Thermodynamics of quantum information scrambling

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(Received 16 December 2016; published 20 June 2017)

Scrambling of quantum information can conveniently be quantified by so-called out-of-time-order correlators (OTOCs), i.e., correlators of the type $\langle [W_\tau, V]^\dagger [W_\tau, V]\rangle$, whose measurements present a formidable experimental challenge. Here we report on a method for the measurement of OTOCs based on the so-called two-point measurement scheme developed in the field of nonequilibrium quantum thermodynamics. The scheme is of broader applicability than methods employed in current experiments and provides a clear-cut interpretation of quantum information scrambling in terms of nonequilibrium fluctuations of thermodynamic quantities, such as work and heat. Furthermore, we provide a numerical example on a spin chain which highlights the utility of our thermodynamic approach when understanding the differences between integrable and ergodic behaviors. We also discuss how the method can be used to extend the reach of current experiments.

DOI: 10.1103/PhysRevE.95.062127

I. INTRODUCTION

How macroscopic irreversibility emerges from reversible microscopic dynamics is one of the most studied problems in statistical mechanics since the early days of the famous debate between Boltzmann and Loschmidt [1]. Popular quantifiers of irreversibility are the Loschmidt echo [2] and the irreversible entropy production [3]. Recently much attention has been devoted to yet another quantifier, namely, the so-called out-of-time-order correlator (OTOC), defined as

$$\begin{aligned} C_{V,W}(\tau) &= \langle [W_\tau, V]^\dagger [W_\tau, V]\rangle \\ &= \langle V^\dagger W_\tau^\dagger W_\tau V\rangle + \langle W_\tau^\dagger V^\dagger V W_\tau\rangle \\ &\quad - 2\,\mathrm{Re}\langle W_\tau^\dagger V^\dagger W_\tau V\rangle, \end{aligned} \tag{1}$$

where V,W are generic Hermitian or unitary operators. The concept was first introduced by Larkin and Ovchinnikov [4] to investigate the instability of semiclassical electron trajectories in superconductors and is currently employed as a tool for diagnosing quantum chaos [5]. The name OTOC reflects the fact that the correlator $C_{V,W}(\tau)$ contains the term,

$$F_{V,W}(\tau) = \langle W_\tau^\dagger V^\dagger W_\tau V\rangle, \tag{2}$$

where the operators are not ordered in time. Such lack of ordering poses a major experimental challenge for which they had been dubbed "unartig Korrelatoren" (i.e., "naughty correlators") [6].

Interest in OTOCs is currently undergoing a revival [7–18] after Kitaev pointed out their relevance in the context of holographic duality [19]. Accordingly a number of experimental schemes have recently been proposed for the measurement of the OTOC, and the first experimental measurements thereof have just been reported. Swingle *et al.* [12] propose an interferometric scheme where the OTOC is encoded into the quantum state of an ancilla, to be implemented in a cold-atom setup. Recent experiments, performed with a trapped ion quantum magnet and NMR [20,21], respectively, report a method for measuring infinite temperature (i.e., $\rho \propto \mathbb{1}$, ρ is the density matrix) OTOCs, i.e., OTOCs of the form $F_{V,W}(t) = \mathrm{Tr}(W_t^\dagger V^\dagger W_t V)$. The fact that ρ does not appear here introduces a great simplification in the implementation of the scheme where now one can access $F_{V,W}(t)$ by preparing the system in the service-state $\rho' \propto V + a\mathbb{1}$ ($\mathbb{1}$ stands for the identity operator) and measure the simpler three-point correlator $\mathrm{Tr}(W_t^\dagger V^\dagger W_t \rho')$ instead (see below for a more detailed discussion of this point).

Here we propose alternative methods to access $F_{V,W}(t)$ and $C_{V,W}(\tau)$ experimentally which do not require the employment of an ancillary system nor of a service state and is applicable to a broad set of states ρ including thermal states. The method is inspired by a scheme that proved extremely successful for the investigation of nonequilibrium quantum thermodynamics, namely, the so-called two-point measurement scheme [22–28]. Two-point measurement schemes are protocols where an observable is measured twice at the beginning and at the end of some nonequilibrium manipulation carried on a system. Their experimental feasibility has recently been demonstrated in Ref. [29]. The statistics of the observed change in the measured values encodes information about the manipulation. In our scheme an observable O is measured projectively at the beginning and at the end of the information scrambling manipulation (which we dub the "wing-flap protocol" due to its connection with the idea of the butterfly effect coming from nonlinear physics), and the change in its value ΔO is recorded. In the wing-flap protocol a system evolves according to some forward evolution $e^{-i\tau H}$ and then goes back with the backward evolution $e^{i\tau H}$. The two evolutions are interrupted by a wing flap (i.e., the application of some unitary operator W) that prevents the system to trace back to where it came from, see Fig. 1. As we will see information about quantum information scrambling is encoded in the statistics of the observed changes in the measured values of O.

