AN ATTEMPT OF DERIVATION OF EQ 4 FROM BUDINI'S ARTICLE

In the interaction picture, density matrices evolve with time due to the interaction hamiltonian, while operators evolve with the system (S) and environment (B) hamiltonian. An arbitrary operator $O \in B(\mathcal{H}_T)$ is represented in this picture by the dependent operator $\hat{O}(t)$ and its time evolution is

$$\hat{O}(t) = e^{i(H_S+H_B)t}\, O\, e^{-i(H_S+H_B)t}. \qquad (1)$$

The time evolution of the total density matrix is given in this picture by

$$\frac{d\hat{\rho}_T(t)}{dt} = -i\lambda\left[H_I(t), \rho_T(t)\right]. \qquad (2) \leftarrow$$

$$H_T = H_S \otimes \mathbb{1}_B + \mathbb{1}_S \otimes H_B + \lambda H_I \overset{\text{notação}}{=} H_S + H_B + \lambda H_I$$

This equation can be easily integrated to give

$$\rho_T(t) = \rho_T(0) - i\lambda \int_0^t ds \left[\hat{H}_I(s), \rho_T(s)\right] \qquad (3)$$

To avoid the problem that we have to integrate the density matrix over all the previous time, we introduce (3) into (2) giving

$$\frac{d\rho_T(t)}{dt} = -i\lambda\left[H_I(t), \rho_T(0)\right] - \lambda^2 \int_0^t ds \left[H_I(t), \left[H_I(s), \rho_T(s)\right]\right] \qquad (4)$$

Introducing again the equation (3) into (4) and making the assumption that $\lambda \ll 1$, so

$$\frac{d\rho_T(t)}{dt} = -i\lambda\left[H_I(t), \hat{\rho}_T(t)\right] - \lambda^2\int_0^t ds\left[\hat{H}_I(t),\left[\hat{H}_I(s), \hat{\rho}_T(t)\right]\right].$$

We are interested in finding an equation of motion for ρ_S, so we trace out over the bath degrees of freedom

$$\frac{d\rho_S(t)}{dt} = -i\lambda \cdot tr_B\left[\hat{H}_I(t), \hat{\rho}_T(t)\right] - \lambda^2 \int_0^\infty ds \cdot tr_B\left\{\left[\hat{H}_I(t),\left[H_I(s), \rho_T(t)\right]\right]\right\}$$

such that

$$-i\lambda \cdot tr_B\left[H_I, \rho_T(0)\right] = -i\sum_i tr_B\left(\lambda V_S^i \otimes V_B^i \cdot \rho_S(0) \otimes \rho_B(0) - \lambda \rho_S(0) \otimes \rho_B(0) \cdot V_S^i \otimes V_B^i\right)$$

$$= \sum_i V_S^i \cdot \rho_S(0) \underbrace{Tr_B\left[V_B^i \cdot \rho_B(0)\right]}_{\langle V_B^i \rangle} - \rho_S(0) \cdot V_S^i \cdot \underbrace{Tr_B\left[\rho_B(0) \cdot V_B^i\right]}_{\langle V_B^i \rangle}$$

where we assume that the integration can be extended to infinity without affecting the result.

Now, taking the expression (4) from Budini's article

$$\frac{d\rho_S(t)}{dt} = -i\left[H_S, \rho_S(t)\right] - \lambda^2 \int_0^\infty d\tau \cdot Tr_B \left\{ \left[H_I, \left[H_I(-\tau), \rho_S(t) \otimes \rho_B^e \right] \right] \right\} \qquad (4)$$

where $H_I(-\tau) = e^{-i(H_B + H_S)\tau} \cdot H_I \cdot e^{+i(H_B + H_S)\tau}$ and ρ_B^e is the equilibrium density matrix of the bath, we introduce the Jacobi identity into eq. (4)

$$\left[X, [Y, Z]\right] + \left[Y, [Z, X]\right] + \left[Z, [X, Y]\right] = 0$$

$$\Rightarrow \left[X, [Y, Z]\right] = \left[[X, Y], Z\right] - \left[Y, [Z, X]\right]$$

