

EQUILIBRIUM AND NONEQUILIBRIUM FORMALISMS MADE UNIFIED

Kuang-chao CHOU, Zhao-bin SU, Bai-lin HAO and Lu YU

Institute of Theoretical Physics, Academia Sinica, P.O. Box 2735, Beijing, China



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Institute of Theoretical Physics, Academia Sinica, P.O. Box 2735, Beijing, China

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* Current address: Department of Physics, City College of New York, New York, NY 10031, U.S.A.

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Abstract:

In this paper we summarize the work done by our group in developing and applying the closed time-path Green function (CTPGF) formalism, first suggested by J. Schwinger and further elaborated by Keldysh and others. The generating functional technique and path integral representation are used to discuss the various properties of the CTPGF and to work out a practical calculation scheme. The formalism developed provides a unified framework for describing both equilibrium and nonequilibrium phenomena. It includes the ordinary quantum field theory and the classical fluctuation field theory as its limiting cases. It is well adapted to consider the symmetry breaking with either constituent or composite order parameters. The basic properties of the CTPGF are described, the two-point functions are discussed in some detail with the transport equation and the time dependent Ginzburg-Landau equation derived as illustrations. The implications of the time-reversal symmetry for stationary states are explored to derive the potential condition and to generalize the fluctuation-dissipation theorem. A system of coupled equations is derived to determine self-consistently the order parameter as well as the energy spectrum, the dissipation and the particle distribution for elementary excitations. The general formalism and the useful techniques are illustrated by applications to critical dynamics, quenched random systems, theory of nonlinear response, plasma, nuclear many-body problem and so on.

1. Introduction

1.1. Why closed time-path?

The field-theoretical technique, introduced into the many-body theory since the late fifties, has proved to be highly successful in studying the ground state, the thermoequilibrium properties and the linear response of the system to the external disturbance [1-3]. However, only limited progress has been made in investigating the nonequilibrium properties beyond the linear response by using the field-theoretical methods. To appreciate the difficulties encountered here, let us recall some basic ingredients of the field-theoretical approach.

The Green function is defined as an average of the time ordered product of Heisenberg field operators over some state which we do not specify for the moment, i.e.,

$$G(t_1, t_2) = -i\langle T(\hat{A}(t_1)\hat{B}(t_2)) \rangle. \quad (1.1)$$

By introducing the interaction picture, (1.1) can be rewritten as

$$G(t_1, t_2) = -i\langle S^\dagger T(\hat{A}_I(t_1)\hat{B}_I(t_2)S) \rangle, \quad (1.2)$$

where the S matrix is defined as

$$S \equiv U(\infty, -\infty) = T \exp\left(-i \int_{-\infty}^{\infty} \mathcal{H}_{\text{int}}^I(t) dt\right), \quad (1.3)$$

with the interacting part of the Hamiltonian $\mathcal{H}_{\text{int}}^I(t)$ in the interaction picture.

If we are interested in the ground-state properties, then

$$S|0\rangle = e^{iL}|0\rangle, \quad \langle 0|S^\dagger = \langle 0|e^{-iL}, \quad (1.4)$$

where L is a phase factor contributed by the vacuum fluctuations and can be set equal to zero, if the renormalized ground state is considered. Therefore, we can easily get rid of S^\dagger in (1.2) so that the powerful arsenal of the quantum field theory can be used without major changes in the many-body theory at zero temperature.

For systems in thermoequilibrium at different from zero temperature, we cannot relate observable quantities directly to the elements of the S matrix, but the density matrix in this case take the following form:

$$\hat{\rho} = \exp[\beta(\mathcal{F} - \hat{\mathcal{H}})], \quad (1.5)$$

where β is the inverse temperature, \mathcal{F} the free energy, $\hat{\mathcal{H}}$ the Hamiltonian. If we consider β as an imaginary time it , $\hat{\rho}$ behaves like an evolution operator $\exp(-i\hat{\mathcal{H}}t)$. The well-known Matsubara technique [4–7] has been successfully developed by making use of this property.

However, it is not easy to handle the S^\dagger term in (1.2), if a general nonequilibrium state is considered. An intelligent way out was suggested by J. Schwinger in 1961 [8]. Let us imagine a time-path p which goes from $-\infty$ to $+\infty$ and then returns back from $+\infty$ to $-\infty$. We can then define a generalized S_p matrix along this closed time-path (as we call it)

$$S_p \equiv T_p \exp\left\{-i \int_p \mathcal{H}_{\text{int}}(t) dt\right\}, \quad (1.6)$$

where T_p is the time-ordering operator along this path p . It is identical to the standard T operator on the positive branch $(-\infty, +\infty)$ and represents \tilde{T} – an anti-time-ordering operator on the negative branch $(+\infty, -\infty)$. Also, any point at the negative branch is considered as a later instant than any time at the positive branch. Equipped with such generalized S_p matrix we can define the Green function along the closed time-path p as

$$\begin{aligned} G_p(t_1, t_2) &= -i\langle T_p(\hat{A}(t_1)\hat{B}(t_2)) \rangle \\ &= -i\langle T_p(\hat{A}_I(t_1)\hat{B}_I(t_2)S_p) \rangle. \end{aligned} \quad (1.7)$$

Although for physical observables the time values t_1, t_2 are on the positive branch, both positive and negative branches will come into play at intermediate steps of calculation if a self-consistent formalism is intended.

The introduction of the closed time-path appears at the first glance as a purely formal trick to restore the mathematical analogy with the quantum field theory. Actually, it has deeper motivation. In particle physics, people are mostly interested in scattering processes for which the S matrix providing the probability of transition from the in-states to the out-states, is the most suitable framework. In statistical physics, however, we are mainly concerned with the expectation value of physical quantities at finite time t . It is thus natural to introduce the S_p matrix along the closed time-path p going from the state at $-\infty$ along t -axis to the $+\infty$ state and returning back to the $-\infty$ state (see S^\dagger in (1.22)). This way we can establish a direct connection of S_p with observable quantities. As we will see later, the great merits of the closed time-path Green function (CTPGF) formalism more than justify the technical complications occurring due to the introduction of the additional negative time branch.

1.2. Few historical remarks

After Schwinger's initiative in 1961 [8], the closed time-path formalism has been elaborated and

developed further by Keldysh and many others [9–19]. Some people used to call it Keldysh formalism. For the recent 20 years, this technique has been used to attack a number of interesting problems in statistical physics and condensed matter theory such as spin system [20], superconductivity [21–24], laser [25], tunneling and secondary emission [26–32], plasma [33, 34], other transport processes [35–38] and so on. For some of these systems like laser, the application of the CTPGF formalism is essential because the standard technique cannot be used directly for far from equilibrium situations, whereas for some of the others the CTPGF approach is used mainly due to its technical convenience. It is our impression, however, that the potential advantages of this formalism have not yet been fully exploited, partly because of its apparent technical complexity.

For the last few years we have combined the generating functional technique and the path integral representation, widely used in the quantum field theory [39], with the CTPGF approach and have developed a unified framework to describe both equilibrium and nonequilibrium systems with symmetry breaking and dynamical coupling between the order parameter and the elementary excitations [40–49]. To check the formalism developed and to explore its potentiality we have applied it to a number of problems including critical dynamics, quenched random systems, nonlinear response theory, superconductivity, laser, plasma, nuclear matter, quasi-one-dimensional conductor, and so on [40–57]. Although most of these problems in principle can be also discussed using other techniques, the logical simplicity and the flexibility, the unified approach to equilibrium and nonequilibrium processes as well as the deep insight one can get make the CTPGF formalism promising and encouraging.

1.3. Outline of the paper

In this paper we would like to summarize some of the results obtained by our group in developing and applying the CTPGF formalism. Because of the limitation of space we will only outline the main features along with some useful techniques of the CTPGF approach and illustrate them by few examples. Since the major part of our papers was published either in Chinese or in not easily accessible English journals, we will attempt to make this article self-contained as much as possible. Nevertheless, we should warn the reader that some part of this review is still descriptive and sketchy. A brief summary of the CTPGF formalism was given by us earlier [58], but this paper is much more extended and complete. Since we are mainly summarizing our own results, the contributions of other authors in developing and applying the CTPGF approach may not be emphasized as they should be. We apologize to them for any possible omissions or underestimates. To keep the integrity of presentation we will not distinguish carefully what was known before and what is new.

The topics to be covered in this review can be seen from the table of contents. We will not repeat them here. Few remarks, however, are in order. Section 2 is mainly tutorial, but the subsection on normalization and causality is important for further discussion. Section 3 is devoted to a detailed discussion of the two-point functions. The differentiation of the micro- and macro-time scales described there is very useful. In section 4 the potential condition and the fluctuation-dissipation theorem (FDT) are discussed from a microscopic point of view. The theory of nonlinear response which may be important for future applications is outlined in section 5. Section 6 is devoted to the consideration of the symmetry breaking and the Ward-Takahashi (WT) identities. We believe that the CTPGF formalism is advantageous in studying systems with broken symmetry. We also mention there the additional way of describing fluctuations available in the CTPGF approach. The unified framework of treating the dynamical coupling between the order parameter and the elementary excitations mentioned before, is given in section 7. We could start from this formalism at the very beginning, but the present more

inductive exposition is probably more convenient for the reader. In section 8 we show that the quenched average may be carried out directly on the generating functional in the CTPGF formalism and the replica trick can be thus avoided. The connections with other formalisms are described in section 9. Readers, familiar with them might have a look at this section before the others. An experienced and busy reader could get a rough idea about the CTPGF approach by a quick scanning of sections 2 and 6–9.

1.4. Notations

Throughout this paper we will use the units $\hbar = k_B = c = 1$ except for few paragraphs where the Planck constant \hbar is written out explicitly to emphasize the quasiclassical nature of expansion.

The metric tensor we use is given by

$$g^{00} = -g^{11} = -g^{22} = -g^{33} = 1 \quad (1.8)$$

with the scalar product and the d'Alembertian

$$\square \equiv \partial_\mu \partial^\mu = \partial_t^2 - \nabla^2 \quad (1.9)$$

defined correspondingly.

The Fourier transformation with respect to the relative coordinates $x-y$ is defined as

$$G(x, y) = G\left[\frac{x+y}{2}, x-y\right] = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \exp[-ip \cdot (x-y)] \tilde{G}\left(\frac{x+y}{2}, p\right), \quad (1.10)$$

where d is the space dimension and $p \cdot x = p_0 t - \mathbf{p} \cdot \mathbf{r}$. The tilde “ $\tilde{\cdot}$ ” will be omitted wherever no confusion occurs.

The formalism presented in this paper can be applied to a broad class of fields including non-Abelian gauge fields, but in most cases we will illustrate it by a real boson field, either relativistic or nonrelativistic (e.g. phonons), and a nonrelativistic complex boson or fermion field. The former will be denoted by $\varphi(x)$, whereas the latter by $\hat{\psi}(x)$ and $\hat{\psi}^\dagger(x)$. Wherever a double sign \pm or \mp appears, the upper case will always correspond to the boson field, while the lower one corresponds to the fermion field.

As a rule, the field operator is not distinguished by the caret “ $\hat{\cdot}$ ” which itself is used in some cases to denote a two-component vector or a 2×2 matrix.

Also, for simplicity we introduce an abbreviated notation for integration

$$J\varphi \equiv \int J(x)\varphi(x) \equiv \int d^d x dt J(x)\varphi(x). \quad (1.11)$$

The form at the right is used only in exceptional cases, while the middle one most frequently.

The Pauli matrices are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.12)$$

2. Basic properties of CTPGF

As mentioned in the Introduction, this section is mainly tutorial. To get familiar with the concepts and notations used in the CTPGF formalism we start from two-point functions (section 2.1) in close contact with the ordinary Green functions. We then define the generating functional and discuss the perturbation theory (section 2.2). The single time representation and the physical representation as well as the transformation from one to another are discussed in section 2.3. Furthermore, the consequences of the normalization and the causality are outlined in section 2.4. Finally, the Lehmann spectral representation is described in section 2.5.

2.1. Two-point functions

The two-point Green functions are most useful in practical applications and hence their properties have been most thoroughly investigated. In this section we first define the two-point CTPGF and then discuss their connection with the ordinary retarded, advanced and correlation functions along with the causality relations. As we will see later, they are special cases of much more general relations following from the normalization condition of the generating functional and the causality. The explicit expressions will be given for free propagators in thermoequilibrium systems.

2.1.1. Definition

The two-point CTPGF for a complex field $\psi(x)$ is defined as

$$\begin{aligned} G(x, y) &\equiv -i \operatorname{Tr}\{T_p(\psi(x)\psi^\dagger(y))\hat{\rho}\} \\ &\equiv -i\langle T_p(\psi(x)\psi^\dagger(y))\rangle, \end{aligned} \quad (2.1)$$

where $\psi(x)$, $\psi^\dagger(y)$ are Heisenberg operators, $\hat{\rho}$ the density matrix, T_p the time ordering operator as discussed in the Introduction. Inasmuch as x, y can assume values on either positive or negative time branches, $G(x, y)$ can be presented as a 2×2 matrix

$$\hat{G}(x, y) \equiv \begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix} \equiv \begin{pmatrix} G_F & G_+ \\ G_- & G_{\bar{F}} \end{pmatrix}, \quad (2.2)$$

with

$$G_F(x, y) \equiv -i\langle T(\psi(x)\psi^\dagger(y))\rangle. \quad (2.3a)$$

$$G_+(x, y) \equiv \mp i\langle \psi^\dagger(y)\psi(x)\rangle, \quad (2.3b)$$

$$G_-(x, y) \equiv -i\langle \psi(x)\psi^\dagger(y)\rangle, \quad (2.3c)$$

$$G_{\bar{F}}(x, y) \equiv -i\langle \tilde{T}(\psi(x)\psi^\dagger(y))\rangle. \quad (2.3d)$$

Here G_F is the usual Feynman causal propagator, whereas the other three are new in the CTPGF formalism. Sometimes $G_{\bar{F}}$ defined as expectation value of anti-time-ordering product, is called anti-

causal propagator. Using the step function

$$\theta(x, y) = \begin{cases} 1, & \text{if } t_x > t_y \\ 0, & \text{otherwise,} \end{cases} \quad (2.4)$$

eqs. (2.3a) and (2.3d) can be rewritten as

$$\begin{aligned} G_F(x, y) &= -i\theta(x, y)\langle\psi(x)\psi^\dagger(y)\rangle \mp i\theta(y, x)\langle\psi^\dagger(y)\psi(x)\rangle, \\ G_{\bar{F}}(x, y) &= -i\theta(y, x)\langle\psi(x)\psi^\dagger(y)\rangle \mp i\theta(x, y)\langle\psi^\dagger(y)\psi(x)\rangle. \end{aligned} \quad (2.5)$$

These four functions are not independent of each other. There is an algebraic identity

$$\begin{aligned} G_{++} + G_{--} &= G_{+-} + G_{-+}, \\ \text{or} \quad G_F(x, y) + G_{\bar{F}}(x, y) &= G_+(x, y) + G_-(x, y), \end{aligned} \quad (2.6)$$

following from the normalization of the step function

$$\theta(x, y) + \theta(y, x) = 1. \quad (2.7)$$

In what follows we will call CTPGFs defined by relations like (2.3) as “single time” representation and denote them by a tensor $G_{\alpha\beta\cdots\rho}(12\cdots n)$ with Greek subscripts $\alpha, \beta, \dots, \rho = \pm$. As a whole, the tensor itself is written as \hat{G} .

2.1.2. Physical representation

The CTPGFs defined above are most convenient for calculations, but more direct contact with measurable quantities is established via the “physical” representation defined as

$$G_r(x, y) \equiv -i\theta(x, y)\langle[\psi(x), \psi^\dagger(y)]_+\rangle, \quad (2.8a)$$

$$G_a(x, y) \equiv i\theta(y, x)\langle[\psi(x), \psi^\dagger(y)]_+\rangle, \quad (2.8b)$$

$$G_c(x, y) \equiv -i\langle\{\psi(x), \psi^\dagger(y)\}\rangle, \quad (2.8c)$$

where G_r , G_a and G_c are retarded, advanced and correlation functions, correspondingly. In this definition,

$$[\psi(x), \psi^\dagger(y)]_+ = \{\psi(x), \psi^\dagger(y)\} = \psi(x)\psi^\dagger(y) + \psi^\dagger(y)\psi(x).$$

It is straightforward to check that these functions are related to the CTPGF in single time representation as follows:

$$G_r = G_F - G_+ = G_- - G_{\bar{F}}, \quad (2.9a)$$

$$G_a = G_F - G_- = G_+ - G_{\bar{F}}, \quad (2.9b)$$

$$G_c = G_F + G_{\bar{F}} = G_+ + G_-. \quad (2.9c)$$

The inverse relations are given by

$$\hat{G} = \frac{1}{2}G_r \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \frac{1}{2}G_a \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \frac{1}{2}G_c \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (2.10)$$

If we introduce two-component vectors

$$\xi \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \xi^\dagger \equiv (1, 1), \quad \eta \equiv \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \eta^\dagger \equiv (1, -1), \quad (2.11)$$

(2.10) can be rewritten as

$$\hat{G} = \frac{1}{2}G_r \xi \eta^\dagger + \frac{1}{2}G_a \eta \xi^\dagger + \frac{1}{2}G_c \xi \xi^\dagger, \quad (2.12a)$$

or in components

$$G_{\alpha\beta} = \frac{1}{2}G_r \xi_\alpha \eta_\beta + \frac{1}{2}G_a \eta_\alpha \xi_\beta + \frac{1}{2}G_c \xi_\alpha \xi_\beta. \quad (2.12b)$$

Sometimes it is convenient to introduce a matrix form for the physical functions

$$\tilde{G} = \begin{pmatrix} 0 & G_a \\ G_r & G_c \end{pmatrix}. \quad (2.13)$$

The transformations (2.10) and (2.12) can then be presented as

$$\hat{G} = Q^{-1} \tilde{G} Q, \quad \tilde{G} = Q \hat{G} Q^{-1}, \quad (2.14)$$

using the orthogonal matrix

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (1 - i\sigma_2), \quad Q^\dagger = Q^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (2.15)$$

which was first introduced by Keldysh [9]. In what follows we will call \tilde{G} the CTPGF in physical representation and denote their components by $G_{ij\dots l}(12\dots n)$ with Latin subscripts $i, j, \dots, n = 1, 2$. In the case of two-point functions,

$$G_{11} = \frac{1}{2}(G_F + G_{\bar{F}} - G_+ - G_-) = 0, \quad (2.16a)$$

$$G_{12} = G_a = \frac{1}{2}(G_F - G_- + G_+ - G_{\bar{F}}), \quad (2.16b)$$

$$G_{21} = G_r = \frac{1}{2}(G_F - G_+ + G_- - G_{\bar{F}}), \quad (2.16c)$$

$$G_{22} = G_c = \frac{1}{2}(G_F + G_{\bar{F}} + G_+ + G_-). \quad (2.16d)$$

We see thus G_{11} is always zero and the other equations of (2.16) are identical to those of (2.9) by virtue of the identity (2.6).

It is obvious from definition (2.8) that

$$G_{12}(x, y) \equiv G_a(x, y) = 0, \quad \text{if } t_x > t_y; \quad G_{21}(x, y) \equiv G_r(x, y) = 0, \quad \text{if } t_y > t_x, \quad (2.17)$$

and also that

$$G_{12}(x, y) \cdot G_{21}(x, y) = 0, \quad (2.18)$$

because

$$\theta(x, y)\theta(y, x) = 0. \quad (2.19)$$

As will be shown later, almost all that has been said here, e.g., $G_{11} = 0$, the transformation of single and physical representations, the causality relations (2.17), (2.18), etc., can be easily generalized to the multi-point functions using the generating functional technique. However, before going on to describe this technique itself we give here the explicit expressions for the free propagators.

2.1.3. Free propagators

The Lagrangian of the free fermion field is given by

$$\mathcal{L}_0 = \int \psi^\dagger(x) \left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) \psi(x), \quad (2.20)$$

where m is the particle mass. The single time CTPGFs are defined by (2.3). To distinguish it from the Bose case we will use the letter S instead of G . If the system is in thermoequilibrium, the free propagator can be evaluated immediately from the definition. In Fourier space the CTPGFs turn out to be

$$\begin{aligned} S_{0p}(p) &= \begin{pmatrix} S_F & S_+ \\ S_- & S_{\bar{F}} \end{pmatrix}, \\ S_F(p) &= \frac{1 - n(p)}{p_0 - p^2/2m + i\epsilon} + \frac{n(p)}{p_0 - p^2/2m - i\epsilon} \\ &= \frac{1}{p_0 - p^2/2m + i\epsilon} + 2\pi i n(p) \delta(p_0 - p^2/2m), \end{aligned} \quad (2.21a)$$

$$S_+(p) = 2\pi i n(p) \delta(p_0 - p^2/2m), \quad (2.21b)$$

$$S_-(p) = -2\pi i (1 - n(p)) \delta(p_0 - p^2/2m), \quad (2.21c)$$

$$\begin{aligned} S_{\bar{F}}(p) &= -\frac{n(p)}{p_0 - p^2/2m + i\epsilon} - \frac{1 - n(p)}{p_0 - p^2/2m - i\epsilon} \\ &= \frac{-1}{p_0 - p^2/2m - i\epsilon} + 2\pi i n(p) \delta(p_0 - p^2/2m). \end{aligned} \quad (2.21d)$$

where

$$n(p) = \frac{1}{\exp[(p_0 - \mu)/T] + 1} \quad (2.22)$$

is the Fermi distribution with μ as the chemical potential. If $n(p)$ is set equal to zero, we recover the propagator for the “pure” vacuum. It is interesting to note that the additional term $2\pi i n(p)\delta(p_0 - p^2/2m)$ is the same for all components of \hat{G} . The reason for this will be clear from the next section.

In accord with (2.9) we find the physical functions to be

$$S_r(p) = \frac{1}{p_0 - p^2/2m + i\epsilon}, \quad (2.23a)$$

$$S_a(p) = \frac{1}{p_0 - p^2/2m - i\epsilon}, \quad (2.23b)$$

$$S_c(p) = -2\pi i(1 - 2n(p))\delta(p_0 - p^2/2m). \quad (2.23c)$$

We note in passing that the retarded and the advanced Green functions S_r , S_a do not depend on the particle distribution $n(p)$.

Similarly, for the Hermitian boson field described by the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2} \int (\partial_\mu \varphi(x) \partial^\mu \varphi(x) - m^2 \varphi^2(x)) \quad (2.24)$$

we have

$$\Delta_p(x, y) \equiv -i\langle T_p(\varphi(x)\varphi(y)) \rangle \equiv \begin{pmatrix} \Delta_F(x, y) & \Delta_+(x, y) \\ \Delta_-(x, y) & \Delta_F(x, y) \end{pmatrix}. \quad (2.25)$$

In the Fourier space the free boson CTPGFs are

$$\Delta_F(p) = \frac{1}{p_0^2 - \omega^2(p) + i\epsilon} - 2\pi i f(p)\delta(p_0^2 - \omega^2(p)), \quad (2.26a)$$

$$\Delta_+(p) = -2\pi i(\theta(p_0) + f(p))\delta(p_0^2 - \omega^2(p)), \quad (2.26b)$$

$$\Delta_-(p) = -2\pi i(\theta(-p_0) + f(p))\delta(p_0^2 - \omega^2(p)), \quad (2.26c)$$

$$\Delta_F(p) = -\frac{1}{p_0^2 - \omega^2(p) - i\epsilon} - 2\pi i f(p)\delta(p_0^2 - \omega^2(p)), \quad (2.26d)$$

where

$$f(p) = \frac{1}{\exp[\omega(p)/T] - 1} \quad (2.27)$$

is the Bose distribution and

$$\omega(p) = \sqrt{p^2 + m^2} \quad (2.28)$$

is the particle energy.

If $f(p) = 0$, we recover the standard boson propagator of the quantum field theory [39]. Also, the additional term proportional to $f(p)$ is the same for all components of \hat{A} .

The corresponding retarded, advanced and correlation functions are given by

$$\Delta_r(p) = \frac{1}{p_0^2 - \omega^2(p) + 2i\epsilon p_0}, \quad (2.29a)$$

$$\Delta_a(p) = \frac{-1}{p_0^2 - \omega^2(p) - 2i\epsilon p_0}, \quad (2.29b)$$

$$\Delta_c(p) = -2\pi i(1 + 2f(p))\delta(p_0^2 - \omega^2(p)). \quad (2.29c)$$

It can be shown [40] that the expressions for fermion and boson propagators (2.21), (2.26) remain the same for inhomogeneous, nonequilibrium systems provided $n(p)$ and $f(p)$ are replaced by their nonequilibrium counterparts – Wigner distributions $n(X, p)$, $f(X, p)$ in the external field, where

$$X = (x + y)/2 \quad (2.30)$$

is the center of mass coordinates.

2.2. Generating functionals

For interacting fields we can construct the perturbation expansion in full analogy with the quantum field theory. The Wick theorem can be generalized to the CTPGF case, most conveniently by using the generating functional. For simplicity, we consider real bosons. The extension to other systems is obvious.

2.2.1. Definition of $Z[J]$

The Lagrangian of the system is given by

$$\mathcal{L} = \mathcal{L}_0(\varphi) + \int (J(x)\varphi(x) - V(\varphi)), \quad (2.31)$$

where $\mathcal{L}_0(\varphi)$ is given by (2.24), $V(\varphi)$ the self-interaction and $J(x)$ the external source. The generating functional for CTPGF is defined as

$$Z[J(x)] \equiv \text{Tr} \left\{ T_p \left[\exp \left(i \int_p J(x) \varphi(x) \right) \right] \hat{\rho} \right\}, \quad (2.32)$$

where the integration path p and the time ordering product along it T_p have been already defined in the Introduction. In general, the external source on the positive branch $J_+(x)$ and the negative branch $J_-(x)$ are assumed to be different. They will be set equal to each other or both to zero at the end of calculation.

The n -point CTPGF is defined as

$$G_p(1 \cdots n) \equiv (-i)^{n-1} \text{Tr} [T_p(\varphi(1) \cdots \varphi(n)) \hat{\rho}] = i(-1)^n \left. \frac{\delta^n Z[J(x)]}{\delta J(1) \cdots \delta J(n)} \right|_{J=0}. \quad (2.33)$$

2.2.2. Generalized Wick theorem

In the incoming interaction picture (2.32) can be rewritten as

$$Z[J(x)] = \text{Tr} \left\{ T_p \left[\exp \left(-i \int_p (V(\varphi_I(x)) - J(x) \varphi_I(x)) \right) \right] \hat{\rho} \right\}, \quad (2.34)$$

where the in-field $\varphi_I(x)$ satisfies the free equation of motion. The interaction term can be then taken from behind the trace operator to obtain

$$Z[J(x)] = \exp \left[-i \int_p V \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right] \text{Tr} \left\{ T_p \left[\exp \left(i \int J(x) \varphi_I(x) \right) \right] \hat{\rho} \right\}. \quad (2.35)$$

It is easy to show by generalizing the Wick theorem that [40]

$$T_p \left[\exp \left(i \int_p J(x) \varphi_I(x) \right) \right] = Z_0[J(x)] : \exp \left[i \int_p J(x) \varphi_I(x) \right] :, \quad (2.36)$$

where $: :$ means normal product and $Z_0[J(x)]$ is the generating functional for the free field

$$Z_0[J(x)] = \exp \left\{ -\frac{i}{2} \int_p \int_p J(x) G_{0p}(x-y) J(y) \right\}, \quad (2.37)$$

G_{0p} being the free propagator given by (2.26) with $f(p) = 0$. Substituting (2.36) into (2.35) we obtain

$$Z[J(x)] = \exp \left[-i \int_p V \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right] Z_0[J(x)] N[J(x)], \quad (2.38)$$

with

$$N[J(x)] = \text{Tr} \left[: \exp \left(i \int J(x) \varphi_1(x) \right) : \hat{\rho} \right] \quad (2.39)$$

as the correlation functional for the initial state. $N[J(x)]$ can be expanded into a series of successive cumulants

$$N[J(x)] = \exp(i W_p^N[J(x)]), \quad (2.40)$$

$$W_p^N[J(x)] = \sum_{n=1}^{\infty} \frac{1}{n!} \int_p \cdots \int_p W_p^n(1 \cdots n) J(1) \cdots J(n), \quad (2.41)$$

where

$$W_p^n(1 \cdots n) = i^{n-1} \text{Tr} [: \varphi_1(1) \cdots \varphi_1(n) : \hat{\rho}]_c. \quad (2.42)$$

It is worthwhile to note that correlation functions contribute to the propagator only on the mass shell because φ_1 satisfies the homogeneous equation

$$(p_0^2 - \omega^2(p))\varphi_1(p) = 0, \quad (2.43)$$

where $\omega(p)$ is the boson energy (2.28). Also, the definition (2.39) is independent of the time branch, i.e., each CTPGF component will get the same additional term as we have seen in the last section on the example of free propagator.

Therefore, the perturbation expansion in the CTPGF approach has a structure identical to that of the quantum field theory except that the time integration is carried out over the closed path consisting of positive and negative branches. Rewritten in single time representation, each n -point function (also the corresponding Feynman diagram) is decomposed into 2^n functions (diagrams). The presence of initial correlations $W_p^n(1 \cdots n)$ which vanish for the vacuum state, constitutes another difference from the ordinary field theory. In principle, all orders of correlations can be taken into account, but in most cases we will limit ourselves to the second cumulant.

It can be shown quite generally that the counter terms of the quantum field theory at zero temperature are enough to remove all ultraviolet divergences for the CTPGFs under reasonable assumption concerning the initial correlations [41]. We will not elaborate further on this point here, but it should be mentioned that near the phase transition point the infrared singularities have to be separated first so that the ultraviolet renormalization for CTPGFs in this case is different from that of the ordinary field theory (see section 9.2).

2.2.3. Connected and vertex generating functionals

The generating functional for the connected CTPGF is defined as

$$W[J(x)] = -i \ln Z[J(x)], \quad (2.44)$$

$$\begin{aligned}
G_p^c(1 \cdots n) &= (-1)^{n-1} \frac{\delta^n W[J(x)]}{\delta J(1) \cdots \delta J(n)} \Big|_{J=0} \\
&= (-i)^{n-1} \langle T_p(\varphi(1) \cdots \varphi(n)) \rangle_c
\end{aligned} \tag{2.45}$$

where $\langle \rangle_c$ stands for $\text{Tr}(\cdots \hat{\rho})$ with the connected parts taken only. The $n = 1$ case corresponds to the expectation value

$$\varphi_c(x) = \delta W[J(x)]/\delta J(x) = \langle \varphi(x) \rangle_J \tag{2.46}$$

of the field operator in the presence of the external source. Therefore, $\varphi_c(x)$ is also a functional of $J(x)$.

Performing the Legendre transformation upon $W[J]$, we obtain the vertex functional

$$\Gamma[\varphi_c(x)] = W[J(x)] - \int_p J(x) \varphi_c(x), \tag{2.47}$$

which depends on $\varphi_c(x)$ explicitly as well as implicitly via $J(x)$ by eq. (2.46). It follows then from (2.46) and (2.47) that

$$\delta \Gamma[\varphi_c(x)]/\delta \varphi_c(x) = -J(x). \tag{2.48}$$

This is the basic equation of the CTPGF formalism from which we will derive a number of important consequences.

The general n -point vertex function, or one particle irreducible (1PI) function, is defined as

$$\Gamma_p(1 \cdots n) = \delta^n \Gamma[\varphi_c(x)]/\delta \varphi_c(1) \cdots \delta \varphi_c(n). \tag{2.49}$$

2.2.4. Dyson equation

As an immediate consequence of the definition for the generating functionals $W[J(x)]$ and $\Gamma[\varphi_c(x)]$ we derive here the Dyson equation.

Taking functional derivative of (2.48) with respect to $J(y)$ and using (2.45) and (2.46)

$$\begin{aligned}
\frac{\delta}{\delta J(y)} \left(\frac{\delta \Gamma[\varphi_c(x)]}{\delta \varphi_c(x)} \right) &= \int_p \frac{\delta \varphi_c(z)}{\delta J(y)} \frac{\delta^2 \Gamma[\varphi_c(x)]}{\delta \varphi_c(z) \delta \varphi_c(x)} \\
&= \int_p \frac{\delta^2 W[J]}{\delta J(y) \delta J(z)} \frac{\delta^2 \Gamma[\varphi_c]}{\delta \varphi_c(z) \delta \varphi_c(x)} = -\frac{\delta J(x)}{\delta J(y)},
\end{aligned}$$

we obtain the Dyson equation

$$\int_p G_p(y, z) \Gamma_p(z, x) = \delta_p(x - y). \tag{2.50a}$$

Similarly, by varying (2.46) with respect to $\varphi_c(y)$ we find

$$\int_p \Gamma_p(x, z) G_p(z, y) = \delta_p(x - y). \quad (2.50b)$$

Here

$$G_p(x, y) \equiv -\delta^2 W[J(x)]/\delta J(x)\delta J(y) \quad (2.51)$$

is the two-point connected Green function (the subscript c is suppressed), whereas

$$\Gamma_p(x, y) \equiv \delta^2 \Gamma[\varphi_c]/\delta \varphi_c(x)\delta \varphi_c(y) \quad (2.52)$$

is the two-point vertex function containing only 1PI part. The δ -function on the closed path δ_p is defined as

$$\int_p \delta_p(x - y)f(y) = f(x). \quad (2.53)$$

In the single time representation

$$\int_p dt = \int_{-\infty}^{\infty} dt_+ - \int_{-\infty}^{\infty} dt_-, \quad (2.54)$$

where the minus sign in the second term comes from the definition of the closed time-path. The negative branch goes from $+\infty$ to $-\infty$. To satisfy eq. (2.53), it should be that

$$\delta_p(x - y) = \begin{cases} \delta(x - y), & \text{if both } x, y \text{ on positive branch,} \\ -\delta(x - y), & \text{if both } x, y \text{ on negative branch,} \\ 0, & \text{otherwise.} \end{cases} \quad (2.55)$$

In matrix notation δ_p can be written as

$$\hat{\delta}(x - y) = \delta(x - y)\sigma_3. \quad (2.56)$$

The Dyson equation (2.50) in single time representation is thus

$$\int \hat{I}(x, z)\sigma_3 \hat{G}(z, y) = \sigma_3 \delta(x - y), \quad \int \hat{G}(x, z)\sigma_3 \hat{I}(z, y) = \sigma_3 \delta(x - y). \quad (2.57)$$

The transformation properties for the two-point connected Green function are the same as those

discussed in section 2.1.2, except that a disconnected part proportional to $-i\varphi_c(x)\varphi_c(y)$ should be subtracted from all \hat{G} functions, whereas a term $-2i\varphi_c(x)\varphi_c(y)$ must be subtracted only from G_c with G_r and G_a remaining the same.

Multiplying (2.57) by matrix Q from left and Q^{-1} (see (2.15)) from right we obtain

$$\int \tilde{\Gamma}(x, z)\sigma_1\tilde{G}(z, y) = \sigma_1\delta(x - y), \quad \int \tilde{G}(x, z)\sigma_1\tilde{\Gamma}(z, y) = \sigma_1\delta(x - y), \quad (2.58)$$

where

$$\tilde{\Gamma} = Q\hat{\Gamma}Q^{-1} = Q \begin{pmatrix} \Gamma_F & \Gamma_+ \\ \Gamma_- & \Gamma_{\bar{F}} \end{pmatrix} Q^{-1}. \quad (2.59)$$

We note in passing that the Pauli matrix σ_3 always accompanies the CTPGF in single time representation \hat{G} , $\hat{\Gamma}$, whereas σ_1 appears together with the CTPGF in physical representation \tilde{G} , $\tilde{\Gamma}$. As seen from eqs. (2.57) and (2.58) $\sigma_3\hat{\Gamma}\sigma_3$ is the inverse of \hat{G} , while $\sigma_1\tilde{\Gamma}\sigma_1$ is that for \tilde{G} .

It is more important to point out that all characteristic features of Green's functions G discussed in section 2.1 are transmitted to vertex functions Γ via the Dyson equations (2.57) and (2.58). In particular, we have

$$\Gamma_{11} = 0, \quad \Gamma_F + \Gamma_{\bar{F}} = \Gamma_+ + \Gamma_-, \quad (2.60)$$

so that

$$\tilde{\Gamma} = \begin{pmatrix} 0 & \Gamma_a \\ \Gamma_r & \Gamma_c \end{pmatrix}, \quad (2.61)$$

with

$$\Gamma_r = \Gamma_F - \Gamma_+ = \Gamma_- - \Gamma_{\bar{F}}, \quad \Gamma_a = \Gamma_F - \Gamma_- = \Gamma_+ - \Gamma_{\bar{F}}, \quad \Gamma_c = \Gamma_F + \Gamma_{\bar{F}} = \Gamma_+ + \Gamma_-. \quad (2.62)$$

The inverse transformation from $\tilde{\Gamma}$ to $\hat{\Gamma}$ is given by

$$\hat{\Gamma} = \frac{1}{2}\Gamma_r\xi\eta^\dagger + \frac{1}{2}\Gamma_a\eta\xi^\dagger + \frac{1}{2}\Gamma_c\xi\xi^\dagger, \quad (2.63)$$

i.e., exactly the same way as Green's function (2.12).

Further discussion on the Dyson equation will be postponed to section 3.1. Meanwhile, we would like to emphasize that the “transmissibility” of the CTPGF characteristics is an evidence of the logical consistency of the formalism itself. More examples along with some useful computation rules were given before [40, 43, 44].

One more remark concerning the generating functional technique itself. Up to now we have considered only CTPGFs for the constituent field $\varphi(x)$, but what has been said for it can be repeated almost word for word for any composite operator $Q[\varphi(x)]$. In the forthcoming discussion we will use the corresponding formulas without repeating their definitions.

2.3. Single time and physical representations

In the CTPGF approach we have to deal with three representations which are equivalent to each other, namely, the closed time-path form G_p used for compact writing of formulas, the single time form \hat{G} and the physical representation \tilde{G} . In section 2.1 we have already discussed these representations and their mutual transformations on the example of two-point functions. In this section we will use the generating functional to consider general n -point functions. The underlying connection here is that different representations of CTPGF are generated by the same generating functional expressed in different functional arguments. The explicit expressions for n -point functions in physical representation will be obtained along with the transformations from \hat{G} to \tilde{G} and vice versa. As we will see later in section 2.4, these formulas are the starting point for discussing the important normalization and causality relations.

2.3.1. Preliminaries

To start with, we need to specify some more notations. The multi-point step function

$$\theta(1, 2, \dots, n) = \begin{cases} 1, & \text{if } t_1 > t_2 \dots > t_n, \\ 0, & \text{otherwise,} \end{cases} \quad (2.64)$$

is a product of two-point step functions

$$\theta(1, 2, \dots, n) = \theta(1, 2)\theta(2, 3)\dots\theta(n-1, n). \quad (2.65)$$

It can be used to define the time ordered product

$$T(A_1(1) \dots A_n(n)) = \sum_{p_n} \theta(\bar{1}, \dots, \bar{n}) A_{\bar{1}}(\bar{1}) \dots A_{\bar{n}}(\bar{n}), \quad (2.66a)$$

or the anti-time-ordered product

$$\tilde{T}(A_1(1) \dots A_n(n)) = \sum_{p_n} \theta(\bar{1}, \dots, \bar{n}) A_{\bar{n}}(\bar{n}) \dots A_{\bar{1}}(\bar{1}). \quad (2.66b)$$

The summation here is carried out over all permutations of n numbers p_n

$$\begin{pmatrix} 1 & 2 \dots n \\ \bar{1} & \bar{2} \dots \bar{n} \end{pmatrix}.$$

These step functions satisfy the normalization condition

$$\sum_{p_n} \theta(\bar{1}, \dots, \bar{n}) = 1, \quad (2.67)$$

and the summation formula

$$\theta(1, 2, \dots, m) = \sum_{p_n(1 \cdots m)} \theta(\bar{1} \cdots \bar{n}), \quad (2.68)$$

where $p_n(1 \cdots m)$ means permutations of n numbers with 1 preceding 2, 2 preceding 3, etc., but the order of the rest is arbitrary. In fact, (2.67) is the special case $m = 0$ of (2.68).

The external source term in the generating functional (2.32) can be presented as

$$I_s = \int_p J(x) \varphi(x) = \int_{-\infty}^{\infty} dt d^d x (J_+(x) \varphi_+(x) - J_-(x) \varphi_-(x)) \equiv \int \hat{J}^\dagger \sigma_3 \hat{\varphi}, \quad (2.69)$$

where

$$\hat{\varphi} = \begin{pmatrix} \varphi_+(x) \\ \varphi_-(x) \end{pmatrix}, \quad \hat{J} = \begin{pmatrix} J_+(x) \\ J_-(x) \end{pmatrix}, \quad (2.70)$$

and also as

$$I_s = \int (J_\Delta \varphi_c + J_c \varphi_\Delta), \quad (2.71)$$

with

$$\begin{aligned} J_\Delta &\equiv \eta^\dagger \hat{J} = J_+ - J_-, & J_c &\equiv \frac{1}{2} \xi^\dagger \hat{J} = \frac{1}{2}(J_+ + J_-), \\ \varphi_\Delta &\equiv \eta^\dagger \hat{\varphi} = \varphi_+ - \varphi_-, & \varphi_c &\equiv \frac{1}{2} \xi^\dagger \hat{\varphi} = \frac{1}{2}(\varphi_+ + \varphi_-). \end{aligned} \quad (2.72)$$

Also, we can express \hat{J} , $\hat{\varphi}$ in terms of J_c , J_Δ , φ_c and φ_Δ as

$$\hat{J} = J_c \xi + \frac{1}{2} J_\Delta \eta, \quad \hat{\varphi} = \varphi_c \xi + \frac{1}{2} \varphi_\Delta \eta. \quad (2.73)$$

The functional derivatives are related with each other by the following equations:

$$\frac{\delta}{\delta J_\alpha(x)} = \frac{1}{2} \xi_\alpha \frac{\delta}{\delta J_c(x)} + \eta_\alpha \frac{\delta}{\delta J_\Delta(x)}, \quad (2.74a)$$

$$\frac{\delta}{\delta J_c(x)} = \xi_\alpha \frac{\delta}{\delta J_\alpha(x)} = \eta_\alpha \frac{\delta}{\delta J(x_\alpha)}, \quad (2.74b)$$

$$\frac{\delta}{\delta J_\Delta(x)} = \frac{1}{2} \eta_\alpha \frac{\delta}{\delta J_\alpha(x)} = \frac{1}{2} \xi_\alpha \frac{\delta}{\delta J(x_\alpha)}, \quad (2.74c)$$

with $\alpha = \pm$ and summation over repeated indices. Here we have introduced a symbolic notation

$$\frac{\delta}{\delta J(x_\alpha)} = \sigma_3^{\alpha\beta} \frac{\delta}{\delta J_\beta(x)}, \quad (2.75)$$

which is useful for a compact writing of the definition for Green's function as seen from (2.76).

A remark concerning the notation $\varphi_c(x)$ is in order. Previously (see (2.46)) we have defined $\varphi_c(x)$ as the expectation value of $\varphi(x)$ on the closed time-path. Hence it is a two-component vector $(\varphi_{c+}(x), \varphi_{c-}(x))$, but we do not make the subscripts $+$, $-$ explicit. Here (see (2.72)) $\varphi_c(x)$ is the linear combination of operators $\varphi_+(x)$ and $\varphi_-(x)$, still, in accord with our convention, no caret is put above it. Later on, the same $\varphi_c(x)$ will denote its expectation value. Hopefully, no confusion will occur, since the meaning of $\varphi_c(x)$ is clear from the context and, moreover, $\varphi_{c+}(x) = \varphi_{c-}(x) = \varphi_c(x)$ for $J_+(x) = J_-(x)$ as seen from (2.105). The same remark is effective with respect to other functions like $Q_c(x)$, $\psi_c(x)$, $\psi_c^\dagger(x)$ and so on, appearing in the future discussion.

