*(S^3)$. The Poisson brackets for the top form a doubled version of the spin algebra, with variables corresponding to generators for the space and body fixed rotations. Further, the top has degrees of freedom corresponding to rotation about its axis and the magnitude of the angular momentum can change. The spin has a *fixed* angular momentum and only its *direction* can change.

Here I is the moment of inertia, and $\boldsymbol{n}$ is a unit vector, the direction of the axis of the rod. The field $\boldsymbol{B}$ tends to align the rod with the field axis. This model would be an appropriate description of the dynamics of a diatomic molecule and is clearly equivalent to the motion of a particle of mass I constrained to lie on the surface of the sphere. The phase space for this system is four-dimensional being $T^*(S^2)$.

The spin, with fixed angular momentum J about its axis, on the other hand has an action which is first-order in time derivatives

$$S = J\int(\dot{\boldsymbol{n}}\cdot\boldsymbol{A} + \boldsymbol{n}\cdot\boldsymbol{B})\,\mathrm{d}t. \tag{3.2}$$

Here $\boldsymbol{A}(n)$ is the vector potential for a fictitious monopole located at the centre of the sphere, e.g. in polar coordinates $A^{\phi} = (1-\cos\theta)$, $A^{\theta} = 0$. It is the balance between the Lorentz force on the particle, due to the fictitious magnetic field of the monopole, and the potential force on the moment, due to the real magnetic field $\boldsymbol{B}$, that leads to the characteristic Larmor precession about the direction of the $\boldsymbol{B}$ field.

If we set $B = 0$, the action (3.2) does not, at first sight, have the rotational invariance expected. The choice of the z-axis as the location of the Dirac string singularity in the gauge field destroys the spherical symmetry. We can write the action more symmetrically as

$$S = J\int_{\Gamma}\boldsymbol{n}\cdot\boldsymbol{B}\,\mathrm{d}t + J(\text{flux through } \Gamma), \tag{3.3}$$

or

$$S = \int_{\Gamma}J\boldsymbol{n}\cdot\boldsymbol{B}\,\mathrm{d}t + J\int_{\Omega}\boldsymbol{n}\cdot(\partial_t\boldsymbol{n}\wedge\partial_\tau\boldsymbol{n})\,\mathrm{d}t\,\mathrm{d}\tau, \tag{3.4}$$

where Ω is the region bounded by Γ. The last term is J multiplied by the area of the region bounded by Γ and is an example of what has come to be called a Wess–Zumino term. We gain the manifest rotational invariance at the apparent expense of having $\boldsymbol{n}$ depend on the two coordinates t, τ which parametrize the area, when we only expect dependence on the physical t. In fact the integral does not depend on the way we extend the function $\boldsymbol{n}(t)$ to $\boldsymbol{n}(t,\tau)$. In particular the variation of this Wess–Zumino term in the action depends only on the values $\boldsymbol{n}$ takes on the boundary Γ

$$\delta\int_{\Omega}\boldsymbol{n}\cdot(\partial_t\boldsymbol{n}\wedge\partial_\tau\boldsymbol{n})\,\mathrm{d}t\,\mathrm{d}\tau = \int_{\Gamma}\boldsymbol{n}\cdot(\delta\boldsymbol{n}\wedge\partial_t\boldsymbol{n})\,\mathrm{d}t. \tag{3.5}$$

There is a genuine ambiguity, however a closed curve Γ on S^2 bounds two distinct surfaces. These surfaces differ in (directed) area by 4π so the ambiguity in the Wess–Zumino action is $4\pi J$. This is a constant and does not effect the classical

equations of motion but, in a path-integral approach to quantum mechanics, this term will appear in the exponent and requiring it to have no effect forces $2J$ to take integer values. This is a common feature: the coefficients of Wess–Zumino terms must be quantized in order for the path integral to be single-valued.

From the variation of the action with respect to $\boldsymbol{n}$ (and preserving the $\boldsymbol{n}^2 = 1$ constraint by restricting ourselves to variations perpendicular to $\boldsymbol{n}$) we find the classical equation for a precessing gyroscope

$$\partial_t \boldsymbol{n} = -(\boldsymbol{n} \wedge \boldsymbol{B}). \tag{3.6}$$

To show that we get the correct commutation relations, or Poisson brackets, we must consider the hamiltonian formalism. Consider a general action which contains $\dot{q}$'s only to the first power

$$L = a(q)_\mu \dot{q}_\mu - E(q). \tag{3.7}$$

From the lagrangian we read off the canonical momenta $p \equiv a(q)$. The p's are just some function of the q's instead of being combinations of the $\dot{q}$'s as usual. In this case the configuration space and the phase space coincide.

To obtain the Poisson brackets we need the abstract formulation of hamiltonian mechanics in terms of differential forms [9]. Phase space is regarded as a symplectic manifold with a two-form ω that is closed i.e. $\mathrm{d}\omega = 0$. Locally we can find coordinates p_i and q_i on the manifold in terms of which ω can be written as $\omega = \sum \mathrm{d}p \wedge \mathrm{d}q$, the exterior derivative of the one-form $\sum p\,\mathrm{d}q$. Unless ω is integrable i.e there exists a globally defined form η such that $\omega = \mathrm{d}\eta$, we can only find such coordinates in patches. In one such patch the velocity in phase space is $\dot{x} = (\dot{p}, \dot{q})$ and Hamilton's equations are written as

$$\mathrm{d}H = \omega(\,, \dot{x}) = -\iota_{\dot{x}}\omega. \tag{3.8}$$

where the symbol $\iota_{\dot{x}}$ is the operation of taking the interior product with $\dot{x}$, that is of substituting $\dot{x}$ in the first slot of the two-form ω. Both sides of the equation are one-forms and taking components in the (p, q) coordinate system gives the equations in the customary form.

To find ω from the lagrangian use $p = a(q)$ and $\omega = \mathrm{d}(\sum p\,\mathrm{d}q)$ to get

$$\omega = \tfrac{1}{2}\omega_{\mu\nu}\,\mathrm{d}q^\mu \wedge \mathrm{d}q^\nu = \tfrac{1}{2}\left(\partial_\mu a_\nu - \partial_\nu a_\mu\right)\mathrm{d}q^\mu \wedge \mathrm{d}q^\nu. \tag{3.9}$$

For the spin system the symplectic two-form ω is J times the area two-form on S^2. There is no globally defined one-form whose exterior derivative is equal to the area form, i.e. it is a generator of the cohomology group $H^2(S^2, R)$, so we cannot make a global decomposition into p's and q's.

Poisson brackets are defined via $\omega^{\mu\nu}$ the inverse matrix to $\omega_{\mu\nu}$ [8,10], by

$$\{f, g\} = \omega^{\mu\nu} \frac{\partial f}{\partial q^{\mu}} \frac{\partial g}{\partial q^{\nu}} . \tag{3.10}$$

We find

$$\{n_1, n_2\} = \frac{1}{J} n_3 \quad \text{etc.} \tag{3.11}$$

and, using the angular momentum $J_i \equiv J n_i$, we obtain the classical Poisson bracket equivalent of the familiar angular-momentum commutation relations

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k . \tag{3.12}$$

We can also find the semiclassical spectrum. The Bohr–Sommerfeld condition, $\oint p \, \mathrm{d}q = 2\pi n\hbar$, requires the orbits to enclose an area of $J/2\pi\hbar$ times an integer, and by Archimedes' famous result on the area of spherical caps, this leads to the quantization of J_3 in units of $\hbar$ in the familiar manner: $J - J_3 = n\hbar$, $J \geqslant J_3 \geqslant -J$.

We can, of course, form a hybrid of the two problems: a rigid rotator with fixed angular momentum J about its axis. For this case

$$S = \int \left(\tfrac{1}{2} I \dot{\boldsymbol{n}}^2 + J(\dot{\boldsymbol{n}} \cdot \boldsymbol{A} + \boldsymbol{B} \cdot \boldsymbol{n}) \right) \mathrm{d}t . \tag{3.13}$$

This problem is appropriate for the motion of an excited diatomic molecule where the electrons are in a state with angular momentum J about the axis joining the nuclei (J is usually called Λ in this case) and is isomorphic to the motion of a massive charged particle on the surface of a sphere with a magnetic monopole at the centre.

