Federal University of Goiás

Institute of Physics

Quantum Pequi Group

Effective Hamiltonian aproach for the Dynamical Casimir Effect

---

Calculations notes

Gustavo de Oliveira

Goiânia, YEAR

Brazil

Contents

Chapter 1

Derivation of the effective Hamiltonian

For $\hat{\rho}_0 = |n_l\rangle \langle n_l|$

$$\int_0^t dt' \hat{H}_I(t')\hat{\rho}_0 = \sum_{kj} \left\{ \beta^*_{kj}\hat{b}^\dagger_k\hat{b}^\dagger_j - \beta_{kj}\hat{b}_k\hat{b}_j + \tilde{\alpha}^*_{kj}\hat{b}^\dagger_k\hat{b}_j - \tilde{\alpha}_{kj}\hat{b}^\dagger_j\hat{b}_k \right\}\hat{\rho}_0 \tag{1.1}$$

$$= \sum_{k(j\neq l)} \beta^*_{kl}\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}_0 - \beta_{ll}\hat{b}_l\hat{b}_l\hat{\rho}_0 + \sum_k \left(\beta^*_{kj}\hat{b}^\dagger_k\hat{b}^\dagger_l\hat{\rho}_0 + 2\tilde{\alpha}^*_{kl}\hat{b}^\dagger_k\hat{b}_l\hat{\rho}_0 \right) \tag{1.2}$$

and

$$\hat{\rho}_0 \int_0^t dt' \hat{H}_I(t') = \hat{\rho}_0 \sum_{kj} \left\{ \beta^*_{kj}\hat{b}^\dagger_k\hat{b}^\dagger_j - \beta_{kj}\hat{b}_k\hat{b}_j + \tilde{\alpha}^*_{kj}\hat{b}^\dagger_k\hat{b}_j - \tilde{\alpha}_{kj}\hat{b}^\dagger_j\hat{b}_k \right\} \tag{1.3}$$

$$= -\sum_{k(j\neq l)} \beta_{kj}\hat{\rho}_0\hat{b}_k\hat{b}_j + \beta^*_{ll}\hat{\rho}_0\hat{b}^\dagger_l\hat{b}^\dagger_l - \sum_k \left(\beta_{lk}\hat{\rho}_0\hat{b}_l\hat{b}_k + 2\tilde{\alpha}_{kl}\hat{\rho}_0\hat{b}^\dagger_l\hat{b}_k \right) \tag{1.4}$$

so

$$\int [H,\rho] = \sum_{k(j\neq l)} \left(\beta^*_{kl}\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}_0 + \beta_{kj}\hat{\rho}_0\hat{b}_k\hat{b}_j \right) - \left(\beta_{ll}\hat{b}_l\hat{b}_l\hat{\rho}_0 + \beta^*_{ll}\hat{\rho}_0\hat{b}^\dagger_l\hat{b}^\dagger_l \right)$$
$$+ \sum_k \left(\beta^*_{kj}\hat{b}^\dagger_k\hat{b}^\dagger_l\hat{\rho}_0 + 2\tilde{\alpha}^*_{kl}\hat{b}^\dagger_k\hat{b}_l\hat{\rho}_0 + \beta_{lk}\hat{\rho}_0\hat{b}_l\hat{b}_k + 2\tilde{\alpha}_{kl}\hat{\rho}_0\hat{b}^\dagger_l\hat{b}_k \right).$$

Now we calculate $\int [H, \int [H, \rho_0]]$

1.1 Dynamical equation for the ladder operators

Let the field equations of motion

$$\partial_t\hat{\phi}(x,t) = \hat{\pi}(x,t) \tag{1.5a}$$

$$\partial_t\hat{\pi}(x,t) = \nabla^2\hat{\phi}(x,t), \tag{1.5b}$$

with

$$\hat{\phi}(x,t) = \sum_j \frac{1}{\sqrt{2\omega_j(t)}} \left[\hat{a}_j(t)e^{-i\Omega_j(t)} + \hat{a}_j^\dagger(t)e^{i\Omega_j(t)}\right] \varphi_j(x,t) \tag{1.6a}$$

$$\hat{\pi}(x,t) = i\sum_j \sqrt{\frac{\omega_j(t)}{2}} \left[\hat{a}_j^\dagger(t)e^{i\Omega_j(t)} - \hat{a}_j(t)e^{-i\Omega_j(t)}\right] \varphi_j(x,t) \tag{1.6b}$$

$$\begin{aligned}\dot{\hat{\phi}}(x,t) &= \sum_j \frac{1}{\sqrt{2\omega_j(t)}} \left[\hat{a}_j(t)e^{-i\Omega_j(t)} + \hat{a}_j^\dagger(t)e^{-i\Omega_j(t)}\right] \left[\dot{\varphi}_j(x,t) - \frac{\dot{\omega}_j}{2\omega_j}\varphi_j(x,t)\right] \\ &\quad + \sum_j \frac{1}{\sqrt{2\omega_j(t)}} \left[\dot{\hat{a}}_j(t)e^{-i\Omega_j(t)} + \dot{\hat{a}}_j^\dagger(t)e^{i\Omega_j(t)}\right] \varphi_j(x,t) + \hat{\pi}(x,t)\end{aligned} \tag{1.6c}$$

$$\begin{aligned}\dot{\hat{\pi}}(x,t) &= i\sum_j \sqrt{\frac{\omega_j(t)}{2}} \left[\hat{a}_j^\dagger(t)e^{i\Omega_j(t)} - \hat{a}_j(t)e^{-i\Omega_j(t)}\right] \left[\dot{\varphi}_j(x,t) + \frac{\dot{\omega}_j}{2\omega_j}\varphi_j(x,t)\right] \\ &\quad + i\sum_j \sqrt{\frac{\omega_j(t)}{2}} \left[\dot{\hat{a}}_j^\dagger(t)e^{i\Omega_j(t)} - \dot{\hat{a}}_j(t)e^{-i\Omega_j(t)}\right] \varphi_j(x,t) + \nabla^2\hat{\phi}(x,t).\end{aligned} \tag{1.6d}$$

comparing (1.5) with (1.6c) and (1.6d), we can isolate the time derivative of the ladder operators by computing

$$\begin{aligned}\int_0^{L(t)} dx\, \varphi_k \left[\dot{\hat{\phi}} - \hat{\pi}\right] &= -\sum_j \frac{1}{\sqrt{2\omega_j}} \left[\hat{a}_j e^{-i\Omega_j} + \hat{a}_j^\dagger e^{i\Omega_j}\right] \left(G_{kj} + \frac{\dot{\omega}_j}{2\omega_j}\delta_{kj}\right) \\ &\quad + i\sum_j \frac{1}{\sqrt{2\omega_j}} \left[\dot{\hat{a}}_j^\dagger e^{i\Omega_j} + \dot{\hat{a}}_j e^{-i\Omega_j}\right] \delta_{kj} = 0\end{aligned} \tag{1.7}$$

