Federal University of Goiás
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# An introduction to the Mathematical Structure of Spacetime

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# Dedication

*I dedicate this work to Professor Lucas Chibebe, my friends at CACP and my girlfriend Giulia...*

*...without them, it would not be possible.*

## Acknowledgments

*I am grateful to all the people who contributed directly or indirectly to the accomplishment of this project...*

*... family, friends and advisor.*

# ABSTRACT

This monograph aims to offer an introduction to the principles of relativity. Starting with Newtonian spacetime, a mathematical approach will be carried out in order to demonstrate the fundamental concepts of such a structure and, also, understanding the definitions of space and time, as well as, how its measurements should be performed. Such structure has certain divergences with other parts of physics. The theory of special relativity, which seeks to resolve such disparities, will be presented and generalized. Lastly, an investigation of the Einstein field equation for certain boundary conditions will also be made, seeking to understand how bodies move in certain regions of spacetime.



**Keywords**: Spacetime. Newtonian spacetime. Special relativity. General relativity. Black holes.

## RESUMO

Esta monografia tem como objetivo fornecer uma introdução aos princípios da relatividade. Começando com a estrutura Newtoniana do espaço-tempo, conceitos matemáticos preliminares serão trabalhados de modo a demonstrar seus fundamentos e, desta forma, entender as formulações básicas de espaço e tempo, assim como, compreender como suas medições devem ser realizadas. Levando em conta que existem certas disparidades entre tal estrutura e outras partes da física, como a eletrodinâmica, a Teoria da Relatividade Especial busca implementar um novo modelo, que será apresentado e generalizado, para resolver tais discrepâncias. Por último, a equação de campo de Einstein será explorada para certas condições de contorno, de modo tal que tenha-se a oportunidade de entender o movimento de objetos em uma região bem definida do espaço-tempo.

**Palavras-Chave**: Spacetime. Newtonian spacetime. Special relativity. General relativity. Black holes.

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# Chapter 1

# Introduction

Classical Mechanics is a field of physics that describes the motion of objects given some initial conditions. But, although the movement being part of everyday life, it is not common to hear people wondering why a pequi always describes a parabolic trajectory when it is thrown upwards.

To understand why it happens it is essential to study the structure which makes the motion to be the way it is. In physics this structure is called spacetime, which is a fundamental property of the universe. It is like a stage where all the things are placed and, therefore, must obey its rules.

Thus, this work will be oriented towards exploring how spacetime can be described, based on mathematical concepts and experimental facts, making a chronological sequence of the theoretical developments. From these considerations, we aim to answer the following question: Given the experimental results, how can one describe the mathematical structure of spacetime?

First, the spacetime structure will be investigated by starting from the Newtonian view. Here the affine spaces will be used to build up the Galilean Structure and so the classical physics we know. A discussion about special relativity will be held, in order to investigate the consequences brought to the classical structure of spacetime after the discovery that the speed of light is, in fact, a finite constant.

Afterwards, the theory of general relativity will be explored, seeking to understand what is gravity and how it is related to spacetime. Einstein's field equations will be presented a discussion about how they can be applied to a static and spherically symmetrical body, which will lead to the development of Schwarzschild black holes.

This project aims to introduce the principles of relativity, investigating mathematically the structure of spacetime according to theoretical developments until the 20Th century. It will be seen that such models have undergone changes that, with the scientific progress, has become necessary to explain certain phenomena that couldn't be explained by the classical model.

# Chapter 2

# Newtonian spacetime

Physics describes nature by means of mathematical models, which are then tested through experiments. Therefore, fundamental concepts as space and time only exist in physics if they can be represented in a measurable form [1].

One of these experiments could be the observation of the trajectory of a pequi when falling from the tree. The pequi was initially at rest on the tree and after being released, it accelerates in a straight line until it hits the ground.

From Newton's point of view, the acceleration occurs due to a force, intrinsic of all bodies, that makes Earth to attract the pequi and vice-versa. Therefore, if the conditions under which the pequi was released are the same, it will describe a similar trajectory independently of the place on Earth it was dropped. This fact leads us to conclude that every point in space must have the same properties.

The same is valid for time, since it flows at equal rates every time the experiment is performed. Thus, we can say that space and time are homogeneous, as long as they present the same properties at any place and instant. If this was not true, we would get different results every time the experiment is carry out in a different place, thus, it would be very difficult to construct a solid mathematical model to describe the involved phenomena.

Also, if we throw the pequi in different directions, we will always observe a parabolic trajectory, whose final point depends only on the initial conditions, i.e, the initial position and velocity of the pequi. This happens because the geometry of the space is the same regardless the direction and, as we saw earlier, position. Thus, we say that space is homogeneous and isotropic, while time, which has only one direction, is homogeneous.

To mathematically describe these physical properties of space and time, Newton needed to use what we call **Affine Spaces** with a **Galilean Structure** defined on it. This is the only kind of space that is compatible with the homogeneity of space and time and the isotropy of space. Also the Galilean structure has the role of shaping the related space according to the observational reality seen by Newton.

In this chapter we briefly describe the mathematical structure of spacetime in Newtonian mechanics. This will pave the way for us to discuss the modern theory of spacetime, the General Theory of Relativity, in the following chapters.

## 2.1 Affine Space

To describe the Newtonian structure we need to introduce the notion of Affine Space. Suppose we have a set $\mathbf{A}$ associated to the vector space $\mathbf{V}$ through a mapping, i.e., an operation by which we associate elements of $\mathbf{A}$ to that of $\vec{v} \in \mathbf{V}$. Mathematically we have [1] [2]

$$f : \mathbf{A} \times \mathbf{A} \to \mathbf{V}, \tag{2.1}$$

thus, two points in $\mathbf{A}$ are connected by a vector in $\mathbf{V}$.

The set $\mathbf{A}$ is called an Affine Space if the following axioms are satisfied.

1. Every pair of ordered points $(P, Q)$ of $\mathbf{A}$ determines a vector $\vec{v}$ in $\mathbf{V}$. So we can write

$$(P, Q) \in \mathbf{A} \to \vec{v} \in \mathbf{V}; \tag{2.2}$$

2. For every vector $\vec{v} \in \mathbf{V}$ associated with an ordered pair $(P, Q)$, there is only one point in $\mathbf{A}$ that satisfies $\overline{PQ} = \vec{v}$. Therefore, if we fix the point $P$, we can define

$$Q \in \mathbf{A} \to \overline{PQ} \in \mathbf{V}, \tag{2.3}$$

$$Q = P + \vec{v}; \tag{2.4}$$

3. $\overline{PQ} = \vec{0}$ if, and only if, $P = Q$;

4. If $\vec{v} = \overline{PQ}$ and $\vec{u} = \overline{QR}$, then $\vec{v} + \vec{u} = \overline{PQ} + \overline{QR} = \overline{PR}$.

Therefore, if $\mathbf{A}$ is an Affine Space, we can construct a vector through a pair of ordered points in $\mathbf{A}$, according to the axiom 1. Also, we can define the sum of a vector with a point $(P + \vec{v} = Q)$, to get another point. Such operation is called *translation* and it is represented by Eq. (2.4).

All the points of $\mathbf{A}$ are indistinguishable, therefore, we cannot define a preferential point in this space to serve as the origin. This situation is different from a vector space, which have a special element that we call *null vector* $\vec{0}$. Such name is because if we sum a vector $\vec{v}$ with $\vec{0}$ we get the same vector $\vec{v}$, i.e., $\vec{v} + \vec{0} = \vec{v}$. It means that $\vec{0}$ does not add anything to $\vec{v}$, thus we can use this element as an origin of $\mathbf{V}$. With the same reasoning, an Affine Space does not have a defined origin, which is a very special property, and it is because of it that it is appropriate to describe the Newtonian spacetime.

If the vector space $\mathbf{V}$ have a finite dimension $n$ the Affine Apace will be an $n-dimensional$ space, since $\mathbf{A}$ is defined over $\mathbf{V}$, so we can write $\mathbf{A}$ as $\mathbf{A}^n$, indicating the dimension, and the same for $\mathbf{V^n}$.

## 2.2 Euclidean Space

Now that we know what an Affine Space is, if we define the *inner product* $< .,. >$ on it, we can built an Euclidean Space with the same dimension. We can say that an Affine Space $\mathbf{A}$, associated with a vector space $\mathbf{V}$, is an Euclidean Epace $\mathbf{E}$ if $\mathbf{V}$ is equipped with the inner product

$$< \vec{v}, \vec{u} >= |\vec{v}||\vec{u}| \cos\theta, \tag{2.5}$$

where $\vec{v}, \vec{u} \in \mathbf{V}$ and $|\vec{x}|$ represents the norm of $\vec{x}$. The inner product is a projection of one vector on another, which makes an angle of $\theta$, and it is a real number. In this case, we have a projection of $\vec{u}$ over $\vec{v}$.

Moreover, calculating the norm of a vector is the same as calculating its length. But before we calculate it, its indispensable that we define time, since, to understand how we measure distances we first need to be clear in what circumstances it can be measured.

We can define time as a linear operator, observed by all inertial frames, which satisfies the relation

$$t(a_1\vec{v_1} + a_2\vec{v_2}) = a_1 t(\vec{v_1}) + a_2 t(\vec{v_2}), \tag{2.6}$$

where $t$ is a function acting on vectors and $a_1$ and $a_2$ are real numbers. We represent this function mathematically as [1] [2]

$$t : \mathbf{A}^4 \rightarrow \Re, \tag{2.7}$$

where $\Re$ is the set of real numbers. Thus, we can define a time interval as a function that connects the occurrence of two events.

When the time interval between two events is equal to zero we say that the events are **simultaneous** independently of the observer — we will explain what is an observer later. If we want to measure the distance between the events $P$ and $Q$ the time interval between them needs to be zero.

To understand why these events need to be simultaneous, suppose we want to measure a length of an object, e.g., a toy car, which is moving with velocity $\vec{v}$ with respect to the ground, as shown on Fig. 2.1. Measuring the position of the end $A$ at a time $t_0$ we get $x_A(t_0)$, and the other end $B$ at the time $t_1$ we get $x_B(t_1)$. So we have

$$L = |x_B(t_1) - x_A(t_0)|, \tag{2.8}$$

where $L$ is the length of the object. But doing this, we will be including all the path traveled by the object during the time interval $t_1 - t_0$ as a part of the length, and this is not true, as we can see in Fig 2.1$b$). So, to measure the real length of the object we need to get the positions $x_A$ and $x_B$ at the same time, thus, the Eq. (2.8) needs to be replaced by [3]

$$L = |x_B(t_0) - x_A(t_0)|. \tag{2.9}$$

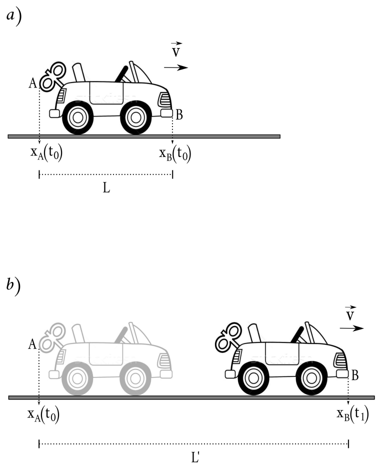


Figure 2.1: To measure the length of a toy car we need to mark the positions of the extremities $A$ and $B$ simultaneously.

Simultaneity is a very important concept regarding our understanding about time, which significantly affects how we make measurements.As already mentioned, time is a linear operator independent of all inertial frames, in other words, it is absolute. If two events are simultaneous for one of these observers, they will be simultaneous to any other. If we connect our reasoning with the above example, we can conclude that time being absolute takes us to an interval of length that is also absolute. This means that any inertial observer needs to measure the same length for the same object [3].

Now that we know how time is deeply connected with measurements of length, we can go back and calculate the norm of a vector $\vec{v} = (A, B)$. We have

$$|\vec{v}| = \sqrt{A^2 + B^2}, \tag{2.10}$$

where $|\vec{v}|$ is the distance between the final point $(A, B)$ and the origin $(0, 0)$. Precisely, in Eq. (2.10) we already defined $Q = (A, B)$ and $P = (0, 0)$, as illustrated in Fig. 2.2.

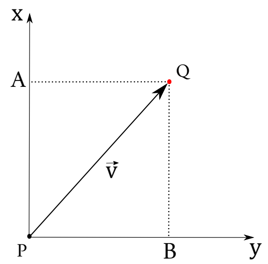


Figure 2.2: Illustration of $P$ and $Q$ at the coordinates axis.

To write the norm according to Eq. (2.4) we need to rewrite $\vec{v}$ as $\vec{v} = Q - P$, so we can define

$$|\vec{v}| = |Q - P| = |(A, B) - (0, 0)|. \tag{2.11}$$

Using these facts, we can define the distance between two arbitrary points, i.e., $P = (C, D)$, as [1]

$$d(P, Q) = |Q - P| = \sqrt{(A - C)^2 + (B - D)^2}. \tag{2.12}$$

## 2.3 Galilean Structure

The Galilean Structure deals with fundamental properties of the universe, time and space. But how can we group all these information? Actually is very simple.

The universe, in this structure, is described by a four-dimensional Affine Space $A^4$, where three of them are euclidean, which are isotropic and homogeneous, and the last one is the time dimension, that is homogeneous. All the points on $A^4$ are called *events*, e.g, $P$ and $Q$ in Fig. 2.2 are both events. Also, we associate two kinds of information about these points, which represent its location on spacetime.

To describe this information we usually use what we call coordinates. We already used it before. In Fig. 2.2 the points $A$ and $B$ are both spacial coordinates of the event $Q$ on the dimensions $x$ and $y$, respectively. These coordinates are defined positions through measurements on a rigid body of reference.

Notice that, event $P$ is described by the coordinates $(0, 0)$ and we build the axes from this point. It means we choose $P$ as the origin of the coordinate system, and we can do it just because we are working on an Affine Space.