Before we discuss quantum mechanics consider, for a moment, the *classical* partition function for the spin in the B field. We want to evaluate

$$\mathscr{Z}_{\text{classical}} = \int_{\text{phase space}} \mathrm{d}p \, \mathrm{d}q \, \mathrm{e}^{-\beta J \boldsymbol{B} \cdot \boldsymbol{n}} . \tag{3.14}$$

For a $2N$-dimensional phase space the Liouville measure, $\mathrm{d}p \, \mathrm{d}q$, is $\omega^N/N!$. In our case this is just the J times the area on S^2. So

$$\mathscr{Z} = J \int \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{e}^{-\beta |B| J \cos\theta} = \frac{2\pi}{\beta |B|} \left(\mathrm{e}^{\beta |B| J} - \mathrm{e}^{-\beta |B| J} \right) . \tag{3.15}$$

Like the quantum version, the classical partition function is given in terms of the two points at which the hamiltonian is stationary. By Hamilton's equations these are

the fixed points of the classical motion in phase space. This is an example of a general principle. In a recent paper Atiyah [11] has drawn attention to a beautiful theorem due to Duistermaat and Heckman [12], set in the context of classical mechanics. This theorem gives circumstances in which the steepest descent, or stationary-phase, approximation to a phase-space integral is exact. We have in eq. (3.15) the simplest non-trivial example of this Duistermaat–Heckman theorem. The theorem states that when we have a classical system, all of whose trajectories have a common period – a characteristic of both harmonic oscillators and Larmor precession, then the classical partition function is given exactly by the method of stationary phase. This is trivial for the oscillator whose hamiltonian is a quadratic function, but it required a conspiracy between the phase-space volume and the energy in order to work for the spin.

There is generalisation of the formula for the spin partition function, due originally to Harish–Chandra [13], and rediscovered, for the case of the group SU(N), by Itzykson and Zuber [14]*. In the form due to Itzykson and Zuber the generalization reads

$$\int \mathrm{d}[U]\exp\bigl(-\beta\,\mathrm{Tr}\bigl(U\Lambda_1 U^{-1}\Lambda_2\bigr)\bigr)$$

$$= \prod_{p=1}^{N-1}(p!)(-\beta)^{N(N-1)/2}\frac{\det\bigl(\exp\bigl(-\beta\lambda_i^{(1)}\lambda_j^{(2)}\bigr)\bigr)}{\Delta(\Lambda_1)\Delta(\Lambda_2)}. \tag{3.16}$$

Here $\mathrm{d}[U]$ is the Haar measure on SU(N) and Λ_1, Λ_2 are diagonal matrices with entries $\lambda_i^{(1)}, \lambda_j^{(2)}$, respectively. $\Delta(\Lambda)$ is the Vandermonde determinant of these entries. It is easy to convince oneself that the expansion of the determinant on the right-hand side of eq. (3.16) consists of sums of terms corresponding to the stationary points of the exponent in the integrand. This is the stationary-phase approximation to the integral and the "approximation" is thus seen to be exact. (Eq. (3.15) is a special case of eq. (3.16) where $N = 2$ and $\Lambda_{1,2}$ are both diagonal matrices proportional to σ_3.)

At first sight, eq. (3.16) does not seem like a phase-space integral since the dimension of the group manifold of SU(N) can be odd dimensional while phase-space integrals are even dimensional – but the integrand is unchanged if the U's are multiplied on the right by diagonal matrices so the integration is really over the coset space G/H where H is the subgroup of diagonal unitary matrices. (The extra integration over this subgroup just provides part of the numerical factor.) The dimension of the coset space is $N^2 - N$ which is even and eq. (3.16) is indeed an example of the Duistermaat–Heckman theorem. This result for the *classical* partition function of the generalized spin problem hints that we may also generalise the

* I am grateful to V.A. Kazakov for pointing out this reference to me.

quantum spin problem in the same manner and still find exactness of the Bohr–Sommerfeld quantization. We will need the coset space G/H to be the phase space of the quantum system, and the dynamics to provide an analog of Larmor precession. The motion of the spin in a B field pointing in the z-direction is given by the action of $\exp(iBJ_3)$ on the two-sphere, so we generalise by taking the dynamics be the action of the group H on the coset space.

A proof of the Duistermaat–Heckman theorem by Berline and Vergne [15] shows the close connection between this result and supersymmetry, while the paper of Atiyah [11] shows that an extension of the theorem to the loop space of a manifold provides a basis for understanding the exactness of the semiclassical approximation for the Dirac index, and thus provides a natural geometric setting for supersymmetry in the lagrangian formalism. It is interesting to speculate on a connection between the classical version of the theorem as it appears in this section and the exactness of the semiclassical approximation for the quantum mechanics of spin. We will see that a supersymmetric extension of the quantum mechanics on G/H provides a route to understanding this exactness.

4. Quantum mechanics of spin

We have seen that the motion of a precessing spin reduces to that of a charged particle around a monopole. The familiar approach to the quantum mechanics of this problem involves selecting a gauge for the magnetic field, which in the monopole case is necessarily singular at the Dirac string located somewhere on the sphere, and then solving for a gauge-dependent wave function. A more powerful procedure, necessary here, is to use the language of fibre bundles and work with all gauges *at once*. In this case the wave function is defined as a *section* of a bundle associated with a principal fibre bundle. Because this method is perhaps unfamiliar, I will devote this section to an exposition of the concepts involved. This account closely follows that in ref. [16].

A principal G-bundle comprises a manifold P called the *total space* together with a projection π to a *base space* B such that the pre-image of every point $x \in \mathrm{B}$, called a *fibre*, is a copy of the group G

$$\pi : \mathrm{P} \to \mathrm{B}, \quad \pi^{-1}(x) = \mathrm{G}. \tag{4.1}$$

Locally P is required to be a product $\mathrm{P} \equiv \mathrm{B} \times \mathrm{G}$ and we can, again locally, parameterise P by ordered pairs (x, g) where $x \in \mathrm{B}$ and $g \in \mathrm{G}$. To define a bundle *associated* with P it is simplest to define the *sections* of the associated bundle. Let $\varphi_i(x, g)$ be a function on the total space P with a set of indices i carrying a representation $\rho(g)$ of G. We say that $\varphi_i(x, g)$ is a section of an associated bundle if it varies in a particular way as we run up and down the fibres, by acting on them from the right with elements of G. We require

$$\varphi_i(x, gh) = \rho_{ij}^{-1}(h)\varphi_j(x, g). \tag{4.2}$$

These functions will be the wave functions of a particle moving on the base space. The representation ρ plays the rôle of the charge and eq. (4.2) are the gauge transformations. To describe the gauge field in which the particle moves we must introduce the notion of a connexion on the bundle. A connexion is a decomposition of the tangent space $T_p(P)$ of P at $p \in P$ into a *horizontal subspace*, $H_p(P)$, and a *vertical subspace*, $V_p(P)$. We require that $V_p(P)$ be the tangent space to the fibres but $H_p(P)$ can be any complementary subspace, i.e the direct sum should be the whole tangent space

$$T_p(P) = H_p(P) \oplus V_p(P). \tag{4.3}$$

The horizontal subspaces must all be images of each other under the action of G on the fibres from the right.

Given a curve $x(t)$ in the base space we can lift it, by solving the equation

$$\dot{g} + \frac{\partial x^\mu}{\partial t} A_\mu g = 0, \tag{4.4}$$

to a curve $(x(t), g(t))$ in the total space, whose tangent is everywhere horizontal.

The A_μ are a set of functions, depending on the choice of horizontal subspace, such that $(dx, -A_\mu dx^\mu g)$ is horizontal for each small displacement dx in the tangent space to B. The directional derivative along the lifted curve is

$$\dot{x}^\mu D_\mu = \dot{x}^\mu \left(\left(\frac{\partial}{\partial x^\mu} \right)_g - A_\mu^a \bar{L}_a \right). \tag{4.5}$$

In eq. (4.5) $\bar{L}_a$ is a right-invariant vector field on G, a differential operator on functions on G. The D_μ are a set of vector fields, the *covariant derivatives*, which span the horizontal subspace at each point. They satisfy the relation

$$[D_\mu, D_\nu] = -F_{\mu\nu}^a \bar{L}_a. \tag{4.6}$$

The curvature, F, is given in terms of the structure constants $[\lambda_a, \lambda_b] = f_{ab}^c \lambda_c$ by

$$F_{\mu\nu}^c = \partial_\mu A_\mu^c - \partial_\nu A_\mu^c + f_{ab}^c A_\mu^a A_\nu^b, \tag{4.7}$$

and the commutator of the covariant derivatives is proportional to $\bar{L}$ and therefore lies entirely in the vertical subspace.