II. TWO-POINT MEASUREMENT

We first focus on the F correlator in the case when V,W are unitary operators. Let us express V as

$$V = e^{iuO}, \tag{3}$$

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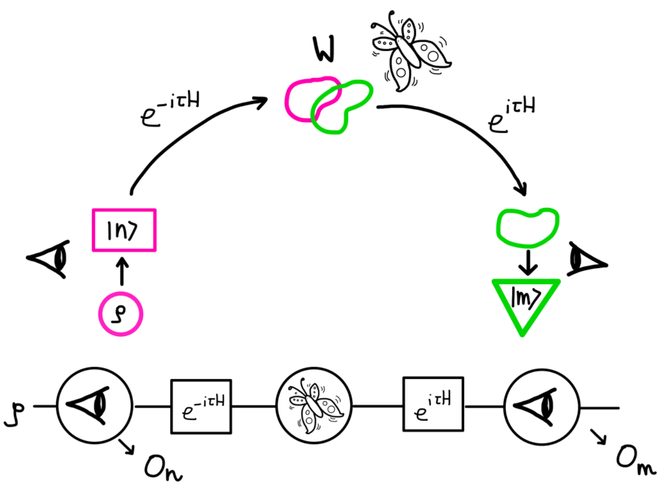

FIG. 1. The wing-flap protocol.

with the appropriate Hermitian operator O and real number u. Let us then consider the following protocol, which we dub the wing-flap protocol, see Fig. 1:

(1) Prepare the system in some state ρ.
(2) Measure O.
(3) Evolve the system with H for a time $t = \tau$.
(4) Apply the wing-flap unitary perturbation W.
(5) Evolve the system with $-H$ for a time $t = \tau$.
(6) Measure O.

The two measurements of O are assumed to be projective giving eigenvalues O_n, O_m, respectively, and collapsing the system on the according eigenstates $|n\rangle, |m\rangle$. For simplicity we assume, without lack of generality, that the eigenvalues of O are nondegenerate. We will call the evolution under H, the forward evolution, and the evolution under $-H$, the backward evolution [30].

In the absence of a wing flap, i.e., when W is the identity operator $\mathbb{1}$, the state of the system at the end of the backward evolution would be exactly the initial state $|n\rangle$ of the forward evolution, and the second measurement would not alter it. When there is wing-flap $W \neq \mathbb{1}$, the state at the end of the backward evolution is in general a linear combination of all eigenstates of O, and the second measurement selects one of them $|m\rangle$, which may differ from $|n\rangle$ due to the information scrambling.

On repeating the wing-flap protocol many times, in each realization the observed eigenvalues O_n, O_m assume random values, and by repeating the protocol an infinite number of times one can build the probability density function (pdf) $p(\Delta O, \tau)$ of observing a change,

$$\Delta O = O_m - O_n, \tag{4}$$

in the quantity O. The statistics $p(\Delta O, \tau)$ contains information about the OTOC Eq. (2). This connection can be established at a formal level as follows. Let

$$G(k,\tau) = \int p(\Delta O, \tau) e^{-ik\,\Delta O} d\,\Delta O \tag{5}$$

be the characteristic function of $p(\Delta O, \tau)$, i.e., its Fourier transform [31]. Our first main result is

$$F_{e^{iuO},W}(\tau) = G(u,\tau). \tag{6}$$

Equation (6) says that the OTOC associated with the operators W, e^{iuO} is identical to the characteristic function of the random variable ΔO in the wing-flap protocol above.

