Effective field equations for expectation values

R. D. Jordan
Department of Physics, University of Texas, Austin, Texas 78712
(Received 16 September 1985)

We discuss functional methods which allow calculation of expectation values, rather than the usual in-out amplitudes, from a path integral. The technique, based on Schwinger's idea of summing over paths which go from the past to the future and then back to the past, provides effective field equations satisfied by the expectation value of the field. These equations are shown to be real and causal for a general theory up to two-loop order, and unitarity is checked to this order. These methods are applied to a simple quantum-mechanical example to illustrate the differences between the new formalism and the standard theory. When applied to the gravitational field, the new effective field equations should be useful for studies of quantum cosmology.

I. INTRODUCTION

The effective action is a useful tool for studying quantum corrections to a classical theory. Not only does it contain the information needed to construct the S matrix, it also provides a dynamical equation satisfied by the *effective field,* i.e., an equation which includes radiative corrections to the classical field equations. In the standard formulation of the method, the effective field is defined to be a matrix element of the field operator between an "in" state and an "out" state (usually the "in" and "out" vacua, respectively). In general, these two states are *not* the same, so that the effective field is *not* an expectation value of the field operator. This leads to difficulties when one attempts to solve the effective field equations in order to find the quantum corrections to the classical dynamics, and it is the resolution of these problems that we shall discuss in this paper.

The usual effective field equations possess several undesirable properties. First, $\langle \text{out,vac} \,|\, \boldsymbol{\phi} \,|\, \text{in,vac} \rangle$ is not real, even if $\boldsymbol{\phi}$ is Hermitian. This problem is particularly acute for the case of the gravitational field, since the complex quantity $\langle \text{out,vac} \,|\, \mathbf{g}_{\mu\nu} \,|\, \text{in,vac} \rangle$ cannot be viewed as a metric for a real manifold, so even after solving the effective field equations one does not know how to interpret the solution. Second, the effective field equations are nonlocal, integrodifferential equations so, unlike the classical equations, they cannot be solved from Cauchy data. In other words, one cannot compute the time evolution of the effective field from initial data and compare it with the classical solution. Instead, the natural boundary conditions are end-point boundary conditions, which means the final state of the system must be specified before the equations can be solved. Again, gravity reveals this problem in its worst form: one of the most important questions to be answered by the quantized theory concerns the occurrence or otherwise of spacetime singularities, so one would like to solve the equations without assuming anything about the final state. A third general problem is that overall features of the system may be hidden by looking at in-out matrix elements rather than expectation values. This will become apparent in Sec. IV, where an example is worked out to demonstrate how much more useful it is to examine expectation values to investigate the general properties of the quantized theory.

The purpose of this paper is to show how to overcome these defects by exploiting an old idea of Schwinger's[1] to modify the definition of the effective action and thereby obtain field equations satisfied by true expectation values $\langle \xi \,|\, \boldsymbol{\phi}(x) \,|\, \xi \rangle$ of the field. Throughout we shall choose the state $|\, \xi \rangle$ to be the in vacuum. The new effective field equations then turn out to be real and causal, and in principle could be solved from initial data, allowing the time evolution of the expectation value to be studied. The equations are still nonlocal, so the field must be specified throughout a spacetime region in the remote past in order to calculate its future evolution, but no assumption need be made about the final state. The main application we have in mind is to use the new formalism to study quantum effects on the evolution of the expectation value of the metric in a cosmological setting, to compare with corresponding results for the in-out matrix elements.[2] In this paper we set up the formalism needed to obtain the desired field equations for a general theory. A simple quantum-mechanical example is worked out to reinforce the motivation behind the work, but the applications to quantum cosmology will be discussed in a later paper.

Section II explains how to modify the boundary conditions on the functional integral to allow computation of expectation values and derives the correspondingly modified Feynman rules. The effective field equations for $\langle \text{in,vac} \,|\, \boldsymbol{\phi}(x) \,|\, \text{in,vac} \rangle$ are obtained in Sec. III and their reality and causality properties are verified to two-loop order. In addition, the formulas obtained include a simple check of unitarity, which can be carried out using the new Feynman rules, once a suitable choice has been made for the measure in the functional integral. The choice made here enables unitarity to be confirmed, again up to two loops. As an example of the new methods, Sec. IV examines a harmonic oscillator with a prescribed, time-dependent spring constant. This system can be solved exactly, thereby showing the ease of interpretation of expectation values over in-out matrix elements. Finally, in Sec. V, we mentioned two alternative ways of looking at the new formalsim and briefly discuss the results obtained.

Although Schwinger's method has not been widely used, it has been studied in various contexts. Hajicek[3] used the idea to investigate a self-interacting scalar field, and found a useful connection with Euclidean field theory as outlined in Sec. V. Fradkin and Gitman[4] applied the technique to QED with external fields creating particles, and with Buchbinder[5] generalized their methods to include a curved spacetime. The Green's functions that arise (see Sec. II) have also found application in statistical physics, as reviewed by Chou, Su, Hao, and Hu in Ref. 6 and references therein. However, in this paper we develop the formalism from scratch in order to find the properties of the effective field equations for a general system.

II. COMPUTATION OF EXPECTATION VALUES

According to the Introduction, the problem of interest is that of a quantum field ϕ in a background which becomes stationary in the remote past (the "in region") but otherwise varies arbitrarily. In this case one can define a set of "in-mode functions" which become positive frequency in the in region and provide a definition of an "in" Fock space with vacuum state $|\text{in,vac}\rangle$. We shall see shortly that if we are studying expectation values of operators in this in-vacuum state we do not need to assume anything about the background in the remote future. This stands in contrast with the standard formulation[7] of quantum field theories on a nonstationary background, in which one assumes that the background also becomes stationary in the remote future (the "out region"). There is then a second set of mode functions, the "out modes," which become positive frequency in the out region, defining an "out" Fock space with vacuum state $|\text{out,vac}\rangle$. All the problems mentioned in Sec. I stem from the fact that, in general

$$|\text{out,vac}\rangle \neq |\text{in,vac}\rangle \ . \tag{2.1}$$

Working throughout in the Heisenberg picture, we shall take ϕ to be a single Bose field with classical action $S[\phi]$. Having solved the problem for this theory it is easy to extend the methods to other systems, including gauge fields. The corresponding results for a fermion field will be stated at the end of this section.

Let us first recall the usual description of the quantum theory based on in-out matrix elements. The starting point is the "in-out generating functional" $W[J]$, which is the vacuum-to-vacuum amplitude in the presence of a source J, expressed as a functional integral:

$$\begin{aligned} e^{iW[J]} &\equiv \langle \text{out,vac} \mid \text{in,vac} \rangle_J \\ &= \text{const} \times \int e^{i(S[\phi]+J\phi)} \mu[\phi]\, d\phi \ , \end{aligned} \tag{2.2}$$

where the integral is over all fields which become pure negative frequency in the remote past and pure positive frequency in the remote future (henceforth to be called "vacuum boundary conditions"). The measure $\mu[\phi]$ is included to guarantee unitarity, and must be chosen before any calculations can be performed, but for the present it will be left unspecified. The functional $W[J]$ does indeed generate in-out matrix elements:

$$\begin{aligned} \frac{\delta W[J]}{\delta J(x)} &= e^{-iW[J]} \frac{\delta}{i\delta J(x)} e^{iW[J]} \\ &= \text{const} \times e^{-iW[J]} \int \phi(x) e^{i(S[\phi]+J\phi)} \mu[\phi]\, d\phi \\ &= \frac{\langle \text{out,vac} \mid \boldsymbol{\phi}(x) \mid \text{in,vac} \rangle_J}{\langle \text{out,vac} \mid \text{in,vac} \rangle_J} \ . \end{aligned} \tag{2.3}$$

If the source is now set equal to zero, (2.3) becomes the usual definition of the effective field.