Moreover, we can write Eq. (2.12) in a general way by using the line element $dS^2$, which determines the distance between two events on spacetime through coordinate differentials and the metric. Thus, in a three-dimensional space we write the line element as [4]

$$dS^2 = dx^2 + dy^2 + dz^2, \tag{2.13}$$

where $dx$, $dy$ and $dz$ are the coordinate differentials of the positions $x$, $y$ and $z$, respectively. The metric components can be written as a diagonal matrix of the functions that multiply the derivatives, in this case, it'll be the coefficients

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1, \end{pmatrix} \tag{2.14}$$

or we can write $g_{ij} = \delta_{ij}$, where $\delta_{ij}$ is the *Kronecker delta*.

The metric is what relates the coordinates differentials to lengths. Once we determined its coefficients it's possible to know the geometry of the spacetime. Since throughout the text we're going to work mostly with spherical coordinates, we can write Eq. (2.13) as

$$dS^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{2.15}$$

Putting all these information together we have the spacetime of classical mechanics, or what we call *Galilean Space*.

## 2.4 Reference frames

Now that we described most of the information about Newtonian spacetime, to finish our elucidation we should ask ourselves: "Who is the actor that makes the measurements in that spacetime?". This question is very important not just for studying Newtonian spacetime, but for all the theory we will approach in this report.

Before answering the question, it is important to know what is a reference frame. This can be characterized as a rigid body, big enough to measure something of interest, that spatially relates all events on spacetime, through three plane structures connected by a common point, that we call the dimensions $(x, y, z)$ of the space, and a line representing time.

Figure 2.1 shows the ground as a reference frame to calculate the length of the car, but not just that, we used the ground as a rigid body of reference also to measure the velocity of the car with respect to it, only then we were able to say that the car is moving with velocity $\vec{v}$. But what happens if we compare the car's velocity to other frame that is moving in a straight line with the same velocity $\vec{v}$? Since there is no movement with respect to each other, they would say that the other car is at rest.

Thereby, we can say that the movement is relative. It means that it depends on the reference frame. But there is a very important detail here, that we can have two types of frames, the inertial and the non-inertial ones. The difference between them lies which in the fact that one is under the effect of acceleration while the other one is not. Inertial frames are not accelerated, and are characterized by Newton's first law, i.e, an object at rest stays at rest and an object in motion stays in motion with the same speed and direction unless it's affected by a force. If a frame agrees to this law we say that it is an inertial frame and it can be referred as an *observer*.

We can have infinitely many observers, but all of them need to respect the following properties [1].

1. The laws of mechanics are the same to all observers;
2. If the initial position and momentum of some inertial frame is known, we can calculate its position and momentum in a future instant of time;
3. Each of these observers carry a clock built in a similar way, such that time flows at the same rate for all of them.

Thus, answering the previous question, the laws that determine the state of motion of a particle are valid just for inertial frames, so we have no choice but to use inertial frames as the ones to make measurements.

### 2.4.1 Galilean Transformations

Suppose we have two observers $S$ and $S'$, which satisfies the previous properties, with $S'$ moving with velocity $\vec{v}$ with respect to $S$. We can describe all the physics as seen by $S'$ in terms of $S$ and, when we do that we get the world lines of the system with new initial conditions. A world line is the trajectory describing the variation of the position with time, and its inclination is related to the velocity.

To transform the measurements of $S$ to $S'$ consider $(t, x, y, z)$ and $(t', x', y', z')$ as the coordinates of $S$ and $S'$, respectively. According to property (3), we have

$$t = t'. \tag{2.16}$$

Then, we do not need to transform time because it will be always the same for all observers.

To transform the spacial coordinates, suppose we want to measure the distance of some event $Q$ which is at rest with respect to $S'$ and it is moving with velocity $\vec{v}$ relative to $S$ in the $x$ direction. If, at instant $t = 0$ the axis $x$ and $x'$ are coincident, then $y = y'$ and $z = z'$.

Thereby, since $Q$ is moving relative to $S$, to measure the distance between $Q$ and $0$ he will need to calculate the distance between $0$ and $0'$ and then, add the length connecting $0'$ to $Q$, which we will call $b$. Therefore, considering $v = |\vec{v}|$ we will have

$$x = b' + vt, \tag{2.17}$$

where $x$ is the position of $Q$ relative to $S$ and $vt$ is the relative distance between $S'$ and $S$. We can generalize this by assuming any value of $b$ as

$$x = x' + vt. \tag{2.18}$$

So, the equations to transform the measurements of $S'$ to $S$ are

$$\begin{aligned} t &= t' & \qquad y &= y' \\ x &= x' + vt & \qquad z &= z'. \end{aligned} \tag{2.19}$$

The set of Eqs. (2.19) are called *Galilean Transformations*. As mentioned before, these transformations are used when we want to describe physics in different inertial frames.

According to Newtonian view of nature, we can say that time and space are intrinsic properties of the universe, which are independent of any observer. Time being independent means that we have a past, a future and a present well defined on the universe, which can be observed by all inertial frames.

# Chapter 3

# Special relativity

The previous chapter was focused on understanding the independent concepts of time and space in Newtonian physics and we saw that both are absolute quantities. But if I ask you, is there something wrong with the sentence "two events $A$ and $B$ occur at the same time independently of the observer"?, you will probably say "no". Indeed it is not a simple question that we could settle at first.

As we discussed before, physics is based on experimental facts, and with the advance of the electrodynamics and optics, it was possible to measure the speed of light in many different situations. According to Maxwell's equations for electrodynamics, the speed of light in vacuum needs to be constant and indeed, according to the measurements it was a constant. It is interesting that its value does not change with the speed of the object that carries it, having the same value independently of the observer [5]. This fact brings us some consequences that will change our mind about the previous question.

To understand that consequences let us work on some examples. Suppose two girls, Kyoshi and Toph, where the first is on a train that is moving in a straight line with a constant velocity near to the speed of light, in direction to a platform where Toph is at rest. Consider also that, there is light coming out from the front of the train, like a headlight. You may think that the train's velocity must influence the speed of light, increasing its velocity, but that's not true. The experiment shows us that, from Toph's perspective light will travel at the same speed $c$ as from Kyoshi's one, whatever being the velocity of the train.

Now consider the same situation, where the train is moving on a straight line towards the platform, but at the moment the middle of the train passes in front of Toph two lightnings hits the ends of the train. Both lightnings will travel the same distance to Toph, that lies at rest exactly in the middle of them, so she will see them simultaneously. On the other hand, if Kyoshi is sitting at the middle of the train, the lightning that struck the front end will travel a shorter distance up to her than the lightning that struck the back end, since she is moving and light always travel at the same finite speed.

Consequently, from Kyoshi's perspective both lightnings are not simultaneous, so we have a "paradox" here. By Newtonian structure we saw that if two events are simultaneous to one observer, then they will be simultaneous to any other one. However, this is not what is happening in our simple mental experiment, where we used the fact that light always have the same speed independently of the observer.

Simultaneity is not independent of the observer, thus, the sentence presented in the first paragraph is inconsistent for observers who have relative velocities close to that of light. Of course, under usual circumstances where the involved velocities are not comparable to the speed of light, we will not be able to distinguish between both point of views and simultaneity will appear absolute.

It is up to us (scientists) to explain what is wrong. So we could think the Galileo's principle of relativity, on which all inertial frames are equivalent and agree with the laws of physics, is invalid, but if so, then there must be some special reference frame at absolute rest, where other frames could be considered inertial when compared to it. The Michelson-Morley's experiment proved the nonexistence of that kind of frame at absolute rest, so the principle of relativity could not be incorrect, what leads us to believe that we needed to develop another structure to describe spacetime.

## 3.1 Einstein Structure of Special Relativity

From the previous arguments we saw why Newton's structure is not a general theory. We will start our description of the mathematical structure of spacetime by considering first inertial reference frames. A general theory will be build up after this, but let us take it easy and focus on how we should describe a new structure for spacetime without the presence of gravity.

We can deduce this structure using two facts that have already been mentioned before. Here we consider such facts as postulates of the Special Theory of Relativity (SR) [6]. They can be stated as follows.

1. *Universality of the speed of light*: The speed of light in vacuum is a constant, denoted by $c$, independent of the state of motion of the source.

2. *Principle of Relativity*: The laws of physics are equivalent to all inertial frames.

These are essential properties that we must not forget. Thus, they act like limiting factors to the theory we are about to describe. The first property also tell us that if the speed of light cannot have its value changed regardless of the state of motion of the source, it will be the greatest speed an observer could measure, what limits the velocity of all massive bodies on the universe in the sense that none of them can exceed this speed.

On the other hand, it is important to highlight that with the Newtonian structure we are able to do great things, among them taking the human being off the planet and launch satellites every day. Therefore, we must take into account that if we have an observer that is moving at low speeds with respect to the speed of light, there must be a consistency that links special relativity with the classical structure.

The second property defines a class of reference bodies that we will work with during the entire development of special relativity. They are the inertial frames, which will be called from now on just observers. Therefore, we must understand how they will be treated and how we can represent them in this theory.

### 3.1.1 Inertial Frames

An observer can be built in a very simple way. Imagine that we have three straight wooden sticks and that one end of each is firmly placed together in such a way that they lie on perpendicular directions. Each one of these sticks are called coordinate axis representing, for instance, the dimensions $(x, y, z)$. Also, this observer carries several synchronized clocks, which runs at the same rate, all over the respective wooden sticks. Such clocks measures the coordinate $t$ according to the observer.

Such structure is called a reference frame and distinct structures represent different frames. Additionally, they need to satisfy one more property in order to be called inertial, the first Newton's law. So we built up a structure to measure distances and, for each of them, we spread synchronized clocks all over it, but which are not synchronized with a different structure. Notice that it leads us to treat the time coordinate $t$ as a part of the frame as well as the coordinates $(x, y, z)$. That is interesting because now we can assume that the particles are not moving in a space where time is apart, but in a spacetime where the time is relative to each observer.

Now, as stated in Chapter 2, the distance between two events $A$ and $B$ can only be measured if these events are simultaneous. However, as each observer is able to measure its own time, we must be careful when defining simultaneous events for different frames, as noted in the introductory example. The notions of distance and time will become more clear when we analyze the spacetime diagrams.

### 3.1.2 Spacetime Diagrams

To understand Einstein's structure of SR it is essential to use the spacetime diagrams because it will allow us to visualize how an observer will measure a set of physical quantities for each event. For practical reasons, we are going to work on a two-dimensional spacetime using the four-dimensional notation

$$(x^0, x^1, x^2, x^3) = (ct, x, y, z), \tag{3.1}$$

to represent the coordinates.

A spacetime diagram with coordinates $(x^0, x^1)$ is illustrated in Fig. 3.1.

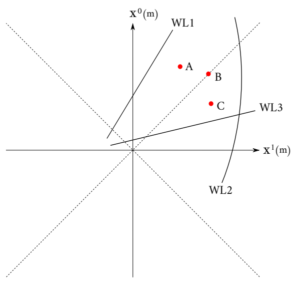


Figure 3.1: Spacetime diagram with events $A$, $B$ and $C$.

As we can see, there is a lot of information on it, we will work on every one of them to facilitate the understanding. First, we only have two coordinate axis, one for the time coordinate and another one for the spatial coordinate. The full coordinate system is composed by the four coordinates $(x^0, x^1, x^2, x^3)$, which is different from the Galilean structure, since there was no need to represent the time axis as it was absolute.

This structure of the spacetime tells us that it must be a four-dimensional vector space, being ideal for us, since we want a space where each observer is able to measure his own time, so we have no choice but to use this structure. If the coordinate $x^0$ is constant, the space can be treated as three-dimensional and agrees with the properties of an Euclidean space and the Galilean structure can be used, so that the theory we are describing is consistent with Newton's one for that limit.

Another very important is the structure delimited by the dashed lines that form an X in the figure, or one cone for each side of the $x^0$ axis (in bigger dimensions). Such structure is called the light cone. We can see it better in Fig. 3.2, where a three-dimensional spacetime diagram is shown.

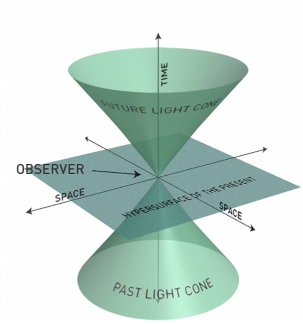


Figure 3.2: Light cone in three dimensions.

The light cone is the heart of Special Relativity, and is fundamental to understand this theory, thus, before we go back to the spacetime diagram, let us understand what a light cone is and what kind of information it carries. There are two cones, one that grows in the same direction of the time axis and another one that grows in the opposite direction, representing the trajectories of light relative to the future and to the past of the observer, respectively. Also, the hyper-surface between the cones constitute the dimensions $(x^1, x^2, x^3)$ of the observer, and for a constant time it can be considered as an Euclidean Space.

The lines that trace the edges of the cone form an angle of $45°$ with the vertical axis, and consequently, to the horizontal lines too, which represents the path through which the light moves with speed $c$. Thus, if we take the tangent of this angle, relative to the vertical axis, we have

$$\tan 45° = \frac{dx^1}{dx^0} = 1, \tag{3.2}$$

where $dx^0$ and $dx^1$ are infinitesimal distances of the axis $x_0$ and $x_1$, respectively. Also note that, the fraction $\frac{dx^1}{dx^0}$ represent a velocity which is, for this angle, the speed of light. So we can write [6]

$$\frac{dx^1}{dx^0} = c = 1. \tag{3.3}$$

Therefore, we are left with the value $c = 1$ as a limit value for any velocity, since nothing can have a greater speed than light, which implies the impossibility of the existence of lines with greater angles.

