Quantum fluctuation relation for a spin-2 quantum field

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More bla bla bla....

I. INTRODUCTION

The deviation of the light due to the Sun gravitational field is predicted by the null geodesic calculation within General Relativity. However, the tensorial character of the field is not taken into account, since the only feature that we need to know is that light, as a massless field, propagates on the null geodesics. Therefore, General Relativity predicts the same deviation angle to every massless field, including the gravitational radiation.

By employing quantum field theory it is possible to take into account the tensorial nature of the field by means of the differential scattering cross section. This method was applied to the case of photons [1], massless scalar bosons [2] and gravitons [3], resulting in the same deviation given by the null geodesic calculation. Such results give support to the conjecture that all massless fields follow the null geodesic. Here we consider the problem of the scattering of a graviton by the classical gravitational potential generated by the Sun from the perspective of quantum thermodynamics. Specifically, by employing the scattering matrix formalism we compute the probability distribution of the work that the Sun performs on the scattered graviton and show that such distribution fulfils a fluctuation relation.

We are interested in the work performed by the Sun's gravitational field on the gravitational wave. Specifically, we consider a weak gravitational wave, that can be described quantum mechanically, being scattered by the Sun. We then compute the scattering amplitude using perturbation theory and then relate such amplitude with the work probability density.

II. GRAVITON SCATTERING

Let us start by writing the spacetime metric as $\bar{g}_{\mu\nu} \approx g_{\mu\nu} + h_{\mu\nu}$, with $g_{\mu\nu}$ being the (classical) gravitational field of the Sun while $h_{\mu\nu}$ stands for the perturbation describing the gravitational wave. We are interested in the quantum description of the gravitational wave, that can be achieved only in the weak field regime, where a well defined notion of the graviton (the excitation of the quantum gravitational field) can be given. We thus consider the following problem. An incident graviton, coming from spatial infinity, is scattered by the Sun, generating an outgoing graviton (also at spatial infinity). We ask for the transition probability density between such states. From this, we can compute the statistics of the work performed by the Sun on the graviton.

Let us consider the radiative solution of Einstein's equations both for the incident and the scattered graviton. In this case, the Minkowiski metric provides a very good description of the background field (as long as the radiation does not exert backaction on the gravitational field), and we can attach a precise physical meaning to the concept of a graviton. This is the well known weak field approximation and the spacetime metric takes the form $\bar{g}_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski (flat spacetime) metric. Note that such expansion is valid only at spatial infinity (with respect to the Sun). In this sense, we can think about $h_{\mu\nu}$ as a quantum spin-2 field propagating on a flat classical spacetime. By keeping terms up to first order in h and choosing the harmonic coordinate system, the vacuum Einstein field equations become (see Appendix A and Ref. [4] for more details).

$$\Box h_{\mu\nu} = 0, \tag{1}$$

where the D'Alambertian operator is defined in terms of the flat spacetime $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$.

In order to proceed with the quantum description, we note that within this low energy regime, the general solutions of Eqs. (1) can be written as a linear combinations of the plane waves

$$h_{\mu\nu} = \sqrt{\frac{\hbar}{2\omega V}}\left[\varepsilon_{\mu\nu}(\sigma,k)e^{ik_\lambda x^\lambda} + \varepsilon^*_{\mu\nu}(\sigma,k)e^{-ik_\lambda x^\lambda}\right], \tag{2}$$

where $\varepsilon_{\mu\nu}(\sigma,k)$ is the polarization tensor (σ is the polarization) and the wave vector satisfies $k_\mu k^\mu = 0$ and $k_\mu \varepsilon^\mu{}_\nu = k_\nu \varepsilon^\mu{}_\mu/2$ (due to the harmonic condition). V is the normalization volume and the coefficient is determined so that the energy associated with the graviton equals $\hbar\omega$ (semiclassical treatment). Within this expansion, we can properly define the timelike Killing vector fields and, by employing well known techniques of quantum field theory we can define the gravitons. However, we do not need the full machinery of the theory since it will be enough a semiclassical approximation in order to compute the scattering amplitudes.