To use Schwinger's method[1] for calculating expectation values in the in vacuum one defines a different generating functional which involves a summation over a complete set of out states:

$$e^{iW[J_+,J_-]} \equiv \sum_\alpha \langle \text{in,vac} \mid \text{out},\alpha \rangle_{J_-} \langle \text{out},\alpha \mid \text{in,vac} \rangle_{J_+} \ . \tag{2.4}$$

This new generating functional depends on two different sources, J_+ and J_-. If these sources are set equal, completeness of the states $|\text{out},\alpha\rangle$ implies that

$$W[J,J]=0 \tag{2.5}$$

corresponding to the normalization

$$\langle \text{in,vac} \mid \text{in,vac} \rangle = 1 \ . \tag{2.6}$$

Equation (2.5) is clearly a statement of unitarity. Strictly speaking, for a complete check of unitarity we should consider arbitrary in states in place of the in vacuum in (2.4), but in this paper we shall stick with $|\text{in,vac}\rangle$ for simplicity. Another useful property of $W[J_+,J_-]$ is

$$W[J_-,J_+] = -W[J_+,J_-]^* \ . \tag{2.7}$$

Taking the complete set of out states $|\text{out},\alpha\rangle$ to be eigenstates of ϕ on some spacelike hypersurface Σ, we can write

$$\begin{aligned} e^{iW[J_+,J_-]} &= \text{const} \times \int d\phi' \int e^{-i(S[\phi_-]+J_-\phi_-)} \mu[\phi_-]\, d\phi_- \\ &\quad \times \int e^{i(S[\phi_+]+J_+\phi_+)} \mu[\phi_+]\, d\phi_+ \end{aligned} \tag{2.8}$$

with vacuum boundary conditions in the past for both ϕ_+ and ϕ_-, and

$$\phi_+ = \phi_- = \phi' \ \text{ on } \ \Sigma \ . \tag{2.9}$$

Here we have assumed that the field, the action, the sources, and the measure are all real; the integration over ϕ' is a functional integral over the field value at each point of Σ, i.e., $d\phi' = \prod_{x\in\Sigma} d\phi(x)$. The functional $W[J_+,J_-]$ does generate the desired expectation values upon differentiation with respect to J_+ or J_- and then setting $J_+ = J_- = J$:

$$\frac{\delta W[J_+,J_-]}{\delta J_+(x)} = \text{const}\times e^{-iW[J_+,J_-]}\int d\phi' \int e^{-i(S[\phi_-]+J_-\phi_-)}\mu[\phi_-]\,d\phi_- \int \phi_+(x)e^{i(S[\phi_+]+J_+\phi_+)}\mu[\phi_+]\,d\phi_+$$

$$= e^{-iW[J_+,J_-]}\sum_\alpha \langle \text{in,vac} \mid \text{out},\alpha\rangle_{J_-}\langle \text{out},\alpha \mid \boldsymbol{\phi}(x) \mid \text{in,vac}\rangle_{J_+}$$

$$\xrightarrow[J_+=J_-=J]{} \langle \text{in,vac} \mid \boldsymbol{\phi}(x) \mid \text{in,vac}\rangle_J \tag{2.10}$$

and similarly

$$\frac{\delta W[J_+,J_-]}{-\delta J_-(x)} \xrightarrow[J_+=J_-=J]{} \langle \text{in,vac} \mid \boldsymbol{\phi}(x) \mid \text{in,vac}\rangle_J\ . \tag{2.11}$$

Either of these two constructions therefore yields the effective field we wish to study, when we finally set $J=0$.

The definition (2.8) thus provides a means of calculating expectation values in the in vacuum. Different choices of boundary conditions in the past would provide expectation values in other in states, as needed for a full check of unitarity. Since the out vacuum is no longer used in the definition of $W[J_+,J_-]$, we do not need to assume the existence of a stationary out region when calculating expectation values. The surface Σ need only be taken to the future of the spacetime points at which the expectation value is to be evaluated.

In view of the boundary conditions imposed on (2.8) and the integration over ϕ', this expression can be written more compactly:

$$e^{iW[J_+,J_-]}$$

$$= \text{const}\times \int \exp[i(S[\phi_+]-S[\phi_-]+J_+\phi_+ - J_-\phi_-)] \times\mu[\phi_+]\mu[\phi_-]\,d\phi_+\,d\phi_- \tag{2.12}$$

with vacuum boundary conditions in the remote past and the constraint that $\phi_+=\phi_-$ on Σ. This can be thought of as a sum over paths which go forward in time from the vacuum in the in region up to Σ, then backward in time (with different sources present) to the remote past; because of this interpretation, $W[J_+,J_-]$ has been called[1,3,6] the "closed-time-loop generating functional." This idea can be confusing, though, and the real meaning of (2.12) is simply the original definition (2.8). To emphasize its use in the present context we shall refer to $W[J_+,J_-]$ as the "in-in generating functional," in contrast with the standard "in-out" functional $W[J]$.

In order to calculate $W[J_+,J_-]$ in perturbation theory we must find the modified Feynman rules corresponding to the unusual boundary conditions in (2.12). The functional integral is now over two fields ϕ_+ and ϕ_- which are conveniently combined into a column matrix:

$$\Phi = \begin{bmatrix}\phi_+\\ \phi_-\end{bmatrix} \tag{2.13}$$

with sources

$$\mathscr{J} = \begin{bmatrix}J_+\\ -J_-\end{bmatrix} \tag{2.14}$$

and action

$$\mathscr{S} = S[\phi_+] - S[\phi_-]\ . \tag{2.15}$$

The propagator is then a negative inverse of the second functional derivative of the action (denoted by a subscript 2):

$$\begin{bmatrix}S_2[\phi_+] & 0\\ 0 & -S_2[\phi_-]\end{bmatrix}\begin{bmatrix}G_{++} & G_{+-}\\ G_{-+} & G_{--}\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} \tag{2.16}$$

and the various elements of the matrix propagator are found from the generating functional in the standard way:

$$G_{++}(x,x') = i\left[\frac{\delta}{i\delta J_+(x)}\right]\left[\frac{\delta}{i\delta J_+(x')}\right]e^{iW[J_+,J_-]}\Bigg|_{J_+=J_-=0}$$

$$= \text{const}\times i\int \phi_+(x)\phi_+(x')e^{i(S[\phi_+]-S[\phi_-])}\mu[\phi_+]\mu[\phi_-]\,d\phi_+\,d\phi_-$$

$$= i\sum_\alpha \langle \text{in,vac} \mid \text{out},\alpha\rangle\langle \text{out},\alpha \mid T\boldsymbol{\phi}(x)\boldsymbol{\phi}(x') \mid \text{in,vac}\rangle$$

$$= i\langle \text{in,vac} \mid T\boldsymbol{\phi}(x)\boldsymbol{\phi}(x') \mid \text{in,vac}\rangle \equiv G_{\text{in}}(x,x')\ . \tag{2.17}$$

Here T standing to the left of an operator time orders it in the usual way:

$$T\boldsymbol{\phi}(x)\boldsymbol{\phi}(x') \equiv \theta(x,x')\boldsymbol{\phi}(x)\boldsymbol{\phi}(x') + \theta(x',x)\boldsymbol{\phi}(x')\boldsymbol{\phi}(x)\ . \tag{2.18}$$