Coming back to Fig. 3.1, observe that there are three points in red on the diagram, that points are elements of the Minkowski's geometry, they are called *events*, similarly to the Newtonian structure. Here we label them $A$, $B$ and $C$, each of their positions gives us different information about them. A line that joins the event $A$ to the origin of the coordinate axis forms an angle, relative to the $x^0$ axis, smaller than a line that joins the event $B$ to the origin (the line on which the light travels), it means that it is a possible path for a massive object, since the tangent of that angle will give us a velocity smaller than the speed of light.

Event $A$ is in the future light cone, while event $B$ happens on the light cone and event $C$ is outside the light cone and there is no causal relation between this event and the other two. If we imagine again a straight line that joins $C$ to the origin, we will have an angle greater than $45°$, and consequently, its tangent will give us a higher speed than $c$. We have special names for the events $A$, $B$ and $C$, but we will see this later, since it is better now to understand the meaning of the line element.

We can represent the trajectories of particles on the spacetime through solid lines drawn on the diagram, that are called *world lines*. Figure 3.1 shows three world lines, $WL1$, $WL2$ and $WL3$. These lines also form angles with the $x^0$ axis, which give us the information about the velocity of the particle that possibly follow such trajectory. For example, the wold line $WL1$ have a smaller angle than the world line of the light (the dashed line forming the light cone) and, as we saw, it means that the particle following such trajectory has a velocity smaller than $c$. We can write $|v^1| < 1$.

The world line $WL3$ has a slope greater than the light cone. Thus, the velocity of a particle following such trajectory would be greater than the speed of light ($|v^3| > 1$), which is not possible. World line $WL2$ has a curved path in the diagram, meaning that its relative velocity to the observer varies in time. Therefore, a particle following such trajectory is accelerated.

The spacetime diagram shown in Fig. 3.1 stands for an observer, let us say $S$. Suppose now that we have a second observer, $S'$, which moves in the $x^1$ direction with velocity $\vec{v}$ relative to $S$. How can we build a spacetime diagram to represent the axes $x'^0$ and $x'^1$, of $S'$, on the diagram of $S$?

We can ask about the world line of a particle at rest with respect to the coordinate system $S'$, in the origin. This will be a straight line with slope $\vec{v}$. Which is illustrated in Fig. 3.4.

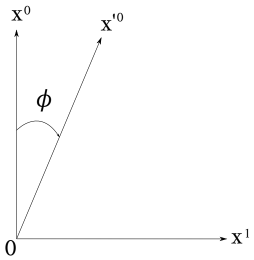


Figure 3.3: World Line of a particle moving in the $x^1$ direction with velocity $\vec{v}$ relative to the observer $S$.

Note that the origin of the frame $S$ and the $x'^0$ axis are common because we are considering that both observers started to be taken into account from the same initial time $x^0 = x'^0 = 0$. Following the same reasoning of the Eq. (3.4) we have

$$\tan\phi = \frac{dx^1}{dx^0} = v, \tag{3.4}$$

since $S'$ has a velocity $\vec{v}$ relative to $S$.

Thus, axis $x'^0$ corresponds to the world line of the observer $S'$ from the point of view of $S$. Also, note that all events with the same time coordinate measured by one observer will be simultaneous. Thus, all events that have a time coordinate $x'^0 = 0$ are simultaneous to $S'$ and draw the $x'^1$ axis as the place of all the events that occur at $x'^0 = 0$, as shown in Fig. 3.4.

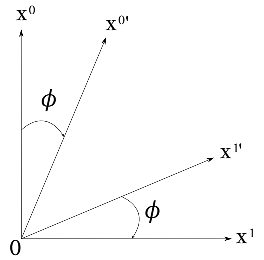


Figure 3.4: Coordinate $x'^1$ of $S'$ from the $S$ perspective.

Therefore, we were able to construct a set of coordinate axes $(x'^0, x'^1)$ of an observer $S'$ relative to another observer $S$, where $x'^0$ act like the world line of origin of $S'$ in the $S$ system, while $x'^1$ is the axis of all simultaneous events that correspond to the time $x'^0 = 0$. Also, note that if we mark a constant value for the time coordinate $x'^0$ and draw a straight line parallel to the axis $x'^1$, that line will represent all simultaneous events for the specified time value.

But if we are looking for the observer $S$ from $S'$ perspective, we will have the opposite situation, in which $S$ moves with velocity $-\vec{v}$ relative to $S'$. Thus, we can draw a spacetime diagram showing how the axes $x^0$ and $x^1$ of $S$ are positioned with respect to $S'$, as shown in Fig. 3.5.

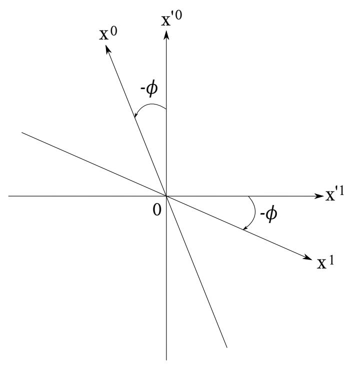


Figure 3.5: Spacetime diagram relative to the observer $S'$.

According to the figure, the velocity of $S$ relative to $S'$ can be written as,

$$\tan(-\phi) = -\tan\phi = -v. \tag{3.5}$$

Through the spacetime diagrams we can see that, unlike the structure built by Newton for the spacetime, we must now specify in detail what each reference system is measuring.

### 3.1.3 Simultaneity

There are some properties that were previously seen which remain valid, such as the homogeneity of space and time and the isotropy of space. The reasoning previously used, where we observed the fall of a pequi, is coherent, however when we are dealing with problems that involve very high velocities, which can be compared with the speed of light, we must keep in mind that the Newtonian structure will not give sufficiently accurate results. And for these cases, we will need to use Einstein's structure.

The structure in which we are working is known for including time as one of its dimensions, giving rise to what we call spacetime, and as we have already mentioned, we will have a Euclidean space only if we consider a constant moment in time, so that we will have a variation of the time coordinate $dx^0$ equal to zero. But this will be discussed in more depth later.

As we have been discussing during the chapter, time is a coordinate that makes part of the spacetime structure as the other coordinates $x^1, x^2$ and $x^3$. The spacetime diagrams must always contain a specific time coordinate, which are different for each observer. So in order to understand what it means to say that each observer measures its own time we are going to analyze a situation where we will have a series of events and their corresponding times measured by each of the observers.

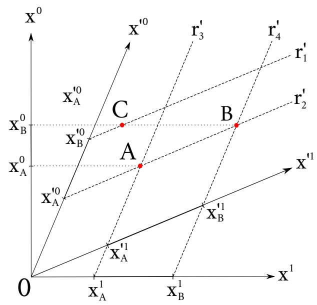


Figure 3.6: Different events measured by different observers.

Let us consider two observers, $S$ and $S'$, where the second one has velocity $\vec{v}$ relative to $S$. The spacetime diagram that represent this situation is drawn at Fig. 3.4, with the tangent of $\phi$ being equal to $v$ as shown in Eq. (3.4). Thus, the same situation is illustrated in Fig. 3.6, in which the angles were removed for clarity. Observe that there are three events, $A$, $B$, and $C$, which are localized at different places over the diagram.

In order to avoid any confusion, let us first analyze events $A$ and $B$. The lines $r'_1$ and $r'_2$ are both parallel to the coordinate axis $x'^1$. Events $A$ and $B$ occur over the line $r'_2$, and as you can see by the figure, both events have specific time coordinates for each observer. The event $A$ has coordinate ${x'}^0{}_A$ measured by observer $S'$ and coordinate $x^0{}_A$ measured by $S$. For event $B$ we have ${x'}^0{}_B$ and $x^0{}_B$ measured by observers $S'$ and $S$, respectively.

Note that, from the viewpoint of observer $S'$, the values of the time coordinates for both events are equal, so we say that the events $A$ and $B$ happened at the same time, thus, they are simultaneous events for $S'$. Now, as we can see, those same events have different time coordinates for $S$, implying that these events are not simultaneous in this frame.

Now, including the event $C$, we can see that it is located on the line $r'_1$ and has temporal coordinates ${x'}^0{}_C$ and $x^0{}_C$ with respect to $S'$ and $S$, respectively. Events located on lines parallel to $x^1$ must be simultaneous to the observer $S$. And indeed, events $B$ and $C$ have the same value for time as measured by $S$, thus being simultaneous events. Okay, but why simultaneous events for $S'$ will not be for $S$ or otherwise?

This is due to the principle of the universality of the speed of light. Since $c$ is constant and equal for everyone, the simultaneity lines for different observers will be distinct.

Physically, there is a relative velocity between the observers, when that velocity is big enough in comparison with $c$, we need to take account all the path traveled by light up to the observer, since he needs to see the event to measure it. As each observer has a different velocity, light must travel different distances to reach them, making time measurements distinct.

### 3.1.4 Line Element

As in the Newtonian physics, in order for us to measure the length of an object it is necessary that the ends of the object are measured at the same time. But, since simultaneity is a concept that depends on the observer, the lengths related to the dimensions $(x^1, x^2, x^3)$, measured by each observer, will provide different values.

For example, in Fig. 3.6 we can see two different lengths measured by the observers $S$ and $S'$, both corresponding to the distance between the events $A$ and $B$, which are $x^1{}_B - x^1{}_A$ and ${x'}^1{}_B - {x'}^1{}_A$, respectively. Physically, the rigid body cannot change its real length, which means that this difference is related to what each observer is measuring. We will discuss the difference between these measured lengths in more depth in the results section. Now let us talk about the invariant length of the object, which is calculated using the line element of Minkowski's space.

When measuring the length between any two events in a four-dimensional spacetime, we use a scalar product between two four-vectors describing the positions of an object in this spacetime. It leads us to

$$dS^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \tag{3.6}$$

that corresponds to the **line element** in Minkowski's space. This quantity defines the infinitesimal distance (or interval) between any two events that are separated in spacetime by the infinitesimal variations $(dx^0, dx^1, dx^2, dx^3)$. Note that, if we consider time as constant the variation $(dx^0)^2$ will be zero and the line element will correspond to Equation (2.13), since $(dx^1, dx^2, dx^3) = (dx, dy, dz)$.

Since $(dx^0)^2$ has a negative sign, we can now have a different case from the Newtonian structure, the line element can also assume a negative sign or be null. First let us analyze the case when $dS^2 = 0$, considering the two-dimensional case where two events $D$ and $E$ lies on the world line described by a light ray, as shown in Fig. 3.7.

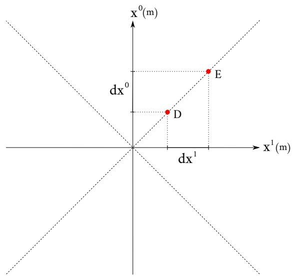


Figure 3.7: Events $D$ and $E$ localized on the world line described by a light ray.

Since the speed of light does not depend on the coordinate system and its world line makes a 45 degree angle with the observer's world line (axis $x^0$ in the Fig. 3.7), we can take the components of the distance between the events $D$ and $E$ on the $x^0$ and $x^1$ axes as equals. In this way, we can write

$$(dx^1)^2 - (dx^0)^2 = dS^2 = 0 \tag{3.7}$$

and then

$$dx^1 = dx^0. \tag{3.8}$$

Equation (3.7) will be true if, and only if, the events considered are on the light's world line. From this, we can define the sign of $dS^2$ based only on where the events to be measured are located. If the events are inside the light cone, its coordinate $(dx_0)^2$ will assume a value greater than $(dx^1)^2 + (dx^2)^2 + (dx^3)^2$, and therefore $dS^2$ will be negative. These events are called **time-like** events. For the events located outside the light cone, the opposite situation holds, thus, $dS^2$ will be positive, and as the spatial coordinates are also greater than time coordinate, we call these events **space-like**.

Note that, for a physical system, $dS^2$ cannot assume values greater than zero, because if so, the events will correspond to a speed greater than that of light. $dS^2$ will be null only for the light itself, since only they can move at the speed $c$.

We can also write the line element in terms of the metric, using Einstein's summation convention, such that

$$dS^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \qquad (\mu, \nu = 0, 1, 2, 3.) \tag{3.9}$$

with $\eta_{\mu\nu}$ being the components of the **metric tensor** describing Minkowski's geometry, and can be written as

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.10}$$

Since the metric tensor needs to be the same to all inertial frames, according to the principle of relativity, we can write the metric as

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{cases} -1 & if \quad \mu = \nu, \\ 1 & if \quad \mu = \nu = i \quad (i = 1, 2, 3), \\ 0 & if \quad \mu \neq \nu. \end{cases} \tag{3.11}$$

Relation $\eta_{\mu\nu} = \eta^{\mu\nu}$ will be satisfied to any flat space, since the metric $\eta_{\mu\nu}$ is exactly the same for all of them. It is also important to note that the three spatial dimensions and the time dimension of the line element are variables which, for each observer, we will have a set of specific values that correspond to these coordinates, but the whole is invariant. We call **proper time** the time which is measured by a clock that is at rest in the frame that actually passes through the events.

We can then say that the line element is a scalar as well as mass, that is, it assumes the same value for any pair of events regardless of the reference body used, since the physical length of the object cannot change. Therefore we can write,

$$dS^2 = dS'^2, \tag{3.12}$$

$$-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = -(dx'^0)^2 + (dx'^1)^2 + (dx'^2)^2 + (dx'^3)^2, \tag{3.13}$$

where, $dS'^2$ is the line element for the observer $S'$ mentioned before.

To understand why the interval must be invariant we can think about the Lorentz Transformations as rotations in Minkowski's four-dimensional spacetime, so the quadratic distance, or the interval $dS^2$, does not change. We must, also, note that Eq. (3.6) is similar to the hyperbola equation, but in four dimensions, since $dS^2$ is a scalar. Therefore, we can say, in fact, that physical distances are given by hyperboloids, and not by spheroids as in the case of the Newtonian spacetime.

## 3.2 Some Results and "Paradoxes"

The invariance of the line element in Minkowski's geometry brings important results that are not possible in Newtonian physics. Imagine a reference system $S'$ that is moving at a constant speed $\vec{v}$ with respect to $S$, whose diagram is shown in Fig. 3.8.

