Information geometric approach to dynamical phase transition

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I. INTRODUCTION

The main question considered in these notes is the following. Is it possible to characterize a dynamical quantum phase transition in terms of geometric quantities? Since the Riemannin structure of the state space is deeply related with thermodynamics, such description could provide a solid physical (in terms of thermodynamic quantities) for the explanation behind the dynamical phase transitions.

Also, another question that could be investigated is the possible connections between dynamical quantum phase transisions and chaos, that can be characterized by the Loschmidt echo. This could also be linked with thermodynamics.

II. INFORMATION GEOMETRY

Here we present an overview of the geometric formulation of the information theory, exploring the deep link between measures of state distinguishability in Hilbert space and the metrics defined on state space.

In general, quantum operations represent physical processes that encode information in the state of the quantum system. Thus, the information processing requires a procedure to discriminate the resulting state of the quantum operation and infer the information encoded in it. A solution of this task is the analysis of distinguishability between the states of the system before and after the implementation of the operation.

In the last decades several distinguishability quantifiers were introduced in the literature in the context of distance measurements in the state space. Examples include the trace distance, the Bures distance, the Hilbert-Schmidt and the relative entropy, among others [1, 2]. This discussion started decades ago with Wooters [3] that linked distinguishability measures and the concept of statistical distance on the probability space. The geometric version of this problem was proposed by Braunstein and Caves [4] through the description of statistical distinguishability by a Riemannian metric. This approach has laid the bases of which is called today information geometry, an area of knowledge that applies the methods of differential geometry in the context of information theory [5].

In this formulation, both classical and quantum state spaces define manifolds endowed with a structure that characterizes distances between its elements. In the classical case, Čencov theorem states that the Fisher information is the only metric in the space of probability distributions that is contractive under stochastic maps [6]. On the other hand, in the quantum case this identification is not trivial, and there is an infinite family of contractive distance measures in the space of quantum states whose characterization was given by Morozova, Čencov and Petz [7, 8] through the so-called MCP theorem.

Let $\mathcal{H}$ be a finite dimensional Hilbert space, $d = \dim\mathcal{H}$, and $\mathcal{S} \subset \mathcal{H}$ a convex set of density operators ρ, which is called the state space. This set defines a differentiable manifold $\mathcal{M}$ endowed with a tangent space $\mathcal{T}_\rho\mathcal{M}$ for each of its points. The tangent space is equipped with an inner product structure, inducing a metric on $\mathcal{M}$, which can be employed to quantify the distance between any of its points or even the norm of an operator.

Consider the spectral decomposition $\rho = \sum_j p_j |j\rangle\langle j|$ of the state ρ on the basis $\{|j\rangle\}_{j=1,\ldots,d}$, with $0 < p_j < 1$ and $\sum_j p_j = 1$. The MCP theorem states that the distance between the two neighbour density operators ρ and $\rho + d\rho$ in state space, i.e. the norm $\|d\rho\|$ of the tangent vector $d\rho \in \mathcal{T}_\rho\mathcal{M}$, is given by [8]

$$ds^2 = \frac{1}{4}\left[\sum_j \frac{(d\rho_{jj})^2}{p_j} + 2\sum_{j<l} C_F(p_j, p_l)|d\rho_{jl}|^2\right], \qquad (1)$$

where $d\rho_{jl} := \langle j|d\rho|L\rangle$, $C_F(x,y) := 1/[yf(x/y)]$ and $f(u)$ is the class of Morozova-Čencov (MC) functions. The first term on the right side of Eq (1), which is common to the whole family of metrics and depends only on the populations p_j of the state ρ, is the classical Fisher metric for the probability distribution p_j. The second term, associated with the non-uniqueness of the metric in the space of quantum states, is due to the coherence of $d\rho$ in the basis defined by the eigenstates of ρ. Such non-uniqueness of the distinguishability measures in state space arises due to injective mapping between Eq. (1) and the MC functions. Of all MC functions there is a minimum function, $f_{\min}(u) = 2u/(1+u)$, and maximum, $f_{\max}(u) = (1+u)/2$, which corresponds to the Fisher information [9, 10]. Thus, an arbitrary MC function $f(u)$ must satisfy $f_{\min}(u) \le f(u) \le f_{\max}(u)$ [11]. This is the case, for example, of the MC function $f_{WY}(u) = (1/4)(\sqrt{u}+1)^2$ associated with the Wigner-Yanase metric.