Similarly,

$$
\begin{aligned}
G_{+-}(x,x') &= i\left[\frac{\delta}{i\delta J_+(x)}\right]\left[\frac{\delta}{-i\delta J_-(x')}\right]e^{iW[J_+,J_-]}\Bigg|_{J_+=J_-=0} \\
&= \text{const}\times i\int \phi_-(x')\phi_+(x)e^{i(S[\phi_+]-S[\phi_-])}\mu[\phi_+]\mu[\phi_-]\,d\phi_+\,d\phi_- \\
&= i\sum_\alpha \langle \text{in,vac}\,|\,\phi(x')\,|\,\text{out},\alpha\rangle\langle\text{out},\alpha\,|\,\phi(x)\,|\,\text{in,vac}\rangle \\
&= i\langle\text{in,vac}\,|\,\phi(x')\phi(x)\,|\,\text{in,vac}\rangle \equiv G_{\text{in}}^{(-)}(x,x')
\end{aligned} \tag{2.19}
$$

and

$$
\begin{aligned}
G_{-+}(x,x') &= i\langle\text{in,vac}\,|\,\phi(x)\phi(x')\,|\,\text{in,vac}\rangle \\
&\equiv -G_{\text{in}}^{(+)}(x,x')\ ,
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
G_{--}(x,x') &= i\langle\text{in,vac}\,|\,\phi(x)\phi(x')T\,|\,\text{in,vac}\rangle \\
&= -G_{\text{in}}^{*}(x,x')\ ,
\end{aligned} \tag{2.21}
$$

where T standing to the right of an operator means that operator is to be time ordered in the opposite way:

$$
\phi(x)\phi(x')T \equiv \theta(x,x')\phi(x')\phi(x)+\theta(x',x)\phi(x)\phi(x')\ . \tag{2.22}
$$

Thus the propagator for calculating $W[J_+,J_-]$ is

$$
\mathscr{G} = \begin{bmatrix} G_{\text{in}} & G_{\text{in}}^{(-)} \\ -G_{\text{in}}^{(+)} & -G_{\text{in}}^{*} \end{bmatrix} \tag{2.23}
$$

and Eq. (2.16) is easily verified.

Viewing the functional integral as over closed time paths, the propagator (2.23) is evidently just a generalization of the usual time-ordering prescription, adapted to the special path. Points on the reverse section of the path are to be considered later in time than points on the forward section so in (2.19) and (2.20) there is no time ordering. Notice that $\mathscr{G}(x,x')$ is symmetric in its arguments, as a consequence of the relation

$$
G_{\text{in}}^{(-)}(x',x) = -G_{\text{in}}^{(+)}(x,x')\ . \tag{2.24}
$$

Other useful properties are

$$
G_{\text{in}}^{(-)*} = G_{\text{in}}^{(+)} \tag{2.25}
$$

and

$$
G_{\text{in}} + G_{\text{in}}^{(+)} - G_{\text{in}}^{(-)} - G_{\text{in}}^{*} = 0\ , \tag{2.26}
$$

the latter following from

$$
T\phi(x)\phi(x') - \phi(x)\phi(x') - \phi(x')\phi(x) + \phi(x)\phi(x')T = 0\ . \tag{2.27}
$$

All these properties hold in an arbitrary in state, not just the in vacuum.

Perturbation theory can now be used directly on the in-in generating functional, using the modified Feynman rules. For example, if the theory is free,

$$
e^{iW[J_+,J_-]} = \text{const}(\det\mathscr{G})^{1/2}\exp(\tfrac{1}{2}i\mathscr{J}^{\sim}\mathscr{G}\mathscr{J})\ , \tag{2.28}
$$

where we use the shorthand

$$
\mathscr{J}^{\sim}\mathscr{G}\mathscr{J} \equiv \int d^4x\int d^4x'\,\mathscr{J}^{\sim}(x)\mathscr{G}(x,x')\mathscr{J}(x')\ . \tag{2.29}
$$

This gives

$$
W[J_+,J_-] = \tfrac{1}{2}\mathscr{J}^{\sim}\mathscr{G}\mathscr{J} + \text{const} \tag{2.30}
$$

and the constant must be thrown away in order to recover (2.5).

For an interacting theory the generating functional can be expanded in a diagrammatic series. It is simpler to abandon the matrix notation and revert to a description in terms of the two fields ϕ_+ and ϕ_-. There are then two kinds of vertices: "plus" vertices from $S[\phi_+]$ and "minus" vertices from $-S[\phi_-]$. The complete configuration-space Feynman rules for a $\lambda\phi^4$ interaction are given in Fig. 1. The diagrams representing $W[J_+,J_-]$ are exactly the same as those for $W[J]$ (see Fig. 2) but each graph represents a sum over all possible combinations of plus and minus for the vertices and for the sources. For example, the third diagram in Fig. 2 gives

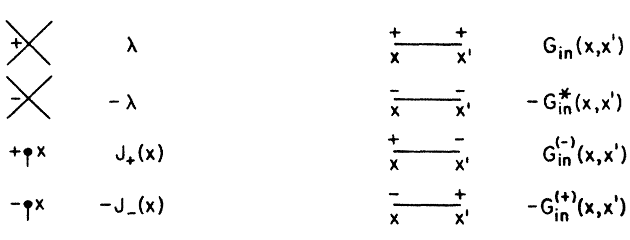

FIG. 1. In-in Feynman rules in configuration space for a scalar field with $\lambda\phi^4$ interaction.

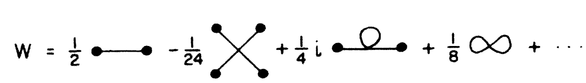

FIG. 2. Diagrammatic expansion of $W[J_+,J_-]$ for $\lambda\phi^4$ interaction.

$$\frac{i\lambda}{4}\int d^4x\int d^4x'\int d^4x''\{J_+(x)[G_{\rm in}(x,x')G_{\rm in}(x',x')G_{\rm in}(x',x'')-G_{\rm in}^{(-)}(x,x')G_{\rm in}^*(x',x')G_{\rm in}^{(+)}(x',x'')]J_+(x'')$$
$$+2J_+(x)[G_{\rm in}^{(-)}(x,x')G_{\rm in}^*(x',x')G_{\rm in}^*(x',x'')-G_{\rm in}(x,x')G_{\rm in}(x',x')G_{\rm in}^{(-)}(x',x'')]J_-(x'')$$
$$+J_-(x)[G_{\rm in}^*(x,x')G_{\rm in}^*(x',x')G_{\rm in}^*(x',x'')-G_{\rm in}^{(+)}(x,x')G_{\rm in}(x',x')G_{\rm in}^{(-)}(x',x'')]J_-(x'')\}\ . \tag{2.31}$$

Note that the unitarity condition (2.5) is not obvious from Fig. 2, because we have ignored the measure. This is correct for a scalar field in flat space but, as we shall see in the next section, to maintain unitarity in a general theory the measure functional must be chosen with care.

Once $W[J_+,J_-]$ has been calculated, it can be used to generate expectation values as follows:

$$\left[\frac{\delta}{i\delta J_+(x_1)}\right]\cdots\left[\frac{\delta}{i\delta J_+(x_n)}\right]\left[\frac{\delta}{-i\delta J_-(y_1)}\right]\cdots\left[\frac{\delta}{-i\delta J_-(y_m)}\right]e^{iW[J_+,J_-]}\Bigg|_{J_+=J_-=0}$$
$$=\langle \text{in,vac}\,|\,\phi(y_1)\cdots\phi(y_m)T\phi(x_1)\cdots\phi(x_n)\,|\,\text{in,vac}\rangle\ , \tag{2.32}$$

where the time-ordering operator acts both to the right and to the left, according to (2.18) and (2.22).

All this formalism can be applied without difficulty to fermion fields. The fields and sources appearing in the functional integral (2.8) are then a numbers, so the order in which they are written becomes important. In particular, the entire ϕ_- integral should stand to the left of the ϕ_+ integral, so formula (2.12) is misleading in the fermion case. Taking careful account of the signs, the propagator is found as before from the generating functional. The result is

$$\mathscr{G}_{\rm fermion}=\begin{bmatrix} G_{\rm in} & -G_{\rm in}^{(-)}\\ -G_{\rm in}^{(+)} & G_{\rm in}^*\end{bmatrix} \tag{2.33}$$

which is now antisymmetric in x and x'. The definitions of $G_{\rm in}$ and $G_{\rm in}^{(\pm)}$ are the same as before, provided we understand the time-ordering operator for fermions to mean

$$T\phi(x)\phi(x')=\theta(x,x')\phi(x)\phi(x')-\theta(x',x)\phi(x')\phi(x) \tag{2.34}$$

and

$$\phi(x)\phi(x')T=\theta(x,x')\phi(x')\phi(x)-\theta(x',x)\phi(x)\phi(x')\ . \tag{2.35}$$

Properties (2.24) and (2.25) remain true, but instead of (2.26) we now have

$$G_{\rm in}+G_{\rm in}^{(-)}+G_{\rm in}^{(+)}+G_{\rm in}^*=0 \tag{2.36}$$

following from

$$T\phi(x)\phi(x')+\phi(x')\phi(x)-\phi(x)\phi(x')-\phi(x)\phi(x')T=0\ . \tag{2.37}$$

For the remainder of this paper we return to the case of a boson field.