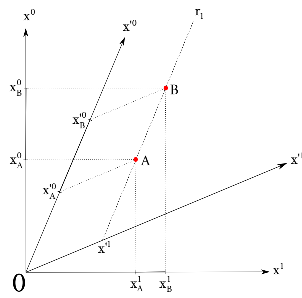

Figure 3.8: Events $A$ and $B$ located on the straight line $r_1$ parallel to $x'^0$.

According to the figure, observer $S'$ passes through the events $A$ and $B$ which has the same spatial coordinate $x'$ and the time coordinates ${x'^0}_A$ and ${x'^0}_B$ for $A$ and $B$, respectively. We can write the line element related to $S'$ as

$$dS'^2 = -({x'^0}_2 - {x'^0}_1)^2 = -(dx'^0)^2 \tag{3.14}$$

where,

$$dx'^1 = 0. \tag{3.15}$$

And the line element of $S$ can be written as in Eq. (3.6), but in two dimensions. Thus, according to Eq. (3.12), we have

$$-(dx'^0)^2 = -(dx^0)^2 + (dx^1)^2. \tag{3.16}$$

Isolating the coordinate $(dx^0)^2$ on the right side of this equation, we get

$$(dx'^0)^2 = (dx^0)^2 \left[1 - \left(\frac{dx^1}{dx^0}\right)^2\right], \tag{3.17}$$

where $\frac{dx^1}{dx^0}$ is the relative velocity $v$ of the observer $S'$. Finally, as was said before, the time measured by an observer such that his world line, or any line parallel to it, such as the line $r_1$, passes through the events specified in the spacetime diagram is called **proper time** of the interval between those events. The proper time can be written as $d\tau$, so Eq. (3.17) becomes

$$d\tau = dx^0\sqrt{1 - v^2}. \tag{3.18}$$

Moreover, note that the expression $\sqrt{1-v^2}$ will always be less than one in vacuum, since $v$ cannot be greater than 1 accordingly to the universality of the speed of light. Thus, $dx'^0 < dx^0$. So, the time interval measured by an observer that is moving with a constant velocity will always be shorter if compared to an observer at rest with respect to the measured system. We can conclude that the faster the observer is, the slower its clock will tick, this phenomenon is called **time dilation**.

If two observers see different times, then they must also see different distances. To facilitate our understanding, suppose that $A$ and $B$ are the end points of a rod that is at rest on the $x'^1$ axis, which moves with velocity $\vec{v}$ relative to $S$, according to Fig. 3.9. The length of the rod is the distance between two simultaneous events at its ends. Note that, by the figure, $A$ and $B$ are already simultaneous to $S'$, since they are on the $x'^1$ axis. So, the length measured by this observer is $L' = {x'^1}_B - {x'^1}_A$.

Now, as the rod is moving relative to $S$, $A$ and $B$ are not simultaneous. Thus, this observer will measure a length for the rod through the coordinates of the world lines $r'_1$ and $r'_2$ of the events $A$ and $B$ on the axis $x^1$, i.e, $x_A$ and $x_B$, since all points on it are simultaneous.

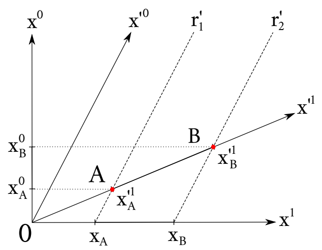


Figure 3.9: Distance between the events $A$ and $B$ as measured by $S$ and $S'$.

If $L = x^1_B - x^1_A$ is the length measured by $S$, then

$$L' = \frac{L}{\sqrt{1-v^2}}, \tag{3.19}$$

or, in a general way, we can write

$$dx'^i = \frac{dx^i}{\sqrt{1-v^2}}, \qquad (i = 1, 2, 3). \tag{3.20}$$

By the same reasoning that we used before, Eq. (3.20) tells us that the length of a rod, as measured by an observer that is moving relative to it, will always be less than the one measured by an observer that is at rest relative to it. Therefore, the greater the relative velocity of the rod, the shorter will be the length of it, based on the measurements of some observer that is at rest relative to that rod. We call this effect **length contraction**.

As has been said before, the rod will not actually change its length, the differences in measurements arise due to the principle of the universality of the speed of light. Observers in relative motion with respect to each other will measure different things, since the ends of the rod need to be measured simultaneously, and as we saw, we cannot have two simultaneous events for observers in relative motion with respect to each other. Thus, one of them will be forced to measure the distance from the extension of the world line of the events analyzed on the line parallel to its spatial axis, so that in this way, the ends of the object can be measured simultaneously.

Time dilation and length contraction are important results in the Special Theory of Relativity, which are not possible to be described by the Newtonian world, and, as a consequence of these phenomena some "paradoxes" arise. One of them is the so called car and garage paradox, that we now describe.

Returning to our characters, Toph and Kyoshi, suppose the following situation. Kyoshi, as a professional driver, is in a car moving at a constant velocity in a straight line towards a garage where Toph is. Before that, they measured the length of the car and the garage at rest, so they both know that at rest the car is longer than the garage. Since the car is moving, from Toph point of view, it will suffer the effect of space contraction, so it will fit in the garage if the speed is high enough. But, for Kyoshi, the car is at rest and the garage is moving towards her. Thus, in her frame the garage undergoes the relativistic effect of space contraction, so will be impossible for the car to fit there.

As the car is moving at a constant speed along a straight line, so as the garage from Kyoshi's point of view, we can consider both as inertial frames, and according to the principle of consistency for the laws of physics, we have an apparent paradox here, since Kyoshi and Toph's conclusions are not consistent to each other when we apply the same laws of physics.

This apparent paradox arises due to the fact that we are thinking about an absolute simultaneity. It is a natural thought, since this absolutist view of the Newtonian world is much more common in our daily lives. To solve the paradox, we need to take into account that simultaneity is relative to each observer, and therefore, Toph and Kyoshi will be measuring different events.

In order for the car to fit the garage, its front and its back need to be inside the garage simultaneously. To know if the car is completely inside the garage, let us suppose that the garage has one entrance and one exit door which opens when the car approaches and closes when the car has passed by. If the entrance door get closed, without the exit door opens, we will have the two doors closed simultaneously, so the car is completely inside the garage. But if the exit door opens before the first one is closed, the car will not be totally inside the garage.

Also, consider that the end points of the car are labeled by the events $A$ and $B$ in Fig. 3.9, which Kyoshi is the observer $S'$ and Toph is the $S$ one. As we said before, these events are not simultaneous to Toph, they have a time gap of $x^0_B - x^0_A$, but from Kyoshi's point of view, they are simultaneous. A similar situation happens with the garage, but the doors will be closed simultaneously to Toph, as it is at rest with respect to her, and they will not be closed at the same time to Kyoshi, which is not at rest. This is a natural situation since both are measuring different things, thus, this is in fact, a situation that is in accordance with Special Relativity and which is expected when we take into account that simultaneity is relative.

This problem leads us to think about a more complicated version of it. What if Toph locks the car in the garage as soon as it passes through the front door? We saw that for her, the car will completely fits inside the garage. Thus, both will need to agree that the car fits, right? Actually, no... When Toph locks the car inside the garage, it will undergoes deceleration, making Kyoshi's frame no longer inertial. Therefore, the laws of physics are not the same for both, so we cannot compare them using Special Relativity, which leads us to the need for a more general theory.

# Chapter 4

# General relativity

In this chapter we will focus on introducing Einstein's field equations, thus providing the basis on which we can describe black holes in the next chapter. As we saw earlier, the SR theory describes the spacetime without the presence of gravity, and now, in order to understand the meaning of the Einstein field equations we need to introduce such element.

To understand how the motion of bodies are affected by gravity, Galileo first performed an experiment using an inclined plane where he left a sphere to freely move between two specific points, say $A$ and $B$. He measured the time it took to the ball to leave point $A$ and reach point $B$. By changing the distance and taking the measurements, Galileo realized that the ball's velocity was increasing constantly.

But is the rate in which velocity increase the same to all the bodies? To answer that, Galileo performed the same experiment using a second sphere with different mass, and the result was the same: its velocity was increasing at a constant rate. But the interesting fact is, indeed, this rate was equal for both spheres, so it does not depend on the mass of the spheres. Therefore, if Galileo release the spheres from the same point $A$ on the inclined plane and let them move towards point $B$, they would arrive at the same time, i.e, if the plane is sufficiently inclined so that the air resistance can be neglected. If these spheres were released in free fall, outside the inclined plane, the air resistance would be sufficient to alter the results and the spheres would not reach the ground at the same time.

Then, by considering the situation where air resistance is neglected, Newton would say that the spheres fall in a constant accelerated motion, independently of its mass, due to a force generated by the Earth's gravitational field. Einstein interpreted the experiment differently. According to him, there is no force at all, and the spheres are in inertial frames, so the observer, i.e, Galileo, on the Earth's surface is a non-inertial frame.

Einstein went even further, he analyzed the experiment as if it were performed inside a capsule completely closed to the outside world. In this way, the observer would not know if the capsule was being accelerated upwards in space or if it was standing on the surface of the Earth. For both cases the spheres would reach the end point in the same instant of time (neglecting the air resistance inside the capsule). This implies that the inertial mass $m_i$, which is a form of resistance to variation in the state of movement of a body due to a force $\vec{F}_i$, is equal to the gravitational mass $m_g$, i.e, the intensity of the response of the sphere to the gravitational "force" $\vec{F}_g$ acting on it. Then, by using Newton's second law and considering that the forces $\vec{F}_i$ and $\vec{F}_g$ are equal to $\vec{F}$, we can write

$$\vec{F} = m_i \vec{a} = -m_g \boldsymbol{\nabla} \Phi, \tag{4.1}$$

where, $\vec{a}$ is an uniform acceleration of the sphere, $\phi$ is the gravitational potential and the minus sign is because of the different directions of the vectors. So, we have that

$$\vec{a} = -\boldsymbol{\nabla} \Phi, \tag{4.2}$$

which means that an uniform acceleration does not depend on the mass of the sphere, but only on the gravitational field.

Moreover, putting together these information with the Special Relativity, Einstein realized that, for sufficiently small regions, the spacetime looks like Minkowski space, i.e, locally, the spacetime agrees with all the laws of Special Relativity. This statement is known as **Equivalence Principle**. More generally, we can say that there is no experiment capable of locally detecting any difference between gravity and an uniform acceleration.

It is important to be clear that when we say "locally" we are referring to a point in spacetime where it is not possible to recognize any curvature. Thus, at that point, spacetime appears to be flat, and the same to local inertial frames. For example, if an ant walks on the surface of a sphere which is much larger than its body, it would not notice the sphere's curvature, for it the surface would be like a large plane. Gravity can also be treated locally, as we did in the experiments at the beginning of the chapter. In that situation the gravitational field looks homogeneous, while, in fact, it is converging to a point on the body's center of mas [7] [4] [6].

Now that we know this principle, let us study the behavior of light in the presence of an homogeneous gravitational field. To do this, imagine that Kyoshi is at the top of a rocket and she sends pulses of light at each minute (or any constant rate), as measured by her clock, to Toph, who is at the base of the rocket, along with a detector to measure the signals and send back a pulse of light at each minute, according to her own clock. If the rocket has a constant acceleration $\vec{a}$ pointing upwards (in some arbitrary coordinate system), as shown in Fig. 4.1, the pulses sent by Kyoshi will arrive faster at the Toph detector than they are emitted. Thus, Toph will receive the pulses at a greater frequency than they were sent. She can say that Kyoshi's clock is running faster than hers. This effect can be easily understood in terms of the constancy of the speed of light.

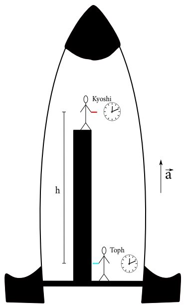


Figure 4.1: Kyoshi sending light pulses to Toph while both are accelerated by a rocket.

If we consider that the entire system is now stationary in a small region on Earth's surface, such that we cannot detect any variation on the gravitational field, the gravitational acceleration can be written as a constant vector $\vec{g}$ that is pointing downwards as shown in Fig. 4.2. Accordingly with the equivalence principle, we must observe the same effect that was observed inside the accelerating rocket, if $\vec{a} = -\vec{g}$. Otherwise it would be possible to locally distinguish gravity and acceleration. This effect is called gravitational time dilation.

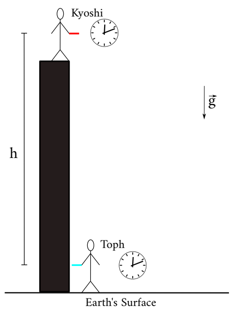


Figure 4.2: Kyoshi sending light pulses from the top of a tower to Toph which is at the ground. Both are under effect of an uniform gravitational field.

The increased frequency of the pulses in the case of the accelerating rocket means that the wave crests get closed to each other, and this effect is called blue-shift [7]. The opposite situation is called red-shift, in which the frequency of the light pulses received at the top of the tower is lower than that sent from below. Note that it is crucial that the gravitational field is homogeneous in order to make both situations equivalent.

Let us now move to a distinct situation. Imagine that Kyoshi is inside the rocket, which has acceleration $\vec{a}$, while Toph, who is outside of it and is carrying a laser with her, is inertial. Toph then turns on the laser, perpendicularly to the movement of the rocket, when it passes in front of her. From Kyoshi's (accelerated) point of view the light beam will pass through the rocket describing a parabolic trajectory, as showed in Fig. 4.3. This can be understood from the postulate that the speed of light is constant and finite.

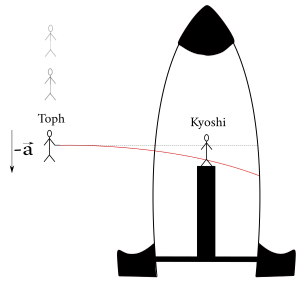


Figure 4.3: A light beam emitted by Toph.

