# Quantum Clocks, Gravitational Time Dilation, and Quantum Interference

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A proper time observable for a quantum clock is introduced and it is found that the proper time read by one clock conditioned on another clock reading a different proper time obeys classical time dilation in accordance with special relativistic kinematical time dilation. Here, we extend this proposal to a weak gravitational field in order to investigate whether the weak equivalence principle holds for quantum matter. We find that for general quantum states the quantum time dilation in a weak gravitational field obeys a similar gravitational time dilation found in classical relativity. However, unlike the special relativistic case, the time dilation involves the external time (a background coordinate time at the observer on the Earth) as well as the proper times of two clocks. We also investigate a quantum time dilation effect induced by a clock in a superposition of wave packets localized in momentum space or in position space and propose the setup to observe the gravitational effect in the quantum interference effect in the time dilation.

#### I. INTRODUCTION

The equivalence principle played the fundamental role in constructing general relativity (GR) by Einstein. The weak equivalence principle (WEP) states that the motion of a (uncharged) test body is independent of its internal structure and composition [1]. With the WEP together with the local Lorentz invariance (independence of the results of nongravitational experiments from the velocity of the local Lorentz frame) and the local position invariance (the independence of the experimental results from the spacetime position), the metric couples universally to matter. However, since this "universal coupling to matter" is the root of the problem of the cosmological constant, the validity of the WEP should be examined further, especially in the quantum regime. Tests of the weak equivalence principle have been performed by measuring relative acceleration between different atoms using atom interferometers [2–8].

There are several attempts to formulate the equivalence principle in the quantum regime [9–15]. In addition to the position of a particle which is measured by interferometric experiments, an important observable quantity regarding the equivalence principle is the proper time of a clock because the gravitational time dilation is the direct consequence of the equivalence principle [1]. Measurements of the gravitational time dilation by comparisons of atomic clocks located at different heights have been performed [16–19].

Recently, a proper time observable for a quantum clock is proposed in [20] and the probability that one clock reads a given proper time conditioned on another clock reading a different proper time is derived. Moreover, it is shown that when the external degrees of freedom of these clock particles are described by Gaussian wave packets localized in momentum space, the clocks observe classical time dilation in accordance with special relativistic kinematical time dilation.

In this paper, we extend this proposal to a weak gravitational field and study the time dilation of quantum particles under the Newtonian gravity in order to investigate whether quantum particles move in the same manner as the classical particles. Moreover, we are able to derive the quantum time dilation between two clocks for general quantum states not limited to Gaussian wave packets. Thanks to the existence of the conserved quantities associated with the Galilean invariance and the translation invariance of the system, it can be shown that the gravitational time dilation depends only on the expectation values of the positions and the velocities of clocks at the initial time. This may be regarded as the weak equivalence principle for quantum system [10].

The paper is organized as follows. In Sec. II, we derive the quantum time dilation formula in a weak gravitational field. In Sec. III, we calculate the gravitational time dilation for a linear gravitational potential. We also consider a quantum time dilation effect induced by a clock in a superposition of wave packets. Sec. IV is devoted to summary. In Appendix A, the time dilation for Gaussina wavepackets is given, and in Appendix B, the initial expectation values for a superposition states are given.

#### II. QUANTUM CLOCK PARTICLES IN SPACETIME

#### A. Classical Particles

We consider a system of N free particles. Each particle whose mass is  $m_n$  (n = 1, ..., N) has a set of internal degrees of freedom, labeled by the configuration variables  $q_n$  and their conjugate momenta  $P_{q_n}[20]$ . These internal degrees of freedom is supposed to represent the quantum clock.

The action of such a system in a curved spacetime with the metric  $g_{\mu\nu}$  is given by

$$S = \sum_{n} \int d\tau_n \left( -m_n c^2 + P_{q_n} \frac{dq_n}{d\tau_n} - H_n^{\text{clock}} \right), \tag{1}$$

where  $\tau_n$  is the proper time of the *n*th particle and  $H_n^{\text{clock}} = H_n^{\text{clock}}(q_n, P_{q_n})$  is the Hamiltonian for its internal degrees of freedom.

Let  $x_n^{\mu}$  denote the spacetime position of the *n*th particle. The trajectory of the *n*th particle  $x_n^{\mu}(t)$  is parametrized by an arbitrary external time parameter t. Noting that  $cd\tau_n = \sqrt{-g_{\mu\nu}\dot{x}_n^{\mu}\dot{x}_n^{\nu}}dt \equiv \sqrt{-\dot{x}_n^2}dt$ , where a dot denotes differentiation with respect to t, the action is rewritten as

$$S = \int dt \sum_{n} \frac{1}{c} \sqrt{-\dot{x}_n^2} \left( -m_n c^2 + P_{q_n} c \frac{\dot{q}_n}{\sqrt{-\dot{x}_n^2}} - H_n^{\text{clock}} \right) =: \int dt \ L. \tag{2}$$

The momentum conjugate to  $x_n^{\mu}$  is given by

$$P_{n\mu} = \frac{\partial L}{\partial \dot{x}_n^{\mu}} = \frac{g_{\mu\nu}\dot{x}_n^{\nu}}{c\sqrt{-\dot{x}_n^2}} \left( m_n c^2 + H_n^{\text{clock}} \right). \tag{3}$$

Then the Hamiltonian associated with the Lagrangian L is constrained to vanish:

$$H = \sum_{n} (P_{n\mu} \dot{x}_{n}^{\mu} + P_{q_{n}} \dot{q}_{n}) - L \approx 0.$$
 (4)

In terms of the momentum, the constraints can be expressed in the form

$$C_{H_n} := g^{\mu\nu} P_{n\mu} P_{n\nu} c^2 + \left( m_n c^2 + H_n^{\text{clock}} \right)^2 \approx 0.$$
 (5)

Using the (3+1) decomposition of the metric in terms of the lapse function  $\alpha$ , the shift vector  $\beta^i$  and the three-metric  $\gamma_{ij}$  such that [21]

$$ds^2 = -\alpha^2 c^2 dt^2 + \gamma_{ij} (dx^i + \beta^i cdt) (dx^j + \beta^j cdt), \tag{6}$$

the constraint is factorized in the form

$$C_{H_n} = -\alpha^{-2} \left( P_{n0} - \beta^i P_{ni} \right)^2 c^2 + \gamma^{ij} P_{ni} P_{nj} c^2 + \left( m_n c^2 + H_n^{\text{clock}} \right)^2 = -\alpha^{-2} C_n^+ C_n^- \approx 0, \tag{7}$$

where  $C_n^{\pm}$  is defined by

$$C_n^{\pm} := \left( P_{n0} - \beta^i P_{ni} \right) c \pm h_n, \tag{8}$$

$$h_n := \alpha \sqrt{\gamma^{ij} P_{ni} P_{nj} c^2 + (m_n c^2 + H_n^{\text{clock}})^2}. \tag{9}$$

Note that we have set  $x^0 = ct$ . Hereafter we assume that the spacetime is static and the shift vector is vanishing,  $\beta^i = 0$ . The coordinates  $x_n^{\mu}$  and their conjugate momenta  $P_{n\mu}$  satisfy the fundamental Poisson brackets:  $\{x_m^{\mu}, P_{n\nu}\} = \delta_{mn}\delta_{\nu}^{\mu}$ . The canonical momentum  $P_{n\mu}$  generates translations in the spacetime coordinate  $x_n^{\mu}$ . Therefore, if  $C_n^{\pm} \approx 0$  and the shift vector vanishes, then  $\pm h_n$  is the generator of translation in the *n*th particle's time coordinate and is the Hamiltonian for both the external and internal degrees of freedom of the *n*th particle.