III. EFFECTIVE FIELD EQUATIONS

We are now in a position to find effective field equations satisfied by the expectation value given by (2.10). As usual we define an effective action from the generating functional by taking a Legendre transform:

$$\Gamma[\bar\phi_+,\bar\phi_-]\equiv W[J_+,J_-]-J_+\bar\phi_++J_-\bar\phi_-\ , \tag{3.1}$$

where

$$\bar\phi_\pm\equiv\pm\frac{\delta W[J_+,J_-]}{\delta J_\pm}\ , \tag{3.2}$$

and we assume that (3.2) is invertible for $J_\pm=J_\pm[\bar\phi_\pm,\bar\phi_-]$ so that (3.1) really does give a functional of $\bar\phi_+$ and $\bar\phi_-$.

$\Gamma[\bar\phi_+,\bar\phi_-]$ will be called the "in-in effective action" to distinguish it from the usual effective action. It satisfies

$$\frac{\delta\Gamma[\bar\phi_+,\bar\phi_-]}{\delta\bar\phi_+}=\frac{\delta W[J_+,J_-]}{\delta J_+}\frac{\delta J_+}{\delta\bar\phi_+}+\frac{\delta W[J_+,J_-]}{\delta J_-}\frac{\delta J_-}{\delta\bar\phi_+}-\frac{\delta J_+}{\delta\bar\phi_+}\bar\phi_+-J_++\frac{\delta J_-}{\delta\bar\phi_+}\bar\phi_-=-J_+ \tag{3.3}$$

using (3.2), and similarly

$$\frac{\delta\Gamma[\bar\phi_+,\bar\phi_-]}{\delta\bar\phi_-}=\frac{\delta W[J_+,J_-]}{\delta J_+}\frac{\delta J_+}{\delta\bar\phi_-}+\frac{\delta W[J_+,J_-]}{\delta J_-}\frac{\delta J_-}{\delta\bar\phi_-}-\frac{\delta J_+}{\delta\bar\phi_-}\bar\phi_++\frac{\delta J_-}{\delta\bar\phi_-}\bar\phi_-+J_-=+J_-\ . \tag{3.4}$$

Now we can finally set $J_+=J_-=0$ which gives, from (2.10) and (2.11)

$$\bar\phi_+(x)=\bar\phi_-(x)=\langle\text{in,vac}\,|\,\phi(x)\,|\,\text{in,vac}\rangle\ . \tag{3.5}$$

Equation (3.3) then becomes the dynamical equation satisfied by this expectation value:

$$\frac{\delta\Gamma[\bar\phi_+,\bar\phi_-]}{\delta\bar\phi_+}\Bigg|_{\bar\phi_+=\bar\phi_-}=0\ . \tag{3.6}$$

As discussed in the Introduction, these equations (called the "in-in effective field equations") are to be used to study quantum corrections to the classical dynamics. Before turning to any specific problems of interest, we shall investigate the properties of (3.6) for a general system.

First note that these equations do not immediately follow from a simple variational principle, in contrast with the in-out equations. We have to introduce the two fields $\bar{\phi}_+$ and $\bar{\phi}_-$, and only when the sources have been eliminated do these become the expectation value of ϕ. If we try to set $\bar{\phi}_+=\bar{\phi}_-$ in the in-in effective action itself, we find that it vanishes identically:

$$\Gamma[\bar{\phi},\bar{\phi}]=W[J,J]-J\bar{\phi}+J\bar{\phi}=0\ . \tag{3.7}$$

So we have effective field equations for $\langle \text{in,vac} \,|\, \phi \,|\, \text{in,vac}\rangle$ but, strictly speaking, no effective action for it. Equation (3.7) is the statement of unitarity in the current language of effective fields, and it is in this form that we shall check unitarity.

The two other properties of (3.6) that must be checked are (a) reality and (b) causality. Clearly the equations should be real if they are to describe the evolution of the expectation value of the Hermitian operator ϕ; and since we have chosen to look at the expectation value in the in vacuum we can expect that, as a function of x, the solution will depend on the background at spacetime points only to the past of x. We can now check that both of these properties hold order by order in perturbation theory.

In the formal expressions that follow it is convenient to label the field with an index: ϕ^i. This label includes the spacetime point at which the field is evaluated, so summations over repeated indices include integrations over spacetime. First, consider the usual effective action defined by

$$\Gamma[\bar{\phi}]\equiv W[J]-J_i\bar{\phi}^i\ , \tag{3.8}$$

where $W[J]$ is given by (2.2) and

$$\bar{\phi}^i\equiv\frac{\delta W[J]}{\delta J_i}\ . \tag{3.9}$$

Then

$$\Gamma_{,i}[\bar{\phi}]\equiv\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}^i}=-J_i \tag{3.10}$$

which enables us to eliminate the sources and obtain a functional-integral equation for the effective action:

$$\begin{aligned}e^{i\Gamma[\bar{\phi}]}&=\text{const}\times\int\exp\{i(S[\phi]+J_i\phi^i-J_i\bar{\phi}^i)\}\mu[\phi]\,d\phi\\&=\text{const}\times\int\exp\{i[S[\phi]+\Gamma_{,i}[\bar{\phi}](\bar{\phi}^i-\phi^i)]\}\\&\qquad\times\mu[\phi]\,d\phi\ .\end{aligned} \tag{3.11}$$

This equation can be solved by iteration, once a choice has been made for the measure functional $\mu[\phi]$. We shall see that the right choice for μ is essential to ensure not only unitarity but also the properties of reality and causality. DeWitt[8] has given the form of μ which suffices for two-loop calculations in nongauge theories:

$$\mu[\phi]=(\det G^+[\phi])^{-1/2}\ , \tag{3.12}$$

where $G^+[\phi]$ is the advanced Green's function for the operator

$$S_{,ij}[\phi]\equiv\frac{\delta^2S[\phi]}{\delta\phi^i\delta\phi^j}\ , \tag{3.13}$$

so in this condensed notation we have

$$S_{,ij}[\phi]G^{+jk}[\phi]=-\delta_i{}^k\ . \tag{3.14}$$

Using (3.12) in (3.11)

$$e^{i\Gamma[\bar{\phi}]}=\text{const}\times\int\exp\left[i\left[S[\phi]+\Gamma_{,i}[\bar{\phi}](\bar{\phi}^i-\phi^i)+\frac{i}{2}\operatorname{tr}\ln G^+[\phi]\right]\right]d\phi\ . \tag{3.15}$$

Now expand the integrand about $\bar{\phi}$:

$$\begin{aligned}e^{i\Gamma[\bar{\phi}]}&=\text{const}\times\int\exp\left[i\left[S[\bar{\phi}]+S_{,i}[\bar{\phi}](\phi^i-\bar{\phi}^i)+\tfrac{1}{2}S_{,ij}[\bar{\phi}](\phi^i-\bar{\phi}^i)(\phi^j-\bar{\phi}^j)+\cdots\right.\right.\\&\qquad\left.\left.+\Gamma_{,i}[\bar{\phi}](\bar{\phi}^i-\phi^i)+\frac{i}{2}\operatorname{tr}\ln G^+[\bar{\phi}]+\frac{i}{2}(\operatorname{tr}\ln G^+[\bar{\phi}])_{,i}(\phi^i-\bar{\phi}^i)+\cdots\right]\right]d\phi\\&=\text{const}\times\exp\left[i\left[S[\bar{\phi}]+\frac{i}{2}\operatorname{tr}\ln G^+[\bar{\phi}]\right]\right]\int\exp\left[\frac{i}{2}S_{,ij}[\bar{\phi}]\phi^i\phi^j\right](1+\cdots)d\phi\end{aligned} \tag{3.16}$$

which is a series of Gaussian integrals. Keeping only the first term gives

$$\Gamma[\bar{\phi}]=S[\bar{\phi}]-\frac{i}{2}\operatorname{tr}\ln G[\bar{\phi}]+\frac{i}{2}\operatorname{tr}\ln G^+[\bar{\phi}]+\cdots\ , \tag{3.17}$$

where $G[\bar{\phi}]$ is the Feynman propagator for $S_{,ij}[\bar{\phi}]$. This is recognized as the one-loop approximation. Substituting this approximation for Γ into the right-hand side of (3.15) and collecting all the two-loop terms gives the next iteration, represented graphically in Fig. 3, where solid lines represent $G[\bar{\phi}]$ and dashed lines represent $G^+[\bar{\phi}]$ (which is not symmetric, as indicated by the arrow). To get the field equations from this effective action one inserts a prong in all possible ways into each diagram, resulting in the graphs of Fig. 4.