We can make contact with the more familiar definitions of covariant derivatives by remembering that right-invariant vector fields are derivatives that involve multiplication from the *left*

$$\bar{L}_a \varphi_i(x, g) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\varphi_i(x, (1 + \varepsilon \lambda_a) g) - \varphi_i(x, g) \right). \tag{4.8}$$

So, for a section of the associated bundle

$$\overline{L}_a\varphi_i(x,g) = \lim_{\varepsilon\to 0}\left(\rho_{ij}\left(g^{-1}(1-\varepsilon\lambda_a)g\right)\varphi_j(x,g) - \varphi_i(x,g)\right)/\varepsilon$$

$$= -\rho_{ij}(g^{-1})(\lambda_a)_{jk}\rho_{kl}(g)\varphi_l(x,g) = \left(\left(-g^{-1}\lambda_a g\right)\varphi\right)_i, \quad (4.9)$$

$$D_\mu\varphi = \left(\partial_\mu\varphi\right)_g + g^{-1}A_\mu g\varphi. \quad (4.10)$$

This still does not look too familiar because the derivatives with respect to x_μ are being taken at *fixed* g. We normally *fix a gauge* by making a choice of g for each x_μ. The usual $\varphi(x)$ is then $\varphi(x, g(x))$. So, using $\varphi(x, g(x)) = g^{-1}\varphi(x,1)$, we have

$$\partial_\mu\varphi = \left(\partial_\mu\varphi\right)_g + \left(\partial_\mu g^{-1}\right)g\varphi = \left(\partial_\mu\varphi\right)_g - \left(g^{-1}\partial_\mu g\right)\varphi. \quad (4.11)$$

From this we get a derivative on functions on the base space, B, which is the usual covariant derivative, now containing gauge fields which are gauge transformations of a fixed A

$$\nabla_\mu = \partial_\mu + \left(g^{-1}A_\mu g + g^{-1}\partial_\mu g\right) = \partial_\mu + \omega_\mu. \quad (4.12)$$

These covariant derivatives on the base space obey

$$\left[\nabla_\mu, \nabla_\nu\right] = g^{-1}F_{\mu\nu}g = \Omega_{\mu\nu}, \quad (4.13)$$

with

$$\Omega = \mathrm{d}\omega + \omega\wedge\omega. \quad (4.14)$$

While these operations and definitions seem rather abstract, we see their power when we construct the monopole bundle and its associated wave functions. Consider the representation matrices $D^{(J)}_{mn}(g)$, where $g\in$ G could be an element of any compact Lie group but for concreteness we will think of G = SU(2). The $D^{(J)}_{mn}(g)$ are functions on G which will be our total space. We will take a subgroup H as the gauge group, and the coset space G/H as the base space. The projection π is just the usual projection G → G/H. The functions $D^{(J)}(g)$ are not, in general, *functions* on the coset space G/H since they depend on the choice of representative. Instead, because of the representation property, they vary with the choice of representative in a well-defined way

$$D^{(J)}_{mn}(gh) = D^{(J)}_{mn'}(g)D^{(J)}_{n'n}(h). \quad (4.15)$$

Since we are dealing with compact groups the representations can be taken to be unitary and

$$\begin{aligned}\left[D^{(J)}_{mn}(gh)\right]^* &= \left[D^{(J)}_{mn'}(g)\right]^* \left[D^{(J)}_{n'n}(h)\right]^* \\ &= D^{(J)}_{nn'}(h^{-1})\left[D^{(J)}_{mn'}(g)\right]^* . \end{aligned} \tag{4.16}$$

This is the correct variation under the right action of the group H for the set of functions $[D^{(J)}_{mn}(gh)]^*$ to be thought of as sections of a bundle associated with the principal fibre bundle $G \to G/H$. The representation ρ is not necessarily that defined by the label (J) because irreducible representations of G may be reducible under H; ρ depends on what representation of H the index n belongs. If ρ is the identity representation then the functions are functions on G/H in the ordinary sense. For G = SU(2) and H the U(1) subgroup generated by J_3, the quotient space is just S^2 and projection is the Hopf map: $S^3 \to S^2$. Parameterising SU(2) with Euler angles

$$D^{(J)}_{mn}(\theta, \phi, \psi) = \langle J, m | e^{-i\phi J_3} e^{-i\theta J_2} e^{-i\psi J_3} | J, n \rangle , \tag{4.17}$$

shows that the Hopf map consists of simply forgetting about ψ, so Hopf: $[(\theta, \phi, \psi) \in S^3] \mapsto [(\theta, \phi) \in S^2]$. Taking $n = 0$ gives us functions independent of ψ and we obtain the well-known identification of the spherical harmonics with representation matrices

$$Y^L_m(\theta, \phi) = \sqrt{\frac{2L+1}{4\pi}} \left[D^{(L)}_{m,0}(\theta, \phi, 0)\right]^* . \tag{4.18}$$

For $n = \Lambda \neq 0$ we get sections of the bundle: these are the monopole harmonics

$$\mathscr{Y}^{(J)}_{m,\Lambda}(\theta, \phi, \psi) = \sqrt{\frac{2J+1}{4\pi}} \left[D^{(J)}_{m,\Lambda}(\theta, \phi, \psi)\right]^* . \tag{4.19}$$

The monopole harmonics have a non-trivial dependence on the choice we make for ψ at each point on S^2 and we cannot make a globally smooth choice; we always encounter a point where there is a string singularity. Apart from being useful for the study of monopoles, these functions occur in molecular physics, as discussed in sect. 3, as the wave functions of excited diatomic molecules (see, for example, the footnote on p. 317 of ref. [17]).

We should show that these harmonics are eigenfunctions of the Schrödinger operator, $-\nabla^2$, containing the gauge field connexion, in the same way that the spherical harmonics are eigenfunctions of the laplacian on the sphere. This is just a

geometrical exercise. Because they are irreducible representations, the $D^{(J)}(g)$ are automatically eigenfunctions of the quadratic Casimir operator

$$\left(J_1^2+J_2^2+J_3^2\right)D^{(J)}(g)=J(J+1)D^{(J)}(g). \tag{4.20}$$

The J_i can be either right- or left-invariant vector fields on G; the quadratic Casimir is the same second-order differential operator in either case, and it is a good guess that it is proportional to the laplacian on the group manifold. Taking a locally geodetic coordinate system (in which the connexion vanishes) confirms this: $J^2=-\nabla^2$ on the three-sphere. The operator in eq. (4.20) is not the laplacian we want, however. We need the ∇^2 on the two-sphere, G/H, including the connexion. This ∇^2 operator differs from that on the total space since it must contain only differential operators lying in the horizontal subspaces. There is a natural notion of orthogonality in the Lie group, deriving from the Killing form, and it is natural to choose the horizontal subspaces to be orthogonal to the fibres of G/H. Since multiplication on the right by the subgroup generated by J_3 moves one up and down the fibres, the orthogonal displacements are obtained by multiplication on the right by infinitesimal elements made by exponentiating J_1 and J_2. The desired ∇^2 is thus made out of the left-invariant vector fields (which act by multiplication on the right), J_1 and J_2 only. The wave operator must be $-\nabla^2=J_1^2+J_2^2=J^2-J_3^2$. Applying this to the $\mathscr{Y}_{m,\Lambda}^{(J)}$ we see that they are eigenfunctions of $-\nabla^2$ on S^2 with eigenvalues $J(J+1)-\Lambda^2$. The energy levels for our monopole problem therefore have energies

$$E_{J,m}=\frac{1}{2I}\left(J(J+1)-\Lambda^2\right)-|B|m. \tag{4.21}$$

For general G/H there will be a similar construction although we will, in general, have to adapt the basis of the representation of G to the required representations of H, else we will not automatically find irreducible representations of H, and thus not get eigenvectors of ∇^2.

5. Coherent states and path integrals for spin

The classical mechanics of a spinning molecule is that of a massive particle with charge Λ moving on the surface of a sphere surrounding a monopole. A conventional Feynman path-integral representation for the partition function of such a quantum mechanical system can be written as

$$\mathscr{Z}=\int \mathrm{d}[\boldsymbol{n}]\delta(\boldsymbol{n}^2-1)\exp\left(-\int_0^\beta \mathrm{d}t\left(\tfrac{1}{2}I\dot{\boldsymbol{n}}^2+i\Lambda\dot{\boldsymbol{n}}\cdot\boldsymbol{A}+\Lambda\boldsymbol{B}\cdot\boldsymbol{n}\right)\right). \tag{5.1}$$

Because of the ambiguity noticed in sect. 3, consistency requires 2Λ be an integer. In terms of the wave functions the partition function is given by a sum over representations J which contain a state of spin Λ

$$\mathscr{Z} = \int \sin\theta \, d\theta \, d\phi \sum_{J \geqslant \Lambda} \sum_{m=-J}^{+J} |D^{(J)}_{m\Lambda}(\theta, \phi, \psi)|^2 e^{-\beta((J(J+1)-\Lambda^2)/2I - m|B|)}$$

$$= \text{const.} \sum_{J \geqslant \Lambda} \sum_{m=-J}^{+J} e^{-\beta((J(J+1)-\Lambda^2)/2I - m|B|)}. \tag{5.2}$$

As I tends to zero all the states with J greater than Λ have their energy pushed up to infinity compared with the lowest state $J = \Lambda$ and, up to normalisation, $\mathscr{Z}$ goes over to the expected partition function for a spin $J = \Lambda$. The exponent in the path integral simultaneously becomes the classical action for the spin. This limit is rather singular, however; the conventional path integral depends on the quadratic $\dot{\boldsymbol{n}}^2$ term to define its measure and this quadratic piece plays a crucial rôle in deriving the Schrödinger equation. It is possible to go directly to a path integral for the pure spin, $I = 0$ case, by using coherent-state path integrals [1, 18].

