Federal University of Goiás

Institute of Physics

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The Unruh effect

Monography

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Goiânia - 2022

Brazil

FEDERAL UNIVERSITY OF GOIÁS
INSTITUTE OF PHYSICS

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Monograph presented to the Institute of Physics of the Federal University of Goiás as partial requirement to obtain the Bachelor's degree in Physics.

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Goiânia - 2022

Acknowledgments

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Luisa Siewerdt Amarilho
Lucas Chibebe Céleri
Maurilho Rodrigues da Rocha
Marcus Vinicius S. O. Silva
Gustavo de Oliveira
Maria de Paulo Sousa
José de Paulo Souza
Tiago Gaspio dos Santos Souza

ABSTRACT

The present work aims to discuss an extraordinary phenomenon predicted through the quantum field theory called the Unruh effect. This effect shows us that an accelerating observer can detect a thermal bath of particles whereas an inertial one observes a vacuum. This effect plays a crucial role in our understanding that the particle content of a quantum field is observer-dependent. The idea will be to motivate and give the basic mathematical tools that are necessary to develop and understand the calculations behind this effect. In addition to the nature of an undergrad thesis, the present text might help other students who decide to address this subject.

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Chapter 1

Introduction

The Unruh effect was discovered by the physicists William Unruh, Paul Davies, and Stephen Fulling in 1976 and it is essentially a theoretical result of quantum field theory that is crucial in our understanding that the particle content of a field theory is observer depends. This effect expresses the fact that uniformly accelerated observers in Minkowski spacetime associate a thermal bath of Rindler particles to the no-particle state of inertial observers. This effect is extremely important and we can see it through the increase in interest in it over the last few decades [1], t is worth mentioning that in parts it happens because of its connection with other phenomena such as the thermal emission of particles from black holes [2], cosmological horizons [3], among others.

The effect itself is supported by the quantum field theory which in turn is an extremely useful and precise framework created to describe the subatomic world. The concept of the field was born as a broader and more abstract idea than those that have been used since Newton to describe forces and particles. Such a tool aims to associate an entity like a number, a vector, a tensor, or some other object with well-defined characteristics, to each point of space; thus helping the process of dealing with several objects of the same type. The first field to be created was the electric field. Faraday did it to generalize the Coulomb force; initially, it was just a mathematical tool but soon, with the discovery of electromagnetic waves, we figure out that it has a physical existence. In the following years, several other fields were created, some of them to be an alternative for interaction force like Einstein's general relativity and others to describe particles like the Klein-Gordon field.

The necessity of a quantum field was raised since some predictions of special relativity were not compatible with quantum mechanics, this fact allied with our difficulty in dealing with systems of many particles bring to us the idea of try of breaking down our systems in the smallest number of equal particles possible and then treat these equal particles as components of the same field. As the electromagnetic field, initially, the field of particles was purely mathematical but today we already have some experiments that verify the physical existence of the field itself, among these experiments it is worth mentioning the Lamoreaux demonstration of the Casimir effect [4] [5].

Chapter 2

Quantum field theory in curved spacetime

This chapter has the intent to present the mathematical formalism that will be used along this monograph to present the Unruh effect on flat spacetime. Since it involves a lot of different and complex topics, it will only be possible to introduce them without going into too many details but I made a point of citing the references for those who want to go deeper.

2.1 Classical mechanics

The most common formulation of quantum field theory appears in the study of continuous systems, which in turn can usually be interpreted as an infinity set of particles, a set of objects with specific characteristics and whose dimensions are irrelevant. There are many theories to deal with particles [6], among them the Lagrangian and Hamiltonian formulations of classical mechanics will be strongly useful here since the study of quantum mechanics and Einstein's relativity usually arise with them. Hence, to start, it makes sense to introduce these theories.

Fortunately, Lagrangian and Hamiltonian descriptions of mechanics are intimately connected to each other and so they share several definitions that will be presented. In this context, the state of a particle will be given by its position $\mathbf{r}$ and momentum $\mathbf{p}$, which are vectors in a Galilean space [7], these knowledge are enough to characterize our particle but it is not sufficient to give us its dynamics. For this, we need to know how these quantities will evolve on time. The time t will be viewed here as an invariant and external entity from which some other is defined, as example we have the velocity $\mathbf{v} \stackrel{\text{def}}{=} \frac{d\mathbf{r}}{dt} \equiv \dot{\mathbf{r}}$, acceleration $\mathbf{a} \stackrel{\text{def}}{=} \frac{d^2\mathbf{r}}{dt^2}$, force $\mathbf{F} \stackrel{\text{def}}{=} \frac{d\mathbf{p}}{dt}$, and so on.

Since the particles have some characteristics that are fundamental to determine their behavior, it is relevant to introduce a few more things. When some characteristic is intrinsic and observer-independent we say that it is a type of "charge"; for example, we have the mass charge m, or just mass, that can be detected in the presence of a gravitational field; the electric charge that creates and interact with the electromagnetic field; in quantum mechanics, there is the spin that is related with the magnetic field; and so on. The interactions of particles with these characteristics were initially described by Newton through the concept of a force $\mathbf{F}$ and the idea of energy E. Some relevant types of energy are the kinetic energy T that is linked with the movement and the potential energy V that came from conservative forces $\mathbf{F} = -\nabla V$.

The equations that relate these parameters allowing us to determine the behavior of a system can be given by Newton, Lagrange or Hamilton mechanics. Starting with the Lagrangian, consider a system composed by N particles. It is possible to get its equation in two different ways. The first one is based on D'Alembert's principle of virtual work: Considering that the position, given by the components of the position vector, can be written as a function of a set of variables, called generalized coordinates, $\mathbf{r} \xrightarrow{S} \{q_k(t)\}$, that are capable of uniquely specifying the system, and assuming the presence of some forces that are conservative and some that are not. We define the Lagrangian function as [8]

$$L = T - V. \tag{2.1}$$

From which it follows the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}. \tag{2.2}$$

So far we have a mechanism able to describe a system formed by a finite set of particles with the aforementioned characteristics; something that is enough to deal with pretty much everything that is classical, things that are not too small and/or are not moving too fast. However, in order to work with fields, it will be necessary to generalize these ideas for continuous bodies and in this process it makes sense to use the second method of obtaining the eq.((2.2)) employing variational calculus and starting from Hamilton's principle of least action [9].

Theorem 2.1.1. *The true evolution $\mathbf{q}(t)$ of a system described by a set of N generalized coordinated $\mathbf{q} = (q_1(t), q_2(t), ..., q_N(t))$ between two specified states $\mathbf{q}(t_1)$ e $\mathbf{q}(t_2)$ at two specified times t_1 and t_2 is a stationary point (a point where the variation is zero $\delta\mathcal{S} = 0$) of the action functional*

$$\mathcal{S}[q_k] \stackrel{\text{def}}{=} \int_{t_1}^{t_2} L(q_k(t), \dot{q}_k(t), t)\, dt. \tag{2.3}$$

Since the action functional is linear, it is possible to write a resultant functional as a sum $L = \sum_i L_i(q_{k,i}, \dot{q}_{k,i}, t)$; it is not hard to notice that the one-dimension continuous analog of this sum will be an integral over the position variable. Imposing that there is only interaction between a particle and its closet 'neighbor' since we want to develop a simple model; we can write the Lagrangian as an integral of a new function called Lagrangian density $\mathscr{L}$ (sometimes just named as Lagrangian) that will depend on the field $\Phi^i \in \{\Phi^0, \Phi^1, ..., \Phi^n\}$ (the continuous analog of generalized coordinates), position, time and the fields' derivative with respect to position and time

$$L[\Phi^i] = \int_{x_1}^{x_2} dx\ \mathscr{L}(\Phi^i, \dot{\Phi}^i, \Phi'^i, x, t) \tag{2.4}$$

with $\dot{\Phi}^i = \dfrac{\partial \Phi^i}{\partial t}$, and $\Phi'^i = \dfrac{\partial \Phi^i}{\partial x}$. Therefore, we have an action for the continuous case

$$\mathcal{S}[\Phi^i] = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx\ \mathscr{L}(\Phi^i, \dot{\Phi}^i, \Phi^{i\prime}, x, t). \tag{2.5}$$

Applying the principle of least action with the following boundary conditions

$$\delta\Phi^i(x, t_1) = \delta\Phi^i(x, t_2) = \delta\Phi^i(x_1, t) = \delta\Phi^i(x_2, t) = 0$$

we get

$$\begin{aligned}
\delta\mathcal{S} &= \delta \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx\ \mathscr{L}(\Phi^i, \dot{\Phi}^i, \Phi'^i, x, t) \\
&= \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left[\frac{\partial \mathscr{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathscr{L}}{\partial \dot{\Phi}^i}\frac{\partial}{\partial t}\delta\dot{\Phi}^i + \frac{\partial \mathscr{L}}{\partial \Phi'^i}\frac{\partial}{\partial x}\delta\Phi'^i\right] \\
&= \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left[\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\Phi}^i}\right) + \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial \Phi'^i}\right) - \left(\frac{\partial \mathscr{L}}{\partial \Phi^i}\right)\right]\delta\Phi^i \\
&= 0
\end{aligned}$$

since the variations $\delta\Phi^i(x, t)$ are completely arbitrary, we must have

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\Phi}^i}\right) + \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial \Phi'^i}\right) - \left(\frac{\partial \mathscr{L}}{\partial \Phi^i}\right) = 0 \tag{2.6}$$

which is the Euler-Lagrange equations for fields.

2.2 Spacetime

2.2.1 Flat spacetime

Until the early 20th century, most of the physics phenomenon was explained in terms of classical mechanics, but with the rise of electromagnetism and Maxwell's equations, a new formulation of mechanics was necessary to resolve an existing inconsistency between these two theories. In the vacuum, whereas classical mechanics allowed different values for the speed of light c, since it could be measured by different observers with different velocities, the electrodynamics categorically stated that the speed of light was the same regardless of the reference frame.

This inconsistency was solved by Einstein in 1905 [10] by using Lorentz transformations instead of Galilean transformations to establish the relationship between two inertial frames. In the Lorentz transformations, we understand the time as a variable such as those used to describe position instead of an external parameter; so it also depends on the observer. As it is of interest to deal with objects in the relativist regime, a few definitions will aid in the discussion. We start with the postulates of the special theory of relativity

Postulate 2.2.1. *The laws of physics take the same form in all inertial frames of reference.*

Postulate 2.2.2. *The speed of light in empty space c is a constant, independent of the observer and the source.*

The space which special relativity aims to explain is the four-dimension spacetime with a Lorentzian metric, called the Minkowski spacetime. A vector here will be constructed through a linear combination of four-dimensional base vectors $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \in \mathbb{V}$. The four-vector position $\mathbf{x}$ that represents a point (an event) on the spacetime, for example, will be written, in canonical base, as

$$\begin{aligned}\mathbf{x} &\stackrel{\text{def}}{=} \sum_{\mu} x^{\mu}\mathbf{e}_{\mu} \equiv x^{\mu}\mathbf{e}_{\mu} \\ &= x^0\mathbf{e}_0 + x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 \\ &= ct\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.\end{aligned} \tag{2.7}$$

In this notation x^{μ}, with $\mu \in \{0, 1, 2, 3\}$, represents an element in such way that we can construct a vector through the Einstein sum. It is remarkable that the first component x^0 is given by ct so that it has a unit of length as the other components.