These are the effective field equations for the in-out amplitude (2.3). The in-in effective field equations can be found by repeating the same steps, but starting with the in-in boundary conditions in (2.8). The structure of the

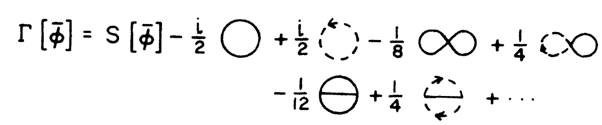

FIG. 3. Graphical representation of the effective action up to two-loop order.

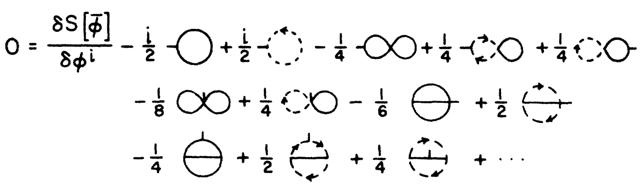

FIG. 4. The two-loop effective field equations.

diagrams is exactly as in Fig. 4 but now the solid lines represent $\mathscr{G}$ and dashed lines $\mathscr{G}^+$, the corresponding advanced Green's function for $\mathscr{S}_2$:

$$\mathscr{G}^+ \equiv \begin{bmatrix} G^+ & \widetilde{G} \\ 0 & -G^- \end{bmatrix}, \tag{3.18}$$

where

$$\begin{aligned} G^+ &\equiv G_{\text{in}} + G_{\text{in}}^{(+)}, \\ G^- &\equiv G_{\text{in}} - G_{\text{in}}^{(-)}, \\ \widetilde{G} &\equiv G_{\text{in}}^{(+)} + G_{\text{in}}^{(-)}, \end{aligned} \tag{3.19}$$

so that G^+ and G^- are the advanced and retarded Green's functions of $S_{,ij}$ and $\widetilde{G}$ is the commutator function. Using the Feynman rules as explained in Sec. II, the inserted prong in Fig. 4 is always a plus vertex, while all others can be either plus or minus. So to one loop the equation is

$$\begin{aligned} 0 &= S_{,i}[\bar\phi] - \frac{i}{2} S_{,ijk}[\bar\phi] G_{\text{in}}^{jk}[\bar\phi] + \frac{i}{2} S_{,ijk}[\bar\phi] G^{+jk}[\bar\phi] \\ &= S_{,i}[\bar\phi] + \frac{i}{2} S_{,ijk}[\bar\phi] G_{\text{in}}^{(+)jk}[\bar\phi] \\ &= S_{,i}[\bar\phi] + \frac{i}{4} S_{,ijk}[\bar\phi] (G_{\text{in}}^{(+)jk}[\bar\phi] - G_{\text{in}}^{(-)jk}[\bar\phi]), \end{aligned} \tag{3.20}$$

where we have used (2.24) and the symmetry of $S_{,ijk}$. The reality of this follows from (2.25). The causality of (3.20) is obvious: the vertex function $S_{,ijk}$ contains two δ functions and the spacetime integrations over indices j and k collapse completely. Moreover the functions $G_{\text{in}}^{(\pm)}$ can be constructed from in-mode functions propagated from the remote past, so they too are causal.

The algebra involved in evaluating the two-loop graphs is considerable, but fortunately the reality and causality properties hold separately for each set of diagrams with a given topology, so we can split the check up into pieces. For example, the sixth and seventh graphs of Fig. 4 give

$$\begin{aligned} -\tfrac{1}{8} S_{,ijklm}[\bar\phi](G_{\text{in}}^{jk}[\bar\phi] G_{\text{in}}^{lm}[\bar\phi] - 2G^{+jk}[\bar\phi] G_{\text{in}}^{lm}[\bar\phi]) &= -\tfrac{1}{8} S_{,ijklm}[\bar\phi](G_{\text{in}}^{jk}[\bar\phi] G_{\text{in}}^{lm}[\bar\phi] - 2\overline{G}^{jk}[\bar\phi] G_{\text{in}}^{lm}[\bar\phi]) \\ &= \tfrac{1}{8} S_{,ijklm}[\bar\phi](\overline{G}^{jk}[\bar\phi] \overline{G}^{lm}[\bar\phi] + \tfrac{1}{4} G_{\text{in}}^{(1)jk}[\bar\phi] G_{\text{in}}^{(1)lm}[\bar\phi]), \end{aligned} \tag{3.21}$$

where

$$\overline{G} \equiv \tfrac{1}{2}(G^+ + G^-) \tag{3.22}$$

and

$$\tfrac{1}{2} G_{\text{in}}^{(1)} \equiv \tfrac{1}{2} i (G_{\text{in}}^{(+)} - G_{\text{in}}^{(-)}) \tag{3.23}$$

are the real and imaginary parts of G_{in}, respectively. The other "figure-eight" graphs contribute

$$\begin{aligned} -\tfrac{1}{4} S_{,ijk} S_{,lmnp} (& G_{\text{in}}^{jl} G_{\text{in}}^{km} G_{\text{in}}^{np} + G_{\text{in}}^{(-)jl} G_{\text{in}}^{(-)km} G_{\text{in}}^{*np} \\ & - G^{+jl} G^{-km} G_{\text{in}}^{np} - G_{\text{in}}^{jl} G_{\text{in}}^{km} G^{+np} \\ & - G_{\text{in}}^{(-)jl} G_{\text{in}}^{(-)km} G^{-np}), \end{aligned} \tag{3.24}$$

where the omitted argument of all vertex functions and Green's functions is $\bar\phi$. After a little algebra this can be recast in the form

$$\tfrac{1}{4} S_{,ijk} S_{,lmnp} G^{-jl} (\tfrac{1}{2} G_{\text{in}}^{(1)km} G_{\text{in}}^{(1)np} - G^{+km} G^{+np}). \tag{3.25}$$

Expressions (3.21) and (3.25) are explicitly real (the advanced and retarded Green's functions are both real) and the retarded propagator G^{-jl} in (3.25) ensures that all the spacetime integrations only involve points to the past of the point associated with the index i.