Of course, from Toph point of view, light will travel following a straight line, but since the rocket is moving, it will reach the opposite walls of the rocket exactly at the points shown in Fig. 4.3. According with the equivalence principle, if we perform the same experiment but by considering the a stationary rocket on Earth's surface, the same result will be obtained. Therefore, under the action of a gravitational field, light follows a curved trajectory. We can say that the light is "freely falling" in the presence of gravity, as well as the whole class of inertial frames. The best way to explain why light "change" its trajectory is to consider that the spacetime is curved by the presence of mass and energy, so that non-inertial frames perceive this curvature.

From this analysis, we can conclude that objects in free fall, i.e, inertial frames, towards a source follow a trajectory, called geodesic, which is determined by the curvature, and the last is generated by the source itself. In this way, we can say that the gravitational field is the curvature of the spacetime. This is Einstein's general relativity.

## 4.1 First Postulates of General Relativity

As already mentioned, SR does not cover all situations and, therefore, we need a more general theory. Starting from the previous experiments, we can explore the postulates that gave rise to the famous Theory of General Relativity (GR), which is a theory that take into account the presence of the non-inertial frames into its structure. GR is based on four postulates. The first two can be stated as follows.

1. *The Equivalence principle*: Locally, the geometry of spacetime looks like Minkowski's, therefore, all the laws of physics must agree with Special Relativity.

2. *The spacetime is a differential manifold*: As a consequence of the equivalence principle we have no choice but to describe spacetime as a manifold, since it is a space that lo-

cally looks like the Euclidean space. Besides that, as the laws of physics are written in differential form, this manifold must be differentiable everywhere.

To calculate things in a space that is not the Euclidean space, $\Re^n$, it was developed the notion of a manifold, which is a space that may be curved, but locally it looks like $\Re^n$ [6], e.g., a $n$-sphere or a $n$-torus. By "looks like" we do not mean that the metric is necessarily the same of Minkowski, but can be some linear combination of it.

More precisely, we can define a manifold as a set of open subsets whose points can be mapped to real space $\Re^n$. A set or a subset is said to be open if, and only if, none of its points are on the border. If $M$ is a manifold and $A$ is an open subset, such that $A \in M$, then we can define the one-to-one mapping $\psi : A \rightarrow \Re^n$, i.e., each element of the image $\psi(A)$ has at most one element of the domain $A$ mapped into it, such that $\psi(A)$ must be also an open set as illustrated in Fig. 4.4.

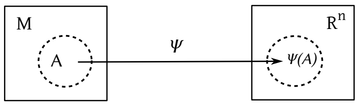


Figure 4.4: The map $\psi : A \rightarrow \Re^n$.

If we have an $n$-dimensional manifold $M$, for each value of $n$ there is a *one-to-one* correspondence between a subset $A_n \in M$ and an open subset of $\Re^n$, through the mapping [8]

$$\psi_n : A_n \rightarrow \Re^n, \tag{4.3}$$

where the number of maps $n$ represent the dimension of the manifold and the functions $\psi_n$ along with the open subsets $A_n$ are known in physics as **coordinate systems**.

The entire manifold is covered by the subsets $A_n$, and these are said to be "smoothly" joined together. In other words, every map $\psi_n$ must be continuous and infinitely differentiable everywhere [6]. This condition is imposed precisely because physical equations are written in terms of derivatives, so the degree of differentiability is adjusted according to the application under consideration.

Put it differently, the sets that form $M$ are smoothly overlapping, as shown in Fig. 4.5. We have two open subsets, $A_\alpha$ and $A_\beta$, which, through the maps $\psi_\alpha$ and $\psi_\beta$, leads to the images $X_\alpha$ and $X_\beta$, respectively. The shaded parts can be related through the *onto* maps $\psi_\beta \circ {\psi_\alpha}^{-1}$ and $\psi_\alpha \circ {\psi_\beta}^{-1}$, where these are defined only in the overlapping region and must be infinitely differentiable in this region by the considerations above [8].

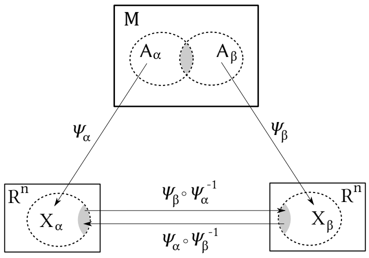


Figure 4.5: The coordinate systems $A_\alpha$ and $A_\beta$ smoothly overlap.

A manifold have as many as overlapping as necessary, so that the space is completely smooth, thus allowing us to work with derivatives of vectors and tensors without problems.

Now that we introduced what we call a reference frame, we are ready to state the third postulate of GR.

3. *The covariance principle*: The laws of physics must be the same in all reference frames.

To describe a more general theory than SR it is necessary that we write the laws of physics not only with respect to inertial coordinate frames, but also to non-inertial ones. Thus, for the physical laws to be invariant under any coordinate transformations we must write such laws using tensors. Such objects will be discussed in the next section.

## 4.2 Tensors and Curvature

As already established by the covariance principle, from now on it is necessary that we understand the basic concept of tensors, as well as what types of tensors we will work with, since we need to write the laws of physics in tensor form, because only them will be invariant under any coordinate transformations.

But, in order to understand the tensors it is necessary that, first of all, we understand what a vector in General Relativity means. In the Newtonian structure and in Special Relativity we did not take care of this because there we did not have to worry about the shape of spacetime, since its metrics keeps its form in the entire space. In other words, for these structures, the shortest distance between two points in spacetime will always be a straight line, and therefore vectors can be defined along it. But in General Relativity this is not true so we have to worry about how to define vectors.

According to the GR's third postulate we cannot build a coordinate system larger than a point in spacetime due to its curvature, so we cannot define a vector as a line segment in space. Vectors must be described locally, and one way to do that is to write them in a tangent vector space to a point in the manifold using directional derivatives.

A directional derivative operating on a function $f(x^\alpha)$ of the curve $x^\alpha(\xi)$, where $\xi$ is the curve parameter, is given by

$$\frac{\mathrm{d}f(x^\alpha)}{\mathrm{d}\xi} = \frac{\partial f(x^\alpha)}{\partial x^\alpha}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\xi}, \tag{4.4}$$

where

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}\xi} = v^\alpha, \tag{4.5}$$

can be identified as the components of the vector $\vec{v}$. Thereby we can define, for any vector $\vec{v}$ tangent to an arbitrary point on the manifold, the relation

$$\vec{v} = v^\alpha \frac{\partial}{\partial x^\alpha}, \tag{4.6}$$

where, $v^\alpha$ are the components of the vector $\vec{v}$ and the partial derivatives can be seen as the coordinates basis.

We can define a tensor as a linear map which takes a set of vectors and dual vectors into real numbers, and the total number of indices determines the rank of tensor. For example, in an $N$-dimensional space, a *0-rank* tensor, or a tensor of type $\binom{0}{0}$, is a scalar and can be represented by $N^0 = 1$ number, while a *1-rank* is a vector defined in the dual space, i.e., a linear functional that associates any vector defined in a vector space to a real number through the scalar product, which is represented by a set of $N^1 = N$ numbers and are commonly called *one-forms*. A general $n$-*rank* tensor, or, a tensor of type $\binom{i}{j}$ where, $i$ is the contravariant index and $j$ is the covariant one, and $i + j = n$, can be represented by $N^n$ numbers.

If $\vec{a}$ and $\vec{b}$ are two one forms, we can write a 2-rank tensor as the tensor product [6]

$$\mathbf{f}(\vec{a}, \vec{b}) := \vec{a} \otimes \vec{b}, \tag{4.7}$$

where the symbol $\otimes$ is called "outer product sign" which is in general not commutative ($\bar{a} \otimes \bar{b} \neq \bar{b} \otimes \bar{a}$). By this definition, we see that the product of two tensors also gives a tensor. If we write these one-forms in a given basis as

$$\begin{cases} \vec{a} = A^\alpha\, \vec{e}_\alpha, & to \quad \alpha = (0, 1, 2, 3) \\ \vec{b} = B^\beta\, \vec{e}_\beta, & to \quad \beta = (0, 1, 2, 3) \end{cases} \tag{4.8}$$

Eq. 4.7 becomes

$$\begin{aligned} \mathbf{f}(\vec{a}, \vec{b}) &= f(\, A^\alpha\, \vec{e}_\alpha, B^\beta\, \vec{e}_\beta \,) \\ &= A^\alpha B^\beta f(\, \vec{e}_\alpha, \vec{e}_\beta \,) \\ &= A^\alpha B^\beta f_{\alpha\beta}. \end{aligned} \tag{4.9}$$

In this equation, $f_{\alpha\beta}$ are the components of the tensor $\mathbf{f}$ in the chosen basis.

The metric is a tensor of type $\binom{0}{2}$, and using its components together with the Einstein summation notation we can raise and lower the indices of any tensor, as well as make contractions of them in some corresponding index. An example of contraction is the interval $dS^2$ itself, where the components $dx^\mu$ and $dx^\nu$ of the vectors $d\vec{x}$ are being contracted in the $\mu$ and $\nu$ indices with $g_{\mu\nu}$, thus resulting in a scalar $dS^2$.

In general, we can raise an index or lower it using a contraction with the metric tensor $g_{\mu\nu}$, for example, to rewrite the one-form $x^\mu$ of type $\binom{1}{0}$ in the form $\binom{0}{1}$, we apply the operation

$$x_\nu = g_{\mu\nu} x^\mu. \tag{4.10}$$

Similarly, we can also transform an one-form $x_\nu$ into $x^\mu$ by making

$$x^\mu = g^{\mu\nu} x_\nu, \tag{4.11}$$

where we can write $g^{\mu\nu}$ as well as $g^{\nu\mu}$, since the metric is symmetric in these two indices.

It is important to note that, the types of slots filled with certain indices must be respected throughout all the equation. For example, if we write an equation with a tensor of type $\binom{3}{1}$, which has three contravariant slots filled and one covariant slot filled, as [8]

$$T^{\mu\nu\gamma}_{\alpha} = S^{\mu\nu} \otimes K^{\gamma}_{\alpha}, \tag{4.12}$$

where $\mu, \nu, \gamma, \alpha = (0, 1, 2, 3)$ and the right side of the equation must also be of type $\binom{3}{1}$. But we can also write

$$T^{\mu\nu\gamma}_{\alpha} = T^{\mu\nu\beta\gamma}_{\alpha\beta}, \tag{4.13}$$

since we are making a *contraction* in the index $\beta$, so it continues to be a tensor of the same type. The operation called contraction is to be performed always between a contravariant and a covariant index, just like the previous equation.

For the development of the mathematical structure GR it is natural that the notions of local operators, such as the partial derivatives just presented, are generalized. In addition, physical equations must be written in terms of derivatives. The derivative is an instantaneous rate of change of a function at a point, thus, the derivative of a vector field describes its variation with respect to a direction in spacetime. This will allow us to compare vectors and tensors defined in different points over the manifold.

We did not care about defining a tangent space or tangent vectors on the manifold in the previous chapters because we knew that spacetime was completely flat, thus comparing vectors for that kind of space is very simple, since we do not have to worry about the shape of spacetime.

In the curved spacetime case, in order to compare any kind of quantity, other than a scalar, it is necessary that they are in the same tangent space. Thus, vectors defined in different tangent spaces must be "transported" to the same tangent space so that they can be compared.

Transporting a vector from one point in space to another is called **parallel transport**. It is performed by the covariant derivative of the vector. Therefore, before understanding what it means to parallel transport a vector we must build up derivatives on curved spacetime.

The rate of change of a vector $\vec{V}(x^\alpha)$, along a curve $x^\alpha$, in the $\mu$ direction, where $\alpha, \mu = 0, 1, 2, 3$, is defined by the gradient of $\vec{V}$ as

$$\boldsymbol{\nabla}_\mu \vec{V} = \frac{\partial V^\alpha}{\partial x^\mu} \vec{e_\alpha} \tag{4.14}$$

where $\vec{e_\alpha}$ are the basis vectors of $\vec{V}$. But when we are dealing with a curved space, or with polar coordinates, the base vectors do not remain constant when we derive a vector field, and this variation is not being taken into account in the case of the above equation. To do so, we need to add this change to the gradient expression. Thus, we will use an operator that takes into account the neighborhood of an arbitrary point in the manifold, and at the same time, it is reduced to a partial derivative in the case where the manifold is completely flat.

Now, taking into account the variation of the basis vectors, we will operate the gradient $\boldsymbol{\nabla}_\mu$ on the vector $\vec{V} = V^\alpha \vec{e}_\alpha$, such that

$$\boldsymbol{\nabla}_\mu \vec{V} = \frac{\partial \vec{V}}{\partial x^\mu} = \frac{\partial V^\alpha}{\partial x^\mu} \vec{e}_\alpha + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\mu}, \tag{4.15}$$

and we can write the partial derivative of the basis vector $\vec{e}_\alpha$ as the linear combination

$$\frac{\partial \vec{e}_\alpha}{\partial x^\mu} = \Gamma^\nu_{\mu\alpha} \vec{e}_\nu. \tag{4.16}$$

Substituting Eq. (4.16) into (4.15) we have

$$\boldsymbol{\nabla}_\mu \vec{V} = \frac{\partial V^\alpha}{\partial x^\mu} \vec{e}_\alpha + V^\alpha \Gamma^\nu_{\mu\alpha} \vec{e}_\nu. \tag{4.17}$$

Therefore, the covariant derivative of the components $V^\alpha$ of the vector $\vec{V}$ is

$$\boldsymbol{\nabla}_\mu V^\alpha = \frac{\partial V^\alpha}{\partial x_\mu} + V^\nu \Gamma^\alpha_{\mu\nu}, \tag{4.18}$$

where the first part on the right side of the equation represents a partial derivative of the field in the direction $\mu$, while the second part is a correction to make the result of the operation covariant, i.e, covariant derivatives are frame independent and holds in any coordinate basis.