As we know, in the Euclidean space the size (or the norm) of a vector $\|\mathbf{v}\|$ is commonly constructed through an inner product (dot product) such we have

$$\|\mathbf{v}\|^2 \stackrel{\text{def}}{=} \mathbf{v} \cdot \mathbf{v}. \tag{2.8}$$

Where I'm using the following definition of inner product

Definition 2.2.1. *Let $\mathbb{V}$ be a vector space over $\mathbb{R}$. An inner product $(\ ,\ )$ is the map $\mathbb{V} \times \mathbb{V} \to \mathbb{R}$ with the following properties:*

1. $\forall\, \mathbf{u} \in V, (\mathbf{u}, \mathbf{u}) \geq 0, \quad (\mathbf{u}, \mathbf{u}) = 0 \iff \mathbf{u} = 0$
2. $\forall\, \mathbf{u}, \mathbf{v} \in V$, *holds* $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$
3. $\forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ *and* $\forall\, a, b \in \mathbb{R}$, *holds* $(a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = a(\mathbf{u}, \mathbf{w}) + b(\mathbf{v}, \mathbf{w})$

in the specific case of Euclidean space, the dot product between standard basis vector is $(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$.

For the Minkowski spacetime, the squared norm of the position vector will, by definition, be given by

$$\|\mathbf{x}\|^2 \stackrel{\text{def}}{=} (ct)^2 - x^2 - y^2 - z^2. \tag{2.9}$$

Clearly, $\mathbf{x}$ can be positive (timelike), negative (spacelike), or zero (lightlike). This is completely inconsistent with our definition of inner product and therefore we will not use it to induce the norm, we will need to build this idea through the concept of covectors.

It is well known that some physical phenomena are the same for all observers, which means that an observable $\mathscr{F}$, usually represented by a norm of a vector, will not vary under a change of reference frame. More formally we can say that some physical measurement must have a well-defined characteristic (represented by a scalar, vector, tensor, spinor, etc.) beyond an isometric group implying in a covariant relation

$$\mathscr{F}(S) = \mathscr{F}(S'). \tag{2.10}$$

This statement restricts the type of transformation that we can use to go from one observer to another, allowing us to determine a unique law of change. The physical phenomenon that is invariant throughout a change of reference frame depends on the theory that we are dealing with. In classical mechanics, we want that the acceleration stays the same to maintain Newton's laws invariant. In special relativity we want the light speed to remain the same to not violate Maxwell's laws. The transformation for the components of the position vector between two inertial frames with relative velocity $\|\mathbf{u}\|$ along $\mathbf{e}_1$ will be given by

$$\Lambda = \begin{bmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.11}$$

with $\gamma = (1 - u^2/c^2)^{-1/2}$ and $\beta = u/c$. This is not the most general transformation that we can have since it does not include rotations and dilations, but it is enough for our purposes.

Imposing that the vectors themselves are invariant; once it is composed of base vectors and vector components, if one transforms in one way it is expected that the other transforms oppositely, such

$$\begin{aligned} \mathbf{x} &= x'^{\mu}\mathbf{e}'_{\mu} \\ &= \Lambda^{\mu}_{\nu}\, x^{\nu}\, (\Lambda^{-1})^{\rho}_{\mu}\, \mathbf{e}_{\rho} \\ &= x^{\nu}\mathbf{e}_{\rho}\, \Lambda^{\mu}_{\nu}\, (\Lambda^{-1})^{\rho}_{\mu} \\ &= x^{\mu}\mathbf{e}_{\mu}. \end{aligned} \tag{2.12}$$

So if one is contravariant ($\_^{\mu}$) the other must be covariant ($\__{\mu}$)

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} x^{\nu} \qquad \mathbf{e}'_{\mu} = \left(\Lambda^{-1}\right)^{\nu}_{\mu} \mathbf{e}_{\nu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \mathbf{e}_{\nu}. \tag{2.13}$$

We will create a dual vector space $\mathbb{V}^*$, associate with $\mathbb{V}$, whose elements have covariant components and contravariant bases vector

$$x'_{\mu} = \left(\Lambda^{-1}\right)^{\nu}_{\mu} x_{\nu} = \frac{\partial x'_{\mu}}{\partial x_{\nu}} x_{\nu} \qquad \mathbf{e}'^{\mu} = \Lambda^{\mu}_{\nu} \mathbf{e}^{\nu} = \frac{\partial x_{\nu}}{\partial x'_{\mu}} \mathbf{e}^{\nu}. \tag{2.14}$$

These objects, the covector, will by definition be able to linearly map the vectors to scalars

$$\begin{aligned} &\mathbf{e}^{\mu} \in \mathbb{V}^* \\ &\mathbf{e}^{\mu} : \mathbb{V} \to \mathbb{R}. \end{aligned} \tag{2.15}$$

And, usually, we will want that this application respect the following condition

$$\mathbf{e}^{\mu}(\mathbf{e}_{\nu}) = \delta^{\mu}_{\nu} \tag{2.16}$$

with δ being the Kronecker delta, it is equal to one when its indexes are equal and zero otherwise. In this way, if we consider that the base of the dual space is given by $\{\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$, we can construct a norm of a vector using its representation in $\mathbb{V}$ and $\mathbb{V}^*$ such

$$\begin{aligned} \|\mathbf{x}\|^2 &\stackrel{\text{def}}{=} x_{\nu} \mathbf{e}^{\nu}(x^{\mu} \mathbf{e}_{\mu}) \\ &= x^{\mu} x_{\mu} \end{aligned} \tag{2.17}$$

So far, so good. With just a little modification in classical mechanics, we got a new one that is four-dimension and consistent with special relativity. In the same line of reasoning, we can adapt Eqs. (2.5) and (2.6) to get their correspondents in this formalism

$$\mathcal{S}[\Phi^i] = \int_{t_1}^{t_2} dt \int_V d^3x \; \mathscr{L}(\Phi^i, \partial_{\mu}\Phi^i, x^{\mu}) \tag{2.18}$$

$$\therefore \qquad \partial_{\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial_{\mu}\Phi^i)}\right) - \frac{\partial \mathscr{L}}{\partial \Phi^i} = 0. \tag{2.19}$$

For the sake of convenience, I'm using a simplified notation of derivative $\partial_t = \dfrac{\partial}{\partial t}$, $\partial_0 = \dfrac{1}{c}\dfrac{\partial}{\partial t}$ and $\partial_i = \dfrac{\partial}{\partial x^i}$ $(i = 1, 2, 3)$.

2.2.2 Tensor

It is natural to extend this concept of vectors and covectors to something even more generic called tensor.

Definition 2.2.2. *A tensor* $\mathbf{T}$ *of rank* (m, n) *is a multilinear map from a collection of* m *covectors and* n *vectors to* $\mathbb{R}$

$$\begin{aligned}\mathbf{T} &\in \underbrace{\mathbb{V}^* \otimes \cdots \otimes \mathbb{V}^*}_{m} \otimes \underbrace{\mathbb{V} \otimes \cdots \otimes \mathbb{V}}_{n} \\ \mathbf{T} &: \underbrace{\mathbb{V}^* \times \cdots \times \mathbb{V}^*}_{n} \times \underbrace{\mathbb{V} \times \cdots \times \mathbb{V}}_{m} \to \mathbb{R}\end{aligned} \tag{2.20}$$

with $\otimes$ *and* $\times$ *representing the tensor product and the Cartesian product, respectively.*

With this new concept, we can write the norm through an application of a tensor that contains information about the space on a Cartesian product formed by the vector we want to know the norm

$$(\eta_{\alpha\beta}\mathbf{e}^{\alpha} \otimes \mathbf{e}^{\beta}) : (x^{\mu}\mathbf{e}_{\mu}, x^{\nu}\mathbf{e}_{\nu}) \to \|\mathbf{x}\|^2$$

$$\begin{aligned}\|\mathbf{x}\|^2 &= (\eta_{\alpha\beta}\mathbf{e}^{\alpha} \otimes \mathbf{e}^{\beta})((x^{\mu}\mathbf{e}_{\mu}, x^{\nu}\mathbf{e}_{\nu})) \\ &= x^{\mu}\eta_{\alpha\beta}x^{\nu}\mathbf{e}^{\alpha}(\mathbf{e}_{\mu})\mathbf{e}^{\beta}(\mathbf{e}_{\nu}) \\ &= x^{\mu}\eta_{\alpha\beta}x^{\nu}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} \\ &= x^{\mu}\eta_{\mu\nu}x^{\nu} \\ &= x^{\mu}x_{\mu}.\end{aligned}$$

By convention, the coefficients of the Minkowski metric tensor η will be

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.21}$$

It is worth commenting that we can build this metric tensor considering that the Minkowski spacetime is equipped with a new type of inner product, a pseudo inner product, that allows negative norm instead of just positive or zero. However, I chose not to do it in this way to have the opportunity to present the covectors which will be very important from here on.

2.2.3 Rindler coordinates

Now that time is a variable whose value depends on the observer, it is important to define something similar to the time, but invariant. The proper time τ will be given by

$$\Delta\tau = \int d\tau \stackrel{\text{def}}{=} \int \frac{\|d\mathbf{x}\|}{c}. \tag{2.22}$$

In the previous section, we have been using an ordinary time derivative to obtain some quantities; now that we are on Minkowski spacetime it makes sense to start using the proper time to compute derivatives to define some similar objects. We define the proper velocity, momentum and acceleration as

$$\mathbf{V} \stackrel{\text{def}}{=} \frac{d\mathbf{x}}{d\tau} = \frac{d(x^\mu \mathbf{e}_\mu)}{d\tau} \tag{2.23}$$

$$\mathbf{P} \stackrel{\text{def}}{=} m\mathbf{V} \tag{2.24}$$

$$\mathbf{A} \stackrel{\text{def}}{=} \frac{d\mathbf{V}}{d\tau}. \tag{2.25}$$

Curiously the norm of the four-velocity is always constant $\|\mathbf{V}\| = c^2$ and the components associated with the $\mathbf{e}_0$ on the four-momentum are related with the energy

$$c\mathbf{P}^0 = \gamma mc^2 = \underbrace{mc^2}_{\text{rest mass energy}} + \underbrace{\frac{m\mathbf{v}^2}{2}}_{\text{Newtonian kinetic energy}} + \underbrace{mc^2\left(\frac{3\mathbf{v}^4}{8c^4} + \frac{5\mathbf{v}^6}{16c^6} + ...\right)}_{\text{new higher-order kinetic energy terms}} \stackrel{\text{def}}{=} E \tag{2.26}$$

what implies that we can get the relativistic Einstein's Energy-momentum relation from it

$$E^2 = (mc^2)^2 + (\gamma\mathbf{p}c)^2. \tag{2.27}$$

All of these definitions work very well for massive objects; however, they can not handle light, since light has no mass although carrying energy and momentum. For light we define

$$|\mathbf{P}_{light}| \stackrel{\text{def}}{=} \frac{E}{c} \tag{2.28}$$

and we know from quantum mechanics that

$$|\mathbf{P}_{light}| = \frac{h}{\lambda} = \hbar|\mathbf{k}| \tag{2.29}$$

Finally, as it will be of interest to analyze objects moving with constant acceleration, it is important to introduce this here. Objects with constant acceleration $\mathbf{A}\cdot\mathbf{A} = -\alpha^2$ always will have the acceleration being orthogonal to the four-velocity $\mathbf{V}\cdot\mathbf{A} = 0$ which means that the acceleration does not change the length of the proper velocity, it can only change its direction.