We begin by defining the coherent states for a general group [19, 20]. Let $D(g)$ be an irreducible representation of the group G. Suppose $|0\rangle$ is some state in the representation space, and define $|g\rangle$ by

$$|g\rangle = D(g)|0\rangle. \tag{5.3}$$

Let $d[g]$ be the Haar measure on the group, then Shur's lemma and the irreducibility of the representation tell us that

$$1 = \text{const.} \int d[g] |g\rangle\langle g|. \tag{5.4}$$

(The constant is the dimension of the representation divided by the volume of the group manifold.) If we want to calculate the thermodynamic partition function

$$\mathscr{Z} = \text{Tr}(e^{-\beta H}), \tag{5.5}$$

where the trace is restricted to the representation space on which $D(g)$ acts, we can break up the Matsubara time-interval β into n parts and form an iterated integral

$$\text{Tr}(e^{-\beta H}) = \int (d[g]\, d[g'] \ldots) \langle g|e^{-\beta H/n}|g'\rangle\langle g'|e^{-\beta H/n} \ldots |g\rangle. \tag{5.6}$$

For short time intervals, using naïve calculus, we expect $g' = g + \delta g$ with $\delta g = O(\delta t)$, so we can approximate

$$\langle g|e^{-\delta t H}|g'\rangle = 1 + \langle g|\delta g\rangle + \langle g|(-\delta t H)|g\rangle + O(\delta t^2). \tag{5.7}$$

Again working to order δt we find formally (ignoring the subleties in the calculus will produce problems that will come back to haunt us later)

$$\mathrm{Tr}(e^{-\beta H}) = \int d[g] \exp\left(\oint \langle g|dg\rangle - \int_0^\beta dt \langle g|H|g\rangle\right), \tag{5.8}$$

where now $d[g]$ is the path-integral measure made from the Haar measure at each time step. The integral is over all periodic paths on G taking "time" β.

Actually, the path integral is over a quotient space of the group. We have to notice that many of the $|g\rangle$ differ by only a phase, and that the integrand is insensitive to this phase. If H is the subgroup of G, obtained by exponentiating a maximal commuting set of generators, i.e. H is a maximal torus, we can take $|0\rangle$ to be an eigenstate of the generators of H – a state of definite weight – and then the $|g\rangle$ in any one coset of G/H are all phase multiples of one another. In other words the coherent states form a bundle over G/H with the maximal torus as the gauge group and the integral is really over paths in G/H.

The coset obtained by quotienting a Lie group by a maximal torus is often called the *flag manifold*. For G = SU(2) the flag manifold and resultant principal bundle is just S^2 with the Hopf fibration and path integral is really an integral over the two-sphere. For SU(N) the flag manifold is the coset space of the integral (3.17) and the present construction seems to be natural extension of that integral.

It is possible to write the integrand in the coherent-state path integral in such a way as to make the independence of choice of representatives transparent. We introduce the projection operators [5]

$$Q(g) = |g\rangle\langle g|. \tag{5.9}$$

These do not have any phase ambiguity and project directly onto the physically distinct states. The term in the integrand, $\langle g|H|g\rangle$, which was manifestly gauge independent, becomes $\mathrm{Tr}(Q(g)H)$, while the term $\oint\langle g|dg\rangle$ can be seen to be gauge invariant by using Stokes theorem

$$\oint_{\Gamma=\partial\Omega} \langle g|dg\rangle = \int_\Omega d\langle g|dg\rangle, \tag{5.10}$$

and the identity

$$d\langle g|dg\rangle = \langle dg|dg\rangle = -\mathrm{Tr}(dQ\,Q\,dQ). \tag{5.11}$$

In this last formula, a wedge product is to be understood so both sides of the equation are two-forms. The path integral becomes

$$\mathrm{Tr}(\mathrm{e}^{-\beta H}) = \int \mathrm{d}[g]\exp\left(-\int_{\Omega}\mathrm{Tr}(\mathrm{d}Q\,Q\,\mathrm{d}Q) - \int_{\Gamma}\mathrm{Tr}(Q(g)H)\,\mathrm{d}t\right). \quad (5.12)$$

The integrand depends only on the elements of G/H so the volume of the gauge group H factors out.

We are rediscovering Berry's phase and the Berry curvature [21]. The one-form $-\langle g|\mathrm{d}g\rangle$ is the Berry connexion on the two-sphere and the two-form $-\langle \mathrm{d}g|\mathrm{d}g\rangle$ is its curvature. This curvature and connexion are the same ones already discovered while working with the wave functions in sect. 4.

For the case G = SU(2) we can take the cyclic vector $|0\rangle$ as any of the states $= |J,m\rangle$. With hamiltonian $\boldsymbol{B}\cdot J$ we find, in terms of spherical polar angles

$$\langle g|\boldsymbol{B}\cdot J|g\rangle = m|B|\cos\theta\,, \quad (5.13)$$

$$\langle \mathrm{d}g|\mathrm{d}g\rangle = im\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,, \quad (5.14)$$

which is m multiplied by the area two-form on the sphere. The path integral for spin becomes

$$\mathrm{Tr}(\mathrm{e}^{(-\beta\boldsymbol{B}\cdot J)})_{\mathrm{spin}\,J} = \int \mathrm{d}[\boldsymbol{n}]\delta(\boldsymbol{n}^2-1)\exp\left(-\int_0^{\beta} m\boldsymbol{n}\cdot\boldsymbol{B} + im(\text{area enclosed by path})\right) \quad (5.15)$$

The exponent in the integrand is not quite that of the conventional Feynman path integral. The surprise is that the coherent state path integral depends on m, the z-component of the spin of the state cyclic $|0\rangle = |J, m\rangle$, and not, as expected, on the representation index J. We get the classical result only if we chose the basic state $|0\rangle$ to be the highest weight state $|J, J\rangle$. This is disconcerting, not just because it means that a set of $2J+1$ different integrands give the same final answer, but, because the same m occurs in many representations J, the same integral is required to give many different answers. Obviously our naïve calculus manipulations concealed problems reminiscent of operator orderings in the more traditional Feynman path integral.

We will not pursue the resolution of this problem here although it is certainly of interest*. We will content ourselves with noting that the coherent states built from states of highest weight have other special properties which single them out from the

* There does not seem to be much literature on the subleties of coherent-state path integrals compared with that for Feynman path integrals. Some discussion is to be found in ref. [29].

rest. They are the coherent states which are most tightly peaked about their location on the sphere, so it is not surprising that they are the best states for semiclassical treatments, but, more importantly, they have *holomorphic* properties.

Let us quickly review some basic facts about semisimple Lie algebras [20]. Recall that the generators of the Lie algebra $\mathscr{G}$ of a Lie group G can be decomposed into a maximally commuting set $\mathscr{H}=\{H_i\}$, called the *Cartan algebra*, and a set of ladder operators, E_α, one for each *root* vector $\alpha \in \mathrm{R}$. The ladder operators, which are the analogues of the $J_\pm$ familiar from SU(2), require the complexification of the algebra to $\mathscr{G}^{\mathrm{c}}$, obtained by allowing the parameters of the group to take complex values. The H_i and the E_α obey

$$[H_i, E_\alpha] = \alpha_i E_\alpha , \tag{5.16}$$

so if $|\lambda\rangle$ is an eigenstate of the H_i with eigenvalues λ_i (the weight vector), $H_i|\lambda\rangle = \lambda_i|\lambda\rangle$, then $E_\alpha|\lambda\rangle = \text{const.}|\lambda+\alpha\rangle$. The roots may be divided into two classes by an arbitrary (except that it must not contain a root) hyperplane in root space. Those on one side are called positive roots (denoted by $\alpha \in \mathrm{R}_+$ and are regarded as increasing the weights and the others are negative roots (denoted by $\alpha \in \mathrm{R}_-$) and decrease the weights. (If $\alpha \in \mathrm{R}_+$ then $-\alpha \in \mathrm{R}_-$.) A state that is annihilated by all E_α, $\alpha \in \mathrm{R}_+$ has *greatest weight*. (I will, in the following, assume that the state of greatest weight is non-degenerate.)