In order to compute the scattering matrix, we need to define the interaction of the graviton with the gravitational potential generated by the Sun, which is described by $g_{\mu\nu}$. The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}T_{\mu\nu}, \tag{3}$$

where $T_{\mu\nu}$ is the energy-momentum describing the graviton. Such quantity can be obtained by the variation of the action

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that can be expanded as $S = {}^{(0)}S + {}^{(1)}S + {}^{(2)}S + \cdots$, with the superscripts denoting the order of perturbation.

The second order term determines the interaction of the graviton with the background field and thus we consider such a term in order to compute the scattering matrix

$$
\begin{aligned}
{}^{(2)}S &= \frac{1}{16\pi}\int \mathrm{d}^4x\,\sqrt{-g}\left[\frac{1}{8}\nabla_\alpha\bar{h}\nabla^\alpha\bar{h}\right.\\
&\left.+\frac{1}{2}\nabla_\beta\bar{h}_{\alpha\beta}\nabla^\gamma\bar{h}^{\alpha\beta}-\frac{1}{4}\nabla_\gamma\bar{h}_{\alpha\beta}\nabla^\gamma\bar{h}^{\alpha\beta}\right],
\end{aligned}
\tag{4}
$$

where quantities without a label refer to the background classical field and we have used the combination $\bar{h}_{\mu\nu} = h_{\mu\nu} - (1/2)\eta_{\mu\nu}h$.

It is more convenient to use the combination $\overline{h}^{(s)}_{\mu\nu} = h^{(s)}_{\mu\nu} - (1/2)\eta_{\mu\nu}h^{(s)}$, where the only nonzero component is $\overline{h}^{(s)}_{00} = -4GM/rc^2$. In this case, $\mathcal{L} = -(1/2)\overline{h}^{(s)}_{\mu\nu}h_{\alpha\beta,}{}^{\mu}h^{\alpha\beta,\nu}$. We now choose the gauge $h_{o\mu} = h^\alpha_\alpha = 0$, implying that only h_{ij} is nonzero ($i, j = 1, 2, 3$). Therefore, we can write

$$
\mathcal{L} = -\frac{1}{2}\overline{h}^{(s)}_{00}h_{ij,}{}^{0}h^{ij,0}. \tag{5}
$$

How this equation is modified when the energy-momentum tensor is written in terms of a curved background. (Regarding formulation 2!)

The probability amplitude associated with the transition from the initial state $(\varepsilon^{(\lambda)}_{ij}, \vec{k})$ to the final one $(\varepsilon^{(\lambda')}_{ij}, \vec{k}')$ is then given by

$$
S = \frac{i}{\hbar}\int \mathcal{L}dVdt = \frac{2i}{\hbar}\int \bar{h}_{00}h'_{ij}h_{ij}\omega\omega' dVdt. \tag{6}
$$

In this expression, h_{ij} describes an incident graviton, with frequency ω, while h'_{ij} represents the scattered one, whose frequency is ω'. Following the standard semiclassical treatment, we find that the term containing $e^{-i\omega t}$ should describe the annihilation of the graviton, while the creation is represented by the term proportional to $e^{i\omega t}$. Therefore, we should have

$$
h_{ij} = \sqrt{\frac{\hbar}{2\omega V}}\varepsilon^{(\lambda)}_{ij}(\hat{k})e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{7}
$$

for the incident graviton and

$$
h'_{ij} = \sqrt{\frac{\hbar}{2\omega' V}}\varepsilon^{(\lambda')}_{ij}(\hat{k}')e^{-i(\vec{k}'\cdot\vec{r}-\omega' t)} \tag{8}
$$

for the scattered one. From this we obtain the probability amplitude

$$
S^{\sigma,\sigma'}_{\vec{k},\vec{k}'} = \frac{i}{V}\int \bar{h}_{00}\sqrt{\omega\omega'}\varepsilon^{(\sigma')}_{ij}\varepsilon^{(\sigma)}_{ij}e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}e^{-i(\omega-\omega')t}dVdt. \tag{9}
$$