The position vector of this object in two dimensions $\mathbb{M}^{(1,1)}$ will be

$$\mathbf{x}(\tau) = \frac{c^2}{\alpha}\sinh\left(\frac{\alpha}{c}\tau\right)\mathbf{e}_0 + \frac{c^2}{\alpha}\cosh\left(\frac{\alpha}{c}\tau\right)\mathbf{e}_1 \tag{2.30}$$

which follows the worldline of a hyperbola on spacetime:

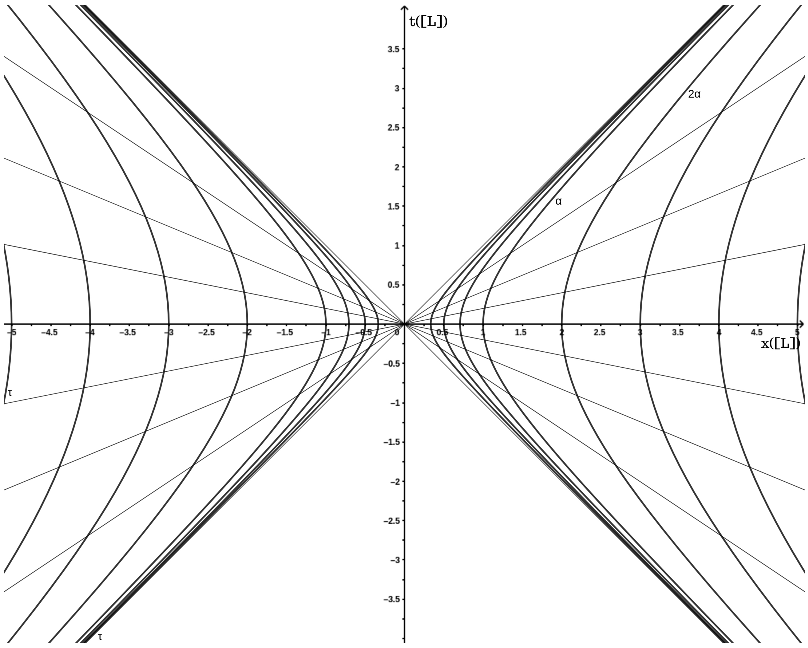

Figure 2.1: Hyperbole generated by several α with its asymptotic curves and some straight lines which gives us a visual idea of how is its metric.

We can not find a coordinates transformation for all the worldlines of a non-inertial frame. However, we can always find it to a specific point. For the special case where the observer S' has a constant proper acceleration like the object which we have dealt with

$$\begin{aligned} \frac{c^2}{\alpha} &\to x' \\ c\tau &\to ct' \end{aligned} \tag{2.31}$$

and so, we get the transformation between the components of a position vector as

$$\begin{aligned} ct &= x'\sinh\left(\frac{\alpha}{c^2}ct'\right), \quad ct' = \frac{c^2}{\alpha}\operatorname{arctanh}\left(\frac{ct}{x}\right) \\ x &= x'\cosh\left(\frac{\alpha}{c^2}ct'\right), \quad x' = \sqrt{(x)^2 - (ct)^2}. \end{aligned} \tag{2.32}$$

(ct', x') is what we call the Rindler coordinates, it represents the measurements of an observer traveling with proper acceleration α and with the Rindler horizon at the origin. The Rindler horizon is the line that separates the events that can and can not affect the observer. We know that nothing can travel faster than light and so the observer while is traveling with constant acceleration will never be reached by any beam of light coming from the left side of the line represented by the light path passing through the origin of the spacetime diagram. Another important thing about these coordinates is the metric. We can examine it by writing the norm of the differential line element

$$(d\mathbf{x})^2 \equiv g_{\mu\nu} dx^\mu dx^\nu \tag{2.33}$$

which in this case will be

$$(d\mathbf{x})^2 = (\alpha x')^2 (dt')^2 - (dx')^2. \tag{2.34}$$

It depends on the position, unlike the Minkowski metric given in eq.((2.34)). This will cause us to no longer be able to set a global time, which will take us to the Unruh effect.

2.2.4 Curved spacetime

The curved spacetime plays an essential role in general relativity. Although we will not work with general relativity here, it will be important to introduce some concepts like the axioms, how we define the bases vectors, and the procedure to generalize the theory from flat spacetime to curved spacetime.

Whereas the flat spacetime is associated with the Minkowski space, the curved spacetime is linked to the pseudo-Riemannian manifold $\mathbb{M}$. The "manifold" means a space that may be curved having a complicated topology, but in local regions looks just like the flat. Here "looks like" does not mean that the metric is the same, but only that more primitive notions like functions and coordinates work similarly. Therefore, the entire manifold is constructed by smoothly sewing together these local regions. To work here it will be necessary to update some definitions.

A base vector here will be defined by a partial derivative

$$\mathbf{e}_\mu \stackrel{\text{def}}{=} \partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{2.35}$$

The covector will be a differential

$$\mathbf{e}^\mu \stackrel{\text{def}}{=} dx^\mu \tag{2.36}$$

such that its application on a vector is as follows

$$dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) \stackrel{\text{def}}{=} \frac{dx^\mu}{dx^\nu} = \delta^\mu_\nu. \tag{2.37}$$

It is easy to verify they transform themselves as vector and covector respecting the relations (2.14) and (2.12).

Once we are using the partial derivative to define a vector, we need to adapt our old concept of the derivative itself. The covariant derivative $\nabla_\mathbf{w}\mathbf{v}$ appears to be the rate of change of a vector field $\mathbf{v}$ in $\mathbf{w}$ direction.

Another thing that we need to adapt is the metric tensor (previously label as η), the metric is not necessarily preserved anymore along a curved space, but we can associate a tangent plan T in a specific point P of $\mathbf{M}$ and in this location the metric will be constant and give by

$$g : T_pM \times T_pM \to \mathbb{R}. \tag{2.38}$$

If this metric does not chance along some direction, we say that this direction is associated with a Killing vector field; in other words, if $\nabla_{\partial_x} g = 0$ then ∂_x is a Killing vector. This vector field will place a crucial rule in the Unruh effect. Just a spoiler, inertial frames in flat space are continuous isometrics and so they agree with the vacuum state, on the other hand, the non-inertial frames on flat space or a frames in curved spaces don't have this property, and thus they do not necessarily agree with the vacuum state.

The most important application (in Physics) of curved spacetimes is in general relativity, formulated in 1915 by Einstein in order to describe gravity as a manifestation of the curvature of the spacetime. I won't go into the details of this theory.

In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments.

There are highly-sophisticated mathematical methods that are used to generalize laws of physics from flat to curved spacetimes. But in this monograph I will present simplest one, the "minimal-coupling principle". The procedure associated with this method consists of taking a law of physics, valid in inertial coordinates in flat spacetime, writing it in a coordinate-invariant (tensorial) form and asserting that the resulting law remains true in curved spacetimes. This is what we will use further.

2.3 Quantum mechanics

Since we are presenting the necessary tools to quantize a field, nothing fairer than introducing quantum mechanics. This theory was developed to deal with events that usually occur on an atomic scale, although it can also explain some macroscopic scale phenomena. The theory was born through an analysis of anomalies that happen on large scale and that could not be explained by classical mechanics. Some experiments that lead us to develop this theory were about black bodies [11], the photoelectric effect [12] and the Stern-Gerlach [13].

Like any other theory, there are many different ways to represent the same thing. The most common formalism of this new theory [14] was developed around 1926 mainly by physicists like Heisenberg, Schrödinger, and Dirac using the classical Hamiltonian formalism together with the algebra of Hilbert space $\mathbb{H}$.

2.3.1 Hamiltonian formalism

The Hamiltonian formalism is indeed really close to the Lagrangian formalism. It is built through the Legendre transformation

$$H(\mathbf{q}, \mathbf{p}, t) \stackrel{\text{def}}{=} \sum_k p_k \dot{q}_k - L(\mathbf{q}, \dot{\mathbf{q}}, t), \tag{2.39}$$

where the canonical momentum is given by $p_k = \dfrac{\partial L}{\partial \dot{q}_k}$. The differential of this function is

$$\begin{aligned} dH &= \sum_k \dot{q}_k \, dp_k - \dot{p}_k \, dq_k - \frac{\partial L}{\partial t} dt \\ &= \sum_k \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt, \end{aligned}$$

which gives us the canonical equations of this formalism

$$\begin{cases} \dot{q}_k = \dfrac{\partial H}{\partial p_k} \\ \\ \dot{p}_k = -\dfrac{\partial H}{\partial q_k} \\ \\ \dfrac{\partial L}{\partial t} = -\dfrac{\partial H}{\partial t}. \end{cases} \tag{2.40}$$

This is completely equivalent to the Lagrangian equation 2.2 and so, in the same way that we have the Lagrangian for a field, we can also have a Hamiltonian for a field. The conjugate momentum for a field is given by

$$\pi(t, \mathbf{x}) \stackrel{\text{def}}{=} \frac{\delta L}{\delta(\partial_0 \phi)}. \tag{2.41}$$

The Hamiltonian density is

$$\mathscr{H}(\phi, \pi, x^\mu) \stackrel{\text{def}}{=} \dot{\phi}\pi - \mathscr{L}(\phi, \partial_\mu \phi, x^\mu) \tag{2.42}$$

and the Hamiltonian itself takes the form

$$H(\phi, \pi, t) = \int d^3x \, \mathscr{H} \tag{2.43}$$

This new function, contrary to the Lagrangian, has a physical interpretation; it represents, in some cases, the total energy of the system; what is perfect once we usually go from energy to quantize a system. However there are differences between classical and quantum: A quantum state is represented by a vector on Hilbert space $\mathbb{H}$ instead of a point $(\mathbf{x}, \mathbf{p})$ in phase space, and a quantum observable will be given by a self-adjoint operator instead of a real-valued function.

2.3.2 Quantum mechanics in Dirac notation

In the most used notation, the Dirac notation [15], a state ψ is represented by a vector $|\psi\rangle$ that belongs to $\mathbb{H}$. The associated linear functional is defined by $\langle\psi| \in \mathbb{H}^*$. An operator $\hat{A}$ acts on this space as

$$\hat{A} : \mathbb{H} \to \mathbb{H} \tag{2.44}$$

and similarly; we define its adjunct, conjugate complex, $\hat{A}^\dagger = (\hat{A}^T)^*$ that acts

$$\hat{A}^\dagger : \mathbb{H}^* \to \mathbb{H}^*. \tag{2.45}$$

In the special case where $\hat{A}^\dagger = \hat{A}$ we say that $\hat{A}$ is a self-adjoint, or Hermitian operator.

The action of the bra is

$$\langle\psi| : \mathbb{H} \to \mathbb{C} \tag{2.46}$$

$$\langle\psi| \left(|\rho\rangle\right) = \langle\psi|\rho\rangle = \langle\rho|\psi\rangle^* = \mathrm{IP}\left(|\psi\rangle\ ,\ |\rho\rangle\right).$$

where $\mathrm{IP}\left(\cdot, \cdot\right)$ denotes the complex internal product of continuous functions.

Definition 2.3.1. *An inner product in the vector space of continuous functions in [a,b], denoted as $V = C([a, b])$, is defined as follows. Given two arbitrary vectors $f(x)$ and $g(x)$, introduce the inner product*

$$(f, g) \stackrel{\text{def}}{=} \int_a^b f^*(x)g(x)dx. \tag{2.47}$$

Through this Hamiltonian structure together with the linear algebra of Hilbert space $\mathbb{H}$ and the concept of operators, Schrödinger postulated the equation that governs the temporal evolution of a quantum system

Postulate 2.3.1. *The time evolution of the state vector $|\Psi(t)\rangle$ is governed by the Schrödinger equation*

$$\hat{H} \left|\Psi(t)\right\rangle = i\hbar\frac{d}{dt} \left|\Psi(t)\right\rangle, \tag{2.48}$$

where $\hat{H}(t)$ is the observable associated with the total energy of the system (called Hamiltonian).