2.3.2. "Physical" representation of the generating functional

As we said in the introductory remarks to this section, the same generating functional will generate CTPGF in different representation provided the external source term is expressed in the corresponding functional arguments. In particular, the generating functional in the form (2.32) will give rise to CTPGF in the closed time-path representation. If, however, the source is written in single time form as given by (2.69), the same generating functional (2.32) can be then expanded as

$$\begin{aligned} Z[J_+(x), J_-(x)] &= \sum_{n=0}^{\infty} \int \frac{1}{n!} \frac{\delta^n Z[J_+, J_-]}{\delta J_\alpha(1) \cdots \delta J_\rho(n)} \Big|_{J=0} J_\alpha(1) \cdots J_\rho(n) \\ &= 1 + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} G_{\alpha \cdots \rho}(1 \cdots n) (\sigma_3 \hat{J})_\alpha \cdots (\sigma_3 \hat{J})_\rho, \end{aligned} \quad (2.76)$$

with

$$G_{\alpha \cdots \rho}(1 \cdots n) \equiv i(-1)^n \frac{\delta^n Z[J_+, J_-]}{\delta J_\alpha(1) \cdots \delta J_\rho(n)}, \quad (2.77)$$

where

$$(\sigma_3 J_3 \hat{J})_\alpha \equiv \sigma_3^{\alpha \beta} J_\beta,$$

and both the space-time coordinates and the dummy indices α, \dots, ρ should be summed over.

Moreover, if the expression (2.71) for the source is used, the same generating functional (2.32) can be expanded as

$$Z[J_\Delta(x), J_c(x)] = \sum_{m, l=0}^{\infty} \frac{1}{m! l!} \frac{\delta^{m+l} Z[J_\Delta, J_c]}{\delta J_\Delta(1) \cdots \delta J_\Delta(m) \delta J_c(\bar{1}) \cdots \delta J_c(\bar{l})} \Big|_{J_c=J_\Delta=0} J_\Delta(1) \cdots J_\Delta(m) J_c(\bar{1}) \cdots J_c(\bar{l}), \quad (2.78)$$

where

$$\begin{aligned} B_n &\equiv \frac{(-i)^n \delta^n Z[J_\Delta, J_c]}{\delta J_\Delta(1) \cdots \delta J_\Delta(m) \delta J_c(m+1) \cdots \delta J_c(n)} \\ &= \langle T_p(\varphi_c(1) \cdots \varphi_c(m) \varphi_\Delta(m+1) \cdots \varphi_\Delta(n)) \rangle \equiv i^{n-1} 2^{-m+1} \tilde{G}_{\underbrace{2 \cdots 2}_{m} \underbrace{1 \cdots 1}_{n-m}}(1 \cdots n). \end{aligned} \quad (2.79)$$

Now we find another expression of the CTPGF in physical representation, namely, in terms of expectation values of nested commutators and anti-commutators.

Using the normalization condition for the step function (2.67), eq. (2.79) can be rewritten as

$$\begin{aligned} B_n &= \sum_{p_n} \theta(\bar{1} \cdots \bar{n}) \langle T_p(\varphi_c(1) \cdots \varphi_c(m) \varphi_\Delta(m+1) \cdots \varphi_\Delta(n)) \rangle \\ &= \sum_{p_n} \theta(\bar{1} \cdots \bar{n}) 2^{-m} \xi^{\alpha_1} \cdots \xi^{\alpha_m} \eta^{\alpha_{m+1}} \cdots \eta^{\alpha_n} \langle T_p(\varphi_{\alpha_1}(1) \cdots \varphi_{\alpha_m}(m) \varphi_{\alpha_{m+1}}(m+1) \cdots \varphi_{\alpha_n}(n)) \rangle. \end{aligned} \quad (2.80)$$

For convenience we introduce a unified notation for ξ and η

$$\zeta^{\alpha_i} = \begin{cases} \xi^{\alpha_i}, & \text{if } 1 \leq i \leq m, \\ \eta^{\alpha_i}, & \text{if } m+1 \leq i \leq n. \end{cases} \quad (2.81)$$

Since the order of operators under T_p can be changed arbitrarily, (2.80) can be also presented as

$$B_n = \sum_{p_n} \theta(\bar{1} \cdots \bar{n}) 2^{-m} \zeta^{\bar{1}} \cdots \zeta^{\bar{n}} \langle T_p(\varphi_{\bar{1}}(\bar{1}) \cdots \varphi_{\bar{n}}(\bar{n})) \rangle, \quad (2.82)$$

where the subscript $\bar{i} \equiv \alpha_{\bar{i}}$. Now let us get rid of T_p in (2.82) step by step for each term of permutation p_n . As far as the θ -function ensures \bar{n} to be the earliest moment on the positive branch and the latest one on the negative branch, we have

$$\begin{aligned} \zeta^{\bar{n}} \langle T_p(\varphi_{\bar{1}}(\bar{1}) \cdots \varphi_{\bar{n}}(\bar{n})) \rangle &= \xi^{\bar{n}} \langle T_p(\varphi_{\bar{1}}(\bar{1}) \cdots \varphi_{\bar{n}}(\bar{n})) \rangle \\ &= \langle T_p(\varphi_{\bar{1}}(\bar{1}) \cdots \varphi_{\bar{n-1}}(\bar{n-1})) \varphi(\bar{n}) \rangle + \langle \varphi(\bar{n}) T_p(\varphi_{\bar{1}}(\bar{1}) \cdots \varphi_{\bar{n-1}}(\bar{n-1})) \rangle \\ &= \langle [T_p(\varphi_{\bar{1}}(\bar{1}) \cdots \varphi_{\bar{n-1}}(\bar{n-1})), \varphi(\bar{n})] \rangle, \end{aligned}$$

if

$$1 \leq \bar{n} \leq m,$$

or

$$\begin{aligned} \zeta^{\bar{n}} \langle T_p(\varphi_{\bar{1}}(\bar{1}) \cdots \varphi_{\bar{n}}(\bar{n})) \rangle &= \eta^{\bar{n}} \langle T_p(\varphi_{\bar{1}}(\bar{1}) \cdots \varphi_{\bar{n}}(\bar{n})) \rangle \\ &= \langle [T_p(\varphi_{\bar{1}}(\bar{1}) \cdots \varphi_{\bar{n-1}}(\bar{n-1})), \varphi_{\bar{n}}(\bar{n})] \rangle, \end{aligned}$$

if

$$m+1 \leq \bar{n} \leq n.$$

Such processes may be continued like “a cicada sloughing its skin” in accord with the Chinese saying, up to the last step to get zero if $m+1 \leq \bar{1} \leq n$, or the expectation value of nested commutators (and/or anti-commutators).

If we introduce a short writing

$$(\ , \varphi(\bar{i})) = \begin{cases} [\ , \varphi(\bar{i})], & \text{when } m+1 \leq \bar{i} \leq n \\ \{ \ , \varphi(\bar{i}) \} & \text{when } 1 \leq \bar{i} \leq m, \end{cases} \quad (2.83)$$

we find finally

$$\tilde{G}_{\underbrace{2 \dots 2}_{m} \underbrace{1 \dots 1}_{n-m}}(1 \dots n) = (-i)^{n-1} \sum' \theta(\bar{1} \dots \bar{n}) \langle (\dots (\varphi(\bar{1}), \varphi(\bar{2})) \dots, \varphi(\bar{n})) \rangle, \quad (2.84)$$

where Σ' means that permutations $m+1 \leq \bar{i} \leq n$ should be excluded from the summation.

As a special case we find for $n = 2$ that

$$G_{11}(12) = 0, \quad G_{21}(12) = -i\theta(1, 2)\langle[\varphi(1), \varphi(2)]\rangle, \quad G_{22}(12) = -i\langle\{\varphi(1), \varphi(2)\}\rangle, \quad (2.85)$$

thus recovering G_{21} , G_{22} as retarded and correlation functions. Using the symmetry

$$G_{21}(12) = G_{12}(21)$$

we obtain the advanced Green function as

$$G_{12}(12) = -i\theta(2, 1)\langle[\varphi(2), \varphi(1)]\rangle.$$

Furthermore, for $n = 3$ case we have

$$\begin{aligned} G_{111} &= 0, & G_{211}(123) &= (-i)^2 \sum_{p \in \binom{23}{ij}} \theta(1ij)\langle[[1, i], j]\rangle, \\ G_{221}(123) &= (-i)^2 \sum_{p \in \binom{12}{ij}} (\theta(ij3)\langle\{i, j\}, 3]\rangle + \theta(i3j)\langle\{i, 3\}, j]\rangle), \\ G_{222}(123) &= (-i)^2 \sum_{p \in p_3} \theta(ijk)\langle\{i, j\}, k]\rangle. \end{aligned} \quad (2.86)$$

Here for short we write i, j instead of $\varphi(i), \varphi(j)$.

Therefore, this way we can exhaust all possible n -point functions. Without resorting to the CTPGF formalism this would be a cumbersome task.

The functional expansion (2.78) and the explicit expression for physical CTPGF (2.84) are very important equations from which a number of far-going implications will be extracted in the next section. In the meantime we note only that the functional derivative $\delta/\delta J_c$ will generate φ_Δ which in turn yields a commutator in the Green function, whereas the functional derivative $\delta/\delta J_\Delta$ gives rise to φ_c leading to an anti-commutator.

Moreover, in the summation (2.82) none of the time variables $m+1 \leq \bar{i} \leq n$ can take a value larger than all of the time arguments $1 \leq \bar{j} \leq m$, because in that case $\bar{1}$ should be one of $(m+1, \dots, n)$ excluded from the summation. This fact will give rise to important causality relations.

2.3.3. Transformation formula

Using the definitions of CTPGFs in single time and physical representations as expansion coefficients of the generating functional (2.77) and (2.79), respectively, we can readily find the transformation from one to another.

In fact, using (2.79) and (2.74) we have

$$\begin{aligned}
 G_{2\dots 21\dots 1}(1\dots n) &= -i(-1)^{n-1}2^{m-1} \frac{\delta^n Z[J_\Delta, J_c]}{\delta J_\Delta(1)\dots\delta J_\Delta(m)\delta J_c(m+1)\dots\delta J_c(n)} \Big|_{J=0} \\
 &= -i(-1)^{n-1}2^{-1}\xi^{\alpha_1}\dots\xi^{\alpha_m}\eta^{\alpha_{m+1}}\dots\eta^{\alpha_n} \frac{\delta^n Z[J_+, J_-]}{\delta J(\alpha_1)\dots\delta J(\alpha_m)\dots\delta J(\alpha_n)} \Big|_{J=0} \\
 &= 2^{-1}\xi^{\alpha_1}\dots\xi^{\alpha_m}\eta^{\alpha_{m+1}}\dots\eta^{\alpha_n} G_{\alpha_1\dots\alpha_n}(1\dots n).
 \end{aligned} \tag{2.87}$$

In a similar way we find the inverse transformation as given by

$$G_{\alpha_1\dots\alpha_n}(1\dots n) = 2^{1-n} \sum_{i_1\dots i_n} \zeta_{\alpha_1}^{i_1}\dots\zeta_{\alpha_n}^{i_n} G_{i_1\dots i_n}(1\dots n), \tag{2.88}$$

where

$$i_1, \dots, i_n = 1, 2, \quad \zeta_\alpha^1 = \eta_\alpha, \quad \zeta_\alpha^2 = \xi_\alpha. \tag{2.89}$$

Using the orthogonal matrix Q defined by (2.15) these formulas can be rewritten as

$$G_{i_1\dots i_n}(1\dots n) = 2^{n/2-1} Q_{i_1\alpha_1}\dots Q_{i_n\alpha_n} G_{\alpha_1\dots\alpha_n}(1\dots n), \tag{2.90a}$$

$$G_{\alpha_1\dots\alpha_n}(1\dots n) = 2^{1-n/2} Q_{\alpha_1 i_1}^T \dots Q_{\alpha_n i_n}^T G_{i_1\dots i_n}(1\dots n). \tag{2.90b}$$

For the case $n = 2$ eqs. (2.12) and (2.16) obtained in section 2.1 are recovered. For $n = 3$ we have

$$\begin{aligned}
 G_{+++}(123) &= (-i)^2 \langle T(123) \rangle, & G_{++-}(123) &= (-i)^2 \langle 3T(12) \rangle, \\
 G_{+-}(123) &= (-i)^2 \langle \tilde{T}(23)1 \rangle, & G_{--}(123) &= (-i)^2 \langle \tilde{T}(123) \rangle.
 \end{aligned} \tag{2.91}$$

The other functions, i.e., G_{+-+} , G_{-++} , G_{-+-} , G_{---} can be obtained by symmetry.

For illustration we also write down some of the transformation formulas such as

$$G_{111} = G^{(+)} - G^{(-)} = 0, \quad G_{222} = G^{(+)} + G^{(-)}, \tag{2.92}$$

with

$$\begin{aligned}
 G^{(+)} &= G_{+++} + G_{+--} + G_{-+-} + G_{---}, & G^{(-)} &= G_{---} + G_{-++} + G_{+-+} + G_{++-}, \\
 G_{211} &= \frac{1}{2}(G_{+..} + G_{-..}),
 \end{aligned} \tag{2.93}$$

with

$$G_{+..} = \sum_{\alpha, \beta = \pm} \alpha \beta G_{+\alpha\beta}, \quad G_{-..} = \sum_{\alpha, \beta = \pm} \alpha \beta G_{-\alpha\beta}.$$

2.3.4. “Physical” representation for $W[J]$ and $\Gamma[\varphi_c]$

We have discussed above different expansions for $Z[J]$ along with some of their consequences. The same thing can be done toward $W[J]$ and $\Gamma[\varphi_c]$. For example, in the physical representation we have

$$W[J_\Delta(x), J_c(x)] \equiv -i \ln Z[J_\Delta(x), J_c(x)], \quad (2.94)$$

$$\Gamma[\varphi_\Delta(x), \varphi_c(x)] \equiv W[J_\Delta(x), J_c(x)] - \int (J_\Delta \varphi_c + J_c \varphi_\Delta), \quad (2.95)$$

where

$$\varphi_c(x) \equiv \frac{\delta W[J_\Delta, J_c]}{\delta J_\Delta(x)}, \quad \varphi_\Delta(x) \equiv \frac{\delta W[J_\Delta, J_c]}{\delta J_c(x)}. \quad (2.96)$$

It follows then from (2.95) and (2.96) that

$$\begin{aligned} \frac{\delta \Gamma[\varphi_\Delta(x), \varphi_c(x)]}{\delta \varphi_c(x)} &= -J_\Delta(x), \\ \frac{\delta \Gamma[\varphi_\Delta(x), \varphi_c(x)]}{\delta \varphi_\Delta(x)} &= -J_c(x). \end{aligned} \quad (2.97)$$

It is obvious that $W[J_\Delta, J_c]$ and $\Gamma[\varphi_\Delta, \varphi_c]$ defined by (2.94) and (2.95) are identical to $W_p[J]$ and $\Gamma_p[\varphi_c]$ as given by (2.44) and (2.47) respectively.

We should note, however, that the explicit form of the connected Green function is different from that of the “total” (connected + disconnected) Green function obtained as expansion coefficient of $Z[J]$. For example,

$$G_{222}^c(123) = (-i)^2 \sum_{p_3} \theta(ijk) [\langle \{i, j\}, k \rangle - \langle \{i, j\} \rangle \langle k \rangle + 2\{\langle i \rangle, \langle j \rangle\}, \langle k \rangle]. \quad (2.98)$$

The only exceptions are the “all retarded” Green functions like G_{21} , G_{211} , G_{2111} etc., for which

$$G_{21\dots 1}^c(1 \dots n) = G_{21\dots 1}(1 \dots n).$$

Therefore, the transformation from \hat{G} to \tilde{G} for the connected Green function and vice versa should also be correspondingly modified.

We note in passing that the “all retarded” functions are nothing but the r -functions used to construct the LSZ field theory [59]. Unlike the zero temperature case, these functions alone are not enough to construct the CTPGF formalism, but they still play a very important role here.

2.4. Normalization and causality

As we emphasized in the Introduction, the normalization and causality relations are essential for applications. In fact, they are already implied by the expansion of the generating functional discussed in

the last section, but we would like to make them explicit here for future reference. We start from the normalization (section 2.4.1), then indicate the consequences of the causality (section 2.4.2) and wind up with few comments on the two aspects of the Liouville problem – dynamical evolution and statistical correlation – naturally embodied in the CTPGF formalism (section 2.4.3).

2.4.1. Normalization

If $J_\Delta(x)$ is set equal to zero, i.e., $J_+(x) = J_-(x)$ in the expansion (2.78), we find

$$\frac{\delta^n Z[J_\Delta, J_c]}{\delta J_c(1) \cdots \delta J_c(n)} \Big|_{J_\Delta=0} = 0, \quad \text{for } n \geq 1, \quad (2.99)$$

because in accord with (2.84)

$$G_{11 \cdots 1}(1 \cdots n) = 0. \quad (2.100)$$

Using the normalization condition of the density matrix

$$\text{Tr}(\hat{\rho}) = 1, \quad (2.101)$$

we find

$$Z[J_\Delta(x), J_c(x)] \Big|_{J_\Delta=0} = 1. \quad (2.102)$$

By definition (2.94) we have

$$W[J_\Delta(x), J_c(x)] \Big|_{J_\Delta=0} = 0, \quad (2.103)$$

or, equivalently,

$$\frac{\delta^n W[J_\Delta, J_c]}{\delta J_c(1) \cdots \delta J_c(n)} \Big|_{J_\Delta=0} = 0 \quad \text{for any } n \geq 0. \quad (2.104)$$

In particular,

$$\varphi_\Delta(x) \Big|_{J_\Delta(x)=0} = \frac{\delta W[J_\Delta, J_c]}{\delta J_c(x)} \Big|_{J_\Delta=0} = 0, \quad (2.105)$$

which leads to

$$\Gamma[\varphi_\Delta(x), \varphi_c(x)] \Big|_{\varphi_\Delta=0} = 0, \quad (2.106)$$

and

$$\frac{\delta^n \Gamma[\varphi_\Delta, \varphi_c]}{\delta \varphi_c(1) \cdots \delta \varphi_c(n)} \Big|_{\varphi_\Delta=0} = 0, \quad (2.107)$$

or, equivalently

$$\Gamma_{11\cdots 1}(1 \cdots n) = 0. \quad (2.108)$$

We see thus the algebraic relations obtained before, such as (2.6), (2.16), (2.60) and (2.92) are special cases of these general conditions following from the normalization.

We would like to emphasize here that the normalization condition for the CTPGF generating functional is different from that of the quantum field theory or the standard many-body formalism. In this case we require only the equality of the external source on the positive and negative branches $J_+(x) = J_-(x)$ instead of its vanishing. We can thus incorporate the external field $J_c(x) = \frac{1}{2}(J_+(x) + J_-(x))$ into the theoretical framework in a natural way. Moreover, this fundamental property will give rise to a number of important consequences which make the CTPGF formalism advantageous in many cases as we will see later.

We note also, that eqs. (2.99), (2.104), (2.107) and (2.108) are valid even in the presence of a finite external field $J_c(x)$.

2.4.2. Causality

As mentioned in section 2.3.2, in the functional expansion (2.78) none of the time variables with $m+1 \leq \bar{i} \leq n$ can take values greater than the time arguments $1 \leq \bar{j} \leq m$, because this would contradict the rule established by (2.84) that terms $m+1 \leq \bar{i} \leq n$ should be excluded from the summation. Put in another way,

$$\frac{\delta^n Z[J_\Delta, J_c]}{\delta J_\Delta(1) \cdots \delta J_\Delta(m) \delta J_c(m+1) \cdots \delta J_c(n)} \Big|_{J_\Delta=J_c=0} = 0 \quad (2.109)$$

provided any t_i , with $m+1 \leq i \leq n$ is greater than all t_j with $1 \leq j \leq m$. This is one of the causality relations we consider here. It is obvious that the causality relation for the two-point function (2.17) is a special case of (2.109) for $m=1$, $n=2$, i.e.,

$$G_{21}(12) = 0, \quad \text{if } t_2 > t_1.$$

In a sense, the algebraic relation (2.99) or (2.100) is also a special case of (2.109) for $m=0$.

Similarly, under the same condition, i.e., the time argument of any J_c , φ_c is greater than that of all J_Δ , φ_Δ , we have for the functional derivatives of $W[J]$,

$$\frac{\delta^n W[J_\Delta, J_c]}{\delta J_\Delta(1) \cdots \delta J_\Delta(m) \delta J_c(m+1) \cdots \delta J_c(n)} \Big|_{J_c-J_\Delta=0} = 0, \quad (2.110)$$

and those of the vertex functional

$$\frac{\delta^n I[\varphi_\Delta, \varphi_c]}{\delta \varphi_\Delta(1) \cdots \delta \varphi_\Delta(m) \delta \varphi_c(m+1) \cdots \delta \varphi_c(n)} \Big|_{\varphi_\Delta=0, \varphi_c=\varphi} = 0. \quad (2.111)$$

In deriving (2.111) we have made use of the relations between the vertex functions and the “amputated” connected Green functions [39, 60].

It is worthwhile to emphasize that a causality sequence is established by (2.109)–(2.111), namely, the space–time points associated with $J_c(x)$, $\varphi_c(x)$ should precede those of $J_\Delta(x)$ and $\varphi_\Delta(x)$, since the former is the cause, whereas the latter is the consequence.

We indicate here also some useful product relations as a generalization of (2.18) for a two-point function. For example, we have for three-point functions

$$G_{211}G_{112} = G_{211}G_{121} = G_{211}G_{122} = 0, \quad G_{112}G_{221} = G_{112}G_{121} = G_{121}G_{212} = 0. \quad (2.112)$$

It is easy to see that the general rule is

$$G_{i_1 \dots i_{m_1} \dots i_{m_2} \dots i_n}(1 \dots n)G_{j_1 \dots j_{m_1} \dots j_{m_2} \dots j_n}(1 \dots n) = 0, \quad (2.113)$$

provided $i_{m_1} = \dots = i_{m_r} = 2$ and the rest are equal to 1, whereas $j_{m_1} = \dots = j_{m_r} = 1$ but the rest can be either 1 or 2.

2.4.3. Dynamical evolution and statistical correlation

Now we discuss the physical meaning of J_Δ , J_c , φ_Δ and φ_c . In addition to (2.105) we have

$$J_\Delta(x)|_{\varphi_\Delta=0} = 0, \quad (2.114)$$

from (2.97) and (2.107), so that the conditions $J_\Delta = 0$ and $\varphi_\Delta = 0$ are equivalent to each other. Also, it follows from (2.91), (2.94) and (2.102) that

$$\begin{aligned} \varphi_c(x)|_{J_\Delta(x)=0} &= \text{Tr} \left\{ T_p \left[\varphi_c(x) \exp \left(i \int J_c(y) \varphi_\Delta(y) \right) \right] \hat{\rho} \right\} \\ &= \text{Tr} \left\{ \left(\tilde{T} \exp \left(-i \int_{-\infty}^t J_c(y) \varphi(y) \right) \right) \varphi(x) \left(T \exp \left(i \int_{-\infty}^t J_c(y) \varphi(y) \right) \right) \hat{\rho} \right\} \\ &= \sum_{n=0}^{\infty} i^n \int_{-\infty}^t \text{d}1 \int_{-\infty}^{t_1} \text{d}2 \dots \int_{-\infty}^{t_{n-1}} \text{d}n J_c(1) \dots J_c(n) \text{Tr} \{ [\dots [[\varphi(x), \varphi(1)], \varphi(2)], \dots], \varphi(n)] \hat{\rho} \}. \end{aligned} \quad (2.115)$$

We see thus $\varphi_c(x)$ under $J_\Delta = 0$ is the expectation value of the field operator, i.e., the order parameter in the presence of the external field J_c , whereas

$$\varphi(x) = \varphi_c(x)|_{J_c(x)=0} = \text{Tr}(\varphi_c(x)\hat{\rho}) \quad (2.116)$$

is the expectation value that might cause symmetry breaking in the vanishing field. Equation (2.115) is a nonlinear expansion of the order parameter in the external field. A detailed discussion of the nonlinear response will be given in section 5.

As we already mentioned in the last section, in accord with (2.78) and (2.84) the functional derivative $\delta/\delta J_c(x)$ generates the expectation value of the commutator of the field variables describing the

dynamical evolution in the quantum mechanical sense, whereas $\delta/\delta J_\Delta(x)$ generates the expectation value of anti-commutator describing the statistical correlation in the statistical mechanical sense. Although the physical observables are defined on the manifold $J_\Delta(x) = 0$ or $\varphi_\Delta(x) = 0$, these functional arguments are needed in addition to $J_c(x)$ and $\varphi_c(x)$ for a complete description of the statistical system. These two complementary aspects of the Liouville problem – dynamical evolution and statistical correlation – have been embodied in the CTPGF formalism in a natural way. It is worthwhile to note that the response and the correlation functions have found their “proper seats” in the CTPGF formalism just because in the external source term (2.71) φ and J are “twisted”, i.e., φ_c is coupled with J_Δ , while φ_Δ with J_c as follows directly from the definition of the closed time-path. As we will see later in section 9, this is one of the advantages for the CTPGF formalism compared with the others.

2.5. Lehmann spectral representation

In this section we study the analytical properties of the Green functions. As in quantum field theory [39] and in the standard Green function technique [1–3], the Lehmann spectral representation is a powerful tool towards this end. We will discuss in some detail the spectral as well as the symmetry properties of Green functions for a nonrelativistic complex (Bose or Fermi) field defined by eqs. (2.3) and (2.8). The modification needed for a real boson field is also briefly mentioned.

2.5.1. Spectral expansion

Assume, the inhomogeneity of the system is caused by the nonuniformity of the state, while the evolution of the Heisenberg operators $\psi(x), \psi^\dagger(x)$ with x is given by the total energy-momentum operator p as

$$\psi(x) = \exp(ip \cdot x)\psi(0)\exp(-ip \cdot x), \quad \psi^\dagger(x) = \exp(ip \cdot x)\psi^\dagger(0)\exp(-ip \cdot x). \quad (2.117)$$

Let $|n\rangle$ be a complete set determined by p_μ and other operators commuting with p_μ . According to (2.117) we have for $G_-(x, y)$ defined by (2.3c)

$$iG_-(x, y) = \sum_{n, m, n'} \langle n|\psi(0)|m\rangle \langle m|\psi^\dagger(0)|n'\rangle \rho_{n'n} \exp[i(p_n - p_m) \cdot x + i(p_m - p_{n'}) \cdot y]. \quad (2.118)$$

Set $Z = x - y$, $X = (x + y)/2$ and take Fourier transform with respect to Z , we obtain

$$iG_-(k, X) = \sum_{n, m, n'} \langle n|\psi(0)|m\rangle \langle m|\psi^\dagger(0)|n'\rangle \rho_{n'n} \exp[i(p_n - p_{n'}) \cdot X] (2\pi)^{d+1} \delta(k - p_m + (p_n + p_{n'})/2). \quad (2.119)$$

If

$$\rho_{n'n} \equiv \langle n'|\hat{\rho}|n\rangle \propto \delta(p_n - p_{n'}), \quad (2.120)$$

$G_-(k, X)$ will not depend on X and the system is homogeneous. In the presence of macroscopic inhomogeneity, $\rho_{n'n}$ is different from zero only for $p_n - p_{n'}$ small compared with k , so that the high orders of $\partial/\partial X$ can be neglected.

2.5.2. Sum rule

For nonrelativistic fields we have the following equal-time (anti-) commutation relation

$$[\psi(x), \psi^\dagger(y)]_\mp \delta(x_0 - y_0) = \delta^{d+1}(x - y), \quad (2.121)$$

which leads to

$$i \int \frac{dk_0}{2\pi} (G_-(k, X) - G_+(k, X)) = 1. \quad (2.122)$$

Introducing the spectral function

$$\rho(k, X) \equiv i(G_-(k, X) - G_+(k, X)), \quad (2.123)$$

we rewrite (2.122) as

$$\int \frac{dk_0}{2\pi} \rho(k, X) = 1. \quad (2.124)$$

For a real boson field we have

$$[\varphi(x), \partial_0 \varphi(y)] \delta(x_0 - y_0) = i \delta^{d+1}(x - y), \quad (2.125)$$

from which one can derive

$$\int \frac{dk_0}{2\pi} k_0 \rho(k, X) = 1, \quad \partial_{x_0} \int \frac{dk_0}{2\pi} \rho(k, X) = 0, \quad (2.126)$$

with $\rho(k, X)$ still defined by (2.123).

2.5.3. Lehmann representation

Presenting the retarded Green function $G_r(x, y)$ defined by (2.8a) as

$$G_r(x, y) = \theta(x_0, y_0)(G_-(x, y) - G_+(x, y)),$$

we find

$$G_r(k, X) = \int \frac{dk'_0}{2\pi} \frac{\rho(k, k'_0, X)}{k_0 - k'_0 + i\epsilon}, \quad (2.127)$$

which is analytic in the upper half-plane of k_0 . Similarly, we have for the advanced function

$$G_a(k, X) = \int \frac{dk'_0}{2\pi} \frac{\rho(k, k'_0, X)}{k_0 - k'_0 - i\epsilon}, \quad (2.128)$$

which is analytic in the lower half-plane of k_0 . Presenting G_{\pm} as

$$\begin{aligned} G_{\pm}(k, X) &= \int dk'_0 G_{\pm}(k, k'_0, X) \delta(k_0 - k'_0) \\ &= i \int \frac{dk'_0}{2\pi} G_{\pm}(k, k'_0, X) \left(\frac{1}{k_0 - k'_0 + i\epsilon} - \frac{1}{k_0 - k'_0 - i\epsilon} \right), \end{aligned} \quad (2.129)$$

we find the spectral form of G_F and $G_{\bar{F}}$,

$$G_F(k, X) = G_r(k, X) + G_+(k, X) = i \int \frac{dk'_0}{2\pi} \left(\frac{G_-(k, k'_0, X)}{k_0 - k'_0 + i\epsilon} - \frac{G_+(k, k'_0, X)}{k_0 - k'_0 - i\epsilon} \right), \quad (2.130)$$

$$\begin{aligned} G_{\bar{F}}(k, X) &= G_r(k, X) + G_-(k, X) \\ &= i \int \frac{dk'_0}{2\pi} \left(\frac{G_+(k, k'_0, X)}{k_0 - k'_0 + i\epsilon} - \frac{G_-(k, k'_0, X)}{k_0 - k'_0 - i\epsilon} \right). \end{aligned} \quad (2.131)$$

Equations (2.127), (2.128), (2.130) and (2.131) are the Lehmann spectral representations we are looking for.

2.5.4. Symmetry relations

It is straightforward to check that for nonrelativistic complex (boson or fermion) field we have

$$G_{\pm}^*(x, y) = -G_{\pm}(y, x), \quad G_F^*(x, y) = -G_{\bar{F}}(y, x), \quad G_r^*(x, y) = G_a(y, x), \quad (2.132)$$

or in Fourier components

$$G_{\pm}^*(k, X) = -G_{\pm}(k, X), \quad G_{\bar{F}}^*(k, X) = -G_F(k, X), \quad G_r^*(k, X) = G_a(k, X), \quad (2.133)$$

whereas for real boson field we have additionally

$$\hat{G}(x, y) = \hat{G}^T(y, x) = -\sigma_1 \hat{G}^*(x, y) \sigma_1 = -\sigma_1 \hat{G}^\dagger(y, x) \sigma_1, \quad (2.134)$$

or in Fourier components

$$\hat{G}(k) = \hat{G}^T(-k) = -\sigma_1 \hat{G}^*(-k) \sigma_1 = -\sigma_1 \hat{G}^\dagger(k) \sigma_1, \quad (2.135)$$

where T means transposition, * means complex conjugation and \dagger Hermitian conjugation.

2.5.5. Two analytic functions

It is obvious from (2.133) that $G_{\pm}(k, X)$ are purely imaginary on the real axis of k_0 . If they vanish as $|k_0| \rightarrow \infty$, we can define two analytic functions on the complex plane of k_0 , namely,

$$G_1(k, Z, X) = i \int \frac{dk'_0}{2\pi} \frac{G_-(k, k'_0, X)}{Z - k'_0}, \quad G_2(k, Z, X) = i \int \frac{dk'_0}{2\pi} \frac{G_+(k, k'_0, X)}{Z - k'_0}. \quad (2.136)$$

In terms of functions G_1 and G_2 we find

$$\begin{aligned}
 G_r(k, X) &= G_1(k, k_0 + i\epsilon, X) - G_2(k, k_0 + i\epsilon, X), \\
 G_a(k, X) &= G_1(k, k_0 - i\epsilon, X) - G_2(k, k_0 - i\epsilon, X), \\
 G_F(k, X) &= G_1(k, k_0 + i\epsilon, X) - G_2(k, k_0 - i\epsilon, X), \\
 G_{\bar{F}}(k, X) &= G_2(k, k_0 + i\epsilon, X) - G_1(k, k_0 - i\epsilon, X), \\
 G_-(k, X) &= G_1(k, k_0 + i\epsilon, X) - G_1(k, k_0 - i\epsilon, X), \\
 G_+(k, X) &= G_2(k, k_0 + i\epsilon, X) - G_2(k, k_0 - i\epsilon, X),
 \end{aligned} \tag{2.137}$$

i.e., all these functions are superpositions of G_1 and G_2 on approaching the real axis from different sides. It follows from (2.133) and (2.136) that

$$G_{1,2}(k, Z, X)^* = G_{1,2}(k, Z^*, X). \tag{2.138}$$

We see thus to ensure the causality, the retarded Green function should be analytic on the upper half-plane of k_0 . If a singularity is found on the upper half of k_0 during the process of solving G_r , it must be located on the second Riemann sheet. The appropriate analytic continuation is to take the integral along a contour in the complex plane of k_0 which circulates the singularity from above.

3. Quasiuniform systems

In this section we will discuss in some detail further properties of two-point Green functions, mainly concentrating on quasiuniform systems. The starting point is the Dyson equation formally derived from the generating functional in the last section. The quasiuniformity can be realized only near some stationary state, either thermoequilibrium or nonequilibrium under steady external conditions. We will derive the stability condition from the analytic properties of Green's functions. In section 3.1 the properties of the Dyson equation are further elaborated, especially for a uniform system. The thermoequilibrium situation is then discussed (section 3.2) mainly for the tutorial purpose. Furthermore, the Dyson equation is used to derive the transport equation (section 3.3). Finally, the multi-time-scale perturbation (section 3.4) and the derivation of the time dependent Ginzburg–Landau (TDGL) equation (section 3.5) are briefly described. The separation of micro- and macro-time scales is the common feature of the last three topics.

3.1. The Dyson equation

3.1.1. An alternative derivation

The Dyson equation and its equivalent forms (2.50), (2.57) and (2.58) have been derived from the generating functional. Here we give another derivation which will shed some light on the structure of the vertex function.

Consider an Hermitian boson field $\varphi(x)$ described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) - \frac{1}{2}m^2\varphi^2(x) - V(\varphi(x)). \tag{3.1}$$

For simplicity we assume $\varphi_c(x) = \text{Tr}(\varphi(x)\hat{\rho}) = 0$ in zero external field $J(x) = 0$. The field operator satisfies the equation of motion

$$\square_x \varphi(x) = j(x) \equiv -\delta V(\varphi(x))/\delta \varphi(x), \quad (3.2)$$

where

$$\square_x \equiv \partial_{\mu x} \partial^{\mu x} + m^2, \quad (3.3)$$

and $j(x)$ is sometimes called the internal source of $\varphi(x)$.

The two-point vertex function defined as

$$\Gamma_p(x, y) \equiv \frac{\delta^2 \Gamma[\varphi_c(x)]}{\delta \varphi_c(x) \delta \varphi_c(y)} \Big|_{\varphi_c(x)=0} \quad (3.4)$$

can be presented as

$$\Gamma_p(x, y) = \Gamma_{0p}(x, y) - \Sigma_p(x, y), \quad (3.5)$$

where

$$\Gamma_{0p}(x, y) = -\square_x \delta_p^{d+1}(x - y) \quad (3.6)$$

is the vertex function in the tree approximation and $\Sigma_p(x, y)$ the self-energy part due to loop corrections. The inverse of Γ_{0p} is Green's function for free field, satisfying

$$\square_x G_{0p}(x, y) = -\delta_p^{d+1}(x - y). \quad (3.7)$$

Using (3.2) and the commutation relation (2.125) we find

$$\square_x \square_y G_p(x, y) = -\square_y \delta_p^{d+1}(x - y) - i \left[\text{Tr}(T_p(j(x)j(y))\hat{\rho}) + i\delta_p^{d+1}(x - y) \text{Tr} \left\{ \frac{\delta^2 V}{\delta \varphi(x) \delta \varphi(y)} \hat{\rho} \right\} \right],$$

or

$$\begin{aligned} \square_x G_p(x, y) &= -\delta_p^{d+1}(x - y) + i \int_p \left[\text{Tr}(T_p(j(x)j(z))\hat{\rho}) + i\delta_p^{d+1}(x - z) \text{Tr} \left\{ \frac{\delta^2 V}{\delta \varphi(x) \delta \varphi(z)} \hat{\rho} \right\} \right] \\ &\quad \times G_{0p}(z, y) d^{d+1}z. \end{aligned} \quad (3.8)$$

Comparing (3.8) with (2.50) we obtain

$$\begin{aligned} &\int \Sigma_p(x, z) G_p(z, y) d^{d+1}z \\ &= -i \int_p \left[\text{Tr}(T_p(j(x)j(z))\hat{\rho}) + i\delta_p^{d+1}(x - z) \text{Tr} \left\{ \frac{\delta^2 V}{\delta \varphi(x) \delta \varphi(z)} \hat{\rho} \right\} \right] G_{0p}(z, y) d^{d+1}z, \end{aligned}$$

which yields

$$\Sigma_p(x, y) = \left(-i \text{Tr}(T_p(j(x)j(y))\hat{\rho}) + \delta_p^{d+1}(x - y) \text{Tr}\left(\frac{\delta^2 V}{\delta\varphi(x)\delta\varphi(y)}\hat{\rho}\right) \right)_{\text{I.P.I.}}. \quad (3.9)$$

This expression will be used later to discuss the transition probability.

3.1.2. Matrix representation

The matrix representation of the Dyson equation as given by (2.57) and (2.58) are very convenient for practical calculations. For example, we find immediately from (2.58) that

$$\Gamma_r = G_r^{-1}, \quad \Gamma_a = G_a^{-1}, \quad \Gamma_c = -G_r^{-1}G_cG_a^{-1}, \quad (3.10a)$$

$$G_r = \Gamma_r^{-1}, \quad G_a = \Gamma_a^{-1}, \quad G_c = -\Gamma_r^{-1}\Gamma_c\Gamma_a^{-1}. \quad (3.10b)$$

Using eqs. (2.9) and (2.62) we find the corresponding relations for \hat{G} as

$$G_{\pm} = -\Gamma_r^{-1}\Gamma_{\pm}\Gamma_a^{-1}, \quad G_F = -\Gamma_r^{-1}\Gamma_F\Gamma_a^{-1}, \quad G_{\bar{F}} = -\Gamma_r^{-1}\Gamma_F\Gamma_a^{-1}. \quad (3.11)$$

The symmetry relations (2.132) and (2.133) valid for G can be also transmitted to Γ to give

$$\begin{aligned} \Gamma_{\pm}^*(x, y) &= -\Gamma_{\pm}(y, x), & \Gamma_F^*(x, y) &= -\Gamma_F(y, x), & \Gamma_r^*(x, y) &= \Gamma_a(y, x), \\ \Gamma_{\pm}^*(k) &= -\Gamma_{\pm}(k), & \Gamma_F^*(k) &= -\Gamma_F(k), & \Gamma_r^*(k) &= \Gamma_a(k). \end{aligned} \quad (3.12a)$$

For real field we have from (2.134) and (2.135)

$$\begin{aligned} \hat{I}(x, y) &= \hat{I}^T(y, x) = -\sigma_1\hat{I}^*(x, y)\sigma_1 = -\sigma_1\hat{I}^*(y, x)\sigma_1, \\ \hat{I}(k) &= \hat{I}^T(-k) = -\sigma_1\hat{I}^*(-k)\sigma_1 = -\sigma_1\hat{I}^*(k)\sigma_1. \end{aligned} \quad (3.12b)$$

3.1.3. Vertex functions

As seen from (3.12a), only three components of \hat{I} are independent. They can be set as

$$\Gamma_F + \Gamma_{\bar{F}} = \Gamma_+ + \Gamma_- = 2iB(k), \quad \Gamma_F - \Gamma_{\bar{F}} = 2D(k), \quad \Gamma_- - \Gamma_+ = 2iA(k), \quad (3.13a)$$

where A , B and D are real functions in accord with (3.12a). In terms of unity and Pauli matrices, (3.13a) can be rewritten as

$$\hat{I} = iB(I + \sigma_1) + A\sigma_2 + D\sigma_3. \quad (3.13b)$$

We then find from (3.13) that

$$\begin{aligned} \Gamma_r(k) &= D(k) + iA(k), & \Gamma_a(k) &= D(k) - iA(k), & \Gamma_{\pm}(k) &= i(B(k) \pm A(k)), \\ \Gamma_F(k) &= D(k) + iB(k), & \Gamma_{\bar{F}}(k) &= -D(k) + iB(k). \end{aligned} \quad (3.14)$$

In what follows we will call $D(k)$ the dispersive part and $A(k)$ the absorptive part of the self-energy in analogy with the quantum field theory.

3.1.4. Green's functions

The expressions for Green's functions follow immediately from (3.10) and (3.11) as

$$\begin{aligned} G_r(k) &= \frac{1}{D(k) + iA(k)}, & G_a(k) &= \frac{1}{D(k) - iA(k)}, & G_{\pm}(k) &= -i \frac{B(k) \mp A(k)}{D^2(k) + A^2(k)}, \\ G_F(k) &= \frac{D(k) - iB(k)}{D^2(k) + A^2(k)}, & G_{\bar{F}}(k) &= \frac{-D(k) - iB(k)}{D^2(k) + A^2(k)}. \end{aligned} \quad (3.15)$$

It follows also from the matrix equation (2.57) and the symmetry relations (2.133) and (3.12) that

$$G_+(k)\Gamma_-(k) = G_-(k)\Gamma_+(k), \quad (3.16)$$

which can be verified directly from (3.14) and (3.15).

By virtue of the definition (3.5) we can express functions $A(k)$, $B(k)$ and $D(k)$ in terms of the self-energy part Σ as

$$\begin{aligned} A(k) &= \frac{1}{2}i(\Sigma_-(k) - \Sigma_+(k)), & B(k) &= \frac{1}{2}i(\Sigma_+(k) + \Sigma_-(k)), \\ D(k) &= k^2 - m^2 - \frac{1}{2}(\Sigma_F(k) - \Sigma_{\bar{F}}(k)). \end{aligned} \quad (3.17)$$

It is unlikely that both $A(k)$ and $D(k)$ have zero on the real axis of k_0 , so the divergence on the mass shell $k^2 = m^2$ can be removed by the renormalization procedure. If there are no zeroes of $D(k) + iA(k)$ in the upper half-plane of k_0 , then the causality is guaranteed and the pole of G_r in the lower half-plane of k_0 will describe a quasiparticle moving in a dissipative medium. On the opposite, if there is a pole a in the upper half-plane, then G_r is analytic only for $\text{Im } k_0 > \text{Im } a$. This pole will describe a quasiparticle moving in an amplifying medium with growing amplitude of the wavefunction. In such a case the original state is unstable with respect to a new coherent state of quasiparticles like the laser system beyond the threshold.

3.2. Systems near thermoequilibrium

The formal solution of the Dyson equation (3.10) and (3.11) as well as the explicit form of the vertex function (3.14) and Green's function (3.15) are valid for any quasiuniform system near equilibrium or nonequilibrium stationary state. In this section we consider the thermoequilibrium system in more detail. The transition probability is first studied (section 3.2.1), the dispersive part is then discussed (section 3.2.2) to show that the thermoequilibrium system is stable and the detailed balance is ensured (section 3.2.3). Furthermore, formulas for nonrelativistic fields are written out explicitly for future reference (section 3.2.4). Finally, the fluctuation-dissipation theorem is derived for the complex boson and fermion field (section 3.2.5).