Denoting by $\vec{q} = \vec{k} - \vec{k}'$ and introducing the Fourier transform of $\overline{h}^{(s)}_{00}(r)$ as $\overline{h}^{(s)}_{00}(q) = \int \overline{h}^{(s)}_{00}(r)e^{i\vec{q}\cdot\vec{r}}dV$, we obtain

$$
S^{\sigma,\sigma'}_{\vec{k},\vec{k}'} = \frac{i\pi}{V}\sqrt{\omega\omega'}\varepsilon^{(\sigma')}_{ij}\varepsilon^{(\sigma)}_{ij}\overline{h}^{(s)}_{00}(q)\delta(\omega-\omega'). \tag{10}
$$

From this expression we can compute the transition probability per unit time for the graviton within the solid angle $\mathrm{d}\Omega$ as $dw = \int |S^{\sigma,\sigma'}_{\vec{k},\vec{k}'}|^2\mathrm{d}\eta/t$, with $\mathrm{d}\eta = V(k')^2\mathrm{d}k'\mathrm{d}\Omega/(2\pi)^3$ counts the number of the corresponding states. The differential cross section is defined by $\mathrm{d}\sigma = dw/J$, with $J = c/V$ is the incident current. Therefore, the angular distribution for non polarized radiation is given by

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{G^2M^2}{c^4\sin^4\left(\frac{\theta}{2}\right)}\left[\cos^8\left(\frac{\theta}{2}\right)+\sin^8\left(\frac{\theta}{2}\right)\right], \tag{11}
$$

with θ being the angle between $\vec{k}$ and $\vec{k}'$, i.e it is the deviation angle. The details of the calculation are given in Appendix C.

III. QUANTUM FIELD THEORY APPROACH

The field equations for the graviton is

$$
\Box h_{\mu\nu} = 0 \tag{12}
$$

and we assume that each component of $h_{\mu\nu}$ obeys the Klein-Gordon equation, with $m = 0$. The momentum-space solutions (i.e. plane waves) are given by

$$
\text{plane wave} = \varepsilon_{\mu\nu}(k_\alpha)\,e^{-ik_\lambda x^\lambda}, \tag{13}
$$

where $\varepsilon_{\mu\nu}(k_\alpha)$ a 2-tensor indicating the gravitons polarization.

We expand $h_{\mu\nu}$ in terms of plane waves (as in Klein-Gordon field) and get

$$
\begin{aligned}
h_{\mu\nu}(x_\alpha) = \int \frac{d^3\vec{k}}{\sqrt{2\omega}}\sum_{\{\rho\sigma\}=00}^{33}&\left[a^{(\rho\sigma)}_{\vec{k}}\varepsilon^{(\rho\sigma)}_{\mu\nu}(k_\alpha)\,e^{-ik_\lambda x^\lambda}\right.\\
&\left.+\,a^{\dagger(\rho\sigma)}_{\vec{k}}\varepsilon^{*(\rho\sigma)}_{\mu\nu}(k_\alpha)\,e^{ik_\lambda x^\lambda}\right],
\end{aligned}
\tag{14}
$$

where the sum runs over the polarization basis $\{\rho\sigma\} = 00, 01, 02, \cdots, 23, 33$.

Scattering process — By adopting the gauge $h_{0\mu} = h^\alpha_\alpha = 0$), the full Hamiltonian is $H = H_0 + H_{\text{int}}$, where H_0 is the energy of the free graviton

$$
H_0 = \int d^3\vec{k}\;\omega\;a^{\dagger(rs)}_{\vec{k}}a^{(rs)}_{\vec{k}}, \tag{15}
$$

while H_{int} is the interaction Hamiltonian characterizing the scattering by the Sun gravitational potential

$$
\begin{aligned}
H_{\text{int}} &= \int d^3\vec{x}\,\frac{\bar{h}_{00}}{2}h_{ij,}{}^{0}(x_\alpha)\,h^{ij,0}(x_\alpha)\\
&= \int d^3\vec{x}\,\frac{\bar{h}_{00}\cdot\omega^2}{2}h_{ij}(x_\alpha)\,h^{ij}(x_\alpha)
\end{aligned}
\tag{16}
$$

Here $\bar{h}_{00}$ plays the role of the coupling constant. Try to justify the definition of the Hamiltonian in this last equation on the basis of Eq. (5) defining the Lagrangean density. Something we need to look for is if the integral is over space (i.e. $d^3\vec{x}$) or space-time (i.e. d^4x). Note that in this paper [Phys. Rev. D 93, 044027 (2016)] certain things related to the interaction

Hamiltonian and its quantization were done with a lot of care. We could consider this.