Unfortunately, this structure presented together with the postulates of quantum mechanics are not enough to characterize the quantum system [14]. It is necessary to choose the Hilbert space and the operators corresponding to the quantities we will deal with. For this, we will do what we call the canonical quantization which is, for bosons, basically go from Poisson-bracket to commutators

$$\widehat{\{f, g\}} \to i\hbar[\hat{f}, \hat{g}], \tag{2.49}$$

with $f(\mathbf{q}, \mathbf{p}, t)$ and $g(\mathbf{q}, \mathbf{p}, t)$ are classical objects. In this way we get the algebraic structure of quantum observables on the Hilbert space [16]. Worth remembering that the commutation and the Poisson bracket are given by

$$[\hat{f}, \hat{g}] \stackrel{\text{def}}{=} \hat{f}\hat{g} - \hat{g}\hat{f} \tag{2.50}$$

and

$$\{f, g\} \stackrel{\text{def}}{=} \sum_{k=1}^{N} \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right), \tag{2.51}$$

respectively.

2.3.3 Quantum harmonic oscillator

Most problems in classical mechanics can be described, in first approximation, as a system with analytical potential energy $V = V(x)$ (in the case of a single spatial dimension). Using the Maclaurin series [17], it is possible to write

$$V(x) = \sum_{n}^{\infty} \frac{1}{n!} \frac{d^n V(x)}{dx^n}\bigg|_{x=0} x^n = V(x=0) + \frac{dV(x)}{dx}\bigg|_{x=0} x + \frac{d^2 V(x)}{dx^2}\bigg|_{x=0} x^2 + \frac{d^3 V(x)}{dx^3}\bigg|_{x=0} x^3 + ...$$

under the following conditions

$$V(x=0) = 0, \quad \frac{dV(x)}{dx}\bigg|_{x=0} = 0 \text{ and } \frac{d^2 V(x)}{dx^2}\bigg|_{x=0} > 0$$

. By ignoring the contribution of terms with a degree greater than two, we obtain

$$V(x) = \frac{d^2 V(x)}{dx^2}\bigg|_{x=0} x^2 = \frac{1}{2} k x^2 = \frac{1}{2} m\omega^2 x^2. \tag{2.52}$$

This is the potential energy of a harmonic oscillator which will be very useful for us latter.

The Hamiltonian and the Lagrangian associated with this energy are

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \tag{2.53}$$

and

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2, \tag{2.54}$$

respectively. The solution, that can be found using Eq. (2.40) or Eq. (2.2), is given by a linear combination of exponential complex functions

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t} \tag{2.55}$$

with A , $B \in \mathbb{R}$.

The starting point for solving most quantum problems is the quantum harmonic oscillator.

The quantum harmonic oscillator can be constructed, in the Schrödinger picture, considering that the states are represented by complex-valued wave function $\psi(x,t)$ or, the way it will be done, using a more generic approach using a vector state $|\psi\rangle$, instead of its projection on the coordinate space.

Imposing the canonical commutation relation Eq. (2.49) between (x,p)

$$[\hat{x},\hat{p}] = i\hbar\hat{1} \tag{2.56}$$

the quantum Hamiltonian can be write as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \tag{2.57}$$

Since the Hamiltonian operator is Hermitian ,there is a set of orthogonal eigenstates $|\varphi_n\rangle$, associated with the eigenvalues E_n

$$\hat{H}\,|\varphi_n\rangle = E_n\,|\varphi_n\rangle \tag{2.58}$$

and any quantum state can be written as a superposition of this eigenstates, in other words $\{|\varphi_n\rangle\}$ is a base of the Hilbert space

$$|\psi\rangle = \sum_n c_n\,|\varphi_n\rangle\,. \tag{2.59}$$

In this way, the temporal evolution of the quantum state is trivially given as

$$|\psi\rangle = \sum_n c_n exp\left\{\frac{-iE_nt}{\hbar}\right\}|\varphi_n\rangle \tag{2.60}$$

So far we quantized the problem, now we need to solve it. With this purpose, we will introduce the creation, annihilation, and number operators. They are defined as

$$\begin{aligned}\hat{a}^\dagger &\stackrel{\text{def}}{=} \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\\ \hat{a} &\stackrel{\text{def}}{=} \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)\\ \hat{n} &\stackrel{\text{def}}{=} \hat{a}^\dagger\hat{a}\end{aligned} \tag{2.61}$$

with the follow commutation relations

$$\begin{aligned}[][\hat{a},\hat{a}^\dagger] &= \hat{1}\\ [\hat{a}^\dagger,\hat{a}^\dagger] &= 0\\ [\hat{a},\hat{a}] &= 0.\end{aligned} \tag{2.62}$$

The number operator respect the eigenvalue equation

$$\hat{n}\,|n\rangle = n\,|n\rangle\,. \tag{2.63}$$

Rewriting the Hamiltonian in terms of $\hat{n}$ results

$$\hat{H} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right) \tag{2.64}$$

and, therefore

$$\hat{H}\,|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle\,. \tag{2.65}$$

In these equations, n is an integer that represents the oscillator excitation level. It is easy to see from this that the lowest energy the system can have is $\hbar\omega/2$, instead of 0. This is called the zero-point energy.

2.4 Klein-Gordon field

Now we will present the field where the Unruh effect will be analyzed. I chose this one because of its simplicity since it has only a single degree of freedom, and even though it is simple, it is possible to generalize it to describe more complex physical phenomena.

The Klein-Gordon field obeys the Klein-Gordon equation which in turn is a relativistic quantum pseudo-scalar equation that has as solution the scalar fields $\phi(x^\mu)$.

$$\phi(x^\mu) : (spacetime) \to \mathbb{R}. \tag{2.66}$$

A scalar field like that give rise to spinless particles like a neutral π-meson or even a massless charged particles like the electric particle of the electric field, whereas complex, vector and tensors fields can give rise to higher-spin particles since they have more degrees of freedom.

There is a lot of ways to justify the Klein-Gordon equation, the most famous is using the relativistic energy-momentum relation Eq. (2.27), but I chose to introduce it through an analogy with a classical harmonic system. It is not hard to believe that the analogy for a field would be something like: $T \propto \hbar^2\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)/c^2$ and $V(\phi) \propto m^2\phi^2$. So we have our Lagrangian density being

$$\mathscr{L} = \frac{1}{2}\frac{\hbar^2}{c^2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2. \tag{2.67}$$

Using Eq. (2.19) we get the Klein-Gordon equation

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0. \tag{2.68}$$

Here $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial^\mu\partial_\mu$ is a differential operator called the d'Alembertian and m will represent, when we quantize it, the mass of the quanta associated withe the field.

It is important to emphasize that $\phi(x^\mu)$, the solution of the Klein-Gordon equation, is not a wave function (in the sense of quantum mechanics). It is just our classic field made of dynamics variables generalizing the single degree of freedom x.

2.5 Quantum field theory

The concept of a quantum field as we know was formulated in the late 1920s and early 1930s, starting with the historical article of Dirac "The Quantum Theory of the Emission and Absorption of radiation" [18]. Its emergence was due to the necessity to explain phenomena that were not covered by quantum mechanics. As much as quantum mechanics was extremely successful in dealing with particle dynamics at low speeds, it was unfortunately unable to deal with processes that occurred in the relativistic regime, besides not being able to describe fundamental things like the interaction between light and matter or any phenomenon that involved the creation or annihilation of particles predicted by Einstein's equations $E = mc^2$. The problem of unifying special relativity with quantum mechanics was corrected through the Klein-Gordon and Dirac equations, but as both were still based on a complex wave function arising from the projection of the system state in the position space $\psi(x,t) = \langle x|\Psi(t)\rangle$, they are not able to deal with a variable number of particles since the Hilbert space in which they are defined is immutable, that is, it is not capable of dealing with a system whose number of degrees of freedom, number of particles, is variable; this problem has been solving introducing the Fock space which consists basically of a superposition of Hilbert spaces.

Chapter 3

The Unruh effect

This chapter aims to analyze the Unruh effect. We will find the solutions of the Klein-Gordon equation, quantize it and establish a relationship between the relevant reference frames. The analysis itself will be done just in flat spacetime since there are many conflicts in trying to join the curved spacetime with quantum mechanics. It is important to highlight from now on we will use the system of natural units where $\hbar = c = k_B = 1$.

3.1 Putting the field in a box

The Unruh effect is essentially a phenomenon that occurs with quantum fields. Therefore, in order to understand this effect, it will be necessary to quantize our field whose dynamical equation is given by

$$(\Box + m^2)\phi = 0. \tag{3.1}$$

In order to do this, we need to solve this equation. Here we have a second order partial differential equation whose uniqueness of the solution can be proved. Since it is linear, its solution can be found using different methods, but we have two in special that worth mentioning: The first is using plane wave expansion in which the field assumes the explicit form of an infinite system of decoupled, time-independent, harmonic oscillators. And the second, without using a plane wave expansion, thereby enabling us to distinguish clearly between the essential input and the inessential input and allows one to make more general choices of a subspace of solutions to this equation. Once that this second one is much more extensive and complex, we will do it in a first way, but the other can be found on [16].

The strategy will be to expand $\phi(x^\mu)$ in terms of a complete set of functions $\{f(x^\mu)\}$ in a interval $[-L, L]$. The set that we will use is composed of exponentials that depend on the wavenumber four-vector $k_\mu = (-\omega_{\mathbf{k}}, \mathbf{k})$ with $\mathbf{k} = 2\pi/L_{\mathbf{k}}(n_1, n_2, n_3) \quad n_i \in \mathbb{N}$ and with the frequency ω depending on the $\mathbf{k}$, since it must satisfy the dispersion relation. So we have

$$\phi(x^\mu) = \sum_{\mathbf{k}} c_{\mathbf{k}} e^{ik_\mu x^\mu} = \sum_{\mathbf{k}} c_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)} \tag{3.2}$$

with the coefficients, in their generic form, written as

$$c_{\mathbf{k}} = \frac{1}{2L_{\mathbf{k}}} \int_{-L_{\mathbf{k}}}^{L_{\mathbf{k}}} \phi(x^\mu) e^{-ik_\mu \cdot x^\mu} d^4x. \tag{3.3}$$

A specific solution can be found by putting initial and boundary conditions on these coefficients. Comparing this solution with the solution of the harmonic oscillator we see that the Klein-Gordon field may be viewed as an infinite collection of decoupled harmonic oscillators. However there is also an important difference: For the oscillator, there is only one independent solution. Because the oscillator has a unique frequency. This is no longer true in field theory, the frequency is determined by spatial wave vector $\mathbf{k}$, at least up to sign. Therefore, instead of a single kind of solution, we have a set of possible values.

Therefore we have an expression for the solution of the Klein-Gordon equation, but unfortunately, it is impossible to quantize this, since it has time dependency and we do not have an operator capable to represent the time itself in quantum mechanics.

3.2 Foliation of spacetime

The way that we will solve this quantization problem is through the foliation of the Minkowski spacetime, we will separate spacetime into space and time components and then we will be able to quantize field. The notation that we will use, the ADM notation introduced in 1962 by the physicists Arnowitt, Deser and Misner was developed to emphasize its field-theoretic, rather than geometric, content. In order to be an approach to General Relativity and other like the gauge theories [19].