3.2.1. Transition probability

It follows from (3.9) that

$$\Sigma_-(x, y) = -i \text{Tr}(j(x)j(y)\hat{\rho})_{1 \text{P.I.}}, \quad \Sigma_+(x, y) = -i \text{Tr}(j(y)j(x)\hat{\rho})_{1 \text{P.I.}}. \quad (3.18)$$

As done in section 2.5, the evolution of $j(x)$ under the space-time translation is given by

$$j(x) = \exp(ip \cdot x)j(0)\exp(-ip \cdot x).$$

Substituting this expression into (3.18) and taking Fourier transformation, we obtain

$$i\Sigma_-(k) = \sum_{l, n} |\langle l | j(0) | n \rangle_{1 \text{P.I.}}|^2 \rho_{nn} (2\pi)^{d+1} \delta^{d+1}(k - p_l + p_n), \quad (3.19a)$$

$$i\Sigma_+(k) = \sum_{l, n} |\langle n | j(0) | l \rangle_{1 \text{P.I.}}|^2 \rho_{ll} (2\pi)^{d+1} \delta^{d+1}(k - p_l + p_n). \quad (3.19b)$$

Here we neglect the off-diagonal elements of the density matrix because the system is uniform. For $k_0 > 0$ each term of (3.19a) corresponds to the probability of transition from the state $|n\rangle$ to the state $|l\rangle$ by absorbing a quasiparticle of momentum k , i.e.,

$$i\Sigma_-(k)_{k_0 > 0} = 2k_0 W_a(k), \quad (3.20a)$$

while each term of (3.19b) corresponds to the probability of emitting a quasiparticle

$$i\Sigma_+(k)_{k_0 > 0} = 2k_0 W_e(k). \quad (3.20b)$$

Since $E_l > E_n$ for both cases, $\rho_{nn} > \rho_{ll}$ in thermoequilibrium, so that

$$i(\Sigma_-(k) - \Sigma_+(k)) = 2A(k) > 0 \quad (3.21a)$$

for $k_0 > 0$, i.e., the probability of absorbing a particle is greater than that of emission.

Using the relation

$$\Gamma_+(k) = \Gamma_-(-k),$$

following from (3.12b) we find for $k_0 < 0$,

$$i\Sigma_-(k, -|k_0|) = 2|k_0| W_e(-k), \quad (3.20c)$$

$$i\Sigma_+(k, -|k_0|) = 2|k_0| W_a(-k), \quad (3.20d)$$

so that

$$i(\Sigma_-(k) - \Sigma_+(k)) = 2A(k) < 0. \quad (3.21b)$$

Therefore, we can write

$$A(k) = k_0 \gamma(k), \quad (3.22)$$

where $\gamma(k)$ is always positive for systems near thermoequilibrium.

3.2.2. Dispersive part

The dispersive part $D(k)$ can be written as

$$D(k) = k^2 - m^2 - \delta m^2(k), \quad (3.23)$$

where $\delta m^2(k)$ comes from loop correction. Assuming $\gamma(k)$ and $\delta m^2(k)$ to be small, we find the pole of the retarded Green function located at

$$k_0 = \omega(\mathbf{k}) - \frac{1}{2}iZ_\varphi\gamma(\mathbf{k}, \omega(\mathbf{k})), \quad (3.24)$$

where

$$\omega(\mathbf{k}) = \omega_0(\mathbf{k}) \left(1 + \frac{\delta m^2(\mathbf{k}, \omega_0(\mathbf{k}))}{2\omega_0^2(\mathbf{k})} \right) \quad (3.25)$$

with

$$\omega_0(\mathbf{k}) = \pm \sqrt{\mathbf{k}^2 + m^2}, \quad Z_\varphi^{-1} = \frac{\partial D}{\partial k_0^2} \Big|_{k_0 = \omega(\mathbf{k})}. \quad (3.26)$$

Here Z_φ^{-1} is the wavefunction renormalization which is close to 1 provided δm^2 is small. However, it can drastically deviate from 1, even become negative for Coulomb field $\varphi(x)$ in a plasma. As seen from (3.24) the pole is located in the lower half-plane for $\gamma(k) > 0$, so that the quasiparticle decays with γ^{-1} as its life time.

3.2.3. Detailed balance

The causal Green function G_F can be written as

$$G_F(k) = \frac{\bar{a}}{k^2 - m^2 - \delta m^2(k) + i|k_0|\gamma(k)} + \frac{\bar{b}}{k^2 - m^2 - \delta m^2(k) - i|k_0|\gamma(k)}, \quad (3.27)$$

where

$$a = \frac{1}{2}(1 + B(k)/|A(k)|) = W_a(k)/[W_a(k) - W_e(k)], \quad (3.28a)$$

$$b = \frac{1}{2}(-1 + B(k)/|A(k)|) = W_e(k)/[W_a(k) - W_e(k)]. \quad (3.28b)$$

For a system in equilibrium

$$W_a(k)/W_e(k) = \Sigma_-(k)/\Sigma_+(k) = \rho_{nn}/\rho_{ll} = \exp(\beta k_0), \quad k_0 > 0, \quad (3.29)$$

so that

$$b = n_{th}(k) = \frac{1}{\exp(\beta k_0) - 1}. \quad (3.30)$$

It can be shown also that (3.30) still holds for the case $k_0 < 0$ if k_0 is replaced by $|k_0|$.

Therefore, the detailed balance condition

$$a(k)/b(k) = W_a(k)/W_e(k) = [1 + n(k)]/n(k) \quad (3.31)$$

is fulfilled for equilibrium systems. Also, it follows from (3.20), (3.12) and (3.29) that

$$\gamma(k) = W_a(k)[1 - \exp(-\beta|k_0|)] . \quad (3.32)$$

One more remark. Green's functions obtained in this section reduce to those for free field discussed in section 2.1, provided δm^2 and γ are ignored.

3.2.4. Complex field

Up to now we have discussed mainly the Hermitian field in this section. A complex boson or fermion field with conserved particle number can be treated in a similar way. The Lagrangian is written as

$$\mathcal{L} = \psi^\dagger(x) \left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} - E_0 \right) \psi(x) - V(\psi^\dagger(x), \psi(x)) + J^\dagger(x) \psi(x) + \psi^\dagger(x) J(x) ,$$

where $J^\dagger(x)$, $J(x)$ are anti-commuting c numbers for the fermion case, E_0 a constant.

The connected two-point Green function is defined as

$$\begin{aligned} G_p(x, y) &= -\delta^2 W[J^\dagger, J]/\delta J^\dagger(x) \delta J(y) \\ &= -i(\text{Tr}[T_p(\psi(x)\psi^\dagger(y))\hat{\rho}] - \psi_c(x)\psi_c^\dagger(y)) , \end{aligned}$$

where

$$\psi_c(x) = \delta W/\delta J^\dagger(x), \quad \psi_c^\dagger(x) = \pm \delta W/\delta J(x) . \quad (3.33)$$

Here the functional derivative is acting from the left.

Introducing the vertex generating functional

$$\Gamma[\psi_c^\dagger(x), \psi_c(x)] = W[J^\dagger(x), J(x)] - \int (J^\dagger \psi_c + \psi_c^\dagger J) ,$$

we have

$$\delta\Gamma[\psi_c^\dagger, \psi_c]/\delta\psi_c^\dagger(x) = -J(x), \quad \delta\Gamma[\psi_c^\dagger, \psi_c]/\delta\psi_c(x) = \mp J^\dagger(x) . \quad (3.34)$$

The two-point vertex function is given by

$$\begin{aligned} \Gamma_p(x, y) &= \delta^2 \Gamma[\psi_c^\dagger, \psi_c]/\delta\psi_c^\dagger(x) \delta\psi_c(y) \\ &= (i\partial/\partial t + \nabla^2/2m - E_0) \delta_p^{d+1}(x - y) - \Sigma_p(x, y) , \end{aligned} \quad (3.35)$$

where the self-energy part $\Sigma_p(x, y)$ is defined as

$$\Sigma_p(x, y) = -i \operatorname{Tr}[T_p(j(x)j^\dagger(y))\hat{\rho}]_{\text{1 P.I.}}$$

with

$$j(x) = \delta V/\delta\psi^\dagger(x), \quad j^\dagger(x) = \delta V/\delta\psi(x). \quad (3.36)$$

Expressing \hat{I} in the form of (3.13b) we have

$$\begin{aligned} D(k, X) &= k_0 - \frac{1}{2m} \mathbf{k}^2 - E_0 - \Delta E(k, X), \quad A(k, X) = \frac{i}{2}(\Sigma_-(k, X) - \Sigma_+(k, X)), \\ B(k, X) &= \frac{i}{2}(\Sigma_-(k, X) + \Sigma_+(k, X)) = \frac{i}{2}(\Sigma_F(k, X) + \Sigma_{\bar{F}}(k, X)), \\ \Delta E(k, X) &= \frac{1}{2}(\Sigma_{\bar{F}}(k, X) - \Sigma_F(k, X)) \end{aligned} \quad (3.37)$$

in the Fourier representation of the relative coordinates.

3.2.5. Fluctuation-dissipation theorem (FDT)

We should mention that in general the components of \hat{G} and \hat{I} are not scalars, especially when we consider the multicomponent field in the coordinate representation. For example, the causal Green function

$$G_{Fij}(x, y) \equiv -i \operatorname{Tr}\{T_p(\psi_i(x)\psi_j^\dagger(y))\hat{\rho}\}$$

should be considered as a matrix with subscripts ix and jy , where i, j are indices of internal degrees of freedom. Therefore, different components of \hat{G} , \hat{I} as well as A, B, D in (3.13b) do not commute with each other. However, they are commuting scalars for single component field in k space provided these functions do not depend on the center-of-mass coordinates. We have thus

$$N \equiv B/A = [\Sigma_-(k) + \Sigma_+(k)]/[\Sigma_-(k) - \Sigma_+(k)] \equiv 1 \pm 2n(k), \quad (3.38)$$

i.e.,

$$n(k) = \pm \Sigma_+(k)/(\Sigma_-(k) - \Sigma_+(k)). \quad (3.39)$$

Using the same technique as in the case of real boson field we can show that in thermoequilibrium

$$\Sigma_+(k) = \pm \Sigma_-(k) \exp[-\beta(k_0 - \mu)], \quad (3.40)$$

where μ is the chemical potential. It then follows from (3.39) and (3.40) that the particle distribution

$$n(k) = \frac{1}{\exp[\beta(k_0 - \mu)] \mp 1}. \quad (3.41)$$

Using (3.13), (3.38) and (3.40) we find

$$\begin{aligned}\Gamma_c &= \Gamma_+ + \Gamma_- = \coth[\frac{1}{2}\beta(k_0 - \mu)](\Gamma_- - \Gamma_+) \\ &= \coth[\frac{1}{2}\beta(k_0 - \mu)](\Gamma_r - \Gamma_a),\end{aligned}\quad (3.42a)$$

for a boson system, and

$$\begin{aligned}\Gamma_c &= \tanh[\frac{1}{2}\beta(k_0 - \mu)](\Gamma_- - \Gamma_+) \\ &= \tanh[\frac{1}{2}\beta(k_0 - \mu)](\Gamma_r - \Gamma_a),\end{aligned}\quad (3.42b)$$

for a fermion system. This is the well-known fluctuation-dissipation theorem (FDT) for equilibrium systems. Using (3.10), it can be rewritten for the Green functions as

$$G_c = \coth[\frac{1}{2}\beta(k_0 - \mu)](G_r - G_a), \quad (3.43a)$$

$$G_c = \tanh[\frac{1}{2}\beta(k_0 - \mu)](G_r - G_a), \quad (3.43b)$$

for Bose and Fermi cases, respectively.

3.3. Transport equation

We show in this section that the usual transport equation for the quasiparticles can be derived from the Dyson equation for the quasuniform system (sections 3.3.1 and 3.3.2). As illustrative examples we derive the equation of weak turbulence in a plasma (section 3.3.3) and a generalized Leonard–Balescu equation for charge carriers (section 3.3.4).

3.3.1. Quasiparticle approximation

According to (3.10), (3.11) and (3.13), the correlation function G_c can be presented as

$$\begin{aligned}G_c(k, X) &\equiv G_F + G_{\bar{F}} = -\Gamma_r^{-1}(\Gamma_F + \Gamma_{\bar{F}})\Gamma_a^{-1} \\ &= -2i\Gamma_r^{-1}(k, X)B(k, X)\Gamma_a^{-1}(k, X),\end{aligned}\quad (3.44)$$

which can be rewritten as

$$G_c(k, X) = \Gamma_r^{-1}N - N\Gamma_a^{-1}, \quad (3.45)$$

where the matrix N satisfies the equation

$$ND - DN - i(NA + AN) = -2iB. \quad (3.46)$$

The energy spectrum of the quasiparticle is determined by putting zero for Γ_r , i.e., from the equation

$$D(k, X) = 0, \quad (3.47)$$

if the dissipation can be neglected. As in the equilibrium case, we assume

$$N|_{\text{pole}} = 1 \pm 2n, \quad (3.48)$$

then (3.46) can be rewritten as

$$nD - Dn - i(nA + An) = \mp i(B - A) = \pm \Sigma_+, \quad (3.49a)$$

or

$$nD - Dn = \pm \frac{1}{2}[(1 \pm n)\Sigma_+ + \Sigma_+(1 \pm n)] - \frac{1}{2}(\Sigma_- n + n\Sigma_-). \quad (3.49b)$$

3.3.2. Quasiclassical approximation

As shown in the preceding sections, Σ_+ , Σ_- are proportional to the probability of emitting and absorbing quasiparticles, respectively, so the right-hand side of (3.49b) is related to the collision term, whereas the left-hand side to the drift term of the transport equation. It is worthwhile to note that we cannot entirely ignore the noncommutativity of n and D on the left-hand side. To the leading order of gradients

$$\begin{aligned} (Dn - nD)(k, X) &= \int \exp[i\mathbf{k} \cdot (\mathbf{x} - \mathbf{z})][D(\mathbf{x}, \mathbf{y})n(\mathbf{y}, \mathbf{z}) - n(\mathbf{x}, \mathbf{y})D(\mathbf{y}, \mathbf{z})] d\mathbf{y} d(\mathbf{x} - \mathbf{z}) \\ &= i \left\{ \frac{\partial D(k, X)}{\partial k_\mu} \frac{\partial n(k, X)}{\partial X_\mu} - \frac{\partial n(k, X)}{\partial k_\mu} \frac{\partial D(k, X)}{\partial X_\mu} \right\}, \end{aligned} \quad (3.50)$$

where

$$X = \frac{1}{2}(\mathbf{x} + \mathbf{z}).$$

To convince oneself of the validity of (3.50), one can either consider the Poisson bracket as the quasiclassical approximation to the commutator or check it by a straightforward calculation. As we mentioned before, the microscopic scale $\mathbf{x} - \mathbf{y}$ is a fast variable with respect to which the Fourier transformation is carried out, whereas the center of mass coordinate is a slow variable. Expanding D , n in (3.50) as

$$F(k, (\mathbf{x} + \mathbf{y})/2) = F(k, \frac{1}{2}(\mathbf{x} + \mathbf{z})) + \frac{1}{2}(\mathbf{y} - \mathbf{z}) \partial F / \partial X + \dots$$

with

$$F = D, n,$$

and integrating over \mathbf{k}' by parts, one can easily verify eq. (3.50). Such separation of micro- and macro-time scales and the replacement of the commutator by its quasiclassical counterpart – the Poisson bracket, will be used frequently in our future discussion.

Assume that the solution of (3.47) is given by

$$\mathbf{k}_0 = \omega(\mathbf{k}, X),$$

then we have

$$\frac{\partial D}{\partial k_0} \Big|_{k_0=\omega(\mathbf{k}, X)} \nabla_{\mathbf{k}} \omega(\mathbf{k}, X) + \nabla_{\mathbf{k}} D(\mathbf{k}, X) = 0, \quad \frac{\partial D}{\partial k_0} \Big|_{k_0=\omega(\mathbf{k}, X)} \frac{\partial \omega(\mathbf{k}, X)}{\partial X_{\mu}} + \frac{\partial D(\mathbf{k}, X)}{\partial X_{\mu}} = 0. \quad (3.51)$$

Using eqs. (3.20), (3.50) and (3.51) we have for $k_0 = \omega(\mathbf{k}, X)$,

$$\begin{aligned} \frac{\partial n(\mathbf{k}, X)}{\partial t} + \mathbf{v} \cdot \nabla n(\mathbf{k}, X) + \frac{\partial \omega(\mathbf{k}, X)}{\partial X_{\mu}} \frac{\partial n(\mathbf{k}, X)}{\partial k_{\mu}} \\ = \frac{2\omega(\mathbf{k}, X)}{(\partial D/\partial k_0)_{k_0=\omega(\mathbf{k}, X)}} \{W_e(\mathbf{k}, X)(1 \pm n(\mathbf{k}, X)) - W_a(\mathbf{k}, X)n(\mathbf{k}, X)\}, \end{aligned} \quad (3.52)$$

where

$$\mathbf{v} = \nabla_{\mathbf{k}} \omega(\mathbf{k}, X) \quad (3.53)$$

is the group velocity. If the renormalization of the wavefunction is neglected, then

$$\partial D/\partial k_0^2 = 1,$$

and (3.52) can be simplified as

$$\frac{\partial n}{\partial t} + \mathbf{v} \cdot \nabla n + \frac{\partial \omega}{\partial X_{\mu}} \frac{\partial n}{\partial k_{\mu}} = W_e(1 \pm n) - W_a n, \quad (3.54)$$

which is the transport equation for the quasiparticle distribution in the phase space. The last term on the left-hand side of (3.54) comes from the force $\partial \omega(\mathbf{k}, X)/\partial X_{\mu}$ due to the variation of the energy $\omega(\mathbf{k}, X)$ with the coordinate. If the space-time dependence of D is unimportant, this term can be neglected and the standard transport equation is recovered.

3.3.3. Plasma equation

As an illustration consider the transport equation for plasma. Let $\psi_i(x)$, $i = 1, 2, \dots$, be the charged fermion fields (electron and ion). The equation of motion for the Coulomb field is given by

$$\nabla^2 \varphi(x) = - \sum_i e_i \psi_i^\dagger(x) \psi_i(x). \quad (3.55)$$

To the one-loop approximation of free fermions we find the two-point vertex function to be

$$\Gamma_{\varphi}(x, y) = -\nabla^2 \delta_p^4(x - y) - i \sum_j e_j^2 S_{pj}(x, y) S_{pj}(y, x), \quad (3.56)$$

where $S_{pj}(x, y)$ is the propagator of the ψ_j field

$$S_{pj}(x, y) = -i \text{Tr}\{T_p(\psi_j(x) \psi_j^\dagger(y)) \rho\}. \quad (3.57)$$

In the Fourier representation of the relative coordinates the retarded vertex function for the Coulomb field can be determined as

$$\begin{aligned}\Gamma_{r\varphi}(k) &= \mathbf{k}^2 + i \sum_j e_j^2 \int \frac{d^4 l}{(2\pi)^4} \text{Tr}\{i_2(1 + \sigma_3)S_j(l)\sigma_3S_j(l - k)\} \\ &= \mathbf{k}^2 - i \sum_j e_j^2 \int \frac{d^4 l}{(2\pi)^4} (n_j(l) - n_j(l - k))\Gamma_{rj}^{-1}(l)\Gamma_{aj}^{-1}(l - k),\end{aligned}\quad (3.58)$$

where

$$\Gamma_{rj}(l) = l_0 - \frac{1}{2m_j} l^2 + i\epsilon, \quad \Gamma_{aj}(l - k) = l_0 - k_0 - \frac{1}{2m_j} (l - k)^2 - i\epsilon, \quad (3.59)$$

and $n_j(l)$ the fermion distribution. Integration of (3.58) over l_0 yields

$$\Gamma_{r\varphi}(k) = \mathbf{k}^2 \epsilon(k), \quad (3.60)$$

where

$$\epsilon(k) = 1 + \mathbf{k}^{-2} \sum_j e_j^2 \int \frac{d^3 l}{(2\pi)^3} \frac{n_j(l) - n_j(l - k)}{k_0 + (1/2m_j)[(l - k)^2 - l^2] + i\epsilon} \quad (3.61)$$

is the susceptibility. It then follows from the expression

$$\Gamma_{r\varphi} = D_\varphi + iA_\varphi,$$

that

$$D_\varphi = \mathbf{k}^2 \text{Re}(\epsilon(k)), \quad (3.62)$$

$$A_\varphi = \mathbf{k}^2 \text{Im}(\epsilon(k)) = -\pi \sum_j e_j^2 \int \frac{d^3 l}{(2\pi)^3} \delta(k, l)(n_j(l) - n_j(l - k)), \quad (3.63)$$

where

$$\delta(k, l) = \delta\left(k_0 + \frac{1}{2m_j}[(l - k)^2 - l^2]\right). \quad (3.64)$$

Equation (3.63) is the well-known Landau formula of dissipation for plasmas in thermoequilibrium.

If $n_j(l) > n_j(l - k)$ for $l^2 > (l - k)^2$, then $A_\varphi < 0$, which means the pole of the retarded Green function moves into the upper half-plane of k_0 and an instability of the plasma occurs. However, near thermoequilibrium we always have $A_\varphi > 0$, so that the plasma oscillation decays in time.

Using the expressions for the free fermion propagator (2.21) we can also obtain from (3.56) that

$$\begin{aligned} i\Gamma_{-\varphi}(k) &= -2\pi \sum_j e_j^2 \int \frac{d^3 l}{(2\pi)^3} \delta(k, l)(1 - n_j(l))n_j(l - k), \\ i\Gamma_{+\varphi}(k) &= -2\pi \sum_j e_j^2 \int \frac{d^3 l}{(2\pi)^3} \delta(k, l)n_j(l)(1 - n_j(l - k)). \end{aligned} \quad (3.65)$$

The Boltzmann equation for the plasmon distribution $N(k, X)$ can then be derived from (3.52) and (3.65). The plasmon energy is determined from

$$\operatorname{Re} \varepsilon(k_0, \mathbf{k}, X) = 0,$$

whereas the wavefunction renormalization

$$\frac{\partial D}{\partial k_0} = \mathbf{k}^2 \frac{\partial \operatorname{Re}(\varepsilon(k, X))}{\partial k_0}$$

cannot be set equal to 1 in this case. Therefore, the resulting transport equation for the plasmon is

$$\begin{aligned} \frac{\partial N}{\partial t} + \mathbf{v} \cdot \nabla N + \frac{\partial \omega}{\partial X_\mu} \frac{\partial N}{\partial k^\mu} &= \frac{1}{\mathbf{k}^2 (\partial \operatorname{Re} \varepsilon(k, X) / \partial k_0)_{k_0=\omega(\mathbf{k})}} 2\pi \sum_j e_j^2 \int \frac{d^3 l}{(2\pi)^3} \delta(k, l) [n_j(l)(1 - n_j(l - k)) \\ &\quad \times (1 + N(k, X)) - (1 - n_j(l))n_j(l - k)N(k, X)]|_{k_0=\omega(\mathbf{k})} \end{aligned} \quad (3.66)$$

which describes the weak turbulence of the plasma.

In eq. (3.66) we take into account only the absorption and emission of plasmons by charge carriers. If the high order self-interaction of the Coulomb field is included, we will have in addition the plasmon-plasmon collision term (wave-wave interaction).

3.3.4. Equation for charge carriers

Now we discuss the transport equation for the charge carriers. First we find from (3.11) Green's function

$$\begin{aligned} iG_+(k) &= \frac{\pi}{\mathbf{k}^4 |\varepsilon(k)|^2} \sum_j e_j^2 \int \frac{d^3 l}{(2\pi)^3} \delta(k, l)(1 - n_j(l))n_j(l - k), \\ iG_-(k) &= \frac{\pi}{\mathbf{k}^4 |\varepsilon(k)|^2} \sum_j e_j^2 \int \frac{d^3 l}{(2\pi)^3} \delta(k, l)n_j(l)(1 - n_j(l - k)). \end{aligned} \quad (3.67)$$

We must separate from (3.67) the contribution of plasmons which can be written as

$$iG_-(k) = 2\pi \delta(k_0 - \omega(k)) \frac{1 + N(k, X)}{\mathbf{k}^2 \partial \operatorname{Re} \varepsilon / \partial k_0}, \quad iG_+(k) = 2\pi \delta(k_0 - \omega(k)) \frac{N(k, X)}{\mathbf{k}^2 \partial \operatorname{Re} \varepsilon / \partial k_0}. \quad (3.68)$$

To the one-loop approximation the two-point vertex function is given by

$$\Gamma_{pj}(x, y) = -i \left(\frac{\partial}{\partial t} + \frac{1}{2m_j} \nabla^2 \right) \delta_p^4(x - y) + ie_j^2 S_{pj}(x, y) G_p(x, y), \quad (3.69)$$

where $S_{pj}(x, y)$ is the fermion propagator (3.57), while $G_p(x, y)$ is the plasmon propagator. After separating the contribution of the plasmon pole we find

$$\begin{aligned} -i\Gamma_{-j}(k) &= 2\pi e_j^2 \int \frac{d^3 l}{(2\pi)^3} \delta(k, l, \omega) \frac{1}{l^2 \partial \operatorname{Re} \epsilon / \partial l_0} (1 - n_j(l+k)) [\theta(l_0)N(l) + \theta(-l_0)(1+N(l))] \\ &\quad - 2\pi e_j^2 \sum_{j'} e_{j'}^2 \int \frac{d^3 l \, d^3 l'}{(2\pi)^6} \delta(k, l, l') \left[\frac{1}{l^4 |\epsilon(l)|^2} \right]^{\text{np}} (1 - n_j(l-k))(1 - n_{j'}(l'-l)) n_{j'}(l'), \\ -i\Gamma_{+j}(k) &= 2\pi e_j^2 \int \frac{d^3 l}{(2\pi)^3} \delta(k, l, \omega) \frac{n_j(l+k)}{l^2 \partial \operatorname{Re} \epsilon / \partial l_0} [\theta(l_0)(1+N(l)) + \theta(-l_0)N(-l)] \\ &\quad - 2\pi e_j^2 \sum_{j'} e_{j'}^2 \int \frac{d^3 l \, d^3 l'}{(2\pi)^6} \delta(k, l, l') \left[\frac{1}{l^4 |\epsilon(l)|^2} \right]^{\text{np}} n_j(l+k) n_{j'}(l'-l) (1 - n_{j'}(l')), \end{aligned} \quad (3.70)$$

where

$$\begin{aligned} \delta(k, l, \omega) &= \delta \left(k_0 + \omega(l) - \frac{1}{2m_j} (l+k)^2 \right), \\ \delta(k, l, l') &= \delta \left(k_0 + \frac{1}{2m_j} [l'^2 - (l'-l)^2] - \frac{1}{2m_j} (l+k)^2 \right), \end{aligned}$$

and $[\]^{\text{np}}$ means non-pole contribution.

Taking into account the relation between Γ_{\pm} and $W_{c,a}$, we can readily write down the transport equation for the charge carriers. The first term in (3.70) describes the interaction of the charge carriers with the plasmon, whereas the second term describes the mutual interaction of the charged particles via the screened Coulomb field. If only the second term is retained, the usual Leonard–Balescu equation is recovered. The transport equation for plasmas has been also discussed by DuBois et al. [33, 34] using the CTPGF technique.

3.4. Multi-time-scale perturbation

As repeatedly emphasized before, we must distinguish the microscopic (relative) and macroscopic (center-of-mass) space–time scales. However, in the mean field approximation $\varphi_c(x)$ depends only on a single variable x which contains both micro- and macro-scales. To distinguish these two types of change we use the multi-time-scale perturbation theory.

The mean field $\varphi_c(x)$ satisfies the following equation:

$$\delta\Gamma[\varphi_c]/\delta\varphi_c(x) = 0,$$

an expansion of which in powers of $\varphi_c(x)$ can be written as

$$\frac{\delta \Gamma}{\delta \varphi_c(x)} = \int_p \Gamma_p(x, y) \varphi_c(y) + \dots = \int \Gamma_r(x, y) \varphi_c(y) d^4y + E[\varphi_c(x)], \quad (3.71)$$

where $E[\varphi_c(x)]$ contains high-order terms. Separating the diagonal part of Γ_r from (3.71),

$$\Gamma_r(x, y) = D_0(i\partial_x) \delta^4(x - y) + \Delta \Gamma_r(x, y),$$

we find

$$D_0(i\partial_x) \varphi_c(x) + \int \Delta \Gamma_r(x, y) \varphi_c(y) d^4y + E[\varphi_c(x)] = 0. \quad (3.72)$$

Assume both $\Delta \Gamma_r$ and $E[\varphi_c(x)]$ to be small, then (3.72) has a solution

$$\varphi_c(x) = \varphi_c \exp(-ik \cdot x), \quad (3.73)$$

where k satisfies the dispersion relation

$$D_0(k) = 0, \quad (3.74)$$

from which we find

$$k_0 = \omega(k) = \text{a real number}.$$

Now consider the influence of $\Delta \Gamma_r$. If its imaginary part is less than zero and, in addition, $Z_\varphi^{-1} = \partial D_0 / \partial k_0^2 > 0$ provided (3.74) is satisfied, then $\varphi_c(x)$ will grow in time to form a laser-type state with its amplitude being limited by nonlinear term $E[\varphi_c(x)]$. Near the critical point when such instability occurs, $\Delta \Gamma_r$ is a small quantity and φ_c changes with time rather slowly. Assume that the approximate solution of (3.72) can be presented as

$$\varphi_c(x) = \varphi_c(x, \varepsilon x) = \varphi_c^{(0)}(x, \varepsilon x) + \varepsilon \varphi_c^{(1)}(x, \varepsilon x) + \dots,$$

where ε is a small parameter which should be set equal to 1 by the end of the calculation, εx describes the slowly varying part. Set $\tilde{x} = \varepsilon x$, assume both $\Delta \Gamma_r$ and E to be of order ε , the differentiation with respect to x can be written as $\partial_x + \varepsilon \partial_{\tilde{x}}$ so that (3.72) becomes

$$D_0(i\partial_x + i\varepsilon\partial_{\tilde{x}})(\varphi_c^{(0)} + \varepsilon\varphi_c^{(1)} + \dots) + \varepsilon \int \Delta \Gamma_r(x, y) \varphi_c^{(0)}(y, \tilde{x}) d^4y + \varepsilon E[\varphi_c^{(0)}(x, \tilde{x})] = 0.$$

So far as \tilde{x} is a slow variable, we neglect the difference of \tilde{x} and \tilde{y} in the last two terms. To the first two leading orders we have

$$\begin{aligned} D_0(i\partial_x) \varphi_c^{(0)}(x, \tilde{x}) &= 0, \\ D_0(i\partial_x) \varphi_c^{(1)}(x, \tilde{x}) + iD_{0\mu}(i\partial_x) \partial_{\tilde{x}}^\mu \varphi_c^{(0)}(x, \tilde{x}) + \int \Delta \Gamma_r(x, y) \varphi_c^{(0)}(y, \tilde{x}) d^4y + E[\varphi_c^{(0)}(x, \tilde{x})] &= 0, \end{aligned} \quad (3.75)$$

where

$$D_{0\mu}(k) = \partial D_0(k) / \partial k^\mu. \quad (3.76)$$

As seen from (3.75), the solution is given by

$$\varphi_c(x, \tilde{x}) = \varphi_k(\tilde{x}) \exp(-ik \cdot x), \quad (3.77)$$

where k is determined from (3.74). If we require that $\varphi_c^{(1)}$ does not contain a term proportional to $\varphi_c^{(0)}$, then the second equation of (3.75) after Fourier transformation becomes

$$iD_{0\mu}(k) \partial_{\tilde{x}}^\mu \varphi_k(\tilde{x}) + \Delta \Gamma_r(k, \tilde{x}) \varphi_k(\tilde{x}) + \exp(ik \cdot x) E[\varphi_k(\tilde{x}) \exp(-ik \cdot x)] = 0, \quad (3.78)$$

where we have also replaced the center-of-mass coordinates $\frac{1}{2}(x + y)$ in $\Gamma_r(x, y)$ by \tilde{x} . This is an equation satisfied by the oscillating mode of the mean field.

We have used this technique to discuss the laser system coupled with two-energy-level electrons [40, 41]. We will not reproduce the calculation here, but it should be mentioned that a stable laser state allowed in the classical theory, is unstable in the quantum case. In the quantum theory we must consider the fluctuation of the photon number. Since the laser system is described by a coherent state with fixed phase, the fluctuation of the photon number diverges. This divergence can be removed by a renormalization procedure which leads to the decay of the laser state. Similarly, the soliton solution of $\varphi_c(x)$ is also unstable due to the quantum fluctuation.

It is worthwhile to note that such multi-time-scale perturbation technique is quite useful. In fact, we have already made use of its basic idea in deriving the transport equation in the last section. It is also the key point in obtaining the TDGL equation which we are going to discuss now.

3.5. Time dependent Ginzburg–Landau equation

The concept of macrovariableness is very useful in critical dynamics, hydrodynamics, and many other fields [61]. Usually, the set of macrovariables includes both order parameters and conserved quantities. As a rule, their microscopic counterparts are composite operators. In this section we use the equation for the vertex functional (2.48) to derive the TDGL equation [61] for their expectation value. As seen from the later discussion, the term TDGL equation is used here in a much more general sense.

Let $Q_i(x)$, $i = 1, 2, \dots$, be the set of composite operators corresponding to macrovariables. Without loss of generality, we assume them to be Hermitian Bose operators. The order parameter $Q_c(x)$ is determined from the equation for the vertex generating functional

$$\delta\Gamma/\delta Q_{ci}(x) = -J_i(x). \quad (3.79a)$$

Suppose, $Q_{ci}(x, \tau)$ is known for the moment τ . At the time t following τ , the left-hand side of (3.78) can be expanded as

$$-J_i(x, t) = \frac{\delta\Gamma}{\delta Q_{ci}} \bigg|_{Q_{ci}=Q_{ci}(\tau)} + \int \Gamma_{rij}(x, y) [Q_j(y, t_y) - Q_j(y, \tau)] dy, \quad (3.79b)$$

which is true for t located either on the positive or negative time branches. So far as Q varies slowly with time, we can write

$$Q_j(\mathbf{y}, t_y) - Q_j(\mathbf{y}, \tau) = (t_y - \tau) \frac{\partial Q_j(\mathbf{y}, \tau)}{\partial \tau}.$$

Substituting this expression back into (3.79) and taking into account that in the limit $t \equiv t_x \rightarrow \tau$,

$$\begin{aligned} \gamma_{ij}(\mathbf{x}, \mathbf{y}, \tau) &\equiv -\lim_{t_x \rightarrow \tau} \int dt_y (t_y - t_x) \Gamma_{rij}(\mathbf{x}, t_x, \mathbf{y}, t_y) \\ &= i \frac{\partial}{\partial k_0} \Gamma_{rij}(\mathbf{x}, \mathbf{y}, k_0, \tau) \Big|_{k_0=0}, \end{aligned} \quad (3.80)$$

where $\Gamma_{rij}(\mathbf{x}, \mathbf{y}, k_0, \tau)$ is the Fourier transform of $\Gamma_{rij}(\mathbf{x}, t_x, \mathbf{y}, t_y)$ with respect to $t_x - t_y$, taken at $T = \frac{1}{2}(t_x + t_y) \approx \tau$, we obtain in the matrix form

$$\gamma(t) \frac{\partial Q(t)}{\partial t} = \frac{\delta \Gamma}{\delta Q_{c+}} \Big|_{Q_{c+} = Q_{c-} = Q} + J(t), \quad (3.81)$$

where we change τ for t .

For the moment let

$$I_i(\mathbf{x}, t) \equiv \frac{\delta \Gamma}{\delta Q_{c+}} \Big|_{Q_{c+} = Q_{c-} = Q},$$

and calculate the functional derivative of I_i with respect to $Q(\mathbf{x}, t)$ as a function of three-dimensional argument

$$\begin{aligned} \frac{\delta I_i(\mathbf{x}, t)}{\delta Q_j(\mathbf{y}, t)} &= \int dz d\tau_z \left\{ \frac{\delta^2 \Gamma}{\delta Q_{i+}(\mathbf{x}) \delta Q_{k+}(z)} \frac{\delta Q_{k+}(z)}{\delta Q_j(\mathbf{y})} - \frac{\delta^2 \Gamma}{\delta Q_{i+}(\mathbf{x}) \delta Q_{k-}(z)} \frac{\delta Q_{k-}(z)}{\delta Q_j(\mathbf{y})} \right\} \\ &= \Gamma_{Fij}(\mathbf{x}, \mathbf{y}, k_0 = 0, t) - \Gamma_{+ij}(\mathbf{x}, \mathbf{y}, k_0 = 0, t), \\ \frac{\delta I_j(\mathbf{y}, t)}{\delta Q_i(\mathbf{x}, t)} &= \Gamma_{Fji}(\mathbf{y}, \mathbf{x}, k_0 = 0, t) - \Gamma_{+ji}(\mathbf{y}, \mathbf{x}, k_0 = 0, t) \\ &= \Gamma_{Fij}(\mathbf{x}, \mathbf{y}, -k_0 = 0, t) - \Gamma_{-ij}(\mathbf{x}, \mathbf{y}, -k_0 = 0, t), \end{aligned}$$

where in the last step we have used the symmetry relation (3.12b). The difference

$$\frac{\delta I_i(\mathbf{x}, t)}{\delta Q_j(\mathbf{y}, t)} - \frac{\delta I_j(\mathbf{y}, t)}{\delta Q_i(\mathbf{x}, t)} = \lim_{k_0 \rightarrow 0} (\Gamma_{-ij}(\mathbf{x}, \mathbf{y}, -k_0, t) - \Gamma_{+ij}(\mathbf{x}, \mathbf{y}, k_0, t)) \quad (3.82)$$

vanishes due to the relation

$$\Gamma_+ = \Gamma_- \exp(-\beta k_0), \quad (3.83)$$

following from (3.29) for a system in equilibrium. Therefore, a free energy functional $\mathcal{F}[Q(x, t)]$ exists such that

$$I_i(x, t) = -\delta\mathcal{F}/\delta Q_i(x, t). \quad (3.84)$$

Equation (3.81) can then be rewritten as

$$\gamma(t) \frac{\partial Q(t)}{\partial t} = -\frac{\delta\mathcal{F}}{\delta Q(t)} + J(t). \quad (3.85)$$

If the macrovariables $Q(t)$ do not change with time in the external field J , then

$$\delta\mathcal{F}/\delta Q = J. \quad (3.86)$$

Hence \mathcal{F} is actually the free energy of the system and (3.86) is the Ginzburg–Landau equation to determine the stationary distribution of macrovariables.

For nonequilibrium systems the potential condition, i.e., the vanishing of (3.84) can be realized if

$$\lim_{k_0 \rightarrow 0} A(x, y, k_0, t) = 0, \quad (3.87)$$

where A is the absorptive part of G_r . In the next section we will show that (3.87) is fulfilled for nonequilibrium stationary state (NESS) obeying time reversal symmetry.

It is usual to multiply eq. (3.85) by $\gamma^{-1}(t)$ to obtain

$$\frac{\partial Q(t)}{\partial t} = \gamma^{-1}(t) \left\{ -\frac{\delta\mathcal{F}}{\delta Q(t)} + J(t) \right\}. \quad (3.88)$$

This is the generalized TDGL equation we would like to derive. If a random source term is added to the right-hand side of (3.88), it will appear like a Langevin equation. However, there is an important difference. Equation (3.88) includes the renormalization effects. Also, the way of describing the fluctuations in CTPGF formalism is very special as we will see in section 6.

4. Time reversal symmetry and nonequilibrium stationary state (NESS)

It is well known that the principle of local equilibrium and the Onsager reciprocity relations are the two underlying principles on which the thermodynamics of irreversible processes is constructed [62]. This is true near thermoequilibrium. Within the framework of statistical mechanics a successful theory of linear response has been developed by Kubo and others [63–65]. The two fluctuation–dissipation theorems, namely, the relation between the response and the correlation functions as given by (3.43) and the generalization of the Einstein relation $D = \mu kT$ with D as the diffusion constant and μ as the

mobility, follow immediately from the linear response theory [63] along with the Onsager reciprocal relations.

Many attempts have been made to generalize this theory to far from equilibrium systems such as hydrodynamics, laser, chemical reactions and so on [66–73]. Recently there has been great interest to study the fluctuation effects in NESS in connection with the light scattering experiments in fluids with temperature gradient [74, 75]. However, in most of these treatments a phenomenological approach based on the Fokker–Planck equation or Langevin equation, at most a mesoscopic (semi-phenomenological) method using the master equation or the transport equation, were adopted. In particular, the existence condition for a free energy type generalized potential in NESS were discussed in [69–73] using the Fokker–Planck equation.

In this section we will explore the consequences of the microscopic time reversal symmetry for NESS, derive the potential condition along with nonequilibrium FDT and Onsager relations, and also decompose the inverse relaxation matrix into symmetrical and antisymmetrical parts within the framework of CTPGF approach [45].

4.1. Time inversion and stationarity

The time reversal symmetry is well known and is discussed in detail in text books [76]. Here we recall some basic points to specify our notations.

4.1.1. Time inversion in the Schrödinger picture

Suppose the system is conservative, being described by the Hamiltonian

$$\mathcal{H}[J] = \mathcal{H} - JQ, \quad (4.1)$$

where Q is a multi-component real boson field, either order parameter or conserved quantity, and J the corresponding external source. In general, J is time dependent, but here it is assumed to be constant in time.

The wavefunction at moment t is given by

$$\psi_{t_0}(1, J, \lambda) = S^J(t, t_0)\psi_0(\lambda), \quad (4.2)$$

where

$$S^J(t, t_0) \equiv \exp\{-i\mathcal{H}[J](t - t_0)\}, \quad (4.3)$$

and $\psi_0(\lambda) = \psi_{t_0}(t_0, J, \lambda)$ with λ as parameters specifying the initial state. Under time inversion,

$$\begin{aligned} J_i &\rightarrow \varepsilon_i J_i, \quad i = 1, 2 \dots n, \\ \lambda_a &\rightarrow \varepsilon_a \lambda_a, \quad a = 1, 2 \dots g, \end{aligned}$$

where $\varepsilon_i, \varepsilon_a$ are ± 1 depending on the signature of the quantity considered under time inversion.

It is well known that the time inversion in quantum mechanics is implemented by an antiunitary operator R such that

$$Q_i \rightarrow RQ_iR^\dagger = \varepsilon_i Q_i, \quad \mathcal{H}[J] \rightarrow R\mathcal{H}[J]R^\dagger = \mathcal{H}[\varepsilon J], \quad (4.4)$$

$$\psi_{t_0}(t, J, \lambda) \rightarrow R\psi_{t_0}(t, J, \lambda) = S^{\varepsilon J}(-t, -t_0)\tilde{\psi}_{-t_0} = S^{\varepsilon J}(-t, t_0)\tilde{\psi}_{t_0}. \quad (4.5)$$

If

$$\tilde{\psi}_{t_0} = \psi_0(\varepsilon\lambda), \quad (4.6)$$

the state is considered to be time reversal invariant, i.e.,

$$R\psi_{t_0}(t, J, \lambda) = \psi_{t_0}(-t, \varepsilon J, \varepsilon\lambda). \quad (4.7)$$

Analogously, for a statistical ensemble the density matrix

$$\rho_{t_0}(t, J, \lambda) = S^J(t, t_0)\rho_0(\lambda)S^J(t_0, t) \quad (4.8)$$

transforms as

$$\rho_{t_0}(t, J, \lambda) \rightarrow R\rho_{t_0}(t, J, \lambda)R^\dagger = S^{\varepsilon J}(-t, t_0)\tilde{\rho}_{t_0}S^{\varepsilon J}(t_0, -t). \quad (4.9)$$

It is time reversal invariant if

$$\tilde{\rho}_{t_0} = \rho_0(\varepsilon\lambda). \quad (4.10)$$

4.1.2. Time inversion in Heisenberg picture

Now we turn to the Heisenberg picture. Suppose it coincides with the Schrödinger picture at $t = t_0$, then we have

$$Q_{t_0}^J(t) \rightarrow RQ_{t_0}^J(t)R^\dagger = \varepsilon Q_{-t_0}^{\varepsilon J}(-t) = \varepsilon S^{\varepsilon J}(-t_0, t_0)Q_{t_0}^{\varepsilon J}(-t)S^{\varepsilon J}(t_0, -t_0). \quad (4.11)$$

The density matrix does not change with time in the Heisenberg picture. Set $t = t_0$ in (4.9) and use (4.10), we find

$$\rho_{t_0} \rightarrow R\rho_{t_0}R^\dagger = S^{\varepsilon J}(-t_0, t_0)\rho_0(\varepsilon\lambda)S^{\varepsilon J}(t_0, -t_0). \quad (4.12)$$

The expectation value

$$Q_{t_0}(t, J, \lambda) \equiv \text{Tr}\{\rho_{t_0}Q_{t_0}^J(t)\} = \varepsilon Q_{t_0}(-t; \varepsilon J, \varepsilon\lambda) \quad (4.13)$$

by virtue of (4.11), (4.12) and antiunitarity of R .