To prove the statement consider the formal expression of $p(\Delta O, \tau)$,

$$p(\Delta O, \tau) = \sum_{n,m} \delta[\Delta O - O_m + O_n] P_\tau[m|n] p_n, \tag{7}$$

where $p_n = \mathrm{Tr}\, \Pi_n \rho$ is the probability of observing O_n in the first measurement and $\delta[x]$ stands for Dirac's δ function. For simplicity, in the following we will be restricted to the case of $[\rho, O] = 0$. This is the only restriction we have on ρ. The symbol $P_\tau[m|n]$ stands for the probability of observing O_m in the second measurement given that O_n was observed in the first measurement,

$$P_\tau[m|n] = |\langle m|U_\tau|n\rangle|^2, \tag{8}$$

where U_τ is the unitary describing the evolution between the two measurements. It reads

$$U_\tau = e^{i\tau H} W e^{-i\tau H}. \tag{9}$$

Note that U_τ is the operator W at time τ in the Heisenberg representation: $U_\tau = W_\tau$. Using (5), (7)–(9), the resolution of identity $\sum_m |m\rangle\langle m| = \mathbb{1}$ and $[\rho, O] = 0$, we obtain

$$\begin{aligned} G(u,\tau) &= \sum_{n,m} e^{-iuO_m} e^{iuO_n} P_\tau[m|n] p_n \\ &= \sum_{n,m} \langle n|U^\dagger(\tau)|m\rangle e^{-iuO_m} \langle m|U_\tau|n\rangle e^{iuO_n} p_n \\ &= \sum_n \langle n|U_\tau^\dagger e^{-iuO} U_\tau e^{iuO}|n\rangle p_n \\ &= \mathrm{Tr}\, U_\tau^\dagger e^{-iuO} U_\tau e^{iuO} \rho \\ &= \mathrm{Tr}\, W_\tau^\dagger V^\dagger W_\tau V \rho. \end{aligned} \tag{10}$$

Thus the characteristic function of the statistics $p(\Delta O, \tau)$ of changes in ΔO is the OTOC between e^{iuO} and W. Practically it means that one can experimentally access $F_{e^{iuO},W}(\tau)$ by measuring the two-point statistics of O and then Fourier transform it. In a typical experimental scenario O might be, for example, a local spin operator, and W might be a local rotation at a different site as in Ref. [21]. Such a scheme is well within the reach of current experimental setups.

III. THERMODYNAMICS OF INFORMATION SCRAMBLING

The realization that OTOCs can be recast as two-point measurement protocols allows us to directly connect with nonequilibrium quantum thermodynamics. Consider the case when the measured quantity O is the Hamiltonian H_0 of a quantum system $O = H_0$. In this case Eq. (6) implies that $F_{e^{iuH_0},W}(\tau)$ is the characteristic function of work $G_{\mathrm{work}}(u,\tau)$, namely, the Fourier transform of the work probability distribution function,

$$p(w,\tau) = \sum_{n,m} \delta[w - e_m + e_n] P_\tau[m|n] p_n. \tag{11}$$

Here w denotes work, i.e., the difference of final and initial measured system eigenenergies e_k. In absence of the wing

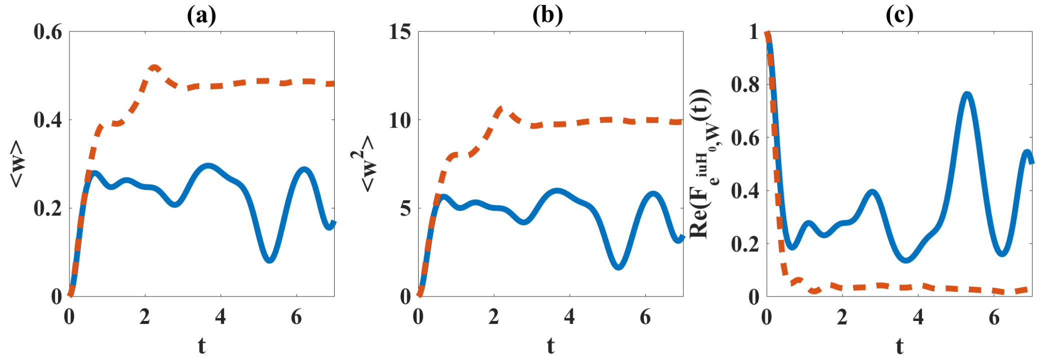

FIG. 2. Shown here are (a) the time dependent mean and (b) second moment of the work distribution for the quench protocol described in the text. (c) shows the real part of the OTOC $F_{e^{iuO},W}(\tau)$ for $u = 1$. Here we focus on a chain of nine spins at inverse temperature $\beta = 0.1$ contrasting both integrable (blue solid line) and nonintegrable (orange dashed line) dynamics. We use the model and parameters specified in Ref. [39] ($J = 1,\ g = 0.904\,508\,49, h = 0.809\,0169$).