$$\begin{aligned}\int_0^{L(t)} dx\, \varphi_k \left[\dot{\hat{\pi}} - \nabla^2\hat{\phi}\right] &= \sum_j \sqrt{\frac{\omega_j}{2}} \left[\hat{a}_j^\dagger e^{i\Omega_j} - \hat{a}_j e^{-i\Omega_j}\right] \left[G_{jk} + \frac{\dot{\omega}_j}{2\omega_j}\delta_{kj}\right] \\ &\quad + i\sum_j \sqrt{\frac{\omega_j}{2}} \left[\dot{\hat{a}}_j^\dagger e^{i\Omega_j} - \dot{\hat{a}}_j e^{-i\Omega_j}\right] \delta_{kj} = 0\end{aligned} \tag{1.8}$$

$$\tag{1.9}$$

where it was used $G_{kj} \coloneqq -\int_0^{L(t)} \varphi_k\dot{\varphi}_j$. We therefore, obtain

$$\dot{\hat{a}}_k^\dagger e^{i\Omega_k} + \dot{\hat{a}}_k e^{-i\Omega_k} = \sum_j \mu_{kj}(t)\left(\hat{a}_j e^{-i\Omega_j} + \hat{a}_j^\dagger e^{i\Omega_j}\right), \tag{1.10}$$

$$\dot{\hat{a}}_k^\dagger e^{i\Omega_k} - \dot{\hat{a}}_k e^{-i\Omega_k} = \sum_j \mu_{jk}(t)\left(\hat{a}_j e^{-i\Omega_j} - \hat{a}_j^\dagger e^{i\Omega_j}\right), \tag{1.11}$$

From the last system is easy to isolate $\dot{\hat{a}}_k$ and $\dot{\hat{a}}_k^\dagger$ as

$$\dot{\hat{a}}_k = \sum_j \left(\mathcal{A}_{kj}(t) a_j + \mathcal{B}_{kj}(t) a_j^\dagger \right) \tag{1.12a}$$

$$\dot{\hat{a}}_k^\dagger = \sum_j \left(\mathcal{A}_{kj}^*(t) a_j^\dagger + \mathcal{B}_{kj}^*(t) a_j \right) \tag{1.12b}$$

with

$$\mathcal{A}_{kj}(t) = \frac{1}{2} \left(\mu_{k,j} - \mu_{j,k} \right) e^{i[\Omega_k(t) - \Omega_j(t)]} \tag{1.13a}$$

$$\mathcal{B}_{kj}(t) = \frac{1}{2} \left(\mu_{k,j} + \mu_{j,k} \right) e^{i[\Omega_k(t) + \Omega_j(t)]} \tag{1.13b}$$

1.1.1 Effective Hamiltonian

To find the effective Hamiltonian that generates the dynamical equations (1.13) we begin by considering the most general quadratic operator (in terms of the creation and annihilation operator) which is both hermitian and invariant under an index change

$$\hat{H} = \sum_{kl} \left(B_{kl} \hat{a}_k^\dagger \hat{a}_l^\dagger + A_{kl} \hat{a}_k^\dagger \hat{a}_l + A_{kj}^* \hat{a}_l^\dagger \hat{a}_k + B_{kl}^* \hat{a}_k \hat{a}_l \right),$$

where $B_{kj} = B_{jk}$, $B_{kj}^* = B_{jk}^*$, $A_{kj} = A_{jk}^*$ and $A_{jk} = A_{kj}^*$.

The correspondent Heisenberg equation of motion is therefore

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} a_k(t) &= i \left[\hat{H}, \hat{a}_k(t) \right] \qquad (1.14)\\
&= i \sum_{jl} \left(B_{jl} \left[\hat{a}_j^\dagger \hat{a}_l^\dagger, a_k(t) \right] + A_{jl} \left[\hat{a}_j^\dagger \hat{a}_l, a_k(t) \right] + A_{jl}^* \left[\hat{a}_l^\dagger \hat{a}_j, a_k(t) \right] + B_{jl}^* [\hat{a}_j \hat{a}_l, a_k(t)] \right) \\
&= -i \left(\sum_{jl} B_{jl} \delta_{lk} \hat{a}_j^\dagger + \sum_{jl} B_{jl} \delta_{jk} \hat{a}_l^\dagger + \sum_{jl} A_{jl} \delta_{jk} \hat{a}_l + \sum_{jl} A_{jl}^* \delta_{lk} \hat{a}_j \right) \\
&= -i \sum_j \left[\left(B_{jk} + B_{kj} \right) \hat{a}_j^\dagger + \left(A_{kj} + A_{jk}^* \right) \hat{a}_j \right] \qquad (1.15)
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\mathrm{d}a_k^\dagger}{\mathrm{d}t} = i\left[\hat{H}, \hat{a}_k^\dagger\right] = i\sum_{jl}\Big(B_{jl}\left[\hat{a}_j^\dagger\hat{a}_l^\dagger, a_k^\dagger\right] + A_{jl}\left[\hat{a}_j^\dagger\hat{a}_l, a_k^\dagger\right] + A_{jl}^*\left[\hat{a}_l^\dagger\hat{a}_j, a_k^\dagger\right] + B_{jl}^*\left[\hat{a}_j\hat{a}_l, a_k^\dagger\right]\Big) \\
= i\left(\sum_{jl} B_{jl}^*\hat{a}_j\delta_{lk} + \sum_{jl} B_{jl}^*\delta_{jk}\hat{a}_l + \sum_{jl} A_{jl}\delta_{lk}\hat{a}_j^\dagger + \sum_{jl} A_{jl}^*\delta_{jk}\hat{a}_l^\dagger\right) \\
= i\sum_j\left[\left(B_{jk}^* + B_{kj}^*\right)\hat{a}_j + \left(A_{jk} + A_{kj}^*\right)\hat{a}_j^\dagger\right]
\end{aligned}
\tag{1.16}
$$

Comparing (B.12a) with (B.10) and (B.8) with (B.11), we obtain therefore we obtain instead the following system

$$
\begin{aligned}
-i\left(B_{jk} + B_{kj}\right) = -2iB_{kj} = \mathcal{B}_{kj} \\
-i\left(A_{jk}^* + A_{kj}\right) = -2iA_{kj} = \mathcal{A}_{kj} \\
i\left(B_{jk}^* + B_{kj}^*\right) = 2iB_{kj}^* = \mathcal{B}_{kj}^* \\
i\left(A_{jk} + A_{kj}^*\right) = 2iA_{kj}^* = \mathcal{A}_{kj}.
\end{aligned}
$$