At any fixed time t_0 we can write

$$h_{ij}(\vec{x}) = \int \frac{d^3\vec{k}}{\sqrt{2\omega}} \sum_{\{rs\}=11}^{33} \left[a_{\vec{k}}^{(rs)} \varepsilon_{ij}^{(rs)}(\vec{k})\, e^{-i\vec{k}\vec{x}} + a_{\vec{k}}^{\dagger(rs)} \varepsilon_{ij}^{*(rs)}(\vec{k})\, e^{i\vec{k}\vec{x}} \right], \tag{17}$$

which depends only on space position $\vec{x}$.

In the interaction picture we get

$$h_{ij}(x_\alpha) = e^{iH(x_0-t_0)}\, h_{ij}(\vec{x})\, e^{-iH(x_0-t_0)}, \tag{18}$$

where $x_\alpha = (x_0, \vec{x})$. The graviton field now is defined on space-time position x_α, evolving due to the whole Hamiltonian (free + interaction). For $\bar{h}_{00} = 0$, the full Hamiltonian is reduced to the free one, leading to

$$h_{ij}(x_\alpha)|_{\bar{h}_{00}=0} = e^{iH_0(x_0-t_0)}\, h_{ij}(\vec{x})\, e^{-iH_0(x_0-t_0)} \equiv h_{ij}^I \tag{19}$$

the full expression of which is

$$h_{ij}^I(x_\alpha) = \int \frac{d^3\vec{k}}{\sqrt{2\omega}} \sum_{\{rs\}=11}^{33} \left[a_{\vec{k}}^{(rs)} \varepsilon_{ij}^{(rs)}(\vec{k})\, e^{-ik_\lambda x^\lambda} + a_{\vec{k}}^{\dagger(rs)} \varepsilon_{ij}^{*(rs)}(\vec{k})\, e^{ik_\lambda x^\lambda} \right]\Bigg|_{t=x_0-t_0} \tag{20}$$

In this picture, the interaction Hamiltonian takes the form

$$\begin{aligned} H_I(t) &= e^{iH_0(x_0-t_0)}\, H_{\text{int}}\, e^{-iH_0(x_0-t_0)} \\ &= \int d^3\vec{x}\, \frac{\bar{h}_{00}\cdot\omega^2}{2}\, h_{ij}^I(x_\alpha)\, h^{I\,ij}(x_\alpha), \end{aligned} \tag{21}$$

leading to the unitary evolution

$$U(x_0, t_0) = T\left\{ \exp\left[-i \int_{t_0}^{x_0} dt\, H_I(t) \right] \right\}, \tag{22}$$

where T stands for time ordered (always holds $x_0 \geq t_0$).

If $|k\rangle = \sqrt{2\omega} a_k^{\dagger ij} |0\rangle$ is a one graviton state of momentum k (and since it is massless, also energy), then the transition amplitudes between the asymptotically before and after the scattering momentum states is

$$_{\text{after}}\langle p|k\rangle_{\text{before}} = \lim_{T\to\infty} \langle p|\, U(T, -T)\, |k\rangle = \lim_{T\to\infty} \langle p|\, e^{-iH_I\, 2T}\, |k\rangle\,. \tag{23}$$

The transition probabilities are

$$P_{\text{before}\to\text{after}} = \left| _{\text{after}}\langle p|k\rangle_{\text{before}} \right|^2 \tag{24}$$

Can we prove the relation between this equation and Eq. (11) for the differential cross section?

IV. WORK

At time t_0, before the scattering, the graviton is on state $|k\rangle$ and its energy is measured, resulting in the outcome k with probability $P_i^0 = e{-}\beta k/Z_0$. After this the process takes place and the energy of the system (graviton) is measured for the second time, resulting in p with conditional probability (24), $P_{i|j} = \left| _j\langle p|k\rangle_i \right|^2$. The work in each run of the process is a random variable and is defined as $W_{i,j} = p_j - k_i$, associated with the joint probability distribution $P_{i,j} = P_i^0 P_{i|j}$. From these definitions (almost) we can prove the Jarzynsky equality.