#### B. Quantization

We canonically quantize the system of N particles by promoting the phase space variables to operators acting on appropriate Hilbert spaces:  $x_n^0$  and  $P_{n0}$  become canonically conjugate self-adjoint operators acting on the Hilbert space  $\mathcal{H}_n^0 \simeq L^2(\mathbb{R})$  associated with the nth particle's temporal degree of freedom; operators  $x_n^i$  and  $P_{ni}$  acting on the Hilbert space  $\mathcal{H}_n^{\text{ext}} \simeq L^2(\mathbb{R}^3)$  associated with the particle's external degrees of freedom; operators  $q_n$  and  $P_{q_n}$  acting on the Hilbert space  $\mathcal{H}_n^{\text{clock}}$  associated with the particle's internal degrees of freedom. Then the Hilbert space describing the nth particle is  $\mathcal{H}_n \simeq \mathcal{H}_n^0 \otimes \mathcal{H}_n^{\text{ext}} \otimes \mathcal{H}_n^{\text{clock}}$ .

The constraint equations (7) now become operator equations restricting the physical state of the theory,

$$C_n^+ C_n^- |\Psi\rangle = 0, \quad \forall n,$$
 (10)

where  $|\Psi\rangle\rangle \in \mathcal{H}_{\mathrm{phys}}$  is a physical state of a clock C and a system S and lives in the physical Hilbert space  $\mathcal{H}_{\mathrm{phys}}$ . To specify  $\mathcal{H}_{\mathrm{phys}}$ , following Page and Wooters [22, 23] (see also [24–26]), the normalization of the physical state in  $\mathcal{H}_{\mathrm{phys}}$  is performed by projecting a physical state  $|\Psi\rangle\rangle$  onto a subspace in which the temporal degree of freedom of each particle (clock C) is in an eigenstate  $|t_n\rangle$  of the operator  $x_n^0$  associated with the eigenvalue  $t\in\mathbb{R}$  in the spectrum of  $x_n^0$ :  $x_n^0|t_n\rangle=ct|t_n\rangle$ . The state of S by conditioning  $|\Psi\rangle\rangle$  on C reading the time t is then given by

$$|\psi_S(t)\rangle = \langle t|\otimes I_S|\Psi\rangle,\tag{11}$$

where  $|t\rangle = \otimes_n |t_n\rangle$  and  $I_S$  is the identity on  $\mathcal{H} \simeq \bigotimes_n \mathcal{H}_n^{\text{ext}} \otimes \mathcal{H}_n^{\text{clock}}$ . We demand that the state  $|\psi_S(t)\rangle$  is normalized as  $\langle \psi_S(t)|\psi_S(t)\rangle = 1$  for  $\forall t \in \mathbb{R}$  on a spacelike hypersurface defined by all N particles' temporal degree of freedom being in the state  $|t_n\rangle$ . The physical state  $|\Psi\rangle\rangle$  is thus normalized with respect to the inner product [20]:

$$\langle \langle \Psi | \Psi \rangle \rangle_{PW} := \langle \langle \Psi | | t \rangle \langle t | \otimes I_S | \Psi \rangle \rangle = \langle \psi_S(t) | \psi_S(t) \rangle = 1, \tag{12}$$

and the physical state  $|\Psi\rangle$  can be written as

$$|\Psi\rangle\rangle = \int dt |t\rangle\langle t| \otimes I_S |\Psi\rangle\rangle = \int dt |t\rangle |\psi_S(t)\rangle.$$
 (13)

Hereafter, we consider physical states that satisfy

$$C_n^+|\Psi\rangle\rangle = (P_{n0}c + h_n)|\Psi\rangle\rangle = 0, \quad \forall n,$$
 (14)

where  $h_n$  is the operator corresponding to Eq. (9). This implies that the conditioned state  $|\psi_S(t)\rangle$  has positive energy as measured by  $h_n$ . It can be shown that the conditioned state  $|\psi_S(t)\rangle$  satisfies the Schrödinger equation with t as a time parameter [20]:

$$i\hbar \frac{d}{dt}|\psi_S(t)\rangle = H_S|\psi_S(t)\rangle,$$
 (15)

where  $H_S = \sum_n h_n \otimes I_{S-n}$  with  $I_{S-n}$  being the identity on  $\bigotimes_{m \neq n} \mathcal{H}_m^{\text{ext}} \otimes \mathcal{H}_m^{\text{clock}}$ . Therefore,  $|\psi_S(t)\rangle$  can be regarded as the time-dependent state of the N-particles with the Hamiltonian  $H_S$  evolved with the external time t.

# C. Proper Time Observables

In [20], a clock is defined to be the quadrupole  $\{\mathcal{H}^{\text{clock}}, \rho_C, H^{\text{clock}}, T_C\}$ , where  $\rho_C$  is a fiducial state and  $T_C$  is proper time observable. The proper time observable is defined as a POVM (positive operator valued measure)

$$T_C := \left\{ E_C(\tau) \ \forall \tau \in G \text{ s.t.} \int_G d\tau E_C(\tau) = I_C \right\},\tag{16}$$

where  $E_C(\tau) = |\tau\rangle\langle\tau|$  is a positive operator on  $\mathcal{H}^{\text{clock}}$ , G is the group generated by  $H^{\text{clock}}$ , and  $|\tau\rangle$  is a clock state associated with a measurement of the clock yielding the time  $\tau$ . The clock state  $|\tau\rangle$  evolves according to  $U_C(\tau) = e^{-iH^{\text{clock}}\tau/\hbar}$  as

$$|\tau + \tau'\rangle = U_C(\tau')|\tau\rangle.$$
 (17)

#### D. Probabilistic Time Dilation

Consider two clock particles A and B with internal degrees of freedom,  $\{\mathcal{H}_A^{\text{clock}}, \rho_A, H_A^{\text{clock}}, T_A\}$  and  $\{\mathcal{H}_B^{\text{clock}}, \rho_B, H_B^{\text{clock}}, T_B\}$ . To probe time dilation effects between two clocks, we consider the probability that clock A reads the proper time  $\tau_A$  conditioned on clock B reading the proper time  $\tau_B$  [22, 23]. This conditional probability is given in terms of the physical state as

$$\operatorname{Prob}[T_A = \tau_A | T_B = \tau_B] = \frac{\langle \langle \Psi | E_A(\tau_A) E_B(\tau_B) | \Psi \rangle \rangle}{\langle \langle \Psi | E_B(\tau_B) | \Psi \rangle \rangle}. \tag{18}$$

Consider the case where two clock particles A and B are moving in a curved spacetime. Suppose that initial conditioned state is unentangled,  $|\psi_S(0)\rangle = |\psi_{S_A}\rangle|\psi_{S_B}\rangle$ , and that the external and internal degrees of freedom of both particles are unentangled,  $|\psi_{S_n}\rangle = |\psi_n^{\text{ext}}\rangle|\psi_n^{\text{clock}}\rangle$ . Then, from Eq. (13), the physical state takes the form

$$|\Psi\rangle\rangle = \int dt |t\rangle|\psi_S(t)\rangle = \int dt \bigotimes_{n \in \{A,B\}} e^{-ih_n t/\hbar} |\psi_n^{\text{ext}}\rangle|\psi_n^{\text{clock}}\rangle.$$
 (19)