From the metric given in Eq. (1) we can define the Riemannian curvature tensor as

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\gamma,\delta} - \Gamma^\alpha{}_{\beta\delta,\gamma} + \Gamma^\mu{}_{\beta\gamma}\Gamma^\alpha{}_{\mu\delta} - \Gamma^\mu{}_{\beta\delta}\Gamma^\alpha{}_{\mu\gamma}, \qquad (2)$$

where

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\mu\nu}\left[g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}\right] \qquad (3)$$

are the Christoffel symbols while $g^{\mu\nu}$ are the components of the metric tensor. The Ricci tensor can be computed as $R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$ and the scalar curvature is $R = g^{\mu\nu}R_{\mu\nu}$.

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III. DYNAMICAL QUANTUM PHASE TRANSITION

The dynamics of closed quantum many-body systems has been the subject of considerable interest in the last decade. After a sudden quench, for instance, the support of a local operator will in general spread through all Hilbert space. The precise way through which this takes place reveals important information about the basic mechanisms underlying many-body dynamics. A particularly interesting example are the so-called dynamical phase transitions (DPTs), first discovered in Ref. [12], and subsequently explored in distinct situations. The fact that, in general, there is no connection between the regular equilibrium phase transitions and dynamical one was shown in Ref. [13]. See the recent reviews [14–16] regarding this subject.

The central quantity in the theory of DPT, which usually occur in a quenched quantum system, is the Loschmidt echo. The basic scenario consists in initially preparing a system in the ground-state $|\psi_0\rangle$ of some Hamiltonian H_0. At $t = 0$ the system is then quenched to evolve according to a different Hamiltonian H. The Loschmidt echo is defined as

$$\mathcal{L}(t) = |\langle\psi_0|\psi_t\rangle|^2 = |\langle\psi_0|e^{-iHt}|\psi_0\rangle|^2, \tag{4}$$

and therefore quantifies the overlap between the initial state and the evolved state at any given time. Put it differently, it measures how the support of the wavefunction spreads through the many-body Hilbert space.

In quantum critical systems the Loschmidt echo (4) is characterized by sharp recurrences (see Fig. 1(a) for an example). The nature of these recurrences is more clearly seen in terms of the rate function,

$$r(t) = -\frac{1}{d}\log\mathcal{L}(t), \tag{5}$$

where d is the system size. In the thermodynamic limit ($d \to \infty$), the rate $r(t)$ presents non-analyticities (kinks) at certain instants of time, as shown in Fig. 1, which are the hallmark of DPTs. The Loschmidt echo (4) presents a formal relation with a thermal partition function at imaginary time, which allows one to link these non-analyticities to to the Lee-Yang/Fisher [17, 18] zeros of $\mathcal{L}(t)$ (see Ref. [14] for more details).

A. Example: The Lipkin-Meshkov-Glick model

As an example, we consider the Lipkin-Meshkov-Glick (LMG) model [19–21], described by the Hamiltonian

$$H = -hJ_z - \frac{1}{2j}\gamma_x J_x^2, \tag{6}$$

where j is the total angular momentum, $h \geq 0$ is the magnetic field and $\gamma_x > 0$ (critical field $h_c = \gamma_x$). This model can be viewed as the fully connected version of a system of $d = 2j$ spin-1/2 particles (it therefore presents mean-field exponents).

