Thermodynamic entropy production in the dynamical Casimir effect

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We address the question of entropy production in the context of the dynamical Casimir effect. Specifically, we consider a one-dimensional ideal cavity with one of its mirrors describing a prescribed trajectory. Inside the cavity we have a scalar quantum field and we ask about the changes in the thermodynamic entropy of the field induced by the non-trivial boundary conditions imposed by the moving mirror. By employing an effective Hamiltonian approach, we compute the entropy production and show that it scales with the number of particles created in the short-time limit. Moreover, such approach allows us to demonstrate that this entropy is directly related to the developments of quantum coherences in the mode basis of the field. A distinct approach, based on the time evolution of Gaussian states, allows us to study the long-time limit of the entropy production in single mode of the field. This result is a relation between the thermodynamic entropy production in the field mode with the entanglement between the considered mode and all the other modes. In this way, we link the entropy production in the field due to the dynamical Casimir effect with two fundamental features of quantum mechanics, coherence and entanglement.

I. INTRODUCTION

Although the fundamental laws of physics are invariant under time-reverse symmetry, irreversible phenomena can be found everywhere around us when the system under consideration is complex enough. Classically, such irreversibility is characterized by the second law of thermodynamics, which states that the entropy of a closed system always increases in time [1]. When fluctuations enter the game, stronger relations known as fluctuation theorems emerge [2, 3], and irreversible processes are those for which average entropy increases.

In the case of quantum systems, several approaches emerged in the search of understanding thermodynamics from the microscopic dynamics. Among such developments we can mention information theory [4], statistical physics [5] and axiomatic theories [6], just to mention a few. We point the reader to Ref. [7] (and references therein) for a deep discussion on entropy production in classical and quantum systems.

In the present work we are interested in the thermodynamics of closed quantum systems. In this case, time evolution is unitary, thus implying that the von Neumann entropy is constant. Therefore, such quantity is not a suitable candidate for the quantum thermodynamic entropy since it clashes with the established experimental fact that, in general, spontaneous processes occur in a direction where entropy increases. Moreover, it does not satisfy the fundamental thermodynamic relation. An interesting proposal appeared in Ref. [8], where the so called diagonal entropy

$$S_d(\hat{\rho}) = -\sum_n p_n \ln p_n, \tag{1}$$

with p_n being the diagonal elements of the system density matrix $\hat{\rho}$ in the energy eigenbasis, was proposed as the thermodynamic entropy for closed quantum systems. Such quantity has many interesting properties like extensivity, positivity and it vanishes at zero temperature. More important, it increases for every unitary process that induces transitions in the energy eigenbasis. Only when the Hamiltonian of the system changes slowly the entropy will not change, as we expect from the classical notion of thermodynamic entropy [9, 10]. We note that a closed related quantity, called observational entropy, is defined as a coarse-grained version of this entropy [11].

Interestingly, based on the notion of information, a gauge theory for thermodynamics was recently proposed in which heat is presented even in closed quantum systems [12]. In this theory, physical quantities are those which are invariant under the action of the gauge group and the emergent notion of entropy is the diagonal entropy. Reversible processes are those that do not cause transitions. This result puts the proposal of Ref. [8] under the solid umbrella of the gauge principle.

We choose here a quantum scalar field confined in a one dimensional cavity whose mirrors are in relative motion. This is the scenario known as the dynamical Casimir effect [13–16] that predicts, under certain circumstances, that particles will be created from the vacuum due to the dynamical nature of the boundary conditions imposed on the field. Many developments have been achieved in this field over the last five decades or so, including effects of imperfect mirrors [17–20], distinct geometries [21–25], the effect of the gravitational field on the number of particle production [26, 27], nonlinear interactions [28–30] and the dynamics of entanglement [31]. The interested reader should check Ref. [32] for a recent review on the subject.

Despite all of the developments, the irreversible dynamics of the quantum field in this scenario was not considered to date as far as we are aware. The purpose of the present work is start filling this gap. Specifically, we consider the irreversibility, as measure by the increase of the quantum thermodynamics entropy, associated with the field dynamics. In other words, how much entropy

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is produced in the field due to the nonstationary boundary conditions imposed by the motion of the cavity mirrors? We answer this questions considering two distinct approaches. First, by using an effective Hamiltonian theory based on Ref. [33], we are able to compute the entropy of the total field in the short-time regime. This is shown to be deeply related with the generation of quantum coherences in the energy eigenbasis of the system, a result in full agreement with the gauge theory developed in Ref. [12]. In the second part of the paper we consider a distinct approach in order to obtain the long-time dynamics of the field, which allows us to compute the diagonal entropy for a single mode, which is shown to be determined by the entanglement entropy between the considered mode and all the others. In this way we link irreversibility of the field dynamics with two linked quantum features, coherences and entanglement.

II. THE DYNAMICAL CASIMIR EFFECT

Let us consider a one-dimensional ideal cavity whose mirrors are located at positions $x = 0$ and $x = L(t)$, with $L(t)$ being an externally prescribed trajectory. Confined by this cavity we have a massless real scalar field $\phi(x,t)$ described by the Lagrangian density

$$\mathcal{L}(x,t) = \frac{1}{2}\left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right]. \tag{2}$$

Since we consider ideal mirrors, the boundary conditions we impose on the field take the Dirichlet form

$$\phi(0,t) = \phi(L(t),t) = 0. \tag{3}$$

The Lagrangian (2) implies that the field must obey the wave equation

$$\left(\partial_t^2 - \partial_x^2\right)\phi(x,t) = 0. \tag{4}$$

The set of complex value solutions $\{\phi_i\}$ to Eq. (4) under the restrictions imposed by the non-stationary boundary conditions (3) spams a linear vector space $\mathscr{S}$ with an invariant bilinear form

$$(\phi_1,\phi_2) = i\int_0^{L(t)} \mathrm{d}x\ \left[\phi_1^*\partial_t\phi_2 - \phi_2\partial_t\phi_1^*\right] \tag{5}$$

satisfying all the properties of an inner product with the sole exception of positive definiteness. This last difficult prevents the use of Eq. (5) to directly decompose the field in terms of orthonormal solutions on $\mathscr{S}$. Nevertheless, we can always proceed by choosing instead, any subspace $\mathscr{S}^+ \subset \mathscr{S}$, as long as it satisfies the following properties: (i) the product (5) is positive definite on $\mathscr{S}^+$; (ii) $\mathscr{S} = \mathscr{S}^+ \oplus \overline{\mathscr{S}^+}$ —the bar designates the complex conjugated of the space— and (iii) for all $f^+ \in \mathscr{S}^+$ and $f^- \in \overline{\mathscr{S}^+}$, we have $(f^+, f^-) = 0$ [34]. With these considerations the classical field can then be expanded as

$$\phi(x,t) = \sum_k \left[a_k f_k(x,t) + a_k^* f_k^*(x,t)\right], \tag{6}$$

since the set $\{f_k(x,t)\}$ is an orthonormal basis on $\mathscr{S}^+$ and $\{a_k\}$ is a set of complex coefficients.