Further suppose that  $\mathcal{H}_n^{\text{clock}} \simeq L^2(\mathbb{R})$  so that we may consider an ideal clock such that  $P_n = H_n^{\text{clock}}/c$  and  $cT_n$  are the momentum and position operators on  $\mathcal{H}_n^{\text{clock}}$ . The canonical commutation relation yields  $[cT_n, P_n] = [T_n, H_n^{\text{clock}}] = i\hbar$ . Then, the clock states are orthogonal  $\langle \tau | \tau' \rangle = \delta(\tau - \tau')$  and satisfy the covariance relation  $|\tau + \tau'\rangle = U_C(\tau')|\tau\rangle$ . The conditional probability (18) becomes

$$\operatorname{Prob}[T_A = \tau_A | T_B = \tau_B] = \frac{\int dt \operatorname{tr}[E_A(\tau_A)\rho_A(t)]\operatorname{tr}[E_B(\tau_B)\rho_B(t)]}{\int dt \operatorname{tr}[E_B(\tau_B)\rho_B(t)]},$$
(20)

where  $\rho_n(t)$  is the reduced state of the internal clock degrees of freedom defined as [20]

$$\rho_n(t) = \operatorname{tr}_{\mathcal{H}_S \setminus \mathcal{H}_n^{\text{clock}}} \left( e^{-iH_S t/\hbar} |\psi_{S_n}\rangle \langle \psi_{S_n} | e^{iH_S t/\hbar} \right)$$
(21)

with the trace over the complement of the clock Hilbert space.

We assume that the fiducial states of the internal clock degrees of freedom are the Gaussian wave packets centered at  $\tau = 0$  with width  $\sigma$ :

$$|\psi_n^{\text{clock}}\rangle = \frac{1}{\pi^{1/4}\sigma^{1/2}} \int d\tau \ e^{-\frac{\tau^2}{2\sigma^2}} |\tau\rangle. \tag{22}$$

# E. Gravitational Time Dilation

In order to investigate the effect of gravity on the quantum time dilation, consider a Newtonian approximation of spacetime and adopt the metric,  $g_{00} = -\alpha^2 = -(1 + 2\Phi(\mathbf{x})/c^2), \gamma_{ij} = \delta_{ij}$ , where  $\Phi(\mathbf{x})$  is the Newtonian gravitational potential. Then the Hamiltonian (9) is expanded according to the number of the inverse power of  $c^2$  as

$$h_{n} = \alpha m_{n} c^{2} \sqrt{\frac{\gamma^{ij} P_{ni} P_{nj}}{m_{n}^{2} c^{2}} + \left(1 + \frac{H_{n}^{\text{clock}}}{m_{n} c^{2}}\right)^{2}}$$

$$\simeq m_{n} c^{2} + H_{n}^{\text{clock}} + H_{n}^{\text{ext}} + H_{n}^{\text{int}} + O(c^{-4}), \tag{23}$$

where the rest-mass energy term  $m_n c^2$  is a constant and can be disregarded in  $h_n$ . The external Hamiltonian  $H_n^{\text{ext}}$  and the interaction Hamiltonian  $H_n^{\text{int}}$  are given by

$$H_n^{\text{ext}} := \frac{\delta^{ij} P_{ni} P_{nj}}{2m_n} + m_n \Phi_n \equiv \frac{\mathbf{P}_n^2}{2m_n} + m_n \Phi_n, \tag{24}$$

$$H_n^{\text{int}} := -\frac{H_n^{\text{ext}} H_n^{\text{clock}}}{m_n c^2} + 2 \frac{m_n \Phi_n (H_n^{\text{clock}} + H_n^{\text{ext}})}{m_n c^2} - \frac{(H_n^{\text{ext}})^2}{2m_n c^2} - 2 \frac{(m_n \Phi_n)^2}{m_n c^2},$$
 (25)

where  $\Phi_n := \Phi(\mathbf{x}_n)$ .

The reduced state of the internal clock is then given by

$$\rho_{n}(t) = \operatorname{tr}_{\mathcal{H}_{S} \backslash \mathcal{H}_{n}^{\operatorname{clock}}} \left[ e^{-iH_{S}t/\hbar} | \psi_{S_{n}} \rangle \langle \psi_{S_{n}} | e^{iH_{S}t/\hbar} \right] 
= \bar{\rho}_{n}(t) - it \operatorname{tr}_{\operatorname{ext}} \left( [H_{n}^{\operatorname{int}}, \bar{\rho}_{n}^{\operatorname{ext}}(t) \otimes \bar{\rho}_{n}(t)] + O((H_{n}^{\operatorname{int}}t)^{2}) \right) 
= \bar{\rho}_{n}(t) + it \left( \frac{\langle H_{n}^{\operatorname{ext}} \rangle}{m_{n}c^{2}} - 2 \frac{\langle \Phi_{n} \rangle}{c^{2}} \right) [H_{n}^{\operatorname{clock}}, \bar{\rho}_{n}(t)] + O(c^{-4}),$$
(26)

where  $\bar{\rho}_n(t) = e^{-iH_n^{\mathrm{clock}}t/\hbar}\rho_n e^{iH_n^{\mathrm{clock}}t/\hbar}$  and  $\bar{\rho}_n^{\mathrm{ext}}(t) = e^{-iH_n^{\mathrm{ext}}t/\hbar}\rho_n^{\mathrm{ext}}e^{iH_n^{\mathrm{ext}}t/\hbar}$ . The conditional probability (18) is evaluated to leading relativistic order as

$$\operatorname{Prob}[T_A = \tau_A | T_B = \tau_B] = \frac{e^{-\frac{(\tau_A - \tau_B)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \left[ 1 + \left( \frac{\langle H_A^{\text{ext}} \rangle}{2m_A c^2} - \frac{\langle H_B^{\text{ext}} \rangle}{2m_B c^2} - \frac{\langle \Phi_A \rangle}{c^2} + \frac{\langle \Phi_B \rangle}{c^2} \right) \left( 1 - \frac{\tau_A^2 - \tau_B^2}{\sigma^2} \right) \right], \tag{27}$$

where  $\langle H_n^{\text{ext}} \rangle = \langle \psi_n^{\text{ext}} | H_n^{\text{ext}} | \psi_n^{\text{ext}} \rangle$ . Then the average proper time read by clock A conditioned on clock B indicating the time  $\tau_B$  is

$$\langle T_A \rangle = \int d\tau \operatorname{Prob}[T_A = \tau | T_B = \tau_B] \tau$$

$$= \tau_B \left( 1 - \left( \frac{\langle H_A^{\text{ext}} \rangle}{m_A c^2} - 2 \frac{\langle \Phi_A \rangle}{c^2} \right) + \left( \frac{\langle H_B^{\text{ext}} \rangle}{m_B c^2} - 2 \frac{\langle \Phi_B \rangle}{c^2} \right) \right). \tag{28}$$

Since the external Hamiltonian is defined by Eq. (24),  $\langle T_A \rangle$  is written in a more suggestive form as

$$\langle T_A \rangle = \tau_B \left( 1 - \left( \frac{\langle \mathbf{P}_A^2 \rangle}{2m_A^2 c^2} - \frac{\langle \Phi_A \rangle}{c^2} \right) + \left( \frac{\langle \mathbf{P}_B^2 \rangle}{2m_B^2 c^2} - \frac{\langle \Phi_B \rangle}{c^2} \right) \right). \tag{29}$$

This is one of the main results of this paper and represents the quantum analog of time dilation formula in the Newtonian gravity.