Repeat the same calculations, but for photons. I am not sure which calculation here is referred to, but stuff not related to work was worked out by my PhD advisor [Phys. Rev. A 77, 052103 (2008)]. What is the difference? What is the relation between the geodesics and the transition probabilities? Can we make some sensible statement regarding the fluctuations of the the light cones? (since the gravitons walk on null surfaces).

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Appendix A: Weak field approximation of Einstein's field equations

In this appendix we briefly discuss the derivation of the wave equation (1). We refer the interested reader to the book [4]. Let us start by defining the Riemann curvature tensor

$$\bar{R}^\rho{}_{\mu\sigma\nu} = \bar{\Gamma}^\rho_{\mu\nu,\sigma} - \bar{\Gamma}^\rho_{\sigma\mu,\nu} + \bar{\Gamma}^\rho_{\sigma\lambda}\bar{\Gamma}^\lambda_{\mu\nu} - \bar{\Gamma}^\rho_{\nu\lambda}\bar{\Gamma}^\lambda_{\sigma\mu}, \tag{A1}$$

with the affine connection being given by

$$\bar{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}\bar{g}^{\delta\lambda}\left[\bar{g}_{\mu\delta,\nu} + \bar{g}_{\nu\delta,\mu} - \bar{g}_{\mu\nu,\delta} \right]. \tag{A2}$$

We have employed the notation $\partial_\mu f = \partial f/\partial x^\mu \equiv f_{,\mu}$ and the summation convention. $\bar{g}_{\mu\nu}$ is the spacetime metric.

Einstein's field equations in vacuum read $\bar{G}_{\mu\nu} = 0$, where $\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \bar{g}_{\mu\nu}\bar{R}/2$ and the Ricci tensor is

$$\bar{R}_{\mu\nu} = \bar{g}^{\lambda\delta}\bar{R}_{\lambda\mu\delta\nu} = \bar{\Gamma}^\lambda_{\mu\nu,\lambda} - \bar{\Gamma}^\lambda_{\lambda\mu,\nu} + \bar{\Gamma}^\rho_{\rho\lambda}\bar{\Gamma}^\lambda_{\mu\nu} - \bar{\Gamma}^\rho_{\nu\lambda}\bar{\Gamma}^\lambda_{\mu\rho}, \tag{A3}$$

while $\bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu}$ stands for the curvature scalar.

In the weak field approximation the metric takes the form $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski flat spacetime metric while $|h_{\mu\nu}| \ll 1$ is a small perturbation. Up to first order in the perturbation, the connection takes the form

$${}^{(1)}\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\eta^{\delta\lambda}\left[h_{\mu\delta,\nu} + h_{\nu\delta,\mu} - h_{\mu\nu,\delta} \right]. \tag{A4}$$

Since this is already written to order, the expression for the Ricci tensor becomes

$$\begin{aligned} {}^{(1)}R_{\mu\nu} &= {}^{(1)}\Gamma^\lambda_{\mu\lambda,\nu} - {}^{(1)}\Gamma^\lambda_{\mu\nu,\lambda} \\ &= \frac{1}{2}\left[\Box h_{\mu\nu} - h^\lambda{}_{\mu,\nu\lambda} - h^\lambda{}_{\nu,\mu\lambda} + h^\lambda{}_{\lambda,\mu\nu} \right], \end{aligned} \tag{A5}$$

where the D'Alambertian operator is defined in terms of the flat spacetime $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. Given these expressions, the field equations become

$$\Box h_{\mu\nu} - h^\lambda{}_{\mu,\nu\lambda} - h^\lambda{}_{\nu,\mu\lambda} + h^\lambda{}_{\lambda,\mu\nu} = 0. \tag{A6}$$

Due to the gauge invariance of the field equations, we can choose a coordinate system to work with. The most convenient one for the present case (and that leaves the field weak) is the harmonic system, for which $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$. This coordinate condition is the closest one to the inertial frames in nonrelativistic theory and guarantees that the coordinate functions x^{μ} fulfills d'Alembert's equation. To first order in h we have $h^{\lambda}{}_{\mu,}{}^{\mu} = h^{\mu}{}_{\mu,}{}^{\lambda}/2$. By substituting this into the wave equation written above, we directly obtain Eq. (1) of the main text.