There are a certain special subgroups associated with this decomposition of the Lie algebra. The *Borel* groups $\mathrm{B}_\pm$ are made by exponentiating the algebras $\mathscr{B}_\pm$ spanned by the E_α, H_j, ($\alpha \in \mathrm{R}_\pm$, $H_j \in \mathscr{H}^{\mathrm{c}}$). Other groups, important to us here, are the $\mathrm{Z}_\pm$ obtained by exponentiating the E_α, ($\alpha \in \mathrm{R}_\pm$, respectively) and the maximal torus, which we have already met, obtained by exponentiating $\mathscr{H}^{\mathrm{c}}$.

Almost any $g \in \mathrm{G}$ can be factored as a gaussian decomposition

$$g = \zeta_- h \zeta_+ , \quad \zeta_- \in \mathrm{Z}_- , \quad \zeta_+ \in \mathrm{Z}_+ , \quad h \in \mathrm{H}^c , \tag{5.17}$$

where, for example,

$$\zeta_- = \exp\Big(\sum_{\alpha \in \mathrm{R}_-} z^\alpha E_\alpha \Big) . \tag{5.18}$$

If we apply the representation matrix $D(g)$ to a greatest weight, then the factor ζ_+ acts as the identity and the set of physically distinct states we obtain is in one-to-one correspondence with the coset space $\mathrm{G}^{\mathrm{c}}/\mathrm{B}_+$. The z^α serve as complex coordinates for $\mathrm{G}^{\mathrm{c}}/\mathrm{B}_+$. Let us see how this works in the case of $\mathrm{SU}(2)^{\mathrm{c}}$. The factorization reads

$$g = \mathrm{e}^{zJ_-}\mathrm{e}^{i\theta J_3}\mathrm{e}^{z'J_+} . \tag{5.19}$$

Working in the spin-$\frac{1}{2}$ representation

$$D(g)=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}\begin{pmatrix}1 & z'\\ 0 & 1\end{pmatrix}, \tag{5.20}$$

we read off that $z=\gamma/\alpha$ and $z'=\beta/\alpha$. So except for elements with $\alpha=0$, the gaussian decomposition exists (even for the uncomplexified group ε is not a pure phase.) The quotient space G^c/B_+ is therefore isomorphic to the complex plane with coordinate z. G^c acts on this space by

$$g\zeta_-=\zeta'_-h\zeta_+, \tag{5.21}$$

where ζ'_- contains z'

$$z'=\frac{\delta z+\gamma}{\beta z+\alpha}. \tag{5.22}$$

In the original uncomplexified SU(2) we have $\beta=-\gamma^*$, $\delta=\alpha^*$ and $|\alpha|^2+|\gamma|^2=1$ but this provides no constraint on z which can be any complex number. Because of this, the set of physically distinct coherent states obtained by applying elements of SU(2) to the state of greatest weight $|J,J\rangle$ is both G^c/B_+ and G/H. The two coset spaces are isomorphic. In general

$$G^c/B_+\cong G/H. \tag{5.23}$$

In the case of SU(2) the complex plane G^c/B_+ is mapped on to the compact $G/H\equiv S^2$ by a stereographic projection. The states

$$|g\rangle=|J,m\rangle D^{(J)}_{m,J}(g) \tag{5.24}$$

are thus parameterised by the complex number z and the quotients $D^{(J)}_{m,J}(g)/D^{(J)}_{J,J}(g)$ are functions of z only. Coherent states built on non-greatest weight states will be functions of $\bar{z}$ as well and so are not holomorphic.

6. Kähler manifolds and $N=2$ supersymmetry

In sect. 5 the quantum mechanics of spin was obtained by taking the massless limit of the motion of a particle around a monopole. In this limit all the unwanted states had their energies pushed off to infinity. By introducing a fermionic superpartner for the dynamical variable $\boldsymbol{n}$ it is possible to cancel the contribution to the partition function of all the unwanted states whatever the value of the mass – then, after the cancellation, we can deform the problem *without altering the answer* into a parameter range where the semiclassical approximation is arbitrarily accurate. This implies the exactness of the semiclassical approximation.

The key to the supersymmetry is the fact that the two-sphere S^2, and in general all the flag manifolds, are complex manifolds with the additional property of being *Kähler*. For a manifold to be *complex* one needs to be able to choose complex coordinates on the manifold with holomorphic functions describing the change of coordinates between different patches. Then, in terms of these complex coordinates, the metric on the space will be in the form

$$ds^2 = g_{z^i\bar{z}^j}\, dz^i d\bar{z}^j . \tag{6.1}$$

We associate a two-form $\omega = g_{z^i\bar{z}^j}\, dz^i \wedge d\bar{z}^j$ with the metric and the manifold is *Kähler* if ω is closed, i.e. $d\omega = 0$. The main consequence of this seemingly obscure condition is that it is the necessary and sufficient condition for the existence of complex geodetic coordinates. To see the necessity assume such coordinates exist at a point p. Then at p we have $g_{z^i\bar{z}^j} = \delta_{ij}$ with all the first partial derivatives of $g_{z^i,\bar{z}^j}$ vanishing. This obviously implies that $d\omega$ vanishes at p. Since p was arbitrary, $d\omega \equiv 0$. For sufficiency see, for example, ref. [22].

The existence of geodetic coordinates allows us to perform many calculations as if the space were flat. Since we are mostly interested in the two-sphere, we will use this freedom and for the rest of this section restrict ourselves to the complex plane. To extend the results to an arbitrary complex Riemann manifold requires inserting terms proportional to the curvature whenever we take second derivatives – but we will never need the exact form of these terms.

For the complex plane we have $ds^2 = dz\, d\bar{z}$ so $g_{z\bar{z}} = \frac{1}{2} = g_{\bar{z}z}$ and $g^{z\bar{z}} = 2 = g^{\bar{z}z}$. Tensors have their indices raised and lowered according to $A^z = 2A_{\bar{z}}$ etc.

In terms of $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ we have

$$\nabla^2 = 4\partial_z \partial_{\bar{z}} . \tag{6.2}$$

Following the ideas of Witten [23] we extend the Hilbert space and introduce fermionic creation operators $\psi^{x\dagger}$ and $\psi^{y\dagger}$ to correspond to the differential forms dx and dy

$$dx \to \psi^{x\dagger}|0\rangle , \qquad dy \to \psi^{y\dagger}|0\rangle . \tag{6.3}$$

More precisely $\psi^{x\dagger}$ corresponds to the operation of exterior multiplication by dx and the adjoint, ψ_x, to the operation of interior multiplication, or contraction, by the vector dual to dx, i.e. ∂_x

$$\begin{aligned} \iota(\partial_x) &\to \psi_x , \quad \varepsilon(dx) \to \psi^{x\dagger} , \\ \iota(\partial_y) &\to \psi_y , \quad \varepsilon(dy) \to \psi^{y\dagger} . \end{aligned} \tag{6.4}$$

Then, we have

$$\{\psi_x, \psi^{x\dagger}\} = 1, \qquad \{\psi_x, \psi^{y\dagger}\} = 0. \tag{6.5}$$

With these definitions in mind we set

$$\tfrac{1}{2}(\psi_x - i\psi_y) = \psi_z = \tfrac{1}{2}\psi^{\bar{z}}, \qquad \tfrac{1}{2}(\psi_x + i\psi_y) = \psi_{\bar{z}} = \tfrac{1}{2}\psi^{z}, \tag{6.6}$$

together with the hermitian conjugate relations. There is a warning needed here. Hermitian conjugation changes the "i" in the z's so $(\psi^z)^\dagger = \psi^{\dagger\bar{z}}$!

We want to use these ingredients to establish a supersymmetry. Recall Witten's observation that supersymmetric quantum mechanics on a riemannian manifold is just the exterior calculus of the de-Rham complex [23]. For the complex Kähler manifold the supersymmetry we want will be a similar object but with a more refined structure.