The idea is breaking down the spacetime in (D+1)-dimensions, into one-parameter family of D dimensional spacelike hypersuperficie which are parameterized by a time function t, respecting the equivalence principle of Einstein. The hypersuperficie Σ_t will have timelike normal vectors and spacelike tangent vectors. It is important to emphasize that not necessarily any curved spacetime can be foliated, but we are assuming that the spacetime we are working with is globally hyperbolic, what means that the fundamental condition that generates the Lorentzian manifold is $\tau^2 = t^2 - \mathbf{r}^2$.

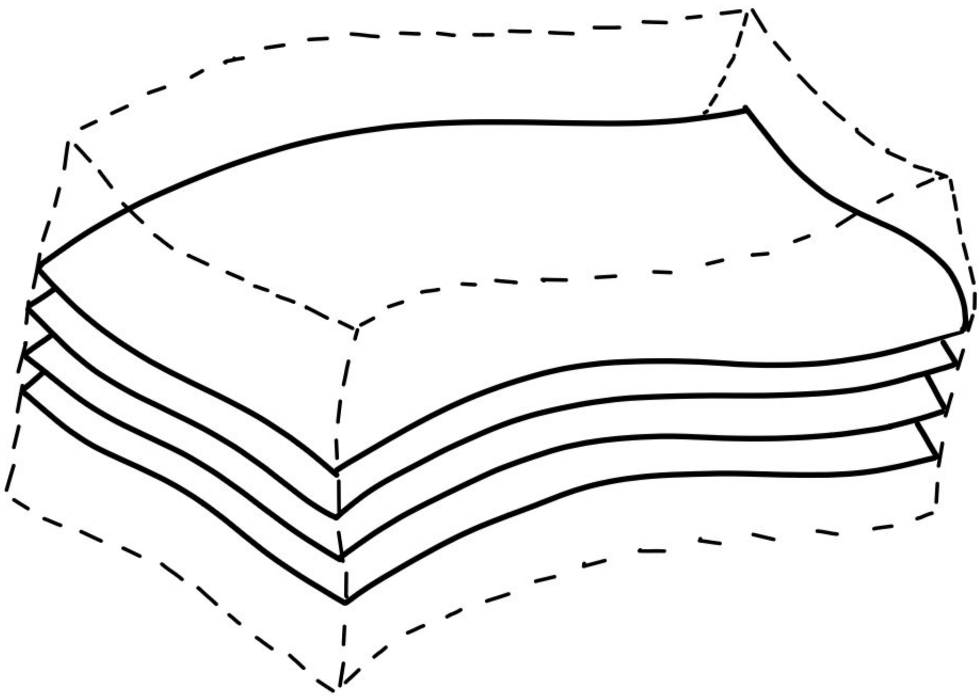

Figure 3.1: Visual representation of the foliation of spacetime

3.3 A solution that can be quantized

3.3.1 Flat spacetime

In order to find an equivalent solution for the Klein-Gordon equation, but now a one that can be quantized, we will write down the solution through a complete and orthonormal set of modes composed of the functions that build the space. We will seize the idea of expressing the field ϕ in terms of complex exponentials, but now we will take $L \to \infty$. Considering that the functions $\phi(x^\mu)$ and $\pi(x^\mu)$ on Σ_t that build our space compose a function space $\mathbf{M}$ such that every point of $\mathbf{M}$ uniquely determines a solution to the Klein-Gordon equation

$$\mathbf{M} = \{[\phi, \pi] \mid \phi : \Sigma_0 \to \mathbb{R},\ \pi : \Sigma_0 \to \mathbb{R};\ \phi, \pi \in C_0^\infty(\Sigma_0)\}. \tag{3.4}$$

The inner product on this space will be, by definition, given by an integral over a constant-time hypersurface Σ_t, $\Omega : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$

$$\Omega\{[\phi_i, \pi_i], [\phi_j, \pi_j]\} \stackrel{\text{def}}{=} -i \int_{\Sigma_t} d^D x \, (\phi_i \pi_j^* - \phi_j^* \pi_i), \tag{3.5}$$

once that the conjugate momentum for the Klein-Gordon field is $\pi_i = \partial_t \phi_i$, we have its inner product like

$$(\phi_i, \phi_j) = \Omega\{[\phi_i, \phi_j]\} = -i \int_{\Sigma_t} d^D x \, (\phi_i \partial_t \phi_j^* - \phi_j^* \partial_t \phi_i). \tag{3.6}$$

Fortunately, this expression is independent of the hypersurface Σ_t over which the integral is taken. We can prove this using the Dirac delta function

$$\int e^{i\mathbf{k}\cdot\mathbf{x}} d^D x = (2\pi)^D \delta^D(\mathbf{k}) \tag{3.7}$$

with

$$\delta(x - x_0) \stackrel{\text{def}}{=} \begin{cases} \infty & \text{for } x = x_0 \\ 0 & \text{for } x \neq x_0 \end{cases} \tag{3.8}$$

Proof.

$$\begin{aligned}
(e^{ik_1^\mu x_\mu}, e^{ik_2^\mu x_\mu}) &= -i \int_{\Sigma_t} \left(e^{i(\mathbf{k}_1\cdot\mathbf{x} - \omega_{\mathbf{k}_1} t)} \partial_t e^{-i(\mathbf{k}_2\cdot\mathbf{x} - \omega_{\mathbf{k}_2} t)} - e^{-i(\mathbf{k}_2\cdot\mathbf{x} - \omega_{\mathbf{k}_2} t)} \partial_t e^{i(\mathbf{k}_1\cdot\mathbf{x} - \omega_{\mathbf{k}_1} t)}\right) d^D x \\
&= (\omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_1}) e^{-i(\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})t} \int_{\Sigma_t} e^{i(\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{x}} d^D x \\
&= (\omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_1}) e^{-i(\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})t} (2\pi)^D \delta^D(\mathbf{k}_1 - \mathbf{k}_2)
\end{aligned} \tag{3.9}$$

□

From this, we get an orthonormal set of mode solutions

$$f_{\mathbf{k}}(x^\mu) = \frac{e^{ik_\mu x^\mu}}{[(2\pi)^D 2\omega_{\mathbf{k}}]^{1/2}} \tag{3.10}$$

with the inner product given by

$$(f_{\mathbf{k}_1}, f_{\mathbf{k}_2}) = \delta^D(\mathbf{k}_1 - \mathbf{k}_2). \tag{3.11}$$

We can introduce its complex conjugates $f^*_{\mathbf{k}}(x^\mu)$, which allows us to keep $\omega_{\mathbf{k}}$ always a positive number. In this way, the $f_{\mathbf{k}}$ modes are said to be positive-frequency. These now complex conjugate modes will be orthogonal to the original modes

$$(f_{\mathbf{k}_1}, f^*_{\mathbf{k}_2}) = 0 \tag{3.12}$$

and orthonormal to each other but with a negative norm

$$(f^*_{\mathbf{k}_1}, f^*_{\mathbf{k}_2}) = -\delta^D(\mathbf{k}_1 - \mathbf{k}_2). \tag{3.13}$$

Therefore, the modes $f_{\mathbf{k}}$ and $f^*_{\mathbf{k}}$ will form a complete set, in terms of which we can expand any solution to the Klein-Gordon equation.

3.3.2 Curved spacetime

Although we will not discuss the Unruh effect in curved spacetimes, it is interesting to find a valid solution in this context too. The Lagrangeqn density of a scalar field in curved spacetime will take the form

$$\mathscr{L} = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2 - \xi R\phi^2\right), \tag{3.14}$$

where we have the metric $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$ and an added term that takes a direct coupling between the field and the Ricci curvature scalar R. This term is parameterized by a constant ξ. g is the determinant of the metric. This will lead us to a new expression for the Klein-Gordon equation

$$\Box\phi - m^2\phi - \xi R\phi = 0. \tag{3.15}$$

The conjugate momentum is defined as

$$\pi = \frac{\partial\mathcal{L}}{\partial\left(\nabla_0\phi\right)} = \pi = \sqrt{-g}\nabla_0\phi. \tag{3.16}$$

The product here will be defined in a spacelike hypersurface Σ, like we have done earlier, but now we need to introduce a metric to normalize the system and a unit normal vector n^μ. We thus have

$$(\phi_1, \phi_2) = -i\int_\Sigma \left(\phi_1\nabla_\mu\phi_2^* - \phi_2^*\nabla_\mu\phi_1\right) n^\mu\sqrt{G}\, d^Dx \tag{3.17}$$

and again it is independent of the choice of Σ with its covariant derivative being perpendicular to Σ.

We could now introduce a set of positive- and negative-frequency modes forming a complete basis for the solution of Eq. (3.14), expand the field in terms of these modes, and then associate its coefficients to the creation and annihilation operators. Unfortunately, this procedure does not work here, since there will not be any timelike Killing vector in general. We are not able to find solutions to the wave equation that separates into time-dependent and space-dependent factors, and consequently can not classify modes as positive- or negative-frequency. So we will stop the resolution here and direct our focus, throughout the rest of this monograph, to quantize the field just in flat spacetime.

3.4 Quantizing the Klein-Gordon field

The system will be quantized in the canonical quantization scheme by promoting our classical variables ϕ and π to operators acting on a Hilbert space, and imposing the canonical commutation relations, Eq. (2.49), on the hypersuperface Σ_t. We have

$$\begin{aligned}
&[\hat{\phi}(t,\mathbf{x}),\hat{\phi}(t,\mathbf{x}')] = \hat{0} \\
&[\hat{\pi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x}')] = \hat{0} \\
&[\hat{\phi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x}')] = i\delta^D(\mathbf{x}-\mathbf{x}')\hat{1}.
\end{aligned} \tag{3.18}$$

This canonical commutation relations tells that the field and momentum commute with themselves throughout space while the delta function on the last equation implies that operators at equal times commute everywhere except at coincident spatial points, thus respecting the causality and the Heisenberg uncertainty.

Therefore, we will write the $\hat{\phi}$ as a expansion in terms of modes $f_{\mathbf{k}}(x^\mu)$ and $f^*_{\mathbf{k}}(x^\mu)$ with the coefficients being the operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$

$$\hat{\phi}(t,\mathbf{x}) = \int d^D k\, [\hat{a}_{\mathbf{k}} f_{\mathbf{k}}(t,\mathbf{x}) + \hat{a}^\dagger_{\mathbf{k}} f^*_{\mathbf{k}}(t,\mathbf{x})]. \tag{3.19}$$

We can find the commutation relation between $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ putting this expression into Eq. (3.18), resulting in

$$\begin{aligned}
&[\hat{a}_{\mathbf{k}},\hat{a}_{\mathbf{k}'}] = 0 \\
&[\hat{a}^\dagger_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}] = 0 \\
&[\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}] = \delta^D(\mathbf{k}-\mathbf{k}')\hat{1},
\end{aligned} \tag{3.20}$$

which are the same as the usual ones for the creation and annihilation operators with the difference that now there is an infinite number of such operators indexed by $\mathbf{k}$. The positive-frequency modes are coefficients of annihilation operators while negative-frequency modes are coefficients of creation operators. As we did for the harmonic oscillator, we will use these operators to define a basis for the Hilbert space in which the basis states are eigenstates of the number operator.