The correspondent effective Hamiltonian reads then

$$
\hat{H}_{\text{eff}}(t) = \frac{i}{2}\sum_{j,k}\left[\mathcal{A}_{kj}(t)\hat{a}_k^\dagger(t)\hat{a}_j(t) - \mathcal{A}_{kj}^*(t)\hat{a}_j^\dagger(t)\hat{a}_k(t) + \mathcal{B}_{kj}(t)\hat{a}_k^\dagger(t)\hat{a}_j^\dagger(t) - \mathcal{B}_{kj}^*(t)\hat{a}_k(t)\hat{a}_j(t)\right]. \tag{1.17}
$$

1.1.2 Bogoliubov Coefficients

In this section we aim to calculate the Bogoliubov coefficients $\alpha_{kj}(t)$ and $\beta_{kj}(t)$ responsible to connect the creation and annihilation operators at instant t with the ones at instant $t = 0$, satisfying

$$
\hat{a}_k(t) = \sum_j\left[\alpha_{kj}(t)\hat{b}_j + \beta_{kj}(t)\hat{b}_j^\dagger\right] \tag{1.18a}
$$

$$
\hat{a}_k^\dagger(t) = \sum_j\left[\alpha_{kj}^*(t)\hat{b}_j^\dagger + \beta_{kj}^*(t)\hat{b}_j\right] \tag{1.18b}
$$

inserting (1.18) into (1.19) and equating with the time derivative of (1.18)

$$\dot{\hat{a}}_k = \sum_{jj'} \left(\left[\mathcal{A}_{kj'}(t)\alpha_{jj'}(t) + \mathcal{B}_{kj}(t)\beta^*_{jj'}(t)\right] \hat{b}_{j'} + \left[\mathcal{B}_{kj'}(t)\alpha^*_{jj'}(t) + \mathcal{A}_{kj'}(t)\beta_{jj'}(t)\right] \hat{b}^\dagger_{j'} \right) \tag{1.19a}$$
$$= \sum_j \left[\dot{\alpha}_{kj}(t)\hat{b}_j + \dot{\beta}_{kj}(t)\hat{b}^\dagger_j\right]$$

$$\dot{\hat{a}}^\dagger_k = \sum_{jj'} \left(\left[\mathcal{A}^*_{kj'}(t)\alpha^*_{jj'}(t) + \mathcal{B}^*_{kj'}(t)\beta_{jj'}(t)\right] \hat{b}^\dagger_{j'} + \left[\mathcal{B}^*_{kj'}(t)\alpha_{jj'}(t) + \mathcal{A}^*_{kj'}(t)\beta^*_{jj'}(t)\right] \hat{b}_{j'} \right) \tag{1.19b}$$
$$= \sum_j \left[\dot{\alpha}^*_{kj}(t)\hat{b}'_j + \dot{\beta}^*_{kj}(t)\hat{b}_j\right],$$

meaning that

$$\dot{\alpha}_{kj}(t) = \sum_{j'} \left[\mathcal{A}_{kj'}(t)\alpha_{jj'}(t) + \mathcal{B}_{kj'}(t)\beta^*_{jj'}(t)\right], \tag{1.20a}$$
$$\dot{\beta}_{kj}(t) = \sum_{j'} \left[\mathcal{B}_{kj'}(t)\alpha^*_{jj'}(t) + \mathcal{A}_{kj'}(t)\beta_{jj'}(t)\right]. \tag{1.20b}$$

Using the initial conditions that $\alpha_{jk}(0) = \delta_{jk}$ and $\beta_{jk}(0) = 0$. Considering that $\mathcal{A}_{kj}(t)$ and $\mathcal{B}_{jk}(t)$ are in general of order $\mathcal{O}(\dot{L}/L)$, we expand the Bogoliubov coefficients in terms of $\dot{L}/L$. The zero-th order solution is

$$\alpha^{(0)}_{kj}(t) = \alpha_{kj}(0) = \delta_{kj} \tag{1.21a}$$
$$\beta^{(0)}_{kj}(t) = \beta_{kj}(0) = 0. \tag{1.21b}$$

Substituting (1.21) into the right side of Eq. (1.21)

$$\dot{\alpha}^{(1)}_{kj}(t) = \sum_{j'} \left[\mathcal{A}_{kj'}(t)\alpha^{(0)}_{jj'}(t) + \mathcal{B}_{kj'}(t)\beta^{(0)*}_{jj'}(t)\right], \tag{1.22a}$$
$$\dot{\beta}^{(1)}_{kj}(t) = \sum_{j'} \left[\mathcal{B}_{kj'}(t)\alpha^{(0)*}_{jj'}(t) + \mathcal{A}_{kj'}(t)\beta^{(0)}_{jj'}(t)\right]. \tag{1.22b}$$

obtaining

$$\alpha^{(1)}_{kj}(t) = \int_0^t dt'\, \mathcal{A}_{kj'}(t'), \tag{1.23}$$
$$\beta^{(1)}_{kj}(t) = \int_0^t dt'\, \mathcal{B}_{kj'}(t'). \tag{1.24}$$

This means, the first order solution for the Bogoliubov coefficients are

$$\alpha_{kj}(t) = \delta_{jk} + \int_0^t dt'\ \mathcal{A}_{kj'}(t') + \mathcal{O}(\dot{L}^2(t)), \tag{1.25a}$$

$$\beta_{kj}(t) = \int_0^t dt'\ \mathcal{B}_{kj'}(t') + \mathcal{O}(\dot{L}^2(t)). \tag{1.25b}$$

From this last expression, we can find the following solutions for $\hat{a}_k(t)$ and $\hat{a}_k^\dagger(t)$ in first order of $\dot{L}/L$

$$\hat{a}_k(t) = \hat{b}_k + \sum_j \left(\tilde{\alpha}_{kj}(t) b_j + \beta_{kj}(t) b_j^\dagger\right) + \mathcal{O}(\dot{L}^2(t)),$$

with $\tilde{\alpha}_{kj}(t) = \int_0^t dt' \mathcal{A}_{kj}(t')$

Chapter 2

Density operator

In this appendix we aim to calculate the expression for the density operator in second-order in $\dot{L}(t)$

$$\hat{\rho}(t) = \hat{\rho}(0) - i\int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right] - \frac{1}{2}\int_0^t dt'' \int_0^t dt' \left[\hat{H}_I(t''), \left[\hat{H}_I(t'), \hat{\rho}(0)\right]\right] + \mathcal{O}(\dot{L}^2(t)). \tag{2.1}$$

with the effective Hamiltonian

$$\hat{H}_{\text{eff}} = \frac{i}{2}\sum_{j,k}\left[\mathcal{B}_{kj}(t)\hat{a}_k^\dagger(t)\hat{a}_j^\dagger(t) - \mathcal{B}_{kj}^*(t)\hat{a}_k(t)\hat{a}_j(t) + \mathcal{A}_{kj}(t)\hat{a}_k^\dagger(t)\hat{a}_j(t) - \mathcal{A}_{kj}^*(t)\hat{a}_j^\dagger(t)\hat{a}_k(t)\right]. \tag{2.2}$$