Appendix B: Graviton energy-momentum tensor

The idea is to define a perturbative expansion of the action and, by employing calculus of variations, defining the energy-momentum tensor associated with the graviton.

Let us start by writing the Lagrangian density as $\mathcal{L} = {}^{(0)}\mathcal{L} + {}^{(1)}\mathcal{L} + {}^{(2)}\mathcal{L} + \cdots$. An upper index represents the order of the perturbation in the field $h_{\mu\nu}$, with respect to the background field $g_{\mu\nu}$. By plugging the expansion of the metric $\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ into Eq. (A3) we obtain to zeroth order

$$ {}^{(0)}R_{\mu\nu} = {}^{(0)}\Gamma^{\lambda}_{\mu\nu,\lambda} - {}^{(0)}\Gamma^{\lambda}_{\lambda\mu,\nu} + {}^{(0)}\Gamma^{\rho}_{\rho\lambda}\,{}^{(0)}\Gamma^{\lambda}_{\mu\nu} - {}^{(0)}\Gamma^{\rho}_{\nu\lambda}\,{}^{(0)}\Gamma^{\lambda}_{\mu\rho} \quad \text{(B1)} $$

with

$$ {}^{(0)}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\delta\lambda}\left[g_{\mu\delta,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta}\right]. \quad \text{(B2)} $$

We also need the expansion of the square root of the determinant of the metric, which, to second order, is given by

$$ \sqrt{-\bar{g}} \approx \sqrt{-g}\left[1 + \frac{1}{2}h^{\mu}{}_{\mu} + \frac{1}{8}h^{\mu}{}_{\mu}h^{\nu}{}_{\nu} - \frac{1}{4}h^{\mu}{}_{\nu}h^{\nu}{}_{\mu}\right], \quad \text{(B3)} $$

with g being the determinant of $g_{\mu\nu}$. This result can be derived by using the identities $\det(g+h) = \det g\, \det(1-g^{-1}h)$, $\log\det = \mathrm{Tr}\log$ and the power expansions of the log and exp functions.

From these equations, it follows the zeroth order term of the Lagrangian density ${}^{(0)}\mathcal{L} = \sqrt{-g}g^{\mu\nu}\,{}^{(0)}R_{\mu\nu}$. The variation of the action —with respect to $g_{\mu\nu}$— associated with this zeroth order results in the field equations ${}^{(0)}G_{\mu\nu} = 0$, with ${}^{(0)}G_{\mu\nu}$ being the Einstein tensor defined in terms of the background metric $g_{\mu\nu}$.

Moving to the first order term, the connection takes the form

$$ \begin{aligned} {}^{(1)}\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2} g^{\delta\lambda}\left[h_{\mu\delta,\nu} + h_{\nu\delta,\mu} - h_{\mu\nu,\delta}\right] \\ &\quad - \frac{1}{2} h^{\delta\lambda}\left[g_{\mu\delta,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta}\right], \end{aligned} \quad \text{(B4)} $$

leading to the following expression for the first order Ricci tensor

$$ \begin{aligned} {}^{(1)}R_{\mu\nu} &= {}^{(1)}\Gamma^{\lambda}_{\mu\nu,\lambda} - {}^{(1)}\Gamma^{\lambda}_{\lambda\mu,\nu} + {}^{(0)}\Gamma^{\rho}_{\rho\lambda}\,{}^{(1)}\Gamma^{\lambda}_{\mu\nu} + {}^{(1)}\Gamma^{\rho}_{\rho\lambda}\,{}^{(0)}\Gamma^{\lambda}_{\mu\nu} \\ &\quad - {}^{(0)}\Gamma^{\rho}_{\nu\lambda}\,{}^{(1)}\Gamma^{\lambda}_{\mu\rho} - {}^{(1)}\Gamma^{\rho}_{\nu\lambda}\,{}^{(0)}\Gamma^{\lambda}_{\mu\rho}. \end{aligned} \quad \text{(B5)} $$