The remaining two-loop diagrams are treated similarly. After some simplification using the kinematical relations between the various Green's functions as well as the symmetry of the vertex functions, we find that the eighth and ninth diagrams of Fig. 4 give

$$\begin{aligned} -\tfrac{1}{6} S_{,ijkl} S_{,mnp} G^{-lp} (& -\tfrac{3}{4} G^{+jm} G^{+kn} \\ & + \tfrac{1}{4} G^{-jm} G^{-kn} - \tfrac{3}{2} G^{-jm} G^{+kn} \\ & - \tfrac{3}{4} G_{\text{in}}^{(1)jm} G_{\text{in}}^{(1)kn}) \end{aligned} \tag{3.26}$$

and the last three graphs contribute

$$\begin{aligned} -\tfrac{1}{4} S_{,ijk} S_{,lnp} S_{,mqr} (& \tfrac{1}{4} G^{-jl} G^{-km} \widetilde{G}^{nq} \widetilde{G}^{pr} \\ & - 2G^{+jl} G^{-km} G^{+nq} \overline{G}^{pr} \\ & - \tfrac{1}{4} G^{-jl} G^{-km} G_{\text{in}}^{(1)nq} G_{\text{in}}^{(1)pr} \\ & - G_{\text{in}}^{(1)jl} G^{-km} G^{+nq} G_{\text{in}}^{(1)pr}). \end{aligned} \tag{3.27}$$

These expressions are real and in each case the retarded propagator appears in such a way as to ensure that the dependence of the equations on the background $\bar\phi$ is causal.

Turning now to the in-in effective action itself, the diagrammatic expansion in Fig. 3 can be used to formally check unitarity in the form of equation (3.7). Again, the two-loop graphs of a given topology cancel when we set $\bar\phi_+ = \bar\phi_-$. For example, the two figure-eight diagrams give

$$-\tfrac{1}{8}S_{,ijkl}(G^{ij}_{\rm in}G^{kl}_{\rm in}-G^{*ij}_{\rm in}G^{*kl}_{\rm in}-2G^{+ij}G^{kl}_{\rm in}+2G^{-ij}G^{*kl}_{\rm in})=-\tfrac{1}{8}S_{,ijkl}[(G^{ij}_{\rm in}+G^{*ij}_{\rm in})(G^{kl}_{\rm in}-G^{*kl}_{\rm in})-2G^{+ij}G^{kl}_{\rm in}+2G^{-ij}G^{*kl}_{\rm in}]$$

$$=-\tfrac{1}{8}S_{,ijkl}[2\bar{G}^{ij}(G^{kl}_{\rm in}-G^{*kl}_{\rm in})-2\bar{G}^{ij}(G^{kl}_{\rm in}-G^{*kl}_{\rm in})]=0\ . \quad (3.28)$$

The successful checks on unitarity, causality, and reality are an indication that the form (3.12) is a good choice for the measure.

Finally, a note concerning renormalizability. Because of the off-diagonal elements of the propagator the in-in effective action will contain terms which involve both $\bar{\phi}_+$ and $\bar{\phi}_-$, and some of these terms may be divergent. These infinities cannot be renormalized because the classical action $S[\bar{\phi}_+]-S[\bar{\phi}_-]$ does not contain such terms, but this does not matter. We are interested in the effective field equations, with $\bar{\phi}_+$ set equal to $\bar{\phi}_-$, and it is these equations that must be renormalized, rather than the effective action itself. Provided the in-out theory is renormalizable, it will be possible to absorb all the infinities in the equations into the bare field equations, using standard regularization techniques.

IV. A SIMPLE EXAMPLE

The discussion so far has been very general but also very formal, so we now work out an example to show how the theory works. The example is a simple quantum-mechanical system which nevertheless has different in and out "vacua." Moreover the system is exactly solvable so the in-in calculations can be compared to the in-out results; we hope that this example will persuade the unconvinced reader that one can gain greater insight into the nature of the system by examining expectation values rather than in-out amplitudes.

The action is that for a harmonic oscillator with a prescribed, time-dependent spring "constant:"

$$S[q]=\int[-\tfrac{1}{2}\dot{q}^2(t)+\tfrac{1}{2}\omega^2(t)q^2(t)]dt\ , \quad (4.1)$$

where

$$\omega^2(t)=A+B\tanh\lambda t\ . \quad (4.2)$$

The function $\omega^2(t)$ is shown in Fig. 5. It becomes constant in the remote past and the remote future:

$$\omega^2(t)\underset{t\to\pm\infty}{\longrightarrow}A\pm B\equiv\omega_{\substack{\rm out\\ \rm in}}{}^2\ . \quad (4.3)$$

We can expect to find a complete set of in-mode functions by propagating the harmonic-oscillator mode functions forward in time using the equations of motion from (4.1):

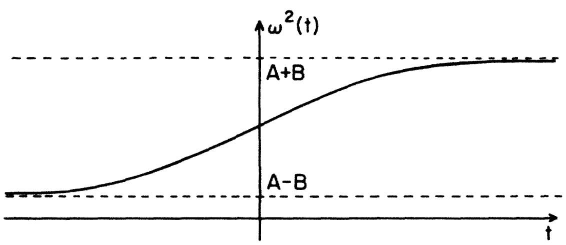

FIG. 5. Time dependence of the spring "constant" $\omega^2(t)$.

$$\ddot{q}(t)+\omega^2(t)q(t)=0\ . \quad (4.4)$$

Actually we do not have to do this, because it is not hard to solve Eq. (4.4) in terms of hypergeometric functions. The solution which becomes positive frequency in the remote past is

$$u_{\rm in}(t)=\frac{1}{(2\omega_{\rm in})^{1/2}}\exp\left[-i\omega_+t-\frac{i\omega_-}{\lambda}\ln(2\cosh\lambda t)\right]\times{}_2F_1\left[1+\frac{i\omega_-}{\lambda},\frac{i\omega_-}{\lambda};1-\frac{i\omega_{\rm in}}{\lambda};\ \tfrac{1}{2}(1+\tanh\lambda t)\right]\ , \quad (4.5)$$

where

$$\omega_\pm\equiv\tfrac{1}{2}(\omega_{\rm out}\pm\omega_{\rm in})\ . \quad (4.6)$$

Then as $t\to-\infty$

$$u_{\rm in}(t)\sim\frac{1}{(2\omega_{\rm in})^{1/2}}e^{-i\omega_+t+i\omega_-t}=\frac{1}{(2\omega_{\rm in})^{1/2}}e^{-i\omega_{\rm in}t}\ . \quad (4.7)$$

To work out the in-out theory (as we do here for comparison) we also need mode functions which are pure positive frequency in the remote future:

$$u_{\rm out}(t)=\frac{1}{(2\omega_{\rm out})^{1/2}}\exp\left[-i\omega_+t-\frac{i\omega_-}{\lambda}\ln(2\cosh\lambda t)\right]\times{}_2F_1\left[1+\frac{i\omega_-}{\lambda},\frac{i\omega_-}{\lambda};1+\frac{i\omega_{\rm out}}{\lambda};\ \tfrac{1}{2}(1-\tanh\lambda t)\right] \quad (4.8)$$

$$\underset{t\to\infty}{\sim}\frac{1}{(2\omega_{\rm out})^{1/2}}e^{-i\omega_+t-i\omega_-t}=\frac{1}{(2\omega_{\rm out})^{1/2}}e^{-i\omega_{\rm out}t}\ . \quad (4.9)$$

These mode functions are related by a Bogoliubov transformation:

$$u_{\rm in}(t)=\alpha u_{\rm out}(t)+\beta u^*_{\rm out}(t)\ ,$$
$$u_{\rm out}(t)=\alpha^* u_{\rm in}(t)-\beta u^*_{\rm in}(t)\ , \quad (4.10)$$

where

$$\alpha=\left[\frac{\omega_{\rm out}}{\omega_{\rm in}}\right]^{1/2}\frac{\Gamma\left[1-\frac{i\omega_{\rm in}}{\lambda}\right]\Gamma\left[-\frac{i\omega_{\rm out}}{\lambda}\right]}{\Gamma\left[-\frac{i\omega_+}{\lambda}\right]\Gamma\left[1-\frac{i\omega_+}{\lambda}\right]}\,,$$

$$\beta=\left[\frac{\omega_{\rm out}}{\omega_{\rm in}}\right]^{1/2}\frac{\Gamma\left[1-\frac{i\omega_{\rm in}}{\lambda}\right]\Gamma\left[\frac{i\omega_{\rm out}}{\lambda}\right]}{\Gamma\left[\frac{i\omega_-}{\lambda}\right]\Gamma\left[1+\frac{i\omega_-}{\lambda}\right]}\,. \tag{4.11}$$