The quantum description of the field is obtained by means of the usual field quantization prescription. The Fourier coefficients a_k and a_k^* are promoted to instantaneous annihilation and creation operators $\hat{a}_{k(t)}$ and $\hat{a}_{k(t)}^\dagger$ satisfying the standard commutation relations

$$\left[\hat{a}_{k(t)}, \hat{a}_{j(t)}^\dagger\right] = \delta_{kj},$$
$$\left[\hat{a}_{k(t)}, \hat{a}_{j(t)}\right] = \left[\hat{a}_{k(t)}^\dagger, \hat{a}_{j(t)}^\dagger\right] = 0.$$

The reason for the word instantaneous here is related to the time-dependent boundary conditions given in Eq. (3) and its physical meaning will become clear latter. In terms of these operators the field operator takes the form

$$\hat{\phi}(x,t) = \sum_k \left[f_{k(t)}(x,t)\hat{a}_{k(t)} + f_{k(t)}^*(x,t)\hat{a}_{k(t)}^\dagger\right]. \tag{7}$$

From this we can define the instantaneous vacuum state $|0;t\rangle$ as the state annihilated by all $\hat{a}_{k(t)}$, whereas a instantaneous particle state can be constructed by the application of the creation operator $\hat{a}_{k(t)}^\dagger$ on this vacuum state

$$|\boldsymbol{n};t\rangle = |n_{k_1}, n_{k_2}, \ldots; t\rangle = \prod_i \frac{1}{\sqrt{n_{k_i}!}}\left[a_{k_i(t)}^\dagger\right]^{n_{k_i}} |0;t\rangle\,,$$

with n_{k_i} representing the number of particles in the k_i-th mode in a given instant of time t.

Note that, since we imposed time-dependent boundary conditions on the field, the interpretation of the number of particles in a given instant of time is not physically justified. When the cavity is at rest, we can use the time translation symmetry of the wave equation as a natural criterion to select $\mathscr{S}^+$ as the subspace of solutions which oscillate with purely positive frequencies. In contrast, when the time-dependence of the boundary condition (3) is taken into account, the last criterion is no longer available and there is no unambiguous choice for $\mathscr{S}^+$. Consequently, during the cavity motion, the expansion of the field in terms of creation and annihilation operators becomes arbitrary, implying the nonexistence of a preferred choice for a vacuum state. Thus, unless we can specify a measurement process, the usual notion of particle loses its well-defined meaning and only when the cavity comes at rest one can associate a definite reality to the quanta described by theses operators [33].

For this reason, we consider the cavity to be in a static configuration with constant size L_0 for instants of time $t \leq 0$ (referred to as the *in* interval) and with size L_T for $t \geq T$ (the *out* interval).

For these intervals, the space of solutions $\mathscr{S}^+$ is then naturally selected as the ones spanned by the set of orthonormal basis with purely positive frequencies

$$f_k^{\mathrm{in}}(x,t) = \frac{1}{\sqrt{\pi k}}\sin(\omega_k^{\mathrm{in}}x)e^{-i\omega_k^{\mathrm{in}}t}, \qquad \text{for} \quad t \leq 0, \tag{8a}$$

$$f_k^{\mathrm{out}}(x,t) = \frac{1}{\sqrt{\pi k}}\sin(\omega_k^{\mathrm{out}}x)e^{-i\omega_k^{\mathrm{out}}t}, \quad \text{for} \quad t \geq T, \tag{8b}$$

where $\omega_k^{\rm in} = k\pi/L_0$ and $\omega_k^{\rm out} = k\pi/L_T$ are respectively, the frequency of the *in* and *out* intervals for the k-th mode. We employed the notation $f_{k'}^{\rm in} := f_{k'(0)}$ and $f_k^{\rm out} := f_{k(T)}$.

Let us denote the *in* field operators by $\hat{b}_k$ and $\hat{b}_k^\dagger$, while the *out* ones will be denoted by $\hat{a}_k$ and $\hat{a}_k^\dagger$. Under these considerations, the field operators read

$$\hat{\phi}(x,t\le 0) = \sum_k \left[f_k^{\rm in}(x,t)\hat{b}_k + (f_{k'}^{\rm in}(x,t))^*\hat{b}_k^\dagger\right]$$
$$\hat{\phi}(x,t\ge T) = \sum_{k'} \left[f_{k'}^{\rm out}(x,t)\hat{a}_{k'} + (f_{k'}^{\rm out}(x,t))^*\hat{a}_{k'}^\dagger\right].$$

Observe that the vacuum defined by $\hat{a}_k$ and $\hat{b}_k$ are nonequivalent in general. Both sets of operators are linked by a Bogoliubov transformation

$$\hat{a}_{k'} = \sum_k \left[\alpha_{kk'}\hat{b}_k + \beta_{kk'}^*\hat{b}_k^\dagger\right]. \tag{10}$$

This decomposition readily tell us that the former vacuum state (as seen after the motion of the mirror) now contains a non-vanishing number of particles with respect to the initial vacuum $|0;{\rm in}\rangle$

$$N = \sum_k \langle 0;{\rm in}|\hat{a}_k^\dagger\hat{a}_k|0;{\rm in}\rangle = \sum_{k,k'}|\beta_{kk'}|^2. \tag{11}$$

In general $\beta_{kk'}$ is non zero when time-dependent boundary conditions are imposed on the field. This last result characterizes the DCE as the quantum field phenomena of particle creation from the vacuum due to the time-dependent nature of the imposed boundary conditions.

The next section introduces an effective Hamiltonian approach [20, 33, 35, 36] to describe the field dynamics. This will be important for us to compute the evolved state and consequently, the entropy generated by the particle creation process.

III. EFFECTIVE HAMILTONIAN APPROACH

In this section we introduce an effective Hamiltonian for the DCE based on Ref. [33]. The utilized method, also called instantaneous basis approach, is based on the introduction of a set of instantaneous basis functions

$$\varphi_{k(t)}(x) = \sqrt{\frac{2}{L(t)}}\sin\left(\omega_{k(t)}x\right), \tag{12}$$

with eigenfrequencies $\omega_{k(t)} = k\pi/L(t)$. In this description the correspondent mode function in Eq. (7) is defined as

$$f_{k(t)}(x,t) = \frac{1}{\sqrt{2\omega_{k(t)}}}\varphi_{k(t)}(x)e^{-i\Omega_k(t)}, \tag{13}$$

where $\Omega_k(t) = \int_0^t dt'\omega_{k'(t)}$. Equation (13) differs from Ref. [33] in respect to the exponential term with explicit time dependence. As we will see, this particular choice is equivalent to describing the resulting effective Hamiltonian in the interaction picture, where the quantum state and the operators share explicit time dependence.