flap no work is performed on the system, whereas in general, when a wing-flap operator W is present, work is performed on the system, and the distribution drifts and spreads. The spread, namely, the second moment of the work distribution can be cast in the form $\langle w^2\rangle = \int dw\ p(w,\tau)w^2 = \mathrm{Tr}(W_\tau^\dagger H_0 W_\tau - H_0)^2\rho$. Simple manipulations show that it is indeed the square of the commutator between W_τ and H_0,

$$\langle w^2\rangle = \langle [W_\tau, H_0]^\dagger [W_\tau, H_0]\rangle = C_{H_0,W}(\tau). \tag{12}$$

In the case of an initial thermal equilibrium $\rho = e^{-\beta H_0}/Z$ (β denotes the inverse temperature, and $Z = \mathrm{Tr}\, e^{-\beta H_0}$ denotes the partition function), the first moment $\langle w\rangle = \mathrm{Tr}(W_\tau^\dagger H_0 W_\tau - H_0)\rho$ of the work distribution can be written in terms of the relative entropy $S[\rho_\tau \parallel \rho]$ between the initial state ρ and its evolved ρ_τ [32,33],

$$\langle w\rangle = \beta^{-1} S[\rho_\tau \parallel \rho] = \beta^{-1}\mathrm{Tr}(\rho_\tau \ln \rho_\tau - \rho_\tau \ln \rho). \tag{13}$$

The quantum relative entropy is a measure of the distinguishability of the two quantum states $\rho, \rho_\tau = W_\tau \rho W_\tau^\dagger$, and its non-negativity is a consequence of the second law of thermodynamics [34]. It accordingly provides a meaningful quantifier of information scrambling occurring as a consequence of wing flapping [35].

The characteristic function of work, the second moment of the work distribution, and the first moment all encode various aspects of information scrambling. We remark that all of them can be inferred from the work pdf $p(w,\tau)$, however the last two can also be accessed without measuring $p(w,\tau)$. In fact, in order to measure the first or second moment $\langle w^a\rangle$, $a = 1,2$, one can measure the expectation of H_0^a in the initial state ρ and its expectation in the final state ρ_τ and then take their difference, which is a much simpler procedure. It is worth stressing that, if after the wing-flap protocol the system is brought back in contact with a thermal bath as to reestablish its initial thermal state, the quantity $\beta^{-1} S[\rho_\tau \parallel \rho]$ is equal to the average heat $\langle q\rangle$ that the system releases into the bath [36]. Thus information scrambling might be accessed not only through work measurements, but also through heat measurements. Calorimetric measurement schemes being developed for low temperature solid state devices [37,38] could be used for this purpose.

IV. ILLUSTRATIVE EXAMPLE

In order to illustrate our results, we consider a spin chain of length L described by the Hamiltonian,

$$H_0 = g\sum_{i=1}^{L}\sigma_x^i \tag{14}$$

prepared in the thermal state $\rho = e^{-\beta H_0}/Z$. At time $t = 0$ we turn on a perturbation H_i so that the system evolves with the Hamiltonian $H = H_0 + H_i$. At time $t = \tau/2$ we instantaneously apply the wing-flap operator (a rotation about the x axis) $W = e^{-i\theta\sigma_x^k}$ with $\theta = \pi/2$. The system then evolves with $-H$ until time τ. We consider the following forms of evolutions generated by Hamiltonians H_i:

$$H_1 = J\sum_{i=1}^{L-1}\sigma_z^i\sigma_z^{i+1}, \tag{15}$$

$$H_2 = J\sum_{i=1}^{L-1}\sigma_z^i\sigma_z^{i+1} + h\sum_{i=1}^{L-1}\sigma_z^i + (h-J)\left(\sigma_z^1 + \sigma_z^L\right), \tag{16}$$