### GRAVITATIONAL TIME DILATION OF QUANTUM CLOCKS

To evaluate the time dilation, for simplicity, we consider a one-dimensional problem in the vertical direction and assume that the gravitational potential is approximated by

$$\Phi = gx, \tag{30}$$

where q is the gravitational acceleration. x is the vertical coordinate from the surface of the Earth so that the external time coordinate t corresponds to the proper time of an observer at rest on the Earth. Here it is to be noted that unlike the case with the external Hamiltonian being  $\mathbf{P}_n^2/2m_n$  [20], the averages of  $P_n^2$  and  $x_n$  evolve in time since  $P_n$ does not commute with  $H_n^{\text{ext}}$ , (24). The external Hamiltonian (24),  $H_n^{\text{ext}} = P_n^2/2m_n + m_n g x_n$ , can be derived from the Lagrangian  $L_n$ 

$$L_n = \frac{1}{2}m_n\dot{x}_n^2 - m_n g x_n. {31}$$

Since  $L_n$  does not depend on the external time t explicitly,  $H_n^{\text{ext}}$  is a constant. Moreover, there are two additional conserved quantities (Noether charges) associated with the invariance of  $L_n$  (up to total derivative). (i) Noether charge  $Q_G$  associated with the Galilean transformation, s.t.  $x_n \to x_n + vt$ ,

$$Q_G = tP_n - m_n x_n + \frac{1}{2} m_n g t^2, (32)$$

and (ii) Noether charge  $Q_x$  associated with the spatial translation, s.t.  $x_n \to x_n + a$ 

$$Q_x = P_n + m_n gt. (33)$$

These three conserved quantities facilitate expressing the time dilation at the external time t for a general quantum state  $|\psi_n^{\text{ext}}\rangle$ . For example, since the expectation value of  $H_n^{\text{ext}}$  is conserved, the expectation value evaluated at t,  $\langle H_n^{\rm ext} \rangle_t$ , is the same as its initial value:

$$\langle H_n^{\text{ext}} \rangle_t = \frac{\langle P_n^2 \rangle_t}{2m_n} + m_n g \langle x_n \rangle_t = \langle H_n^{\text{ext}} \rangle_0 = \frac{\langle P_n^2 \rangle_0}{2m_n} + m_n g \langle x_n \rangle_0.$$
(34)

Similarly, we have

$$\langle Q_G \rangle_t = t \langle P_n \rangle_t - m_n \langle x_n \rangle_t + \frac{1}{2} m_n g t^2 = -m_n \langle x_n \rangle_0, \tag{35}$$

$$\langle Q_x \rangle_t = \langle P_n \rangle_t + m_n g t = \langle P_n \rangle_0. \tag{36}$$

From these, we obtain

$$\langle x_n \rangle_t = \langle x_n \rangle_0 + \frac{\langle P_n \rangle_0}{m_n} t - \frac{1}{2} g t^2, \tag{37}$$

$$\langle P_n \rangle_t = \langle P_n \rangle_0 - m_n gt. \tag{38}$$

Namely, the evolution of the average of the position is identical to the evolution of a classical particle with the initial position  $\langle x_n \rangle_0$  and the initial velocity  $\langle P_n \rangle_0/m_n$ . In this sense, the trajectory is independent of the mass of the clock particle. This may be regarded as the weak equivalence principle for quantum particles. [10] observed that the probability distribution of the position for a free-falling clock is the same as that of a free clock with the shift of its mean which is independent of its mass.

Plugging Eq. (34) and Eq. (37) into Eq. (29), the observed average time dilation between two clocks is given by

$$\langle T_A \rangle = \tau_B \left( 1 - \left( \frac{\langle \mathbf{P}_A^2 \rangle}{2m_A^2 c^2} - \frac{\langle \Phi_A \rangle}{c^2} \right) + \left( \frac{\langle \mathbf{P}_B^2 \rangle}{2m_B^2 c^2} - \frac{\langle \Phi_B \rangle}{c^2} \right) \right)$$

$$= \tau_B \left( 1 - \left( \frac{\langle \mathbf{P}_A^2 \rangle_0}{2m_A^2 c^2} - \frac{\langle \mathbf{P}_B^2 \rangle_0}{2m_B^2 c^2} \right) + \frac{g}{c^2} \left( \langle x_A \rangle_0 + 2 \frac{\langle P_A \rangle_0}{m_A} t - \langle x_B \rangle_0 - 2 \frac{\langle P_B \rangle_0}{m_B} t \right) \right). \tag{39}$$

Note that a  $\frac{1}{2}gt^2$  term disappears because it is independent of mass (the universality of free fall). It turns out that the differences of the initial average position and velocity between the two clocks will contribute to the gravitational effect on the time dilation. This is another main result of this paper. This time dilation formula holds for a general quantum state, while an evaluation of the time dilation for Gaussian wave packets is given in Appendix A. The time dilation in general depends on the external time t as well as on the initial average of the position and the momentum squared of each clock. Only if the average of the initial velocity for each clock is equal, i.e.,  $\langle P_A \rangle_0/m_A = \langle P_B \rangle_0/m_B$ , the time dilation between the two clocks depends only on the initial average position and is independent of the external time t, that is, the duration of the experiment.

### A. Time Dilation for a Clock in a Superposition

However, for a state in a superposition, the time dilation would involve a term different from the classical time dilation. In order to investigate the effect of the superposition of states, we consider two clocks A and B and suppose that initially clock A is prepared as a superposition of two Gaussian wave packets with average momenta  $\overline{p}_A$  and  $\overline{p}'_A$  with spread  $\Delta_A$  and with mean positions  $\overline{x}_A$  and  $\overline{x}'_A$  generalizing the situation considered in [20, 27]:

$$\langle P_A | \psi_A^{\text{ext}} \rangle = \frac{1}{N^{1/2} (\pi \Delta_A^2)^{1/4}} \left( \cos \theta e^{-i\frac{\overline{x}_A (P_A - \overline{p}_A)}{\hbar}} e^{-\frac{(P_A - \overline{p}_A)^2}{2\Delta_A^2}} + \sin \theta e^{i\phi} e^{-i\frac{\overline{x}_A' (P_A - \overline{p}_A')}{\hbar}} e^{-\frac{(P_A - \overline{p}_A')^2}{2\Delta_A^2}} \right), \tag{40}$$

where  $\theta \in [0, \pi/2)$  and  $\phi \in [0, \pi]$  is the relative phase, and

$$N = 1 + \sin 2\theta \exp\left(-\frac{(\overline{p}_A - \overline{p}_A')^2}{4\Delta_A^2} - \frac{(\overline{x}_A - \overline{x}_A')^2}{4\sigma_{xA}^2}\right) \cos\left(\phi - \frac{(\overline{x}_A + \overline{x}_A')(\overline{p}_A - \overline{p}_A')}{2\hbar}\right)$$
(41)

is a normalization factor and  $\sigma_{xA} = \hbar/\Delta_A$ . Further, clock B is prepared in a Gaussian wave packet with average momentum  $\overline{p}_B$  and spread  $\Delta_B$  and with mean position  $\overline{x}_B$ .