The generating functional for this linear system can be calculated exactly. The in-out theory is obtained from

$$e^{iW[J]} = \text{const}\times\int\exp\left[i\int_{-\infty}^{\infty}(-\tfrac{1}{2}\dot{q}^2+\tfrac{1}{2}\omega^2q^2+Jq)dt\right]dq \tag{4.12}$$

with the usual in-out vacuum boundary conditions, leading to

$$e^{iW[J]}=\text{const}\times(\det G_F)^{1/2}e^{(i/2)JG_FJ}\,. \tag{4.13}$$

Here G_F is the usual Feynman propagator:

$$G_F(t,t')=i\frac{\langle \text{out,vac}\mid T\mathbf{q}(t)\mathbf{q}(t')\mid \text{in,vac}\rangle}{\langle\text{out,vac}\mid\text{in,vac}\rangle} \tag{4.14}$$

$$=i\theta(t-t')u_{\rm out}(t)\alpha^{-1}u^*_{\rm in}(t')$$
$$+i\theta(t'-t)u^*_{\rm in}(t)\alpha^{-1}u_{\rm out}(t')\,. \tag{4.15}$$

From (4.13)

$$W[J]=\tfrac{1}{2}JG_FJ+\text{const}\,. \tag{4.16}$$

The in-out effective field generated by $W[J]$ is

$$\bar{q}(t)=\frac{\delta W[J]}{\delta J(t)}=\frac{\langle\text{out,vac}\mid\mathbf{q}(t)\mid\text{in,vac}\rangle_J}{\langle\text{out,vac}\mid\text{in,vac}\rangle_J} \tag{4.17}$$

$$=\int_{-\infty}^{\infty}G_F(t,t')J(t')dt'\,. \tag{4.18}$$

This can easily be inverted

$$J(t)=-\left[\frac{d^2}{dt^2}+\omega^2\right]\bar{q}(t)\,, \tag{4.19}$$

and the in-out effective action is

$$\Gamma[\bar{q}]=W[J]-J\bar{q}=\tfrac{1}{2}JG_FJ+\text{const}-JG_FJ$$
$$=\tfrac{1}{2}\int_{-\infty}^{\infty}\bar{q}(t)\left[\frac{d^2}{dt^2}+\omega^2\right]\bar{q}(t)dt+\text{const}$$
$$=S[\bar{q}]+\text{const}\,. \tag{4.20}$$

This could also have been obtained from the functional-integral expression for Γ as in Sec. III. Either way we discover that the in-out effective field equations are identical to the classical field equations, but the boundary conditions dictate a complex solution [(4.17) and 4.18)] employing the Feynman propagator. The effective field therefore depends on the source at all times, while in the absence of sources $\bar{q}(t)=0$.

This is to be compared with the results of the in-in theory, which starts from the generating functional:

$$e^{iW[J_+,J_-]}=\text{const}\times\int\exp\left[i\int_{-\infty}^{T}(-\tfrac{1}{2}\dot{q}_+{}^2+\tfrac{1}{2}\omega^2q_+{}^2+J_+q_+)dt-i\int_{-\infty}^{T}(-\tfrac{1}{2}\dot{q}_-{}^2+\tfrac{1}{2}\omega^2q_-{}^2+J_-q_-)dt\right]dq_+dq_- \tag{4.21}$$

with the in-in boundary conditions as given in Sec. II. T is the "turn-around" time for the paths in (4.21) and henceforth we take $T=\infty$. These boundary conditions yield, as before

$$e^{iW[J_+,J_-]}=\text{const}\times(\det\mathcal{G})^{1/2}e^{(i/2)\mathcal{J}\tilde{\mathcal{G}}\mathcal{J}}\,, \tag{4.22}$$

where we are using the matrix notation introduced in Sec. II. This gives rise to effective fields:

$$\bar{q}_+(t)=\frac{\delta W[J_+,J_-]}{\delta J_+(t)}=\int_{-\infty}^{\infty}[G_{\rm in}(t,t')J_+(t')-G_{\rm in}^{(-)}(t,t')J_-(t')]dt'\,,$$
$$\bar{q}_-(t)=\frac{\delta W[J_+,J_-]}{-\delta J_-(t)}=\int_{-\infty}^{\infty}[-G_{\rm in}^{(+)}(t,t')J_+(t')+G^*_{\rm in}(t,t')J_-(t')]dt' \tag{4.23}$$

so the expectation value is

$$\langle\text{in,vac}\mid\mathbf{q}(t)\mid\text{in,vac}\rangle_J=\bar{q}_\pm(t)\mid_{J_+=J_-=J}=\int_{-\infty}^{\infty}G^-(t,t')J(t')dt'\,. \tag{4.24}$$

For the in-in effective action we must first invert (4.23),

$$\begin{bmatrix}J_+\\-J_-\end{bmatrix}=\begin{bmatrix}-S_2&0\\0&S_2\end{bmatrix}\begin{bmatrix}\bar{q}_+\\\bar{q}_-\end{bmatrix} \tag{4.25}$$

giving

$$\Gamma[\bar{q}_+,\bar{q}_-]=W[J_+,J_-]-J_+\bar{q}_+ +J_-\bar{q}_- =\tfrac{1}{2}\mathscr{J}^{\sim}\mathscr{G}\mathscr{J}+\text{const}-\mathscr{J}^{\sim}\mathscr{G}\mathscr{J}$$

$$=\tfrac{1}{2}\int_{-\infty}^{\infty}(\bar{q}_+\ \bar{q}_-)\begin{bmatrix}S_2 & 0\\ 0 & -S_2\end{bmatrix}\begin{bmatrix}\bar{q}_+\\ \bar{q}_-\end{bmatrix}dt=S[\bar{q}_+]-S[\bar{q}_-]+\text{const}\ . \tag{4.26}$$

The in-in effective field equations are also just the classical equations, but this time the boundary conditions ensure that the solution (4.24) is real and depends causally on the source because it involves the retarded Green's function instead of the Feynman propagator. This is just as was expected. The weakness of this example, however, is that both in-out and in-in effective fields vanish identically in the absence of sources. To compare the two formalisms for the case of zero source we must look at a quadratic operator:

$$\left[\frac{\delta}{i\delta J_+(t)}\right]\left[\frac{\delta}{-i\delta J_-(t')}\right]e^{iW[J_+,J_-]}\Big|_{J_+=J_-=0}=\langle \text{in,vac}\,|\,\mathbf{q}(t')\mathbf{q}(t)\,|\,\text{in,vac}\rangle$$

$$=-iG_{\text{in}}^{(-)}(t,t')=u_{\text{in}}(t')u_{\text{in}}^*(t)$$

$$\underset{t,t'\to-\infty}{\sim}\frac{1}{2\omega_{\text{in}}}e^{i\omega_{\text{in}}(t-t')}\ . \tag{4.27}$$

This result is useful physically, because it reproduces the harmonic-oscillator result in the remote past. Deviations from that value as time proceeds can easily be studied in this object. On the other hand, the in-out generating functional yields matrix elements that are hard to interpret in such a way:

$$\left[\frac{\delta}{i\delta J(t)}\right]\left[\frac{\delta}{i\delta J(t')}\right]e^{iW[J]}\Big|_{J=0}=\frac{\langle \text{out,vac}\,|\,T\mathbf{q}(t)\mathbf{q}(t')\,|\,\text{in,vac}\rangle}{\langle \text{out,vac}\,|\,\text{in,vac}\rangle}=-iG_F(t,t')$$

$$\underset{t,t'\to-\infty}{\sim}\theta(t-t')\frac{1}{2\omega_{\text{in}}}e^{i\omega_{\text{in}}t'}\left[\frac{\alpha^*}{\alpha}e^{-i\omega_{\text{in}}t}-\frac{\beta}{\alpha}e^{i\omega_{\text{in}}t}\right]$$

$$+\theta(t'-t)\frac{1}{2\omega_{\text{in}}}e^{i\omega_{\text{in}}t}\left[\frac{\alpha^*}{\alpha}e^{-i\omega_{\text{in}}t'}-\frac{\beta}{\alpha}e^{i\omega_{\text{in}}t'}\right]\ . \tag{4.28}$$

Since this does not even reduce to the harmonic-oscillator result in the remote past (nor the remote future) it is difficult to know what physical information it yields when studied in the time-dependent region. Certainly if one wishes to learn about the time evolution of the system one is better off looking at expectation values.