and

$$\mathcal{A}_{kj}(t) = \frac{1}{2}\left[\mu_{kj}(t) - \mu_{jk}(t)\right] e^{i[\Omega_k(t) - \Omega_j(t)]} \tag{2.3a}$$

$$\mathcal{B}_{kj}(t) = \frac{1}{2}\left[\mu_{kj}(t) + \mu_{jk}(t)\right] e^{i[\Omega_k(t) + \Omega_j(t)]} \tag{2.3b}$$

where in general $\mathcal{A}_{kj}(t)$ and $\mathcal{B}_{kj}(t)$ have a first order dependence in $\dot{L}(t)$ whereas the time dependent annihilation (and creation) operators $\hat{a}(t)$ can be expanded as

$$\hat{a}_k(t) = \hat{b}_k + \sum_j \left(\tilde{\alpha}_{kj}(t)b_j + \beta_{kj}(t)b_j^\dagger\right) + \mathcal{O}(\dot{L}^2),$$

with $\tilde{\alpha}_{kj}(t) = \alpha_{kj}(t) - \delta_{kj}$, with

$$\alpha_{kj}(t) = \delta_{kj} + \int_0^t dt' \mathcal{A}_{kj}(t') + \mathcal{O}\left(\dot{L}^2\right), \tag{2.4a}$$

$$\beta_{kj}(t) = \int_0^t dt' \mathcal{B}_{kj}(t') + \mathcal{O}(\dot{L}^2. \tag{2.4b}$$

2.1 Computing the first term

To compute expression (A.1) we begin with the first term

$$\int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right] = \int_0^t dt' \hat{H}_I(t')\hat{\rho}(0) - \hat{\rho}(0) \int_0^t dt' \hat{H}_I(t'),$$

more specifically, by computing the expression

$$\int_0^t dt' \hat{H}_I(t').$$

To do so, we start expanding the quadratic terms involving $\hat{a}_k(t)$ and $\hat{a}_k^\dagger(t)$ in terms of $\dot{L}(t)$

2.1.1 Computing the quadradic terms

$\hat{a}_k^\dagger \hat{a}_j^\dagger$ term

$$\begin{aligned}
\int dt\, \mathcal{B}_{kj} \hat{a}_k^\dagger \hat{a}_j^\dagger &= \left[\hat{b}_k^\dagger + \sum_l \left(\tilde{\alpha}_{kl}^* b_l^\dagger + \beta_{kl}^* b_l\right)\right] \left[\hat{b}_j^\dagger + \sum_n \left(\tilde{\alpha}_{jn}^* b_n^\dagger + \beta_{jn}^* b_n\right)\right] \\
&= \int dt\, \mathcal{B}_{kj} \left[\hat{b}_k^\dagger \hat{b}_j^\dagger + \sum_l \left(\tilde{\alpha}_{kl}^* b_l^\dagger \hat{b}_j^\dagger + \beta_{kl}^* b_l \hat{b}_j^\dagger\right) + \sum_n \left(\tilde{\alpha}_{jn}^* \hat{b}_k^\dagger b_n^\dagger + \beta_{jn}^* \hat{b}_k^\dagger b_n\right)\right] \\
&= \int dt\, \mathcal{B}_{kj} \left[\hat{b}_k^\dagger \hat{b}_j^\dagger + \sum_l \left(\tilde{\alpha}_{kl}^* b_l^\dagger \hat{b}_j^\dagger + \tilde{\alpha}_{jl}^* \hat{b}_k^\dagger b_l^\dagger + \beta_{kl}^* b_l \hat{b}_j^\dagger + \beta_{jl}^* \hat{b}_k^\dagger b_l\right)\right] \\
&= \beta_{kj} \hat{b}_k^\dagger \hat{b}_j^\dagger + \frac{1}{2} \sum_l \left(\beta_{kj} \tilde{\alpha}_{kl}^* b_l^\dagger \hat{b}_j^\dagger + \beta_{kj} \tilde{\alpha}_{jl}^* \hat{b}_k^\dagger b_l^\dagger + \beta_{kj} \beta_{kl}^* b_l \hat{b}_j^\dagger + \beta_{kj} \beta_{jl}^* \hat{b}_k^\dagger b_l\right)
\end{aligned}$$

$\hat{a}_k \hat{a}_j$ term

$$\begin{aligned}
\int dt\, \mathcal{B}_{kj}^* \hat{a}_k \hat{a}_j &= \int dt\, \mathcal{B}_{kj}^* \left[\hat{b}_k + \sum_l \left(\tilde{\alpha}_{kl} b_l + \beta_{kl} b_l^\dagger\right)\right] \left[\hat{b}_j + \sum_n \left(\tilde{\alpha}_{jn} b_n + \beta_{jn} b_n^\dagger\right)\right] \\
&= \int dt\, \mathcal{B}_{kj}^* \left[\hat{b}_k \hat{b}_j + \sum_l \left(\tilde{\alpha}_{kl} b_l \hat{b}_j + \beta_{kl} b_l^\dagger \hat{b}_j\right) + \sum_n \left(\tilde{\alpha}_{jn} \hat{b}_k b_n + \beta_{jn} \hat{b}_k b_n^\dagger\right)\right] \\
&= \int dt\, \mathcal{B}_{kj}^* \left[\hat{b}_k \hat{b}_j + \sum_l \left(\tilde{\alpha}_{kl} b_l \hat{b}_j + \tilde{\alpha}_{jl} \hat{b}_k b_l + \beta_{kl} b_l^\dagger \hat{b}_j + \beta_{jl} \hat{b}_k b_l^\dagger\right)\right] \\
&= \beta_{kj}^* \hat{b}_k \hat{b}_j + \frac{1}{2} \sum_l \left(\beta_{kj}^* \tilde{\alpha}_{kl} b_l \hat{b}_j + \beta_{kj}^* \tilde{\alpha}_{jl} \hat{b}_k b_l + \beta_{kj}^* \beta_{kl} b_l^\dagger \hat{b}_j + \beta_{kj}^* \beta_{jl} \hat{b}_k b_l^\dagger\right)
\end{aligned}$$