Then, the average time read by A conditioned on B is given by

$$\langle T_{A} \rangle = \tau_{B} \left\{ 1 - \frac{1}{Nc^{2}} \left[ \left( \frac{(\overline{p}_{A} - m_{A}gt)^{2}}{2m_{A}^{2}} - g\left(\overline{x}_{A} + \frac{\overline{p}_{A}t}{m_{A}}\right) \right) \cos^{2}\theta + \left( \frac{(\overline{p}'_{A} - m_{A}gt)^{2}}{2m_{A}^{2}} - g\left(\overline{x}'_{A} + \frac{\overline{p}'_{A}t}{m_{A}}\right) \right) \sin^{2}\theta \right. \\
+ \frac{\Delta_{A}^{2}}{4m_{A}^{2}} + \frac{1}{2}g^{2}t^{2} \right] - \frac{\sin 2\theta}{Nc^{2}}e^{-\frac{(\overline{p}_{A} - \overline{p}'_{A})^{2}}{4\Delta_{A}^{2}} - \frac{(\overline{x}_{A} - \overline{x}'_{A})^{2}}{4\sigma_{x}^{2}A}} \left[ \left( \frac{(\overline{p}_{A} + \overline{p}'_{A} - m_{A}gt)^{2}}{2m_{A}^{2}} - \frac{\Delta_{A}^{2}(\overline{x}_{A} - \overline{x}'_{A})^{2}}{8m_{A}^{2}\sigma_{xA}^{2}} + \frac{\Delta_{A}^{2}}{4m_{A}^{2}} \right. \\
- \frac{1}{2}g(\overline{x}_{A} + \overline{x}'_{A}) - \frac{gt}{2m_{A}}(\overline{p}_{A} + \overline{p}'_{A}) + \frac{1}{2}g^{2}t^{2} \right) \cos\left(\phi - \frac{(\overline{x}_{A} + \overline{x}'_{A})(\overline{p}_{A} - \overline{p}'_{A})}{2\hbar}\right) \\
- \left( \frac{\hbar(\overline{x}_{A} - \overline{x}'_{A})}{2m_{A}^{2}\sigma_{xA}^{2}} \left( \frac{\overline{p}_{A} + \overline{p}'_{A}}{2} - m_{A}gt \right) - g\frac{\hbar t(\overline{x}_{A} - \overline{x}'_{A})}{2m_{A}\sigma_{xA}^{2}} + g\frac{\hbar(\overline{p}_{A} - \overline{p}'_{A})}{2\Delta_{A}^{2}} \right) \sin\left(\phi - \frac{(\overline{x}_{A} + \overline{x}'_{A})(\overline{p}_{A} - \overline{p}'_{A})}{2\hbar} \right) \right] \\
+ \frac{(\overline{p}_{B} - m_{B}gt)^{2}}{2m_{B}^{2}c^{2}} + \frac{\Delta_{B}^{2}}{4m_{B}^{2}c^{2}} - \frac{g}{c^{2}}\left( -\frac{1}{2}gt^{2} + \overline{x}_{B} + \frac{\overline{p}_{B}t}{m_{B}} \right) \right\}. \tag{42}$$

Note that for the state given by (40) each initial expectation value,  $\langle P_A \rangle_0$ ,  $\langle P_A^2 \rangle_0$ , and  $\langle x_A \rangle_0$ , is evaluated in Appendix B. It coincides with the expression given in [20] when putting g=0 and  $\overline{x}_A=\overline{x}_A'=\overline{x}_B=0$ . It also reproduces the result by [27] when putting  $\phi=0$  and  $\overline{p}_A=\overline{p}_A'$ . The terms proportional to  $\sin 2\theta$  represent a genuin quantum time dilation effect.

To make the effect of quantum time dilation manifest, we divide the time dilation formula Eq. (42) into the classical dilation part  $K_C$  and the quantum dilation part  $K_Q$  as  $\langle T_A \rangle = \tau_B (1 - K_C - K_Q)$ .  $K_C$  is given by the contribution of a statistical mixture of the wave packets with momentum  $\overline{p}_A$  and the position  $\overline{x}_A$  with weight  $\cos^2 \theta$  and  $\overline{p}'_A$  and  $\overline{x}'_A$  with weight  $\sin^2 \theta$ :

$$K_{C} = \frac{(\overline{p}_{A} - m_{A}gt)^{2}}{2m_{A}^{2}c^{2}}\cos^{2}\theta + \frac{(\overline{p}_{A}' - m_{A}gt)^{2}}{2m_{A}^{2}c^{2}}\sin^{2}\theta - \frac{(\overline{p}_{B} - m_{B}gt)^{2}}{2m_{B}^{2}c^{2}} + \frac{\Delta_{A}^{2}}{4m_{A}^{2}c^{2}} - \frac{\Delta_{B}^{2}}{4m_{B}^{2}c^{2}}$$
$$-\frac{g}{c^{2}}\left(\left(\overline{x}_{A} + \frac{\overline{p}_{A}t}{m_{A}}\right)\cos^{2}\theta + \left(\overline{x}_{A}' + \frac{\overline{p}_{A}'t}{m_{A}}\right)\sin^{2}\theta - \overline{x}_{B} - \frac{\overline{p}_{B}t}{m_{B}c^{2}}\right)$$
(43)

and  $K_Q$  is the rest of the time dilation and is given by

$$K_{Q} = \frac{\sin 2\theta}{8m_{A}^{2}c^{2}N}e^{-\frac{(\overline{p}_{A}-\overline{p}_{A}^{\prime})^{2}}{4\Delta_{A}^{2}} - \frac{(\overline{x}_{A}-\overline{x}_{A}^{\prime})^{2}}{4\sigma_{xA}^{2}}} \left\{ \left[ 2\left(\overline{p}_{A}^{\prime2}-\overline{p}_{A}^{2}+4m_{A}gt(\overline{p}_{A}-\overline{p}_{A}^{\prime})+2gm_{A}^{2}(\overline{x}_{A}-\overline{x}_{A}^{\prime})\right)\cos 2\theta \right. \\ \left. - (\overline{p}_{A}-\overline{p}_{A}^{\prime})^{2} - \frac{\Delta_{A}^{2}}{\sigma_{xA}^{2}}(\overline{x}_{A}-\overline{x}_{A}^{\prime})^{2} \right]\cos\left(\phi - \frac{(\overline{x}_{A}+\overline{x}_{A}^{\prime})(\overline{p}_{A}-\overline{p}_{A}^{\prime})}{2\hbar}\right) \\ \left. - \left[ \frac{2\hbar}{\sigma_{xA}^{2}}(\overline{p}_{A}+\overline{p}_{A}^{\prime}-4m_{A}gt)(\overline{x}_{A}-\overline{x}_{A}^{\prime}) + \frac{4\hbar m_{A}^{2}g}{\Delta_{A}^{2}}(\overline{p}_{A}-\overline{p}_{A}^{\prime}) \right]\sin\left(\phi - \frac{(\overline{x}_{A}+\overline{x}_{A}^{\prime})(\overline{p}_{A}-\overline{p}_{A}^{\prime})}{2\hbar}\right) \right\}.$$
 (44)

Positive  $K_Q$  implies the enhanced time dilation. We evaluate  $K_Q$  for two cases: (1) the superposition in momentum space:  $\overline{x}_A = \overline{x}_A' = 0$  and  $\overline{p}_A \neq \overline{p}_A'$ , and (2) the superposition in position space:  $\overline{x}_A \neq \overline{x}_A'$  and  $\overline{p}_A = \overline{p}_A'$ .

### 1. Momentum Superposition

For the superposition in momentum space,  $K_Q$  is given by

$$K_{Q} = \frac{\sin 2\theta}{8m_{A}^{2}c^{2}N}e^{-\frac{(\overline{p}_{A}-\overline{p}_{A}')^{2}}{4\Delta_{A}^{2}}} \left\{ \left[ 2(\overline{p}_{A}'^{2}-\overline{p}_{A}^{2}+4m_{A}gt(\overline{p}_{A}-\overline{p}_{A}'))\cos 2\theta - (\overline{p}_{A}-\overline{p}_{A}')^{2} \right]\cos \phi - \frac{4\hbar m_{A}^{2}g}{\Delta_{A}^{2}}(\overline{p}_{A}-\overline{p}_{A}')\sin \phi \right\}.$$

$$(45)$$

The coefficients of  $\cos \phi$  and  $\sin \phi$  involve the effect of g. The coefficient of  $\cos \phi$  depends linearly on t, while the coefficient of  $\sin \phi$  depends only on g and is independent of the external time t.