It is important to note that we need the time invariance for both dynamical variable and initial state to get the invariance for the expectation value. The external source is introduced here for mathematical treatment. In the final answer we usually set $J = 0$, so the stationary state is described by $\rho_0(\lambda)$ which does not depend on J . However, in the process of calculation using the generating functional the dynamics are determined by $\mathcal{H}[J]$. To ensure the time translational invariance we have to set $t_0 = -\infty$ and

switch on the external source adiabatically. In fact, there are two implicit assumptions. First, the limit

$$\lim_{t_0 \rightarrow -\infty} \exp(-i\mathcal{H}[J](\tau - t_0)) \rho_0(\lambda) \exp(i\mathcal{H}[J](\tau - t_0)) \quad (4.14)$$

exists in NESS, moreover, it does not depend on τ . Second, the limiting procedure $t_0 \rightarrow -\infty$ commutes with R . It follows then,

$$R\rho(\lambda, J)R^\dagger = \rho(\varepsilon\lambda, \varepsilon J). \quad (4.15)$$

4.1.3. Implications for Green's functions

The correlation function of Heisenberg operators defined as

$$F_{12\dots l}(12 \dots l, J, \lambda) \equiv \lim_{t_0 \rightarrow -\infty} \text{Tr}\{\rho_0(\lambda) Q_{1t_0}^J(1) \dots Q_{lt_0}^J(l)\}, \quad (4.16)$$

transforms as

$$F_{12\dots l}(12 \dots l, J, \lambda) = \varepsilon_1 \varepsilon_2 \dots \varepsilon_l F_{l\dots 21}(-l \dots -2 - 1, \varepsilon J, \varepsilon \lambda) \quad (4.17)$$

under time inversion as follows from (4.10)–(4.12), (4.14) and (4.15). Note that the order of arguments on the RHS of (4.17) is reversed due to the antiunitarity of R . Here 1 stands for t_1 , etc.

It is ready to check that Green's functions for NESS are time translationally invariant and transform under time reversal as

$$Q_i(J) = \varepsilon_i Q_i(\varepsilon J), \quad (4.18a)$$

$$G_{rij}(t_1 - t_2, J) = \varepsilon_i \varepsilon_j G_{rij}(-t_1 + t_2, \varepsilon J), \quad (4.18b)$$

$$G_{cij}(t_1 - t_2, J) = \varepsilon_i \varepsilon_j G_{cij}(-t_1 + t_2, \varepsilon J), \quad (4.18c)$$

in accord with (4.17). Hereafter we drop the parameters for the initial state λ for simplicity. We can solve for J from (4.18a) to obtain

$$J_i(Q) = \varepsilon_i J_i(\varepsilon Q). \quad (4.19)$$

Using the equation for the vertex functional (3.78) and taking functional derivative of (4.19) in accord with the formula

$$\frac{\delta}{\delta Q_i} (E[Q(t)])|_{Q_i(t)=Q_i} = \left[\int dt' \frac{\delta E[Q(t)]}{\delta Q_i(t')} \right]_{Q_i(t)=Q_i},$$

we find that

$$\Gamma_{rij}(k_0 = 0, Q) = \varepsilon_i \varepsilon_j \Gamma_{rij}(k_0 = 0, \varepsilon Q). \quad (4.20)$$

Also, it follows from the Dyson equation (3.10) and the relation (4.18) that

$$\Gamma_{rij}(t_1 - t_2, Q) = \varepsilon_i \varepsilon_j \Gamma_{aij}(-t_1 + t_2, \varepsilon Q). \quad (4.21)$$

Equations (4.18)–(4.21) are the implications of the time reversal symmetry needed for our further discussion.

4.2. Potential condition and generalized FDT

4.2.1. Potential condition

We have shown in section 3.5, that the potential condition can be satisfied if the zero frequency limit of the absorptive part $A(k, X)$ vanishes as given by (3.87).

Taking the Fourier transform of (4.21) we obtain

$$\Gamma_{rij}(k_0, Q) = \varepsilon_i \varepsilon_j \Gamma_{aij}(-k_0, \varepsilon Q).$$

Comparing its zero frequency limit with (4.20) we find

$$\Gamma_{rij}(k_0 = 0, Q) = \Gamma_{aij}(k_0 = 0, Q), \quad (4.22)$$

which will yield the potential condition (3.87) if combined with the definition of D and A in (3.14). Thus we have shown that one can construct a free-energy-like generalized potential for NESS satisfying the microscopic time reversal symmetry, i.e., one can construct functionals \mathcal{F} , \mathcal{G} such that

$$-\frac{1}{2}\xi_\alpha \frac{\delta \Gamma[Q]}{\delta Q(x_\alpha)} \Big|_{Q_+ = Q_- = Q} = \frac{\delta \mathcal{F}[Q(x)]}{\delta Q(x)}, \quad (4.23a)$$

$$-\frac{1}{2}\xi_\alpha \frac{\delta W[J]}{\delta J(x_\alpha)} \Big|_{J_+ = J_- = J} = \frac{\delta \mathcal{G}[J(x)]}{\delta J(x)}. \quad (4.23b)$$

If the macrovariables $Q_i(t)$ vary slowly with time as discussed in section 3.5, we should assume the time reversal symmetry in the “local” sense, i.e., the system is invariant for a macroscopically short and microscopically long time scale so that

$$A_{ij}(k_0 = 0, t, Q) = 0 \quad (4.24)$$

at each moment of t .

4.2.2. Generalized FDT

Now we discuss the nonequilibrium FDT. First we split the relaxation matrix defined by (3.81) into symmetrical $\gamma_{(i, j)}$ and antisymmetrical $\gamma_{[i, j]}$ parts as

$$-a_{ij} \equiv \gamma_{(i, j)} = \frac{1}{2}(\gamma_{ij} + \gamma_{ji}) = -\frac{\partial A_{ij}(\omega, t, Q)}{\partial \omega} \Big|_{\omega=0}, \quad (4.25a)$$

$$id_{ij} = \gamma_{[i,j]} = \frac{1}{2}(\gamma_{ij} - \gamma_{ji}) = i \frac{\partial D_{ij}(\omega, t, Q)}{\partial \omega} \Big|_{\omega=0}. \quad (4.25b)$$

As follows from (3.10) and (3.45), the correlation vertex function Γ_c can be expressed in terms of function N (cf. (3.46)) as

$$\begin{aligned} \Gamma_{cij}(\omega, Q) &= \Gamma_{rik}(\omega, Q)N_{kj}(\omega, Q) - N_{ik}(\omega, Q)\Gamma_{akj}(\omega, Q) \\ &= D_{ik}N_{kj} - N_{ik}D_{kj} + i(A_{ik}N_{kj} + N_{ik}A_{kj}). \end{aligned} \quad (4.26)$$

In thermal equilibrium

$$N_{ij}(\omega) = N(\omega)\delta_{ij} = \coth(\omega/2T)\delta_{ij} \approx (2T/\omega)\delta_{ij},$$

as follows from (3.38). Hence the FDT (3.42a) can be rewritten as

$$-i\Gamma_{cij}(\omega = 0, Q) = \lim_{\omega \rightarrow 0} \frac{2A_{ij}(\omega, Q)T}{\omega} = 4Ta_{ij} \quad (4.27)$$

in the low frequency limit, where a_{ij} is the symmetrical part of the relaxation matrix defined by (4.23).

If we assume the zero frequency limit of $N_{ij}(\omega, Q)$ to be finite then we find from (4.26) that

$$i \sum_j \Gamma_{cij}(\omega = 0, Q) = 0$$

which contradicts the positive definiteness of the quantum fluctuation. It is therefore more natural to assume that

$$\lim_{\omega \rightarrow 0} N_{ij}(\omega, Q) \rightarrow (2/\omega)T^{\text{eff}}\delta_{ij}, \quad (4.28)$$

where T^{eff} is the effective temperature. Using once again the potential condition (4.24) we find

$$-i\Gamma_{cij}(\omega = 0, Q) = 4T^{\text{eff}}a_{ij}(Q). \quad (4.29)$$

This is the low frequency limit of the FDT for NESS. Substituting (4.29) into (3.10) and carrying out the inverse Fourier transformation, we obtain the FDT for Green's function as

$$i\partial G_c/\partial t = 2T^{\text{eff}}(G_r - G_a) \quad (4.30)$$

which has the same form as that used in critical dynamics [77].

4.3. Generalized Onsager reciprocity relations

4.3.1. Kinetic matrix

As seen from eq. (3.85) the matrix $\gamma_{ij}(t)$ describes the relaxation of the order parameter. Using the

definition (3.81), the symmetry relation under the time reversal (4.21) and the basic relation

$$\Gamma_{rij}(\omega, Q) = \Gamma_{aji}(-\omega, Q) \quad (4.31)$$

following from the definition (cf. (3.12b)), we can easily find the symmetry relation for γ_{ij} under time inversion

$$\gamma_{ij}(Q) = \varepsilon_i \varepsilon_j \gamma_{ji}(\varepsilon Q). \quad (4.32)$$

4.3.2. Diffusion matrix

The standard Langevin equation is written as [61]

$$\partial Q_i / \partial t = -\gamma_{ij}^{-1}(\delta \mathcal{F} / \delta Q_j) + V_i(Q) + \xi_i \quad (4.33)$$

where $V_i(Q)$ is the mode coupling term. The random source ξ_i obeys a Gaussian distribution such that

$$\langle \xi_i \rangle = 0, \quad \langle \xi_i(t) \xi_j(t') \rangle = 2\sigma_{ij} \delta(t - t'), \quad (4.34)$$

where σ_{ij} is the diffusion matrix, appearing in the Fokker–Planck equation. We will show later (section 6.4) that within the CTPGF formalism σ_{ij} can be expressed as (cf. (6.105))

$$\sigma_{ij}(Q) = -\frac{1}{2}i\gamma_{ik}^{-1} \Gamma_{ckl}(\omega = 0, Q) (\gamma_{lj}^T)^{-1}, \quad (4.35)$$

where T means transposition. Using (3.10), (4.18) and (4.22) we can find

$$\Gamma_{cij}(\omega = 0, Q) = \varepsilon_i \varepsilon_j \Gamma_{cij}(\omega = 0, \varepsilon Q). \quad (4.36)$$

Substitution of (4.32) and (4.36) into (4.35) will yield

$$\sigma_{ij}(Q) = \varepsilon_i \varepsilon_j \sigma_{ij}(\varepsilon Q), \quad (4.37)$$

provided γ_{ij} is symmetrical. If not, (4.37) can then be easily derived using the FDT (4.29) and the symmetrized γ_{ij} as defined by (4.25a). Equation (4.37) is the symmetry relation for the diffusion matrix first obtained by Van Kampen et al. [67–70].

4.3.3. Response matrix

Now we turn to the response of the system in NESS to an external disturbance. Consider the density of conserved quantities $Q_\alpha(\mathbf{x}, t)$, $\alpha = 1, 2, \dots$, and the corresponding sources $\tilde{J}_\alpha(\mathbf{x}, t)$. The rest of the external sources J are stationary in time. The NESS is described by the Hamiltonian (4.1) and the density matrix $\rho(\lambda, J)$ (4.15) invariant under time inversion. The coupling of $Q_\alpha(\mathbf{x}, t)$ to the source $\tilde{J}_\alpha(\mathbf{x}, t)$ is treated perturbatively. Let $j^i(\mathbf{r}, t)$ be the current density satisfying the Heisenberg equation of motion

$$\frac{\partial Q_\alpha(\mathbf{x}, t)}{\partial t} + \frac{\partial j_\alpha^i(\mathbf{x}, t)}{\partial x^i} = 0, \quad (4.38)$$

where $i = 1, 2, 3$. Using the generating functional (2.32) it is straightforward to find

$$\delta\langle j_\alpha^i(x, t)\rangle = -i \int d^4y \text{Tr}\{\rho(J, \lambda)\theta(t_x - t_y)[j_\alpha^i(x), Q_\beta(y)]\}\tilde{J}_\beta(y) \quad (4.39)$$

to the linear order in \tilde{J}_α near NESS. The retarded Green function for the current density is defined as

$$G_{\alpha\beta}^{ij}(x - y, J, \lambda) \equiv -i \text{Tr}\{\rho(J, \lambda)\theta(t_x - t_y)[j_\alpha^i(x), j_\beta^j(y)]\}, \quad (4.40)$$

whereas $\tilde{J}_\alpha(x)$ in the interaction Lagrangian varies slowly with coordinates so that the external force

$$-\nabla\tilde{J}_\alpha(x, t) = X_\alpha = \text{const.} \quad (4.41)$$

It is then ready to derive from (4.38)–(4.40) by use of the Lehmann representation discussed in section 2.5, that

$$\delta\langle j_\alpha^i(x)\rangle = -\mathcal{L}_{\alpha\beta}^{ij}(J, \lambda)X_\beta^j, \quad (4.42)$$

where the response matrix $\mathcal{L}_{\alpha\beta}^{ij}$ is defined as

$$\mathcal{L}_{\alpha\beta}^{ij}(J, \lambda) \equiv -i \frac{\partial G_{\alpha\beta}^{ij}(\omega, p)}{\partial \omega} \bigg|_{\substack{p=0 \\ \omega=0}}. \quad (4.43)$$

The current operator $j_\alpha^i(x)$ in the Schrödinger picture transforms under time inversion as

$$j_\alpha^i(x) \rightarrow R j_\alpha^i(x) R^\dagger = -\varepsilon_\alpha j_\alpha^i(x). \quad (4.44)$$

So far as $j_\alpha^i(x)$ is similar to Q_i , whereas the definitions of $G_{\alpha\beta}^{ij}$ (4.40) and $\mathcal{L}_{\alpha\beta}^{ij}$ are analogous to their counterparts for the order parameter Q , we can readily find that

$$\mathcal{L}_{\alpha\beta}^{ij}(J, \lambda) = \varepsilon_\alpha \varepsilon_\beta \mathcal{L}_{\beta\alpha}^{ji}(\varepsilon J, \varepsilon \lambda). \quad (4.45)$$

Equations (4.42), (4.43) and (4.45) are the generalization of the Onsager theorem to the NESS case, whereas in the literature eqs. (4.32), (4.37) and (4.45) are known as generalized Onsager relations.

4.4. Symmetry decomposition of the inverse relaxation matrix

In this section we will decompose the inverse relaxation matrix γ^{-1} , first introduced in deriving the TDGL equation (3.88), into symmetric and antisymmetric parts and find their explicit expressions, i.e., to complete the derivation of the generalized TDGL equation with both dissipative and mode coupling terms.

4.4.1. Symmetric part

Using eq. (4.35) the FDT for NESS (4.29) can be cast into another equivalent form. In fact,

substituting (4.29) into (4.35) yields

$$\begin{aligned}\sigma_{ij}(Q) &= 2\gamma_{ik}^{-1}a_{kl}(Q)\gamma_{li}^{T^{-1}}T^{\text{eff}} \\ &= T^{\text{eff}}(\gamma^{-1} + \gamma^{T^{-1}})_{ij},\end{aligned}\quad (4.46)$$

i.e., the symmetric part of γ^{-1} is nothing but the diffusion matrix divided by twice the effective temperature. Equation (4.46) is another form of FDT in close analogy with the Einstein relation.

4.4.2. Antisymmetric part

The antisymmetric part of γ_{ij} is more complicated. According to the discussion in section 2.5, the Lehmann representation can be written as

$$G_{rij}(k_0) = \int \frac{dk'_0}{2\pi i} \frac{C_{ij}(k'_0)}{k'_0 - k_0 - i\varepsilon}, \quad (4.47)$$

where

$$C_{ij}(k_0) = -i \int \text{Tr}\{[Q_i(x_0), Q_j(0)]\hat{\rho}\} \exp(ik_0 x_0) dx_0. \quad (4.48)$$

Integrating (4.48) over k_0 gives the expectation value of the equal time commutator

$$\int C_{ij}(k_0) \frac{dk_0}{2\pi} = -i \text{Tr}\{[Q_i(x_0), Q_j(0)]\hat{\rho}\}|_{x_0=0} \equiv f_{ij}(Q). \quad (4.49)$$

We then obtain from (4.47) and (4.49) that

$$\lim_{k_0 \rightarrow \infty} k_0 G_{rij}(k_0, Q) = i f_{ij}(Q), \quad (4.50)$$

i.e., $G_{rij}(k_0)$ increases linearly with k_0 as $k_0 \rightarrow \infty$. We can thus write down a subtracted dispersion relation as

$$f_{ij} \Gamma_{rjk}(k_0) = -ik_0 \delta_{ik} + f_{ij} \Gamma_{rjk}(k_0 = 0) + k_0 \int \frac{dk'_0}{2\pi} \frac{f_{ij} \Delta_{jk}(k'_0)}{(k'_0 - k_0 - i\varepsilon)k'_0}. \quad (4.51)$$

Differentiating (4.51) with respect to k_0 and setting $k_0 \rightarrow 0$, we find that

$$f_{ij} d_{jk} = -i \delta_{ik} + \Delta_{ik}, \quad (4.52)$$

where

$$\left. \frac{\partial \Gamma_r}{\partial k_0} \right|_{k_0=0} = d + ia, \quad (4.53)$$

$$\begin{aligned}
\Delta_{ik} &= P \int_{-\infty}^{\infty} \frac{dk'_0}{2\pi} \frac{f_{ij} A_{jk}(k'_0)}{k'^2_0} \\
&= f_{ij} \int_0^{\infty} \frac{dk'_0}{\pi} \frac{A_{jk}(k'_0) - A_{kj}(k'_0)}{k'^2_0} = f_{ij} \bar{\Delta}_{jk}.
\end{aligned} \tag{4.54}$$

In deriving (4.52) we have made use of the consequence of the potential condition

$$\lim_{k_0 \rightarrow 0} A_{jk}(k_0) = k_0 a_{jk}(k_0). \tag{4.55}$$

Since $a_{jk}(k_0)$ is a symmetric matrix, $\bar{\Delta}_{jk}$ is finite.

Solving $(d^{-1})_{ij}$ from (4.52) we obtain

$$(d^{-1})_{ij} = \{(I + if\bar{\Delta})^{-1}if\}_{ij}. \tag{4.56}$$

The antisymmetric part of γ^{-1} can thus be written as

$$\begin{aligned}
\gamma_{[i,j]}^{-1} &= -(a - id)_{[i,j]}^{-1} \approx -id_{ij}^{-1} + O(a^2) \\
&= -i\{(I + if\bar{\Delta})^{-1}if\}_{ij} + O(a^2) = f_{ij} + O(a^2, \bar{\Delta}).
\end{aligned} \tag{4.57}$$

Usually f_{ij} itself is considered as the antisymmetric part of γ^{-1} . However, as seen from our discussion, this approximate result is true if both high order effects of dissipation a^2 and the dispersion $\bar{\Delta}$ are neglected.

4.4.3. Generalized TDGL equation

Substituting (4.46) and (4.57) into the TDGL equation (3.88) and setting $J = 0$, we obtain

$$\begin{aligned}
\frac{\partial Q_i(t)}{\partial t} &= -\gamma_{(i,j)}^{-1} \frac{\delta \mathcal{F}}{\delta Q_j} - \gamma_{(i,j)}^{-1} \frac{\delta \mathcal{F}}{\delta Q_j} \\
&= -\frac{1}{2T^{\text{eff}}} \sigma_{ij} \frac{\delta \mathcal{F}}{\delta Q_j} - f_{ij} \frac{\delta \mathcal{F}}{\delta Q_j},
\end{aligned} \tag{4.58}$$

where the first term is associated with the irreversible dissipation, whereas the second term is the reversible part due to canonical motion. In many practical problems Q_i are either conserved quantities or a linear representation of some Lie group. In both cases

$$f_{ij}(Q) = f_{ij}^k Q_k, \tag{4.59}$$

where f_{ij}^k are either structure constants of the symmetry group or elements of the representation matrix. The TDGL equation (4.58) derived in the CTPGF formalism has the same form as that used in critical

dynamics and other related fields [61, 68]. The second term in (4.58) is usually called the mode-coupling term. Previously, we derived the explicit form of (4.58) by comparing the TDGL equation with the Ward-Takahashi identities. Nonetheless, the present derivation is more straightforward.

In closing this section we note that as consequences of microscopic reversibility near NESS, many concepts valid for systems in thermoequilibrium can be generalized to these cases. The potential condition, the FDT as well as the reciprocal relations for diffusion and relaxation matrices which constitute the basis for a semiphenomenological consideration of nonequilibrium processes, can be justified within the CTPGF formalism.

5. Theory of nonlinear response

As we mentioned in the last section, the linear response theory near thermoequilibrium [63–65], centered on the FDT and the Onsager reciprocity relations, belongs to one of the most successful chapters of nonequilibrium statistical mechanics. For the last twenty years it has been generalized in two directions, namely, to linear response near NESS as we discussed in the last section and to nonlinear response near thermoequilibrium. In spite of few formal developments [78–82], the latter issue has not become an active field of research. In fact there are some reasons for such slow-footed advance in nonlinear response theory.

First, the nonlinear response depends not only on the intrinsic properties of the system under consideration, but also on the boundary conditions to remove the heat generated in the nonlinear processes. Second, except for nonlinear optics, there was no urgent need for nonlinear response theory from the experimental point of view. Third, the formulation of the nonlinear theory became tedious due to lack of appropriate framework.

However, things are changing. Progress in pico-, even femtosecond pulse technique and multichannel data acquisition and processing make the detection of multi-time response available. Such measurements will certainly yield much more detailed information on the intrinsic properties of the system provided the nonlinear effects are essential. On the other hand, the development of the CTPGF formalism has furnished a suitable theoretical framework for such nonlinear analysis. This problem has been studied by Hao et al. [50, 51]. Although the discussion for the time being is still rather formal and is restricted to the case of “mechanical” disturbance, i.e., expressible by an additional term in the Hamiltonian, it will serve as a good starting point for further development. As we will summarize in this section, many relations which in principle can be obtained by other, more sophisticated, techniques, appear rather simple and natural in the CTPGF approach.

The general expressions for nonlinear response are presented in section 5.1, whereas different relations among these functions including algebraic, KMS [63, 83], time-reversal and spectral, are discussed in section 5.2. A plausible generalization of FDT in nonlinear case is sketched in section 5.3. In this section we do not write out explicitly the factor $(-i)^{n-1}$ in the definition of Green’s functions.

5.1. General expressions for nonlinear response

5.1.1. Model

As mentioned in section 2, the CTPGF generating functional can include the physical field $J(x) = J_c(x)$ by setting $J_\Delta(x) = 0$, i.e. $J_+(x) = J_-(x)$. Therefore, the high order response functions are contained in the expansions of the generating functional (2.76) and (2.78). Still, for convenience we will write down here the explicit expressions for nonlinear response.

Assume the system was in equilibrium state described by the Hamiltonian \mathcal{H}_0 and the density matrix

$$\rho_0 = Z^{-1} \exp(-\beta \mathcal{H}_0), \quad Z = \text{Tr}(\exp(-\beta \mathcal{H}_0)) \quad (5.1)$$

in the remote past $t_0 = -\infty$ and then a time-dependent external field $J(t)$ has been switched on adiabatically to derive the system out from equilibrium. The field $J(t)$ is coupled to the dynamical variable Q which might be a composite operator, so the total Hamiltonian becomes

$$\mathcal{H} = \mathcal{H}_0 - J(t)Q. \quad (5.2)$$

As before, we use in this section the abbreviated notation

$$JQ \equiv \sum_i \int d\mathbf{x} J_i(\mathbf{x}, t) Q_i(\mathbf{x}), \quad (5.3)$$

sometimes making explicit only the time variable. Here we consider the linear coupling, but, obviously, other cases such as $(\mathbf{n} \cdot \mathbf{E})^2$, $(\mathbf{n} \cdot \mathbf{H})^2$ coupling in the liquid crystal or $E_i E_j P_{ij}$ (P_{ij} being the polarization tensor) coupling in second-order light scattering, can be treated as well.

We should mention that the subscript “0” for \mathcal{H}_0 and ρ_0 does not mean free of interaction between particles of the system. In this section we consider the perturbation expansion in powers of the external field, whereas all interaction effects within the system are included in all Green functions appearing later like Δ_{21} , Δ_{211} , etc.

According to the very spirit of statistical mechanics, an average should be taken over the initial distribution, whereas the evolution of the system is described by the dynamical equation. If we know the various average values $\langle Q_i(t) \rangle$, $\langle Q_i(1)Q_j(2) \rangle$ and so on, we would have more and more detailed information of the nonequilibrium properties for the system.

So far about the classical system. There is an additional problem for the quantum case. Not every product of operators here corresponds to a physical observable. The question of operator ordering occurs. According to Dirac [84], to present an observable (i) the product must be a real operator (ii) which has a complete set of eigenstates and (iii) satisfies certain supplementary physical conditions. We will consider Hermitian operators and use the Hermicity as a guide line for operator product, but in general it is hard to say anything about the completeness. For example, the combinations $AB + BA$ and $i(AB - BA)$ are both Hermitian, while their expectation values are correspondingly $G_c = G_{22}$ and $G_{21} - G_{12} = G_r - G_a$. Since the problem of operator ordering in the quanto-classical correspondence has not yet been solved, an assumption is made in [50, 51] that the quantum counterpart of the average for the product of dynamical variables is just Green’s function \tilde{G} with all subscripts equal to 2. In the $n = 2$ case, G_{22} is the fully symmetrized correlation function, whereas for $n > 2$ cases they are only partially symmetrized averages.

5.1.2. Analytic expansion of the generating functional

In this section (section 5.1) we denote Green’s functions with external field by G , whereas those without field by Δ . Using the analytic expansion of the generating functional $W[J]$,

$$W[J] = \sum_{n=1}^{\infty} \frac{1}{n!} \Delta_p(1 \cdots n) J(1) \cdots J(n), \quad (5.4)$$

where

$$\Delta_p(1 \cdots n) = \frac{\delta^n W[J]}{\delta J(1) \cdots \delta J(n)} \Big|_{J=0} \quad (5.5)$$

are connected propagators without external field. It then immediately follows that

$$\begin{aligned} G_p(1) &\equiv \langle T_p Q(1) \rangle = \Delta_p(1) + \Delta_p(12)J(2) + \frac{1}{2}\Delta_p(123)J(2)J(3) + \cdots \\ G_p(12) &\equiv \langle T_p(Q(1)Q(2)) \rangle = \frac{\delta G_p(1)}{\delta J(2)} = \Delta_p(12) + \Delta_p(123)J(3) + \cdots \\ G_p(123) &\equiv \langle T_p(Q(1)Q(2)Q(3)) \rangle = \frac{\delta G_p(12)}{\delta J(3)} = \Delta_p(123) + \Delta_p(1234)J(4) + \cdots. \end{aligned} \quad (5.6)$$

To transform (5.6) into physical representation we need to insert the Pauli matrix σ_1 [44] (cf. (2.76)) to obtain

$$\begin{aligned} \tilde{G}(1) &= \tilde{\Delta}(1) + \tilde{\Delta}(12)(\sigma_1 J)(2) + \frac{1}{2!}\tilde{\Delta}(123)(\sigma_1 J)(2)(\sigma_1 J)(3) + \cdots, \\ \tilde{G}(12) &= \tilde{\Delta}(12) + \tilde{\Delta}(123)(\sigma_1 J)(3) + \frac{1}{2!}\tilde{\Delta}(1234)(\sigma_1 J)(3)(\sigma_1 J)(4) + \cdots, \\ \tilde{G}(123) &= \tilde{\Delta}(123) + \tilde{\Delta}(1234)(\sigma_1 J)(4) + \cdots. \end{aligned} \quad (5.7)$$

5.1.3. Higher order response

In accord with the discussion at the beginning of this section we need only retain the “all 2” components of (5.7) and set $J_+ = J_-$ to get the general expressions for nonlinear response

$$\begin{aligned} G_2(1) &= \Delta_{21}(1) + \Delta_{21}(12)J(2) + \frac{1}{2!}\Delta_{211}(123)J(2)J(3) + \cdots, \\ G_{22}(12) &= \Delta_{22}(12) + \Delta_{221}(123)J(3) + \frac{1}{2!}\Delta_{2211}(1234)J(3)J(4) + \cdots, \\ G_{222}(123) &= \Delta_{222}(123) + \Delta_{2221}(1234)J(4) + \cdots. \end{aligned} \quad (5.8)$$

The first expression is a short writing of (2.115). The first two formulas of (5.8) were obtained in [78] by explicit manipulation of integrals. In the CTPGF formalism the structure of high order terms is obvious. It is worthwhile to note that the causality is guaranteed in each step of derivation using CTPGF as emphasized in section 2.4. Terms contradicting the causality drop out in accord with (2.109)–(2.111).

Following the convention set in the literature [78–82], Δ_{21} , Δ_{211} , Δ_{2111} should be called response functions of the averaged physical observable to external field, while their Fourier transforms are the admittance functions of various order. Accordingly, G_{22} , G_{222} , \dots , and Δ_{22} , Δ_{222} , \dots , are called, respectively, nonequilibrium and equilibrium fluctuations of different order, whereas Δ_{221} , Δ_{2211} , \dots , are response functions of these fluctuations to the external field.

Other components of (5.7) like nonequilibrium retarded function

$$G_{21}(12) = \Delta_{21}(12) + \Delta_{211}(123)J(3) + \frac{1}{2!} \Delta_{2111}(1234)J(3)J(4) + \dots, \quad (5.9)$$

which is the straightforward extension of the usual response function

$$\Delta_{21}(12) = \left. \frac{\delta \langle Q(1) \rangle}{\delta J(2)} \right|_{J=0}, \quad (5.10)$$

to

$$G_{21}(12) = \left. \frac{\delta \langle Q(1) \rangle}{\delta J(2)} \right|_{J=0}, \quad (5.11)$$

by including the higher order effects of the external field, may be named as nonlinear response functions. In fact, by integrating (5.9) over $J(2)$ we recover the first equation of (5.8). Therefore, no additional information is contained in (5.9) and in terms disappearing in going from (5.7) to (5.8) by setting $J_+ = J_-$.

5.1.4. Physically observable functions

Independent functions which in principle can be measured are those listed in the following table and their higher order extensions.

	Average	Two-point correlation	Three-point correlation	Four-point correlation
Without external field	Δ_2	Δ_{22}	Δ_{222}	Δ_{2222}
Linear response	Δ_{21}	Δ_{221}	Δ_{2221}	Δ_{22221}
Second-order response	Δ_{211}	Δ_{2211}	Δ_{22211}	Δ_{222211}
Third-order response	Δ_{2111}	Δ_{22111}	Δ_{222111}	$\Delta_{2222111}$

In this table the function on the oblique line from lower left to upper right are components of the same $\tilde{\Delta}$, so that the relations indicated in [78, 80, 81] look very natural.

To sum up, the observables in nonlinear response theory are partially symmetrized correlation (fluctuation) functions $G_2, G_{22}, G_{222}, \dots$, etc., as functions of the external field, in particular, their zero field derivatives

$$\Delta_{\underbrace{2 \dots 2}_{k} \underbrace{1 \dots 1}_{l}} = \frac{\delta^l}{\delta J^l} G_{22 \dots 2}(12 \dots k). \quad (5.12)$$

The possibility of detecting them in practice depends on the nonlinearity inherent in the system, i.e., the magnitude of functions (5.12) and the strength of the external field.

5.2. General considerations concerning multi-point functions

The n -point \tilde{G} has 2^n components, but not all of them are independent of each other. There are many constraints following from the normalization of the generating functional and the causality, due to the canonical distribution and the time reversal symmetry, etc. Some of these constraints (due to normalization and causality) were described before in section 2.4. Here we will elaborate further on these relations and discuss the others one by one.

5.2.1. Exact algebraic relations

First of all, let us recall that the “all one” components such as G_{11} , G_{111} , …, vanish in accord with (2.99). Furthermore, eqs. (2.109)–(2.113) are also exact. Using the transformation formulas (2.87), (2.88) and (2.90) for \hat{G} and \tilde{G} functions we can easily rewrite (2.99) in different forms. For example, in the single time representation it can be written as (cf. (2.92))

$$\sum_{\alpha_1 \cdots \alpha_n = \pm} (1 + \alpha_1 \cdots \alpha_n) G_{\alpha_1 \cdots \alpha_n} = \sum_{\alpha_1 \cdots \alpha_n = \pm} (1 - \alpha_1 \cdots \alpha_n) G_{\alpha_1 \cdots \alpha_n} \quad (5.13)$$

in particular, for three-point functions we have

$$G_{+++} + G_{+--} + G_{-+-} + G_{---} = G_{---} + G_{-++} + G_{+-+} + G_{++-}.$$

Equivalently, the same relation for three- and four-point functions can be transformed in the “retarded” combination as

$$\frac{1}{2}(G_{211} + G_{121} + G_{221}) = G_{+++} - G_{++-}, \quad (5.14a)$$

$$\frac{1}{2}(G_{2111} + G_{1211} + G_{2211}) = G_{++++} + G_{++--} - G_{+++-} - G_{+-+-}, \quad (5.14b)$$

or into the “correlation” combination as

$$\frac{1}{2}(G_{112} + G_{122} + G_{212} + G_{222}) = G_{+++} + G_{++-}, \quad (5.15a)$$

$$\frac{1}{2}(G_{1122} + G_{1222} + G_{2122} + G_{2222}) = G_{++++} + G_{++--} + G_{+++-} + G_{+-+-}. \quad (5.15b)$$

Equations (5.14a) and (5.15a) have been derived in [81] by tedious calculations without realizing its connection with (2.99). By careful inspection of (5.14) and (5.15) one can easily uncover the general rule to write down analogous formulas for higher order functions. We would like to emphasize, however, these formulas do not contain any new information in addition to (2.99).

Another set of exact algebraic relations follows from the properties of commutator (anti-commutator) and θ -function [51] on which we will not elaborate any further here. Nevertheless, we indicate here a symmetry relation following directly from the definition of CTPGF as

$$G_{\dots 1 \dots 2 \dots}(\dots i \dots j \dots) = G_{\dots 2 \dots 1 \dots}(\dots j \dots i \dots). \quad (5.16)$$

5.2.2. KMS condition [63, 83]

As mentioned before, in the response theory the system under consideration is assumed to be in thermal equilibrium for $t_0 = -\infty$. Introducing a time-dependent external field, the expectation value of the operator product in the interaction picture (with respect to the external field) satisfies the following equation:

$$\langle Q_j(t)Q_i(t_1) \rangle = \langle Q_i(t_1)Q_j(t + i\beta) \rangle = \exp(i\beta\partial/\partial t)\langle Q_i(t_1)Q_j(t) \rangle. \quad (5.17)$$

Owing to the time translational invariance of equilibrium state, (5.17) can be rewritten as

$$\langle Q_j(t)Q_i(0) \rangle = \exp(i\beta\partial/\partial t)\langle Q_i(0)Q_j(t) \rangle. \quad (5.18)$$

By taking Fourier transformation (5.18) becomes

$$\langle Q_j(\omega)Q_i(0) \rangle = e^{\beta\omega}\langle Q_i(0)Q_j(\omega) \rangle. \quad (5.19)$$

This is the so-called KMS [63, 83] condition as emphasized in the mathematical treatment of statistical mechanics [85], because (5.19) still holds even when the density matrix is ill-defined for systems with infinite degrees of freedom. It follows immediately from (5.19) that

$$\langle\{Q_j(\omega), Q_i\}\rangle = \coth(\beta\omega/2)\langle[Q_j(\omega), Q_i]\rangle, \quad (5.20)$$

which is nothing but the FDT as given by (3.43a).

Now consider the multi-point functions. First of all, for any function invariant under time translation

$$F(t_1, t_2, \dots, t_n) = F(t_1 - t_n, t_2 - t_n, \dots, 0), \quad (5.21)$$

we have

$$\sum_{i=1}^n \frac{\partial}{\partial t_i} F = 0,$$

or symbolically after Fourier transformation

$$\sum_{i=1}^n \omega_i = 0. \quad (5.22)$$

Next, consider the averaged product of n operators. Transposing the leftmost operator to the rightmost one by one, we get

$$\begin{aligned} \langle Q_1(1)Q_2(2) \cdots Q_n(n) \rangle &= \exp(i\beta\partial_1)\langle Q_2(2) \cdots Q_n(n)Q_1(1) \rangle \\ &= \exp[i\beta(\partial_1 + \partial_2)]\langle Q_3(3) \cdots Q_1(1)Q_2(2) \rangle \\ &\quad \dots \\ &= \exp(-i\beta\partial_n)\langle Q_n(n)Q_1(1) \cdots Q_{n-1}(n-1) \rangle, \end{aligned}$$

where

$$\partial_i \equiv \partial/\partial t_i. \quad (5.23)$$

This process can be stopped at any step, say, i th. We introduce the following two functions

$$\begin{aligned} F^{(-)}(1 \cdots i, i+1 \cdots n) &\equiv \langle Q_1(1) \cdots Q_i(i) Q_{i+1}(i+1) \cdots Q_n(n) \rangle, \\ F^{(+)}(1 \cdots i, i+1 \cdots n) &\equiv \langle Q_{i+1}(i+1) \cdots Q_n(n) Q_1(1) \cdots Q_i(i) \rangle, \end{aligned} \quad (5.24)$$

and write

$$F^{(-)}(1 \cdots n) = \exp[i\beta(\partial_1 + \cdots + \partial_i)] F^{(+)}(1 \cdots n) \quad (5.25)$$

which yields after Fourier transformation the generalized KMS condition

$$F^{(-)}(\omega_1 \cdots \omega_n) = \exp[\beta(\omega_1 + \cdots + \omega_i)] F^{(+)}(\omega_1 \cdots \omega_n). \quad (5.26)$$

Defining two more functions

$$\begin{aligned} F^{(c)} &\equiv F^{(-)} + F^{(+)} = \langle \{Q_1(1) \cdots Q_i(i), Q_{i+1}(i+1) \cdots Q_n(n)\} \rangle, \\ F^{(a)} &\equiv F^{(-)} - F^{(+)} = \langle [Q_1(1) \cdots Q_i(i), Q_{i+1}(i+1) \cdots Q_n(n)] \rangle, \end{aligned}$$

we find then

$$F^{(c)}(\omega_1 \cdots \omega_i, \omega_{i+1} \cdots \omega_n) = \coth[\beta(\omega_1 + \cdots + \omega_i)/2] F^{(a)}(\omega_1 \cdots \omega_i, \omega_{i+1} \cdots \omega_n) \quad (5.27)$$

which looks like a generalization of the FDT but actually it is not, because it connects only the symmetric and antisymmetric parts of the same function rather than different functions as in the two-point function case. Therefore, the KMS condition itself is not enough to give rise to the FDT. For example, the four-point function G_{++--} can be represented as

$$G_{++--}(1234) = \frac{1}{2}(-i)^3 [\langle \tilde{T}(34)T(12) \rangle + \langle T(12)\tilde{T}(34) \rangle] + \frac{1}{2}(-i)^3 [\langle \tilde{T}(34)T(12) \rangle - \langle T(12)\tilde{T}(34) \rangle], \quad (5.28)$$

where the first term corresponds to $F^{(c)}$ whereas the second term to $F^{(a)}$. We can thus derive a relation similar to (5.27) which, as we said, is not the FDT.

5.2.3. Time reversal invariance

The implications of the time reversal symmetry for macrosystems have been discussed in section 4. Here we restrict ourselves to the situation when the conditions (4.4), (4.6) and (4.10) are satisfied. It has been shown there also that the average of product from Heisenberg operators transforms under time inversion as (cf. (4.17))

$$\begin{aligned} F_{1 \cdots n}(1 \cdots n; J, \lambda) &= \varepsilon_1 \cdots \varepsilon_n F_{n \cdots 1}(-n \cdots -1; \varepsilon J, \varepsilon \lambda) \\ &= \varepsilon_1 \cdots \varepsilon_n F_{1 \cdots n}^*(-1 \cdots -n; \varepsilon J, \varepsilon \lambda), \end{aligned} \quad (5.29)$$

where λ is a parameter specifying the initial state. The second line of (5.29) is based on the following equality

$$\text{Tr}(AB \cdots K) = \text{Tr}(K \cdots BA)^* \quad (5.30)$$

valid for Hermitian operators. It also follows from the Hermicity of Q and ρ_0 , the density matrix, that

$$F_{1 \cdots n}(1 \cdots n; J, \lambda) = F_{n \cdots 1}^*(n \cdots 1; J, \lambda). \quad (5.31)$$

Since the CTPGF are linear combinations of averaged operator products, so that for systems with time reversal symmetry we can easily write down their transformations under time inversion using (5.28). For example,

$$\langle 3T(12) \rangle = \varepsilon_1 \varepsilon_2 \varepsilon_3 \langle T(-1-2)3 \rangle, \quad \langle T(12)3 \rangle = \varepsilon_1 \varepsilon_2 \varepsilon_3 \langle -3T(-1-2) \rangle.$$

Hereafter we do not specify the change of J and λ explicitly. We can split three-point function G_{++-} into symmetric and antisymmetric parts under time inversion (do not confuse it with $F^{(c)}$ and $F^{(a)}$ considered in the last subsection)

$$G_{++-}(123) = G_{++-}^S(123) + G_{++-}^A(123), \quad (5.32)$$

where

$$\begin{aligned} G_{++-}^S(123) &= G_{++-}^S(-1-2-3) = \frac{1}{2}(-i)^2(\langle 3T(12) \rangle + \varepsilon^3 \langle T(12)3 \rangle), \\ G_{++-}^A(123) &= -G_{++-}^A(-1-2-3) = \frac{1}{2}(-i)^2(\langle 3T(12) \rangle - \varepsilon^3 \langle T(12)3 \rangle) \end{aligned} \quad (5.33)$$

with

$$\varepsilon^3 \equiv \varepsilon_1 \varepsilon_2 \varepsilon_3. \quad (5.34)$$

Applying the KMS condition to (5.34) leads to

$$G_{++-}^S(\omega_1 \omega_2 \omega_3) = \frac{1 - \varepsilon^3 \exp(\beta \omega_3)}{1 + \varepsilon^3 \exp(\beta \omega_3)} G_{++-}^A(\omega_1 \omega_2 \omega_3). \quad (5.35)$$

5.2.4. Fourier transform and spectral representation

It is known that the Fourier transformation itself does not bring about any new information, but it is more convenient to incorporate the KMS condition and the time reversal symmetry in the Fourier space.

The Fourier transform of a time translationally invariant function can be written as

$$F(\omega_1 \cdots \omega_n) = 2\pi \delta(\omega_1 + \cdots + \omega_n) F_1(\omega_1 \cdots, t_k = 0, \cdots \omega_n), \quad (5.36)$$

where all time arguments in F_1 are transformed except for $t_k = 0$ with k being any of 1 to n . In other words, the Fourier transformation of the n -point function appears as

$$F(t_1 \cdots t_n) = \int \frac{d\omega_1 \cdots d\hat{\omega}_k \cdots d\omega_n}{(2\pi)^{n-1}} \exp\{-i[\omega_1(t_1 - t_k) + \cdots + \hat{\omega}_k + \cdots + \omega_n(t_n - t_k)]\},$$

$$\times F(\omega_1 \cdots t_k = 0 \cdots \omega_n), \quad (5.37)$$

where the ω variable with caret is missing. We remind the reader that the Fourier transform in this paper is not distinguished by any special symbols, the meaning being clear from the arguments of functions.