and identifying X with H_I, Y with $H_I(-\tau)$ and Z with $\rho_S \otimes \rho_B^e$,

$$2\left[H_I, \left[H_I(-\tau), \rho_S \otimes \rho_B^e\right]\right] = \left[H_I, \left[H_I(-\tau), \rho_S \otimes \rho_B^e\right]\right] + \left[\left[H_I, H_I(-\tau)\right], \rho_S \otimes \rho_B^e\right] + \left[H_I(-\tau), \left[H_I, \rho_S \otimes \rho_B\right]\right]$$

$$\therefore \left[H_I, \left[H_I(-\tau), \rho_S \otimes \rho_B^e\right]\right] = \frac{1}{2}\left\{ \left[H_I, \left[H_I(-\tau), \rho_S \otimes \rho_B^e\right]\right] + \left[\left[H_I, H_I(-\tau)\right], \rho_S \otimes \rho_B^e\right] + \left[H_I(-\tau), \left[H_I, \rho_S \otimes \rho_B\right]\right] \right\},$$

such that eq (4) turns into

$$\frac{d\rho_s(t)}{dt} = -i[H_S, \rho(t)] - \frac{\lambda^2}{2}\int_0^\infty d\tau . \mathrm{Tr}_B\left\{[[H_I, H_I(-\tau)], \rho_S\otimes\rho_B^e] + [H_I, [H_I(-\tau), \rho_S\otimes\rho_B]] + [H_I(-\tau), [H_I, \rho_S\otimes\rho_B^e]]\right\}. \quad (5)$$

Now, using that

$$[X, [Y, Z]] + [Y, [X, Z]] = \{\{X, Y\}^+, Z\}^+ - 2(XZY + YZX)$$

→ where $\{X, Y\}^+ = XY + YX$ is the anticommutator

the equation (5) reduces to

$$\dot{\rho}_s = -i[H_S, \rho_s(t)] - \frac{\lambda^2}{2}\int_0^\infty d\tau.\ \mathrm{Tr}_B\Big\{\underbrace{[[H_I, H_I(-\tau)], \rho_S\otimes\rho_B^e]}_{①} + \underbrace{\{\{H_I, H_I(-\tau)\}^+, \rho_S\otimes\rho_B^e\}^+}_{②} - 2\big(H_I . \rho_S\otimes\rho_B^e . H_I(-\tau) + H_I(-\tau).\rho_S\otimes\rho_B^e . H_I\big)\Big\}$$

$$①+②: \quad \mathrm{Tr}_B\left\{[H_I, H_I(-\tau)]\rho_S\otimes\rho_B^e - \rho_S\otimes\rho_B^e[H_I, H_I(-\tau)] + \{H_I, H_I(-\tau)\}^+\rho_S\otimes\rho_B^e + \rho_S\otimes\rho_B^e\{H_I, H_I(-\tau)\}^+\right\} =$$

$$= .\mathrm{Tr}_B\left([H_I, H_I(-\tau)].\rho_B^e\right)\rho_S - \rho_S . \mathrm{Tr}_B\left([H_I, H_I(-\tau)]\rho_B^e\right) + \mathrm{Tr}_B\left(\{H_I, H_I(-\tau)\}^+\rho_B^e\right)\rho_S + \rho_S . \mathrm{Tr}_B\left(\{H_I, H_I(-\tau)\}^+\rho_B^e\right)$$

$$= \underbrace{\left[\mathrm{Tr}_B\left([H_I, H_I(-\tau)]\rho_B^e\right), \rho_S\right]}_{③} + \left\{\mathrm{Tr}_B\left(\{H_I, H_I(-\tau)\}^+\rho_B^e\right), \rho_S\right\}^+$$