We define two supercharges

$$\begin{aligned} Q_1 &= \partial = \psi^{\dagger z}\,\partial_z, \quad Q_1^\dagger = \delta = -\psi^{\bar{z}}\,\partial_{\bar{z}}, \\ Q_2 &= \bar{\partial} = \psi^{\dagger\bar{z}}\,\partial_{\bar{z}}, \quad Q_2^\dagger = \bar{\delta} = -\psi^{z}\,\partial_z. \end{aligned} \tag{6.7}$$

Using these operators, and the commutation relation for the fermions, it is easy to compute

$$\begin{aligned} (\partial\delta + \delta\partial) &= \left(Q_1 Q_1^\dagger + Q_1^\dagger Q_1\right) = -\tfrac{1}{2}\nabla^2, \\ (\bar{\partial}\bar{\delta} + \bar{\delta}\bar{\partial}) &= \left(Q_2 Q_2^\dagger + Q_2^\dagger Q_2\right) = -\tfrac{1}{2}\nabla^2. \end{aligned} \tag{6.8}$$

The cross terms, such as

$$(\partial\bar{\delta} + \bar{\delta}\partial) = \left(Q_1 Q_2^\dagger + Q_2^\dagger Q_1\right) = 0, \tag{6.9}$$

all vanish.

The sum, $Q = Q_1 + Q_2$, of the two supercharges is the ordinary exterior derivative, d, and is the supercharge of $N = 1$ supersymmetric quantum mechanics. The Witten index is the index of d

$$\text{Index (d)} = \text{Tr}\left((-1)^F \mathrm{e}^{-t(Q+Q^\dagger)^2}\right), \tag{6.10}$$

and is, in this case, the Euler number of the manifold. For our purposes we only want one of the supercharges, $Q_2 = \bar{\partial}$. The operator $\bar{\partial}$ acts on forms in $\Lambda^{p,q}(\mathrm{M})$ containing $p\,\mathrm{d}z$'s and $q\,\mathrm{d}\bar{z}$'s.

$$\bar{\partial}\colon \overset{p,q}{\bigwedge}(\mathrm{M}) \to \overset{p,q+1}{\bigwedge}(\mathrm{M}). \tag{6.11}$$

The case $p \doteq 0$ forms the *Dolbeault complex* [16, 22]. We define cohomology groups as the space of closed, but not exact, forms

$$H^{p,q}(\mathrm{M}) = \mathrm{Ker}\,\bar{\partial}/\mathrm{Im}\,\bar{\partial}\,. \tag{6.12}$$

For a manifold of real dimension n (and hence complex dimension $\frac{1}{2}n$), the index of the Dolbeault complex is then

$$\mathrm{Index}\,(\bar{\partial}) = \sum_{q=0}^{n/2} (-1)^q \dim H^{0,q}(\mathrm{M}) = \mathrm{Tr}\big((-1)^F \mathrm{e}^{-t(Q_2+Q_2^\dagger)^2}\big)\,. \tag{6.13}$$

This index is called the arithmetic genus of the complex manifold and the Riemann–Roch theorem [16] gives an expression for it in terms of the Todd class of the manifold.

We are going to be more interested in the *twisted* Dolbeault complex obtained by replacing $\bar{\partial} = \psi^{\dagger\bar{z}}\,\partial_{\bar{z}}$ by $\bar{\nabla} = \psi^{\dagger\bar{z}}\nabla_{\bar{z}}$. Here $\nabla_{\bar{z}}$ is a covariant derivative containing a connexion on a U(1) bundle. To see the origin of the connexion we need a deeper study of the geometry of the coherent states. Recall that the coherent states built from a cyclic vector, $|0\rangle$, which is a greatest weight state, are holomorphic. More precisely, using the factorisation $g = \zeta_- h \zeta_+$, where ζ_- is parameterized by the variables z, we can write

$$\widetilde{|z\rangle} = N(z, \bar{z})|z\rangle\,. \tag{6.14}$$

Here $\widetilde{|z\rangle} = \zeta_- h \zeta_+ |0\rangle$ is a normalized state which contains both z's and $\bar{z}$'s because of the factor h. The state $|z\rangle = \zeta_-|0\rangle$ is unnormalised but contains only z's. The normalisation factor $N(z, \bar{z})$ can be expressed as

$$N(z, \bar{z}) = \langle 0|g|0\rangle = \langle 0|h|0\rangle\,. \tag{6.15}$$

As we have seen, the Berry connexion is the natural connexion on G/H. It is $A = -\widetilde{\langle z|}\mathrm{d}\widetilde{|z\rangle}$ and can be expressed in terms of the normalization factor N

$$A = -\widetilde{\langle z|}\mathrm{d}\widetilde{|z\rangle} = \frac{\partial \log N}{\partial z}\,\mathrm{d}z - \frac{\partial \log N}{\partial \bar{z}}\,\mathrm{d}\bar{z}\,. \tag{6.16}$$

The curvature is

$$F = \mathrm{d}A = 2\frac{\partial^2 \log N}{\partial \bar{z}\,\partial z}\,\mathrm{d}\bar{z} \wedge \mathrm{d}z\,. \tag{6.17}$$

Since $N = \langle 0|h|0\rangle$ depends on the weight of the cyclic state, we have a different connexion for each greatest weight state and thus each representation. For SU(2), the monopole charge is proportional to J.

There is also a natural metric on the space of coherent states and thus on G/H. We define the length of a displacement in G/H by the extent to which it moves the state $\widetilde{|z\rangle}$ in a direction perpendicular to itself, i.e. define

$$ds^2 = \|\mathrm{d}\widetilde{|z\rangle} - \widetilde{|z\rangle\langle z|}\,\mathrm{d}\widetilde{|z\rangle}\|^2 . \tag{6.18}$$

A little arithmetic shows that

$$ds^2 = -2\frac{\partial^2 \log N}{\partial\bar{z}\,\partial z}\,\mathrm{d}\bar{z}\,\mathrm{d}z , \tag{6.19}$$

so the space is Kähler and the curvature F is actually the Kähler form associated with the metric ds^2. (When the metric is written like this, $\log N$ is called the Kähler potential.)

With the connexion A and $\nabla_{\bar{z}} = \partial_{\bar{z}} + A_{\bar{z}}$ we find that the components of the normalized coherent states, the functions $D^{(J)}_{mJ}$, are all annihilated by $\overline{\nabla}$ and thus by $\psi^{\dagger\bar{z}}\overline{\nabla}$. The form a multiplet of zero-energy eigenstates of the hamiltonian of the twisted Dolbeault complex. For SU(2) this is the supersymmetric extension of eq. (5.1) with $B = 0$. To include a non-zero B we add some group generators to the hamiltonian. The supertrace then becomes the "G-index".

7. The G-index theorem

If we have a compact riemannian manifold M together with an action of the group on that manifold $\mathrm{G}:\mathrm{M}\to\mathrm{M}$, then we can induce a representation of the group on functions on M by

$$T(g)f(x) = f\big(g^{-1}x\big) . \tag{7.1}$$

This representation will be infinite dimensional and highly reducible. How can we define and evaluate its character?

Suppose the functions $f_n(x)$ are eigenfunctions of $-\nabla^2$ with eigenvalues $\lambda_n > 0$, and form a complete orthonormal set on M. Then there exists an infinite matrix $A_{nm}(g)$ such that

$$f_n\big(g^{-1}x\big) = \sum_m f_m(x)\,A_{mn}(g) . \tag{7.2}$$

We want to know if the sum $\chi(g) = \Sigma A_{nn}(g)$ converges, and if so, to what. Consider the Green function for the heat equation on M

$$G(x, x'; t) = \mathrm{e}^{-t(-\nabla^2)} = \sum_n f_n(x) f_n(x')^* \mathrm{e}^{-\lambda_n t} . \tag{7.3}$$

An expression for the desired sum, with a built in convergence factor, is given by

$$\chi(g) = \lim_{t\to 0} \sum A_{nn}(g)\, \mathrm{e}^{-t\lambda_n} = \lim_{t\to 0} \int_{\mathrm{M}} \mathrm{d}x\, G\big(g^{-1}x, x; t\big). \tag{7.4}$$

As $t \to 0$ the Green function goes rapidly to zero unless $x, g^{-1}x$ are nearly coincident; the only contributions to the integral come from the neighbourhood of *fixed points* of the action of G on M.

Near the fixed points we have a good approximation, in locally geodetic coordinates, to $G(x, x'; t)$

$$G(x, x'; t) \approx \frac{1}{2^d \pi^{d/2} t^{d/2}}\, \mathrm{e}^{-(x-x')^2/4t}. \tag{7.5}$$

At the same time the action of G becomes linear

$$\big(g^{-1}(x)\big)^{\mu} = R^{\mu}_{\nu}\big(x^{\nu} - x^{\nu}_{\text{fixed}}\big) + x^{\mu}_{\text{fixed}}. \tag{7.6}$$

Performing the integral over M then gives us an expression for the character

$$\chi(g) = \sum_{\text{fixed points}} \det{}^{-1/2}\big((1-R)(1-R)^{\mathrm{T}}\big) = \sum_{\text{fixed points}} \frac{1}{|\det(1-R)|}. \tag{7.7}$$

The occurrence of the determinant in the denominator has a nice interpretation. Near the fixed point we can restrict ourselves to considering the partial derivatives $\partial_{ijk} \ldots f$ of the functions on the manifold. These transform under the action of the group as symmetric powers of the matrix R. Then, the determinant occurs in the denominator as a generating function of the symmetric representations in exactly the same way that a determinant occurs when one takes symmetric powers of a single particle space – i.e. Bose statistics.