corresponding to integrable and ergodic dynamics, respectively. This model was recently used by Kim and Huse in order to understand the phenomenology of entanglement growth in ergodic and integrable systems [39]. We choose identical parameters here. Figure 2 shows the temporal behavior of the real part of the OTOC $F_{e^{iuH_0},W}(\tau)$, the mean and second moment of the associated work distributions for nine spins. In all cases the wing-flap operator is acting on the central spin ($k = 5$), and both integrable and ergodic evolutions are displayed. Note how, as also shown in a closely related experiment [21], the integrable case is characterized by oscillations, i.e., recurrences, whereas no recurrence is observed in the time span over which the simulation is carried in the ergodic case. This reflects the irreversible information scrambling occurring in the nonintegrable case. Note also that, despite nonintegrability, no exponential behavior is observed in the time dependence of the OTOC; this is in agreement with previous numerical [40] and experimental findings [21]. The plots reveal a small mean work as compared to the second moment $\langle w\rangle^2 \ll \langle w^2\rangle$ and a proportionality between them in accordance with linear response theory $\langle w\rangle \simeq \beta(\langle w^2\rangle - \langle w\rangle^2)/2 \simeq \beta\langle w^2\rangle/2$ [41,42].

V. EXTENDING THE REACH OF CURRENT EXPERIMENTS

Our analysis allows for extending the reach of existing experiments, e.g., Refs. [20,21]. These experiments, as mentioned in the Introduction, measure infinite temperature OTOCs by means of a service-state ρ'. Let us now comment on the experiment in Ref. [20]. There V is proportional to $S_x = \sum_i \sigma_x^i$, that is, the x component of the total magnetization of a system of N spins. The system is prepared in the factorized service-state ρ' with all spins pointing up in the x direction. It then evolves under some wing-flap protocol W_τ of the type described above. The authors measure the expectation $\langle S_x\rangle$ at the end of the protocol. Interpreting S_x as the initial Hamiltonian, i.e., $H_0 = S_x$, that is, in fact a measurement of the change in expectation of H_0, namely, a measurement of average work $\langle w\rangle$. Due to the special preparation $\rho' = H_0 + (N/2)\mathbb{1}$, the latter assumes the form of an infinite temperature OTOC:

$$\langle w\rangle = \mathrm{Tr}(W_\tau^\dagger H_0 W_\tau - H_0)\rho' = \mathrm{Tr}\, W_\tau^\dagger H_0 W_\tau H_0 - \mathrm{Tr}\, H_0^2.$$

The same measurement of $\langle S_x\rangle \propto \langle w\rangle$ carried on a generic finite temperature thermal state $\rho \propto e^{-\beta H_0}$ would give the relative entropy $S[\rho_\tau \parallel \rho]$ according to Eq. (13). Under the provision of linear response theory that would also give directly the second moment of the work distribution $\langle w^2\rangle$, hence the average square commutator $\langle |[W_\tau, H_0]|^2\rangle$ according to Eq. (12). We thus see that the thermodynamic connection pointed out above directly allows to extend the reach of current experiments, allowing the so far elusive measurement of finite temperature OTOCs.

VI. CONCLUSIONS

To summarize we have put forward an alternate method for measuring information scrambling. The alternate method consists of a two time measurement scheme using projective measurements or simpler measurements of expectations. At variance with previously proposed schemes, the present one does not require the employment of an ancillary system nor the preparation of a service state and is applicable to generic finite temperature conditions. Accordingly, its scope of applicability is broader than current methods. The scheme not only offers a practical alternative for the measurement of OTOCs as compared to existing proposals, but also reveals a fundamental connection between nonequilibrium fluctuations of fundamental thermodynamics quantities, namely, work and heat and scrambling of information.

ACKNOWLEDGMENTS

M.C. was supported by Unicredit Bank and by a Marie Curie Intra European Fellowship within the 7th European Community Framework Programme through the Project NeQuFlux Grant No. 623085. J.G. and M.C. were partially supported by the COST action Grant No. MP1209 “Thermodynamics in the quantum regime”. J.G. first heard of the topic of OTOC at Kitaev’s lecture for the presentation of the 2015 Dirac Medal at ICTP [43] and would like to thank V. Varma for convincing him that it was interesting. J.G. thanks A. Silva and G. Guarnieri for useful comments.

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sign of all momenta and keeps positions unaltered). (ii) Let the system evolve for a time $t = \tau$ under H, and (iii) apply the antiunitary time reversal operator Θ [44]. The overall effect is $\Theta^\dagger e^{-i\tau H}\Theta = e^{i\tau H}$. If a magnetic field B is present in the forward evolution, i.e., $H = H(B)$, one should reverse it in the backward evolution, that is, in point (ii) above the evolution should occur under $H(-B)$ [26].

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