B. Equilibrium quantum phase transition

Before describing the dynamical phase transition, let us first discuss the regular quantum phase transition for this model, which occurs in the thermodynamic limit, $j \to \infty$. In order to do this, it is convenient to define the so-called spin coherent state

$$|\Omega\rangle = e^{-i\phi J_z}e^{-i\theta J_y}|j\rangle, \tag{7}$$

where $|j\rangle$ is the eigenstate of J_z with eigenvalue j and $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ are polar coordinates. These states represent the closest quantum analog of a classical angular momentum vector of fixed length j, in the sense that the expectation value of the spin operators in the state (7) takes the form

$$(\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle) = j(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Moreover, expectation values of higher powers, such as $\langle J_x^2\rangle$, differ from $\langle J_x\rangle^2$ only by terms which become negligible in the limit of large j. As a consequence, it can be shown that to leading order the ground-state of the LMG model in the thermodynamic limit will be a spin coherent state for certain values of θ and ϕ [22]. The energy in this limit can then be computed as $E = \langle\Omega|H|\Omega\rangle$, resulting in [23]

$$\frac{E}{j} = -h\cos\theta - \frac{\gamma_x}{2}\sin^2\theta\cos^2\phi. \tag{8}$$

The ground-state energy is found by minimizing Eq. (8) over θ and ϕ, leading to the set of equations

$$\sin\theta(h - \gamma_x\cos\theta\cos^2\phi) = 0,$$
$$\gamma_x\sin^2\theta\cos\phi\sin\phi = 0.$$

For $h > \gamma_x$ the only solution is $\theta = 0$, in which case ϕ is arbitrary. For $h < \gamma_x$, however, two new solutions appear, corresponding to

$$\cos\theta = \frac{h}{\gamma_x}, \tag{9}$$

and $\phi = 0$ or $\phi = \pi$. The magnetization $m = \cos\theta$ therefore serves as the order parameter of the model, being identically zero for $h > \gamma_x$ or given by Eq. (9) otherwise. The appearance of these new solutions identifies the critical field $h_c = \gamma_x$.

C. Dynamical phase transition

Here we focus on quenches in the field h. The system is prepared in the ground state $|\psi_0\rangle$ of $H_0 = H(h_0)$ and at $t = 0$ is put to evolve under the final Hamiltonian $H = H(h)$.

To quantify the DPT we use the Loschmidt echo defined in Eq. (4) and the corresponding rate in Eq. (5), with $d = 2j$. In this model, a subtlety arises because the ground-state is two-fold degenerate. However, this introduces effects in the rate function which become negligible in the thermodynamic limit. For this reason, we henceforth focus only on the analysis starting from one of the ground-states.