For example, take M to be S^2 and G the rotation group SO(3). Consider a rotation through an angle θ about the z-axis. The character is given by the sum

$$\chi(\theta) = \sum_{L=0}^{\infty} \sum_{m=-L}^{+L} \mathrm{e}^{im\theta} = \sum_{L=0}^{\infty} \chi_L(\theta). \tag{7.8}$$

This sum is convergent, at least in a distributional sense (the late terms oscillate rapidly as functions of θ but do not diminish in magnitude), to the expression

$$\chi(\theta) = \frac{1}{2\sin^2(\theta/2)}. \tag{7.9}$$

On the other hand, near the poles, the rotation R has the form

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{7.10}$$

and, by eq. (7.7), each pole contributes $1/(4\sin^2(\theta/2))$ to the integral, resulting in the same answer.

If we take sections of a bundle over the manifold, then we get similar formulae by using the heat equation with the appropriate connexion built in. At each fixed point, the group induces a representation on the indices carried by the section and this produces numerators for the terms in eq. (7.7) which are the characters of this representation.

In the SU(2) case just considered all the representations of the rotation group appeared in the sum. We wish to pick out one of the representations yet still keep an expression for the character of the chosen representation in terms of behaviour at the fixed points. The tool for performing this selection requires the introduction of superpartners for the bosonic states and is the G-index theorem.

The setting of the general G-index theorem is as follows: suppose $Q, Q^\dagger$ are supercharges mapping us from a space of bosonic states to a space of fermionic states and vice versa. Suppose further that there is a Lie group G generated by G_i which commutes with $Q, Q^\dagger$, i.e

$$\left[G_i, G_j\right] = if_{ij}^k G_k, \tag{7.11}$$

$$[Q, G_i] = 0, \qquad \left[Q^\dagger, G_i\right] = 0, \tag{7.12}$$

so G also commutes with the hamiltonian

$$H = \left(Q + Q^\dagger\right)^2, \qquad [G_i, H] = 0. \tag{7.13}$$

Then we can form a supercharacter as a supertrace of the group elements

$$\Xi\left(e^{i\theta^i G_i}\right) = \mathrm{Tr}\left((-1)^F e^{i\theta^i G_i} e^{-tH}\right). \tag{7.14}$$

Eq. (7.14) is to be compared with a rewriting of eq. (7.4) in similar terms

$$\chi = \lim_{t\to 0} \mathrm{Tr}\left(e^{-\theta^i L_i} e^{-t(-\nabla^2)}\right). \tag{7.4a}$$

($G_i = -iL_i$ where the L_i are differential operators generating the action of G on M. On functions they are ordinary derivatives but on forms they would be Lie derivatives.) The ordinary trace requires a limit $t \to 0$ to be taken but eq. (7.14) is automatically independent of t; just as in the ordinary index, the non-zero energy

levels give contributions which cancel in pairs between the bosonic and fermionic sectors. Only contributions from the zero-energy states are left and these are t independent. The energy levels form multiplets under the action of G so

$$\Xi(g) = \sum_i (-1)^F \chi_i(g), \tag{7.15}$$

where the χ_i are the characters of the representations comprising the zero-energy multiplet.

The character-valued index is a topological invariant under deformations of the operators $Q, Q^\dagger$. On deforming $Q \to Q_s$ whole representations may move in and out of the zero-energy multiplets – but they will always do so in Bose–Fermi pairs leaving $\Xi(g)$ unchanged. Thus Ξ is independent of both s and t.

Such deformations can simplify the calculation of the supertrace. As an example [24], consider the supersymmetry generator Q of the ordinary $N = 1$ supersymmetric quantum mechanics on a manifold. This Q is just the exterior derivative d acting on differential forms. In locally geodetic coordinates

$$Q = \mathrm{d} = \psi^{\mu\dagger}\, \partial_\mu . \tag{7.16}$$

We replace this by the operator

$$Q_s = \mathrm{d} + s\iota_K = \psi^{\mu\dagger}\, \partial_\mu + sK^\mu \psi_\mu , \tag{7.17}$$

where K^μ is a Killing vector field, i.e., the vector field of displacements associated with an isometry. Using the identity for the Lie derivative on differential forms, $\iota_\xi \mathrm{d} + \mathrm{d}\iota_\xi = \mathscr{L}_\xi$, we find that

$$(Q_s)^2 = s\mathscr{L}_K . \tag{7.18}$$

A byproduct of eq. (7.18) is the result $[Q_s, \mathscr{L}_K] = 0$. For an isometry $\mathscr{L}_K^\dagger = -\mathscr{L}_K$ so we also have $[Q^\dagger, \mathscr{L}_K] = 0$.

The hamiltonian H_s becomes (ignoring curvature terms which will be unimportant at large s)

$$-\partial^2 + s^2|K|^2 - \tfrac{1}{2}s\left[\psi_\mu^\dagger, \psi_\nu^\dagger\right] \partial_\mu K_\nu - \tfrac{1}{2}s\left[\psi_\mu, \psi_\nu\right] \partial_\nu K_\mu . \tag{7.19}$$

At large s the $|K|^2$ term ensures that the wave functions are concentrated near the fixed points, $K = 0$, justifying the ommision of any terms reflecting the curvature of the manifold. For two dimensions, with a fixed point at the origin, we can take $K_x = -y$, $K_y = x$ so this operator becomes

$$-\partial^2 + s^2\left(x^2 + y^2\right) - 2s\psi_x^\dagger \psi_y^\dagger + 2s\psi_x \psi_y . \tag{7.20}$$

We want to find the zero-energy states of eq. (7.20). The space the hamiltonian acts on consists of states of the form

$$f(x,y)|0\rangle, \quad f_x(x,y)\psi_x^\dagger|0\rangle, \quad f_y(x,y)\psi_y^\dagger|0\rangle, \quad f_{xy}(x,y)\psi_x^\dagger\psi_y^\dagger|0\rangle. \tag{7.21}$$

In the fermionic sector the Fermi terms have no effect and all the states have energy that becomes large as $s \to \infty$, because of the zero-point energy. On the bosonic space, spanned by the first and last states in the list, we have

$$H_s = -\partial^2 + s^2(x^2+y^2) - 2s\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{7.22}$$

with eigenvalues

$$E_{nm}^{\pm} = ((2n+1)+(2m+1))|s| - 2s(\pm 1). \tag{7.23}$$

There is always one zero-energy state, φ_0, in the Bose sector, with $m = n = 0$, and, because of the nature of the ground state of the harmonic oscillator, we know it is rotationally symmetric about the origin, i.e. $\mathscr{L}_K\varphi_0 = 0$. Therefore, each fixed point contributes unity to both the character-valued index (where we have to replace L_i by the relevant Lie derivative $\mathscr{L}_K$ to take into account the action of the isometry on forms as well as functions)

$$\Xi = \mathrm{Tr}\left((-1)^F \mathrm{e}^{-\theta\mathscr{L}_K}\mathrm{e}^{-tH_s}\right), \tag{7.24}$$

and the ordinary Witten index, here the index of the de-Rham complex, and equal to the Euler number. We therefore have the classical Lefshetz theorem that the number of fixed points of an isometry is equal to the Euler number.

Now we want to apply these ideas to pick out a single representation of the group. The appropriate complex for the application of the G-index theorem is the Dolbeault complex of sect. 6. This is because the sections of coherent states defined by greatest weights are holomorphic and therefore zero modes of $\bar\partial = \psi^{\dagger\bar z}\partial_{\bar z}$ or rather the version of this containing a connexion.