$\hat{a}_k^\dagger \hat{a}_j$ term

$$
\begin{aligned}
\int dt\ \mathcal{A}_{kj}\hat{a}_k^\dagger\hat{a}_j &= \int dt\ \mathcal{A}_{kj}\left[\hat{b}_k^\dagger + \sum_l \left(\tilde{\alpha}_{kl}^* b_l^\dagger + \beta_{kl}^* b_l\right)\right]\left[\hat{b}_j + \sum_n \left(\tilde{\alpha}_{jn} b_n + \beta_{jn} b_n^\dagger\right)\right] \\
&= \int dt\ \mathcal{A}_{kj}\left[\hat{b}_k^\dagger\hat{b}_j + \sum_l \left(\tilde{\alpha}_{kl}^* b_l^\dagger \hat{b}_j + \beta_{kl}^* b_l \hat{b}_j\right) + \sum_n \left(\tilde{\alpha}_{jn}\hat{b}_k^\dagger b_n + \beta_{jn}\hat{b}_k^\dagger b_n^\dagger\right)\right] \\
&= \int dt\ \mathcal{A}_{kj}\left[\hat{b}_k^\dagger\hat{b}_j + \sum_l \left(\tilde{\alpha}_{kl}^* b_l^\dagger \hat{b}_j + \tilde{\alpha}_{jl}\hat{b}_k^\dagger b_l + \beta_{kl}^* b_l \hat{b}_j + \beta_{jl}\hat{b}_k^\dagger b_l^\dagger\right)\right] \\
&= \tilde{\alpha}_{kj}\hat{b}_k^\dagger\hat{b}_j + \frac{1}{2}\sum_l \left(\tilde{\alpha}_{kj}\tilde{\alpha}_{kl}^* b_l^\dagger \hat{b}_j + \tilde{\alpha}_{kj}\tilde{\alpha}_{jl}\hat{b}_k^\dagger b_l + \tilde{\alpha}_{kj}\beta_{kl}^* b_l \hat{b}_j + \tilde{\alpha}_{kj}\beta_{jl}\hat{b}_k^\dagger b_l^\dagger\right)
\end{aligned}
$$

$\hat{a}_j^\dagger \hat{a}_k$ term

$$
\begin{aligned}
\int dt\ \mathcal{A}_{kj}^*\hat{a}_j^\dagger\hat{a}_k &= \int dt\ \mathcal{A}_{kj}^*\left[\hat{b}_j^\dagger + \sum_n \left(\tilde{\alpha}_{jn}^* b_n^\dagger + \beta_{jn}^* b_n\right)\right]\left[\hat{b}_k + \sum_l \left(\tilde{\alpha}_{kl} b_l + \beta_{kl} b_l^\dagger\right)\right] \\
&= \int dt\ \mathcal{A}_{kj}^*\left[\hat{b}_j^\dagger\hat{b}_k + \sum_n \left(\tilde{\alpha}_{jn}^* b_n^\dagger \hat{b}_k + \beta_{jn}^* b_n \hat{b}_k\right) + \sum_l \left(\tilde{\alpha}_{kl}\hat{b}_j^\dagger b_l + \beta_{kl}\hat{b}_j^\dagger b_l^\dagger\right)\right] \\
&= \int dt\ \mathcal{A}_{kj}^*\left[\hat{b}_j^\dagger\hat{b}_k + \sum_l \left(\tilde{\alpha}_{jl}^* b_l^\dagger \hat{b}_k + \tilde{\alpha}_{kl}\hat{b}_j^\dagger b_l + \beta_{jl}^* b_l \hat{b}_k + \beta_{kl}\hat{b}_j^\dagger b_l^\dagger\right)\right] \\
&= \tilde{\alpha}_{kj}^*\hat{b}_j^\dagger\hat{b}_k + \frac{1}{2}\sum_l \left(\tilde{\alpha}_{kj}^*\tilde{\alpha}_{jl}^* b_l^\dagger \hat{b}_k + \tilde{\alpha}_{kj}^*\tilde{\alpha}_{kl}\hat{b}_j^\dagger b_l + \tilde{\alpha}_{kj}^*\beta_{jl}^* b_l \hat{b}_k + \tilde{\alpha}_{kj}^*\beta_{kl}\hat{b}_j^\dagger b_l^\dagger\right)
\end{aligned}
$$

2.1.2 Getting back to the calculating

$$
\begin{aligned}
\int_0^t dt' \hat{H}_I(t') = \sum_{kj}\Bigg\{ & \beta_{kj}\hat{b}_k^\dagger\hat{b}_j^\dagger - \beta_{kj}^*\hat{b}_k\hat{b}_j + \tilde{\alpha}_{kj}\hat{b}_k^\dagger\hat{b}_j - \tilde{\alpha}_{kj}^*\hat{b}_j^\dagger\hat{b}_k \\
& +\frac{1}{2}\sum_l \Bigg(\beta_{kj}\tilde{\alpha}_{kl}^* b_l^\dagger\hat{b}_j^\dagger + \beta_{kj}\tilde{\alpha}_{jl}^*\hat{b}_k^\dagger b_l^\dagger + \beta_{kj}\beta_{kl}^* b_l\hat{b}_j^\dagger + \beta_{kj}\beta_{jl}^*\hat{b}_k^\dagger b_l \\
& \qquad -\beta_{kj}^*\tilde{\alpha}_{kl} b_l\hat{b}_j - \beta_{kj}^*\tilde{\alpha}_{jl}\hat{b}_k b_l - \beta_{kj}^*\beta_{kl} b_l^\dagger\hat{b}_j - \beta_{kj}^*\beta_{jl}\hat{b}_k b_l^\dagger \\
& \qquad +\tilde{\alpha}_{kj}\tilde{\alpha}_{kl}^* b_l^\dagger\hat{b}_j + \tilde{\alpha}_{kj}\tilde{\alpha}_{jl}\hat{b}_k^\dagger b_l + \tilde{\alpha}_{kj}\beta_{kl}^* b_l\hat{b}_j + \tilde{\alpha}_{kj}\beta_{jl}\hat{b}_k^\dagger b_l^\dagger \\
& \qquad -\tilde{\alpha}_{kj}^*\tilde{\alpha}_{jl}^* b_l^\dagger\hat{b}_k - \tilde{\alpha}_{kj}^*\tilde{\alpha}_{kl}\hat{b}_j^\dagger b_l - \tilde{\alpha}_{kj}^*\beta_{jl}^* b_l\hat{b}_k - \tilde{\alpha}_{kj}^*\beta_{kl}\hat{b}_j^\dagger b_l^\dagger \Bigg)\Bigg]
\end{aligned}
\tag{2.5}
$$

considering $\hat{b}_k\hat{\rho}(0) = \hat{\rho}(0)\hat{b}_k^\dagger = 0$ and using $\tilde{\alpha}_{kj} = -\tilde{\alpha}_{jk}^*$ and $\beta_{kj} = \beta_{jk}$

$$\int_0^t dt' \hat{H}_I(t')\hat{\rho}_0 = \sum_{kj} \Bigg[\beta_{kj}\hat{b}_k^\dagger \hat{b}_j^\dagger \hat{\rho}_0 + \frac{1}{2}\left(\beta_{kj}\beta_{kj}^* b_j \hat{b}_j^\dagger \hat{\rho}_0 - \beta_{kj}^*\beta_{jk}\hat{b}_k b_k^\dagger \hat{\rho}_0 \right) \tag{2.6}$$