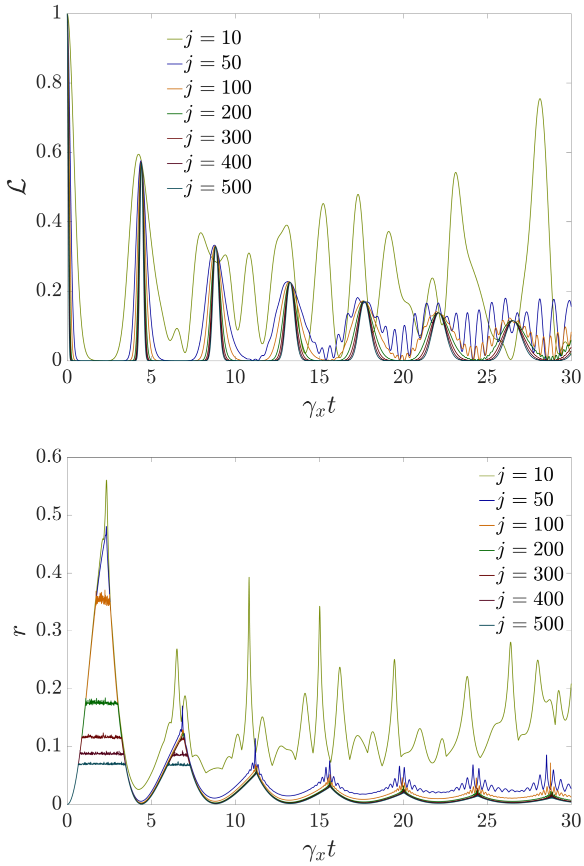

Figure 1. **Dynamical phase transition**. The top panel shows the Loschmidt echo (4) while the bottom one displays the dynamical behaviour of the rate function (5) for a quench in the LMG model, from $h_0 = 0$ to $h = 0.8$. Different curves correspond to different values of j. The plateaus that occur in $r(t)$ at short times is a numerical artifact. They happen because $\mathcal{L}(t)$ becomes exponentially small in these regions (even though this could be fixed with increasing the precision, this is not viable in practice because of the exponential dependence).

Figure 1 shows the DPT for quench from $h_0 = 0$ to $h = 0.8$ for several values of j. The echo (top panel in Fig. 1) is vanishingly small for certain periods of time, but presents sharp periodic revivals at certain instants. This is convoluted with a damping causing the magnitude of $\mathcal{L}(t)$ to decay in time. The presence of the DPTs becomes visible in the rate function (bottom panel in Fig. 1), which presents kinks at certain instants of time, the critical times t_c.

IV. INFORMATION GEOMETRIC APPROACH FOR EQUILIBRIUM PHASE TRANSITIONS

Considering classical systems, it was shown that a non-interacting many-body system has a flat geometry ($R = 0$), while R diverges at the critical point of an interacting one [24]. For this case, the metric is defined as the classical Fisher information matrix

$$g_{\mu\nu} = -\mathbb{E}\left[\frac{\partial^2 \ln p(x|\theta)}{\partial\theta_\mu \partial\theta_\nu}\right], \tag{10}$$

where θ is the set of relevant parameters (like magnetic field and temperature, for instance) and $p(x|\theta)$ is the classical probability distribution. $\mathbb{E}$ denotes expectation value. It is interesting that, in one dimensional Potts model, it was found that the curvature scalar also diverges at the Lee-Yang zeros, that are deeply connected with the DPTs.

Reference [25] presents a review of the information geometry techniques and also their applications on classical and quantum phase transitions. The authors describe how to define measures of complexity in both classical and, whenever available, quantum physical settings, from the tools of information geometry and how such a measures can be applied to the physics of phase transitions. All the analysis is focused on the scalar curvature of the parameter manifold, that diverges at the quantum phase transition. It is worth noting that the Kullback-Leibler divergence plays a central role in both the complexity approach to phase transitions and thermodynamics.

In Refs. [26, 27] the author employed the Uhlmann fidelity in order to study equilibrium and DPT at zero and finite temperature. This is an interesting paper since it relates a geometric quantity (Uhlmann connection) with the Loschmidt echo, a key quantity in the theory of DPT.

Reference [28] study the quantum phase transition of the LMG model by employing the fidelity and its susceptibility. A detailed analysis of the ground state Riemannian geometry induced by the metric tensor for the quantum XY chain in a transverse field is presented in Ref. [29], and the singularity of the curvature scalar was observed near the phase transition. The case of the Dicke model was considered in Ref. [30].

Another interesting study considers the geodesics on the parameter manifold for systems exhibiting second order classical and quantum phase transitions. It was concluded that geodesics are confined to a single phase and exhibit turning behavior near critical points [31].

The geometric critical exponents for systems that undergo continuous second-order classical and quantum phase transitions were defined in Ref. [32], while in Ref. [33] it was argued that the singularities in the informational metric are in correspondence with the quantum phase transitions.

In all of these studies, and in many others that can be found in literature, no mention of the scalar curvature is made. Therefore, the question posed in the introductory section still holds.

V. INFORMAITON GEOMETRIC DESCRIPTION OF DYNAMICAL PHASE TRANSITIONS

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Appendix A: Appendix I

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