As shown in section 2.3, the CTPGF can be presented as a linear combination of products of the θ -function and n -point function. For example,

$$F_R(123) = \theta(123)F(123). \quad (5.38)$$

Factorizing the θ -function as

$$\theta(123) = \theta(12)\theta(23)$$

and using the integral representation

$$\theta(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega}{\Omega + i\varepsilon} \exp(-i\Omega t), \quad (5.39)$$

we can write

$$F_{1R}(\omega_1 \omega_2 t_3 = 0) = \frac{1}{(2\pi)^2} \int \frac{d\Omega_1 d\Omega_2 F_1(\omega_1 \omega_2 t_3 = 0)}{(\omega_1 - \Omega_1 + i\varepsilon)(\omega_1 + \omega_2 - \Omega_2 + i\varepsilon)}. \quad (5.40)$$

It is easy to read out from (5.40) the general rule to write down the spectral representation for any CTPGF. In particular, for the three-point retarded function

$$G_{211}(123) = -\theta(123)\langle [1, 2], 3] \rangle - \theta(132)\langle [1, 3], 2] \rangle, \quad (5.41)$$

by use of the KMS condition and the simplified notation

$$\langle 123 \rangle = \langle 1 - 2, 2 - 3 \rangle = \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \exp[-i\omega_1(t_1 - t_2) - i\omega_2(t_2 - t_3)] \langle \omega_1, \omega_2 \rangle,$$

$$\langle 321 \rangle = \langle 123 \rangle^* = \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \exp[-i\omega_1(t_1 - t_2) - i\omega_2(t_2 - t_3)] \langle -\omega_1, -\omega_2 \rangle^*, \quad (5.42)$$

we find that

$$G_{211}(\omega_1 \omega_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\Omega_1 d\Omega_2}{(\omega_1 - \Omega_1 + i\varepsilon)(\omega_1 + \omega_2 - \Omega_2 + i\varepsilon)} \{ [1 - \exp(-\beta\Omega_2)] \langle \Omega_1, \Omega_2 - \Omega_1 \rangle$$

$$+ [1 - \exp(\beta\Omega_2)] \langle -\Omega_1, -\Omega_2 + \Omega_1 \rangle^* \} + \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\Omega_1 d\Omega_2}{(\omega_1 - \Omega_1 + i\varepsilon)(-\omega_2 - \Omega_2 + i\varepsilon)}$$

$$\times \{ [\exp(-\beta\Omega_1) - \exp[\beta(\Omega_2 - \Omega_1)]] \langle \Omega_1, -\Omega_2 \rangle$$

$$+ [\exp(\beta\Omega_1) - \exp[-\beta(\Omega_2 - \Omega_1)]] \langle -\Omega_1, \Omega_2 \rangle^* \}. \quad (5.43)$$

We see thus $G_{211}(\omega_1\omega_2)$ is analytic in the upper half-plane of ω_1 . There is one pole in the upper half-plane of ω_2 depending on the position of the pole in ω_1 . As functions of several complex variables, the analytic properties of multi-point functions are much more complicated compared with the two-point case.

5.3. Plausible generalization of FDT

Some authors claimed previously [78, 82] that they had already found the nonlinear generalization of the FDT, but indeed this was not the case. As we have seen in the last section, any of the three kinds of relations, namely, the algebraic, the KMS or the time reversal invariance, will not provide by themselves the generalization of the FDT needed. However, their combination might give what we would like to have.

5.3.1. Even and odd combinations

The transformation formula from \tilde{G} to \hat{G} as given by (2.88) can be rewritten as

$$G_{\alpha_1 \dots \alpha_n}(1 \dots n) = 2^{1-n} \sum_{i_1 \dots i_n = 1, 2} (\alpha_1)^{i_1} \dots (\alpha_n)^{i_n} G_{i_1 \dots i_n}(1 \dots n). \quad (5.44)$$

Under the change $\alpha_i \rightarrow -\alpha_i$ the T -product changes into \tilde{T} -product and a factor $(-1)^{i_1 + \dots + i_n}$ appears on the right-hand side of (5.44). Therefore, it is natural to distinguish the “even” and “odd” components of \tilde{G} as G_{n, l_e} and G_{n, l_o} , where l_σ , $\sigma = e, o$ is the number of “1”s among i_k . The even components do not change sign under $\alpha_i \rightarrow -\alpha_i$ whereas the odd components do. As we will see later in this subsection, there are simple algebraic relations among the same type of G_{n, l_σ} , while the connection between these two types of components established by using the KMS condition and the time reversal invariance is the plausible generalization of the FDT.

First consider the “all +” and “all -” components of \hat{G}

$$G_{++\dots+}(12 \dots n) = 2^{1-n} \sum_{i_1 \dots i_n = 1, 2} G_{i_1 \dots i_n}(1 \dots n), \quad (5.45a)$$

$$G_{--\dots-}(12 \dots n) = 2^{1-n} \sum_{i_1 \dots i_n = 1, 2} (-1)^{i_1 + \dots + i_n} G_{i_1 \dots i_n}(12 \dots n). \quad (5.45b)$$

If we define the symmetrized combination of G_{n, l_σ} as

$$G_{n, l_\sigma}^s \equiv \sum'_{i_1 \dots i_n = 1, 2} \frac{1}{2}[1 + (-1)^{l_\sigma + i_1 + \dots + i_n}] G_{i_1 \dots i_n}(12 \dots n), \quad (5.46)$$

where ' means that the number of “1”s should be equal to l_σ , then

$$\sum_{0 \leq l_e \leq n} G_{n, l_e}^s = 2^{n-2} [G_{++\dots+}(12 \dots n) + G_{--\dots-}(12 \dots n)],$$

$$\sum_{1 \leq l_o \leq n} G_{n, l_o}^s = 2^{n-2} [G_{++\dots+}(12 \dots n) - G_{--\dots-}(12 \dots n)]. \quad (5.47)$$

Making use of the relation following from the time reversal invariance, we have

$$\varepsilon_1 \cdots \varepsilon_n G_{\alpha\alpha \cdots \alpha}(-1 - 2 \cdots - n) = G_{\alpha\alpha \cdots \alpha}(12 \cdots n) \quad (5.48)$$

for $\alpha = \pm$. It follows then immediately from (5.47) that

$$\sum_{0 \leq l_\sigma \leq n} (G_{n, l_\sigma}^S - \varepsilon_1 \cdots \varepsilon_n G_{n, l_\sigma}^{(-)S}) = 0, \quad (5.49)$$

where

$$G_{n, l_\sigma}^{(-)S} \equiv G_{n, l_\sigma}^S(-1 \cdots - n).$$

After Fourier transformation (5.49) becomes

$$\sum_{0 \leq l_\sigma \leq n} (G_{n, l_\sigma}^S(\omega) - \varepsilon_1 \cdots \varepsilon_n G_{n, l_\sigma}^S(-\omega)). \quad (5.50)$$

Since the components of \tilde{G} contain the factor $(-i)^{n-1}$ and an average of $(n-1)$ nested commutator and/or anticommutator, the G_{n_o, l_e} and G_{n_e, l_o} components are real, whereas G_{n_e, l_e} and G_{n_o, l_o} are imaginary with their Fourier components related with each other by the following equations

$$\begin{aligned} G_{n_o, l_e}^*(\omega) &= G_{n_o, l_e}(-\omega), & G_{n_e, l_o}^*(\omega) &= G_{n_e, l_o}(-\omega), \\ G_{n_e, l_e}^*(\omega) &= -G_{n_e, l_e}(-\omega), & G_{n_o, l_o}^*(\omega) &= -G_{n_o, l_o}(-\omega). \end{aligned} \quad (5.51)$$

It is straightforward to see from (5.50) that the “even” and “odd” components are linearly dependent among themselves but not with each other, as we mentioned at the beginning of this section.

5.3.2. Nonlinear generalization of FDT

Now consider the other algebraic relations following from (5.44) as

$$\begin{aligned} G_{\alpha_1 \cdots \alpha_n}(12 \cdots n) &\pm G_{-\alpha_1 \cdots -\alpha_n}(12 \cdots n) \\ &= 2^{1-n} \sum_{i_1 \cdots i_n=1, 2} [1 \pm (-1)^{i_1+ \cdots + i_n}] (\alpha_1)^{i_1} \cdots (\alpha_n)^{i_n} G_{i_1 \cdots i_n}(12 \cdots n). \end{aligned} \quad (5.52)$$

If $\{j\}$, i.e., (j_1, \dots, j_m) from α_k subscripts are “-” while the rest are “+”, (5.52) can be rewritten as

$$\begin{aligned} G_{\{j\}}(12 \cdots n) &\pm G_{\{j\}}(12 \cdots n) \\ &= 2^{1-n} \sum_{i_1 \cdots i_n=1, 2} (1 \pm (-1)^{i_1+ \cdots + i_n}) (-1)^{j_1+ \cdots + j_m'} G_{i_1 \cdots i_n}(12 \cdots n) \end{aligned} \quad (5.53)$$

with $j'_r \equiv i_{j_r}$.

Defining

$$G_{l_e, \{j\}}(12 \cdots n) \equiv \sum' \frac{1}{2} (1 + (-1)^{i_1 + \cdots + i_n}) (-1)^{j_{i_1} + \cdots + j_{i_n}} G_{i_1 \cdots i_n}(1 \cdots n),$$

$$G_{l_o, \{j\}}(12 \cdots n) \equiv \sum' \frac{1}{2} (1 - (-1)^{i_1 + \cdots + i_n}) (-1)^{j_{i_1} + \cdots + j_{i_n}} G_{i_1 \cdots i_n}(1 \cdots n), \quad (5.54)$$

we rewrite (5.53) as

$$\sum_{0 \leq l_e \leq n} G_{l_e, \{j\}} = 2^{n-2} (G_{\{j-\}} + G_{\{j+\}}), \quad \sum_{1 \leq l_o \leq n} G_{l_o, \{j\}} = 2^{n-2} (G_{\{j-\}} - G_{\{j+\}}). \quad (5.55)$$

Consider the combination

$$G_{\{j-\}}(1 \cdots n) + \varepsilon^n G_{\{j-\}}(-1 \cdots -n) \pm G_{\{j+\}}(1 \cdots n) \pm \varepsilon^n G_{\{j+\}}(-1 \cdots -n)$$

and make use of the following relations:

$$G_{\{j-\}}(1 \cdots n) \equiv (-i)^{n-1} \langle \tilde{T}(j_1 \cdots j_m) T(i_1 \cdots i_{n-m}) \rangle,$$

$$G_{\{j-\}}(-1 \cdots -n) = \varepsilon^n \exp \left[-i\beta \sum_{i=1}^m \partial_{j_i} \right] G_{\{j-\}}(1 \cdots n), \quad (5.56)$$

with

$$\partial_{j_i} \equiv \partial / \partial t_{j_i},$$

obtained from the KMS and the time reversal invariance conditions (5.25) and (5.29), we find

$$(1 \pm 1 - e^{-A} \mp e^A) \sum_{l_e} G_{l_e, \{j\}} + \varepsilon^n (1 \pm 1 - e^A \mp e^{-A}) \sum_{l_e} G_{l_e, \{j\}}^{(-)}$$

$$= (-1 \pm 1 + e^{-A} \mp e^A) \sum_{l_o} G_{l_o, \{j\}} + \varepsilon^n (-1 \pm 1 + e^A \mp e^{-A}) \sum_{l_o} G_{l_o, \{j\}}^{(-)}, \quad (5.57)$$

where

$$A \equiv i\beta \sum_{i=1}^m \frac{\partial}{\partial t_{j_i}}, \quad (5.58)$$

$$G^{(-)} \equiv G(-1 \cdots -n). \quad (5.59)$$

After Fourier transformation we get finally

$$\sum_{0 \leq l_e \leq n} [G_{l_e, \{j\}}(\omega) + \varepsilon^n G_{l_e, \{j\}}(-\omega)] = \coth \frac{1}{2} \beta (\omega_{j_1} + \cdots + \omega_{j_m}) \sum_{1 \leq l_o \leq n} [G_{l_o, \{j\}}(\omega) - \varepsilon^n G_{l_o, \{j\}}(-\omega)],$$

$$(5.60a)$$

$$\sum_{0 \leq l_e \leq n} [G_{l_e, \{j\}}(\omega) - \varepsilon^n G_{l_e, \{j\}}(-\omega)] = \tanh \frac{1}{2} \beta (\omega_{j_1} + \dots + \omega_{j_m}) \sum_{1 \leq l_o \leq n} [G_{l_o, \{j\}}(\omega) + \varepsilon^n G_{l_o, \{j\}}(-\omega)]. \quad (5.60b)$$

This is a plausible generalization of FDT to the nonlinear case. For $n = 2$, (5.60a) is the usual FDT, while (5.60b) is an identity. For arbitrary n , we have $2^{n-1} - 1$ algebraic relations of type (5.52) (excluding all $\alpha_k = +$ cases) leading to $2^{n-1} - 1$ pairs of relations given by (5.60) in combination with the KMS and the time reversal invariance conditions.

To illustrate (5.60) we write down explicitly the corresponding expressions for $n = 3$ case

$$\begin{aligned} G_{0, \{1\}} &= G_{0, \{2\}} = G_{0, \{3\}} = G_{222}, \\ G_{1, \{1\}} &= -G_{122} + G_{212} + G_{221}, \quad G_{1, \{2\}} = G_{122} - G_{212} + G_{221}, \quad G_{1, \{3\}} = G_{122} + G_{212} - G_{221}, \\ G_{2, \{1\}} &= -G_{112} - G_{121} + G_{211}, \quad G_{2, \{2\}} = -G_{112} + G_{121} - G_{211}, \quad G_{2, \{3\}} = G_{112} - G_{121} - G_{211}, \end{aligned} \quad (5.61)$$

$$\begin{aligned} G_{2, \{j\}}(\omega) + \varepsilon^3 G_{2, \{j\}}(-\omega) + G_{222}(\omega) + \varepsilon^3 G_{222}(-\omega) &= \coth(\beta \omega_j/2) [G_{1, \{j\}}(\omega) - \varepsilon^3 G_{1, \{j\}}(-\omega)], \\ G_{2, \{j\}}(\omega) - \varepsilon^3 G_{2, \{j\}}(-\omega) + G_{222}(\omega) - \varepsilon^3 G_{222}(-\omega) \\ &= \tanh(\beta \omega_j/2) [G_{1, \{j\}}(\omega) + \varepsilon^3 G_{1, \{j\}}(-\omega)], \quad j = 1, 2, 3. \end{aligned} \quad (5.62)$$

The question whether (5.60) is a correct generalization of FDT should be settled by further studies of nonlinear phenomena. We should mention, however, that Tremblay et al. [38] have considered the heating effects in electric conduction processes using CTPGF formalism. These authors do not discuss the general relations as we do here. In their opinion, the FDT should be model dependent in the nonlinear case.

6. Path integral representation and symmetry breaking

The generating functional $Z[J(x)]$ for CTPGF can be presented as a Feynman path integral. In terms of the eigenstate $|\varphi'(x)\rangle$ of the operator $\varphi(x, t = -\infty)$ the density matrix can be written as

$$\hat{\rho} = \int d\varphi'(x) d\varphi''(x) |\varphi'(x)\rangle \rho_{\varphi' \varphi''} \langle \varphi''(x)|, \quad (6.1)$$

so that the generating functional is given by

$$Z[J(x)] = \int d\varphi'(x) d\varphi''(x) \rho_{\varphi' \varphi''} \langle \varphi''(x)| U(t_- = -\infty, t_+ = -\infty) |\varphi'(x)\rangle, \quad (6.2)$$

where

$$U(t_- = -\infty, t_+ = -\infty) \equiv S_p = T_p \exp \left(i \int_p J(x) \varphi(x) \right) \quad (6.3)$$

is the evolution operator defined along the closed time-path.

It is known in the quantum field theory [39] that

$$\langle \varphi_2(\mathbf{x}) | U(t_2, t_1) | \varphi_1(\mathbf{x}) \rangle = N \int [d\varphi(\mathbf{x})] \exp \left(i \int_{t_1}^{t_2} \mathcal{L}(\varphi(x)) d^{d+1}x \right) \delta(\varphi(\mathbf{x}, t_2) - \varphi_2(\mathbf{x})) \delta(\varphi(\mathbf{x}, t_1) - \varphi_1(\mathbf{x})), \quad (6.4)$$

where N is a constant. The path integral representation (6.4) is valid for any t_1, t_2 , so (6.2) can be rewritten as

$$\begin{aligned} Z[J(x)] &= N \int [d\varphi(\mathbf{x})] \delta(\varphi(\mathbf{x}, t_+ = -\infty) - \varphi'(\mathbf{x})) \rho_{\varphi' \varphi''} \delta(\varphi(\mathbf{x}, t_- = -\infty) - \varphi''(\mathbf{x})) \\ &\times \exp \left[i \int_p (\mathcal{L}(\varphi(x)) + J(x)\varphi(x)) \right], \end{aligned} \quad (6.5)$$

where the integration in the exponent is carried out over the closed time-path p . Since the functional dependence of $\rho_{\varphi' \varphi''}$ upon $\varphi'(\mathbf{x})$ and $\varphi''(\mathbf{x})$ is rather complicated, in general (6.5) is not very suitable for practical calculation. However, it is useful for discussing the symmetry properties of the generating functional so far as the total Lagrangian of the system appears in the exponent. We will use this representation to discuss the Ward–Takahashi (WT) identities and the Goldstone theorem following from the symmetry (section 6.3).

The path integral representation would be well adapted to the practical calculation if the contribution of the density matrix can be expressed in terms of effective Lagrangian in certain simplifying cases. This possibility will be considered in section 6.1. In section 6.2 we briefly discuss the properties of the order parameter and describe two different types of phase transitions. Finally, in section 6.4, the path integral representation is used to consider the fluctuation effects.

6.1. Initial correlations

In this section we derive two equivalent expressions for the generating functional to incorporate the effects of the initial correlation in a convenient way [46].

6.1.1. Model

Consider multi-component nonrelativistic field $\psi_b^\dagger, \psi_b, b = 1, 2 \dots n$ which may be either boson or fermion. The action of the system is given by

$$I[\psi^\dagger, \psi] = I_0[\psi^\dagger, \psi] + I_{\text{int}}[\psi^\dagger, \psi], \quad (6.6)$$

where the free part can be written as

$$I_0[\psi^\dagger, \psi] = \int_p d1d2 \psi^\dagger(1) S_0^{-1}(1, 2) \psi(2) \equiv \psi^\dagger S_0^{-1} \psi, \quad (6.7)$$

with

$$\begin{aligned} S_0^{-1} &= \frac{1}{2}(\xi S_{0r}^{-1} \eta^\dagger + \eta S_{0a}^{-1} \xi^\dagger + \xi S_{0c}^{-1} \xi^\dagger), \\ S_{0r}^{-1} &= S_{0a}^{-1} = [i\partial_t + (1/2m)\nabla^2]\delta^{d+1}(1-2), \quad S_{0c}^{-1} = 0, \end{aligned} \quad (6.8)$$

in accord with (2.23), (2.63) and (3.10).

6.1.2. First expression for W_p^N

The generating functional

$$Z_p[J^\dagger J] \equiv \text{Tr}\{\hat{\rho} T_p \exp[i(J^\dagger \psi + \psi^\dagger J)]\} \quad (6.9)$$

can be rewritten as

$$Z_p[J^\dagger J] = \exp\left\{iI_{\text{int}}\left[\mp i\frac{\delta}{\delta J}, -i\frac{\delta}{\delta J^\dagger}\right]\right\} \exp\{i[-J^\dagger S_0 J + W_p^N(J^\dagger, J)]\} \quad (6.10)$$

in the incoming picture by using the Wick theorem generalized to the CTPGF case as done in section 2.2 for the Hermitian boson field. Here S_0 is the bare propagator satisfying the equation

$$\int_p dx S_0(1, x) S_0^{-1}(x, 2) = \int_p dx S_0^{-1}(1, x) S_0(x, 2) = \delta_p(1-2), \quad (6.11)$$

and

$$\exp\{iW_p^N[J^\dagger, J]\} = \text{Tr}\{\hat{\rho} : \exp[i(J^\dagger \psi_1 + \psi_1^\dagger J)] : \}, \quad (6.12)$$

i.e.,

$$W_p^N[J^\dagger, J] = \sum_{m, n=1}^{\infty} \frac{1}{m! n!} \int_p d1 \cdots dm d\bar{1} \cdots d\bar{n} J^\dagger(1) \cdots J^\dagger(m) W_p^{(m, n)}(1 \cdots m, \bar{n} \cdots \bar{1}) J(\bar{n}) \cdots J(\bar{1}), \quad (6.13)$$

$$W_p^{(m, n)}(1 \cdots m, \bar{n} \cdots \bar{1}) = i^{m+n-1} \text{Tr}\{\hat{\rho} : \psi_1(m) \cdots \psi_1(1) \psi_1^\dagger(\bar{1}) \cdots \psi_1^\dagger(\bar{n}) : \}, \quad (6.14)$$

where ψ_1, ψ_1^\dagger are operators in the incoming picture and $: :$ means normal product.

Since the time ordering does not have any effects under normal product, we can rewrite (6.12) as

$$\exp\{iW^N[J_\Delta^\dagger, J_\Delta]\} = \exp\{iW_p^N[J^\dagger, J]\} = \text{Tr}\{\hat{\rho} : \exp[i(J_\Delta^\dagger \psi_1 + \psi_1^\dagger J_\Delta)] : \}, \quad (6.15)$$

where

$$J_\Delta = J_+ - J_-, \quad J_\Delta^\dagger = J_+^\dagger - J_-^\dagger. \quad (6.16)$$

We can then write down an expansion equivalent to (6.13) as

$$W^N[J_\Delta^\dagger, J_\Delta] = \sum_{m, n=1}^{\infty} \frac{1}{m! n!} (J_\Delta^\dagger)^m W^{(m, n)} (J_\Delta)^n \quad (6.17)$$

with

$$W^{(m, n)} = i^{m+n-1} \text{Tr}[\hat{\rho} : \psi_i(m) \cdots \psi_i(1) \psi_i^\dagger(\bar{1}) \cdots \psi_i^\dagger(\bar{n}) :] \quad (6.18)$$

We note that W_p^N and $W_p^{(m, n)}$ are defined on the closed time-path, whereas W^N and $W^{(m, n)}$ are defined on the ordinary time axis.

Taking into account that in the incoming picture the field operators satisfy the free field equation, we have

$$\begin{aligned} \int_p di' \tilde{S}_0^{-1}(i, i') W_p^{(m, n)}(1 \cdots i' \cdots m, \bar{n} \cdots \bar{1}) &= 0, \\ \int_p di' W_p^{(m, n)}(1 \cdots m, \bar{n} \cdots i' \cdots \bar{1}) \tilde{S}_0^{-1}(i', \bar{i}) &= 0, \end{aligned} \quad (6.19)$$

or equivalently,

$$\begin{aligned} \int di' \tilde{S}_{0r}^{-1}(i, i') W^{(m, n)}(1 \cdots i' \cdots m, \bar{n} \cdots \bar{1}) &= 0, \\ \int di' W^{(m, n)}(1 \cdots m, \bar{n} \cdots i' \cdots \bar{1}) \tilde{S}_{0a}^{-1}(i', \bar{i}) &= 0. \end{aligned} \quad (6.20)$$

As far as the initial condition is fixed at $t = -\infty$, we are not allowed to integrate by parts arbitrarily with respect to $\partial/\partial t$. The correct direction of acting $\partial/\partial t$ is indicated by the arrow.

Substituting (6.15) into (6.10) we get the first expression for $Z[J]$ we would like to derive as

$$Z[J^\dagger J] = \exp \left\{ i I_{\text{int}} \left[\mp i \frac{\partial}{\partial J}, -i \frac{\partial}{\partial J^\dagger} \right] \right\} \exp \{i(-J^\dagger S_0 J + W^N[J_\Delta^\dagger, J_\Delta])\}, \quad (6.21)$$

from which we can obtain the generalized Feynman rule. We see from (6.21) that the density matrix affects only the correlation generated by J_Δ^\dagger and J_Δ . So far as $W^{(m, n)}$ satisfy eq. (6.20), the contribution of the density matrix can be expressed in terms of the initial (sometimes called boundary) condition for Green's function.

6.1.3. Second expression for W_p^N

Now we derive another expression for the CTPGF generating functional. Using the following identity (up to an unimportant constant)

$$\exp(-iJ^\dagger S_0 J) = \int_p [d\psi^\dagger][d\psi] \exp\{i(\psi^\dagger S_0^{-1} \psi + J^\dagger \psi + \psi^\dagger J)\}, \quad (6.22)$$

it is easy to show that

$$\begin{aligned} & \exp\{i(-J^\dagger S_0 J + W_p^N [J^\dagger, J])\} \\ &= \int_p [d\psi^\dagger][d\psi] \exp\{i(J^\dagger \psi + \psi^\dagger J)\} \exp\left\{iW_p^N \left[\pm i\frac{\delta}{\delta\psi}, i\frac{\delta}{\delta\psi^\dagger}\right]\right\} \exp(i\psi^\dagger S_0^{-1} \psi), \end{aligned} \quad (6.23)$$

if the path integration is taken by parts. Taking into account that

$$\begin{aligned} \frac{\delta}{\delta\psi} \exp(i\psi^\dagger S_0^{-1} \psi) &= \exp(i\psi^\dagger S_0^{-1} \psi) \left(\frac{\delta}{\delta\psi} \pm i\psi^\dagger \tilde{S}_0^{-1}\right), \\ \frac{\delta}{\delta\psi^\dagger} \exp(i\psi^\dagger S_0^{-1} \psi) &= \exp(i\psi^\dagger S_0^{-1} \psi) \left(\frac{\delta}{\delta\psi^\dagger} + i\tilde{S}_0^{-1} \psi\right), \end{aligned} \quad (6.24)$$

(6.23) can be transformed into

$$\begin{aligned} & \exp\{i(-J^\dagger S_0 J + W_p^N [J^\dagger, J])\} = \int_p [d\psi^\dagger][d\psi] \exp\{i(J^\dagger \psi + \psi^\dagger J + \psi^\dagger S_0^{-1} \psi)\} \\ & \times \exp\left\{iW_p^N \left[\pm i\frac{\delta}{\delta\psi} - \psi^\dagger \tilde{S}_0^{-1}, i\frac{\delta}{\delta\psi^\dagger} - \tilde{S}_0^{-1} \psi\right]\right\}. \end{aligned} \quad (6.25)$$

Using (6.8) and the convention agreed in (2.69)–(2.74) we find that

$$\eta^\dagger \frac{\delta}{\delta\psi(x_\sigma)} = \frac{\delta}{\delta\psi_c(x)}, \quad \eta^\dagger \frac{\delta}{\delta\psi^\dagger(x_\sigma)} = \frac{\delta}{\delta\psi_c^\dagger(x)}, \quad (6.26)$$

$$\begin{aligned} & \int dy \psi^\dagger(y) \sigma_3 \tilde{S}_0^{-1}(y, x) \eta = \int dy \psi_\Delta^\dagger(y) \tilde{S}_{0r}^{-1}(y, x), \\ & \eta^\dagger \int dy \tilde{S}_0^{-1}(x, y) \sigma_3 \psi(y) = \int dy \tilde{S}_{0a}^{-1}(x, y) \psi_\Delta(y), \end{aligned} \quad (6.27)$$

and obtain from (6.15) that

$$\begin{aligned} \exp\left\{iW_p^N \left[\pm i\frac{\delta}{\delta\psi} - \psi^\dagger \tilde{S}_0^{-1}, i\frac{\delta}{\delta\psi^\dagger} - \tilde{S}_0^{-1} \psi\right]\right\} &= \exp\{iW^N [-\psi_\Delta^\dagger \tilde{S}_{0r}^{-1}, -\tilde{S}_{0a}^{-1} \psi]\} \\ &= \exp\{iW_p^N [-\psi^\dagger \tilde{S}_0^{-1}, \tilde{S}_0^{-1} \psi]\}. \end{aligned} \quad (6.28)$$

Substituting (6.28) into (6.25) and the resulting expression into (6.21), we find

$$Z_p[J^\dagger, J] = \int_p [d\psi^\dagger][d\psi] \exp\{i(I_0[\psi^\dagger, \psi] + I_{\text{int}}[\psi^\dagger, \psi] + J^\dagger\psi + \psi^\dagger J)\} \exp\{iW_p^N[-\psi^\dagger \tilde{S}_0^{-1}, -\tilde{S}_0^{-1}\psi]\} \quad (6.29)$$

as the second path integral representation for the generating functional. It is easy to rederive (6.21) from (6.29), so these two expressions are equivalent to each other. Note that this expression is different from that given by (6.5) in so far as the contribution of the density matrix appears here as an additional term W_p^N in the action. According to (6.28), this term does not depend on field variables ψ_c^\dagger, ψ_c describing the dynamical evolution, but does depend on $\psi_\Delta^\dagger, \psi_\Delta$ describing the statistical correlation. It is also obvious that $W_p^N[-\psi^\dagger \tilde{S}_0^{-1}, -\tilde{S}_0^{-1}\psi]$ has nonvanishing contribution to the path integral only at the end points because of (6.19).

6.1.4. Two-step strategy

For a general nonequilibrium process, (6.29) can be rewritten as

$$Z_p[J^\dagger, J] = \exp\left\{iW_p^N\left[\pm i\frac{\delta}{\delta J} \tilde{S}_0^{-1}, i\tilde{S}_0^{-1}\frac{\delta}{\delta J^\dagger}\right]\right\} Z_p^0[J^\dagger, J], \quad (6.30)$$

where

$$Z_p^0[J^\dagger, J] = \int_p [d\psi^\dagger][d\psi] \exp\{i(I_0 + I_{\text{int}} + J^\dagger\psi + \psi^\dagger J)\} \quad (6.31)$$

is the generating functional for the ground state. Since Z_p^0 has exactly the same structure on the closed time-path as that of the standard quantum field theory, we can first calculate Z_p^0 and then “put into” it the statistical information via (6.30). Such “two-step” strategy is well known in solving the Liouville problem in classical statistical mechanics.

Many interesting nonequilibrium phenomena can be described by a Gaussian process, i.e.,

$$W_p^{(m, n)}(1 \cdots m, \bar{n} \cdots \bar{1}) = 0 \quad \text{except for} \quad W_p^{(1, 1)}(1, \bar{1}) \neq 0, \quad (6.32)$$

for which the contribution of the density matrix reduces to replacing the bare propagator S_{0p} by

$$G_{0p}(x, y) = S_{0p}(x, y) - W_p^{(1, 1)}(x, y). \quad (6.33)$$

For the thermoequilibrium case eq. (6.33), after Fourier transformation, is identical to (2.21). A more rigorous derivation of the diagrammatic expansion for thermoequilibrium will be given in section 9.1.

Another possibility of simplification comes about when the state is stationary due to the microscopic time reversal invariance, the generalized FDT then holds as shown in section 4.2. As seen from (6.20) and (6.21), $W_p^{(m, n)}$ as solutions of the homogeneous equation can be specified by the FDT.

To sum up, the determination of the CTPGF generating functional can be divided into two steps: To first “forget” about the density matrix in calculating the generating functional without the statistical information and then “put it into” the generating functional at the second step. In the general case this can be done using (6.30), but a significant simplification results if the initial correlation is Gaussian or a generalized FDT holds.

If we are interested in some order parameter $Q(x)$ which is a composite operator of the constituent field, we introduce an additional term $h(x)Q(x)$ in the action. The generating functional for the order parameter is given by

$$Z_p[h; J^\dagger, J] = \exp \left\{ i h Q \left[\mp i \frac{\delta}{\delta J}, -i \frac{\delta}{\delta J^\dagger} \right] \right\} Z_p[J^\dagger, J] \quad (6.34)$$

in terms of $Z_p[J^\dagger, J]$ for the constituent field. The extension of (6.21), (6.29) and (6.30) to this case is obvious.

6.2. Order parameter and stability of state

It is well known that the vertex functional $\Gamma[\varphi_c]$ is most suitable for describing the symmetry breaking, inasmuch as it is expressed explicitly in terms of the order parameter φ_c . In section 2.2 we have derived an equation (2.48) satisfied by it. Before going on with the discussion of the order parameter, we rewrite this basic equation of the CTPGF formalism in another equivalent form.

6.2.1. Functional form for the vertex equation

Let the total action of the system be presented as

$$I_t = I + I_s = \int \mathcal{L}(\varphi(x)) + \int J(x)\varphi(x). \quad (6.35)$$

The operator equation satisfied by $\varphi(x)$ is then

$$\frac{\delta I[\varphi(x)]}{\delta \varphi(x)} = -J(x). \quad (6.36)$$

Here we use the Heisenberg picture including the external source. Multiplying (6.36) by the density matrix and taking trace, we find

$$-J(x) = \text{Tr} \left\{ \frac{\delta I[\varphi(x)]}{\delta \varphi(x)} \hat{\rho} \right\}.$$

Transforming into the Heisenberg picture without external source, we obtain

$$-J(x) = \frac{1}{Z[J(x)]} \text{Tr} \left\{ T_p \left(\frac{\delta I[\varphi(x)]}{\delta \varphi(x)} \exp(iJ\varphi) \right) \hat{\rho} \right\} = \frac{1}{Z[J(x)]} \frac{\delta I}{\delta \varphi(x)} \left[\varphi(x) = -i \frac{\delta}{\delta J(x)} \right] Z[J(x)]. \quad (6.37)$$

Using the commutation relation

$$-i \frac{\delta}{\delta J(x)} Z[J(x)] = Z[J(x)] \left(\varphi_c(x) - i \frac{\delta}{\delta J(x)} \right), \quad (6.38)$$

(6.37) can be rewritten as

$$-J(x) = \frac{\delta I}{\delta \varphi(x)} \left[\varphi_c(x) - i \frac{\delta}{\delta J(x)} \right]. \quad (6.39)$$

Comparing (6.39) with (2.48) we find

$$\frac{\delta \Gamma[\varphi_c]}{\delta \varphi_c(x)} = \frac{\delta I}{\delta \varphi} \left[\varphi_c(x) - i \frac{\delta}{\delta J(x)} \right], \quad (6.40)$$

which is a formal relation to derive the generating functional from the action.

As in the quantum field theory, the term $i(\delta/\delta J(x))$ comes from the loop correction. Hence in the tree approximation

$$\frac{\delta \Gamma[\varphi_c(x)]}{\delta \varphi_c(x)} = \frac{\delta I[\varphi(x)]}{\delta \varphi(x)} \Big|_{\varphi = \varphi_c}, \quad (6.41)$$

$$\Gamma[\varphi_c(x)] = \int_p \mathcal{L}(\varphi_c(x)) d^{d+1}x. \quad (6.42)$$

We note in passing that we did not emphasize the validity of (6.36) on the closed time-path, but it is true, so that we can put p in eq. (6.42).

6.2.2. Two types of phase transitions

Coming back to the order parameter itself, $\varphi_c(x)$ should be a solution of equation

$$\delta \Gamma / \delta \varphi_c(x) = 0 \quad (6.43)$$

in the absence of the external field. The $\varphi_c(x) = 0$ solution corresponds to the “normal” state, whereas the nonzero solution corresponds to the symmetry broken state. Then the question arises which state is more stable. Let us consider the fluctuation around a homogeneous solution φ_{c0} of (6.43)

$$\varphi_c(x) = \varphi_{c0} + \varphi_k \exp(-ik \cdot x). \quad (6.44)$$

Put (6.44) into (6.43) and obtain the linear equation satisfied by φ_k as

$$\int_p \frac{\delta^2 \Gamma}{\delta \varphi_c(x) \delta \varphi_c(y)} \Big|_{\varphi_c = \varphi_{c0}} (\varphi_c(y) - \varphi_{c0}) dy = \int \Gamma_r(x, y) \varphi_k \exp(-ik \cdot y) dy = \Gamma_r(k) \varphi_k = 0. \quad (6.45)$$

As follows from (6.45), φ_k is zero unless k is a solution of the equation

$$\Gamma_r(k) = 0. \quad (6.46)$$

Assume the solution of (6.46) to be

$$k_0 = \omega(\mathbf{k}), \quad (6.47)$$

the fluctuation will decay in time, i.e., the state is stable if $\text{Im } \omega(\mathbf{k}) < 0$. Otherwise, the state is unstable.

According to (3.14)

$$\Gamma_r(k) = D(k) + iA(k), \quad (3.14')$$

where

$$A(k) = -\frac{1}{2}i(\Gamma_-(k) - \Gamma_+(k)). \quad (3.13')$$

Near the critical point where $\text{Im } \omega(\mathbf{k})$ changes sign, $\text{Im } \omega(\mathbf{k})$, and probably $A(k)$ is a small quantity. We discuss here two possible situations.

(1) The equation $D(\mathbf{k}, k_0) = 0$ has real solution $k_0 = \text{Re } \omega(\mathbf{k})$ for all \mathbf{k} , it then follows from (6.46) and (3.14') that

$$\text{Im } \omega(\mathbf{k}) = -A(k) \Big/ \frac{\partial D}{\partial k_0} \Big|_{k_0=\text{Re } \omega(\mathbf{k})}. \quad (6.48)$$

If $\text{Im } \omega(\mathbf{k}) > 0$ for some \mathbf{k} , then an instability with this \mathbf{k} occurs to form a new space-time structure. However, as discussed in section 3.2, in thermoequilibrium we have

$$A(k) = -\frac{1}{2}i\Gamma_-(k)[1 - \exp(-\beta k_0)] = k_0 W a(k)[1 - \exp(-\beta k_0)] > 0,$$

and $\partial D / \partial k_0 > 0$ for $k_0 > 0$, whereas both of them change sign for $k_0 < 0$, so that such instability cannot occur in an equilibrium system. In fact, it usually appears in far-from-equilibrium systems under certain special conditions, for example, in a laser system the $\varphi_c = 0$ solution is unstable above the threshold of pumping.

(2) $A(k)$ is not small compared with $D(k)$ as $\mathbf{k} \rightarrow 0$. For an equilibrium system we can write

$$A(k) = k_0 \gamma, \quad \gamma > 0, \quad D(k) = D_0 + a k_0 + \dots$$

Up to the first order of k_0 , the solution of (6.46) is

$$k_0 = -D(0)/(a + i\gamma), \quad (6.49)$$

with

$$\text{Im } k_0 = \frac{\gamma}{a^2 + \gamma^2} D(0).$$

Hence the phase transition occurs at $D(0) = 0$. This is the ordinary second-order phase transition if the nontrivial solution grows continuously from zero. Otherwise, the point $D(0) = 0$ will correspond to supercooling or superheating temperature.

6.3. Ward–Takahashi (WT) identity and Goldstone theorem

In this section we derive the WT identity satisfied by the CTPGF from the invariance of the Lagrangian of the system with respect to global transformations of a Lie group \mathbb{G} .

6.3.1. Group transformations

Let $\varphi(x)$ be the constituent field and $Q(x)$ the order parameter. Each of them has several components forming by themselves bases of unitary representations. Under the infinitesimal transformation of \mathbb{G}

$$\begin{aligned}\varphi(x) &\rightarrow \varphi'(x) = \varphi(x) + \delta\varphi(x), \\ \delta\varphi(x) &= \zeta_\alpha (iI_\alpha^{(0)} - X_\alpha^\mu(x)\partial_\mu)\varphi(x) = iI_\alpha\varphi(x)\zeta_\alpha,\end{aligned}\quad (6.50)$$

$$\begin{aligned}Q(x) &\rightarrow Q'(x) = Q(x) + \delta Q(x), \\ \delta Q(x) &= \zeta_\alpha (iL_\alpha^{(0)} - X_\alpha^\mu(x)\partial_\mu)Q(x) = iL_\alpha Q(x)\zeta_\alpha,\end{aligned}\quad (6.51)$$

where ζ_α are a total of n_G infinitesimal parameters for group \mathbb{G} and $I_\alpha^{(0)}$, $L_\alpha^{(0)}$ representation matrices for the generators of \mathbb{G} . X_α^μ are associated with the transformation of coordinates

$$x^\mu \rightarrow x^{\mu'} = x^\mu + X_\alpha^\mu(x)\zeta_\alpha. \quad (6.52)$$

It can be easily shown that the Lagrangian function transforms in this case as

$$\mathcal{L}(\varphi'(x)) \frac{d^4x}{d^4x'} = \mathcal{L}(\varphi(x')) + i \left(\frac{\delta\mathcal{L}}{\delta\varphi(x)} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi(x)} \right) I_\alpha\varphi(x)\zeta_\alpha(x) + \partial_\mu (j_\alpha^\mu(x)\zeta_\alpha(x)), \quad (6.53)$$

where

$$j_\alpha^\mu(x) = i \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi(x)} I_\alpha\varphi(x) - \mathcal{L}X_\alpha^\mu(x) \quad (6.54)$$

is the current in the α direction and $\zeta_\alpha(x)$ is an arbitrary infinitesimal function. If the Lagrangian is invariant under the global transformation of \mathbb{G} , it then follows that

$$\mathcal{L}(\varphi'(x)) \frac{d^4x}{d^4x'} - \mathcal{L}(\varphi(x)) = i \left(\frac{\delta\mathcal{L}}{\delta\varphi(x)} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi(x)} \right) I_\alpha\varphi(x)\zeta_\alpha + \zeta_\alpha \partial_\mu j_\alpha^\mu(x) = 0,$$

or equivalently,

$$\partial_\mu j_\alpha^\mu(x) = i \left(\partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi(x)} - \frac{\delta\mathcal{L}}{\delta\varphi(x)} \right) I_\alpha\varphi(x), \quad (6.55)$$

i.e., the current is conserved provided $\varphi(x)$ is the solution of the Euler–Lagrangian equation. Substituting (6.55) into (6.53) yields

$$\mathcal{L}(\varphi'(x)) \frac{d^4 x}{d^4 x'} = \mathcal{L}(\varphi(x')) + j_\alpha^\mu(x) \partial_\mu \zeta_\alpha(x), \quad (6.56)$$

which is the change of \mathcal{L} under local transformation of \mathcal{G} if it is invariant under the global action of the same group.

6.3.2. WT identities

The path integral representation for the generating functional (6.5) can be written as

$$\begin{aligned} Z[J(x), h(x)] = N \int [d\varphi(x)] \exp \left\{ i \int_p (\mathcal{L}(\varphi(x)) + J(x)\varphi(x) + h(x)Q(x)) \right\} \\ \times \langle \varphi(x, t_+ = -\infty) | \hat{\rho} | \varphi(x, t_- = -\infty) \rangle. \end{aligned} \quad (6.5')$$

Performing a local transformation of $\varphi(x)$ in (6.5') with $\zeta_\alpha(x)$ satisfying the following boundary conditions

$$\zeta_\alpha(x, t_\pm = -\infty) = \lim_{|x| \rightarrow \infty} \zeta(x, t) = 0, \quad (6.57)$$

and taking into account that the measure $[d\varphi(x)]$ does not change under unitary transformations, we obtain from the invariance of the generating functional that

$$\langle \partial_\mu j_\alpha^\mu(x) \rangle = -iJ(x)I_\alpha\varphi_c(x) - ih(x)L_\alpha Q_c(x). \quad (6.58)$$

On the other hand

$$\langle \partial_\mu j_\alpha^\mu(x) \rangle = Z^{-1} \partial_\mu j_\alpha^\mu \left(\varphi(x) = -i \frac{\delta}{\delta J(x)} \right) Z[J(x), h(x)], \quad (6.59)$$

from which it follows that

$$\partial_\mu j_\alpha^\mu \left(\varphi_c - i \frac{\delta}{\delta J(x)} \right) = -iJ(x)I_\alpha\varphi_c(x) - ih(x)L_\alpha Q_c(x) \quad (6.60)$$

by use of (6.38). Using the generating functional W for the connected Green function, the WT identities (6.60) can be rewritten as

$$\partial_\mu j_\alpha^\mu \left(\frac{\delta W}{\delta J(x)} - i \frac{\delta}{\delta J(x)} \right) = -iJ(x)I_\alpha \frac{\delta W}{\delta J(x)} - ih(x)L_\alpha \frac{\delta W}{\delta h(x)}. \quad (6.61)$$

Taking functional derivatives of (6.61) with respect to $J(y)$ and then setting $J(y) = 0$, we obtain WT identities satisfied by CTPGFs of different order.