Therefore, an operation performed by $\boldsymbol{\nabla}_\mu$ transforms a tensor of type $(1,0)$ into a tensor of type $(1,1)$, or more generally, a tensor of type $(m,n)$ will be transformed into a tensor of type $(m,n+1)$.

The $\Gamma^\alpha_{\mu\nu}$ is called connection, and, as the name suggests, it works in the sense of connecting an arbitrary point of the manifold with neighboring points. Although it looks like components of a tensor, it is not. Only the covariant derivative is a tensor.

In General Relativity, the connection is built according to the metric, since it tell us how is the curvature of the manifold and, therefore, how the neighborhood of the points belonging to it will be. We can write it as

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \quad \rho, \sigma, \mu, \nu = (0,1,2,3) \tag{4.19}$$

where $g_{\mu\nu}$ are the components of the metric tensor. This connection has a special name, the *Christoffel symbols* and we will use it always when we have a curved spacetime.

Note that, if the spacetime is flat, or if we are dealing with local inertial frames, we can write $g_{\mu\nu} = \eta_{\mu\nu}$, and since $\eta_{\mu\nu}$ remain constant, the Christoffel symbols will be zero. This implies that

$$\boldsymbol{\nabla}_\mu V^\alpha = \frac{\partial V^\alpha}{\partial x^\mu}, \tag{4.20}$$

which agrees with Eq. (4.14) for local inertial frames.

Now, in order for a vector to be parallel transported it is necessary that we impose the condition that the rate of change of the vector, for infinitely close points, along the considered curve must be zero. This means that the vector does not change during the transport.

In Minkowski spacetime we can parallel transport a vector $\vec{V}$ along a curve $x^\mu(\xi)$ if its components $V^\alpha$ remain constant all over the path, that is

$$\frac{\mathrm{d}V^\alpha}{\mathrm{d}\xi} = 0, \tag{4.21}$$

and by the chain rule, we can write

$$\frac{\mathrm{d}V^\alpha}{\mathrm{d}\xi} = \frac{\partial V^\alpha}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\xi} = 0. \tag{4.22}$$

To generalize it to curved spacetimes we replace the partial derivative by the covariant one, since the result must be invariant for an arbitrary change of coordinates. We are then left with

$$\frac{\mathrm{d}V^\alpha}{\mathrm{d}\xi} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}\boldsymbol{\nabla}_\mu V^\alpha = 0, \tag{4.23}$$

working explicitly with the gradient we have

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}\frac{\partial V^\alpha}{\partial x^\mu} + \frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}\Gamma^\alpha_{\mu\nu}V^\nu = 0, \tag{4.24}$$

where the derivatives $\frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}$ are the components of the tangent vector. Equation (4.24) is known as the parallel transport equation, and, as we said, keeps the vector's (or the tensor of arbitrary rank) information constant while it is being transported along a curve.

For curved manifolds, like the one we are working on, different paths taken by the parallel transport of a vector will result in different vector. This happens in curved spaces due to the connection, since different curves will result in different components of the connection.

In general, we can also parallel transport the tangent vectors of a curve in the same way we did with the previous vector field. If we substitute the components of the vector $\vec{V}$ to the components of the tangent vector $\frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}$, we will have

$$\frac{d}{d\xi}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}\right) + \frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}\Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\xi} = 0. \tag{4.25}$$

This equation represents the curve whose tangent vectors are parallel transported, thus, it describes the path where free particles move, and this path is as straight as possible (note that it is not a simple straight line, we can see it as a great circle on a sphere surface, for example). Such curves, for which Eq. (4.25) holds, are called geodesics and Eq. (4.25) is the **geodesic equation**. For flat spacetime the connection is zero, so the geodesic equation is simply

$$\frac{d}{d\xi}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}\right) = 0, \tag{4.26}$$

whose solution is given by

$$x^\mu(\xi) = a\xi + b, \tag{4.27}$$

which is a straight-line ($a$ and $b$ are real numbers). Thus, for flat spacetimes, free particles will always move along straight lines. But in the general case, the trajectory of free particles is described by Eq. (4.25) which give us a curve that depends on the curvature of the spacetime.

Now, if we compare the final result of a parallel transported tensor through two different curves, we realize that the final result will be different for each of them. For example, on the surface of a sphere, the vector $\vec{V}$ was parallel transported in two different paths, note that the resulting vectors are different from each other, as shown in Fig. 4.6.

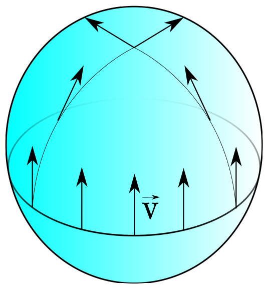


Figure 4.6: A vector $\vec{V}$ parallel transported along two different paths over a sphere.

If we calculate the difference between two infinitesimal paths, say $\boldsymbol{\nabla}_\mu\boldsymbol{\nabla}_\nu$ and $\boldsymbol{\nabla}_\nu\boldsymbol{\nabla}_\mu$, which parallel transports $V^\alpha$, we will have

$$\boldsymbol{\nabla}_\mu\boldsymbol{\nabla}_\nu V^\alpha - \boldsymbol{\nabla}_\nu\boldsymbol{\nabla}_\mu V^\alpha, \tag{4.28}$$

where $\boldsymbol{\nabla}_\mu\boldsymbol{\nabla}_\nu$ described a path and $\boldsymbol{\nabla}_\nu\boldsymbol{\nabla}_\mu$ the opposite path, thus forming an infinitesimal looping. This operation can be written as

$$[\boldsymbol{\nabla}_\mu, \boldsymbol{\nabla}_\nu]V^\alpha = \boldsymbol{\nabla}_\mu\boldsymbol{\nabla}_\nu V^\alpha - \boldsymbol{\nabla}_\nu\boldsymbol{\nabla}_\mu V^\alpha, \tag{4.29}$$

where $[\boldsymbol{\nabla}_\mu, \boldsymbol{\nabla}_\nu]$ is called the commutator of the covariant derivatives. If we compute Eq. (4.29) using Eq. (4.18) and the fact that

$$\boldsymbol{\nabla}_\mu(\boldsymbol{\nabla}_\nu V^\alpha) = \frac{\partial(\boldsymbol{\nabla}_\nu V^\alpha)}{\partial x^\mu} - \Gamma^\lambda_{\mu\nu}\boldsymbol{\nabla}_\lambda V^\alpha + \Gamma^\alpha_{\mu\sigma}\boldsymbol{\nabla}_\mu V^\alpha, \tag{4.30}$$

we will get

$$[\boldsymbol{\nabla}_\mu, \boldsymbol{\nabla}_\nu]V^\alpha = \left(\frac{\partial\Gamma^\alpha_{\nu\sigma}}{\partial x^\mu} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \frac{\partial\Gamma^\alpha_{\mu\sigma}}{\partial x^\nu} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right)V^\sigma + \left(\frac{\partial V^\alpha}{\partial x^\lambda} + \Gamma^\alpha_{\lambda\sigma}V^\sigma\right)(\Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu}), \tag{4.31}$$

where $\Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu}$ is a torsion tensor $-T^\lambda_{\mu\nu}$. The first term in parenthesis is the *Riemann curvature tensor* $R^\alpha_{\lambda\mu\nu}$. So, the last equation can be written as

$$[\boldsymbol{\nabla}_\mu, \boldsymbol{\nabla}_\nu]V^\alpha = R^\alpha_{\sigma\mu\nu}V^\sigma - T^\lambda_{\mu\nu}\boldsymbol{\nabla}_\lambda V^\alpha, \tag{4.32}$$

with

$$R^{\alpha}_{\sigma\mu\nu} = \left( \frac{\partial \Gamma^{\alpha}_{\nu\sigma}}{\partial x^{\mu}} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \frac{\partial \Gamma^{\alpha}_{\mu\sigma}}{\partial x^{\nu}} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \right). \tag{4.33}$$

Note that, if the connection $\Gamma^{\lambda}_{\mu\nu}$ is symmetric, we have $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$ thus the torsion tensor will be zero and Eq. (4.32) becomes

$$[\nabla_{\mu}, \nabla_{\nu}]V^{\alpha} = R^{\alpha}_{\sigma\mu\nu}V^{\sigma}. \tag{4.34}$$

Since the metric tensor is symmetric, i.e, $g_{\mu\nu} = g_{\nu\mu}$ in GR, the connection $\Gamma^{\lambda}_{\mu\nu}$ will also be and, consequently, the torsion tensor will always be zero. Note that Eq. (4.34), as we said before, is the relation between two different paths through which the components $V^{\alpha}$ are parallel transported, thus, the Riemann tensor describes how much these paths change with the curvature of the spacetime. If there is no curvature, the Riemann tensor will vanish, meaning that the metric is constant and all the laws of physics will be reduced to those of SR.

Moreover, the result of Eq. (4.29) will be different if we take first the path $\nabla_{\nu}\nabla_{\mu}$ and them $\nabla_{\mu}\nabla_{\nu}$, since we are going on the opposite direction of the loop. Thus, the result must be the negative of what we got. It means that interchanging the indices $\mu$ and $\nu$ we will get different answers. This implies that the tensor $R^{\alpha}_{\sigma\mu\nu}$, which depends on the path, must be anti-symmetric in these last two indices, i.e, $R^{\alpha}_{\sigma\mu\nu} = -R^{\alpha}_{\sigma\nu\mu}$.

We are now ready to state the fourth postulate of General Relativity.

## 4.3 Einstein Field Equations

By the equivalence principle, we saw that, locally, a gravitational field cannot be distinguished from constant acceleration, and therefore, it cannot be detected by any local inertial frame that performs experiments in a short period of time. This is because none of these frames perceive the inhomogeneity of the gravitational field, which does not happen when we consider larger regions in space, as has already been explained.

We also saw that spacetime is a differential manifold which locally agrees with Minkowski geometry, such that it is not possible to detect any curvature. In this way, we were able to relate such postulates, saying that gravity is nothing more than the manifestation of spacetime curvature.

This idea differs from the Newtonian structure, where gravity is treated as a force between bodies due to their masses, as we can see from the Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho, \tag{4.35}$$

where $\Phi$ is the **gravitational potential**, $G$ is the **Newton's gravitational constant** and $\rho$ is the **mass density** of the body.

In Newtonian physics, the potential $\Phi$ is generated by the mass density $\rho$, but when we talk about the Einsteins's GR, this equation must be generalized. According to what we saw, the gravitational field is the curvature of the spacetime itself, and therefore, it must also be generated by a mass density.

Moreover, we need to take into account that, differently from the newton structure, mass does not conserve by itself. For the conservation condition to be satisfied it is necessary that we use the **relativistic energy-momentum relation**, which tell us that the total energy of some local system is related to the mass of the system through the equation

$$E^2 = (cp)^2 + (mc^2)^2, \tag{4.36}$$

where, $E$ is the energy, $p$ is the momentum of the system and $m$ is the mass of the system, consequently, $mc^2$ represents the so called rest energy. However, the mass of the system can be obtained by the relation

$$m = \sqrt{\left(\frac{E}{c^2}\right)^2 - \left(\frac{p}{c^2}\right)^2}, \tag{4.37}$$

and it is the same for all local inertial frames (it is a scalar).

In this way, we can say that the total energy and momentum of the system modify the spacetime in order to create a curvature and that, in turn, it manifest itself as gravity, which describes the movement of the particles on spacetime.

In general, we can represent the source of the gravitational field by a tensor that represents the distribution of energy and momentum in a considered region of spacetime, in order to agree with the third postulate of GR. The tensor that describes such a distribution is called **Energy-Moment tensor T**.

The components of this tensor are written as $T_{\mu\nu}$, and as we can see from the number of indices, it is a rank-2 tensor and, therefore, can be represented by a $4 \times 4$ matrix with 10 independent components, since it is symmetric. In this way, we can write

$$[T_{\mu\nu}] = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix} = [T_{\nu\mu}]. \tag{4.38}$$

Note that, writing this tensor as $T_{\mu\nu}$ means that we already coupled it with the components of the metric tensor, since $T_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}$. The form of the energy-moment tensor will depend on the system we are dealing with. It can take different forms for different particle configurations. Moreover, the component $T^{00}$ is the energy density of the local rest frame of the body which represents the flux of the spatial component $p^0$ of the four-momentum $p^\mu = (p^0, p^1, p^2, p^3)$ over a surface of constant $x^0$.

The components $T^{i0}$ for $i = 1, 2, 3$ represent the flux of the momentum $p^i$, with $p^i$ being the spatial components of the four-moment $p^\mu$, over a surface where $x^0$ is constant. This is the momentum density. $T^{0j}$, for $j = 1, 2, 3$, is the flux of energy through a surface of constant $x^j$. The last components $T^{ij}$, for $i, j = 1, 2, 3$, are the flow of the $i$ momentum component in the $j$-direction.

Let us now consider the conservation law in the general form. By using Maxwell's equations we are able to show that the change in the electric charge density is balanced by the flow of current density in a certain region where the Minkowski geometry holds, i.e, to local regions in spacetime. The equation that describes this relation is called **continuity equation** and it can be written as

$$\sum_{\mu=0}^{3} \frac{\partial J^\mu}{\partial x^\mu} = 0 \tag{4.39}$$

where, $J^0 = \rho$ is the electric charge density and $J^i$ to $(i = 1, 2, 3)$ are the components of the current density.

Equation (4.39) represents the local conservation law of the electric charge, and the same must happens to the energy and momentum when considering local frames. Thus, we expect that the energy-momentum tensor must be conserved by

$$\sum_{\mu=0}^{3} \frac{\partial T_{\mu\nu}}{\partial x^\mu} = 0, \tag{4.40}$$

where we can write it in General Relativity as

$$\sum_{\mu} \boldsymbol{\nabla}^\mu T_{\mu\nu} = 0. \tag{4.41}$$

Remembering that raising and lowering indices are performed according to Eq. (4.11), $\boldsymbol{\nabla}^\mu = g^{\mu\alpha} \boldsymbol{\nabla}_\alpha$. Therefore, this equation represents the local conservation of energy and momentum.