This example, involving a linear theory and only a single degree of freedom, avoids many of the complications that arise in an interacting field theory. It is very closely related, however, to the problem of scalar particle creation in a Friedman-Robertson-Walker universe with the scale factor behaving like (4.2). The time dependence of the field equations in that setting is exactly the same as the equations of motion (4.4) and all the calculations outlined here can be used to study expectation values of the fields in that background. Details of the in-out theory of that system can be found in the literature.[9]

V. DISCUSSION

There is another interpretation for the generating functional which avoids the notion of paths going backward in time. Keeping the turn-around time T finite, one can consider a system with action

$$S[\phi]=\int dt\,d^3x[\theta(T-t)-\theta(t-T)]L(\phi,\partial_\mu\phi,\dots)\ . \tag{5.1}$$

The negative sign for $t>T$ corresponds to the reverse half of the time loop in the previous interpretation. The advantage of this approach is that the boundary conditions on the path integral are now the usual Feynman ones so it can be treated by the standard methods. It is not hard to find mode functions for this system and from them to construct the propagator, which turns out to be

$$\begin{aligned}\mathscr{G}(x,x')=&\,\theta(T-t)\theta(T-t')G_{\text{in}}(x,x')\\ &+\theta(T-t)\theta(t'-T)G_{\text{in}}^{(-)}(x,x')\\ &-\theta(t-T)\theta(T-t')G_{\text{in}}^{(+)}(x,x')\\ &-\theta(t-T)\theta(t'-T)G_{\text{in}}^*(x,x')\ .\end{aligned} \tag{5.2}$$

This is recognized as the old matrix propagator (2.23), the different matrix elements now being labeled by the various step functions. It is then obvious how to translate from the language of the closed time path to that of the step functions, and with this translation the calculational rules are identical.

Yet another viewpoint has been suggested by Hajicek.[3] He points out that the unusual signs in the action (5.1) can be obtained from the Euclidean theory via an unusual rotation of the time contour:

$$\tau=\begin{cases}it, & t<0\\ -it, & t>0\ ,\end{cases} \tag{5.3}$$

where we are now setting $T=0$. The effect of (5.3) is to take the (Euclidean) time contour along the imaginary t axis and "fold" it to produce a contour from the past up to $t=0$ and back to the past. It is straightforward to see that this rotation applied to the Euclidean Green's function yields the propagator $\mathcal{G}(x,x')$ once again. Thus, whichever viewpoint one prefers, one is led to the same results in any calculation.

This route from the Euclideanized theory to the in-in effective field equations sheds light on an alternative proposal of Frolov and Vilkovisky[10] for obtaining such equations. They suggested that the effective field equations be calculated in Euclidean theory, and argued[11] that it should be possible to write the equations in the form

$$0=\sum G_E R G_E R \cdots G_E R \ , \tag{5.4}$$

i.e., curvatures linked by Euclidean propagators into a "tree structure." Their proposal to obtain the in-in equations is to replace G_E everywhere in (5.4) by the retarded propagator G^-. Certainly this gives a result that is real and causal, and it is in fact the correct procedure; we now show by induction that the effect of the contour rotation (5.3) on a structure like (5.4) is indeed to replace G_E by G^-.

The first term in (5.4) is

$$\int G_E(x,x')R(x')dx'=\left[\int_{-\infty}^{0} d\tau'+\int_{0}^{\infty} d\tau'\right] \times \int d\mathbf{x}' G_E(x,x')R(x') \ , \tag{5.5}$$

which, on rotating as in (5.3) and setting $g_{\mu\nu}(t,\mathbf{x})=g_{\mu\nu}(-t,\mathbf{x})$, gives

$$\int_{-\infty}^{0} dt' \int d\mathbf{x}'[G_{\text{in}}(x,x')-G_{\text{in}}^{(-)}(x,x')]R(x') = \int dx' G^-(x,x')R(x') \ . \tag{5.6}$$

Similarly the effect of (5.3) on a tree structure T_{n+1} with n propagators is

$$\begin{aligned} T_{n+1}=T_n G_E R \underset{(5.3)}{\longrightarrow} & \ T_n^{++}G_{\text{in}}R_+ - T_n^{+-}G_{\text{in}}^{(+)}R_+ \\ & -T_n^{++}G_{\text{in}}^{(-)}R_- + T_n^{+-}G_{\text{in}}^{*}R_- \\ \xrightarrow[g_{\mu\nu_+}=g_{\mu\nu_-}]{} & (T_n^{++}+T_n^{+-})G^-R \end{aligned} \tag{5.7}$$

$$=T_n G^- R \tag{5.8}$$

so if the replacement of G_E by G^- works for T_n, it also works for T_{n+1}, proving the result.

Thus the Frolov-Vilkovisky prescription is valid *once the Euclidean field equations are put in the form (5.4);* the replacement of G_E by G^- cannot be performed in diagrammatic rules.[12]

In summary, the formalism presented here provides a general framework for finding expectation values as well as setting down modified Feynman rules for use in calculations. The rules found in this paper agree with those of Ref. 5 for the case of QED, but we showed in Sec. III that for a more general theory additional terms, arising from the measure in the functional integral, must be included. The main result, the effective field equations up to two loops, provides the sought-after quantum corrections to the classical equations and should be useful in applications such as quantum cosmology. The causality of the equations means it should be possible to solve them numerically by a differencing scheme, starting from a classical solution (or something close to a classical solution) in the remote past. The reality property means the solution can then be interpreted as an effective metric during the quantum era.

ACKNOWLEDGMENTS

It is a pleasure to thank Bryce DeWitt for many informative discussions throughout the course of this work, as well as Richard Woodward for helpful conversations. Thanks are also due to G. A. Vilkovisky for clarifying his classical causality conjecture. This work was supported in part by National Science Foundation Grant No. PHY83-18635.

[1]J. Schwinger, J. Math. Phys. **2**, 407 (1961); *Particles, Sources and Fields* (Addison-Wesley, Reading, Mass., 1970).

[2]M. V. Fischetti, J. B. Hartle, and B. L. Hu, Phys. Rev. D **20**, 1757 (1979); J. B. Hartle and B. L. Hu, *ibid.* **20**, 1772 (1979); **21**, 2756 (1980); J. B. Hartle, *ibid.* **22**, 2091 (1980); **23**, 2121 (1981).

[3]P. Hajicek, University of Berne report, 1978 (unpublished); in *Proceedings of the Second Marcel Grossman Meeting on General Relativity,* edited by R. Ruffini (North-Holland, Amsterdam, 1982).

[4]E. S. Fradkin and D. M. Gitman, Fortschr. Phys. **29**, 381 (1981).

[5]I. L. Buchbinder, E. S. Fradkin, and D. M. Gitman, Fortschr. Phys. **29**, 187 (1981).

[6]K. Chou, Z. Su, B. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985).

[7]B. S. DeWitt, in *Relativity, Groups and Topology II,* edited by B. S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984).

[8]B. S. DeWitt, Phys. Rev. **162**, 1195 (1967).

[9]C. Bernard and A. Duncan, Ann. Phys. (N.Y.) **107**, 201 (1977); N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[10]V. P. Frolov and G. A. Vilkovisky, in *Quantum Gravity,* proceedings of the Second Moscow Seminar, edited by M. A. Markov and P. C. West (Plenum, New York, 1983).

[11]G. A. Vilkovisky, in *Quantum Theory of Gravity,* edited by S. M. Christensen (Adam Hilger, Bristol, 1984).

[12]G. A. Vilkovisky (private communication).