Incorporating the term ③ in the von-Neumann term, we can write the result in the form

$$\frac{d\rho_S(t)}{dt} = -i\left[H_{eff}, \rho_S(t)\right] - \left\{D, \rho_S(t)\right\}^{+} + F\left[\rho_S(t)\right] \quad (6)$$

where

$$H_{eff} = H_S - i\frac{\lambda^2}{2}\int_0^{\infty} d\tau \cdot Tr_B\left(\left[H_I, H_I(-\tau)\right]\rho_B^e\right) \quad (7)$$

$$D = \frac{\lambda^2}{2}\int_0^{\infty} d\tau \cdot Tr_B\left(\left\{H_I, H_I(-\tau)\right\}^{+} \cdot \rho_B^e\right) \quad (8)$$

$$F\left[\rho_S(t)\right] = \lambda^2\int_0^{\infty} d\tau \, Tr_B\left(H_I \cdot \rho_S \otimes \rho_B^e \cdot H_I(-\tau) + H_I(-\tau) \cdot \rho_S \otimes \rho_B^e \cdot H_I\right). \quad (9)$$

Therefore, eq (4) has a Kossakowsky - Lindblad (KL) form, in which

$$D = \frac{1}{2}\sum_{\alpha,\gamma=1}^{N^2-1} a_{\alpha\gamma} V_\gamma^{\dagger} V_\alpha \qquad F[\bullet] = \sum_{\alpha,\gamma=1}^{N^2-1} a_{\alpha\gamma} V_\alpha \bullet V_\gamma^{\dagger},$$

D is called the dissipative operator and F the fluctuating superoperator. In order to obtain the matrix $a_{\alpha\gamma}$, let's consider

$$H_I = \sum_{\beta=1}^{n} V_\beta^S \otimes V_\beta^B \quad , \quad n \leq N^2-1. \quad (10)$$

Using the fact that H_I is hermitian and introducing the notation

$$\chi_{\alpha\beta}(-\tau) \equiv Tr_B\left(\rho_B^e \cdot V_\alpha^{B\dagger} \cdot V_\beta^B(-\tau)\right) \qquad (11)$$

we have

$$H_{eff} = H_S - \frac{i\lambda^2}{2}\int_0^\infty d\tau \; Tr_B\left(\left[H_I, H_I(-\tau)\right]\rho_B^e\right)$$

$$H_S - \frac{i\lambda^2}{2}\int_0^\infty d\tau \cdot Tr_B\left(H_I \cdot H_I(-\tau) \cdot \rho_B^e - H_I(-\tau) H_I \cdot \rho_B^e \right)$$

$$H_S - \frac{i\lambda^2}{2}\int_0^\infty d\tau \cdot Tr_B\left(\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \left(V_\alpha^S \otimes V_\alpha^B\right)\cdot\left(V_\beta^S(-\tau) \otimes V_\beta^B(-\tau)\right)\cdot \rho_B^e - \left(V_\beta^S(-\tau) \otimes V_\beta^B(-\tau)\right)\cdot\left(V_\alpha^S \otimes V_\alpha^B\right)\cdot \rho_B^e \right)$$

$$H_S - \frac{i\lambda^2}{2}\int_0^\infty d\tau \sum_{\alpha,\beta=1}^{n} Tr_B\left(V_\alpha^B \cdot V_\beta^B(-\tau)\cdot \rho_B^e\right) V_\alpha^S \cdot V_\beta^S(-\tau) - Tr_B\left(V_\beta^B(-\tau)\cdot V_\alpha^B \cdot \rho_B^e\right) V_\beta^S(-\tau)\, V_\alpha^S$$

$$H_S - \frac{i\lambda^2}{2}\sum_{\alpha,\beta=1}^{n}\int_0^\infty d\tau \left(\chi_{\alpha\beta}\cdot V_\alpha^S \cdot V_\beta^S(-\tau) - \chi_{\alpha\beta}\cdot V_\beta(-\tau)\, V_\alpha^S\right)$$

$$H_I^\dagger = H_I \Rightarrow \begin{array}{l} V_\beta^{S\dagger} = V_\beta^S \\ V_\beta^{B\dagger} = V_\beta^B \end{array}$$

$$H_S - \frac{i\lambda^2}{2}\sum_{\alpha,\beta=1}^{n}\int_0^\infty d\tau \left(\chi_{\alpha\beta}\, V_\alpha^{S\dagger}\, V_\beta^S(-\tau) - \chi_{\alpha\beta}^*\, V_\beta^{S\dagger}(-\tau) V_\alpha^S\right) \qquad (12)$$

$$\rightarrow \quad Tr_B\left(\{H_I, H_I(-\tau)\}^{+}\rho_B^e\right) =$$

$$= Tr_B\left(H_I, H_I(-\tau)\rho_B^e + H_I(-\tau)\cdot H_I\cdot\rho_B^e\right)$$