The field operator and its conjugated momenta $\hat{\pi} = \partial_t\hat{\phi}$ take the form

$$\hat{\phi} = \sum_k \frac{1}{\sqrt{2\omega_{k(t)}}}\left[\hat{a}_{k(t)}e^{-i\Omega_k(t)} + {\rm h.c.}\right]\varphi_{k(t)}(x), \tag{14a}$$
$$\hat{\pi} = i\sum_k \sqrt{\frac{\omega_{k(t)}}{2}}\left[\hat{a}_{k(t)}^\dagger e^{i\Omega_k(t)} - {\rm h.c.}\right]\varphi_{k(t)}(x), \tag{14b}$$

where h.c. stands for hermitian conjugated. Taking the time derivative of Eqs. (14) and, after some algebra (see Appendix A for details), one obtain the following set of differential equation for the annihilation operator

$$\frac{{\rm d}\hat{a}_{j(t)}}{{\rm d}t} = -\sum_k \left[A_{kj}(t)\hat{a}_{k(t)} + B_{kj}^*(t)\hat{a}_{k(t)}^\dagger\right]. \tag{15}$$

The equation for the creation operator is obtained by simply taking the transpose complex conjugate of this last equation. In this equation we defined the coefficients

$$\left.\begin{matrix} A_{kj}(t) \\ B_{kj}(t)\end{matrix}\right\} = \frac{1}{2}\left(\mu_{kj(t)} \mp \mu_{jk(t)}\right)e^{-i[\Omega_k(t)\mp\Omega_j(t)]} \tag{16}$$

with

$$\mu_{kj(t)} := \left(\sqrt{\frac{j}{k}}g_{kj} + \frac{1}{2}\delta_{kj}\right)\frac{\dot{L}(t)}{L(t)} \tag{17}$$

and

$$g_{kj} = \begin{cases} (-1)^{j-k}\frac{2kj}{j^2-k^2}, & k\ne j \\ 0, & k=j. \end{cases} \tag{18}$$

Identifying Eq. (15) as the Heisenberg dynamical equation of motion for the annihilation operator, it is straightforward to write down the effective Hamiltonian

$$\hat{H}(t) = \frac{i}{2}\sum_{jk}\left[A_{kj}(t)\hat{a}_{j(t)}^\dagger\hat{a}_{k(t)} - B_{kj}^*(t)\hat{a}_{k(t)}^\dagger\hat{a}_{j(t)}^\dagger - {\rm h.c.}\right]. \tag{19}$$

Here we can clearly see the existence of two different contributions. The terms containing the coefficients B_{kj}^* and B_{kj} govern the process of creation and annihilation of pairs of particles while the ones proportional to A_{kj}^* and A_{kj} are responsible for scattering of particles between two distinct modes.

Since we have a Hamiltonian description of our system, we can compute the time-evolution of its density matrix and, from this, the thermodynamic entropy.

A. The density operator

To investigate the entropy production within the proposed scheme, one first needs to obtain an explicit expression for the system's density operator $\hat{\rho}$ after the cavity returns to its stationary configuration. This can be achieved by finding solutions for the dynamical equation

$$\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = -i\left[\hat{H}(t), \hat{\rho}(t)\right]. \tag{20}$$

On the other hand, the complicated form of the effective Hamiltonian presents inherent difficulties in solving Eq. (20). To circumvent this problem, we restrict attention to the class of systems obeying the following form of the equation of motion of the cavity mirror

$$L(t) = L_0\left[1 + \epsilon l(t)\right], \tag{21a}$$

with $l(t)$ being an arbitrary function of order unity while $\epsilon \ll 1$ is a small amplitude.

Since the coefficients in Eq. (17) are proportional to $\dot{L}(t)/L(t)$, it is straightforward to see that the Hamiltonian coefficients given in Eqs. (16) are proportional to ϵ. As a result, a formal solution to Eq. (20) up to second order in ϵ reads

$$\hat{\rho}(T) = \hat{\rho}(0) - i\int_0^T dt'\left[\hat{H}(t'), \hat{\rho}(0)\right] - \int_0^T dt'\int_0^{t'} dt''\left[\hat{H}(t'), \left[\hat{H}(t''), \hat{\rho}(0)\right]\right]. \tag{22}$$

We are interested in the particular case of initial vacuum state $\hat{\rho}(0) = |0;\mathrm{in}\rangle\langle\mathrm{in};0|$, since we want to study the thermodynamics of the particle creation process. It is convenient to write the evolved state in terms of the initial operator $\hat{b}_k$ and $\hat{b}_k^\dagger$, which are related to the operator $\hat{a}_{k(t)}$ and $\hat{a}_{k(t)}^\dagger$ by the Bogoliubov coefficients α_{kj} and β_{kj}.

By substituting the transformations (10) into the set of differential equations (15), we obtain a recursive relation for the Bogoliubov coefficients in terms of powers of ϵ. Up to first order, the resulting coefficients are given by

$$\alpha_{kj}(T) = \delta_{kj} + \int_0^T dt' A_{kj}(t'), \tag{23a}$$

$$\beta_{kj}(T) = \int_0^T dt'\; B_{kj}(t'), \tag{23b}$$

which implies

$$\hat{a}_{k(t)} = \hat{b}_k + \sum_j\left(\tilde{\alpha}_{jk}(t) b_j + \beta^*_{jk}(t) b_j^\dagger\right),$$

where $\tilde{\alpha}_{kj}(t) = \int_0^T dt' A_{kj}(t')$. A direct calculation from Eq. (22) leads us to the following expression for the system's density operator up to second order in ϵ

$$\begin{aligned}\hat{\rho}(T) = \hat{\rho}(0) - \tfrac{1}{2}\sum_{kj}\Bigg\{&\beta^*_{kj}\left(\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\right) - \tfrac{1}{4}\sum_{nm}\Bigg[\beta_{mn}\beta^*_{kj}\left(\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\hat{b}_m\hat{b}_n\right) - \beta_{mn}\beta^*_{kj}\left(\hat{b}_m\hat{b}_n\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\right)\\ &+ \beta^*_{mn}\beta^*_{kj}\left(\hat{b}_m^\dagger\hat{b}_n^\dagger\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\right) + 2\tilde{\alpha}^*_{mn}\beta^*_{kj}\left(\hat{b}_m^\dagger\hat{b}_n\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\right)\Bigg] + \mathrm{h.c.}\Bigg\}.\end{aligned} \tag{24}$$

Considering the initial vacuum state, the number of particles created inside the cavity due do the DCE takes the form

$$N(T) \;=\; \mathrm{Tr}\left\{\sum_k \hat{\rho}(T)\hat{b}_k^\dagger\hat{b}_k\right\} = \sum_{kj}|\beta_{kj}|^2, \tag{25}$$

in agreement with Eq. (11), thus showing the consistency of our calculations.

We are now ready to discuss the entropy production due to the particle creation process.