However, the coefficient of  $\sin \phi$  is negligibly small for an atomic particle. For an atomic particle,  $\sigma_{xA} \sim 10^{-10} \mathrm{m}$  and  $m_A \sim 10^{-25} \mathrm{kg}$ , and  $\Delta_A/m_A c = \hbar/(m_A c \sigma_{xA}) \sim 10^{-8}$ . Therefore,  $\hbar g(\overline{p}_A - \overline{p}_A')/\Delta_A^2 c^2 \sim (\hbar/\Delta_A)(g/c^2) \sim g \sigma_{xA}/c^2 \sim 10^{-26}$ . On the other hand,  $((\overline{p}_A - \overline{p}_A')/m_A c)^2 \sim (\Delta_A/m_A c)^2 \sim 10^{-16}$ .

To compute  $K_Q$  for the momentum superposition, as in [20], consider two clock particles to be  $^{87}$ Rb atoms with a mass of  $m=1.4\times 10^{-25}$ kg and atomic radius of  $\sigma_{xA}=\hbar/\Delta_A=2.5\times 10^{-10}$ m. Suppose these clock particles are moving at average velocities of  $\overline{v}_A'=\overline{p}_A'/m_A=15$  m/s and  $\overline{v}_A=\overline{p}_A/m_A=5$  m/s. We further assume t=1 s. Maintaing a superposition of wavepackets during  $t\sim 1$ s is achieved in [28]. The strength of the quantum time dilation effect  $K_Q$  for several  $\phi$  as a function of  $\theta$  is shown in Fig. 1.

 $K_Q$  with gravity has generically opposite sign to  $K_Q$  without gravity. The magnitude of  $K_Q$  with gravity for  $\theta < \pi/4$  is larger than  $K_Q$  without gravity. The quantum time dilation effect,  $\sim 10^{-17} \mathrm{s}$  for  $t \sim 1 \mathrm{s}$ , can be measured with state-of-the-art optical lattice clocks whose frequency measurement uncertainty recently reaches less than  $10^{-20}$  [18], although the effect for  $\phi = \pi/2$  is still unmeasurably small. For  $\theta = \pi/8$ ,  $K_Q$  ( $K_Q$  without g) is  $-2.2 \times 10^{-17}$  ( $1.1 \times 10^{-17}$ ),  $-1.6 \times 10^{-17}$  ( $7.7 \times 10^{-18}$ ),  $2.0 \times 10^{-27}$  (0),  $1.7 \times 10^{-17}$  ( $-8.1 \times 10^{-18}$ ),  $2.4 \times 10^{-17}$  ( $-1.2 \times 10^{-17}$ ) for  $\phi = 0, \pi/4, \pi/2, 3\pi/4, \pi$ , respectively.

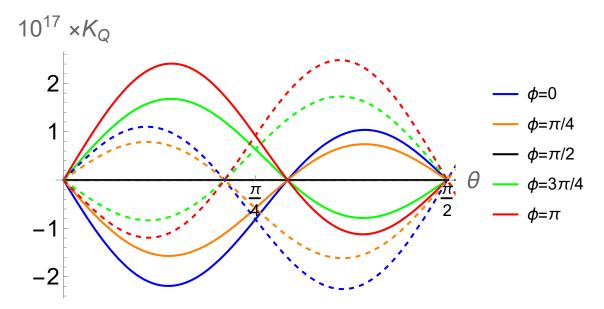


FIG. 1:  $K_Q$  for momentum superposition in a gravitational background (solid curves) and in a non-gravitational background (dashed curves). Curves are for  $\phi = 0$  (blue),  $\phi = \pi/4$  (orange),  $\phi = \pi/2$  (black),  $\phi = 3\pi/4$  (green) and  $\phi = \pi$  (red). For  $\phi = \pi/2$ ,  $K_Q$  is of order  $O(10^{-27})$ .

### 2. Spatial Superposition

For the superposition in position space,  $K_Q$  is given by

$$K_{Q} = \frac{\sin 2\theta}{8m_{A}^{2}c^{2}N}e^{-\frac{(\overline{x}_{A}-\overline{x}_{A}')^{2}}{4\sigma_{xA}^{2}}}\left\{\left[4m_{A}^{2}g(\overline{x}_{A}-\overline{x}_{A}')\cos 2\theta - \frac{\Delta_{A}^{2}}{\sigma_{xA}^{2}}(\overline{x}_{A}-\overline{x}_{A}')^{2}\right]\cos\phi - \frac{4\hbar(\overline{p}_{A}-2m_{A}gt)}{\sigma_{xA}^{2}}(\overline{x}_{A}-\overline{x}_{A}')\sin\phi\right\}.$$

$$(46)$$

Because of the exponential suppression factor  $\exp\left(-(\overline{x}_A - \overline{x}_A')^2/4\sigma_{xA}^2\right)$  in  $K_Q$ , the quantum interference effect is possible only when two wave packets overlap. The coefficients of  $\cos\phi$  and  $\sin\phi$  involve the effect of g, and this time the coefficient of  $\cos\phi$  is independent of t, while the coefficient of  $\sin\phi$  depends linearly on t. However, the effect of g in the coefficient of  $\cos\phi$  is negligibly small for an atomic particle. The first term in  $K_Q$  is of order  $g(\overline{x}_A - \overline{x}_A')/c^2 \sim g\sigma_{xA}/c^2 \sim 10^{-26}$ , while  $(\Delta_A/m_Ac)^2((\overline{x}_A - \overline{x}_A')/\sigma_{xA})^2 \sim (\Delta_A/m_Ac)^2 \sim 10^{-16}$ . The observability of the quantum interference effect through interference patterns is also discussed in [29].

 $K_Q$  for the position superposition is shown in Fig. 2. Here we assumed  $\overline{v}_A = \overline{p}_A/m_A = 10 \text{ m/s}, \overline{x}_A - \overline{x}_A' = 2\sigma_{xA}$  and t = 1s. For  $\phi = 0, \pi$ , solid curves  $(g \neq 0)$  coincide with dotted curves (g = 0) because the effect of  $g \neq 0$  is

negligible. For  $\theta=\pi/4$ ,  $K_Q$  ( $K_Q$  without g) is  $-1.3\times10^{-17}$  ( $-1.3\times10^{-17}$ ),  $5.6\times10^{-17}$  ( $-7.9\times10^{-17}$ ),  $1.2\times10^{-16}$  ( $-1.2\times10^{-16}$ ),  $1.3\times10^{-16}$  ( $-1.0\times10^{-16}$ ),  $2.9\times10^{-17}$  ( $2.9\times10^{-17}$ ) for  $\phi=0,\pi/4,\pi/2,3\pi/4,\pi$ , respectively. It is seen that the sign of  $K_Q$  with gravity is generically positive and is opposite to  $K_Q$  without gravity for  $0<\phi<\pi$ , which implies that the quantum interference effect with gravity facilitates the time dilation of clocks.

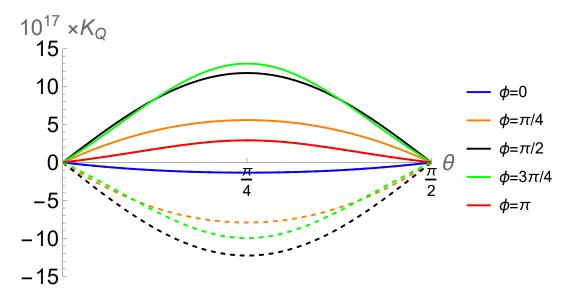


FIG. 2:  $K_Q$  for spatial superposition in a gravitational background (solid curves) and in a non-gravitational background (dashed curves). Curves are for  $\phi = 0$  (blue),  $\phi = \pi/4$  (orange),  $\phi = \pi/2$  (black),  $\phi = 3\pi/4$  (green) and  $\phi = \pi$  (red). For  $\phi = 0, \pi$ , the effect of  $g \neq 0$  is negligible and two curves overlap.