$$+\frac{1}{2}\sum_l \left(\beta_{kj}\tilde{\alpha}_{kl}^* b_l^\dagger \hat{b}_j^\dagger \hat{\rho}_0 - \tilde{\alpha}_{kj}^*\beta_{kl}\hat{b}_j^\dagger b_l^\dagger \hat{\rho}_0 + \beta_{kj}\tilde{\alpha}_{jl}^*\hat{b}_k^\dagger b_l^\dagger \hat{\rho}_0 + \tilde{\alpha}_{kj}\beta_{jl}\hat{b}_k^\dagger b_l^\dagger \hat{\rho}_0 \right) \Bigg]$$

$$= \sum_{kj} \beta_{kj}\hat{b}_k^\dagger \hat{b}_j^\dagger \hat{\rho}_0 \tag{2.7}$$

whereas

$$\hat{\rho}_0 \int_0^t dt' \hat{H}_I(t') = -\sum_{kj} \Bigg[\beta_{kj}^*\hat{\rho}_0 \hat{b}_k \hat{b}_j - \frac{1}{2}\left(\beta_{kj}\beta_{kj}^* \hat{\rho}_0 b_j \hat{b}_j^\dagger + \beta_{kj}^*\beta_{jk}\hat{\rho}_0 \hat{b}_k b_k^\dagger \right) \tag{2.8}$$

$$+\frac{1}{2}\sum_l \left(\beta_{kj}^*\tilde{\alpha}_{kl}\hat{\rho}_0 b_l \hat{b}_j - \tilde{\alpha}_{kj}\beta_{kl}^*\hat{\rho}_0 b_l \hat{b}_j + \beta_{kj}^*\tilde{\alpha}_{jl}\hat{\rho}_0 \hat{b}_k b_l + \tilde{\alpha}_{kj}^*\beta_{jl}^*\hat{\rho}_0 b_l \hat{b}_k \right) \Bigg]$$

$$= -\sum_{kj} \beta_{kj}^*\hat{\rho}_0 \hat{b}_k \hat{b}_j \tag{2.9}$$

meaning that

$$\int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0) \right] = \sum_{kj} \left(\beta_{kj}\hat{b}_k^\dagger \hat{b}_j^\dagger \hat{\rho}_0 + \beta_{kj}^*\hat{\rho}_0 \hat{b}_k \hat{b}_j \right) \tag{2.10}$$

which is exactly the expression we have found consider only the zero-th term on $\hat{a}_k(t)$ and $\hat{a}_k^\dagger(t)$

2.2 Computing second order for β

Let $L(t) = L_0\left[1 + \lambda l(t)\right]$, therefore, in first order in λ we have

$$\Omega_k(t) = \int_0^t dt' \frac{k\pi}{L(t')} = \int_0^t dt' \frac{k\pi}{L_0)}\left[1 - \lambda l(t')\right] = \omega_k \left[t - \lambda \xi(t)\right] \tag{2.11}$$

with $\xi(t) = \int_0^t dt' l(t')$. For $\mu_{(k,j)}(t) = \frac{1}{2}\left[\mu_{kj}(t) + \mu_{jk}(t)\right]$ we obtain

$$\beta_{kj} = \int dt'\ \mu_{(k,j)}(t) e^{i\Omega_k(t)} e^{i\Omega_j(t)} = \int dt'\ \mu_{(k,j)}(t) e^{i\omega_k[t-\lambda\xi(t)]} e^{i\omega_j[t-\lambda\xi(t)]} \tag{2.12}$$

$$= \int dt'\ \mu_{(k,j)}(t) e^{i(\omega_k+\omega_j)t}\left[1 - i\lambda\omega_k\xi(t)\right]\left[1 - i\lambda\omega_j\xi(t)\right]$$

$$= \int dt'\ \mu_{(k,j)}(t) e^{i(\omega_k+\omega_j)t}\left[1 - i\lambda\left(\omega_k + \omega_j\right)\xi(t)\right]$$

$$= \beta_{kj}^{(1)} - i\lambda\left(\omega_k + \omega_j\right)\int dt'\ \mu_{(k,j)}(t)\xi(t) e^{i(\omega_k+\omega_j)t}.$$

For the special case where $l(t) = \sin(\Theta t)$ we have $\xi(t) = -\frac{1}{\Theta}[\cos(\Theta t) - 1]$. So using $\omega \equiv \omega_k + \omega_j$ we have

$$\beta_{kj} = \beta_{kj}^{(1)} + i\lambda\frac{\omega}{\Theta}\beta_{kj}^{(1)} - i\lambda\frac{\omega}{\Theta}\int dt'\, \mu_{(k,j)}(t)\cos(\Theta t)e^{i(\omega_k+\omega_j)t} \tag{2.13}$$

using $\mu_{(k,j)}(t) = \Lambda_{kj}\frac{\dot{L}}{L} = \lambda\Lambda_{kj}\Theta\cos\Theta t$

Considering

$$\hat{H}_{\text{eff}} = -\frac{i}{2}\sum_{j,k}\left[\mathcal{B}_{kj}^*(t)\hat{a}_k^\dagger(t)\hat{a}_j^\dagger(t) - \mathcal{B}_{kj}(t)\hat{a}_k(t)\hat{a}_j(t) + \mathcal{A}_{kj}^*(t)\hat{a}_k^\dagger(t)\hat{a}_j(t) - \mathcal{A}_{kj}(t)\hat{a}_j^\dagger(t)\hat{a}_k(t)\right]. \tag{2.14}$$

so

$$\int_0^t dt'\hat{H}_I(t')\hat{\rho}_0 = -\frac{i}{2}\sum_{kj}\beta_{kj}^*\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}_0 \tag{2.15}$$

$$\hat{\rho}_0\int_0^t dt'\hat{H}_I(t') = \frac{i}{2}\sum_{kj}\beta_{kj}\hat{\rho}_0\hat{b}_k\hat{b}_j \tag{2.16}$$

thus

$$\int_0^t dt'\left[\hat{H}_I(t'), \hat{\rho}(0)\right] = -\frac{i}{2}\sum_{kj}\left(\beta_{kj}^*\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}_0 + \beta_{kj}\hat{\rho}_0\hat{b}_k\hat{b}_j\right) \tag{2.17}$$