B. Entropy production

We are interested in the irreversibility associated with the DCE. As discussed earlier we consider the diagonal entropy [8]

$$S_d(\hat{\rho}) = -\sum_{\boldsymbol{n}}\rho^{(\boldsymbol{n})}_{\mathrm{diag}}\ln\rho^{(\boldsymbol{n})}_{\mathrm{diag}}, \tag{26}$$

as the main figure of merit, where $\rho^{(n)}_{\mathrm{diag}} = \langle\mathrm{in};\boldsymbol{n}|\,\hat{\rho}\,|\boldsymbol{n};\mathrm{in}\rangle$ are the diagonal elements of the system's density operator in the initial energy eigenbasis.

From the expression of the density operator shown in

Eq. (24), the diagonal entropy can be directly computed, resulting in

$$S_d(T) = -\left[1-\tfrac{1}{2}N(T)\right]\ln\left[1-\tfrac{1}{2}N(T)\right] - \sum_{kj}\tfrac{1}{2}|\beta_{kj}(T)|^2\ln\tfrac{1}{2}|\beta_{kj}(T)|^2 \qquad (27)$$

We first observe that the entropy production depends on the number of particles created inside the cavity. Secondly, we note that this entropy production is exactly equal to the creation of coherence in the energy eigenbasis of the field. To see this, let us consider the relative entropy of coherence [37]

$$C(\hat{\rho}) = S(\hat{\rho}_d) - S(\hat{\rho}),$$

which is a measure of coherence in a given basis. Here $S(\hat{\rho}) = -\operatorname{Tr}\hat{\rho}\ln\hat{\rho}$ designate the von Neumman entropy for the system's density operator $\hat{\rho}$ while $\hat{\rho}_d$ is built from the diagonal elements of $\hat{\rho}$ in the chosen basis. Since we are interested in the amount of entropy produced during time evolution, we pick up the initial energy eigenbasis to measure coherences. This is consistent with the definition of the entropy. Under this choice, we directly see that $S(\hat{\rho}_d) = S_d(\hat{\rho})$. Since our evolution is unitary and the initial state is pure, we have $S(\hat{\rho}) = 0$, thus implying that

$$C(\hat{\rho}) = S_d(T). \qquad (28)$$

Note that, differently from Eq. (27), such a result is a general one, independent of the perturbation theory used here.

This result implies that we will observe irreversibility (positive entropy production) for every process that creates coherence in the system. Therefore, reversible processes must be the ones that are performed slowly enough in order to not induce transitions among the energy eigenstates. This result is in agreement with the discussions presented in Refs. [8–10, 12], where both entropy production and heat are associated with processes that generates coherences.

In order to illustrate our results, let us consider that the moving mirror performs harmonic oscillations of the form

$$l(t) = \sin(p\omega_1 t), \qquad (29)$$

where p is an integer while ω_1 is the first unperturbed field frequency.

For simplicity, we define the small dimensionless time $\tau = \epsilon\omega_1 T/2$ and assume the case in which the mirror returns to its initial position at time $t = T$ after performing a certain number of complete cycles ($p\omega_1 T = 2\pi m$ with $m = 1, 2, \dots$). Using Eqs. (18) and (29), we directly obtain

$$|\beta_{kj}(\tau)| = \begin{cases} \sqrt{kj}\;\tau, & \text{if } p = k+j \\ \frac{2\sqrt{kj}\epsilon p}{p^2-(k+j)^2}\sin\left[\frac{2(k+j)\tau}{\epsilon}\right], & \text{if } p \neq k+j. \end{cases} \qquad (30)$$

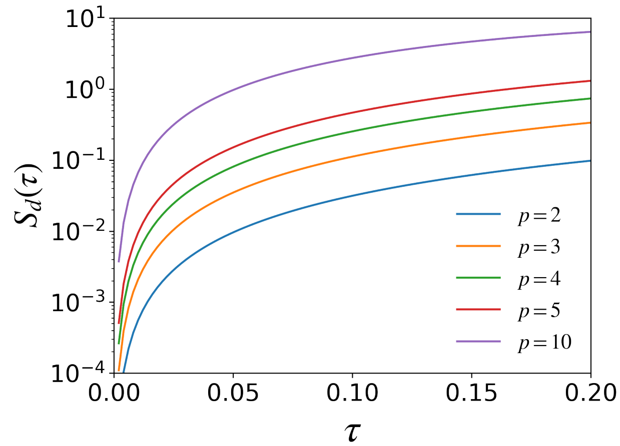

Figure 1. **Entropy production**. Entropy as a function of τ for distinct values of the mirror frequency.

By dropping the rapid oscillating terms, the number of created particles takes the form

$$N(\tau) = \frac{1}{6}p(p^2-1)\tau^2, \qquad (31)$$

in agreement with Ref. [38]. Note that the above expression is valid under perturbation theory involving time and, therefore, it is a good approximation only when $\tau \ll 1$.

In this case, the diagonal entropy, our focus of interest here, reduces to

$$S_d(\tau) = \frac{1}{2}N(\tau)\left[1-\ln\frac{1}{2}N(\tau) + \ln\frac{p(p^2-1)}{6} - \frac{6\,\mathrm{v}(p)}{p(p^2-1)}\right], \qquad (32)$$

with

$$\mathrm{v}(p) = \sum_{k=1}^{p-1}(p-k)k\ln(p-k)k.$$

Figure 1 shows the diagonal entropy for this particular case. As it is clear from the figure, entropy will be produced in the field for every value of the mirror frequency p, except for $p = 1$, where the number of created particles vanishes.

The present approach, based on the effective Hamiltonian, allowed us to compute the entropy production in the system by means of the time evolution of the density operator. This leads to a direct connection between entropy production and the generation of coherences in the field. In the next section we rely on the Heisenberg picture and compute the entropy production in terms of the time evolution of Gaussian states. This allows us to investigate the contribution to the entropy of the entanglement between a single mode and the rest of the field.

IV. GAUSSIAN STATE APPROACH

The last section presented an analysis of the entropy production constrained to the short-time regime of the entire system. Here we present a distinct technique that allows us to analyzed the entropy production in a given mode for all times and also to relate such dynamics with that of entanglement between the considered mode and all the others.

To do this, we proceed with the instantaneous basis approach in the Heisenberg picture, where during the system's evolution, all time dependence is contained in the set of mode functions $\{f_{k(t)}\}$, while the quantum state and the operators $\hat{a}_{k(t)}$ and $\hat{a}^{\dagger}_{k(t)}$ are fixed at time $t = 0$. Following now the developments presented in Ref. [38], we write the mode functions in terms of a series with respect to the instantaneous basis $\varphi_{j(t)}$ as

$$f_{k(t)}(x,t) = \frac{1}{\sqrt{2\omega_k^{\text{in}}}} \sum_j \varphi_{j(t)}(x) Q_j^{(k)}(t),$$

where the time-dependent coefficients $Q_j^{(k)}$ need to satisfy the initial conditions

$$Q_j^{(k)}(0) = \delta_{jk}, \qquad \dot{Q}_j^{(k)}(t) = -i\omega_k^{\text{in}}\delta_{kj}.$$

The wave equation (4) results in the following set of coupled differential equations for these coefficients

$$\begin{aligned}\ddot{Q}_j^{(k)} &+ \omega_{j(t)}^2 Q_j^{(k)} \\ &= 2\lambda(t)\sum_l g_{kl}\dot{Q}_l^{(k)} + \dot{\lambda}(t)\sum_l g_{kl} Q_l^{(k)} + \mathcal{O}(\lambda^2),\end{aligned} \tag{33}$$

where $\lambda(t) = \dot{L}(t)/L(t)$ and the coefficients g_{kl} are given in Eq. (18).