In terms of vertex generating functional $\Gamma[\varphi_c]$ (6.61) can be expressed as

$$\begin{aligned} \partial_\mu j_\alpha^\mu & \left(\varphi_c(x) - i \int_p \left(\frac{\delta^2 \Gamma}{\delta \varphi_c(x) \delta \varphi_c(y)} \right)^{-1} \frac{\delta}{\delta \varphi_c(y)} d^4 y \right) \\ & = \pm i \frac{\delta \Gamma}{\delta \varphi_c(x)} L_\alpha \varphi_c(x) + i \frac{\delta \Gamma}{\delta Q_c(x)} L_\alpha Q_c(x). \end{aligned} \quad (6.62)$$

Here we allow $\varphi(x)$ to be either a boson or fermion field. Taking the functional derivative $\delta/\delta \varphi_c(y)$ of (6.62) and putting $\varphi_c(y) = \varphi_{c0}$, the symmetry breaking in the absence of $J(x)$, we obtain WT identities for different vertex functions.

6.3.3. Goldstone theorem

Now we use the WT identity to discuss the symmetry breaking after phase transition.

Suppose the equations

$$\delta \Gamma / \delta \varphi_c(x) = 0, \quad \delta \Gamma / \delta Q_c(x) = 0$$

have solutions $\varphi_c(x) = 0$, $Q_{c+}(x) = Q_{c-}(x) \neq 0$. Differentiating (6.62) with respect to $Q_c(y)$, setting $J(x) = h(x) = 0$ and integrating over x , we obtain

$$\int_p \frac{\delta^2 \Gamma}{\delta Q_c(y) \delta Q_c(x)} L_\alpha Q_c(x) d^4 x = 0,$$

which can be rewritten in the single time representation as

$$\int \Gamma_r(y, x) L_\alpha Q_c(x) d^4 x = 0. \quad (6.63a)$$

In matrix form (6.63a) appears as

$$\Gamma_r \cdot L_\alpha Q_c(x) = 0, \quad (6.63b)$$

i.e., $L_\alpha Q_c(x)$ is the eigenvector of Γ_r with zero eigenvalue.

Assume $Q_c(x)$ is invariant under a subgroup \mathbb{H} of \mathbb{G} with n_H as its dimension. Therefore,

$$L_\alpha Q_c = 0, \quad \alpha = 1, \dots, n_H$$

if α belongs to the generators of \mathbb{H} . On the contrary, if α belongs to the coset \mathbb{G}/\mathbb{H} , $L_\alpha Q_c \neq 0$, then (6.63b) shows that Γ_r has $n_G - n_H$ eigenvectors with zero eigenvalue. Suppose the representation to be

real, then taking complex conjugation of (6.63b) we obtain

$$Q_c L_\alpha \cdot \Gamma_a = 0, \quad (6.64)$$

due to the orthogonality of L_α . Separating the real and imaginary parts we find

$$Q_c L_\alpha \cdot D = D \cdot L_\alpha Q_c = 0, \quad Q_c L_\alpha \cdot A = A \cdot L_\alpha Q_c = 0, \quad (6.65)$$

i.e., $L_\alpha Q_c$ are zero eigenvalue eigenstates for both D and A . It follows from the Dyson equation (3.10) that the retarded Green function G_r has $n_G - n_H$ nondissipative elementary excitations called Goldstone modes. If Q_c does not depend on coordinates, in Fourier representation of $x - y$ such excitation occurs at zero energy and momentum and is called Goldstone particle.

6.3.4. Applications

The Goldstone bosons considered above have important consequences in the symmetry broken state. For example, in laser system the U(1) symmetry is broken, so the corresponding Goldstone boson leads to the divergence of the fluctuation which in turn makes the classical solution unstable. This phenomenon in the CTPGF approach was observed by Korenman [25] and was analyzed by us in [40, 52]. The WT identity is used to derive a generalized Goldstone theorem in a slowly varying in time system. As its consequence the pole of the Green function splits into two with equal weight, equal energy but different dissipation. Combined with the order parameter (average value of the vector potential) these two quanta (one of which is the Goldstone boson) provide a complete description of the order-disorder transition of the phase symmetry in the saturation state of the laser.

We have also used the WT identity in combination with order parameter expansion (cf. section 3.5) to derive the generalized TDGL equation [43, 47]. We will apply the same identity to discuss the localization problem in section 8.3 [56].

We should mention that the transformations given by (6.50) and (6.51) are linear. We can consider nonlinear transformations under \mathbb{G} as we did in [43, 47]. In that case we need to take into account the Jacobian of transformation for the path integral. The result thus obtained turns out to be the same as the nonlinear mode-mode coupling introduced phenomenologically by Kawasaki [67].

6.4. Functional description of fluctuation

6.4.1. Stochastic functional

It is known that the Gaussian stochastic process $\xi_i(t)$ appearing in the Langevin equation (cf. (4.33))

$$\partial Q_i / \partial t = K_i(Q) + \xi_i(t) \quad (6.66)$$

can be presented by a stochastic integral [86]. Equation (6.66) can then be considered as a nonlinear mapping of the Gaussian process on to a more complicated process $Q_i(t)$. Realization of such mapping actually results in a functional description of $Q_i(t)$ [72]. Nevertheless, such functional description can be achieved by a more straightforward way [87–89].

Consider the normalization of the δ -function under path integration

$$\int [dQ] \delta \left(\frac{\partial Q}{\partial t} - K(Q) - \xi \right) \Delta(Q) = 1, \quad (6.67)$$

where the Jacobian $\Delta(Q)$ appears because the argument of the δ -function is not Q itself but a rather complicated expression. Neglecting multiplicative factors, $\Delta(Q)$ turns out to be [72]

$$\Delta(Q) = \exp \left\{ -\frac{1}{2} \int \delta K(Q)/\delta Q \right\}. \quad (6.68)$$

Using the integral representation of δ -function

$$\delta(f) = \int \left[\frac{d\hat{Q}}{2\pi} \right] \exp \left(i \int \hat{Q} f \right), \quad (6.69)$$

(6.67) can be rewritten as

$$\int [dQ] \left[\frac{d\hat{Q}}{2\pi} \right] \exp \left\{ \int \left[i\hat{Q} \left(\frac{\partial Q}{\partial t} - K(Q) - \xi \right) - \frac{1}{2} \frac{\delta K}{\delta Q} \right] \right\} = 1. \quad (6.70)$$

Inserting the source term

$$\exp \left(i \int (JQ + \hat{J}\hat{Q}) \right),$$

one obtains the generating functional

$$Z_\xi[J, \hat{J}] = \int [dQ] \left[\frac{d\hat{Q}}{2\pi} \right] \exp \left\{ \int \left[i\hat{Q} \left(\frac{\partial Q}{\partial t} - K(Q) - \xi \right) - \frac{1}{2} \frac{\delta K}{\delta Q} + i(JQ + \hat{J}\hat{Q}) \right] \right\}, \quad (6.71)$$

with the normalization condition

$$Z_\xi[0, 0] = 1.$$

Averaging over the random noise distribution

$$W[\xi] \propto \exp(-\frac{1}{2}\xi\sigma^{-1}\xi) \quad (6.72)$$

with σ as the diffusion matrix, one obtains Lagrangian formulation of the generating functional for the statistical fluctuation

$$Z[J, \hat{J}] = \int [dQ] \left[\frac{d\hat{Q}}{2\pi} \right] \exp \left\{ \int \left[-\frac{1}{2} \hat{Q} \sigma \hat{Q} + i\hat{Q} \left(\frac{\partial Q}{\partial t} - K(Q) \right) - \frac{1}{2} \frac{\delta K}{\delta Q} + i(JQ + \hat{J}\hat{Q}) \right] \right\}. \quad (6.73)$$

The Gaussian integration over \hat{Q} can be carried out to yield

$$Z[J] = \int [dQ] \exp \left\{ \int \left[-\frac{1}{2} \left(\frac{\partial Q}{\partial t} - K(Q) - \hat{J} \right) \sigma^{-1} \left(\frac{\partial Q}{\partial t} - K(Q) - \hat{J} \right) - \frac{1}{2} \frac{\delta K}{\delta Q} + iJQ \right] \right\}. \quad (6.74)$$

Historically, the theory of noncommutative classical field was first suggested by Martin, Siggia and Rose (MSR hereafter) [90]. This theory has been extensively applied to critical dynamics [61] and has been later reformulated in terms of a Lagrangian field theory [88, 89] as presented by (6.73) and (6.74).

6.4.2. Effective action

We will show now that such description occurs within the CTPGF formalism in a natural way [43, 47], postponing the comparison with the MSR field theory to section 9.3.

Let $Q_i(x)$ be composite operators of the constituent fields $\varphi_i(x)$. Both of them are taken to be Hermitian Bose operators. Assuming the randomness of the initial phase, the density matrix is diagonal at moment $t = t_0$, i.e.,

$$\langle \varphi'(x, t_0) | \hat{\rho} | \varphi''(x, t_0) \rangle = P(\varphi'(x), t_0) \delta(\varphi''(x, t_0) - \varphi'(x, t_0)). \quad (6.75)$$

The initial distribution of the macrovariables $Q_i(x)$ is then given by

$$\begin{aligned} P(Q_i(x), t_0) &= \text{Tr}(\delta(Q_i(x) - Q_i(\varphi(x))) \hat{\rho}) \\ &= \int [d\varphi(x)] \delta(Q_i(x) - Q_i(\varphi(x))) P(\varphi(x), t_0). \end{aligned} \quad (6.76)$$

The generating functional for Q_i can be written as

$$\begin{aligned} Z[J(x)] &= \exp(iW[J(x)]) = \text{Tr} \left\{ T_p \left[\exp \left(i \int_p J(x) Q(\varphi(x)) \right) \right] \hat{\rho} \right\} \\ &= N \int [d\varphi(x)] \exp \left(i \int [\mathcal{L}(\varphi(x)) + JQ(\varphi(x))] \right) \delta(\varphi_+ - \varphi_-), \end{aligned} \quad (6.77)$$

where

$$\delta(\varphi_+ - \varphi_-) \equiv \int d\varphi'(x) \delta(\varphi(x, t_+ = t_0) - \varphi'(x)) \delta(\varphi(x, t_- = t_0) - \varphi'(x)) P(\varphi'(x), t_0). \quad (6.78)$$

Multiplying (6.77) by the normalization factor of the δ -function

$$\int [dQ] \delta(Q_+ - Q_-) \delta(Q(x) - Q(\varphi(x))) = 1, \quad (6.79)$$

changing the order of integration to replace $Q(\varphi(x))$ by $Q(x)$ and using the δ -function representation (6.69) with \hat{Q} changed for I , we can rewrite (6.77) as

$$Z[J] = N \int [dQ] \exp \left(i S_{\text{eff}} + i \int_p JQ \right) \delta(Q_+ - Q_-), \quad (6.80)$$

where

$$\exp(iS_{\text{eff}}[Q]) = \int [dI/2\pi] \exp \left(iW[I] - i \int_p IQ \right). \quad (6.81)$$

Here we have performed direct and inverse Fourier transformations of the path integral. So far as a continuous integration is taken over $I(x)$, $W[I]$ can be considered as a generating functional in the random external field. Calculating the functional integral in the one loop approximation which is equivalent to the Gaussian average, we can obtain the effective action $S_{\text{eff}}[Q]$ for $Q(x)$.

So far we have discussed the case when macrovariables are composite operators. The same is true if part or all of macrovariables are constituent fields themselves. A new “macro” field can be also introduced by use of the δ -function. However, one should carry out the path integration simultaneously in spite of the fact that the initial correlations are multiplicative, because in general the Lagrangian itself is not additive in terms of these variables.

Before going on to calculate the integral (6.81) we first discuss the basic properties of the effective action $S_{\text{eff}}[Q]$.

It is ready to check that apart from the normalization condition (2.103) the generating functional for the Hermitian boson field also satisfies the relation

$$W^*[J_+(x), J_-(x)] = -W[J_-(x), J_+(x)]. \quad (6.82)$$

It then follows from (6.81) that

$$S_{\text{eff}}^*[Q_+(x), Q_-(x)] = -S_{\text{eff}}[Q_-(x), Q_+(x)]. \quad (6.83)$$

Hence S_{eff} is purely imaginary for $Q_+(x) = Q_-(x)$. Setting $Q_{\pm}(x) = Q + \Delta Q_{\pm}$ and taking successive functional derivatives of (6.83) near Q , we obtain

$$\delta S/\delta Q(x_+) = [\delta S/\delta Q(x_-)]^*, \quad (6.84)$$

$$S_{Fij}(x, y) = S_{Fji}(y, x) = -S_{Fji}^*(y, x), \quad S_{\pm ij}(x, y) = S_{\mp ji}(y, x) = -S_{\mp ji}^*(y, x), \quad (6.85)$$

where

$$S_{ij} = \frac{\delta^2 S_{\text{eff}}}{\delta Q_i \delta Q_j} \Big|_{Q_+ = Q_- = Q}.$$

We see that S_{ij} respect the same symmetry as the two-point Green functions (cf. (2.134)) and vertex functions (3.12b).

If the system is invariant under a symmetry group \mathbb{G} , i.e., both the Lagrangian and the initial distribution do not change under

$$\varphi_i(x) \rightarrow \varphi_i^g(x) = U_{ij}(g)\varphi_j(x), \quad Q_i(\varphi) \rightarrow Q_i^g(\varphi) = V_{ij}(g)Q_j(\varphi),$$

then

$$W[J^g(x)] = W[J(x)], \quad J_i^g = J_i(x) V_{ji}^\dagger(g), \\ S_{\text{eff}}[Q^g(x)] = S_{\text{eff}}[Q(x)], \quad Q_i^g = V_{ij}(g) Q_j(\varphi).$$

The above said is true if the effective action S_{eff} is calculated exactly. However, the symmetry properties of S_{eff} , being related to those of the Lagrangian, may be different from the latter due to the average procedure.

If the lowest order of WKB, i.e., the tree approximation is taken in (6.81) we find that

$$Q = \delta W / \delta I, \quad (6.86)$$

$$S_{\text{eff}}[Q] = \Gamma[Q]. \quad (6.87)$$

In this case, S_{eff} inherits all properties of the generating functional $\Gamma[Q]$, e.g.,

$$S_{\text{eff}}[Q, Q] = 0, \quad (6.88)$$

$$\frac{\delta S_{\text{eff}}}{\delta Q_+(x)} \Big|_{Q_+ = Q_- = Q} = \frac{\delta S_{\text{eff}}}{\delta Q_-(x)} \Big|_{Q_+ = Q_- = Q}, \quad (6.89)$$

$$S_F + S_{\bar{F}} = S_+ + S_-, \quad (6.90)$$

$$\frac{\delta^m S_{\text{eff}}}{\delta Q_i(1) \cdots \delta Q_i(m)} = i^{m-1} \langle T_p [Q_i(1) \cdots Q_i(m)] \rangle_{\text{1 P.I.}}. \quad (6.91)$$

In accord with (6.87), (3.5) and (3.19) we have

$$-iS_{\pm}(k) > 0 \quad (6.92)$$

after taking Fourier transformation. Near thermoequilibrium we find from (3.40)

$$S_{-ij} - S_{+ij} \xrightarrow{k_0 \rightarrow 0} -\beta k_0 S_{-ij}(k). \quad (6.93)$$

6.4.3. Saddle point approximation

Up to now we have discussed only the general properties of $S_{\text{eff}}[Q]$. In principle, S_{eff} can be derived from the microscopic generating functional $W[I]$ by averaging over the random external field, it can be also constructed phenomenologically in accord with the symmetry properties required. We now calculate (6.81) in the one-loop approximation. Near the saddle point given by (6.86) we expand the exponential factor in (6.81)

$$E = W - \int_p Q I = \Gamma - \frac{1}{2} \int_p \Delta I G^{(2)} \Delta I = \Gamma - \frac{1}{2} \int_p \Delta \hat{I}^T \sigma_3 \hat{G} \sigma_3 \Delta \hat{I}, \quad (6.94)$$

where \hat{G} is the two-point connected functions, $\Delta\hat{I}^T = (\Delta I_+, \Delta I_-)$. Up to a numerical constant, the result of the Gaussian integration is

$$\exp(iS_{\text{eff}}[Q]) = \exp(i\Gamma[Q])|\det \sigma_3 \hat{G} \sigma_3|^{-1/2}. \quad (6.95)$$

It then follows from the Dyson equation (2.57) that

$$iS_{\text{eff}}[Q] = i\Gamma[Q] + \frac{1}{2} \text{Tr} \ln \hat{I}. \quad (6.96)$$

By use of the transformation formula (2.59) we have

$$|\det \hat{I}| = |\det \tilde{I}| = |\det \Gamma_r| |\det \Gamma_a| = |\det \Gamma_r|^2,$$

where

$$\Gamma_r(x, y) = \frac{\delta^2 \Gamma}{\delta Q_\Delta(x) \delta Q_c(y)}.$$

As shown in section 3.5,

$$\frac{\delta \Gamma}{\delta Q_\Delta(x)} = \frac{1}{2} \left[\frac{\delta \Gamma}{\delta Q(x_+)} + \frac{\delta \Gamma}{\delta Q(x_-)} \right] \Big|_{Q_\Delta=0} = -\gamma \frac{\partial Q}{\partial t} - \frac{\delta \mathcal{F}}{\delta Q(x)}. \quad (6.97)$$

Comparing (6.97) with (6.67) we find that $\delta^2 \Gamma / \delta Q_\Delta(x) \delta Q_c(y)$ is just the transformation matrix up to the numerical factor γ^{-1} . Therefore, we can calculate the Jacobian in the same way to get

$$iS_{\text{eff}}[Q] = i\Gamma[Q] - \frac{1}{2} \int \delta K / \delta Q, \quad (6.98)$$

with

$$K = -\gamma^{-1} \delta \mathcal{F} / \delta Q.$$

In the path integral (6.80) the most plausible path is given by

$$\delta S_{\text{eff}}[Q] / \delta Q(x \pm) = -J_\pm(x), \quad (6.99)$$

$$Q(x, t_+ = t_0) = Q(x, t_- = t_0). \quad (6.100)$$

In the tree approximation of (6.81)

$$\delta S_{\text{eff}} / \delta Q = -J = -\gamma \partial Q / \partial t - \delta \mathcal{F} / \delta Q, \quad (6.101)$$

which is nothing but the TDGL equation derived in section 3.5 (cf. (3.85)).

6.4.4. Role of fluctuations

We now discuss fluctuations around the most plausible path. In the CTPGF approach there is an additional way of describing the fluctuation: To allow field variables to take different values on the positive and negative time branches. Changing variables in (6.80) to $Q = Q_c = \frac{1}{2}(Q_+ + Q_-)$, $Q_\Delta = Q_+ - Q_-$, the effective action can be expanded as

$$S_{\text{eff}}[Q_+(x), Q_-(x)] = S_{\text{eff}}[Q, Q] + \frac{1}{2} \int [\delta S_{\text{eff}}/\delta Q(x+) + \delta S_{\text{eff}}/\delta Q(x-)] Q_\Delta(x) \\ + \frac{1}{8} \int Q_\Delta(x) (S_{++} + S_{--} + S_{+-} + S_{-+})(x, y) Q_\Delta(y) + \dots$$

Denoting

$$\frac{1}{4}i(S_{++} + S_{--} + S_{+-} + S_{-+})(x, y) = -\gamma(x)\sigma(x, y)\gamma(y) \quad (6.102)$$

and using (6.88), (6.98), (6.101), we obtain

$$e^{iW[J(x)]} = \int [dQ(x)][dQ_\Delta(x)] \exp \left[-\frac{1}{2} \int Q_\Delta(x) \gamma(x) \sigma(x, y) \gamma(y) Q_\Delta(y) \right. \\ \left. - i \int \left(\gamma(x) \frac{\partial Q}{\partial t} + \frac{\delta \mathcal{F}}{\delta Q} \right) Q_\Delta(x) - \frac{1}{2} \int \frac{\delta K}{\delta Q} + i \int (J_\Delta Q + J_c Q_\Delta) \right] \delta(Q_\Delta(x)). \quad (6.103)$$

If we take $J_\Delta = J$ and $\gamma(x)Q_\Delta(x) \rightarrow \hat{Q}$, $J_c\gamma^{-1} \rightarrow \hat{J}$ the stochastic generating functional (6.73) is retrieved. Carrying out integration over $Q_\Delta(x)$ will lead to an equation identical to (6.74). It is important to note here, that in the CTPGF approach, J_c , the counterpart of \hat{J} in MSR theory, is the physical external field, whereas $J_\Delta = J_+ - J_-$, the counter part of J in MSR theory is the fictitious field. It is clear by comparing (6.103) and (6.73) that $\sigma(x, y)$ is the correlation matrix for the random force. If Q is a smooth function of x , σ can be taken as a constant

$$\sigma = -\frac{1}{4}i\gamma^{-1}(S_F + S_{\bar{F}} + S_+ + S_-)(k=0)\gamma^{-1}, \quad (6.104)$$

which reduces to

$$\sigma = -\frac{1}{2}i\gamma^{-1}\Gamma_c(k=0)\gamma^{-1} \quad (6.105)$$

by virtue of (6.87) and (6.90). This is the expression we have used in section 4 (cf. (4.35)) to discuss the symmetry properties of the kinetic coefficients, if it is generalized to the multi-component case.

According to the definition of γ given by (3.81)

$$\gamma = \lim_{k_0 \rightarrow 0} \frac{\partial A}{\partial k_0} = -\frac{i}{2} \lim_{k_0 \rightarrow 0} \frac{\partial}{\partial k_0} (\Gamma_- - \Gamma_+) \approx -\frac{1}{2}i\beta\Gamma_- = -\frac{1}{4}i\beta(S_+ + S_-). \quad (6.106)$$

Comparison of (6.106) with (6.105) yields the Einstein relation (FDT)

$$\sigma = 2/\beta\gamma \quad (6.107)$$

for the diffusion coefficient in the case of single macrovariable.

For simplicity we consider only one component order parameter in this section, but we consciously write some of formulas in such a way, so that the generalization to the multi-component case is obvious.

To sum up, the MSR field theory of stochastic functional is retrieved in the CTPGF approach if the one-loop approximation in the random field integration and the second cumulant expansion in $Q_\Delta(x)$ are taken. The possibility to go beyond such approximation is apparent.

7. Practical calculation scheme using CTPGF

As we have seen, the CTPGF provides us with a unified approach to both equilibrium and nonequilibrium systems. However, to make it practically useful we need a unified, flexible enough calculation scheme. Such scheme has been already worked out by us [48, 49]. In fact, most of the calculations carried out by us so far using CTPGF [40, 46, 52–57] can be cast into this framework.

Consider a typical situation when fermions $\psi(x), \psi^\dagger(x)$ are coupled to the order parameter $Q(x)$ which might be a constituent field like the vector potential $A_\mu(x)$ in the laser case, or a composite operator like

$$\chi(x) = \psi^\dagger(x)\psi(x)$$

in the theory of superconductivity, or

$$S = \psi^\dagger(x)\frac{1}{2}\sigma\psi(x)$$

in the case of itinerant ferromagnetism, where σ are Pauli matrices. The boson field $Q(x)$ via which the fermions interact with each other, may be nonpropagating at the tree level like the Coulomb field. However, the radiative correction will in general make $Q(x)$ a dynamical variable and the fluctuations around the mean field $Q_c(x)$ will propagate and form collective excitations. Therefore, the system is characterized by the mean field $Q_c(x)$ and the two kinds of quasiparticles – constituent fermions and collective excitations with their own energy spectrum, dissipation and distribution. Such a way of description has been found useful in condensed matter physics [1–3, 21–24, 91], plasma physics [33, 34] as well as in the nuclear many-body theory [92–96].

In this section we first (section 7.1) derive a system of coupled equations satisfied by the order parameter and the two kinds of Green's functions using the generating functional with two-point source terms. Next (section 7.2), the technique of Cornwall, Jackiw and Tomboulis (CJT) [97], developed in the quantum field theory to calculate the effective potential for composite operators is generalized to the CTPGF case and is used as a systematic way of computing the self-energy part by a loop expansion. In thermoequilibrium when the dissipation is negligible, the mean field $Q_c(x)$ and the energy spectrum of the fermion field are determined to the first approximation by the Bogoliubov–de Gennes (BdeG) [98] equation, in which the single particle fermion wavefunction satisfies the Hartree type self-consistent equation without Fock exchange term. In section 7.3 we discuss the generalization of the BdeG equation in the four-fermion problem, whereas the free energy in various approximations is calculated explicitly in section 7.4 by directly integrating the functional equation for it. Some problems related to those discussed in this section were considered by Kleinert using the functional integral approach [99].

7.1. Coupled equations of order parameter and elementary excitations

7.1.1. Model

Consider a fermion field $\psi(x)$ interacting via a boson field $Q(x)$ with the action given by

$$I = I_0[\psi^\dagger, \psi] + I_0[Q] + I_{\text{int}}[\psi^\dagger, \psi, Q], \quad (7.1)$$

where

$$I_0[\psi^\dagger, \psi] = \int_p \psi^\dagger(x) S_0^{-1}(x, y) \psi(y), \quad (7.2)$$

$$I_0[Q] = \frac{1}{2} \int_p Q(x) \Delta_0^{-1}(x, y) Q(y), \quad (7.3)$$

with $S_0^{-1}(x, y)$, $\Delta_0^{-1}(x, y)$ as inverse fermion and boson propagators respectively.

For a system with four-fermion interaction only, we can use the Hubbard–Stratonovich (HS) transformation [100] to introduce the effective fermion–boson interaction.

Let the generating functional for the order parameter $Q(\psi^\dagger(x), \psi(x))$ be defined as

$$Z_p[h(x)] = \int [d\psi^\dagger(x)][d\psi(x)] \exp\{i[I_0[\psi^\dagger, \psi] + I_{\text{int}}[\psi^\dagger, \psi] + hQ(\psi^\dagger, \psi)]\}. \quad (7.4)$$

Using the Gaussian integral identity, i.e. the HS transformation, eq. (7.4) can be presented as

$$Z_p[h] = \exp[\frac{1}{2}i h M_0 h] \tilde{Z}_p[h], \quad (7.5)$$

$$\tilde{Z}_p[h] = \int_p [d\psi^\dagger][d\psi][dQ] \exp\{i[I_0[\psi^\dagger, \psi] + I_0[Q] + I_{\text{int}}[\psi^\dagger, \psi, Q] + hQ]\}, \quad (7.6)$$

where

$$I_0[Q] = \frac{1}{2} Q \Delta_0^{-1} Q,$$

$I_{\text{int}}[\psi^\dagger, \psi, Q]$ being the nonlinear interaction. It is important to note that M_0, Δ_0^{-1} are two-point functions, independent of either field variables or external source. Therefore, up to an additive constant M_0 the Green functions of the original system are the same as those of the effective system described by \tilde{Z} . The formal ambiguity in defining the $[dQ(x)]$ integration [96] can be avoided by imposing the condition

$$\frac{\delta Z_p[h]}{\delta h(x)} \Big|_{h=0} = \frac{\delta \tilde{Z}_p[h]}{\delta h(x)} \Big|_{h=0} \quad (7.7)$$

We see thus the four-fermion interaction can be considered on an equal footing with system of fermions interacting through a constituent boson field $Q(x)$.

7.1.2. Two-point source

The generating functional with a two-point source is defined as

$$\begin{aligned} Z_p[h, J^\dagger, J, M, K] = \int_p [d\psi^\dagger][d\psi][dQ] \exp\{i[I_0[\psi^\dagger, \psi] + I_0[Q] + I_{\text{int}}[\psi^\dagger, \psi, Q]] \\ + W_p^N[\psi^\dagger, \psi, Q] + hQ + J^\dagger\psi + \psi^\dagger J + \frac{1}{2}QM\psi + \psi^\dagger K\psi]\}, \end{aligned} \quad (7.8)$$

where $W_p^N[\psi^\dagger, \psi, Q]$ takes care of contribution from the density matrix as discussed in section 6.1. Here we adopt the abbreviated notation and $M(x, y)$, $K(x, y)$ are external sources to generate the second-order CTPGFs.

Introducing the generating functional for the connected CTPGF as usual

$$W[h, J^\dagger, J, M, K] = -i \ln Z[h, J^\dagger, J, M, K], \quad (7.9)$$

it then follows that

$$\delta W_p / \delta h(x) = Q_c(x), \quad (7.10a)$$

$$\delta W_p / \delta J^\dagger(x) = \psi_c(x), \quad (7.10b)$$

$$\delta W_p / \delta J(x) = -\psi_c^\dagger(x), \quad (7.10c)$$

$$\delta W_p / \delta M(y, x) = \frac{1}{2}(Q_c(x)Q_c(y) + i\Delta(x, y)), \quad (7.10d)$$

$$\delta W_p / \delta K(y, x) = -(\psi_c(x)\psi_c^\dagger(y) + iG(x, y)). \quad (7.10e)$$

In case of vanishing sources $Q_c(x)$, $\psi_c(x)$ and $\psi_c^\dagger(x)$ become expectation values of the corresponding fields $Q(x)$, $\psi(x)$ and $\psi^\dagger(x)$, whereas $\Delta(x, y)$, $G(x, y)$ are the second-order CTPGF for the boson field $Q(x)$ and the fermion field $\psi(x)$, $\psi^\dagger(x)$ respectively.

7.1.3. Coupled equations

The generating functional for the vertex CTPGF is defined as the Legendre transform of W_p ,

$$\begin{aligned} \Gamma_p[Q_c, \psi_c^\dagger, \psi_c, \Delta, G] = W[h, J^\dagger, J, M, K] - hQ_c - \psi_c^\dagger J \\ - J^\dagger\psi_c - \frac{1}{2}\text{Tr}[M(Q_cQ_c + i\Delta)] - \text{Tr}[K(\psi_c\psi_c^\dagger + iG)], \end{aligned} \quad (7.11)$$

where

$$\text{Tr}[M(Q_c Q_c + i\Delta)] \equiv \int_p M(x, y)(Q_c(y)Q_c(x) + i\Delta(y, x)),$$

$$\text{Tr}[K(\psi_c \psi_c^\dagger + iG)] \equiv \int_p K(x, y)(\psi_c(y)\psi_c(x) + iG(y, x)).$$

Using (7.10) it is straightforward to deduce from (7.11) that

$$\frac{\delta \Gamma_p}{\delta Q_c(x)} = -h(x) - \int_p M(x, y)Q_c(y), \quad (7.12a)$$

$$\frac{\delta \Gamma_p}{\delta \psi_c^\dagger(x)} = -J(x) - \int_p K(x, y)\psi_c(y), \quad (7.12b)$$

$$\frac{\delta \Gamma_p}{\delta \psi_c(x)} = J^\dagger(x) + \int_p \psi_c^\dagger(y)K(y, x), \quad (7.12c)$$

$$\frac{\delta \Gamma_p}{\delta \Delta(x, y)} = \frac{1}{2i} M(y, x), \quad (7.12d)$$

$$\frac{\delta \Gamma_p}{\delta G(x, y)} = iK(y, x). \quad (7.12e)$$

Equations (7.12) form a set of self-consistent equations to determine the order parameters $Q_c(x)$, $\psi_c^\dagger(x)$, $\psi_c(x)$ as well as the second-order CTPGF $\Delta(x, y)$ and $G(x, y)$ provided Γ_p is known as a functional of these arguments. In almost all cases of practical interest the condensation of the fermion field is forbidden in the absence of the external source, i.e., $\psi_c^\dagger = \psi_c = 0$ for $J^\dagger = J = 0$. On the other hand, the condensation of the boson field (elementary or composite) is described by the order parameter $Q_c(x)$. Since the energy spectrum, the dissipation and the particle number distribution are determined by the second-order CTPGF, eqs. (7.12a, d, e) are just those equations we are looking for. The only question remaining is how to construct the vertex functional Γ_p . In the next section a systematic loop expansion will be developed for this purpose.

7.2. Loop expansion for vertex functional

7.2.1. CJT rule [97]

Without loss of generality in what follows we will set $J^\dagger = J = \psi_c^\dagger = \psi_c = 0$. The vertex functional

$$\Gamma_p[Q_c(x), \Delta, G] \equiv \Gamma_p[Q_c(x), \psi_c^\dagger(x), \psi_c(y), \Delta, G]|_{\psi_c^\dagger = \psi_c = 0}$$

is the generating functional in Q for the two-particle irreducible (2 PI) Green's functions expressed in terms of propagators Δ and G . To derive a series expansion for Γ_p we note that after absorbing W_p^N into the effective action, i.e.,

$$I_{\text{eff}}[\psi^\dagger, \psi, Q] = I_0[\psi^\dagger, \psi] + I_0(Q) + I_{\text{int}}[\psi^\dagger, \psi, Q] + W_p^N[\psi^\dagger, \psi, Q], \quad (7.13)$$

the only difference remaining between the CTPGFs and the ordinary Green functions in the quantum field theory is the range of the time axis. For CTPGF the time integration is taken for both positive and negative branches. Hence the loop expansion technique for the vertex functional and its justification developed by CJT [97] in the quantum field theory can be easily extended to the CTPGF formalism provided the difference in the definition of the time axis is properly taken into account. Here we shall simply state the result as

$$\begin{aligned} \Gamma_p[Q_c, \Delta, G] = & \bar{I}[Q_c] - \frac{1}{2}i\hbar \text{Tr}\{\ln[\Delta_0^{-1}\Delta] - \Delta_0^{-1}\Delta + 1\} \\ & + i\hbar \text{Tr}\{\ln[S_0^{-1}G] - G_0^{-1}G + 1\} + \Gamma_{2p}[Q_c, \Delta, G], \end{aligned} \quad (7.14)$$

where

$$\bar{I}[Q_c] \equiv I_{\text{eff}}[\psi^\dagger, \psi, Q] \Big|_{\substack{\psi = \psi^\dagger = 0 \\ Q = Q_c}}, \quad (7.15a)$$

$$\Delta_0^{-1}(x, y) = \frac{\delta^2 I_{\text{eff}}}{\delta Q(x) \delta Q(y)} \Big|_{\substack{\psi = \psi^\dagger = Q = 0}}, \quad (7.15b)$$

$$S_0^{-1}(x, y) = -\frac{\delta^2 I_{\text{eff}}}{\delta \psi^\dagger(x) \delta \psi(y)} \Big|_{\substack{\psi = \psi^\dagger = Q = 0}}, \quad (7.15c)$$

$$G_0^{-1}(x, y) = -\frac{\delta^2 I_{\text{eff}}}{\delta \psi^+(x) \delta \psi(y)} \Big|_{\substack{\psi = \psi^\dagger = 0 \\ Q = Q_c}}. \quad (7.15d)$$

Note that G_0^{-1} is different from S_0^{-1} and

$$S_0^{-1}(x, y) = G_0^{-1}(x, y) \Big|_{Q_c=0}. \quad (7.16)$$

The quantity $\Gamma_{2p}[Q_c, \Delta, G]$ appearing in (7.14) is computed as follows. First shift the field $Q(x)$ in the effective action $I_{\text{eff}}[\psi^\dagger, \psi, Q]$ by $Q_c(x)$ and keep only terms cubic and higher in ψ^\dagger, ψ and Q as interaction vertices which depend on Q_c . The Γ_{2p} is then calculated as a sum of all 2PI vacuum diagrams constructed by vertices described above with $\Delta(x, y), G(x, y)$ as propagators. In (7.14) the trace, the products $\Delta_0^{-1}\Delta$, etc., as well as the logarithm are taken in the functional sense with both internal indices and space-time coordinates summed over.

7.2.2. Coupled equations

The self-energy parts for the fermion and the boson propagators are defined to be

$$\Sigma(x, y) = \frac{-i}{\hbar} \frac{\delta \Gamma_{2p}}{\delta G(y, x)}, \quad (7.17a)$$

$$\Pi(x, y) = \frac{2i}{\hbar} \frac{\delta \Gamma_{2p}}{\delta \Delta(y, x)}. \quad (7.17b)$$

Hereafter in section 7 we restore the Planck constant \hbar in formulas to show explicitly the order of magnitude.

The equations for the order parameter $Q_c(x)$ and the second-order CTPGF $\Delta(x, y)$ and $G(x, y)$ for a physical system can be obtained from eqs. (7.12) by switching off the external sources. We have thus:

$$\frac{\delta \Gamma_p}{\delta Q_c(x)} = \frac{\delta \bar{I}[Q_c]}{\delta Q_c(x)} - i\hbar \text{Tr} \left\{ \frac{\delta G_0^{-1}}{\delta Q_c(x)} G \right\} + \frac{\delta \Gamma_{2p}}{\delta Q_c(x)} = 0, \quad (7.18)$$

$$\frac{2i}{\hbar} \frac{\delta \Gamma_p}{\delta \Delta(y, x)} = \Delta^{-1}(x, y) - \Delta_0^{-1}(x, y) + \Pi(x, y) = 0, \quad (7.19)$$

$$\frac{-i}{\hbar} \frac{\delta \Gamma_p}{\delta G(y, x)} = G^{-1}(x, y) - G_0^{-1}(x, y) + \Sigma(x, y) = 0. \quad (7.20)$$

Rewritten in the ordinary time variable in accord with the rule set in section 2.3, eq. (7.18) becomes the generalized TDGL equation for the order parameter, whereas (7.19) and (7.20) are the Dyson equations for the retarded and advanced Green functions along with the transport equation for the quasiparticle distribution. Equation (7.18) can be rewritten in a symmetric form as

$$\frac{1}{2} \xi_\sigma \frac{\delta \Gamma_p}{\delta Q_c(x_\sigma)} = \frac{1}{2} \xi_\sigma \left\{ \frac{\delta \bar{I}[Q]}{\delta Q_c(x_\sigma)} - i\hbar \text{Tr} \left(\frac{\delta G_0^{-1}}{\delta Q(x_\sigma)} G \right) + \frac{\delta \Gamma_{2p}}{\delta Q(x_\sigma)} \right\} = 0. \quad (7.18')$$

The retarded, advanced and correlation Green functions are related to the matrix \hat{A} as

$$A_r = \frac{1}{2} \xi^\dagger \hat{A} \eta, \quad A_a = \frac{1}{2} \eta^\dagger \hat{A} \xi, \quad A_c = \frac{1}{2} \xi^\dagger \hat{A} \xi, \quad (7.21)$$

in accord with (2.12). Therefore, the Dyson equations for the retarded propagators take the form

$$\int d^{d+1}y [\Delta_{0r}^{-1}(x, y) - \Pi_r(x, y)] \Delta_r(y, z) = \delta^{d+1}(x - z), \quad (7.22a)$$

$$\int d^{d+1}y [G_{0r}^{-1}(x, y) - \Sigma_r(x, y)] G_r(y, z) = \delta^{d+1}(x - z). \quad (7.22b)$$

The corresponding equations for the advanced functions can be obtained by taking the Hermitian conjugation of (7.22), whereas equations for correlation functions appear in the matrix form as

$$\Delta_c = -\Delta_r(\Delta_{0c}^{-1} - \Pi_c)\Delta_a, \quad (7.23)$$

$$G_c = -G_r(G_{0c}^{-1} - \Sigma_c)G_a. \quad (7.24)$$

As shown in section 3.3, the latter equations reduce to transport equations for the quasiparticle distribution.

7.2.3. Summary

To sum up, we have derived seven equations to determine seven functions $Q_c(x)$, Δ_r , Δ_a , Δ_c , G_r , G_a and G_c , from which the order parameter as well as the energy spectrum, the dissipation and the distribution function for the corresponding quasiparticles can be calculated.

Up to now we have not yet made any approximations. As is well known [97], the loop expansion is actually a series expansion in \hbar . Therefore, for systems which can be described by quasiclassical approximation one needs only the first few terms of this expansion. In fact, one recovers the mean field result if the contribution from Γ_{2p} is neglected altogether. In some other cases like in the theory of critical phenomena, one needs to partially resum the most divergent diagrams.

For most cases of practical interest including thermoequilibrium, the initial correlations expressed in terms of W_p^N are Gaussian. As shown in section 6.1, in such cases the statistical information can be included in the free propagators Δ_0 , S_0 by FDT, so that W_p^N term drops out from the effective action. Hence the analogy with the quantum field theory can be carried through even further for such systems.

As seen from the derivation, this calculation scheme can be applied to both equilibrium and nonequilibrium systems. It is particularly useful when the dynamical coupling between the order parameter and the elementary excitations is essential. We note in passing that the logical simplicity of the present formalism comes partly from introducing the two-point sources $M(x, y)$, $K(x, y)$ and performing Legendre transformation with respect to them.

7.2.4. Comparison with earlier formalism

To make contact with the generating functional introduced before (marked by a prime), we note that

$$Z'_p[h] = Z_p[h, M, K]|_{M=K=0}, \quad (7.25a)$$

$$W'_p[h] = W_p[h, M, K]|_{M=K=0}, \quad (7.25b)$$

$$\Gamma'_p[Q_c] = \Gamma_p[Q_c, \Delta, G]|_{\delta\Gamma_p/\delta\Delta = \delta\Gamma_p/\delta G = 0}, \quad (7.25c)$$

$$\frac{\delta\Gamma'_p[Q_c]}{\delta Q_c(x)} = \left[\frac{\delta\Gamma_p[Q_c, \Delta, G]}{\delta Q_c(x)} \Big|_{\Delta, G} \right] \Big|_{\delta\Gamma_p/\delta\Delta = \delta\Gamma_p/\delta G = 0}. \quad (7.25d)$$

Previously, an effective action method was introduced by us in the third paper of reference [46] to calculate Γ'_p explicitly. The disadvantage of that technique compared with the present formalism lies in the difficulties connected with fermion renormalization when the fermion degrees of freedom were integrated out at the very beginning.

7.2.5. Applications

We have already applied the present formalism to study the weak electromagnetic field in super-

conduction [53] as well as the nonequilibrium superconductivity in general [54], the dynamical behaviour of quenched random systems and a long-ranged spin glass model in particular [55], the quantum fluctuations in the quasi-one-dimensional conductors [57] and the exchange correction in systems with four-fermion interaction [49]. The last topic will be discussed in the following two sections, whereas the random systems are considered in section 8.

7.3. Generalization of Bogoliubov–de Gennes (BdeG) equation

As mentioned in the introductory remarks to this section, the self-consistent equations for the order parameter $Q_c(x)$ and the complete set of single-particle fermion wavefunctions $\psi_n(x)$ are known as the BdeG equations which are of Hartree-type without exchange effects being accounted for. There have been some attempts of extending these equations to include the correlation effects with limited success [96]. These authors emphasize the nonuniqueness of the HS [100] transformation and make use of it to derive various approximations. As mentioned in section 7.1, such ambiguity can be avoided by using the generating functional technique with given definition of the order parameter. As we will show in this section, the successive approximations can be derived in a systematic way using the formalism developed in the preceding two sections.