Now to calculate the second term in (A.1) we consider the first order term in (2.5) and expression (5.4)

$$\begin{aligned}
&\int_0^t dt''\int_0^t dt'\left[H(t''), \left[\hat{H}_I(t'), \rho(0)\right]\right] \\
&= \int_0^t dt'' H(t'')\int_0^t dt'\left[\hat{H}_I(t'), \rho(0)\right] - \int_0^t dt'\left[\hat{H}_I(t'), \rho(0)\right]\int_0^t dt'' H(t'')
\end{aligned}$$

which can be expressed as

$$\begin{aligned}
&= \frac{i}{2}\sum_{nl}\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n + \tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n - \tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]\int_0^{t''} dt'\left[\hat{H}_I(t'), \hat{\rho}(0)\right] \\
&- \frac{i}{2}\sum_{nl}\int_0^t dt'\left[\hat{H}_I(t'), \hat{\rho}(0)\right]\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n + \tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n - \tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) + \tilde{\beta}_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\right]\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n + \tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n - \tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right] \\
&- \frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n + \tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n - \tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]\left[\tilde{\beta}_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) + \tilde{\beta}_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{kj}\sum_{nl}\tilde{\beta}_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n + \tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n - \tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right] \\
&+ \frac{1}{4}\sum_{kj}\sum_{nl}\tilde{\beta}_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n + \tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n - \tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right] \\
&- \frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n + \tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n - \tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]\tilde{\beta}_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) \\
&- \frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n + \tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n - \tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]\tilde{\beta}_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{kj}\tilde{\beta}_{ln}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{kj}\tilde{\beta}_{ln}^*\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l\hat{a}_n + \tilde{\beta}_{kj}\tilde{\alpha}_{ln}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l^\dagger\hat{a}_n - \tilde{\beta}_{kj}\tilde{\alpha}_{ln}^*\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_n^\dagger\hat{a}_l\right] \\
&+ \frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{kj}^*\tilde{\beta}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{kj}^*\tilde{\beta}_{ln}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n + \tilde{\beta}_{kj}^*\tilde{\alpha}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n - \tilde{\beta}_{kj}^*\tilde{\alpha}_{ln}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_n^\dagger\hat{a}_l\right] \\
&- \frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{ln}\tilde{\beta}_{kj}\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) - \tilde{\beta}_{ln}^*\tilde{\beta}_{kj}\hat{a}_l\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) + \tilde{\alpha}_{ln}\tilde{\beta}_{kj}\hat{a}_l^\dagger\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) - \tilde{\alpha}_{ln}^*\tilde{\beta}_{kj}\hat{a}_n^\dagger\hat{a}_l\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right] \\
&- \frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{ln}\tilde{\beta}_{kj}^*\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{\rho}(0)\hat{a}_k\hat{a}_j - \tilde{\beta}_{ln}^*\tilde{\beta}_{kj}^*\hat{a}_l\hat{a}_n\hat{\rho}(0)\hat{a}_k\hat{a}_j + \tilde{\alpha}_{ln}\tilde{\beta}_{kj}^*\hat{a}_l^\dagger\hat{a}_n\hat{\rho}(0)\hat{a}_k\hat{a}_j - \tilde{\alpha}_{ln}^*\tilde{\beta}_{kj}^*\hat{a}_n^\dagger\hat{a}_l\hat{\rho}(0)\hat{a}_k\hat{a}_j\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{kj}\sum_{nl}\left[-\tilde{\beta}_{kj}\tilde{\beta}_{ln}^*\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l\hat{a}_n\right] \\
&+ \frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{kj}^*\tilde{\beta}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{kj}^*\tilde{\beta}_{ln}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n + \tilde{\beta}_{kj}^*\tilde{\alpha}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n - \tilde{\beta}_{kj}^*\tilde{\alpha}_{ln}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_n^\dagger\hat{a}_l\right] \\
&\frac{1}{4}\sum_{kj}\sum_{nl}\left[-\tilde{\beta}_{ln}\tilde{\beta}_{kj}\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) + \tilde{\beta}_{ln}^*\tilde{\beta}_{kj}\hat{a}_l\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) - \tilde{\alpha}_{ln}\tilde{\beta}_{kj}\hat{a}_l^\dagger\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) + \tilde{\alpha}_{ln}^*\tilde{\beta}_{kj}\hat{a}_n^\dagger\hat{a}_l\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right] \\
&\frac{1}{4}\sum_{kj}\sum_{nl}\left[-\tilde{\beta}_{ln}\tilde{\beta}_{kj}^*\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{\rho}(0)\hat{a}_k\hat{a}_j\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{kj}\sum_{nl}\Bigg[-\tilde{\beta}_{kj}\tilde{\beta}^*_{ln}\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\hat{a}_l\hat{a}_n \\
&+\tilde{\beta}^*_{kj}\tilde{\beta}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_l\hat{a}^\dagger_n - \tilde{\beta}^*_{kj}\tilde{\beta}^*_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n + \tilde{\beta}^*_{kj}\tilde{\alpha}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_l\hat{a}_n - \tilde{\beta}^*_{kj}\tilde{\alpha}^*_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_n\hat{a}_l \\
&-\tilde{\beta}_{ln}\tilde{\beta}_{kj}\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0) + \tilde{\beta}^*_{ln}\tilde{\beta}_{kj}\hat{a}_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0) - \tilde{\alpha}_{ln}\tilde{\beta}_{kj}\hat{a}^\dagger_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0) + \tilde{\alpha}^*_{ln}\tilde{\beta}_{kj}\hat{a}^\dagger_n\hat{a}_l\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0) \\
&-\tilde{\beta}_{ln}\tilde{\beta}^*_{kj}\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{\rho}(0)\hat{a}_k\hat{a}_j\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{4}\sum_{kj}\sum_{nl}\Bigg[&\tilde{\beta}^*_{ln}\tilde{\beta}_{kj}\left(\hat{a}_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) + \tilde{\beta}^*_{kj}\tilde{\beta}_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_l\hat{a}^\dagger_n\right) \\
&+\tilde{\alpha}^*_{ln}\tilde{\beta}_{kj}\left(\hat{a}^\dagger_n\hat{a}_l\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) + \tilde{\beta}^*_{kj}\tilde{\alpha}_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_l\hat{a}_n\right) \\
&-\tilde{\beta}_{ln}\tilde{\beta}_{kj}\left(\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) - \tilde{\beta}^*_{kj}\tilde{\beta}^*_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n\right) \\
&-\tilde{\alpha}_{ln}\tilde{\beta}_{kj}\left(\hat{a}^\dagger_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) - \tilde{\beta}^*_{kj}\tilde{\alpha}^*_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_n\hat{a}_l\right) \\
&-\tilde{\beta}_{kj}\tilde{\beta}^*_{ln}\left(\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\hat{a}_l\hat{a}_n\right) - \tilde{\beta}_{ln}\tilde{\beta}^*_{kj}\left(\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{\rho}(0)\hat{a}_k\hat{a}_j\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{4}\sum_{kj}\sum_{nl}\Bigg[&\tilde{\beta}^*_{ln}\tilde{\beta}_{kj}\left(\hat{a}_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) - \tilde{\beta}_{kj}\tilde{\beta}^*_{ln}\left(\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\hat{a}_l\hat{a}_n\right) + \tilde{\alpha}^*_{ln}\tilde{\beta}_{kj}\left(\hat{a}^\dagger_n\hat{a}_l\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) \\
&-\tilde{\beta}_{ln}\tilde{\beta}_{kj}\left(\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) - \tilde{\alpha}_{ln}\tilde{\beta}_{kj}\left(\hat{a}^\dagger_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) + \text{h.c.}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho}(t) = \hat{\rho}(0) - &\frac{1}{2}\sum_{kj}\left[\tilde{\beta}_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) + \tilde{\beta}_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\right] \\
-\frac{1}{8}\sum_{kj}\sum_{nl}&\left[\tilde{\beta}_{ln}^*\tilde{\beta}_{kj}\left(\hat{a}_l\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) + \tilde{\beta}_{kj}^*\tilde{\beta}_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n^\dagger\right)\right. \\
&- \tilde{\beta}_{kj}\tilde{\beta}_{ln}^*\left(\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l\hat{a}_n\right) - \tilde{\beta}_{ln}\tilde{\beta}_{kj}^*\left(\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{\rho}(0)\hat{a}_k\hat{a}_j\right) \\
&+ \tilde{\alpha}_{ln}^*\tilde{\beta}_{kj}\left(\hat{a}_n^\dagger\hat{a}_l\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) + \tilde{\beta}_{kj}^*\tilde{\alpha}_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n\right) \\
&- \tilde{\beta}_{ln}\tilde{\beta}_{kj}\left(\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) - \tilde{\beta}_{kj}^*\tilde{\beta}_{ln}^*\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n\right) \\
&\left.- \tilde{\alpha}_{ln}\tilde{\beta}_{kj}\left(\hat{a}_l^\dagger\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) - \tilde{\beta}_{kj}^*\tilde{\alpha}_{ln}^*\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_n^\dagger\hat{a}_l\right)\right]
\end{aligned}
$$