Here we will concentrate in the parametric amplification of the fundamental cavity mode. Therefore, we impose the following equation of motion for the moving mirror

$$L(t) = L_0\left[1 + \epsilon\cos(2\omega_1 t)\right]. \tag{34}$$

If the mirror returns to its initial position L_0 after some interval of time T, then $\omega_k^{\text{in}} = \omega_k^{\text{out}} = \omega_k$ and the right-hand side of Eq. (33) vanishes and we can write (for $t \geq T$)

$$Q_j^{(k)}(t) = \sqrt{\frac{\omega_k}{\omega_j}}\left(\alpha_{kj}e^{-i\omega_j t} + \beta_{kj}e^{i\omega_j t}\right), \tag{35}$$

where α_{kj} and β_{kj} are the Bogoliubov coefficients defined in Eq. (10).

Since we impose the field to be weakly perturbed by the mirror oscillations (34), is natural to search for solutions to $Q_j^{(k)}$ by allowing the Bogoliubov coefficients in Eq. (35) to be functions that vary slowly in time, i.e., $\dot{\alpha}_{kj}, \dot{\beta}_{kj} \sim \epsilon$. Then by substituting Eq. (35) into (33) and employing the usual prescriptions of the method of slowly varying amplitudes [40], it is possible to obtain a set of coupled first order differential equations with time independent coefficients in terms of α_{kj} and β_{kj}. For $k = 1$, we can write them as [39]

$$\dot{\alpha}_{1j} = -\sqrt{3}\alpha_{3j} - \beta_{1j} \tag{36a}$$

$$\dot{\beta}_{1j} = -\alpha_{1j} - \sqrt{3}\beta_{3j} \tag{36b}$$

whereas for $k > 2$ we obtain

$$\dot{\alpha}_{kj} = \sqrt{k(k-2)}\alpha_{(k-2),j} - \sqrt{k(k+2)}\alpha_{(k+2),j}, \tag{37a}$$

$$\dot{\beta}_{kj} = \sqrt{k(k-2)}\beta_{(k-2),j} - \sqrt{k(k+2)}\beta_{(k+2),j}, \tag{37b}$$

with the upper dot designating a derivative in respect to the dimensionless time variable $\tau = \epsilon\omega_1 T/2$. Because of the initial conditions $\alpha_{kj}(0) = \delta_{kj}$ and $\beta_{kj}(0) = 0$, all the coefficients with at least one even index vanish.

Complete solutions to the set of equations (36) and (37) was obtained in Ref. [38] in terms of the hypergeometrical function. Nonetheless, in this section we will be interested in computing the diagonal entropy generated in particular modes of the field in the regime of parametric oscillations (34). As a result, for reasons that will became clear later, it will be sufficient to pay attention only to the asymptotic behavior of the Bogoliubov coefficients with the first index equal to 1.

For $\tau \ll 1$, their expressions reads

$$\alpha_{1(2\mu+1)} = (\mu+1)K_\mu J_\mu\ \tau^\mu + \mathcal{O}(\tau^{\mu+2}), \tag{38a}$$

$$\beta_{1(2\mu+1)} = -K_\mu J_\mu\ \tau^{\mu+1} + \mathcal{O}(\tau^{\mu+3}), \tag{38b}$$

with $J_\mu = (2\mu)!/2^\mu(\mu!)^2$ and $K_\mu = (-1)^\mu\sqrt{2\mu+1}/(\mu+1)$, whereas for $\tau \gg 1$

$$\alpha_{1(2\mu+1)} \approx \frac{2}{\pi}\frac{(-1)^\mu}{\sqrt{2\mu+1}}, \tag{39a}$$

$$\beta_{1(2\mu+1)} \approx -\frac{2}{\pi}\frac{(-1)^\mu}{\sqrt{2\mu+1}}, \tag{39b}$$

with $\mu = 0, 1, 2, \ldots$.

Now we are ready to write down the reduced density operator for the considered mode and to address the question of the dynamics of entropy production and its relation to entanglement.

A. Reduced density operator

If $\hat{\rho}$ denotes the total density operator of the field, we define the reduced density operator of mode m by

$$\hat{\rho}_m = \text{Tr}_{\{k\}/m}\,\hat{\rho}, \tag{40}$$

where $\text{Tr}_{\{k\}/m}$ means the trace over all the modes except the m-th one.

In order to compute the last expression, we remember that, as seen in the previous section, the time evolution of the system can be described by a quadratic time-dependent Hamiltonian. Also, the time evolution governed by quadratic Hamiltonian's transforms any Gaussian state into another Gaussian state, which is completely characterized by the covariance matrix.

As the vacuum state belongs to this class of Gaussian states, it is in fact possible to describe our initial state in terms of a simple Wigner function for the m-th mode

$$W_m(\mathbf{q}) = \frac{1}{\sqrt{2\pi \det \mathbf{\Sigma}_m}} e^{-\frac{1}{2}(\mathbf{q}-\langle\mathbf{q}\rangle)\mathbf{\Sigma}_m^{-1}(\mathbf{q}-\langle\mathbf{q}\rangle)},$$

where $\mathbf{q} = (\hat{q}_m, \hat{p}_m)$ is the system's quadrature operator with components

$$\hat{q}_m = \frac{1}{\sqrt{2}}\left(\hat{a}_m^\dagger + \hat{a}_m\right), \tag{41a}$$

$$\hat{p}_m = \frac{i}{\sqrt{2}}\left(\hat{a}_m^\dagger - \hat{a}_m\right) \tag{41b}$$

while $\mathbf{\Sigma}_m$ stands for the covariance matrix

$$\mathbf{\Sigma}_m \equiv \begin{pmatrix} \sigma_m^q & \sigma_m^{qp} \\ \sigma_m^{qp} & \sigma_m^p \end{pmatrix} \tag{42}$$

with

$$\sigma_m^q = \langle \hat{q}_m^2\rangle - \langle\hat{q}_m\rangle^2, \tag{43a}$$

$$\sigma_m^p = \langle \hat{p}_m^2\rangle - \langle\hat{p}_m\rangle^2, \tag{43b}$$

$$\sigma_m^{qp} = \frac{1}{2}\langle \hat{p}_m\hat{q}_m + \hat{q}_m\hat{p}_m\rangle - \langle\hat{q}_m\rangle\langle\hat{p}_m\rangle. \tag{43c}$$

Since we are interested in the diagonal entropy, we focus on the diagonal components of the density operator in the energy eigenbasis. For the special case of an initially prepared vacuum state $|0; in\rangle$, these diagonal terms are given in terms of the variances as [38]

$$\rho_m^{(n)} = \frac{2\left[\left(2\sigma_m^q - 1\right)\left(2\sigma_m^p - 1\right)\right]^{n/2}}{\left[\left(2\sigma_m^q + 1\right)\left(2\sigma_m^p + 1\right)\right]^{(n+1)/2}} \times \mathrm{P}_n\left(\frac{4\sigma_m^q\sigma_m^p - 1}{\sqrt{(4(\sigma_m^q)^2 - 1)(4(\sigma_m^p)^2 - 1)}}\right), \tag{44}$$

where P_n is the Legendre polynomial of order n and $\rho_m^{(n)} = \langle \mathrm{in}; n|\,\hat{\rho}_m\,|n; \mathrm{in}\rangle$ are the diagonal elements of the reduced density operator in the initial energy eigenbasis.