In order to commute with the complex structure, the isometries to which the Dolbeault complex version of the G-index theorem can be applied must be holomorphic. Near a fixed point there will be complex coordinates so that

$$z^i \mapsto z'^i = \mathrm{e}^{ir_i\theta}z^i. \tag{7.25}$$

We will continue to use the two-dimensional case to illustrate the theorem. In complex coordinates the Killing field becomes $K = -y\partial_x + x\partial_y = i(z\partial_z - \bar z\partial_{\bar z})$. The deformed $N = 1$ supercharge is decomposed into its holomorphic and antiholomorphic parts according to $\mathrm{d} = \partial + \bar\partial$

$$Q_s = Q_{s1} + Q_{s2} = \left(\psi^{\dagger z}\partial_z + s\psi_z(+iz)\right) + \left(\psi^{\dagger\bar z}\partial_{\bar z} + s\psi_{\bar z}(-i\bar z)\right). \tag{7.26}$$

In order to get well-behaved operators it is convenient to replace s by is, then

$$H_{is} = \left(Q_{is2} + Q_{is2}^{\dagger}\right)^2 = -2\partial_{\bar{z}z}^2 + |s|^2|z|^2 + s\left[\psi^{\dagger\bar{z}}, \psi_{\bar{z}}\right] + s(\bar{z}\partial_{\bar{z}} - z\partial_z). \quad (7.27)$$

This operator has eigenvalues given by

$$E_{nml}^{\pm} = |s|\left(\left(n + \tfrac{1}{2}\right) + \left(m + \tfrac{1}{2}\right)\right) + sl + s(\pm 1). \quad (7.28)$$

The first term comes from the harmonic oscillator. The second is proportional to l, the eigenvalue of $\mathscr{L}_K = \bar{z}\partial_{\bar{z}} - z\partial_z$. The ± 1 is the eigenvalue of $[\psi^{\dagger\bar{z}}, \psi_{\bar{z}}]$. Depending on whether s is greater or less than zero we have different cases to consider in order to get zero eigenvalues.

(i) $s > 0$. We need the minus eigenvalue for $[\psi^{\dagger\bar{z}}, \psi_{\bar{z}}]$ so we are in the Bose sector. We can get zero eigenvalues for $l = 0, -1, -2, \ldots$. Therefore, the contribution to

$$\Xi(\theta) = \mathrm{Tr}\left((-1)^F \mathrm{e}^{-\theta\mathscr{L}_K} \mathrm{e}^{-tH_{is}}\right). \quad (7.29)$$

is

$$\sum_{n=0}^{\infty} \mathrm{e}^{-in\theta} = \frac{1}{1 - \mathrm{e}^{-i\theta}} \quad (7.30)$$

(ii) $s < 0$. We need $[\psi^{\dagger\bar{z}}, \psi_{\bar{z}}]$ to be equal to 1 therefore $F = +1$. We have zero modes for $l = 0, 1, 2, \ldots$. However, in this case the index l is not the whole story of how the wave function transforms under the isometry. The $\psi^{\bar{z}}$ also transforms and provides an extra factor of $\mathrm{e}^{i\theta}$. This would have been clear if we had rewritten the operator H_{is} in terms of the Lie derivative on forms. The contribution to

$$\Xi(\theta) = \mathrm{Tr}\left((-1)^F \mathrm{e}^{-\theta\mathscr{L}_K} \mathrm{e}^{-tH_{is}}\right) \quad (7.31)$$

is

$$-\mathrm{e}^{i\theta}\left(\sum_{n=0}^{\infty} \mathrm{e}^{in\theta}\right) = \frac{-\mathrm{e}^{i\theta}}{1 - \mathrm{e}^{i\theta}} = \frac{1}{1 - \mathrm{e}^{-i\theta}}. \quad (7.32)$$

Thus, either positive or negative s give the same contribution. For the higher dimensional cases, with the holomorphic isometries given by eq. (7.25), and representations $\chi_p(\theta)$ induced on the states at each fixed point p, we get a product of factors from each fixed point

$$\Xi = \sum_{\text{fixed points, } p} \frac{\chi_p(\theta)}{\prod_n (1 - \mathrm{e}^{i\theta r_n})}. \quad (7.33)$$

The holomorphic isometries we want to consider arise from the action of the group G on the flag manifold G/H. We wish to compute the character of an element of the maximal torus (any element in G is conjugate to an element of H)

$$\chi = \mathrm{Tr}(e^{i\theta^i H_i}) = \mathrm{Tr}(h). \tag{7.34}$$

h acts on a point $gh' \in G/H$ by $h(gh') = (g'h'')$. The fixed points of this action occur when $g = g'$, a condition that can be equivalently written as

$$g^{-1}Hg = H,$$

i.e the element g maps the maximal torus into itself under conjugation. Such g form a group called the *normaliser*, N(H), of H. H itself is a normal subgroup of N(H) and the set of mappings $H \to H$ they produce is the Weyl group, $W = N(H)/H$. The fixed points of the action of H on G/H coincide with this group.

One of the fixed points is $g =$ identity and, while h acts on this point to leave it fixed in G/H, it moves the point, considered as an element of G, so its action on a representation with weight λ_i produces a factor of $e^{i\lambda_i\theta^i}$. This is the term that appears in the numerator of the contribution at the identity.

Points in the immediate neighbourhood of the identity can be parameterised as in eq. (5.18)

$$\zeta_- = \exp\Big(\sum_{\alpha\in R_-} z^\alpha E_\alpha\Big), \tag{7.35}$$

and, since $[H_i, E_\alpha] = \alpha_i E_\alpha$ we have

$$e^{i\theta^i H_i}\exp\Big(\sum_{\alpha\in R_-} z^\alpha E_\alpha\Big) = \exp\Big(\sum_{\alpha\in R_-} z'^\alpha E_\alpha\Big) e^{i\theta^i H_i}, \tag{7.36}$$

where

$$z'^\alpha = e^{i\theta^i\alpha_i} z^\alpha. \tag{7.37}$$

This is a holomorphic isometry of the form of eq. (7.25) so the root vectors determine the contribution to the denominator.

The other fixed points are all images of the identity under the action of the Weyl Group, so using $w\cdot$ to denote the action of this group, the character can be written as

$$\chi_\lambda(\theta) = \sum_{w\in W} w\cdot\left(\frac{e^{i\lambda_i\theta^i}}{\Pi_{\alpha\in R_-}(1-e^{i\alpha_i\theta^i})}\right), \tag{7.38}$$

and eq. (2.2) is an example of eq. (7.38) with $\alpha = -1$, $\lambda = J$.

Actually, there is one more step to the argument that eq. (7.38) is exactly the character of the representation in the no-fermion sector. Let us recap what we have achieved so far. We selected the greatest weight λ of the desired representation. This choice determines the connexion and hence the hamiltonian. We know that the set of wave functions $\varphi_\nu = D^{(J)}_{\nu\lambda}$, analagous to the monopole harmonics, are zero modes of the hamiltonian if λ is a greatest weight of the representation labeled J. The other bosonic wave functions for higher representations containing states with the same, but no longer greatest, weight λ are eigen-modes, but not zero-modes so only the desired states contribute to the supertrace in the bosonic sector. We must now argue that there are no further zero-modes in the other, fermion-containing, sectors. This can be proven using the Kodeira vanishing theorems (see ref. [22] p. 232). We will not attempt to describe these theorems here. That there are no zero modes in the one-fermion sector for the two-sphere may be checked directly and the general case is similar to the proof that the Dirac operator can have no zero modes on a manifold with everywhere positive curvature scalar R. (One observes that $(i\gamma^\mu\partial_\mu)^2 = -\nabla^2 + \frac{1}{4}R$ and that $-\nabla^2$ has non-negative eigenvalues. Thus every trial wave function gives an expectation that is strictly positive.) With this result we have established the semiclassical formula for the generalised spin problem.

8. Conclusions and speculations

By introducing a fermionic partner for a particle orbiting a monopole, we have been able to map the problem of computing the partition function, or the character, for spin J to that of the supertrace of this supersymmetric system. Such supertraces are invariant under a wide class of deformations, including deformations that change the system into one which is accurately described semiclassically. This shows that the original problem is also exactly described by semiclassical quantum mechanics.

In the course of this exercise we have introduced a number of interesting concepts, such as coherent-state path integrals and holomorphic sets of coherent states. For spin systems, these ideas have quite practical applications [5, 6, 25] and the same may be true of the supersymmetry. Extensions to field theories would be very interesting and are quite possible. The Weyl formula itself has a generalisation to (1 + 1)-dimensional field theories in the form of the Kac character formula for Kac–Moody algebras, the Lie algebras of the infinite-dimensional loop groups [27]. The continuous Heisenberg spin chain, which is the natural extension of the single spin, has a semiclassical spectrum which is essentially exact [28] and it would nice to obtain it by supersymmetric arguments.

Finally, the whole area of lagrangian supersymmetry raises interesting questions. The hamiltonian formalism of supersymmetric quantum mechanics has a satisfactory geometric interpretation – but the situation at the lagrangian level is less clear. For $N = \frac{1}{2}$ the results of ref. [11] provide a geometric setting and for $N = 1$ we have

the Nicolai map [26]. It would be interesting to understand how the $N = 2$ supersymmetry is to be understood in the lagrangian formalism. Is there a nice geometric interpretation of lagrangian $N = 2$ supersymmetry?

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