## 3. The Effect of Gravity in Quantum Interference

We have seen that the effect of nonzero g in the quantum time dilation can be potentially observed for clocks prepared in a superposition either in momentum space or in position space. In both cases, the measurable contributions originate from the initial expectation value of momentum,  $\langle P_A \rangle_0$ , in Eq. (39). Moreover, we can maximize the effect of g on  $K_Q$  by minimizing the effect of g on  $K_C$ . This is achieved by arranging  $\overline{p}_A$  and  $\overline{p}'_A$  so that the center of mass velocity of clock A coincides with the velocity of clock  $B^1$ :

$$\frac{\overline{p}_A \cos^2 \theta + \overline{p}_A' \sin^2 \theta}{m_A} = \frac{\overline{p}_B}{m_B}.$$
 (47)

Then the time-dependent terms proportional to gt in Eq. (43) disappear and  $K_C$  becomes independent of time:

$$K_C = \frac{\overline{p}_A^2}{2m_A^2 c^2} \cos^2 \theta + \frac{\overline{p}_A'^2}{2m_A^2 c^2} \sin^2 \theta - \frac{\overline{p}_B^2}{2m_B^2 c^2} + \frac{\Delta_A^2}{4m_A^2 c^2} - \frac{\Delta_B^2}{4m_B^2 c^2} - \frac{g}{c^2} \left( \overline{x}_A \cos^2 \theta + \overline{x}_A' \sin^2 \theta - \overline{x}_B \right), \tag{48}$$

while  $K_Q$  involves a term proportional to gt. Therefore, the effect of g in  $K_Q$  would manifest as the time dependence of  $K_Q$ . In principle, the longer the duration of the experiment, the more the gravitational effect is enhanced.

# IV. SUMMARY

In this paper, extending the proper time observable proposed by [20] to a weak gravitational field, we derived a formula of the average proper time read by one clock conditioned on another clock reading a different proper time [Eq. (29)]. The time dilation measured by these quantum clocks has the same form as that in classical relativity.

<sup>&</sup>lt;sup>1</sup> For the case without gravity,  $K_C$  can be made to vanish by arranging  $\bar{p}_A$  and  $\bar{p}'_A$  so that the center of mass kinetic energy of clock A consides with the kinetic energy of clock B [20].

By considering a linear gravitational potential, we found for a general quantum state that the evolution of the average of the position of a quantum clock is identical to the evolution of a classical particle falling under the gravitational force with the initial position  $\langle x_n \rangle_0$  and the initial velocity  $\langle P_n \rangle_0/m_n$ , which may be regarded as the quantum analog of the weak equivalence principle. We also derived the time dilation between clocks at the external time t Eq. (39), which holds not only for Gaussian wave packets but also for general quantum states.

We then considered the case in which the state of one clock is in a superposition of Gaussian wave packets. We found that the effect arising from quantum interference appears in the time dilation. The time dilation is of the order of  $10^{-17}$ s for superpositions of wavepacket during about 1 s. The effect of gravity on such quantum time dilation can be potentially observed for optical lattice clocks prepared in a superposition either in momentum space or in position space. We also identified the condition which maximizes the effect of gravity in the quantum time dilation.

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### Appendix A: Time Dilation for Wave Packets

Here we give the time dilation for Gaussian wave packets.

First, we consider the situation where the external degrees of freedom of two clocks are initially prepared in a Gaussian state localized around an average momentum  $\overline{p}_n$  with spread  $\Delta_n$  and around an average position  $\overline{x}_n$  with spread  $\sigma_{xn} = \hbar/\Delta_n$  generalizing the situation considered in [20, 27]. [20] considered wave packets localized in momentum space and [27] considered wave packets localized in position space.

The external state in the momentum representation is then given by

$$\langle P_n | \psi_n^{\text{ext}} \rangle = \frac{1}{(\pi \Delta_n^2)^{1/4}} e^{-i\frac{\overline{x}_n (P_n - \overline{p}_n)}{\hbar}} e^{-\frac{(P_n - \overline{p}_n)^2}{2\Delta_n^2}}, \tag{A1}$$

and the initial average of  $P_n^2$  is then given by

$$\langle P_n^2 \rangle_0 = \langle \psi_n^{\text{ext}} | P_n^2 | \psi_n^{\text{ext}} \rangle = \int dp \ p^2 |\langle p | \psi_n^{\text{ext}} \rangle|^2 = \overline{p}_n^2 + \frac{\Delta_n^2}{2} \,. \tag{A2}$$

The external state in the position representation is given by

$$\langle x_n | \psi_n^{\text{ext}} \rangle = \int dp \langle x_n | p \rangle \langle p | \psi^{\text{ext}} \rangle$$

$$= \frac{1}{(\pi \sigma_{xn}^2)^{1/4}} e^{\frac{i}{\hbar} x_n \overline{p}_n} e^{-\frac{(x_n - \overline{x}_n)^2}{2\sigma_{xn}^2}}, \qquad (A3)$$

where  $\sigma_{xn} = \hbar/\Delta_n$ , and the initial average of the position is  $\overline{x}_n$ ,  $\langle x_n \rangle_0 = \int dx \ x |\langle x|\psi_n^{\text{ext}}\rangle|^2 = \overline{x}_n$ . Plugging  $\langle \mathbf{P}_n^2 \rangle_0$  into Eq. (39), the time dilation between two clocks is given by

$$\langle T_A \rangle = \tau_B \left( 1 - \left( \frac{\langle \mathbf{P}_A^2 \rangle_0}{2m_A^2 c^2} - \frac{\langle \mathbf{P}_B^2 \rangle_0}{2m_B^2 c^2} \right) + \frac{g}{c^2} \left( \langle x_A \rangle_0 + 2 \frac{\langle P_A \rangle_0}{m_A} t - \langle x_B \rangle_0 - 2 \frac{\langle P_B \rangle_0}{m_B} t \right) \right) \\
= \tau_B \left[ 1 - \left( \frac{(\overline{p}_A - m_A g t)^2}{2m_A^2 c^2} - \frac{(\overline{p}_B - m_B g t)^2}{2m_B^2 c^2} + \frac{\Delta_A^2}{4m_A^2 c^2} - \frac{\Delta_B^2}{4m_B^2 c^2} \right) + \frac{g}{c^2} \left( \overline{x}_A + \frac{\overline{p}_A t}{m_A} - \overline{x}_B - \frac{\overline{p}_B t}{m_B} \right) \right] . (A4)$$

Therefore, supposing  $\Delta_A/m_A = \Delta_B/m_B$ , the time dilation between two quantum clocks agrees with the classical relativistic time dilation.