Expressing the quadrature operators (41) in terms of the initial operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ in Eq. (10), the variances can be directly computed, resulting in

$$\sigma_m^q = \frac{1}{2}\sum_k \left|\alpha_{km} + \beta_{km}\right|^2, \tag{45a}$$

$$\sigma_m^p = \frac{1}{2}\sum_k \left|\alpha_{km} - \beta_{km}\right|^2 \tag{45b}$$

where m is an odd integer and the cross term σ_m^{qp} is identically zero for the choice of the initial state.

By taking the time derivatives of these last equations and inserting the recursive relations (36) and (37), one can shown that the final expression can be simplified to the form

$$\dot{\sigma}_m^q = -\left[\alpha_{1m} + \beta_{1m}\right]^2 \tag{46a}$$

$$\dot{\sigma}_m^p = +\left[\alpha_{1m} - \beta_{1m}\right]^2, \tag{46b}$$

which depends only on Bogoliubov coefficients with the first index equals to 1 (as we have pointed out in the beginning of the section). Moreover, because the definitions (41), the differential equations (46) need to satisfy the initial conditions $\sigma_m^q(0) = \sigma_m^p(0) = 1/2$.

We now analyze the solutions of these equations in two distinct regimes, the short-time and the long-time.

B. Short-time regime

In the short time limit, $\tau \ll 1$, by inserting Eqs. (38) into Eqs. (46) and integrating in τ, one obtain

$$\left.\begin{matrix}\sigma_{2\mu+1}^q \\ \sigma_{2\mu+1}^p\end{matrix}\right\} = \frac{1}{2} \mp \tau^{2\mu+1} J_\mu^2\left[1 \mp K_\mu^2\tau + \mathcal{O}(\tau^2)\right],$$

with J_μ and K_μ defined in Eq. (38).

By introducing expressions (47) into Eq. (44) we obtain the following expression for the diagonal components of the density operator

$$\rho_{2\mu+1}^{(n)} = (-1)^n i^n J_\mu^n \tau^{n(2\mu+1)}\left(1 - K_\mu^4\tau^2\right)^{n/2} \times \left[1 - (n+1)J_\mu^2\tau^{2\mu+2}\left(K_\mu^2 - \frac{1}{2}J_\mu^2\tau^{2\mu}\right)\right] \times P_n\left[i\tau\left(K_\mu^2 - J_\mu^2\tau^{2\mu}\right)\right] + \mathcal{O}(\tau^{2\mu+3}). \tag{47}$$

Using the last expression, we compute the diagonal entropy at short-times for the $(2\mu+1)$-th mode. Up to second order in τ we obtain, for $\mu = 0$, the following result

$$S_d^1(\tau \ll 1) = \frac{1}{2}N_1(\tau)\left[1 - \ln\frac{1}{2}N_1(\tau)\right],$$

while for any other value of μ, we have

$$S_d^{2\mu+1}(\tau \ll 1) = N_{2\mu+1}(\tau)\Big[1 - \ln N_{2\mu+1}(\tau)\Big] + \mathcal{O}(\tau^{2\mu+3}),$$

where $N_{2\mu+1}(\tau) = K_\mu^2 J_\mu^2 \tau^{2\mu+2} + \mathcal{O}(\tau^{2\mu+3})$ is the number of particles created for the corresponding mode.

Therefore, at short times, the entropy for each mode, grows directly proportional to the number of created particles. This is fully consistent with the results presented in the previous section.

As anticipated, the present approach allows us to investigate the long-time behavior of the entropy production and we proceed with such an analysis in what follows.

C. Long-time regime

In the long-time limit, $\tau \gg 1$, by substituting Eqs. (39) into Eqs. (46), we obtain the time derivatives of the system's quadrature variances as

$$\dot{\sigma}^q_{2\mu+1} \approx 0 \tag{48a}$$

$$\dot{\sigma}^p_{2\mu+1} \approx \frac{16}{\pi^2(2\mu+1)}. \tag{48b}$$

The specific constant of integration for Eqs. (48a) varies for each mode and depends on the complete form of the Bogoliubov coefficients [38], but the general behavior is the same: both quadrature variances starts with the same value $1/2$ at $t = 0$ and end up assuming distinct asymptotic behavior at $\tau \gg 1$, with σ^q_m decreasing to a constant value, whereas σ^p_m increases almost linearly in time.

It is now straightforward to compute the single mode reduced density matrix as

$$\rho^{(n)}_m(\tau \gg 1) = C^{(n)}_m \, [\det \mathbf{\Sigma}_m(\tau)]^{-1/2} + \mathcal{O}(1/\tau) \tag{49}$$

where

$$C^{(n)}_m = \frac{1}{\sqrt{1+T_m}} \left(\frac{1-T_m}{\sqrt{1-T_m^2}}\right)^n \mathrm{P}_n\left(\frac{1}{\sqrt{1-T_m^2}}\right) \tag{50}$$

is a positive real coefficient with $T_m = 1/2\sigma^q_m$.

From the above expressions, we can compute the diagonal entropy of the m-th field mode as

$$S^m_d(\tau \gg 1) \approx S^m_R(\tau) + [\det \mathbf{\Sigma}_m(\tau)]^{-1/2}\mathcal{S}_m, \tag{51}$$

where $S^m_R(\tau) = \frac{1}{2}\ln\det \mathbf{\Sigma}_m(\tau)$ is the Rényi-2 entropy of the m-th mode [41] and $\mathcal{S}_m = -\sum_n C^{(n)}_m \ln C^{(n)}_m$. It can be show that the second term in Eq. (51) diverges logarithmically with the system dimension $\mathcal{N}$. This last fact is expect since we are considering a field theory and the number of degrees of freedom of the system is infinite. However, high energy modes are not excited since the energy being inject into the system by the driven of the mirror is finite. Moreover, we must remember that the entropy is defined up to a multiplicative and an additive constant. So, this last term is not fundamental for the dynamical behavior of the entropy.