The vacuum state $|0\rangle$ will be characterized by the fact that it is annihilated by each $\hat{a}_{\mathbf{k}}$

$$\hat{a}_{\mathbf{k}}\,|0\rangle = 0 \qquad \text{for all } \mathbf{k}. \tag{3.21}$$

A state with $n_{\mathbf{k}}$ particles with identical momenta, wavenumber $\mathbf{k}$, is created by the application of $\hat{a}_{\mathbf{k}}^{\dagger}$

$$|n_{\mathbf{k}}\rangle = \frac{1}{\sqrt{n_{\mathbf{k}}!}} \left(\hat{a}_{\mathbf{k}}^{\dagger}\right)^{n_{\mathbf{k}}} |0\rangle \tag{3.22}$$

while a state with n_i excitations of various momenta $\mathbf{k}_i$ will be

$$|n_1, n_2, ..., n_j\rangle = \frac{1}{\sqrt{n_1!n_2!...n_j!}} \left(\hat{a}_{\mathbf{k}_1}^{\dagger}\right)^{n_1} \left(\hat{a}_{\mathbf{k}_2}^{\dagger}\right)^{n_2} ... \left(\hat{a}_{\mathbf{k}_j}^{\dagger}\right)^{n_j} |0\rangle \tag{3.23}$$

acting on such a state.

The creation and annihilation operators change the number of excitations as

$$\begin{aligned} \hat{a}_{\mathbf{k}_i} |n_1, n_2, ..., n_i, ..., n_j\rangle &= \sqrt{n_i}\, |n_1, n_1, ..., n_i - 1, ..., n_j\rangle \\ \hat{a}_{\mathbf{k}_i}^{\dagger} |n_1, n_2, ..., n_i, ..., n_j\rangle &= \sqrt{n_i + 1}\, |n_1, n_1, ..., n_i + 1, ..., n_j\rangle \,. \end{aligned} \tag{3.24}$$

We will define a number operator for each wave vector as

$$\hat{n}_{\mathbf{k}} = \hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}} \tag{3.25}$$

that acts on the space as

$$\hat{n}_{\mathbf{k}_i} |n_1, n_2, ..., n_i, ..., n_j\rangle = n_i\, |n_1, n_2, ..., n_i, ..., n_j\rangle \,. \tag{3.26}$$

The states that are eigenstates of the number operators form a basis for the entire Hilbert space known as the Fock basis. The space that is built through this basis is often called Fock space.

We can investigate how our Fock basis behaves under a Lorentz transformation. Using Eq. (2.11), the time derivative of our mode functions in two inertial frames is

$$\begin{aligned} \partial_{t'} f_{\mathbf{k}} &= \frac{\partial x^{\mu}}{\partial t'} \partial_{\mu} f_{\mathbf{k}} \\ &= \gamma(-i\omega_{\mathbf{k}}) f_{\mathbf{k}} + \gamma \mathbf{u} \cdot (i\mathbf{k}) f_{\mathbf{k}} \\ &= -i\omega'_{\mathbf{k}} f_{\mathbf{k}} \end{aligned} \tag{3.27}$$

with $\omega'_{\mathbf{k}} = \gamma(\omega_{\mathbf{k}} - \mathbf{u} \cdot \mathbf{k})$ being the frequency in the inertial frame S'. From this, we can see that if we have a state described as a collection of particles with certain momenta and we boost it, we get a state described by the same particles, but with a boosted momenta. Thus, the total number operator in the two frames will coincide, and, in particular, the vacuum state will coincide. In this sense, our original choice of the inertial frame was irrelevant. We will see that our ability to find positive and negative frequency solutions can be traced to the existence of a timelike killing vector ∂_t in Minkowski spacetime, while the invariance of the Fock space under changes of basis can be traced to the fact that all such timelike Killing vectors are related by

Lorentz transformations. Consequently, even if the frequency of a mode depends on the choice of inertial frame, the decomposition into positive and negative frequencies is invariant.

Now, we can analyze how the Klein-Gordon Hamiltonian operator will behave. It can be expressed as a function of the field operator as

$$\hat{H} = \int d^D x \, \mathscr{H}(\hat{\phi}, \dot{\hat{\phi}}) = \int d^D x \left[\frac{1}{2}\dot{\hat{\phi}}^2 + \frac{1}{2}(\nabla\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 \right] \tag{3.28}$$

but we can also write it as a function of the creation and annihilation operators as we did for the harmonic oscillator. We have the potential energy term being

$$\begin{aligned} \frac{1}{2}m^2 \int d^D x \, \hat{\phi}^2 &= \frac{1}{2}m^2 \int d^D x \, d^D k \, d^D k' \, (\hat{a}_{\mathbf{k}} f_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}} f^*_{\mathbf{k}})(\hat{a}_{\mathbf{k}'} f_{\mathbf{k}'} + \hat{a}^\dagger_{\mathbf{k}'} f^*_{\mathbf{k}'}) \\ &= \frac{1}{2}m^2 \int d^D x \, d^D k \, d^D k' \, (\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}'} f_{\mathbf{k}} f_{\mathbf{k}'} + \hat{a}^\dagger_{\mathbf{k}'}\hat{a}_{\mathbf{k}} f^*_{\mathbf{k}} f_{\mathbf{k}'} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}'} f_{\mathbf{k}} f^*_{\mathbf{k}'} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}'} f^*_{\mathbf{k}} f^*_{\mathbf{k}'}) \\ &= \frac{1}{2}m^2 \int d^D k \left(\frac{1}{2\omega_{\mathbf{k}}}\right) \left[\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} e^{-2i\omega_{\mathbf{k}} t} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} e^{2i\omega_{\mathbf{k}} t}\right], \end{aligned} \tag{3.29}$$

the kinetic-energy will be

$$\frac{1}{2}\int d^D x \, \dot{\hat{\phi}}^2 = \frac{1}{2}\int d^D k \left(\frac{\omega_{\mathbf{k}}}{2}\right) \left[-\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} e^{-2i\omega_{\mathbf{k}} t} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} e^{2i\omega_{\mathbf{k}} t}\right] \tag{3.30}$$

and the gradient-energy

$$\frac{1}{2}\int d^D x \, (\nabla\hat{\phi})^2 = \frac{1}{2}\int d^D k \left(\frac{\mathbf{k}^2}{2\omega_{\mathbf{k}}}\right) \left[\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} e^{-2i\omega_{\mathbf{k}} t} + \hat{a}^\dagger_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} e^{2i\omega_{\mathbf{k}} t}\right]. \tag{3.31}$$

Using $\omega_{\mathbf{k}} = \mathbf{k}^2 + m^2$ we can put it all together and rewrite the Hamiltonian for the scalar field theory as

$$\begin{aligned} \hat{H} &= \frac{1}{2}\int d^D k \left[\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\right]\omega_{\mathbf{k}} \\ &= \int d^D k \left[\hat{n}_{\mathbf{k}} + \frac{1}{2}\delta^D(0)\right]\omega_{\mathbf{k}}. \end{aligned} \tag{3.32}$$

The factor $\delta^D(0)$ is the field-theory analog of the harmonic oscillator zero-point energy. It means that the Hamiltonian is infinite even when measured in the vacuum state $|0\rangle$. Since the creation an annihilation does not commute, the energy fluctuates even in the vacuum state. The fact that it is an integral over an infinite range of $\mathbf{k}$ of the infinite quantity $\delta^D(0)$ can be translated into the statement that the total energy is an integral over an infinitely big space of an infinite energy density. But the energy density contributed by high-frequency modes is the real problem, not the infinite volume. If we regularized the calculation by performing it in a box of volume L^D, we would find

$$\frac{1}{2}\int d^D k\, \delta^D(0)\omega_{\mathbf{k}} \rightarrow \frac{1}{2}\left(\frac{L}{2\pi}\right)^D \sum_{\mathbf{k}} \omega_{\mathbf{k}}, \tag{3.33}$$

which diverges even for finite L (that appear in the sum as normalization factor) since $\mathbf{k}$ can be arbitrarily large. To correct this problem, we can obtain a finite Hamiltonian for the vacuum state subtracting an infinite constant. It is a very well known technique in quantum field theory, called renormalization. Since the infinities only arise in the relationship between quantum theories and their classical counterparts, not in any observable quantities, there should be nothing deeply disturbing about renormalization.

These excitations in the Fock basis are interpreted as particles. This is how particles arise in a quantum field theory. Its energy eigenstates are collections of particles with definite momenta. Of course, our modes are plane waves that extend throughout space, not the localized tracks in bubble chambers that come to mind when we think about particles. What is worse, in curved spacetimes, the wave equation will not have plane-wave solutions of definite frequency that we can interpret as particles. The solution to both issues is to think operationally, in terms of what would be observed by an experimental apparatus. The best strategy is to define a sensible notion of a particle detector that reduces to our intuitive picture in flat spacetime, and then define "particles" as "what a particle detector detects". For a discussion of particle detectors see [20].

3.5 Unruh effect

The Unruh effect in flat spacetimes states that an accelerating observer in the traditional Minkowski vacuum state will observe a thermal spectrum of particles. This event is a manifestation of the idea that observers with the different notions of positive- and negative-frequency modes will disagree on the particle content of a given state. For a uniformly accelerated observer in Minkowski spacetime, the trajectory will move along orbits of a timelike killing vector, but not that of usual time translation symmetry. We can therefore expand the field in modes appropriate to the accelerated observer and then calculate the number operator in the ordinary Minkowski vacuum where we will find a thermal spectrum of particles, which means that vacuum actually has the character of a thermal state.

In order to facilitate the calculations, we will discard all possible complications that are not essential for our analysis. Considering that our scalar field is massless $(m = 0)$ and it is in two spacetime dimensions $(D = 1)$. The Klein-Gordon equation becomes

$$(\partial_{tt} - \partial_{xx})\phi = 0. \tag{3.34}$$

We can also make some modification in the Rindler coordinates Eq. (2.32). We will rewrite it as a function of new coordinates (η, ξ) to make the proper time τ proportional to the η and the spatial coordinate ξ being a constant

$$\begin{aligned} \eta(\tau) &= \frac{\alpha}{a}\tau \\ \xi(\tau) &= \frac{1}{a}\ln\left(\frac{a}{\alpha}\right), \end{aligned} \tag{3.35}$$

where the parameter a is constant. So we have

$$\begin{aligned} \mathbf{x}(\eta) &= t\mathbf{e}_0 + x\mathbf{e}_1 = \frac{1}{a}e^{a\xi}\sinh(a\eta)\mathbf{e}_0 + \frac{1}{a}e^{a\xi}\cosh(a\eta)\mathbf{e}_1 \qquad x > |t| \\ \mathbf{x}(\eta) &= t\mathbf{e}_0 + x\mathbf{e}_1 = -\frac{1}{a}e^{a\xi}\sinh(a\eta)\mathbf{e}_0 - \frac{1}{a}e^{a\xi}\cosh(a\eta)\mathbf{e}_1 \qquad x < |t| \end{aligned} \tag{3.36}$$

Since we want to deal with Minkowski spacetime and the Rindler observer does not have access to anything inside the Rindler horizon, it is necessary to aplit the Minkowski spacetime in two cones, one which is in $x > 0$ and the other in $x < 0$ sectors.