7.3.1. Model

The effective action of the system is given by

$$I[\psi^\dagger, \psi] = \int \psi_i^\dagger(x) \left(i\hbar \frac{\partial}{\partial t} - H_0 \right)_{ij} \psi_j((x)) - \frac{1}{2}g^2 \int \psi_i^\dagger(x) \psi_j^\dagger(y) \Delta_0(x-y) O_{ik} O_{jl} \psi_l(y) \psi_k(x), \quad (7.26)$$

where i, j are indices of internal degrees of freedom, O_{ik} matrix in this space and

$$\Delta_0(x-y) = \delta(t_x - t_y) V(x-y). \quad (7.27)$$

Using the fermion commutation relation and the matrix notation for O_{ik} , (7.26) can be rewritten as

$$I[\psi^\dagger, \psi] = \int \psi^\dagger(x) \left(i\hbar \frac{\partial}{\partial t} - H_0 + \frac{1}{2}g^2 \hat{O}^2 V(o) \right) \psi(x) - \frac{1}{2}g^2 \int \psi^\dagger(x) \hat{O} \psi(x) \Delta_0(x-y) \psi^\dagger(y) \hat{O} \psi(y). \quad (7.26')$$

The order parameter is defined as

$$Q(x) \equiv g \int \Delta_0(x-y) \psi^\dagger(y) \hat{O} \psi(y) = g \int V(x-y) \psi^\dagger(y, t) \hat{O} \psi(y, t). \quad (7.28)$$

7.3.2. Coupled equations

In accord with the result of the last section, up to the two-loop approximation, the coupled equations for the order parameter $Q_c(x)$ and the second-order CTPGF are the following:

$$-\frac{1}{2}\xi_\alpha \frac{\delta \Gamma_p[Q]}{\delta Q(x_\alpha)} \Big|_{Q_+ = Q_-} = - \int \Delta_0^{-1}(x-y) Q(y) - \frac{1}{2}i\hbar g \text{Sp}\{\hat{O} G_c(x, x)\} = 0, \quad (7.29)$$

$$G_p^{-1}(x, y) = G_{0p}^{-1}(x, y) - \Sigma_p(x, y), \quad (7.30)$$

with

$$\Sigma_p(x, y) = i\hbar g^2 \hat{O}G_p(x, y)\hat{O}\Delta_p(y, x), \quad (7.31)$$

$$\Delta_p^{-1}(x, y) = \Delta_{0p}^{-1}(x, y) - \Pi_p(x, y), \quad (7.32)$$

with

$$\Pi_p(x, y) = -i\hbar g^2 \text{Sp}(\hat{O}G_p(x, y)\hat{O}G_p(y, x)), \quad (7.33)$$

where Sp means taking trace over internal indices only, while Tr remains summing over both internal and space-time coordinates. Here

$$G_{0p}^{-1}(x, y) = S_{0p}^{-1}(x, y) - g\hat{O}Q(x)\delta_p^{d+1}(x - y), \quad (7.34)$$

with

$$S_{0p}^{-1}(x, y) = \left(i\hbar \frac{\partial}{\partial t} - H_0 + \frac{1}{2}g^2 V(\mathbf{o})\hat{O}^2 \right) \delta_p^{d+1}(x - y). \quad (7.35)$$

7.3.3. Spectral representation

For simplicity we consider the stationary states when $Q(x)$ does not depend on time and all Green's functions are time translationally invariant. Also, we assume the dissipation for both fermions and collective excitations to be small, so that they can be considered as quasiparticles to a good approximation. In particular, the Fourier transformed fermion functions can be expanded in terms of the complete set $\{\psi_n(x)\}$ as

$$G_r(p_0, x, y) = \sum_n \psi_n(x) \frac{1}{p_0 - E_n + i\gamma_n} \psi_n^\dagger(y), \quad (7.36a)$$

$$G_a(p_0, x, y) = \sum_n \psi_n(x) \frac{1}{p_0 - E_n - i\gamma_n} \psi_n^\dagger(y), \quad (7.36b)$$

$$G_c(p_0, x, y) = \sum_n \psi_n(x) (1 - 2N_n) \left(\frac{1}{p_0 - E_n + i\gamma_n} - \frac{1}{p_0 - E_n - i\gamma_n} \right) \psi_n^\dagger(y), \quad (7.36c)$$

while the spectral functions $\{\psi_n(x)\}$ satisfy the following equation

$$(E_n - i\gamma_n - H_0 + \frac{1}{2}g^2 \hat{O}^2 V(\mathbf{o}) - g\hat{O}Q(x))\psi_n(x) - \int \Sigma_r(p_0 = E_n, x, y)\psi_n(y) = 0. \quad (7.37)$$

In eqs. (7.36) and (7.37) E_n , γ_n and N_n are energy spectrum, dissipation and particle distribution respectively.

It is ready to check that in combination with FDT written as (cf. (4.29))

$$\Sigma_c(E_n, \mathbf{x}, \mathbf{y}) = \sum_n \psi_n(\mathbf{x}) 2i\gamma_n(1 - 2N_n) \psi_n^\dagger(\mathbf{y}). \quad (7.38)$$

Equations (7.36) and (7.37) are just equivalent to the Dyson equation (7.30) in the limit of weak dissipation. The orthonormalization and the completeness of the $\{\psi_n(\mathbf{x})\}$ set is then obvious.

7.3.4. Hartree approximation

As the first approximation, we neglect the self-energy part altogether, i.e., to set $\Sigma_p(x, y) \rightarrow 0$. It then follows that $\gamma_n \rightarrow 0+$ and the boson propagator Δ will not depend on the order parameter. Substituting (7.36c) into (7.29) and setting $\Sigma_r = 0$ in (7.37), we obtain the well-known BdeG equation

$$\begin{aligned} \int V^{-1}(\mathbf{x} - \mathbf{y}) Q(\mathbf{y}) &= g \sum_n (N_n - \frac{1}{2}) \psi_n^\dagger(\mathbf{x}) \hat{O} \psi_n(\mathbf{x}) \\ &= g \sum_n N_n \psi_n^\dagger(\mathbf{x}) \hat{O} \psi_n(\mathbf{x}) - \frac{1}{2} g \operatorname{Sp}(\hat{O}) \delta^d(\mathbf{x}), \end{aligned} \quad (7.39)$$

$$(E_n - H_0 + \frac{1}{2} g^2 \hat{O}^2 V(\mathbf{o}) - g \hat{O} Q(\mathbf{x})) \psi_n(\mathbf{x}) = 0. \quad (7.40)$$

An interesting and important feature of these equations is that the energy spectrum $\{E_n\}$ should be determined self-consistently with the quasiparticle distribution.

7.3.5. Hartree-Fock (HF) approximation

As the second approximation, we keep $\Sigma_p(x, y)$ but set $\Pi_p(x, y)$ equal to zero. It follows then from (7.31) and (7.32) that

$$\Delta_r(\mathbf{x}, \mathbf{y}) = \Delta_a(\mathbf{x}, \mathbf{y}) = \delta(t_x - t_y) \cdot V(\mathbf{x} - \mathbf{y}), \quad \Delta_c(\mathbf{x}, \mathbf{y}) = 0, \quad (7.41)$$

$$\Sigma_r(p_0, \mathbf{x}, \mathbf{y}) = \frac{ig^2\hbar}{2} \int \frac{dq_0}{2\pi\hbar} \hat{O} G_c(q_0, \mathbf{x}, \mathbf{y}) \hat{O} V(\mathbf{x} - \mathbf{y}), \quad \Sigma_c(\mathbf{x}, \mathbf{y}) = 0, \quad \gamma_n \rightarrow 0+. \quad (7.42)$$

Repeating the procedure carried out in the Hartree case, we find that in the HF approximation the equation for the order parameter $Q(\mathbf{x})$ remains the same as (7.39), but the equation for the spectral functions changes into

$$(E_n - H_0 - g \hat{O} Q(\mathbf{x})) \psi_n(\mathbf{x}) + g^2 \int V(\mathbf{x} - \mathbf{y}) \sum_m \hat{O} \psi_m(\mathbf{x}) N_m \psi_m^\dagger(\mathbf{y}) \hat{O} \psi_n(\mathbf{y}) = 0. \quad (7.43)$$

Comparing (7.43) with (7.40) we find that the unphysical term $\frac{1}{2} g^2 \hat{O}^2 V(\mathbf{o})$ has been cancelled out and a new, Fock exchange term appears here. It is clear from our derivation that the order parameter $Q(\mathbf{x})$ is coupled to the fermion Green functions in the order of \hbar (see (7.29)). Hence one should take into account contributions of the same order from Σ_r in (7.37) no matter what kind of interaction one deals

with. We see thus how the Fock term has been “lost” in some derivations, but “recovered” in the CTPGF approach.

7.3.6. Higher order corrections

We can improve our approximation by keeping also the self-energy part $\Pi_p(x, y)$ in eqs. (7.29)–(7.33). This is the so-called random phase approximation (RPA) if only the leading term in Π is kept. It is straightforward to check that the equation for the order parameter still remains the same as (7.39) but the equation for the spectral function is now coupled to the collective excitation via the fermion self-energy part Σ_p provided the dissipation is still weak. We come back to this approximation in the next section where the free energy is calculated.

As is clear from the presentation, we can in principle continue this systematic process to go on higher order corrections, but we will not elaborate further on them here.

7.3.7. Vacuum fluctuation

Before closing this section a remark on the term $\frac{1}{2}g \text{Sp}(\hat{O})\delta^4(x)$ in (7.39) is in order. For systems with off-diagonal long-range order like superconductivity this term drops out because

$$\text{Sp}(\hat{O}) = 0. \quad (7.44)$$

For systems with diagonal long-range order this term will cancel out the divergent contribution of the Fermi sea. By using the particle-hole symmetry the right-hand side of (7.39) can be rewritten as

$$\sum_n (N_n - \frac{1}{2})\psi_n^\dagger(x)\hat{O}\psi_n(x) = \sum_n N_n^p \psi_n^{p\dagger}(x)\hat{O}\psi_n^p(x) - \sum_n N_n^h \psi_n^{h\dagger}(x)\hat{O}\psi_n^h(x), \quad (7.45)$$

where the first sum is carried out over states above the Fermi level, while the second – below it. The rest of the notation is standard. In the next section we will assume for simplicity that eq. (7.44) is fulfilled.

7.4. Calculation of free energy

7.4.1. Functional equation

As shown in section 4.2, for systems respecting time reversal symmetry, the potential condition is satisfied and the free-energy functional $\mathcal{F}[Q(x)]$ can be defined in accord with (4.23a). Using (7.29) and (7.36c) this equation can be rewritten as

$$\frac{\delta \mathcal{F}[Q]}{\delta Q(x)} = - \int V^{-1}(x - y)Q(y) + g \sum_n (N_n - \frac{1}{2})\psi_n^\dagger(x)\hat{O}\psi_n(x), \quad (7.46)$$

provided the dissipation γ_n is neglected. In the same approximation it follows from (7.37) that

$$\frac{\delta E_n}{\delta Q(x)} = g\psi_n^\dagger(x)\hat{O}\psi_n(x) + \int \psi_n^\dagger(y) \frac{\delta \Sigma_r(E_n, y, z)}{\delta Q(x)} \psi_n(z). \quad (7.47)$$

According to the convention set at the end of the last section, i.e., eq. (7.44) is fulfilled, (7.47) yields

$$\frac{\delta}{\delta Q(x)} \sum_n E_n - \int \sum_n \psi_n^\dagger(y) \frac{\delta \Sigma_r(E_n, y, z)}{\delta Q(x)} \psi_n(z) = 0. \quad (7.48)$$

Insert (7.47) into (7.46), we obtain

$$\begin{aligned} \frac{\delta \mathcal{F}[Q]}{\delta Q(x)} = & - \int V^{-1}(x - y) Q(y) + \sum_n (N_n - \frac{1}{2}) \frac{\delta E_n}{\delta Q(x)} \\ & - \sum_n \int \text{Sp} \left\{ \frac{\delta \Sigma_r(E_n, y, z)}{\delta Q(x)} \psi_n(z) (N_n - \frac{1}{2}) \psi_n^\dagger(y) \right\}, \end{aligned} \quad (7.49)$$

which can be transformed as

$$\begin{aligned} \frac{\delta \mathcal{F}[Q(x)]}{\delta Q(x)} = & - \int V^{-1}(x - y) Q(y) + \sum_n (N_n - \frac{1}{2}) \frac{\delta E_n}{\delta Q(x)} \\ & + \frac{i\hbar}{2} \frac{1}{T} \int d^{d+1}y d^{d+1}z \text{Sp} \left\{ \frac{\delta \Sigma_r(y, z)}{\delta Q(x)} G_c(z, y) \right\} \end{aligned} \quad (7.50)$$

by use of (7.36c) with

$$T \equiv \int_{-T/2}^{T/2} dt.$$

We would like to emphasize here that the functional equation (7.49) or (7.50) for the free energy is valid for both equilibrium and nonequilibrium systems provided the potential condition is satisfied.

7.4.2. Hartree case

In the Hartree approximation when Σ_p is neglected altogether, we find immediately

$$\frac{\delta \mathcal{F}[Q]}{\delta Q(x)} = \frac{\delta}{\delta Q(x)} \left[-\frac{1}{2} \int Q(y) V^{-1}(y - z) Q(z) \right] + \sum_n N_n \frac{\delta E_n}{\delta Q(x)} \quad (7.51)$$

with eq. (7.48) being accounted for. For systems in the thermoequilibrium N_n is the Fermi distribution

$$N_n = (\exp[\beta(E_n - \mu)] + 1)^{-1}, \quad (7.52)$$

the free energy is then given by

$$\mathcal{F}[Q] = -\frac{1}{2} \int d^d y d^d z Q(y) V^{-1}(y - z) Q(z) - \beta^{-1} \sum_n \ln(1 + \exp[-\beta(E_n - \mu)]). \quad (7.53)$$

Taking into account (7.39) and (7.40), the first term in (7.53) can be rewritten as

$$-\frac{1}{2}g^2 \sum_n \sum_m \int N_n \psi_n^\dagger(y) \hat{O} \psi_n(y) V(y-z) N_m \psi_m^\dagger(z) \hat{O} \psi_m(z),$$

which obviously is the fermion interaction energy in the Hartree approximation.

7.4.3. Hartree–Fock case

In the Hartree–Fock approximation we can proceed exactly the same way as in the previous case and end up with an expression for the free energy containing an extra term

$$\mathcal{F}^{\text{HF}} - \mathcal{F}^{\text{H}} = \frac{g^2}{2} \int \sum_{n,m} N_n \psi_n^\dagger(y) \hat{O} \psi_m(y) V(y-z) N_m \psi_m^\dagger(z) \hat{O} \psi_n(z), \quad (7.54)$$

in addition to that given by (7.53). The physical meaning of (7.54) as exchange interaction between quasiparticles is obvious.

7.4.4. Random phase approximation (RPA)

The RPA case is more complicated. Let us decompose the fermion self-energy as

$$\Sigma_p(x, y) = \Sigma_p^{(1)}(x, y) + \Sigma_p^{(2)}(x, y), \quad (7.55)$$

where

$$\Sigma_p^{(1)}(x, y) = i\hbar g^2 \hat{O} G_p(x, y) \hat{O} \Delta_{0p}(y, x), \quad (7.56)$$

$$\Sigma_p^{(2)}(x, y) = i\hbar g^2 \hat{O} G_p(x, y) \hat{O} (\Delta_p(y, x) - \Delta_{0p}(y, x)). \quad (7.57)$$

Correspondingly eqs. (7.48) and (7.50) are rewritten as

$$\frac{\delta}{\delta Q(x)} \left(\sum_n E_n \right) - \frac{1}{2} \int \sum_n \psi_n^\dagger \frac{\delta \Sigma_r^{(1)}}{\delta Q} \psi_n - \frac{1}{2} \int \sum_n \psi_n^\dagger \frac{\delta \Sigma_r^{(2)}}{\delta Q} \psi_n = 0, \quad (7.58)$$

$$\begin{aligned} \frac{\delta \mathcal{F}[Q]}{\delta Q(x)} = & - \int V^{-1} Q + \sum_n \left(N_n - \frac{1}{2} \right) \frac{\delta E_n}{\delta Q} + \frac{i\hbar}{2} \frac{1}{T} \int \text{Sp} \left(\frac{\delta \Sigma_r^{(1)}}{\delta Q} G_c \right) \\ & - \frac{i\hbar}{2} \frac{1}{T} \int \text{Sp} \left(\Sigma_r^{(2)} \frac{\delta G_c}{\delta Q} \right) + \frac{i\hbar}{2} \frac{1}{T} \frac{\delta}{\delta Q} \int \text{Sp} (\Sigma_r^{(2)} G_c), \end{aligned} \quad (7.59)$$

where the abbreviated notation is used. The second term on the left of (7.58) and the second term on the right of (7.59) can be calculated by the same procedure as previously, whereas the third term of (7.58) and the last term of (7.59) turn out to be of higher order [49].

To calculate the contribution from the fourth term of (7.59) we make use of the identity

$$\begin{aligned} & \int \left(\Sigma_r^{(2)}(y, z) \frac{\delta G_c(z, y)}{\delta Q(x)} + \Sigma_c^{(2)}(y, z) \frac{\delta G_a(z, y)}{\delta Q(x)} \right) \\ & = -\frac{1}{2} \int \left(\frac{\delta \Pi_r(y, z)}{\delta Q(x)} (\Delta_c(z, y) - \Delta_{0c}(z, y)) + \frac{\delta \Pi_c(y, z)}{\delta Q(x)} (\Delta_a(z, y) - \Delta_{0a}(z, y)) \right), \end{aligned} \quad (7.60)$$

which can be proved using the ξ, η vectors and the definition of the self-energy parts. Neglecting terms like Σ_c and Π_c which are proportional to the dissipation, taking into account eqs. (7.32) and (7.38) as well as the time translational invariance, the fourth term of (7.59) can be presented as

$$\begin{aligned} M &= -\frac{i\hbar}{2} \frac{1}{T} \int \text{Sp} \left(\Sigma_r^{(2)}(y, z) \frac{\delta G_c(z, y)}{\delta Q(x)} \right) \\ &= -\frac{i\hbar}{4} \int \frac{dq_0}{2\pi\hbar} \int d^d y \, d^d z \frac{\delta \Delta_r^{-1}(q_0, y, z)}{\delta Q(x)} \Delta_c(q_0, z, y). \end{aligned} \quad (7.61)$$

To extract the information about the collective excitations we use the approximate spectral representation for Δ as

$$\begin{aligned} \Delta_r(q_0, y, z) &= \sum_s Z_s^{-1} \phi_s(y) \left[\frac{(q_0 + i\eta)^2}{\hbar^2 c^2} - \frac{\omega_s^2}{c^2} \right]^{-1} \phi_s(z), \\ \Delta_c(q_0, y, z) &= \sum_s Z_s^{-1} \phi_s(y) (1 + 2N(q_0)) \left[\left(\frac{(q_0 + i\eta)^2}{\hbar^2 c^2} - \frac{\omega_s^2}{c^2} \right)^{-1} - \left(\frac{(q_0 - i\eta)^2}{\hbar^2 c^2} - \frac{\omega_s^2}{c^2} \right)^{-1} \right] \phi_s(z), \end{aligned} \quad (7.62)$$

where ω_s , η , Z_s are, respectively, the frequency, dissipation and wavefunction renormalization for the quasiparticle. Making use of (7.62), the expression (7.61) can be presented as

$$M = \sum_s \frac{\delta(\hbar\omega_s)}{\delta Q} (N_s + \frac{1}{2}), \quad (7.63)$$

which is the extra contribution due to the collective excitation. In thermoequilibrium,

$$N_s = [\exp(\beta\hbar\omega_s) - 1]^{-1}, \quad (7.64)$$

the additional term in the free energy for the RPA compared with that of HF is given by

$$\mathcal{F}^{\text{RPA}} - \mathcal{F}^{\text{HF}} = \beta^{-1} \sum_s \ln[1 - \exp(-\beta\hbar\omega_s)] + \frac{1}{2} \sum_s \hbar\omega_s, \quad (7.65)$$

where the last term is the contribution from the zero point motion.

7.4.5. Summary

To summarize we note that the calculation scheme presented in this section is rather practical and complete. We cannot only determine the order parameter as well as the energy spectrum, the dissipation and the quasiparticle distribution for both fermion and collective excitation, but also calculate explicitly the free-energy functional in successive approximations without invoking any extra assumptions. It is also interesting to note that for systems with symmetry breaking we can directly integrate out the vertex equation without introducing the symmetry broken ground state in advance. This technique may be helpful for more complicated problems.

8. Quenched random systems

In quenched random systems some degrees of freedom associated with impurities are frozen into a nonequilibrium but random configuration. Such situation can be created, say, by sudden cooling of a sample in thermal equilibrium to a state with much lower temperature. The impurities are thus frozen into a configuration separated by high potential barriers from the equilibrium one. Diffusion through these barriers will cause the nonequilibrium state to vary slowly in time.

As pointed out by Brout [101], the space average of an observable A in a quenched random system can be replaced by an ensemble average over the impurity degrees of freedom J .

$$\bar{A} = \int [dJ] A(J) P(J), \quad (8.1)$$

where $P(J)$ is the distribution function. Most of previous workers considered quenched random systems as if they were static [102]. In this approach one has to average the free energy proportional to the logarithm of the partition function, over the random configuration of impurities. It is a formidable task and a special n -replica trick has been invented. This method is extensively used to study random systems like spin glass [103–110].

Recently, several authors [111–120] have proposed dynamical theories of spin glass based on the MSR [90] statistical field theory. The advantage of the dynamical theory is the possibility of taking quenched average without resorting to the unphysical replica trick. The results obtained so far can be reproduced in most cases by the replica method with special pattern of replica symmetry breaking [121]. Although a kind of plausible physical interpretation of replica has been suggested very recently [122–124], as far as we understand the whole problem is still controversial. Therefore, a systematic dynamical theory is certainly needed to provide a proper description corresponding to the real experiments.

In this section we will show that the CTPGF formalism might be one of the candidates to provide such a dynamical description. As we have seen already, this formalism is suitable for studying slowly varying in time processes, so far as the causality and the FDT are built in the formalism itself. In the CTPGF approach the quenched average can be carried out over the generating functional itself and the counterpart of the Edwards–Anderson(EA) [103] order parameter q appears naturally as an integral part of the second-order CTPGF. As a consequence, a Dyson equation can be derived for q to describe its slow variation using the quasiclassical approximation.

In section 8.1 we outline the basic features of the CTPGF dynamic theory for quenched systems, whereas in section 8.2 we apply it to discuss the infinite-ranged Ising spin glass, i.e. the Sherrington–Kirkpatrick (SK) model [104]. A boundary line of stability is found on the plane $q - |\chi|$, where χ is the susceptibility. It is argued that the spin glass phase is characterized by the fixed point located at the stability boundary. The magnetization is calculated in perturbation and is found to be in good agreement with those predicted by the projection hypothesis [125, 126].

In section 8.3, we discuss the disordered electron system within the CTPGF framework. The WT identities following from the symplectic symmetry $Sp(2)$ respected by the Lagrangian of the system as well as the localization properties are considered without resorting to the replica trick.

8.1. Dynamic formulation

8.1.1. Model

Suppose the action of the random system is given by

$$I = \int_p \sigma(x) \Gamma_p^{(0,0)}(x, y) \sigma(y) - \int_p V(\sigma(x), J_i) + \int_p (\sigma(x)h(x) + \sigma(x)j(x)) + I_h, \quad (8.2)$$

where $h(x)$ is the external field coupled to the dynamical variable $\sigma(x)$, $J_i(x)$ being random variables with given distribution. The $\sigma(x)j(x)$ term represents the interaction of the dynamical variable with heat bath described by I_h . $\sigma(x)$ may have one or several components.

After integrating over the reservoir degrees of freedom which might be considered as a set of harmonic oscillators, we obtain the effective action

$$I_{\text{eff}} = \int_p \sigma(x) \Gamma_p^{(0)}(x, y) \sigma(y) - \int_p V(\sigma(x), J_i) + \int_p \sigma(x)h(x), \quad (8.3)$$

where

$$\Gamma_p^{(0)}(x, y) = \Gamma_p^{(0,0)}(x, y) - \Sigma_p^{(0)}(x, y), \quad (8.4)$$

with the self-energy part $\Sigma_p^{(0)}$ contributed by the interaction term $\sigma(x)j(x)$. Suppose the system is prepared by sudden cooling down to temperature β^{-1} at moment t_0 , then $\Gamma_p^{(0)}$ satisfies the FDT given by (3.42a).

8.1.2. Averaged generating functional

The generating functional averaged over the random distribution of J_i is

$$\bar{Z}[h(x)] = \int [dJ] P(J) Z[h(x), J] = \langle Z[h(x), J] \rangle_J, \quad (8.5)$$

where

$$Z[h(x), J] = \int [d\sigma] \exp(iI_{\text{eff}}) \langle t_+ = t_0 | \hat{\rho} | t_- = t_0 \rangle. \quad (8.6)$$

Introducing the generating functional for the connected Green functions as usual

$$\bar{W}[h(x)] = -i \ln \bar{Z}[h(x)], \quad (8.7)$$

$$W[h(x), J] = -i \ln Z[h(x), J], \quad (8.8)$$

we obtain the average of the field variable

$$\bar{\sigma}(x) = \delta \bar{W} / \delta h(x), \quad (8.9)$$

and the connected CTPGF as

$$\overline{G_p}(1 \cdots n) = (-1)^{n-1} \frac{\delta^n \bar{W}}{\delta h(1) \cdots \delta h(n)}, \quad (8.10)$$

along with the corresponding terms for $W[h(x), J]$. It follows from (8.5) that

$$\bar{Z}\bar{\sigma}(x) = \{Z\sigma(x, J)\}_J,$$

which by virtue of the normalization condition (2.102) reduces to

$$\bar{\sigma}(x) = \langle \sigma(x, J) \rangle_J \quad (8.11)$$

for the physical limit $h_+(x) = h_-(x)$. Equation (8.11) is what is required for the quenched average. Hereafter we denote all quantities derived from the averaged generating functional by a bar above them, whereas expectation values directly averaged over the ensemble $P(J)$ are presented as $\langle \cdot \cdot \cdot \rangle_J$.

8.1.3. Order parameter

Differentiating (8.10) with respect to $h(y)$ and taking the physical limit, we obtain

$$\langle G_p(x, y; J) \rangle_J = \bar{G}_p(x, y) + iq(x, y), \quad (8.12)$$

where

$$q(x, y) = \langle \sigma(x, J) \sigma(y, J) \rangle_J - \bar{\sigma}(x) \bar{\sigma}(y). \quad (8.13)$$

The expectation value $\sigma(x, J)$ for an Hermitian operator σ is a real function with equal values on the two-time branches. Hence the matrix is real, symmetric and independent of time branch, i.e.

$$q(x, y) = q(y, x) = q^*(x, y), \quad (8.14)$$

$$q(x+, y+) = q(x+, y-) = q(x-, y+) = q(x-, y-). \quad (8.15)$$

Edwards and Anderson [103] have introduced the following order parameter

$$q_{EA} = \lim_{t \rightarrow \infty} \langle \sigma(0, J) \sigma(t, J) \rangle$$

which looks similar to what is defined by (8.13).

It follows from (8.12) that

$$\langle G_r(x, y; J) \rangle_J = \overline{G_r}(x, y), \quad (8.16a)$$

$$\langle G_a(x, y; J) \rangle_J = \overline{G_a}(x, y), \quad (8.16b)$$

$$\langle G_c(x, y; J) \rangle_J = \overline{G_c}(x, y) + 2iq(x, y). \quad (8.16c)$$

It is easy to check that (8.16a) and (8.16b) are also true for higher order retarded and advanced functions, i.e.,

$$\langle G_{21\dots 1}(1 \dots n, J) \rangle_J = \bar{G}_{21\dots 1}(1 \dots N),$$

etc. The appearance of the matrix $q(x, y)$ is a consequence of quenched average over the random variable J_i . Hence it characterizes the behaviour of the quenched random system.

8.1.4. Free energy

After a sufficiently long time the system is expected to reach a steady, yet not necessarily equilibrium state, when the expectation value $\sigma(x, J)$ is no longer time dependent. As follows from the discussion of section 4.2, there exists a free-energy functional such that

$$\sigma(h, J) = -\delta \mathcal{F}[h(x), J] / \delta h(x), \quad (8.17)$$

if

$$\text{Im} \int G_r(\frac{1}{2}t, -\frac{1}{2}t, J) dt = 0. \quad (8.18)$$

In accord with (8.16a), the same is true for the function with bar, i.e.,

$$\bar{\sigma}(h) = -\delta \bar{\mathcal{F}} / \delta h(x). \quad (8.19)$$

From (8.17) and (8.19) we find that

$$\delta \bar{\mathcal{F}} / \delta h = \langle \delta \mathcal{F} / \delta h \rangle_J. \quad (8.20)$$

For a smooth function of distribution $P(J)$ with finite moments the order of differentiation and averaging can be changed. Integrating (8.20) yields

$$\bar{\mathcal{F}}[h] = \langle \mathcal{F}[h, J] \rangle_J + \text{terms independent of } h.$$

So far as our formalism is flexible enough to incorporate various composite operators, in particular, the energy density as a conjugate variable of temperature. This way we can exhaust all variables in the free energy and get the equality

$$\bar{\mathcal{F}}[h] = \langle \mathcal{F}[h, J] \rangle_J, \quad (8.21)$$

up to an unimportant constant.

We see, therefore, the averaged expectation values of physical variables like magnetic moment, free

energy, etc. can be calculated directly from the averaged generating functional $\bar{Z}[h, J]$. Since the technique of deriving various consequences from a well defined $\bar{Z}[h, J]$ is highly developed, the predictions of the dynamic theory are unambiguous.

8.1.5. FDT and Fischer law

After performing Fourier transformation with respect to the relative coordinates $x-y$, the FDT for G before average is written as (cf. eq. (3.43a))

$$G_c(k, X, J) = 2i \coth(\beta k_0/2) \operatorname{Im} G_r(k, X, J), \quad (8.22)$$

whereas for the quench-averaged \bar{G} we have

$$\begin{aligned} \bar{G}_c(k, X) &= 2i \coth(\beta k_0/2) \operatorname{Im} \bar{G}_r(k, X) - 2iq(k, X) \\ &\approx \frac{4i}{\beta k_0} \operatorname{Im} \bar{G}_r(k, X) - 2iq(k, X). \end{aligned} \quad (8.23)$$

The retarded function G_r is analytic in the upper half-plane of k_0 . If $G_r(k, X)$ vanishes as $k_0 \rightarrow \infty$, the real and imaginary parts of G_r satisfy an unsubtracted dispersion relation

$$\operatorname{Re} \bar{G}_r(k_0, \mathbf{k}, X) = \frac{1}{\pi} \int \frac{\operatorname{Im} \bar{G}_r(k'_0, \mathbf{k}, X)}{k'_0 - k_0} dk'_0. \quad (8.24)$$

Making use of (8.23) we find in the high temperature limit

$$\begin{aligned} \operatorname{Re} \bar{G}_r(k_0 = 0, \mathbf{k}, X) &= \bar{G}_r(k_0 = 0, \mathbf{k}, X) \\ &= -\beta \int \frac{dk_0}{2\pi} (i\bar{G}_c(k_0, \mathbf{k}, X) - q(k_0, \mathbf{k}, X)). \end{aligned} \quad (8.25)$$

For a long-ranged Ising spin model when the space dependence of Green's function can be neglected (8.25) becomes Fischer's relation [127]

$$\chi = -\bar{G}_r(k_0 = 0, t) = \beta(1 - q(t, t) - \bar{\sigma}^2(t)). \quad (8.26)$$

We see thus that the validity of Fischer relation depends crucially on the high-frequency behaviour of the retarded Green function as well as the FDT.

8.1.6. Dynamic equation for q

Now we derive the dynamic equation satisfied by the matrix q . The Dyson equation for the quench-averaged function can be written as (cf. (3.10))

$$\bar{\Gamma}_r \bar{G}_r = 1, \quad (8.27a)$$

$$\bar{\Gamma}_a \bar{G}_a = 1, \quad (8.27b)$$

$$\bar{\Gamma}_r \bar{G}_c = -\bar{\Gamma}_c \bar{G}_a. \quad (8.27c)$$

Introducing a new matrix

$$Q \equiv i[\frac{1}{2}\bar{\Gamma}_c - i \coth(\beta k_0/2) \operatorname{Im} \bar{\Gamma}_r], \quad (8.28)$$

we directly find the Dyson equation for q ,

$$\bar{\Gamma}_r q = -Q \bar{G}_a. \quad (8.29)$$

The Hermitian conjugation of eq. (8.29) is given by

$$q \bar{\Gamma}_a = -\bar{G}_r Q. \quad (8.30)$$

Separating the Hermitian and anti-Hermitian parts of (8.29) and (8.30) we obtain

$$\bar{\Gamma}_r q + q \bar{\Gamma}_a = -Q \bar{G}_a - \bar{G}_r Q, \quad (8.31)$$

$$\bar{\Gamma}_r q - q \bar{\Gamma}_a = -Q \bar{G}_a + \bar{G}_r Q. \quad (8.32)$$

In the quasiclassical approximation we replace the product of two matrices A and B ,

$$AB = (AB + BA)/2 + (AB - BA)/2,$$

by the classical expression

$$\tilde{A}\tilde{B} - \frac{i}{2}\{\tilde{A}, \tilde{B}\}_{\text{P.B.}} \equiv \tilde{A}\tilde{B} - \frac{i}{2}\left(\frac{\partial \tilde{A}}{\partial k^\mu} \frac{\partial \tilde{B}}{\partial X_\mu} - \frac{\partial \tilde{A}}{\partial X_\mu} \frac{\partial \tilde{B}}{\partial k^\mu}\right), \quad (8.33)$$

where \tilde{A}, \tilde{B} are Fourier transforms. As seen from the discussion in section 3.3, this approximation is controlled by the inequality

$$\left| \frac{1}{\tilde{O}} \frac{\partial^2 \tilde{O}}{\partial k^\mu \partial X_\mu} \right| \ll 1, \quad (8.34)$$

where \tilde{O} may be either \tilde{A} or \tilde{B} . In this approximation eqs. (8.31) and (8.32) become (with “~” dropped)

$$\begin{aligned} q + Q|\bar{G}_r|^2 &= \frac{1}{2|\bar{G}_r|^2} \left\{ \left(\frac{\partial q}{\partial X_\mu} - |\bar{G}_r|^2 \frac{\partial Q}{\partial X_\mu} \right) \left(\operatorname{Im} \bar{G}_r \frac{\partial \operatorname{Re} \bar{G}_r}{\partial k^\mu} - \operatorname{Re} \bar{G}_r \frac{\partial \operatorname{Im} \bar{G}_r}{\partial k^\mu} \right) \right. \\ &\quad \left. - \left(\frac{\partial q}{\partial k^\mu} - |\bar{G}_r|^2 \frac{\partial Q}{\partial k^\mu} \right) \left(\operatorname{Im} \bar{G}_r \frac{\partial \operatorname{Re} \bar{G}_r}{\partial X_\mu} - \operatorname{Re} \bar{G}_r \frac{\partial \operatorname{Im} \bar{G}_r}{\partial X_\mu} \right) \right\}, \end{aligned} \quad (8.35)$$

$$\frac{\partial |\bar{G}_r|^2}{\partial k^\mu} \left(\frac{\partial q}{\partial X_\mu} - |\bar{G}_r|^2 \frac{\partial Q}{\partial X_\mu} \right) = \frac{\partial |\bar{G}_r|^2}{\partial X_\mu} \left(\frac{\partial q}{\partial k^\mu} - |\bar{G}_r|^2 \frac{\partial Q}{\partial k^\mu} \right). \quad (8.36)$$

Equations (8.35) and (8.36) are transport equations to determine the time evolution of q . They are written here for single dynamic variable. It can be easily, though tediously extended to the multi-component case.

For a homogeneous system in steady state all functions do not depend on the macro-space-time X , then eqs. (8.35) and (8.36) reduce to a single equation

$$q = Q|\bar{G}_r|^2. \quad (8.37)$$

Using the field theoretical technique Q and $|\bar{G}_r|^2$ can be calculated once the Lagrangian is specified. They are functions of q . In some cases

$$Q = \lambda q \quad (8.38)$$

in the first order of perturbation. A nontrivial solution with $q \neq 0$ exists if the condition

$$\lambda|\bar{G}_r|^2 = 1 \quad (8.39)$$

could be satisfied. For the model to be discussed in the next section (8.39) cannot be fulfilled. Hence either the spin glass is not in a steady state or it cannot be characterized by a nonvanishing q .

8.2. Infinite-ranged Ising spin glass

8.2.1. Model

In this section we apply the formalism developed in the previous section to the infinite-ranged Ising spin glass, i.e. the SK [102] model. For simplicity we consider the soft spin version described by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2} \sum_{i \neq j} J_{ij} \sigma_i \sigma_j + \sum_i \left(\frac{1}{2} r_0 \sigma_i^2 + u \sigma_i^4 - \sigma_i h_i \right), \quad (8.40)$$

where the value of spin σ_i is not limited to ± 1 . The exchange integrals J_{ij} are random variables with Gaussian distribution

$$P(J_{ij}) = (2\pi N/J^2)^{-1/2} \exp\{-NJ_{ij}^2/2J^2\}, \quad (8.41)$$

where N is the number of spins interacting with the given one.

Taking into account the interaction with the heat bath and averaging over J_{ij} , we can write the CTPGF generating functional as

$$Z[h_i(t)] = \int [d\sigma_i] \exp(iS_{\text{eff}}) \langle t_+ = t_0 | \hat{\rho} | t_- = t_0 \rangle, \quad (8.42)$$

with

$$S_{\text{eff}} = \sum_j \int_p \{ \sigma_j(t) \Gamma_{0p}(t, t') \sigma_j(t') - u \sigma_j^4(t) + \sigma_j(t) h_j(t) \} + i \frac{J^2}{4N} \sum_{i \neq j} \int_p \sigma_i(t) \sigma_j(t) dt \int_p \sigma_i(t') \sigma_j(t') dt'. \quad (8.43)$$

The notation used here is the same as in the last section, but with the bar over various quantities dropped for simplicity. After Fourier transformation the low-frequency approximation for Γ_0 is given by

$$\bar{\Gamma}_{0r}(\omega, t) = -\bar{r}_0 + i\omega/\bar{\Gamma}_0, \quad (8.44a)$$

$$\bar{\Gamma}_{0c}(\omega, t) = 2i/\beta\bar{\Gamma}_0. \quad (8.44b)$$

8.2.2. Self-energy part

In the infinite range limit when $N \rightarrow \infty$ the matrix $G_{ij}(t, t')$ can be approximated by $\delta_{ij}G(t, t')$. In this case the second-order vertex function can be calculated by the diagrammatic expansion. It is found for small q that

$$\bar{\Gamma}_r(\omega, t) = -\bar{r} + i\omega/\bar{\Gamma} - J_r^2(q)G_r(\omega, t) - \Sigma_r(\omega, t), \quad (8.45)$$

where $\bar{r}, \bar{\Gamma}$ are renormalized quantities which might depend on temperature. To the lowest order perturbation in u we find

$$J_r^2(q) = J^2 + 288u^2(q^2 + 2q\sigma^2), \quad (8.46)$$

where J is also renormalized and $q(t, t)$ is the order parameter. In eq. (8.45) terms proportional to ω and G_r are subtracted from the self-energy part, i.e.,

$$\Sigma_r(0, t) = 0. \quad (8.47)$$

In obtaining (8.46) we assumed $q(\omega, t)$ to be peaked at $\omega = 0$.

To the same approximation the correlated vertex function is calculated to be

$$\bar{\Gamma}_c(\omega, t) = \frac{4i}{\beta\omega} \text{Im } \bar{\Gamma}_r(\omega, t) + 2iJ_c^2(q)q(\omega, t) + 2iJ^2\Delta(\omega, t) - \Sigma_c(\omega, t), \quad (8.48)$$

where

$$\Delta(\omega, t) = \int d\tau e^{i\omega\tau} \sigma(t + \frac{1}{2}\tau) \sigma(t - \frac{1}{2}\tau). \quad (8.49)$$

The expectation value $\sigma(t)$ might be different from zero when a magnetic field is applied. It can be shown that $\Delta(\omega, t)$ is peaked at $\omega = 0$, whereas Σ_c is the smooth part of the self-energy. To the lowest order in u we find

$$J_c(q) = J^2 + 96u^2(q^2 + 2q\sigma^2). \quad (8.50)$$

In the low-frequency limit the function defined by (8.28) turns out to be

$$Q(\omega, t) = -J_c^2(q)q(\omega, t) - J^2\Delta(\omega, t), \quad (8.51)$$

provided only terms peaked at $\omega = 0$ are retained. It is worthwhile noting that $J_r^2(q)$ is greater than $J_c^2(q)$ for all values of q as evident from (8.46) and (8.50). This fact is essential for our later discussion.

8.2.3. Stability

Now we study the stability of the system. As seen from (8.45), in the $J \rightarrow 0$, i.e., the pure limit r is the inverse susceptibility which itself is proportional to the temperature and increases as the magnetization increases. We assume that such qualitative behaviour holds also for random systems. In the zero-frequency limit the Dyson equation for the retarded Green function (8.27a) becomes

$$\chi^{-1} = r - J_r^2(q)\chi, \quad (8.52)$$

in accord with (8.45), where the magnetic susceptibility

$$\chi = -G_r(\omega = 0). \quad (8.53)$$

Equation (8.52) can be solved to yield

$$\chi = \frac{1}{2J_r^2(q)} [r - \sqrt{r^2 - 4J_r^2(q)}]. \quad (8.54)$$

Thus the magnetic susceptibility increases as r decreases and reaches its maximum at $r = 2J_r(q)$. Further decrease of r will make χ complex and the system unstable. Therefore, the stable region is bounded by the inequality

$$\chi J_r(q) < 1. \quad (8.55)$$

It is clear from (8.54) that in the unstable region

$$|\chi|^2 J_r^2(q) = 1, \quad (8.56)$$

which is a curve on the plane $q - |\chi|$. On this plane all stable points are located in the region bounded from above by the curve (8.56) consisting of marginally stable and unstable points. In the stable region q and χ are related by the Fischer relation (8.26). Hence the physical state of the system can evolve either in the stable region or on the boundary (8.56).

8.2.4. Time evolution of $q(t, t)$

Before discussing the time evolution of $q(t, t)$ we first note that in the low-frequency limit

$$G_r(\omega, t) = -\chi + \alpha(t)|\omega|^\nu (\coth(\pi\nu/2) - i \operatorname{sgn} \omega), \quad (8.57)$$

where $\alpha(t)$ and ν are positive quantities to be determined. An analysis similar to that of ref. [117] using (8.45) shows that $\nu \leq 1/2$ if the state is marginally stable and $\nu = 1$ otherwise.

To study the time evolution of q we start from the Dyson equations in the quasiclassical approximation (8.35) and (8.36). Using (8.51) in the absence of the external field we have

$$\frac{\partial \tilde{q}}{\partial t} = \left\{ \frac{1 - J_r^2(q)|\tilde{G}_r|^2}{1 + J_r^2(q)|\tilde{G}_r|^2} \frac{\partial |\tilde{G}_r|^2}{\partial t} \right\} \left/ \left(\frac{\partial \operatorname{Re} \tilde{G}_r}{\partial \omega} \frac{\partial \operatorname{Im} \tilde{G}_r}{\partial t} - \frac{\partial \operatorname{Im} \tilde{G}_r}{\partial \omega} \frac{\partial \operatorname{Re} \tilde{G}_r}{\partial t} \right) \right\} \tilde{q}. \quad (8.58)$$

In the low-frequency limit eq. (8.58) can be simplified and after Fourier transformation becomes [55]

$$\frac{\partial q}{\partial t} = -2 \frac{1 - J^2 \beta^2 (1 - q)^2}{1 + J^2 \beta^2 (1 - q)^2} \frac{\Gamma}{\beta (1 - q)} q, \quad (8.59)$$

by use of the Fischer relation valid in the stable region. Above the critical temperature $\beta_c^{-1} = J$, the only fixed point of (8.59) is $q = 0$. Near this point the order parameter decays exponentially

$$q = q_{\text{in}} \exp(-t/\tau_0), \quad (8.60)$$

with

$$\tau_0 = \frac{\beta_c^2 + \beta^2}{2(\beta_c^2 - \beta^2)} \frac{\beta}{\Gamma}. \quad (8.61)$$

Below T_c there is another fixed point q_1 on the stability boundary

$$q_1 = 1 - \beta_c/\beta. \quad (8.62)$$

The order parameter approaches this point also exponentially with characteristic time

$$\tau_1 = \beta_c^2/2(\beta - \beta_c)\Gamma. \quad (8.63)$$

It is worthwhile noting that both τ_0 and τ_1 have a pole at $\beta = \beta_c$, giving rise to a kind of critical slowing down.

However, q_1 is no more a fixed point if higher order terms are taken into account. As seen from (8.58), the nonlinear fixed point is determined by the intersection of Fischer line with

$$1 = J_c^2(q)|\chi|^2. \quad (8.64)$$

Since $J_r^2(q) > J_c^2(q)$ as we noted before, this curve is outside of the stable region and cannot be reached. In fact, what happens is that after hitting the stability boundary at q_1 , the order parameter further decays along the marginal stability line down to $q = 0$. By inserting (8.57) and (8.64) into (8.58) we obtain

$$\frac{\partial \tilde{q}(\omega, t)}{\partial t} = -\frac{4}{3} \frac{q^2}{\alpha \nu} |\omega|^{1-\nu} \tilde{q}(\omega, t). \quad (8.65)$$

Hereafter we use the units $J = T_c = 1$ and take $u = 1/12$. It is obvious from (8.65) that $\tilde{q}(0, t)$ does not change with time whereas $\tilde{q}(\omega \neq 0, t)$ tends to zero as t goes to infinity.