For the resonant mode $m = 1$, we obtain $\sigma^q_1 \to 2/\pi^2$ [38] and $\sigma^p_1 \to 16\tau/\pi^2$, leading to the Réniy-2 entropy

$$S^1_R(\tau) \approx \frac{1}{2}\ln\frac{32}{\pi^4}\tau,$$

which is in agreement with Ref. [31] [1]. In the case of the subsequent mode $m = 3$, now $\sigma^q_3 \to 38/9\pi^2$ and $\sigma^p_3 \to 16\tau/3\pi^2$, so we obtain

$$S^3_R(\tau) \approx \frac{1}{2}\ln\frac{608}{27\pi^4}\tau.$$

Now, since the global state of the field is pure —initial pure state under unitary evolution—, $S^m_R(\tau)$ quantifies the amount of entanglement between the m-th mode and all the remaining ones. Therefore, what Eq. (51) is saying to us is that the asymptotic behavior of the diagonal entropy is fundamentally determined by the entanglement.

V. CONCLUSIONS

The present article address the problem of the thermodynamic entropy production in the context of the dynamical Casimir effect. Two distinct approaches are considered. The first one, based on an effective Hamiltonian description of the field dynamics allowed us to understand, at the short-time limit, the relations between the entropy production and the generation of quantum coherences in the mode basis of the field. The second approach, based on the reduced density operator of a single mode, allowed us to link the growth of entropy with the growth of entanglement at all times.

In summary, the production of thermodynamic entropy in the field due to the dynamical Casimir effect is governed by the generation of quantum coherences and entanglement between the modes. The non-trivial boundary conditions imposed by the motion of the mirror causes the coupling between the field modes, thus inducing transitions among such modes. These transitions are in the root of the generation of quantum coherence and quantum entanglement. Although the evolution is unitary, irreversibility, which is characterized by the entropy production, arises due to these transitions, as discussed in Refs. [8–10, 12]. Reversible processes are defined in the limit where the motion is so slow that there is no particle creation and no scattering and, thus, no entropy production. Note that in the considered context, in which we have a resonant cavity trapping the field, there are motions in which no particles will be created and, thus, no entropy production.

An important point here is that, since the evolution is unitary, the entropy cannot decrease. The process is fundamentally irreversible. This is a basic property of the diagonal entropy [8].

Our study is important in order to deepen the understanding of the thermodynamics of quantum fields under non-trivial boundary conditions and the role of quantum coherence and entanglement in such a behavior. However, several questions are still open.

A natural question that arises here is the split of the energy in terms of work and heat, with this last one being associated with the irreversible part of the process, while the first one should be linked with the energy that can be extracted from the field after the process ends [43, 44]. Another related question is the statistical description of

[1] Here, the argument in the Réniy-2 entropy differs from Ref. [31] by a factor of 4. This occurs because the variances defined in the last reference are twice as large as the ones in Eq. (43).

the field in terms of stochastic entropy production and the fluctuation relations theorems [45]. Also, what is the role of multiple quantum coherences and multipartite entanglement in the entropy production? How about the thermalization properties of the field dynamics? Finally, a very interesting question is if heat and work, in the sense mentioned in the last paragraph, properly obeys the equivalence principle [46]. These are relevant questions that will be addressed in future works.

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Appendix A: Derivation of the effective Hamiltonian

1. Dynamical equations for the instantaneous creation and annihilation operators

From Eq. (4), the dynamical equation of motion for a quantum scalar field and its conjugated momentum can be written as

$$\partial_t\hat{\phi}(x,t) = \hat{\pi}(x,t) \tag{A1a}$$
$$\partial_t\hat{\pi}(x,t) = \nabla^2\hat{\phi}(x,t). \tag{A1b}$$

By defining $\hat{\mathscr{A}}_k := \hat{a}_k e^{-i\Omega_k}$, the expression for the fields and theirs correspondent time derivatives can be putted as

$$\hat{\phi} = \sum_k \frac{1}{\sqrt{2\omega_k}}\left(\hat{\mathscr{A}}_k + \hat{\mathscr{A}}_k^\dagger\right)\varphi_k; \tag{A2a}$$

$$\hat{\pi} = i\sum_k \sqrt{\frac{\omega_k}{2}}\left(\hat{\mathscr{A}}_k^\dagger - \hat{\mathscr{A}}_k\right)\varphi_k; \tag{A2b}$$

$$\dot{\hat{\phi}} = \sum_k \frac{1}{\sqrt{2\omega_k}}\left(\hat{\mathscr{A}}_k + \hat{\mathscr{A}}_k^\dagger\right)\left(\dot{\varphi}_k - \frac{\dot{\omega}_k}{2\omega_k}\varphi_k\right); \tag{A2c}$$
$$+\sum_k \frac{1}{\sqrt{2\omega_k}}\left(\dot{\hat{a}}_k e^{-i\Omega_k} + \dot{\hat{a}}_k^\dagger e^{i\Omega_k}\right)\varphi_k + \hat{\pi}$$

$$\dot{\hat{\pi}} = i\sum_k \sqrt{\frac{\omega_k}{2}}\left(\hat{\mathscr{A}}_k^\dagger - \hat{\mathscr{A}}_k\right)\left(\dot{\varphi}_k + \frac{\dot{\omega}_k}{2\omega_k}\varphi_k\right) \tag{A2d}$$
$$+\,i\sum_k \sqrt{\frac{\omega_k}{2}}\left(\dot{\hat{a}}_k^\dagger e^{i\Omega_k} - \dot{\hat{a}}_k e^{-i\Omega_k}\right)\varphi_k + \partial_x^2\hat{\phi},$$

where for conciseness we have suppressed the notation of time and spatial dependence in all terms in (A2) and the upper dot convey time derivative. Comparing (A1) with (A2c) and (A2d), we can isolate the time derivative of the ladder operators by computing

$$\int_0^L \mathrm{d}x\varphi_j\left(\dot{\hat{\phi}} - \hat{\pi}\right) = \sum_k \frac{1}{\sqrt{2\omega_k}}\left(\dot{\hat{a}}_k^\dagger e^{i\Omega_k} + \dot{\hat{a}}_k e^{-i\Omega_k}\right)\delta_{kj}$$
$$+\sum_k \frac{1}{\sqrt{2\omega_k}}\left(\hat{\mathscr{A}}_k + \hat{\mathscr{A}}_k^\dagger\right)\left(G_{kj} - \frac{\dot{\omega}_k}{2\omega_k}\delta_{kj}\right) = 0 \tag{A3}$$

$$\int_0^L \mathrm{d}x\varphi_j\left(\dot{\hat{\pi}} - \partial_x^2\hat{\phi}\right) = i\sum_k \sqrt{\frac{\omega_k}{2}}\left(\dot{\hat{a}}_k^\dagger e^{i\Omega_k} - \dot{\hat{a}}_k e^{-i\Omega_k}\right)\delta_{kj}$$
$$+i\sum_k \sqrt{\frac{\omega_k}{2}}\left(\hat{\mathscr{A}}_k - \hat{\mathscr{A}}_k^\dagger\right)\left(G_{jk} - \frac{\dot{\omega}_k}{2\omega_k}\delta_{kj}\right) = 0 \tag{A4}$$