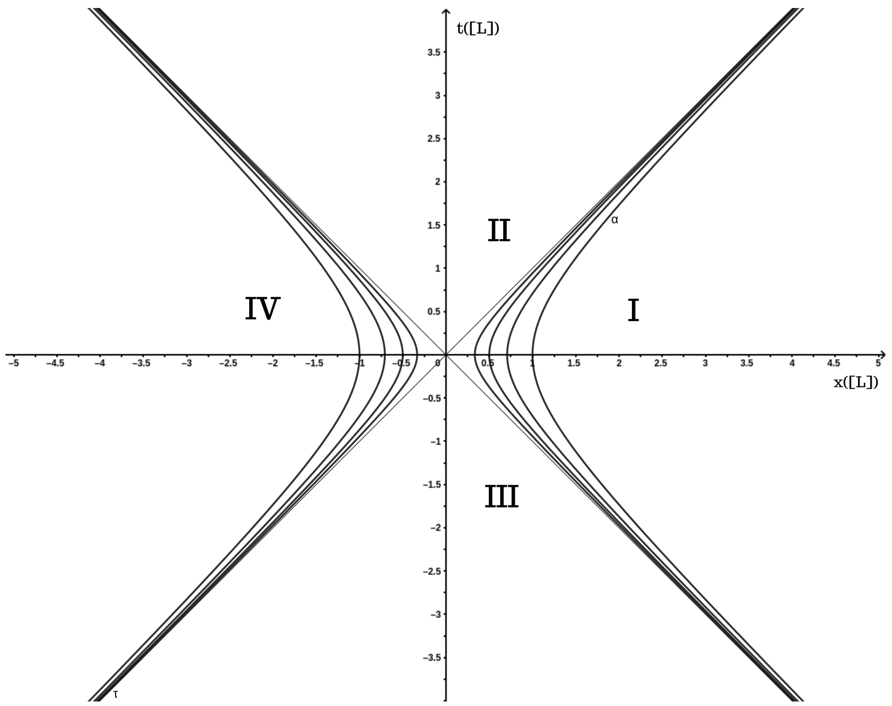

Figure 3.2: Representation of regions of interesting

The differential line element that gives us information about the metric in these new coordinates takes the form

$$d\mathbf{x}^2 = e^{2a\xi}(d\eta^2 - d\xi^2). \tag{3.37}$$

The metric components here are independent of η since $\nabla_{\partial_\eta} g = 0$. So we immediately know that ∂_n is a Killing vector field. This vector can therefore be used to define a set of positive- and negative-frequency modes, on which we can build our Fock basis.

Rewriting Eq. (3.34) using the new coordinates results in

$$(\partial_{tt} - \partial_{xx})\phi = e^{2a\xi}(\partial_\eta^2 - \partial_\xi^2)\phi = 0, \tag{3.38}$$

whose solutions are the normalized plane wave-like $g_k = (4\pi\omega)^{-1/2}e^{-i\omega\eta+ik\xi}$, with $\omega = |k|$. This equation apparently has positive-frequency in the sense that $\partial_\eta g_k = -i\omega g_k$. But we need our modes to be positive-frequency concerning a future-directed Killing vector, and in the region IV that role is played by $\partial_{(-\eta)} = -\partial_\eta$ rather than ∂_η . To deal with this annoyance, we introduce two sets of modes, one with support in the region I and the other in the region IV

$$\begin{aligned} g_k^{(1)} &= \begin{cases} \dfrac{1}{\sqrt{4\pi\omega}}e^{-i\omega\eta+ik\xi} & \text{I} \\ 0 & \text{IV} \end{cases} \\ g_k^{(2)} &= \begin{cases} 0 & \text{I} \\ \dfrac{1}{\sqrt{4\pi\omega}}e^{+i\omega\eta+ik\xi} & \text{IV} \end{cases} \end{aligned} \tag{3.39}$$

Each set of modes being positive-frequency concerning its appropriate future-directed timelike Killing vector

$$\begin{aligned} \partial_\eta g_k^{(1)} &= -i\omega g_k^{(1)} \\ \partial_{(-\eta)} g_k^{(2)} &= -i\omega g_k^{(2)}, \quad \omega > 0. \end{aligned} \tag{3.40}$$

The inner product is defined as

$$\begin{aligned} \left(g_{k_1}^{(1)}, g_{k_2}^{(1)}\right) &= \delta\left(k_1 - k_2\right) \\ \left(g_{k_1}^{(2)}, g_{k_2}^{(2)}\right) &= \delta\left(k_1 - k_2\right) \\ \left(g_{k_1}^{(1)}, g_{k_2}^{(2)}\right) &= 0, \end{aligned} \tag{3.41}$$

and similarly for the conjugate modes.

With this two sets and its conjugates we have a complete set of basis modes from which our wave equation can be expressed. So we have

$$\hat{\phi} = \int dk \left(\hat{b}_k^{(1)} g_k^{(1)} + \hat{b}_k^{(1)\dagger} g_k^{(1)*} + \hat{b}_k^{(2)} g_k^{(2)} + \hat{b}_k^{(2)\dagger} g_k^{(2)*}\right), \tag{3.42}$$

which is equivalent to our Eq. (3.19) but now for a Rindler observer with the mentioned features.

Now that we have two sets of modes, Minkowski and Rindler, with which we can expand solutions to the Klein-Gordon equation, we can compare them. Although the Hilbert space will be the same in either representation, its interpretation as a Fock space will be different. In particular, the vacuum states will be different. The Minkowski vacuum $|0_M\rangle$, satisfying

$$\hat{a}_k \left|0_M\right\rangle = 0 \tag{3.43}$$

will be described as a multi-particle state in the Rindler representation. At the same way, the Rindler vacuum $|0_R\rangle$, satisfying

$$\hat{b}_k^{(1)} \left|0_R\right\rangle = \hat{b}_k^{(2)} \left|0_R\right\rangle = 0 \tag{3.44}$$

will be described as a multi-particle state in the Minkowski representation. The difference arises because an individual Rindler mode can never be written as a sum of positive-frequency Minkowski modes. At $t = 0$ the Rindler modes only have support on the half-line, and such a function cannot be expanded in purely positive-frequency plane waves. Therefore, the Rindler annihilation operators used to define $|0_R\rangle$ are necessarily written as a superposition of Minkowski creation and annihilation operators, so the two vacuums cannot coincide. At a practical level, we can say that the Rindler horizon cuts off its access to certain fundamental frequency modes of the quantum vacuum which leads to the creation of particles in that accelerated frame, those particles should have a thermal spectrum which means that the vacuum should appear warm.

Now that we know that a Rindler observer can detect particles that do not exist for an inertial observer, we need to figure out what kind of particles will be detected. We can do this by calculating the Bogolubov coefficients that relate the Minkowski and Rindler modes and use them to determine the expectation value of the Rindler number operator in the Minkowski vacuum. But I chose to do it differently. We will start with the Rindler modes, analytically extend them to the entire spacetime and then express this extension in terms of the original Rindler modes, since we have the relationships between the Minkowski coordinates (t, x) and Rindler coordinates (η, ξ) in regions I and IV.

$$\begin{aligned} e^{-a(\eta-\xi)} &= \begin{cases} a(-t+x) & \text{I} \\ a(t-x) & \text{IV} \end{cases} \\ e^{a(\eta+\xi)} &= \begin{cases} a(t+x) & \text{I} \\ a(-t-x) & \text{IV} \end{cases} \end{aligned} \tag{3.45}$$

We can thus express the spacetime dependence of a mode $g_k^{(1)}$ with $k > 0$ in terms of Minkowski coordinates in the region I as

$$
\begin{aligned}
\sqrt{4\pi\omega} g_k^{(1)} &= e^{-i\omega\eta + ik\xi} \\
&= e^{-i\omega(\eta-\xi)} \\
&= a^{i\omega/a}(-t+x)^{i\omega/a}.
\end{aligned}
\tag{3.46}
$$

And for the region IV we have

$$
\begin{aligned}
\sqrt{4\pi\omega} g_k^{(2)} &= e^{+i\omega\eta + ik\xi} \\
&= e^{+i\omega(\eta+\xi)} \\
&= a^{-i\omega/a}(-t-x)^{-i\omega/a},
\end{aligned}
\tag{3.47}
$$

what is clearly difference from Eq. (3.46). The behavior of $g_k^{(2)}$ is different than that of $g_k^{(1)}$. In order to fix it, we can take the complex conjugate and reverse the wavenumber

$$
\begin{aligned}
\sqrt{4\pi\omega} g_{-k}^{(2)*} &= e^{-i\omega\eta + ik\xi} \\
&= e^{-i\omega(\eta-\xi)} \\
&= a^{i\omega/a}(t-x)^{i\omega/a} \\
&= a^{i\omega/a}\left[e^{-i\pi}(-t+x)\right]^{i\omega/a} \\
&= a^{i\omega/a}e^{\pi\omega/a}(-t+x)^{i\omega/a}.
\end{aligned}
\tag{3.48}
$$

Matching them, we have

$$
\sqrt{4\pi\omega}\left(g_k^{(1)} + e^{-\pi\omega/a} g_{-k}^{(2)*}\right) = a^{i\omega/a}(-t+x)^{i\omega/a}
\tag{3.49}
$$

A normalized version of this mode can be written as

$$
h_k^{(1)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}}\left(e^{\pi\omega/2a} g_k^{(1)} + e^{-\pi\omega/2a} g_{-k}^{(2)*}\right),
\tag{3.50}
$$

which is an appropriate analytic extension of the $g_k^{(1)}$ modes. We can get a complete set, including the extensions of the $g_k^{(2)}$ modes, which by an analogous argument is

$$
h_k^{(2)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}}\left(e^{\pi\omega/2a} g_k^{(2)} + e^{-\pi\omega/2a} g_{-k}^{(1)*}\right)
\tag{3.51}
$$

We can verify the normalization, for example, for $h_k^{(1)}$

$$\begin{aligned}
\left(h_{k_1}^{(1)}, h_{k_2}^{(1)}\right) =& \frac{1}{2\sqrt{\sinh\left(\frac{\pi\omega_1}{a}\right)\sinh\left(\frac{\pi\omega_2}{a}\right)}}\left[e^{\pi(\omega_1+\omega_2)/2a}\left(g_{k_1}^{(1)}, g_{k_2}^{(1)}\right) + e^{-\pi(\omega_1+\omega_2)/2a}\left(g_{-k_1}^{(2)*}, g_{-k_2}^{(2)*}\right)\right] \\
=& \frac{1}{2\sqrt{\sinh\left(\frac{\pi\omega_1}{a}\right)\sinh\left(\frac{\pi\omega_2}{a}\right)}}\left[e^{\pi(\omega_1+\omega_2)/2a}\delta\left(k_1-k_2\right) + e^{-\pi(\omega_1+\omega_2)/2a}\delta\left(-k_1+k_2\right)\right] \\
=& \frac{e^{\pi\omega_1/a}-e^{-\pi\omega_1/a}}{2\sinh\left(\frac{\pi\omega_1}{a}\right)}\delta\left(k_1-k_2\right) \\
=& \delta\left(k_1-k_2\right)
\end{aligned} \tag{3.52}$$

Now, rewriting our field in terms of these modes

$$\hat{\phi} = \int dk\left(\hat{c}_k^{(1)}h_k^{(1)} + \hat{c}_k^{(1)\dagger}h_k^{(1)*} + \hat{c}_k^{(2)}h_k^{(2)} + \hat{c}_k^{(2)\dagger}h_k^{(2)*}\right) \tag{3.53}$$

we can, by comparison, get the operator $\hat{b}_k^{(1,2)}$ in terms of $\hat{c}_k^{(1,2)}$

$$\begin{aligned}
\hat{b}_k^{(1)} &= \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}}\left(e^{\pi\omega/2a}\hat{c}_k^{(1)} + e^{-\pi\omega/2a}\hat{c}_{-k}^{(2)\dagger}\right) \\
\hat{b}_k^{(2)} &= \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}}\left(e^{\pi\omega/2a}\hat{c}_k^{(2)} + e^{-\pi\omega/2a}\hat{c}_{-k}^{(1)\dagger}\right)
\end{aligned} \tag{3.54}$$

what allow us to express the Rindler number operator in region I

$$\hat{n}_R^{(1)}(k) = \hat{b}_k^{(1)\dagger}\hat{b}_k^{(1)}. \tag{3.55}$$

From the previous discussion that $h_k^{(1,2)}$ can be expressed purely in terms of positive-frequency Minkowski modes f_k and so they share the same vacuum state $|0_M\rangle$, We finally have

$$\begin{aligned}
\langle 0_M|\hat{n}_R^{(1)}(k)|0_M\rangle &= \langle 0_M|\hat{b}_k^{(1)\dagger}\hat{b}_k^{(1)}|0_M\rangle \\
&= \frac{1}{2\sinh\left(\frac{\pi\omega}{a}\right)}\left\langle 0_\mathrm{M}\left|e^{-\pi\omega/a}\ \hat{c}_{-k}^{(1)}\hat{c}_{-k}^{(1)\dagger}\right|0_\mathrm{M}\right\rangle \\
&= \frac{e^{-\pi\omega/a}}{2\sinh\left(\frac{\pi\omega}{a}\right)}\delta(0) \\
&= \frac{1}{e^{2\pi\omega/a}-1}\delta(0).
\end{aligned} \tag{3.56}$$

This result, according to the black-body radiation theory presented by Max Planck in 1914 [21], represent a Planck spectrum with temperature

$$T = \frac{a}{2\pi}. \tag{3.57}$$

Hence we can affirm that: An observer moving with uniform acceleration through the Minkowski vacuum observes a thermal spectrum of particles. This is the Unruh effect!