As in Eq. (4.35) it is possible to write a proportionality relationship between the energy-momentum tensor and another one which is related with the spacetime curvature, but with the condition that it needs to be a rank-2 tensor which is symmetric and it also need to be conserved.

The Riemann tensor $R^\alpha_{\mu\nu\beta}$ is symmetric in its two first indices and asymmetric in the last two, as discussed earlier, but it is not a rank-2 tensor. Actually, there is another tensor that agrees with the conditions we have imposed, it is called the **Einstein Tensor** and can be written as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \qquad \mu, \nu = (0, 1, 2, 3), \tag{4.42}$$

where $R_{\mu\nu}$ is known as the Ricci tensor, $g_{\mu\nu}$ is the metric and $R$ is the curvature scalar. Note that, we do not specify the basis and Eq. (4.42) has a basis independent meaning, which makes this equation a tensor equation of type $(0, 2)$. The coordinate basis are chosen according to the symmetries of a particular spacetime in hand, as will be the case of the next chapter.

The Ricci tensor and the curvature scalar are both contractions of the Riemann tensor. They are constructed as

$$R_{\mu\nu} = \sum_{\alpha} R^{\alpha}_{\mu\nu\alpha} = R^{0}_{\mu\nu 0} + R^{1}_{\mu\nu 1} + R^{2}_{\mu\nu 2} + R^{3}_{\mu\nu 3}, \tag{4.43}$$

The curvature scalar is the contraction of the Ricci tensor with the inverse of the metric

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{4.44}$$

Moreover, since the Riemann tensor ($R^{\alpha}_{\mu\nu\beta}$) is symmetric in $\mu\nu$, the Ricci tensor is also symmetric. And as we already know that the metric is also symmetric, we can conclude that Einstein's tensor $G_{\mu\nu}$ will be symmetric as well. Now we have to prove that this tensor is also preserved under the four-divergence operation. In order to do that, we will use the Bianchi relations, which are written as [8]

$$\boldsymbol{\nabla}_{\alpha} R_{\rho\mu\nu\beta} + \boldsymbol{\nabla}_{\rho} R_{\mu\alpha\nu\beta} + \boldsymbol{\nabla}_{\mu} R_{\alpha\rho\nu\beta} = 0, \tag{4.45}$$

where we use the relation $R_{\rho\mu\nu\beta} = g_{\rho\alpha} R^{\alpha}_{\mu\nu\beta}$. Written in this way this tensor is asymmetric in its first two indices and in the last two ones, so we have that

$$R_{\rho\mu\nu\beta} = R_{\nu\mu\rho\beta} = -R_{\mu\rho\nu\beta} \tag{4.46}$$

and

$$R_{\rho\mu\nu\beta} = R_{\rho\beta\nu\mu} = -R_{\rho\mu\beta\nu}. \tag{4.47}$$

It is symmetric in its first and third indices, as well as in its second and fourth ones, as showed above. Contracting the relation (4.45) twice with the inverse of the metric we obtain

$$g^{\beta\mu} g^{\nu\alpha} \left( \boldsymbol{\nabla}_{\alpha} R_{\rho\mu\nu\beta} + \boldsymbol{\nabla}_{\rho} R_{\mu\alpha\nu\beta} + \boldsymbol{\nabla}_{\mu} R_{\alpha\rho\nu\beta} \right) = 0, \tag{4.48}$$

applying the product and using the asymmetry properties results in

$$g^{\beta\mu}g^{\nu\alpha}\nabla_\alpha R_{\rho\mu\nu\beta} + g^{\beta\mu}g^{\nu\alpha}(-\nabla_\rho R_{\alpha\mu\nu\beta}) + g^{\beta\mu}g^{\nu\alpha}\nabla_\mu R_{\alpha\rho\nu\beta} = 0. \tag{4.49}$$

Now, if we make the contractions with the equal indices, and use that $g^{\nu\alpha}\nabla_\alpha = \nabla^\nu$ and $g^{\beta\mu}\nabla_\mu = \nabla^\beta$, we will get

$$\nabla^\nu R_{\beta\nu} - \nabla_\rho R + \nabla^\beta R_{\rho\beta} = 0. \tag{4.50}$$

Finally, we can do $\nu = \beta$ and get the relation

$$\nabla^\beta R_{\rho\beta} = \frac{1}{2}\nabla_\rho R. \tag{4.51}$$

Now, we can make the covariant derivative of the Einstein tensor written with the indices $\rho$ and $\beta$, since this does not change the tensor, obtaining

$$\nabla_\alpha G_{\rho\beta} = \nabla_\alpha \left(R_{\rho\beta} - \frac{1}{2}g_{\rho\beta}R\right) \tag{4.52}$$

multiplying both sides by $g^{\beta\alpha}$,

$$g^{\beta\alpha}\nabla_\alpha G_{\rho\beta} = g^{\beta\alpha}\nabla_\alpha \left(R_{\rho\beta} - \frac{1}{2}g_{\rho\beta}R\right), \tag{4.53}$$

we can write it as

$$\nabla^\beta G_{\rho\beta} = \nabla^\beta R_{\rho\beta} - \frac{1}{2}g_{\rho\beta}\nabla^\beta R. \tag{4.54}$$

Since the covariant derivative of the metric vanishes in any direction. Using the relation (4.51) we get

$$\nabla^\beta G_{\rho\beta} = \frac{1}{2}\nabla_\rho R - \frac{1}{2}\nabla_\rho R = 0, \tag{4.55}$$

which proves that the four-divergence of Einstein's tensor vanishes. Therefore, we can write a proportionality relation between this and the energy-momentum ones as

$$G_{\mu\nu} = kT_{\mu\nu}, \tag{4.56}$$

where $k$ is a proportionality constant.

In order to determine the value of $k$, we take into account that General Relativity must agree with the Newtonian physics in the appropriate limit, i.e, when the particles are moving slowly when compared with the speed of light, the gravitational field is very weak and static. We find that the constant $k$ will be equal to $8\pi G$, where $G$ is the Newton's gravitational constant. Thus, the **Einstein Field Equations** are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{4.57}$$

Before discussing the above equation, it is necessary that we understand the physical meaning behind each term. First, let's talk about the Ricci tensor $R_{\mu\nu}$. This is a contraction of the Riemann tensor that represents the variation of an infinitesimal volume with respect to what it would be obtained in flat spacetime, when it is parallel transported along a geodesic in curved spacetime.

The curvature constant, or Ricci scalar $R$ represents how much the curvature varies with the distance from a given source. For example, in the Newtonian physics, we know that the gravitational field must decrease with $r^2$, where $r$ is the distance from the center of the source to some region of interest.

Therefore, the left side of the equation determines how the spacetime responds to the presence of matter, which is distributed according to the energy-momentum tensor that is in the right side of the equation. And, by the geodesic equation, we can compute how objects can move in spacetime when it is curved. Although Einstein's equations are similar to Maxwell's equations for electromagnetism, which determine the electromagnetic field due to the presence of current and charge densities, they are not simple equations to solve, due to the fact that they are not linear in $g_{\mu\nu}$. However, there are some circumstances in which such equations can be solved. In the next chapter we will show a very important solution of these equations, the so-called Schwarzschild geometry.

# Chapter 5

# Black holes

In this chapter we will study the first and simplest solution of the Einstein's equation. We will consider the vacuum solution of a spherically symmetric spacetime. We will also briefly describe one of its most impressive consequences, the black holes.

## 5.1 Schwarzschild Solution

In Electrodynamics we can calculate the electromagnetic field for a charged spherically symmetric body. Similarly, in General Relativity we can solve Einstein's field equations to calculate the gravitational field for the same kind of geometry. This is a very important and the simplest solution of Eq. (4.57). From the solution it is possible to describe the trajectory of an object that is at some distance $r$ of the field's source. Since our focus is to describe black holes, we will be concerned only with the exterior solution, i.e, in vacuum, so the equation that we need to solve is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{5.1}$$

The most general solution that describes a spherically symmetric gravitational field is, in spherical coordinates $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$, given by

$$dS^2 = -f(r)c^2dt^2 + g(r)dr^2 + h(r)r^2d\Omega^2, \tag{5.2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The chosen coordinate basis is due to the symmetries of the problem, and so, some of the equations we will write may be valid only in this basis.

The functions $f(r)$, $g(r)$ and $h(r)$ are independent of the time coordinate, since we are considering a *static* source. Terms containing one $dt$ (like, for instance, $dxdt$) will not be presented because the solution must be invariant under time reversal, that is, if we substitute $t$

by $-t$ all the metric components must be conserved.

To preserve the spherical symmetry another condition was applied to Eq. (5.2), that the form of $d\Omega^2$ must be invariant under rotations, so the coefficient of $d\phi^2$ must be $\sin^2\theta$ times that of $d\theta^2$, this implies that the functions in Eq. (5.2) cannot depend on $\theta$ or $\phi$.

Our job now is to find the functions $f(r)$, $g(r)$ and $h(r)$ such that the metric (5.2) becomes a solution of Eq. (5.1). In order to facilitate the calculus and since the functions of the equation (5.2) must be positive, in order to maintain the metric signature, we can rewrite it as

$$dS^2 = -e^{2\alpha(r)}c^2dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2d\Omega^2. \tag{5.3}$$

Making the change of coordinates

$$\bar{r} = re^{\gamma(r)}, \tag{5.4}$$

such that

$$d\bar{r} = e^{\gamma(r)}dr + re^{\gamma(r)}d\gamma = \left(1 + r\frac{d\gamma}{dr}\right)e^{\gamma(r)}dr, \tag{5.5}$$

Eq. (5.3) becomes

$$dS^2 = -e^{2\alpha(r)}c^2dt^2 + \left(1 + r\frac{d\gamma}{dr}\right)^{-2}e^{2\beta(r)-2\gamma(r)}d\bar{r}^2 + \bar{r}^2d\Omega^2. \tag{5.6}$$

We can now replace the coefficient of $d\bar{r}^2$ by $e^{2\bar{\beta}}$, resulting in

$$dS^2 = -e^{2\alpha(r)}c^2dt^2 + e^{\bar{\beta}(r)}d\bar{r}^2 + \bar{r}^2d\Omega^2, \tag{5.7}$$

and finally, we can replace the variables with a bar with other variables without it, so Eq. (5.7) is given by

$$dS^2 = -e^{2\alpha(r)}c^2dt^2 + e^{\beta(r)}dr^2 + r^2d\Omega^2. \tag{5.8}$$

In order to find the functions $e^{2\alpha(r)}$ and $e^{\beta(r)}$, we first need to find the Christoffel Symbols given in Eq. (4.19). Using the components of the metric as

$$\begin{cases} g_{tt} &= -e^{2\alpha} \\ g_{rr} &= e^{2\beta} \\ g_{\theta\theta} &= r^2 \\ g_{\phi\phi} &= r^2sen^2\theta \end{cases} \qquad \begin{cases} g^{tt} &= -e^{-2\alpha} \\ g^{rr} &= e^{-2\beta} \\ g^{\theta\theta} &= r^{-2} \\ g^{\phi\phi} &= r^2sen^{-2}\theta, \end{cases} \tag{5.9}$$

we find that the only Christoffel Symbols that do not vanish are given by [8, 4]

$$\begin{aligned}
&\Gamma^t_{tr} = \partial_r\alpha && \Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha && \Gamma^r_{rr} = \partial_r\beta \\
&\Gamma^\theta_{r\theta} = \frac{1}{r} && \Gamma^r_{\theta\theta} = -re^{-2\beta} && \Gamma^\phi_{r\phi} = \frac{1}{r} \\
&\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta && \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta && \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}
\end{aligned} \tag{5.10}$$

We must remember that, beyond these components, we also have the symmetric ones $\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$.

Our next step is to compute the components of the Riemann tensor. In order to do so we proceed in the following way: by substituting the relations (5.10) into (4.33) and using the anti-symmetric relation $R^\alpha_{\sigma\mu\nu} = -R^\alpha_{\sigma\nu\mu}$, the nonzero components of the Riemann tensor will be

$$\begin{aligned}
&R^t_{rtr} = \partial_r\alpha\,\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 && R^t_{\theta t\theta} = -re^{-2\beta}\,\partial_r\alpha \\
&R^t_{\phi t\phi} = -re^{-2\beta}\sin^2\theta\,\partial_r\alpha && R^r_{\theta r\theta} = re^{-2\beta}\,\partial_r\beta \\
&R^r_{\phi r\phi} = re^{-2\beta}\sin^2\theta\,\partial_r\beta && R^\theta_{\phi\theta\phi} = \left(1 - e^{-2\beta}\right)\,\sin^2\theta
\end{aligned} \tag{5.11}$$

Now, taking the contraction, we have

$$\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\,\partial_r\beta + \frac{2}{r}\partial_r\beta \\
R_{\theta\theta} &= e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1 \\
R_{\phi\phi} &= \sin^2\theta\,R_{\theta\theta},
\end{aligned} \tag{5.12}$$

which are the nonzero components of the Ricci tensor. Using Eq. (4.44) along with Eq. (5.9) and Eq. (5.12), we can find the curvature scalar [8]

$$R = -2e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta + \frac{2}{r}(\partial_r\alpha - \partial_r\beta) + \frac{1}{r^2}(1 - e^{2\beta})\right]. \tag{5.13}$$

Lastly, we can calculate the components of the Einstein Tensor by substituting the above equations into Eq. (5.1), but we can also set the Ricci tensor equal to zero, since the Einstein equation in vacuum can be written as

$$R_{\mu\nu} = 0. \tag{5.14}$$

It can be proved by contracting Eq. (4.57) with the metric tensor

$$
\begin{aligned}
& g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = -(8\pi G) g^{\mu\nu} T_{\mu\nu} \\
& R - \frac{1}{2} R = -(8\pi G) T \\
& R = -2(8\pi G) T,
\end{aligned}
\tag{5.15}
$$

and substituting this result again in the equation

$$R_{\mu\nu} = -(8\pi G)(T_{\mu\nu} + g_{\mu\nu} T), \tag{5.16}$$

which, in vacuum, leads us to Eq. (5.14).