where it was used $\int_0^L dx\varphi_k\varphi_j = \delta_k j$ and $G_{kj} := \int_0^L \varphi_j\dot{\varphi}_k$. By defining $\mu_{kj} = \sqrt{\frac{\omega_j}{\omega_k}}G_{kj} - \frac{\dot{\omega}_k}{2\omega_k}\delta_{kj}$ we obtain from (A3) and (A4)

$$\dot{\hat{a}}_j^\dagger e^{i\Omega_j} + \dot{\hat{a}}_j e^{-i\Omega_j} = -\sum_k \mu_{kj}\left(\hat{\mathscr{A}}_k + \hat{\mathscr{A}}_k^\dagger\right), \tag{A5}$$

$$\dot{\hat{a}}_j^\dagger e^{i\Omega_j} - \dot{\hat{a}}_j e^{-i\Omega_j} = -\sum_k \mu_{jk}\left(\hat{\mathscr{A}}_k - \hat{\mathscr{A}}_k^\dagger\right). \tag{A6}$$

From the last system is easy to isolate $\dot{\hat{a}}_j$ and $\dot{\hat{a}}_j^\dagger$ as

$$\dot{\hat{a}}_j = -\sum_k \left(A_{kj}a_k + B_{kj}^* a_k^\dagger\right), \tag{A7a}$$

$$\dot{\hat{a}}_j^\dagger = -\sum_k \left(A_{kj}^* a_k^\dagger + B_{kj}a_k\right), \tag{A7b}$$

with

$$A_{kj} = \frac{1}{2}\left(\mu_{kj} - \mu_{jk}\right)e^{-i(\Omega_k - \Omega_j)}, \tag{A8a}$$

$$B_{kj} = \frac{1}{2}\left(\mu_{kj} + \mu_{jk}\right)e^{-i(\Omega_j + \Omega_k)}. \tag{A8b}$$

Since $\omega_k = k\pi/L$ and using the definition (12) we can calculate

$$G_{kj} = g_{kj}\frac{\dot{L}}{L}, \tag{A9}$$

$$\frac{\dot{\omega}_k}{\omega_l} = -\frac{\dot{L}}{L}, \tag{A10}$$

where g_{kj} has the same form as expressed in (18). So we obtain $\mu_{kj} = \left(\sqrt{\frac{j}{k}}g_{kj} + \frac{1}{2}\delta_{kj}\right)\frac{\dot{L}(t)}{L(t)}$ as in Eq. (17).

2. Effective Hamiltonian

To find the effective Hamiltonian that generates the dynamical equations (A8) we begin by considering the most general quadratic operator

$$\hat{H} = \sum_{kl} \left(\mathcal{A}_{kl} \hat{a}_k^\dagger \hat{a}_l^\dagger + \mathcal{B}_{kl} \hat{a}_k^\dagger \hat{a}_l + \mathcal{C}_{kl} \hat{a}_l^\dagger \hat{a}_k + \mathcal{D}_{kl} \hat{a}_k \hat{a}_l \right),$$

with the hermitian conditions $\mathcal{A}_{kl} = \mathcal{D}_{kl}^*$, $\mathcal{B}_{kl} = \mathcal{C}_{kl}^*$ and the invariance over index change conditions $\mathcal{A}_{kl} = \mathcal{A}_{lk}$, $\mathcal{D}_{kl} = \mathcal{D}_{lk}$, $\mathcal{B}_{kl} = \mathcal{C}_{lk}$ and $\mathcal{B}_{lk} = \mathcal{C}_{kl}$.

The correspondent Heisenberg equation of motion is therefore

$$\begin{aligned} \dot{a}_j = i \left[\hat{H}, \hat{a}_j \right] = i \sum_{kl} \Bigg(& \mathcal{A}_{kl} \left[\hat{a}_k^\dagger \hat{a}_l^\dagger, \hat{a}_j \right] + \mathcal{B}_{kl} \left[\hat{a}_k^\dagger \hat{a}_l, \hat{a}_j \right] \\ & + \mathcal{C}_{kl} \left[\hat{a}_l^\dagger \hat{a}_k, \hat{a}_j \right] + \mathcal{D}_{kl} [\hat{a}_k \hat{a}_l, \hat{a}_j] \Bigg) \\ = -i \sum_k \Big[& (\mathcal{A}_{kj} + \mathcal{A}_{jk}) \hat{a}_k^\dagger + (\mathcal{B}_{jk} + \mathcal{C}_{kj}) \hat{a}_k \Big] \end{aligned} \tag{A11}$$

and

$$\begin{aligned} \dot{a}_j^\dagger = i \left[\hat{H}, \hat{a}_j^\dagger \right] = i \sum_{kl} \Bigg(& \mathcal{A}_{kl} \left[\hat{a}_k^\dagger \hat{a}_l^\dagger, \hat{a}_j^\dagger \right] + \mathcal{B}_{kl} \left[\hat{a}_k^\dagger \hat{a}_l, \hat{a}_j^\dagger \right] \\ & + \mathcal{C}_{kl} \left[\hat{a}_l^\dagger \hat{a}_k, \hat{a}_j^\dagger \right] + \mathcal{D}_{kl} \left[\hat{a}_k \hat{a}_l, \hat{a}_j^\dagger \right] \Bigg) \\ = i \sum_k \Big[& (\mathcal{D}_{kj} + \mathcal{D}_{jk}) \hat{a}_k + (\mathcal{B}_{kj} + \mathcal{C}_{jk}) \hat{a}_k^\dagger \Big] \end{aligned} \tag{A12}$$

Comparing (A7a) with (A11) and (A7b) with (A12), we obtain the following system

$$\begin{aligned} -i \left(\mathcal{A}_{kj} + \mathcal{A}_{jk} \right) &= -2i \mathcal{A}_{kj} = -B_{kj}^* \\ -i \left(\mathcal{C}_{kj} + \mathcal{B}_{jk} \right) &= -2i \mathcal{C}_{kj} = -A_{kj} \\ i \left(\mathcal{D}_{kj} + \mathcal{D}_{jk} \right) &= 2i \mathcal{D}_{kj} = -B_{kj} \\ i \left(\mathcal{B}_{kj} + \mathcal{C}_{jk} \right) &= 2i \mathcal{B}_{kj} = -A_{kj}^*. \end{aligned}$$

The correspondent effective Hamiltonian reads then

$$\hat{H} = \frac{i}{2} \sum_{jk} \left[A_{kj} \hat{a}_j^\dagger \hat{a}_k - A_{kj}^* \hat{a}_k^\dagger \hat{a}_j + B_{kj} \hat{a}_k \hat{a}_j - B_{kj}^* \hat{a}_k^\dagger \hat{a}_j^\dagger \right] \tag{A13}$$

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