Chapter 4

Conclusion

So we figure out that an accelerated observable has a mix of positive- and negative-frequency modes which leads to a creation of particles in that accelerated frame and those particles should have a thermal spectrum. The vacuum should be warm with the temperature being proportion to the acceleration. But, how can we detect it? William Unruh himself proposed some models to detect these particles. It's worth commenting on the Unruh-DeWitt detector [22], which consist of a particle in a box. This particle is coupled to the quantum field of interest, meaning that it can exchange energy with that field and so the particle can be excited into a higher energy quantum state when it encounters a particle associated with that field.

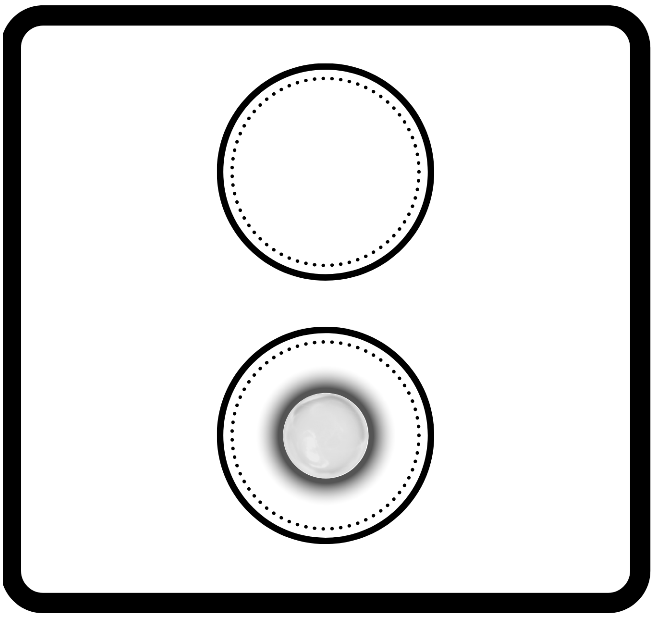

Figure 4.1: The spheres represent the possible energy states that the particle can have and the ball represents the particle itself. In this system as the accelerated box moves, the particle gains energy and can be found in more excited states.

So as the detector accelerates, Unruh particles appear. The detector particle get excited by

Unruh particles causing the detector to click. A deeper discussion of particle detector models can be found in [23].

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Appendices

Appendix A

Electromagnetic field

Most of our knowledge of electromagnetism is contained within Maxwell equations. In 20th century notation, using the natural units

$$\hbar = G = \varepsilon_0 = \mu_0 = c = 1 \tag{A.1}$$

these are

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\partial_t \mathbf{B} \\ \nabla \times \mathbf{B} &= \mathbf{J} + \partial_t \mathbf{E}. \end{aligned} \tag{A.2}$$

From this equations, it is possible to write $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$ with $\mathbf{A}$ and ϕ being the vector and scale potential. Usually, it is easier to work from the potential because they only have in total four components instead of the sex of the electric and magnetic field. Since the electromagnetic forces acting on charged particles is not conservative $\mathbf{F}_{el} \neq \nabla V(\mathbf{r})$, unfortunately, it is not possible to find a simple potential that when put on eq.(**??**) gives us the Lorentz force, but if we look for a potential that depends on the generalized coordinates and their derivatives like:

$$L_{int} = -V(\mathbf{q}, \dot{\mathbf{q}}) = -e\phi + e\mathbf{v} \cdot \mathbf{B} \tag{A.3}$$

$$\Longrightarrow \qquad \mathscr{L}_{int} = -A_\mu j^\mu \tag{A.4}$$

with $A_\mu = (\phi, -\mathbf{A})$ and $j^\mu = (\rho, \mathbf{J})$ being our current. It is possible to obtain the Lagrangian that describes the particle interacting with the field. It is important to highlight that in this context the Lagrangian L in its most complete form that is capable of dealing with particles, fields and the interaction between them is composed by the free Lagrangian L_{fr} which is responsible for the kinetic energy of the particles, the interaction Lagrangian L_{int} which contains information about the particle's interaction with the field and the field Lagrangian L_{field} which treats of field dynamics itself.

Therefore, it is easy to show that replacing the interaction Lagrangian in the **??**, the dynamics of a particle interacting through Lorentz forces in its most complete form with all relativistic corrections are obtained:

$$m\ddot{r}_\mu = eF_{\mu\nu}\dot{r}^\nu \tag{A.5}$$

with

$$F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \tag{A.6}$$

being the electromagnetic tensor.

Since we know that the free Lagrangian of the electromagnetic field needs to be able to generate the Maxwell equations, and knowing that $\partial_\mu F^{\mu\nu} = j^\nu$ contemplates two of these equations while the other two can be adequate from the following identity

$$\partial_{[\mu F_{\nu\sigma}]} = 0 \tag{A.7}$$

derived from the tensor anti-symmetrization. It is not difficult to check that this free Lagrangian will be something like $\mathscr{L} \propto F_{\mu\nu}F^{\mu\nu}$. Imposing the principle of least action we arrive at.

$$\mathscr{L}_{field} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{A.8}$$

and therefore the total Lagrangian will be

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu + L_{fr}. \tag{A.9}$$

As the free Lagrangian part is considerably complex as it involves geodesics and will not be useful here, its expression will not be demonstrated.

1.1 Quantizing the electromagnetic field

In free space without any charges or current and using the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the electric and magnetic fields become

$$\begin{aligned} \mathbf{B}(\mathbf{r}, t) &= \nabla \times \mathbf{A}(\mathbf{r}, t) \\ \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \end{aligned} \tag{A.10}$$

and using the Maxwell equations with the following property $\nabla \times (\nabla \times \quad) = \nabla(\nabla \cdot \quad) - \nabla^2$ we got

$$\nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \tag{A.11}$$

what is a wave equation of vector potential with velocity c. The solution of this equation can be write as planes waves

$$\mathbf{A}_{\mathbf{k},\alpha}(\mathbf{r}, t) = \epsilon_{\mathbf{k},\alpha} A_{\mathbf{k},\alpha} e^{i(\mathbf{kr} - \omega_{\mathbf{k}} t)} \tag{A.12}$$

with $\|\mathbf{k}\| = \frac{2\pi\nu}{c} = \frac{\omega_{\mathbf{k}}}{c}$, the ϵ representing a linear polarization vector and A being the complex amplitude of the field. Imposing periodicity conditions

$$k_i = \frac{2\pi n_i}{L} \quad (i = x, y, z) \tag{A.13}$$

we get that only discrete solutions are allowed for the vector potential for the field modes being compatible with periodicity. From that, it is possible to describe a wave as a superposition of these motions through a Fourier decomposition

$$\mathbf{A}_{\mathbf{k},\alpha}(\mathbf{r}, t) = \sum_{\mathbf{k},\alpha} \epsilon_{\mathbf{k},\alpha} (A_{\mathbf{k},\alpha} e^{i(\mathbf{kr} - \omega_{\mathbf{k}} t)} + A^* \mathbf{k}, \alpha e^{-i(\mathbf{kr} - \omega \mathbf{k} t)}) \tag{A.14}$$

therefore we have that the total energy of radiation field in $V = L^3$ will be

$$H = \sum_{\mathbf{k},\alpha} E_{\mathbf{k},\alpha}$$
$$H = \frac{1}{2} \int_V d^3r \left[\|\mathbf{E}(\mathbf{r}, t)\|^2 + \|\mathbf{B}(\mathbf{r}, t)\|^2 \right]$$
$$H = \frac{1}{2} \int_V d^3r \left[\|\nabla \times \mathbf{A}\|^2 + \left\| \frac{\partial \mathbf{A}}{\partial t} \right\|^2 \right]$$
$$H = \frac{1}{2} \sum_{\mathbf{k},\alpha} \hbar\omega_{\mathbf{k}} (A_{\mathbf{k},\alpha} A^*_{\mathbf{k},\alpha} + A^{\mathbf{k},\alpha} A_{\mathbf{k},\alpha})$$
$$H = \sum_{\mathbf{k},\alpha} \hbar\omega_{\mathbf{k}} (N_{\mathbf{k},\alpha} + \frac{1}{2})$$

which are extremely close to the expression of the quantum harmonic oscillator, equation 2.64. In fact, the process of quantization of this field consists in associate a harmonic oscillator to every radiation mode such that the creation and annihilation operators can change the degree of excitation of the field. For this, we will take the scalars into operators

$$A_{\mathbf{k}} \rightarrow \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_{\mathbf{k}}}} \hat{a}_{\mathbf{k}}, \qquad A^*_{\mathbf{k}} \rightarrow \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_{\mathbf{k}}}} \hat{a}^\dagger_{\mathbf{k}}$$

(explaining some constants that we have assumed to be one due to the balance of units). Therefore, a quantum version of the vector potential $\mathbf{A}_{\mathbf{k},\alpha}$ and consequently the electric and magnetic fields

$$\hat{\mathbf{A}}_{\mathbf{k},\alpha}(\mathbf{r},t) = \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega \mathbf{k}}} \sum_{\mathbf{k},\alpha} \epsilon_{\mathbf{k},\alpha} (\hat{a}_{\mathbf{k},\alpha} e^{i(\mathbf{kr}-\omega_{\mathbf{k}} t)} + \hat{a}_{\mathbf{k},\alpha} e^{-i(\mathbf{kr}-\omega * \mathbf{k} t)}) \tag{A.15}$$

A photon, in this definition [24], will be massless particles of defined energy, momentum, and spin represented by an excitation quantum of the harmonic oscillator associated with a mode on the radiation field, this way it is possible to create and destroy photons through the relation of operators on eigenstates

$$\hat{a}^{\dagger}_{\mathbf{k}} |n_{\mathbf{k}}\rangle = \sqrt{n_{\mathbf{k}} + 1}\, |n_{\mathbf{k}} + 1\rangle \tag{A.16}$$

$$\hat{a}_{\mathbf{k}} |n_{\mathbf{k}}\rangle = \sqrt{n_{\mathbf{k}}}\, |n_{\mathbf{k}} - 1\rangle \tag{A.17}$$