= (proceeding analogous to the last term)

$$= \left(\chi_{\alpha\beta}^{(-\tau)}\cdot V_\alpha^{s\dagger} V_\beta^s(-\tau) + \chi_{\alpha\beta}^{*(-\tau)}\cdot V_\beta^{s\dagger}(-\tau)\cdot V_\alpha^s\right) \checkmark \qquad (13)$$

$$\rightarrow \quad Tr_B\left(H_I\cdot\rho_S\otimes\rho_B^e\cdot H_I(-\tau) + H_I(-\tau)\,\rho_S\otimes\rho_B^e\; H_I\right) =$$

$$= \sum_{\alpha,\beta=1}^{n} Tr_B\left(V_\alpha^S\otimes V_\alpha^B\cdot\rho_S\otimes\rho_B^e\cdot V_\beta^S(-\tau)\otimes V_\beta^B(-\tau) + V_\beta^S(-\tau)\otimes V_\beta^B(-\tau)\cdot\rho_S\otimes\rho_B^e\cdot V_\alpha^S\otimes V_\alpha^B\right)$$

$$= \sum_{\alpha,\beta} Tr_B\left(V_\alpha^B\cdot\rho_B^e\cdot V_\beta^B(-\tau)\right)\cdot V_\alpha^S\,\rho_S\cdot V_\beta^S(-\tau) + Tr_B\left(V_\beta^B(-\tau)\cdot\rho_B^e\cdot V_\alpha^B\right)\cdot V_\beta^S(-\tau)\,\rho_S\cdot V_\alpha^S$$

$$= \sum_{\alpha,\beta} \chi_{\alpha\beta}^{*(-\tau)} V_\alpha^S\cdot\rho_S\, V_\beta^{s\dagger}(-\tau) + \chi_{\alpha\beta}^{(-\tau)}\cdot V_\beta^S(-\tau)\,\rho_S\cdot V_\alpha^{s\dagger} \checkmark$$

$$F[\rho_S(t)] = \lambda^2\sum_{\alpha\beta}\int_0^\infty d\tau\left(\chi_{\alpha\beta}(-\tau)\cdot V_\beta^S(-\tau)\,\rho_S\, V_\alpha^{s\dagger} + \chi_{\alpha\beta}^*(-\tau)\, V_\alpha^s\cdot\rho_S\cdot V_\beta^{s\dagger}(-\tau)\right) \qquad (14)$$

Finally, defining the matrix $C_{\beta\gamma}(-\tau)$ from

$$V_\beta^s(-\tau) \equiv e^{-i\tau H_S} V_\beta^s e^{+i\tau H_S} = \sum_{\gamma=1}^{N^2-1} C_{\beta\gamma}(-\tau) V_\gamma^s \tag{15}$$

and replacing it into eq (12),

$$H_{eff} = H_S - \frac{i\lambda^2}{2} \sum_{\alpha\beta\gamma} \int_0^\infty d\tau \left(\chi_{\gamma\beta}(-\tau)\, C_{\beta\alpha}(-\tau) - \chi^*_{\alpha\beta}(-\tau)\, C^*_{\beta\gamma}(-\tau) \right) V_\gamma^{s\dagger} V_\alpha^s$$

where we have used the fact that the indices in eqs (12), (13) and (14) are dumb. Now, we have the matrix

$$a_{\alpha\gamma} = \lambda^2 \sum_\beta \int_0^\infty d\tau \left(\chi_{\gamma\beta}(-\tau)\, C_{\beta\alpha}(-\tau) + \chi^*_{\alpha\beta}(-\tau) \cdot C^*_{\beta\gamma}(-\tau) \right).$$