LMG Model

$$H = -hJ_z - \frac{1}{2j}\gamma_x J_x^2$$

Considering this model as a fully connected version of $N = 2j$ spin-1/2 particles, then defining

$$J_z = \sum_{i=1}^{N} \sigma_i^z$$

$$J_x = \sum_{i=1}^{N} \sigma_i^x ,$$

we'll have

$$H = -h\sum_{i=1}^{N}\sigma_i^z - \frac{\gamma_x}{N}\left(\sum_{i=1}^{N}\sigma_i^x\right)^2$$

$$\boxed{H = -h\sum_{i=1}^{N}\sigma_i^z - \frac{\gamma_x}{N}\sum_{i,j}^{N}\sigma_i^x\sigma_j^x} .$$

Let's calculate

$$\|[A(t), B]\|$$

where $\|\cdot\|$ is the Hilbert-Schmidt norm, defined as

$$\|O\| := \mathrm{Tr}(O^\dagger O)$$

and

$$A(t) = e^{iHt} A e^{-iHt} .$$

So, at the end we want to evaluate

$$f(t) = \mathrm{Tr}\left([A(t),B]^{\dagger} \cdot [A(t),B] \right)$$

$$= \mathrm{Tr}\left([B, A(t)] \cdot [A(t), B] \right)$$

$$= \mathrm{Tr}\left[\left(B \cdot A(t) - A(t) \cdot B\right)\left(A(t) \cdot B - B \cdot A(t)\right) \right]$$

$$= \mathrm{Tr}\left[B \cdot A(t) \cdot A(t) \cdot B - B \cdot A(t) \cdot B \cdot A(t) - A(t) \cdot B \cdot A(t) \cdot B + A(t) \cdot B \cdot B \cdot A(t) \right].$$

Choosing

$$A = \sigma_1^x$$

$$B = \sigma_N^x$$

and noticing that

$$A(t) \cdot A(t) = e^{iHt} \cdot A \underbrace{e^{-iHt} \cdot e^{iHt}}_{\mathbb{1}} \cdot A \cdot e^{-iHt}$$

$$= e^{iHt} \cdot AA \cdot e^{-iHt}$$

$$= A^2(t)$$

thus

$$f(t) = \mathrm{Tr}\left[\sigma_N^x \cdot \sigma_1^{x^2}(t) \cdot \sigma_N^x - \sigma_N^x \cdot \sigma_1^x(t) \cdot \sigma_N^x \sigma_1^x(t) - \sigma_1^x(t) \cdot \sigma_N^x \cdot \sigma_1^x(t)\, \sigma_N^x + \sigma_1^x(t) \cdot \sigma_N^{x^2} \cdot \sigma_1^x(t) \right].$$

All the Pauli matrices satisfy $(\sigma_i^\alpha)^2 = \mathbb{1}_i$, $\forall i \in \{1,\dots,N\}$ e $\forall \alpha \in \{x,y,z\}$,

hence

$$f(t) = \mathrm{Tr}\left[2 \mathbb{1}_N \mathbb{1}_1 - \sigma_N^x \sigma_1^x(t) \cdot \sigma_N^x \sigma_1^x(t) - \sigma_1^x(t) \sigma_N^x \cdot \sigma_1^x(t) \sigma_N^x \right]$$

$$= 2^{N+1} - \mathrm{Tr}\left[\sigma_N^x \sigma_1^x(t) \cdot \sigma_N^x \cdot \sigma_1^x(t) \right] - \mathrm{Tr}\left[\sigma_1^x(t)\, \sigma_N^x\, \sigma_1^x(t)\, \sigma_N^x \right].$$

We have to remind that

$$\sigma_N^x \sigma_1^x(t)\, \sigma_N^x \sigma_1^x(t) \equiv \sigma_N^x \sigma_1^x(t)\, \sigma_N^x \sigma_1^x(t) \cdot \mathbb{1}_2 \cdot \mathbb{1}_3 \cdots \mathbb{1}_{N-1}$$

$$\sigma_1^x(t)\, \sigma_N^x\, \sigma_1^x(t)\, \sigma_N^x \equiv \sigma_1^x(t)\, \sigma_N^x\, \sigma_1^x(t)\, \sigma_N^x \; \mathbb{1}_2\, \mathbb{1}_3 \cdots \mathbb{1}_{N-1}$$

and the product between the operator is the tensor product. The order will not be important because

$$\mathrm{Tr}(X \otimes Y) = \mathrm{Tr}(X) \cdot \mathrm{Tr}(Y)$$

then, we'll have

$$f(t) = 2^{N+1} - 2^{N-2}\, \mathrm{Tr}\left[\sigma_N^x \sigma_1^x(t)\, \sigma_N^x \cdot \sigma_1^x(t) \right] - 2^{N-2} \cdot \mathrm{Tr}\left[\sigma_1^x(t) \cdot \sigma_N^x \sigma_1^x(t)\, \sigma_N^x \right].$$

Now, we need to calculate

$$\sigma_1^x(t) = e^{iHt} \cdot \sigma_1^x \cdot \sigma^{-iHt}$$

which is, using BCH formula,

$$\sigma_1^x(t) = \sigma_1^x + it\left[H, \sigma_1^x\right] + \frac{(it)^2}{2!}\left[H, \left[H, \sigma_1^x\right]\right] + \ldots$$

that essentially depends on

$$\left[H, \sigma_1^x\right]$$

$$[H, \sigma_1^x] = \left[-h \sum_{i=1}^{N} \sigma_i^z - \frac{\gamma_x}{N} \sum_{i,j=1}^{N} \sigma_i^x \sigma_j^x \ , \ \sigma_1^x \right]$$

$$[H, \sigma_1^x] = -h \cdot 2i\sigma_1^y - \frac{\gamma_x}{N} \underbrace{\left[\sum_j \sigma_1^x \sigma_j^x + \sum_i \sigma_i^x \sigma_1^x \ , \ \sigma_1^x \right]}_{=0 \ , \text{ since } [\sigma_i^x, \sigma_j^x] = 0 \ , \ \forall i,j}$$

$$[H, \sigma_1^x] = -2ih\sigma_1^y$$

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