8.2.5. Susceptibility and q in a small field

It follows from (8.29) and (8.49)–(8.51) that the static fixed point for q is determined from the equation

$$q_0 = |\chi|^2 (q_0 + \sigma^2 + \frac{2}{3} q_0^3 + 2q_0^2 \sigma^2) \quad (8.66)$$

where $q_0 \equiv \lim_{t \rightarrow \infty} q(t, t)$.

For $T > T_c$, the whole Fischer line (8.26) is located inside the stable region, so for small field h we have

$$q_0 = \frac{\beta^2}{1 - \beta^2} \sigma^2, \quad (8.67)$$

$$\chi \equiv \frac{d\sigma}{dh} = \beta - \frac{\beta^3}{1 - \beta^2} h^2. \quad (8.68)$$

At the critical point the Fischer relation (8.26) still holds to yield

$$q_0 = \sigma/\sqrt{2} - \frac{19}{24}\sigma^2 + \dots, \quad (8.69)$$

$$\chi = 1 - h/\sqrt{2} + \frac{1}{24}h^2 + \dots. \quad (8.70)$$

Below T_c , there exists a critical magnetic field h_c above which the static fixed point is still sitting in the stable region. The value of h_c turns out to be

$$h_c^2 = \frac{4}{3}\tau^2(1 + 3\tau + \dots) \quad (8.71)$$

near T_c with $\tau = 1 - \beta^{-1}$.

For $T < T_c$ and $h < h_c$, the intersection of the Fischer line with the fixed point equation (8.66) is outside the physical boundary, so it can never be reached. This false fixed point is just what was found before [104, 105] to yield negative entropy at low temperatures. The only plausible fixed points are those located on the stability boundary (8.56) which in new units appears as

$$\chi = (1 + 2q_0^2 + 4q_0\sigma^2)^{-1/2}. \quad (8.72)$$

Solving (8.66) and (8.72) yields

$$q_0 = (\frac{3}{4}\sigma^2)^{1/3}, \quad (8.73)$$

$$\chi = 1 - (\frac{3}{4}h^2)^{2/3}. \quad (8.74)$$

All results obtained in eqs. (8.67)–(8.71) as well as in eqs. (8.73) and (8.74) agree with those predicted by the projection hypothesis [125, 126].

8.2.6. Summary

To sum up, we have found from a systematic analysis using the CTPGF formalism that a physical boundary exists on the plane $q-|\chi|$. Above T_c , the Fischer line is lying entirely in the stable region and the order parameter approaches the fixed point at this line exponentially in time. Below T_c and h_c there are no fixed points on the Fischer line inside the stable region. In this case the fixed point is located on the stability boundary. In the presence of a persistent magnetic field h the order parameter will first decay exponentially to the intersection point q_1 and then decreases further along the boundary down to

the fixed point q_0 . The magnetization calculated for the state q_0 agrees with what follows from the projection hypothesis. In the absence of magnetic field, $q_0 = 0$. If the $\omega = 0$ component of the order parameter $q(t, t)$ constitutes a finite part of q , say q_{EA} , the system will finally reach a steady state with $q = q_{\text{EA}}$. The value of q_{EA} depends on the dynamic processes with infinite long relaxation time in the $N \rightarrow \infty$ limit.

It is worthwhile to note that the boundary line on the $q-|\chi|$ plane, and hence the fixed point, is temperature independent. As a consequence, the magnetization is also temperature independent, while the entropy does not vary with magnetic field. This is just the assumption of the projection hypothesis [125, 126] which follows from the CTPGF formalism in a natural way.

Comparing the results obtained here with those of Parisi [110] as well as Sompolinsky and Zippelius [117–119], it is natural to conjecture that Parisi's function $q(x)$, $0 \leq x \leq 1$, corresponds to our $q(t, t)$ varying along the boundary from q_1 to q_0 . The present formalism not only elucidates the physical meaning of the order parameter but also provides us with an equation to solve its time evolution.

8.3. Disordered electron system

Anderson first showed in 1958 [128] that if the electron site energies in a solid were sufficiently random, some of the energy eigenstates became localized instead of being extended as they would be in a regular crystal. Such localized states will not contribute to the electric conduction. The nature of the state depends on whether the energy value is located on the “localized” or “extended” side of the “mobility edge” [129].

The drastic change in the behaviour of the wavefunction with energy is reminiscent of phase transition. The great success in applying the field-theoretical technique and the renormalization group to critical phenomena [60] encouraged similar attempts in the localization problem. Wegner [130] suggested a nonlinear σ -model to study the scaling properties of the noninteracting disordered electron system near the mobility edge [131] with conductance playing the role of coupling constant. This model was later derived from the field theory [132, 133] and was further studied by other authors [134–136]. Recently, there is a revival of interest in this problem due to the discovery of the quantized Hall effect in two-dimensional electron system [137, 138]. This is a challenge to the theory since according to the existing theory all electron states in disordered two-dimensional system should be localized in the absence of magnetic field [131], whereas extended states are certainly needed for explaining the observed quantized Hall effect. The extension of the field-theoretical approach to include the external magnetic field was made very recently by Pruisken et al. [139, 140].

However, almost all studies of field-theoretic approach in the localization problem were based on the replica trick. With n -replicated system an $O(n_+, n_-)$ or $U(2n)$ symmetry is used to construct the nonlinear σ -model. The critical behaviour from the extended state side of the mobility edge is described by a Goldstone mode resulting by virtue of the spontaneous breaking of the replica symmetry. The difficulty of the replica trick is the necessity of continuing n , originally defined for integers, to zero to get the physical result. Such process might not be unique as in the case of spin glass.

It turns out that the CTPGF formalism can be also applied to the localization problem without resorting to the replica trick [56]. In this section we describe the symmetry of the model and derive the corresponding WT identities. The order parameter and the symmetry breaking pattern are also briefly sketched.

8.3.1. Model

We are concerned with the effect of disorder on Green's functions of a noninteracting electron gas moving in external fields. The Lagrangian of the system is given by

$$\mathcal{L} = \int \psi^\dagger(x) \left(i \frac{\partial}{\partial t} - L_0 - V \right) \psi(x), \quad (8.75)$$

where

$$L_0 = \frac{1}{2m} (-i\nabla - e\mathbf{A}(\mathbf{x}))^2 + e\varphi(\mathbf{x}). \quad (8.76)$$

and V the random potential.

In the CTPGF formalism the generating functional $Z[J(x), V]$ is specified by the effective action which in the single-time representation can be written as

$$S_{\text{eff}} = \int [\psi^\dagger(x) \sigma_3 \Gamma^0(x, y) \sigma_3 \psi(y) + J^\dagger(x) \sigma_3 \psi(x) + \psi^\dagger(x) \sigma_3 J(x)] \quad (8.77)$$

where $\psi(x)$, $J(x)$ are two-component vectors as usual. The vertex function in the tree approximation is given by

$$\hat{\Gamma}_0(x, y) = \sigma_3 \left(i \frac{\partial}{\partial t} - L_0 - V \right) \delta(x - y) + i\varepsilon(I + \sigma_1 - i\sigma_2) \delta(x - y) \quad (8.78)$$

where ε is a positive infinitesimal constant. To convinve ourselves in the validity of (8.78) we note that for noninteracting fermions

$$\Gamma_{0r} = E - \varepsilon(k) + i\varepsilon, \quad \Gamma_{0a} = E - \varepsilon(k) - i\varepsilon, \quad \Gamma_{0c} = 2i\varepsilon, \quad (8.79)$$

as follows from (2.23) and the Dyson equation (3.10). Equation (8.78) is then a direct consequence of (2.63).

The generating functional can be thus rewritten as

$$Z[J, V] = \int [d\psi][d\psi^\dagger] \exp \left\{ i \int \left[\psi^\dagger(x) \left(\left(i \frac{\partial}{\partial t} - L_0 - V \right) \sigma_3 + i\varepsilon(I - \sigma_1 + i\sigma_2) \right) \psi(x) + \psi^\dagger \sigma_3 J + J^\dagger \sigma_3 \psi \right] \right\}. \quad (8.80)$$

As shown in section 8.1, the quenched average of random potential can be carried out directly on the generating functional. It is, however, more convenient to work in the energy representation for the localization problem. After Fourier transformation the effective action is given by

$$S_{\text{eff}} = \int d^d x dE \{ \psi^\dagger(\mathbf{x}, E) [\sigma_3(E - L_0 - V) + i\varepsilon(I - \sigma_1 + i\sigma_2)] \psi(\mathbf{x}, E) + \psi^\dagger \sigma_3 J(\mathbf{x}, E) + J^\dagger \sigma_3 \psi(\mathbf{x}, E) \}. \quad (8.81)$$

We restrict ourselves to the Wegner model [130] where the correlation of the random potential between different energy shells vanishes, i.e.,

$$\langle V(\mathbf{x}, E) \rangle = 0, \quad \langle V(\mathbf{x}, E) V(\mathbf{y}, E') \rangle = \gamma \delta(\mathbf{x} - \mathbf{y}) \delta(E - E'). \quad (8.82)$$

After averaging over the random potential V the generating functional

$$\bar{Z}[J] = \int [dV] Z[J, V] \exp \left(-\frac{1}{2\gamma} \int V^2(\mathbf{x}) d\mathbf{x} \right) \quad (8.83)$$

is determined by a new effective action S which differs from S_{eff} (8.81) by dropping V in the single-particle Lagrangian and a new term

$$\Delta S = \frac{i\gamma}{2} \int d^d x dE \psi^\dagger(\mathbf{x}, E) \sigma_3 \psi(\mathbf{x}, E) \psi^\dagger(\mathbf{x}, E) \sigma_3 \psi(\mathbf{x}, E). \quad (8.84)$$

The generating functionals $\bar{W}[J]$ and $\bar{I}[\psi_c]$ are defined in the standard manner so that there is no need to write them down explicitly. We would like to mention, nevertheless, that to avoid the possible confusion with sign of the Grassman variables we adopt the convention that $\delta/\delta J(\mathbf{x})$, $\delta/\delta \psi_c(\mathbf{x})$ act from the right, whereas $\delta/\delta J^\dagger(\mathbf{x})$, $\delta/\delta \psi_c^\dagger(\mathbf{x})$ act from the left.

8.3.2. Symmetry properties

Before discussing the specific model under consideration we would like to make a general remark concerning the symmetry properties in the CTPGF formalism. It is well known that the $U(1)$ symmetry of a complex field

$$\psi(\mathbf{x}) \rightarrow e^{i\alpha} \psi(\mathbf{x}), \quad \psi^\dagger(\mathbf{x}) \rightarrow e^{-i\alpha} \psi^\dagger(\mathbf{x})$$

corresponds to the charge conservation. In the CTPGF approach we deal with an action defined on the closed time-path which in single time representation appears as

$$S = \int_{-\infty}^{\infty} dt [\mathcal{L}(\psi(t+)) - \mathcal{L}(\psi(t-))]. \quad (8.85)$$

This action, therefore, respects the $U(1) \times U(1)$ symmetry, i.e., it is invariant under

$$\begin{aligned} \psi_+(\mathbf{x}) &\rightarrow \exp(i\alpha_+) \psi_+(\mathbf{x}), & \psi_+^\dagger(\mathbf{x}) &\rightarrow \exp(-i\alpha_+) \psi_+^\dagger(\mathbf{x}), \\ \psi_-(\mathbf{x}) &\rightarrow \exp(i\alpha_-) \psi_-(\mathbf{x}), & \psi_-^\dagger(\mathbf{x}) &\rightarrow \exp(-i\alpha_-) \psi_-^\dagger(\mathbf{x}), \end{aligned} \quad (8.86)$$

where α_+ , α_- are independent of each other.

However, such $U(1) \times U(1)$ symmetry is always spontaneously broken, because Γ_{0-+} or G_{0-+} is different from zero even for vacuum state average as seen from (2.21), so that only $U(1)$ symmetry is

retained. The question whether there are any physical consequences of such spontaneous breaking, has to be studied.

Coming back to our model, we note that apart from the source terms and terms proportional to ε , the effective action (8.81) (and also the averaged action (8.84)) has a global $\text{Sp}(2)$ symmetry keeping $\psi^\dagger \sigma_3 \psi$ invariant. The function $\psi(x)$ forms a two-dimensional representation of the $\text{Sp}(2)$ group transforming as

$$\psi(x) \rightarrow \psi'(x) = U\psi(x), \quad \psi^\dagger(x) \rightarrow \psi^\dagger(x) = \psi^\dagger(x)U^\dagger, \quad (8.87)$$

where

$$U = \exp\{\sigma_1\lambda_1 + \sigma_2\lambda_2 + i\sigma_3\lambda_3\} \quad (8.88)$$

satisfies the condition

$$U\sigma_3 U^\dagger = U^\dagger \sigma_3 U = \sigma_3. \quad (8.89)$$

Here $\lambda_1, \lambda_2, \lambda_3$ are group parameters corresponding to the three generators.

The term proportional ε does not respect this symmetry. Like a small magnetic field determines the direction of magnetization in an $O(3)$ ferromagnet, the ε term can be considered as a small external field inducing the breakdown of the $\text{Sp}(2)$ symmetry. Actually, the $\text{Sp}(2)$ symmetry is spontaneously broken by dynamic generation of the imaginary part of the retarded (advanced) Green functions.

8.3.3. WT identities

If we make an infinitesimal transformation with group parameters $\lambda_i(E)$ for functions $\psi^\dagger(x), \psi(x)$ in the path integral of $\bar{Z}[J]$ we obtain the following three WT identities corresponding to the three generators of the $\text{Sp}(2)$ group:

$$\begin{aligned} -2i\varepsilon \int d^d x \text{Sp} \left\{ (1 + \sigma_1) \left[-i \frac{\delta^2 \bar{W}}{\delta J^\dagger(x, E) \delta J(x, E)} + \frac{\delta \bar{W}}{\delta J^\dagger(x, E)} \frac{\delta \bar{W}}{\delta J(x, E)} \right] \right\} \\ = \int d^d x \left[J^\dagger(x, E) \sigma_1 \frac{\delta \bar{W}}{\delta J^\dagger(x, E)} + \frac{\delta \bar{W}}{\delta J(x, E)} \sigma_1 J(x, E) \right], \end{aligned} \quad (8.90a)$$

$$-2\varepsilon \int \text{Sp} \left\{ (1 + i\sigma_2) \left[-i \frac{\delta^2 \bar{W}}{\delta J^\dagger \delta J} + \frac{\delta \bar{W}}{\delta J^\dagger} \frac{\delta \bar{W}}{\delta J} \right] \right\} = \int \left[J^\dagger \sigma_2 \frac{\delta \bar{W}}{\delta J^\dagger} + \frac{\delta \bar{W}}{\delta J} \sigma_2 J \right], \quad (8.90b)$$

$$2i\varepsilon \int \text{Sp} \left\{ (\sigma_1 - i\sigma_2) \left[-1 \frac{\delta^2 \bar{W}}{\delta J^\dagger \delta J} + \frac{\delta \bar{W}}{\delta J^\dagger} \frac{\delta \bar{W}}{\delta J} \right] \right\} = \int \left[\frac{\delta \bar{W}}{\delta J} \sigma_3 J - J^\dagger \sigma_3 \frac{\delta \bar{W}}{\delta J^\dagger} \right]. \quad (8.90c)$$

The WT identities for various Green functions can be obtained by differentiating (8.90) with respect to $J^\dagger(x, E), J(x, E)$ and then setting $J^\dagger = J = 0$. As an example, we show how the dynamic generation of the imaginary part breaks the $\text{Sp}(2)$ symmetry.

Using eq. (2.12) we find that

$$G_r = \frac{1}{2} \xi^\dagger \hat{G} \eta = -\frac{1}{2} \text{Sp} \left[(\sigma_3 - i\sigma_2) \frac{\delta^2 \bar{W}}{\delta J^\dagger(\mathbf{x}, E) \delta J(\mathbf{x}, E)} \right]. \quad (8.91)$$

Taking the functional derivative $\delta^2/\delta J^\dagger(\mathbf{y}, E) \delta J(\mathbf{y}, E)$ of both sides of (8.90b) we obtain

$$\begin{aligned} \text{Im } G_r(\mathbf{y}, \mathbf{y}, E) &= \frac{1}{2} \text{Sp} \left\{ \sigma_2 \frac{\delta^2 \bar{W}}{\delta J^\dagger(\mathbf{y}, E) \delta J^\dagger(\mathbf{y}, E)} \right\} \Big|_{J=J^\dagger=0} \\ &= -\frac{\varepsilon}{2} \int d^d x \text{Sp} \left\{ (1 + i\sigma_2) \left[-i \frac{\delta^4 \bar{W}}{\delta J_\alpha^\dagger(\mathbf{y}, E) \delta J_\alpha(\mathbf{y}, E) \delta J^\dagger(\mathbf{x}, E) \delta J(\mathbf{x}, E)} \right. \right. \\ &\quad \left. \left. + \frac{\delta^2 \bar{W}}{\delta J_\alpha(\mathbf{y}, E) \delta J^\dagger(\mathbf{x}, E)} \frac{\delta^2 \bar{W}}{\delta J_\alpha^\dagger(\mathbf{y}, E) \delta J(\mathbf{x}, E)} \right] \right\} \Big|_{J=J^\dagger=0}. \end{aligned} \quad (8.92)$$

As is well known, $\text{Im } G_r(\mathbf{y}, \mathbf{y}, E)$ is proportional to the density of states $\rho(E)$. It is different from zero certainly for extended states and possibly for localized states as $\varepsilon \rightarrow 0$. Therefore, the $\text{Sp}(2)$ symmetry is broken for both cases. McKane and Stone [133] pointed out that there are two ways to satisfy the WT identities. Although their interpretation is given in an entirely different theory based on the replica trick, we expect it applicable to our case as well. For extended states, the dynamic generation of the imaginary part for the retarded Green function caused by the breakdown of the $\text{Sp}(2)$ symmetry leads to Goldstone mode characterized by long-range correlation and governing the critical behaviour from the “extended” side. On the other hand, there is no Goldstone mode with vanishing momentum for localized states, so the integrand on the right-hand side of (8.92) must diverge as $\varepsilon \rightarrow 0$ before integration to satisfy the WT identities in this case.

8.3.4. Order parameter and nonlinear σ -model

As said before, the order parameter breaking the $\text{Sp}(2)$ symmetry is proportional to the imaginary part of the retarded Green function. A Goldstone mode is therefore generated. To describe this mode it is convenient to introduce a composite matrix field

$$q(\mathbf{x}, E) = \psi(\mathbf{x}, E) \psi^\dagger(\mathbf{x}, E), \quad (8.93)$$

the vacuum expectation value of which is connected to the second-order CTPGF.

Under the $\text{Sp}(2)$ group, the field q transforms as

$$q \rightarrow U q U^\dagger, \quad (8.94)$$

where U is given by (8.88). The vacuum expectation value of q is

$$\langle q \rangle = a(1 - \sigma_1 + i\sigma_2) + b\sigma_3, \quad (8.95)$$

where the diagonal part of the first term describes the imaginary part of G_F and $G_{\bar{F}}$, whereas the second term describes their real part. As seen from (8.89) the b term does not break the $\text{Sp}(2)$ symmetry, while the a term does. Hence Goldstone modes will be dynamically generated by the condensation of the q field.

In analogy with the earlier work [133], we can derive a nonlinear σ -model describing such Goldstone modes and carry out the renormalization procedure to study the scaling behaviour near the mobility edge. However, we will not elaborate further on such discussion here.

To summarize section 8, we note that the theoretical framework outlined here for quenched random systems is quite general as well as flexible. It is based on the dynamics of the system itself without resorting to replica trick, so the approximation involved are well under control. Apart from spin glass and disordered electron system discussed in this section, the present formalism can be applied to other quenched random systems as well. In particular, we discussed [141] the controversial problem of the lower critical dimension for an Ising spin system in a random magnetic field [142]. We hope the CTPGF formalism is helpful in solving some of the difficult problems still remaining open in this field.

9. Connection with other formalisms

To save space in this paper we attempted to avoid as much as possible digressing from the main subject and comparing the CTPGF approach with other formalisms in passing. We would like such comparison to be concentrated here. Although not so much new information will be presented to experts, hopefully, this section will help the newcomer entering this field to orient himself in the forest of diversified formalisms. We will mainly discuss the connection of the CTPGF approach with the Matsubara technique (section 9.1), quantum and fluctuation theories as low and high temperature limits of the CTPGF formalism (section 9.2) and also the CTPGF formalism as a plausible microscopic justification of the MSR field theory (section 9.3). There are still many related papers not covered in this review for which we apologize to their authors once again.

9.1. Imaginary versus real time technique

The Matsubara technique [1–7] using the imaginary time for thermoequilibrium is well developed and highly successful. However, there are two difficulties from the technical point of view. One is associated with the fact that in this technique Green's functions are defined on a finite section of the imaginary time axis $(0, -i\beta)$ so the Fourier series expansion in frequency is used instead of Fourier integral. The frequency summation is sometimes quite cumbersome. Another difficulty is connected with the analytic continuation of the frequency (or time) variable in the final answer. Usually, such a process is rather delicate. We see thus in spite of the great success of the Matsubara technique, a convenient real time formalism would be highly desirable. The CTPGF formalism is one of the possible candidates for this purpose.

9.1.1. Real time diagrammatic technique

We have mentioned in section 6.1 that the diagrammatic expansion for thermoequilibrium system at finite temperature is similar to that of the quantum field theory provided the free propagator is given by (6.33). Here we would like to justify this statement using expressions for the effective action derived in section 6.1.

The correlation functional W^N defined by (6.15) becomes in this case

$$\exp(iW_{\text{th}}^N[J_\Delta^\dagger, J_\Delta]) = \text{Tr}\{\hat{\rho}_{\text{th}} : \exp[i(J_\Delta^\dagger \psi_1 + \psi_1^\dagger J_\Delta)] : \} \quad (9.1)$$

where

$$\hat{\rho}_{\text{th}} = \exp(-\Omega - \beta(\mathcal{H} - \mu N)) \quad (9.2)$$

where Ω is the thermodynamic potential, whereas ψ_i^\dagger , ψ_i are operators in the incoming picture. It is known that for the in-field we have [39]

$$e^{i\mathcal{H}\tau} A_i(t) e^{-i\mathcal{H}\tau} = A_i(t + \tau), \quad (9.3)$$

where \mathcal{H} is the total Hamiltonian. This point is essential for our derivation. By analytic continuation $\tau \rightarrow -i\beta$ we find

$$e^{\beta\mathcal{H}} A_i(t) e^{-\beta\mathcal{H}} = A_i(t - i\beta). \quad (9.4)$$

Taking into account that for a complex field the operator of particle number

$$N \sim \int d^d x \psi_i^\dagger(x) \psi_i(x) = \int d^d x \psi_i^\dagger(x) \psi_i(x)$$

is a conserved quantity, it is easy to show that

$$\rho_{\text{th}}^{-1} A_i(t) \rho_{\text{th}} = \exp \left\{ -i\beta \frac{\partial}{\partial t} - \lambda \beta \mu \right\} A_i(t), \quad (9.5)$$

where

$$\lambda = +1, \quad \text{if } A_i(t) = \psi_i^\dagger(x), \quad \lambda = -1, \quad \text{if } A_i(t) = \psi_i(x), \quad \lambda = 0, \quad \text{if } A_i(t) \text{ is Hermitian}. \quad (9.6)$$

Using (9.5) we can apply as done by Gaudin [7] the following identity

$$\begin{aligned} \text{Tr}\{(\rho A(1) \mp (\pm)^n A(1) \rho) A(2) \cdots A(n)\} \\ = \text{Tr}\{\rho [A(1), A(2)]_\mp A(3) \cdots A(n)\} \pm \{\rho A(2) [A(1), A(3)]_\mp A(4) \cdots A(n)\} \\ + \cdots \cdots (\pm)^{n-2} \text{Tr}\{\rho A(2) \cdots A(n-1) [A(1), A(n)]_\mp\} \end{aligned} \quad (9.7)$$

to the right-hand side of (9.1) to obtain

$$\exp\{iW_{\text{th}}^N\} = \text{Tr}\{\rho_{\text{th}}\} \langle 0 | : \exp(\hat{F}_i) \exp[i(J_\Delta^\dagger \psi_i + \psi_i^\dagger J_\Delta)] : | 0 \rangle, \quad (9.8)$$

where

$$\hat{F}_i = \pm \int d^{d+1}x d^{d+1}y \frac{\delta}{\delta \psi_i^\dagger(x)} \left(1 \mp \exp \left(-i\beta \frac{\partial}{\partial x_0} - \beta \mu \right) \right)^{-1} [\psi_i^\dagger(x), \psi_i(y)]_\mp \frac{\delta}{\delta \psi_i(y)}. \quad (9.9)$$

In deriving (9.8) the properties of the normal product and the particle number conservation are taken

into account properly. Note that for nonrelativistic complex fields the operator $\psi_1(x)$ contains only positive frequencies whereas $\psi_1^\dagger(x)$ only negative ones, so that

$$[\psi_1(x), \psi_1^\dagger(y)]_\mp = iS_{0-+}(x, y), \quad (9.10)$$

where S_{0-+} is given by (6.8). Substituting (9.10) into (9.9) we find that

$$\hat{F}_1 = \pm i \int \frac{\delta}{\delta \psi_1(x)} S_0^N(x, y) \frac{\delta}{\delta \psi_1^\dagger(y)} \quad (9.11)$$

with

$$S_0^N(x, y) = \left(1 \mp \exp \left(i\beta \frac{\partial}{\partial x_0} - \beta\mu \right) \right)^{-1} S_{0-+}(x, y). \quad (9.12)$$

It then follows from (9.8) by using (9.11) that

$$W_{\text{th}}^N[J_\Delta^\dagger, J_\Delta] = - \int J_\Delta^\dagger(x) S_0^N(x, y) J_\Delta(y). \quad (9.13)$$

In accord with (6.15) the correlation functional defined on the closed time-path is

$$W_{\text{th}, p}^N[J^\dagger, J] = - \int_p J^\dagger(x) S_{0p}^N(x, y) J(y), \quad (9.14)$$

where

$$S_{0p}^N(x, y) = S_0^N(x, y) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (9.15)$$

Equations (9.13) and (9.14) are the contribution of the density matrix to the generating functional. We see thus the initial correlation is actually a Gaussian process described by the two-point correlation function $S_0^N(x, y)$.

Substituting (9.14) back into (6.10) and (6.29) we obtain the following expressions for the CTPGF generating functional in thermal equilibrium:

$$Z_p[J^\dagger, J] = \exp \left\{ i I_{\text{int}} \left[\mp i \frac{\delta}{\delta J}, -i \frac{\delta}{\delta J^\dagger} \right] \right\} \exp(-i J^\dagger G_0 J), \quad (9.16)$$

$$Z_p[J^\dagger, J] = \int_p [d\psi^\dagger][d\psi] \exp \{ i(\psi^\dagger G_0^{-1} \psi + I_{\text{int}} t[\psi^\dagger, \psi] + J^\dagger \psi + \psi^\dagger J) \}, \quad (9.17)$$

where

$$G_0(12) = S_0(12) + S_{0p}^N(1, 2), \quad (9.18)$$

$$G_0^{-1}(1, 2) = S_0^{-1}(1, 2) - \int_p d\bar{1} d\bar{2} \tilde{S}_0^{-1}(1, \bar{1}) S_{0p}^N(\bar{1}\bar{2}) \tilde{S}_0^{-1}(\bar{2}, 2). \quad (9.19)$$

It is ready to check that G_0 and G_0^{-1} defined by (9.18) and (9.19) are reciprocal to each other and that the FDT (3.43) is satisfied.

Since the Green functions in thermoequilibrium are time translationally invariant, the only role of the second term in (9.19) is to produce $S_{0p}^N(1, 2)$ term in $G_0(1, 2)$ and cannot appear independently in the final result. Hence we can ignore the difference of \tilde{S}_0^{-1} and S_0^{-1} from the very beginning and take

$$G_0^{-1}(1, 2) = S_0^{-1}(1, 2) \quad (9.20)$$

instead of (9.19). In that case the first and the second terms of (9.18) can be considered as superposition of solutions for inhomogeneous equation $S_0^{-1}F = 1$ and homogeneous equation $S_0^{-1}F = 0$. The numerical coefficient of the latter is determined by the FDT.

We see thus (9.16) and (9.17) are the generalization of the Matsubara technique to real time axis. The advantage of using real time variables in some cases more than justifies the technical complications owing to the matrix representation of the propagator.

9.1.2. Other real time formalisms

Several authors previously considered the possible generalization of the Feynman–Wick expansions for the Matsubara functions [143, 144]. However, some of these attempts ended up with very involved formalism, whereas the others were difficult to justify. We believe the incoming picture adopted here is helpful in avoiding these difficulties. Very recently, Niemi and Semenoff [145] proposed a version of real time technique to study the finite temperature field theory. Their work is close to ours but is still different. The time-path in the complex plane they adopt consists of four pieces $(-\infty, +\infty)$, $(+\infty, +\infty - i\beta/2)$, $(+\infty - i\beta/2, -\infty - i\beta/2)$ and $(-\infty - i\beta/2, -\infty - i\beta)$. Their free boson propagator is given by

$$G(k) = \hat{A} \begin{pmatrix} 1/(k^2 - m^2 + i\eta) & 0 \\ 0 & -1/(k^2 - m^2 - i\eta) \end{pmatrix} \hat{A}, \quad (9.21)$$

where

$$\hat{A} = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}, \quad \cosh^2 \theta = \exp(\beta|k_0|)/[\exp(\beta|k_0|) - 1]. \quad (9.22)$$

Obviously, (9.22) is different from that given by (2.26) in our formalism. A detailed comparison of these two versions has to be made by future studies.

9.1.3. Thermo field dynamics

For the last ten years Umezawa and coworkers [91] have developed the “thermo field dynamics” and applied it to a number of interesting problems in condensed matter physics. They have adopted a great

many ideas and techniques from the arsenal of the quantum field theory, especially the operator transformations. So far we have not yet studied all the topics they have covered, and it is hard to make a thorough analysis of the merits as well as the shortcomings of each formalism. It seems to us, however, that the extensive use of the generating functional technique, especially the vertex functional in the CTPGF formalism, is advantageous. As we found from the study of the weak electromagnetic field coupled to the superconductor [53] the ambiguity connected with the dynamic mapping and boson transformation occurring in thermo field dynamics [146], can be avoided in the CTPGF formalism. Another merit of the latter is the unified approach to both equilibrium and nonequilibrium phenomena, whereas the thermo field dynamics is limited to equilibrium systems up to now.

9.1.4. Kadanoff–Baym formalism

We should also mention the Green function formalism developed by Kadanoff and Baym (KB) [147]. These authors do not use the closed time-path, but rather start from the original paper by Martin and Schwinger [83]. There are still many common features of these two formalisms. In fact, the $G^>$ and $G^<$ functions appearing in the KB technique are nothing but G_{-+} and G_{+-} in the CTPGF approach. There are many papers applying the KB formalism to different problems in both equilibrium and nonequilibrium systems [148]. We will not go on to compare these two formalisms in further detail. The interested readers are referred to their excellent book [147].

9.2. Quantum versus fluctuation field theory

In this section we consider the low and high temperature limits of the CTPGF formalism for thermoequilibrium. It is natural to expect that in the zero temperature limit the standard quantum field theory or its equivalent in the many-body systems should be recovered. In fact, if the boson density is set equal to zero in (2.26), Δ_F becomes the usual Feynman propagator. Of course, there is no need to duplicate the time axis in this limit.

9.2.1. Critical phenomena

Now consider the high temperature limit. As seen from the Bose distribution

$$n(p) = \frac{1}{\exp[\varepsilon(p)/T] - 1}$$

for particle number nonconserving system or near the critical point (where the chemical potential $\mu = 0$), the quasiparticle density

$$n \approx T/\varepsilon(p) \gg 1, \quad (9.23)$$

if

$$\varepsilon(p)/T \ll 1. \quad (9.24)$$

In the ordinary units (9.24) can be rewritten as

$$\lambda \gg k/\sqrt{2mk_B T}, \quad (9.25)$$

i.e., the characteristic wavelength of elementary excitation is much greater than the thermal wavelength. Hence near the critical point the thermal fluctuations dominate, whereas the quantum fluctuations are irrelevant. This is the basis of the modern theory of critical phenomena. It is worthwhile noting that this classical limit is not described by the Boltzmann distribution which holds when

$$\exp[(\varepsilon(p) - \mu(T))/T] \gg 1.$$

Therefore, it is more appropriate to call this limit “super Bose” distribution, as far as the expectation value of $a^\dagger a$ is of order $n \gg 1$, so that the noncommutativity of Bose operators can be ignored.

We have thus two types of field theory: quantum field theory at zero temperature and classical field theory near the critical temperature. They have many features in common but differ from each other in some essential aspect.

9.2.2. Finite temperature field theory

For the recent years many authors study the finite temperature field theory and possible phase transitions in such systems [149]. As mentioned before, we have shown [41] that the counter terms introduced in the quantum field theory for $T = 0$ K are enough to remove all ultraviolet divergences for CTPGF at any finite temperature, even in nonequilibrium situation. This has been shown also by other authors [149] for thermoequilibrium without resorting to CTPGF. This result is understandable from a physical point of view since the statistical average does not change the properties of the system at very short distance and hence does not contribute new ultraviolet divergences. What we should like to emphasize is that in considering phase transition-like phenomena one must first separate the leading infrared divergent term and then carry out the ultraviolet renormalization which is different from that for the ordinary quantum field theory.

9.2.3. Leading infrared divergence

To be specific, let us consider the real relativistic scalar boson theory the free propagator of which is given by (2.26). Near the transition point, the mass m vanishes so the energy spectrum is given by $\omega(\mathbf{k}) = |\mathbf{k}|$ (cf. (2.28)). Since terms proportional to the particle density n appear together with the δ -function, i.e., on the mass shell, the integration over frequencies can be carried out immediately to give stronger infrared divergence (k^{-2}) than the other terms (k^{-1}). Therefore, the marginal dimension of renormalizability for φ^4 -theory is $d_c = 4$, whereas for quantum field theory at $T = 0$ K the marginal space dimension is $d_c = 4 - 1$. This is what is usually meant by saying “quantum system in d dimensions corresponds to classical system in $d + 1$ dimensions”. If we keep only the most infrared divergent terms, then all components of \hat{G} become $(k^2 + m^2)^{-1}$, i.e., exactly the same as that used in the current theory of critical phenomena [60].

What has been said above can be checked explicitly by calculating the primitive divergent diagrams for mass, vertex and wavefunction renormalization, carrying out the frequency integration and taking the high temperature limit $T \gg \omega(k)$. The results obtained turned out to be identical to those resulting from the theory of critical phenomena. For example, in φ^4 theory, the primitive mass and wavefunction correction diagrams have quadratic divergence, whereas the vertex correction term diverges logarithmically in four-space dimensions. We know that in the quantum field theory such divergences occur for three-space and one-time dimensions.

9.2.4. High temperature limit in Matsubara formalism

The high temperature limit can be easily taken in the Matsubara technique. For example, the free

propagator for nonrelativistic complex boson field is given by

$$g(\omega_n) = \frac{1}{i\omega_n - k^2 - m^2} \quad (9.26)$$

where $\omega_n = 2\pi nT$, $m^2 \sim T - T_c$. Since $T \gg k^2 + m^2$, in the frequency summation to be carried out later we need to keep only the $\omega_n = 0$ term. Hence the propagator (9.26) reduces to minus the correlation function in the theory of critical phenomena. This fact seemed to be first realized by Landau [150].

Some investigators of finite temperature field theory in the early stage of their work incorrectly used the renormalization constants for $T = 0$ K to study phase transition related phenomena. As far as the high frequency limit, or, equivalently, the leading infrared divergent terms are picked up, both relativistic and quantum effects are irrelevant. The only exception is the phase transition near $T = 0$ K when both thermal and quantum fluctuations are essential so a special consideration is needed. Otherwise, the field-theoretic models (including non-Abelian gauge models) cannot provide anything new beyond the current theory of critical phenomena as far as the phase transition is concerned, i.e., they are classified into the same universality classes as their classical counterparts.

9.3. A plausible microscopic derivation of MSR field theory

We have mentioned already in section 6.4, that Martin, Siggia and Rose (MSR) [90] proposed a field theory to describe the classical fluctuations. There are several peculiar features of this theory: (i) Being a classical field theory, it deals with noncommutative quantities; (ii) A response field $\hat{\varphi}$ is introduced in addition to the ordinary field φ ; (iii) Some components of the Green functions should be zero along with their counterparts – vertex functions. Nevertheless, the general structure of this theory is very close to that of the quantum field theory. These authors originally proposed their theory to consider nonequilibrium fluctuations such as those in hydrodynamics, but it has been extensively used in critical dynamics near thermoequilibrium [61, 77]. In spite of the great success, its microscopic foundation especially the motivation for using noncommutative variables to describe classical fields, was poorly understood.

A few years later, the MSR theory has been reformulated in terms of stochastic functionals as a Lagrangian field theory [87–89]. The noncommutativity of field variables was thus obscured by the continuum integration, whereas the calculation procedure was significantly simplified. Nonetheless, the physical meaning remains not sufficiently clarified.

We would like to note that the CTPGF formalism provides us with a plausible microscopic justification for the MSR field theory. In a sense, the MSR theory is nothing but the physical representation, i.e., in terms of retarded, advanced and correlation functions, of the CTPGFs in the quasiclassical (the low frequency) limit.

9.3.1. Noncommutativity

First consider the operator nature of the field variable. As we discussed in the last section, in the high temperature limit the noncommutativity of operators can be ignored altogether for static critical phenomena, which implies that all components of the Green functions are replaced by correlation functions. This is no more true for dynamic phenomena. The first term in G_{++} and G_{--} (2.26) i.e., $(k_0^2 - k^2 - m^2)^{-1}$ comes from the inhomogeneous term of the Green function equation which in turn is determined by the commutator of operators. If only leading infrared divergent terms are retained, the retarded function

$$G_r = G_{++} - G_{+-} = 0.$$

Therefore, to describe the response of the system to the external disturbance we need to keep the next to leading order of infrared divergence. Put another way, the response is less infrared divergent than the correlation function. In the sense of “super Bose” distribution, the commutator of order 1 can be neglected in the leading order of n but not in the next to leading order. This is the physical interpretation for the noncommutativity of a “purely classical” field variable. We would like to emphasize here that the classical field should be considered as condensation of bosons. This is why the “quantum” or wave nature of the classical field comes into play.

The deep analogy of the quantum and fluctuation field theories is then understandable. Such parallelism is particularly evident in the functional formulation. The classical path in the quantum field theory corresponds to the mean field, or TDGL orbit in the fluctuation theory.

9.3.2. Doubling of degrees of freedom

Next consider the necessity of doubling the degrees of freedom. It has been realized for a long time that to describe the time-dependent phenomena one needs both response and correlation functions. This was, probably, the motivation of introducing the response field $\hat{\varphi}$ and putting together the response and correlation functions into a matrix function by MSR [90]. In the CTPGF formalism we introduce an extra negative time axis, so also double the degrees of freedom, i.e., to use φ_+, φ_- instead of one φ . In fact, the MSR response field

$$\hat{\varphi} = \varphi_\Delta \equiv \varphi_+ - \varphi_-,$$

whereas their physical field

$$\varphi = \varphi_c \equiv \frac{1}{2}(\varphi_+ + \varphi_-).$$

The CTPGF formalism is constructed on the functional manifold φ_+ and φ_- , or, equivalently, φ_Δ and φ_c , but in the final answer one should put $\varphi_+ = \varphi_-$ to get the physical result. As mentioned before, this is an additional way of describing fluctuations, the physical content of which should be further uncovered.

In using the generating functional technique the following external source terms are introduced in the MSR theory [87–89]

$$I_s = \int (J\varphi + \hat{J}\hat{\varphi}),$$

where J is the usual source, \hat{J} the response source. This is rather similar to our generating functional, but with an important difference. As naturally follows from the definition of the closed time path we should set (see eq. (2.71))

$$I_s = \int (J_\Delta \varphi_c + J_c \varphi_\Delta).$$

As seen before, such “twisted” combination is most natural. In fact, the physical source J_c generates the dynamic response in terms of φ_Δ functional, whereas the fluctuation source J_Δ generates the

statistical correlation in terms of φ_c functional. Another advantage of such “twisted” combination is that we do not need to introduce any extra physical field in the Hamiltonian like in the MSR theory [87–89]. In the CTPGF approach J_c is the physical field built in the formalism itself.

9.3.3. Constraints

Finally, a remark concerning various constraints imposed on propagators and vertex functions. In the original formulation of MSR [90] such constraints appeared rather difficult to understand. They became more systematic in the latter Lagrangian formulation [89] but remained not so transparent. Within the CTPGF formalism, as shown in section 2.4, they are natural consequences of the normalization for the generating functional and the causality. While in the MSR theory one needs to explore the implications of the causality order by order [151], within the CTPGF framework it is ensured from the right beginning, so that causality violating terms can never occur.

As seen before, in the low frequency limit when the MSR theory holds, the CTPGF formalism yields the same results in rather low approximations. We believe, therefore, the CTPGF formalism provides us with a plausible microscopic justification for the MSR theory and indicates how to go beyond it.

10. Concluding remarks

It is time now to summarize what has been achieved and what has to be done.

(1) The CTPGF formalism is a rather general as well as flexible theoretical framework to study the field theory and many-body systems. It describes the equilibrium and nonequilibrium phenomena on a unified basis. The ordinary quantum field theory and the classical fluctuation field theory are included in this formalism as different limits. The two aspects of the Liouville problem, i.e., the dynamic evolution and the statistical correlation are incorporated into it in a natural way. The formalism is well adapted to consider systems with symmetry breaking described by either constituent or composite order parameters. If different space-time variation scales can be distinguished, a macroscopic or mesoscopic description can be provided for inhomogeneous systems from the first principles.

(2) The powerful machinery of the quantum field theory including the generating functional technique and the path integral representation can be transplanted and further developed in the CTPGF formalism to study the general structure of the theory. The implications of the normalization condition for the generating functional and the causality are explored. The consequences of the time reversal symmetry such as the potential condition, the generalized fluctuation–dissipation theorem and reciprocity relations for kinetic coefficients are derived. The role of the initial correlation is clarified. The symmetry properties of the system under consideration are studied to derive the Ward–Takahashi identities. Also, a general theory of nonlinear response is worked out.

(3) A practical calculation scheme is worked out which derives a system of coupled, self-consistent equations to determine the order parameter along with the energy spectrum, the dissipation and the particle number distribution for both constituent fermions and collective excitations. A systematic loop expansion is developed to calculate the self-energy parts. The Bogoliubov–de Gennes equation is generalized to include the exchange and correlation effects. A way of computing the free energy by a straightforward integration of the functional equation is found.

(4) The general formalism is applied to a number of physical problems such as critical dynamics, superconductivity, spin system, plasma, laser, quenched random systems like spin glass and disordered electron system, quasi-one-dimensional conductors and so on. Although most of these problems can be and have been discussed using other formalisms, but, as far as we know, the CTPGF approach is the

only one to consider them within a unified framework. Moreover, new results, new insight or significant simplifications are always found by using the CTPGF approach.

(5) In general, systems in stationary state or near it have been studied more thoroughly, whereas the transient processes need further investigation. As far as the formalism itself is concerned, the two-point functions are well under control, but the properties of multi-point functions must be explored further in the future.

Our general impression, or our partiality, is that the potentiality of the CTPGF formalism is still great. One has to overcome the “potential barrier” occurring due to its apparent technical complexity to appreciate its logical simplicity and power. It is certainly not a piece of virgin soil, but the efforts of a dedicated explorer will be more than justified. We hope that in applying this formalism to attack more difficult problems in condensed matter, plasma, nuclear physics as well as particle physics and cosmology, its beauty and power will be uncovered to a greater extent.

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Note added in proof

After our manuscript was submitted we became aware of some more references [152–158] where the CTPGF formalism was applied to various problems. We would appreciate other colleagues to inform us of their work using this approach.

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