The first order Lagrangian density is then given by

$$ \begin{aligned} {}^{(1)}\mathcal{L} &= \sqrt{-g}\left[{}^{(1)}R + \frac{1}{2}h^{\mu}{}_{\mu}\,{}^{(0)}R\right] \\ &= \sqrt{-g}g^{\alpha\beta}\left[{}^{(1)}R_{\alpha\beta} + \frac{1}{2}h^{\mu}{}_{\mu}\,{}^{(0)}R_{\alpha\beta}\right]. \end{aligned} \quad \text{(B6)} $$

Appendix C: Derivation of Eq. (11)

$|\overline{h}^{(s)}_{00}(q)|^2 = 16\pi^2 G^2 M^2/(c^4 k^4 \sin^4(\theta/2))$

Appendix D: Some discussion of gauge invariance

We start from the formula

$$ \delta S = -\frac{i}{2}\mathcal{T}\int d^4x\,\sqrt{|g(x)|}\,[T^{\mu\nu}(x)\,S]\,\delta g_{\mu\nu}(x)\,, \quad \text{(D1)} $$

and the infinitesimal coordinate gauge transformation

$$ x'^{\mu}(x) = x^{\mu}(x) - \epsilon^{\mu}(x)\,, \quad \text{(D2)} $$

which leads to [Phys. Rev. D 55, 658 (1997)]

$$ \delta g_{\mu\nu} = g_{\mu\nu,\sigma}\epsilon^{\sigma} + g_{\mu\sigma}\epsilon^{\sigma}_{,\nu} + g_{\nu\sigma}\epsilon^{\sigma}_{,\mu}. \quad \text{(D3)} $$

The last expression is nothing else than the Lie derivative of the metric and can be written as

$$ \begin{aligned} \delta g_{\mu\nu} &= \epsilon_{\mu;\nu} + \epsilon_{\nu;\mu} \\ &= \epsilon_{\mu,\nu} + \epsilon_{\nu,\mu} - 2\epsilon_{\sigma}\Gamma^{\sigma}_{\mu\nu}. \end{aligned} \quad \text{(D4)} $$

Due to the symmetry of $T^{\mu\nu}$ we have

$$ T^{\mu\nu}\delta g_{\mu\nu} = -2T^{\mu\nu}\left(\epsilon_{\sigma}\Gamma^{\sigma}_{\mu\nu} - \epsilon_{\mu,\nu}\right). \quad \text{(D5)} $$

After integration by parts we get

$$ \delta S = i\mathcal{T}\int d^4x\left(\epsilon_{\sigma}\Gamma^{\sigma}_{\mu\nu} + \epsilon_{\mu}\partial_{\nu}\right)\sqrt{|g|}\,[T^{\mu\nu}S]\,. \quad \text{(D6)} $$

In the last steps we need to use the identity

$$ \partial_{\nu}\sqrt{|g|} = \sqrt{|g|}\partial_{\nu}\ln\sqrt{|g|} \equiv \sqrt{|g|}\Gamma^{\sigma}_{\nu\sigma}, \quad \text{(D7)} $$

and relabel indices in order to get:

$$ \begin{aligned} \delta S &= i\mathcal{T}\int d^4x\left(\epsilon_{\sigma}\Gamma^{\sigma}_{\mu\nu} + \epsilon_{\mu}\partial_{\nu}\right)\sqrt{|g|}\,[T^{\mu\nu}S] \\ &= i\mathcal{T}\int d^4x\,\sqrt{|g|}\left[\epsilon_{\mu}\left(\Gamma^{\mu}_{\sigma\nu}T^{\sigma\nu} + \Gamma^{\sigma}_{\nu\sigma}T^{\mu\nu} + T^{\mu\nu}{}_{,\nu}\right)S\right] \\ &= i\mathcal{T}\int d^4x\,\sqrt{|g|}\left[\epsilon_{\mu}T^{\mu\nu}{}_{;\nu}S\right] \\ &= 0. \end{aligned} $$

The last equality is nothing else than $T^{\mu\nu}{}_{;\nu} = 0$.

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