Therefore, each component of the Ricci tensor must vanish independently, so we can write

$$e^{2(\beta-\alpha)} R_{tt} + R_{rr} = 0, \tag{5.17}$$

which gives

$$\partial_r \alpha + \partial_r \beta = 0. \tag{5.18}$$

Solving the equation we obtain $\alpha = -\beta + c$, where $c$ is a constant which can be set equal to zero. Then we have

$$\alpha = -\beta. \tag{5.19}$$

The $R_{\theta\theta}$ component, along with Eq. (5.19), leads us to

$$e^{2\alpha}(2r\partial_r \alpha + 1) = 1. \tag{5.20}$$

However, $e^{-2\alpha}(2r\partial_r \alpha + 1) = \partial_r(re^{2\alpha})$, therefore

$$\partial_r(re^{2\alpha}) = 1. \tag{5.21}$$

Solving it, we have

$$e^{2\alpha} = 1 + \frac{C}{r}, \tag{5.22}$$

where $C$ is some constant that we set equal to $-R_S$, where $R_S$ is called the **Schwarzschild radius**. This yields

$$e^{2\alpha} = 1 - \frac{R_S}{r}, \tag{5.23}$$

and, consequently

$$e^{-2\beta} = 1 - \frac{R_S}{r}. \tag{5.24}$$

Using Eqs. (5.23) and (5.24) we find the solution for Eq. (5.1), which is written as

$$dS^2 = -\left(1 - \frac{R_S}{r}\right) c^2 dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{5.25}$$

We still need to find the constant $R_S$. In the weak-field limit, i.e, when the gravitational field can be considered as a small perturbation on the flat spacetime, the metric component $g_{tt}$ is given by

$$g_{tt} = -\left(1 + 2\Phi\right), \tag{5.26}$$

where $\Phi = -\frac{GM}{rc^2}$ is the gravitational potential, $G$ is the Newton's gravitational constant and $M$ is the Newtonian mass of the source. Through this equation we can identify the Schwarzschild radius as $R_S = 2\frac{GM}{c^2}$, and so, the spherically symmetric vacuum solution of the Einstein's field equation is given by [8]

$$dS^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{5.27}$$

which is the famous Schwarzschild metric.

It is important to note that, in Eq. (5.27) $M$ is not just the sum of of all the masses that are contributing to the curvature. It also takes into account the **gravitational biding energy** [8], i.e., the minimum energy which must be added to the system to get it out of the gravitationally bound state. This is a consequence of Eq. (5.27) being a solution of Einstein's field equations.

Before proceeding with some consequences of this solution, we note that, if we take the limit $M \to 0$, or to a point far away from the source ($r \to \infty$), Eq. (5.27) recovers, as expected, the Minkowski space

$$dS^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2. \tag{5.28}$$

Such property is called **asymptotic flatness**.

## 5.2 Singularities

From Eq. (5.27), we can see that the metric coefficients are not well behaved if we make the substitutions $r = 0$ or $r = 2\frac{GM}{c^2}$, as the first and second terms become infinite. So what does it mean? There are two possibilities: there is a real singularity at these points or they are just a breakdown of the coordinate system.

The first possibility, a singularity, is a divergence point on the curvature, independently of the coordinate system. We could try to look at the Riemann tensor, which describes the curvature, and substitute the values of $\alpha$ and $\beta$, given by Eqs. (5.23) and (5.24), and see if it is well behaved or not. However, there is a problem with this line of thinking. As we mentioned before, when we choose a coordinate basis the tensor components will depend on it. We need to find something that is independent of the coordinate system.

A breakdown of a specific coordinate system occurs when the components of the tensor are not well behaved in some points, but we can fix this by a change of coordinates. So, to find a singularity we need to test the points to all possible basis, which would be a lot of work. But to our lucky, all the tensors can be contracted to a scalar which is coordinate invariant by definition.

Since the Riemann tensor describes the curvature, contracting it, using the relation $R_{\rho\mu\nu\beta} = g_{\rho\alpha}R^{\alpha}_{\mu\nu\beta}$ and the set of Eqs. (5.11), we get

$$K = R^{\rho\mu\nu\beta}R_{\rho\mu\nu\beta}, \qquad \rho, \mu, \nu, \beta = (t, r, \theta, \phi) \tag{5.29}$$

where $K$ is known as the **Kretschmann scalar**, and it is used to locate real physical singularities since if it is finite in any one coordinate system. This calculation yields

$$K = 48\frac{G^2M^2}{c^4r^6}. \tag{5.30}$$

Therefore, for $r = 2\frac{GM}{c^2}$ we have a breakdown of the specific coordinate system we are using, since Eq. (5.30) remains finite at this point. However, for $r = 0$ we have a divergence, and, therefore, such a point constitute a real singularity. You could ask yourself why we did not use the Ricci scalar, as it is also invariant. The fact is, since we are working outside the source of the field, i.e, in vacuum, we make $R_{\mu\nu} = 0$ and then, the scalar $R$ will also be zero.

The Ricci scalar being zero implies there is no curvature of the spacetime in this region we are working. But, as we know, it this is not true. If the spacetime is not flat, the Riemann tensor will not vanish everywhere and so will be the Kretschmann scalar. This is why the Kretschmann scalar is more appropriate to use in order to study the curvature.

## 5.3 Black Holes

To fix the problem of the surface $r = 2\frac{GM}{c^2}$ we should choose a more appropriate coordinate system, but before that, let us think about this point. Since $G$ is a constant, it will depends only on the quantity $M$, thus we can say that $R_S$ will be very small for bodies with little mass. For

Sun we have $M_\odot = 1,99 \times 10^{30}\ kg$ and $R_\odot = 6,96 \times 10^{8}\ m$, in comparison with $R_S$ we have

$$\frac{R_S}{R_\odot} = \frac{2GM}{c^2}\frac{1}{R_\odot} = 4,24 \times 10^{-6}, \tag{5.31}$$

where we used $c = 3 \times 10^8\ m/s$. Which shows that $R_S$ is far inside the Solar radius. Thus, in this case we will need to solve the Einstein's equation in the interior region to say something meaningful.

But the fact is, there are some objects which $R_S$ larger than $R$. These are known as **Black Holes** and for them, the total solution is given by Eq. (5.27). Thus, in this section we will study the causal structure of spacetime, as defined by the light cones, and explore what happens to it when we approach to the region $r = 2\frac{GM}{c^2}$ and also to smaller radii. Therefore we will consider radial null curves with $\theta$ and $\phi$ constants (spherical symmetry) in such a way that Eq. (5.27) becomes

$$dS^2 = 0 = -\left(1 - \frac{2GM}{rc^2}\right)dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2. \tag{5.32}$$

We then have

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{rc^2}\right)^{-1}, \tag{5.33}$$

Which measures the slope of the radial null curve, of the light cone, with respect to the $r$ axis in the $r - t$ diagram. Computing this at the points $r \to \infty$ and $r = 2\frac{GM}{c^2}$, we obtain

$$\frac{dt}{dr} = \begin{cases} \pm 1, & if \quad r \to \infty \\ \infty, & if \quad r = 2\frac{GM}{c^2} \end{cases} \tag{5.34}$$

The first case lead us to the flat space, the Minkowski geometry. This makes sense since we expect the curvature to decrease as the distance of the source increases. Now, for the second case, something interesting appears: as we approach the point $r = 2\frac{GM}{c^2}$ the light cone closes, meaning that nothing can escape such the region defined by this point. If we imagine that there are two observers, one at infinity and another one free falling towards such a region, the signals emitted by the falling observer will suffer a huge red-shift, in such a way that, at $r = 2\frac{GM}{c^2}$, the signal sent will take an infinity amount of time to reach the outside observer. That is why such a region is called a black hole.

Therefore, the proper time of the frame which dove into the black hole will corresponds to increasingly large intervals of the time interval of the observer who is far away from $r = 2\frac{GM}{c^2}$. In this way, we can suppose that it is impossible to get in there, but in the point of view of who dove in, there should be no problem to reach and cross this radius, since it is freely falling and detects no gravity. To solve the problem we must change our coordinate systems.

Solving Eqs. (5.33) we get

$$t = \pm \left[ r + 2\frac{GM}{c^2}\, ln \left( \frac{rc^2}{2GM} - 1 \right) \right] + C, \tag{5.35}$$

where $C$ is a constant. Then, by making

$$r^* = r + 2\frac{GM}{c^2}\, ln \left( \frac{rc^2}{2GM} - 1 \right), \tag{5.36}$$

where $r^*$ the so-called **tortoise coordinate**, we can rewrite Eq. (5.35) as

$$t = \pm\, r^* + C. \tag{5.37}$$

In this way, we can write the Schwarzschild solution in terms of the tortoise coordinates

$$dS^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right) dr^{*2} + r^2 d\Omega, \tag{5.38}$$

in which the metric components $g_{tt}$ and $g_{r^*r^*}$ are well behaved at $r = 2\frac{GM}{c^2}$, since both become zero. But it does not give us the complete information. To see what happens to the light cone we must use coordinates that are adapted to radial null geodesics, which are known as the **Eddington–Finkelstein coordinates**, that can be defined as

$$t = v - r^*, \tag{5.39}$$

which represents the radial null geodesics toward the black hole to a fixed $v$, and

$$t = u + r^*, \tag{5.40}$$

that is the outgoing radial null geodesics. Here we will work with the first one, since the method is the same. Thus, substituting Eq. (5.39) into Eq. (5.25), we obtain

$$dS^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dv^2 + 2dvdr + r^2 d\Omega^2. \tag{5.41}$$

As we can see, the metric components are completely well behaved at $r = 2GM$. Using this result, we can make the same process we did before in order to study the spacetime causal structure. We must consider the radial null curves with constant $\theta$ and $\phi$. Equation (5.41) leads us to

$$dS^2 = 0 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dv^2 + 2dvdr, \tag{5.42}$$

and then

$$\frac{dv}{dr} = \frac{2}{\left(1 - \frac{2GM}{rc^2}\right)}. \tag{5.43}$$

The above equation represents the tangent of the angle $\phi$ between the null line and the $r$ axis, as illustrated in Fig. 5.1. As we can see, when an observer falls towards the black hole, approaching to the radius $2\frac{GM}{c^2}$, Eq. (5.43) approaches infinity, which means that $\phi \to \frac{\pi}{2}$.

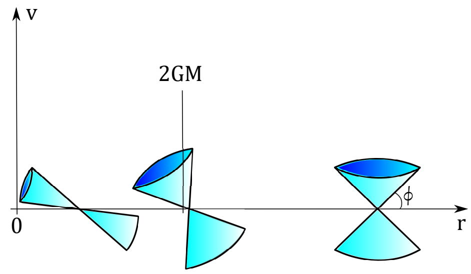


Figure 5.1: Schwarzschild light cones in the $(v, r)$ coordinates.

The surface $r = 2\frac{GM}{c^2}$ is known as the **Event Horizon**, which defines a region where a free particle could never return once it pass through, since it coincides with the null line of the light cone. Therefore, the future directed geodesics will be restricted in the direction of decreasing $r$. Also, once an observer enter in this region he can never prevent himself from moving towards the singularity at $r = 0$, since all the future directed path will lead him to it.

The causal structure of spacetime itself makes impossible to pass through the null surface, i.e., the event horizon, in the direction of increasing $r$, since not even light could cross it. This is why these objects got the name **black holes**.

There are two ways a black hole could be formed and both are related to the evolution of a massive star, i.e., a star whose mass is approximately 10 times larger than the Sun's. A star, which has a radius $R > 2\frac{GM}{c^2}$, in its main sequence transforms hydrogen in helium through nuclear reactions, maintaining a balanced relationship between its mass and luminosity. After this stage, the evolution will depend on the star's mass, but considering that we are dealing with a star with at least $10M_\odot$ the nucleus will collapse increasing the temperature, luminosity and the external layers, and the star will transform helium into carbon and successively until turning the nucleus into iron.

In this final stage there is not energy being produced and so there is nothing to balance the gravitational pressure, thus, the star collapse into a **supernova**. The remaining residue of a supernova is extremely compact and is called **neutron star**, and if its mass is bigger than $3M_\odot$ it will probably collapse into a black hole, since its radius will be less than $2\frac{GM}{c^2}$. The second way is, if the mass of a star exceed $25M_\odot$ the supernova will directly generate a black hole.

# Chapter 6

# Conclusions

In this work, it was discussed how space time is structured in the Newtonian view and how we can generalize it to a Minkowski space and also curved spaces. The generalization of classical theory is something that revolutionized, and continues to revolutionize, the worldview of several people who work in science.

The Newtonian structure is built on absolute space and time, where these quantities are treated separately and independently of who is measuring them, the observers. But a model built in this way collapses with electromagnetism when the last one proves that the speed of light must be invariant. For this, a new structure was built taking this fact into account, and when this is done, we have seen that the causal structure of spacetime is completely modified.

Furthermore, considering that light is a universal constant and that physics must be the same for any reference frame, we had no viable choice but to describe spacetime as a differentiable manifold and the equations of motion in form of tensors. Which led us to a solution where we had the opportunity to learn a little about Schwarzschild's black holes.

The General Theory of Relativity is one of the greatest scientific developments of the 20th century. This project was important for discussing this knowledge in a simple and accessible way, comparing it with a classic structure that lasted for many years. In addition, it was important to improve my critical thinking and prepare me for a stage in academic life where scientific research is exceptional.

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