## Appendix B: Initial expectation values for a superposition of states

We summarize the expressions of expectation values of momentum, momentum squared, and position operators for a superposition of Gaussian wave packets given by Eq. (40):

$$\langle P_A \rangle = \overline{p}_A \cos^2 \theta + \overline{p}_A' \sin^2 \theta - \frac{\sin 2\theta}{N} e^{-\frac{(\overline{p}_A^2 - \overline{p}_A')^2}{4\Delta_A^2} - \frac{(\overline{x}_A - \overline{x}_A')^2}{4\sigma_{xA}^2}}$$

$$\times \left[ \frac{(\overline{p}_A - \overline{p}_A') \cos 2\theta}{2} \cos \left( \phi - \frac{(\overline{x}_A + \overline{x}_A')(\overline{p}_A - \overline{p}_A')}{2\hbar} \right) + \hbar \frac{\overline{x}_A - \overline{x}_A'}{2\sigma_{xA}^2} \sin \left( \phi - \frac{(\overline{x}_A + \overline{x}_A')(\overline{p}_A - \overline{p}_A')}{2\hbar} \right) \right], \quad (B1)$$

$$\langle P_A^2 \rangle = \overline{p}_A^2 \cos^2 \theta + \overline{p}_A'^2 \sin^2 \theta + \frac{\Delta_A^2}{2} - \frac{\sin 2\theta}{N} e^{-\frac{(\overline{p}_A^2 - \overline{p}_A')^2}{4\Delta_A^2} - \frac{(\overline{x}_A - \overline{x}_A')^2}{4\sigma_{xA}^2}}$$

$$\times \left\{ \left[ \frac{\overline{p}_A^2 - \overline{p}_A'^2}{2} \cos 2\theta + \left( \frac{\overline{p}_A - \overline{p}_A'}{2} \right)^2 + \Delta_A^2 \left( \frac{\overline{x}_A - \overline{x}_A'}{2\sigma_{xA}} \right)^2 \right] \cos \left( \phi - \frac{(\overline{x}_A + \overline{x}_A')(\overline{p}_A - \overline{p}_A')}{2\hbar} \right)$$

$$+ \hbar \frac{(\overline{p}_A + \overline{p}_A')(\overline{x}_A - \overline{x}_A')}{2\sigma_{xA}^2} \sin \left( \phi - \frac{(\overline{x}_A + \overline{x}_A')(\overline{p}_A - \overline{p}_A')}{2\hbar} \right) \right\}, \quad (B2)$$

$$\langle x_A \rangle = \overline{x}_A \cos^2 \theta + \overline{x}_A' \sin^2 \theta - \frac{\sin 2\theta}{N} e^{-\frac{(\overline{x}_A^2 - \overline{p}_A')^2}{4\sigma_x^2} - \frac{(\overline{x}_A - \overline{x}_A')^2}{4\sigma_x^2}}$$

$$\times \left[ \frac{\overline{x}_A - \overline{x}_A'}{2} \cos 2\theta \cos \left( \phi - \frac{(\overline{x}_A + \overline{x}_A')(\overline{p}_A - \overline{p}_A')}{2\hbar} \right) - \hbar \frac{\overline{p}_A - \overline{p}_A'}{2\Delta_A^2} \sin \left( \phi - \frac{(\overline{x}_A + \overline{x}_A')(\overline{p}_A - \overline{p}_A')}{2\hbar} \right) \right]. \quad (B3)$$

Note that the terms proportional to  $\frac{\sin 2\theta}{N}$  in the above expectation values represent contributions of quantum interference.

- [1] C. Will, Theory and experiment in gravitational physics, 2nd ed. (Cambridge University Press, Cambridge, 2018).
- [2] A. Peters, K. Y. Chung, and S. Chu, Nature **400**, 849 (1999).
- [3] S. Fray, C. Alvarez Diez, T. W. Hansch, and M. Weitz, Phys. Rev. Lett. 93, 240404 (2004), arXiv:physics/0411052.
- [4] A. Bonnin, N. Zahzam, Y. Bidel, and A. Bresson, Phys. Rev. A 88, 043615 (2013), arXiv:1307.2734 [physics.atom-ph].
- [5] D. Schlippert, J. Hartwig, H. Albers, L. L. Richardson, C. Schubert, A. Roura, W. P. Schleich, W. Ertmer, and E. M. Rasel, Phys. Rev. Lett. 112, 203002 (2014), arXiv:1406.4979 [physics.atom-ph].
- [6] H. Albers et al., Eur. Phys. J. D 74, 145 (2020), arXiv:2003.00939 [physics.atom-ph].
- [7] P. Asenbaum, C. Overstreet, M. Kim, J. Curti, and M. A. Kasevich, Phys. Rev. Lett. 125, 191101 (2020), arXiv:2005.11624 [physics.atom-ph].
- [8] B. Barrett et al., AVS Quantum Sci. 4, 014401 (2022), arXiv:2110.13273 [physics.atom-ph].
- [9] M. Zych and v. Brukner, Nature Phys. 14, 1027 (2018), arXiv:1502.00971 [gr-qc].
- [10] C. Anastopoulos and B.-L. Hu, Class. Quant. Grav. 35, 035011 (2018), arXiv:1707.04526 [quant-ph].
- [11] L. Hardy, in Progress and Visions in Quantum Theory in View of Gravity: Bridging foundations of physics and mathematics (2019) arXiv:1903.01289 [quant-ph].
- [12] A. Roura, Phys. Rev. X 10, 021014 (2020), arXiv:1810.06744 [physics.atom-ph].
- [13] A. Roura, C. Schubert, D. Schlippert, and E. M. Rasel, Phys. Rev. D 104, 084001 (2021), arXiv:2010.11156 [physics.atom-ph]
- [14] F. Giacomini and v. Brukner, AVS Quantum Sci. 4, 015601 (2022), arXiv:2109.01405 [quant-ph].
- [15] C. Cepollaro and F. Giacomini, (2021), arXiv:2112.03303 [quant-ph].
- [16] C. W. Chou, D. B. Hume, T. Rosenband, and D. J. Wineland, Science 329, 1630 (2010).
- [17] M. Takamoto, I. Ushijima, N. Ohmae, T. Yahagi, K. Kokado, H. Shinkai, and H. Katori, Nature Photon. 14, 411 (2020).
- [18] T. Bothwell, C. J. Kennedy, A. Aeppli, D. Kedar, J. M. Robinson, E. Oelker, A. Staron, and J. Ye, Nature 602, 420 (2022), arXiv:2109.12238 [physics.atom-ph].
- $[19] \ X.\ Zheng,\ J.\ Dolde,\ H.\ M.\ Lim,\ \ and\ S.\ Kolkowitz,\ \ (2022),\ arXiv:2207.07145\ \ [physics.atom-ph]\ .$
- [20] A. R. H. Smith and M. Ahmadi, Nature Commun. 11, 5360 (2020), arXiv:1904.12390 [quant-ph].
- [21] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (Princeton University Press, Princeton, N.J., 2017).
- [22] D. N. Page and W. K. Wootters, Phys. Rev. D 27, 2885 (1983).
- [23] W. K. Wootters, International Journal of Theoretical Physics 23, 701 (1984).
- [24] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. D 92, 045033 (2015), arXiv:1504.04215 [quant-ph].
- [25] E. Castro-Ruiz, F. Giacomini, A. Belenchia, and v. Brukner, Nature Commun. 11, 2672 (2020), arXiv:1908.10165 [quant-ph].
- [26] L. Maccone and K. Sacha, Phys. Rev. Lett. 124, 110402 (2020), arXiv:1810.12869 [quant-ph].

- [27] S. Khandelwal, M. P. E. Lock, and M. P. Woods, Quantum 4, 309 (2020), arXiv:1904.02178 [quant-ph] .
- [28] T. Kovachy, P. Asenbaum, C. Overstreet, C. A. Donnelly, S. M. Dickerson, A. Sugarbaker, J. M. Hogan, and M. A. Kasevich, Nature 528, 530 (2015).
- [29] Paczos, Jerzy and Debski, Kacper and Grochowski, Piotr T. and Smith, Alexander R. H. and Dragan, Andrzej, (2022), arXiv:2204.10609 [quant-ph] .