2.3 Computing the second term

Now to calculate the second term in (A.1) we consider the first order term in (2.5) and expression (5.4)

$$
\begin{aligned}
&\int_0^t dt'' \int_0^t dt' \left[H(t''), \left[\hat{H}_I(t'), \rho(0)\right]\right] \\
&= \int_0^t dt'' H(t'') \int_0^t dt' \left[\hat{H}_I(t'), \rho(0)\right] - \int_0^t dt' \left[\hat{H}_I(t'), \rho(0)\right] \int_0^t dt'' H(t'')
\end{aligned}
$$

which can be expressed as

$$
\begin{aligned}
&= \frac{i}{2}\sum_{nl}\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n + \tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n - \tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]\int_0^{t''} dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right] \\
&- \frac{i}{2}\sum_{nl}\int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right]\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger - \tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n + \tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n - \tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]
\end{aligned}
$$

$$= \frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)+\tilde{\beta}_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\right]\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger-\tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n+\tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n-\tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger-\tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n+\tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n-\tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]\left[\tilde{\beta}_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)+\tilde{\beta}_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\right]$$

$$= \frac{1}{4}\sum_{kj}\sum_{nl}\tilde{\beta}_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger-\tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n+\tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n-\tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]$$
$$+\frac{1}{4}\sum_{kj}\sum_{nl}\tilde{\beta}_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger-\tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n+\tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n-\tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger-\tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n+\tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n-\tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]\tilde{\beta}_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger-\tilde{\beta}_{ln}^*\hat{a}_l\hat{a}_n+\tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n-\tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]\tilde{\beta}_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j$$

$$= \frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{kj}\tilde{\beta}_{ln}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l^\dagger\hat{a}_n^\dagger-\tilde{\beta}_{kj}\tilde{\beta}_{ln}^*\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l\hat{a}_n+\tilde{\beta}_{kj}\tilde{\alpha}_{ln}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l^\dagger\hat{a}_n-\tilde{\beta}_{kj}\tilde{\alpha}_{ln}^*\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_n^\dagger\hat{a}_l\right]$$
$$+\frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{kj}^*\tilde{\beta}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n^\dagger-\tilde{\beta}_{kj}^*\tilde{\beta}_{ln}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n+\tilde{\beta}_{kj}^*\tilde{\alpha}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n-\tilde{\beta}_{kj}^*\tilde{\alpha}_{ln}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_n^\dagger\hat{a}_l\right]$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{ln}\tilde{\beta}_{kj}\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)-\tilde{\beta}_{ln}^*\tilde{\beta}_{kj}\hat{a}_l\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)+\tilde{\alpha}_{ln}\tilde{\beta}_{kj}\hat{a}_l^\dagger\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)-\tilde{\alpha}_{ln}^*\tilde{\beta}_{kj}\hat{a}_n^\dagger\hat{a}_l\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right]$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{ln}\tilde{\beta}_{kj}^*\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{\rho}(0)\hat{a}_k\hat{a}_j-\tilde{\beta}_{ln}^*\tilde{\beta}_{kj}^*\hat{a}_l\hat{a}_n\hat{\rho}(0)\hat{a}_k\hat{a}_j+\tilde{\alpha}_{ln}\tilde{\beta}_{kj}^*\hat{a}_l^\dagger\hat{a}_n\hat{\rho}(0)\hat{a}_k\hat{a}_j-\tilde{\alpha}_{ln}^*\tilde{\beta}_{kj}^*\hat{a}_n^\dagger\hat{a}_l\hat{\rho}(0)\hat{a}_k\hat{a}_j\right]$$

$$= \frac{1}{4}\sum_{kj}\sum_{nl}\left[-\tilde{\beta}_{kj}\tilde{\beta}_{ln}^*\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l\hat{a}_n\right]$$
$$+\frac{1}{4}\sum_{kj}\sum_{nl}\left[\tilde{\beta}_{kj}^*\tilde{\beta}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n^\dagger-\tilde{\beta}_{kj}^*\tilde{\beta}_{ln}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n+\tilde{\beta}_{kj}^*\tilde{\alpha}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n-\tilde{\beta}_{kj}^*\tilde{\alpha}_{ln}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_n^\dagger\hat{a}_l\right]$$
$$\frac{1}{4}\sum_{kj}\sum_{nl}\left[-\tilde{\beta}_{ln}\tilde{\beta}_{kj}\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)+\tilde{\beta}_{ln}^*\tilde{\beta}_{kj}\hat{a}_l\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)-\tilde{\alpha}_{ln}\tilde{\beta}_{kj}\hat{a}_l^\dagger\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)+\tilde{\alpha}_{ln}^*\tilde{\beta}_{kj}\hat{a}_n^\dagger\hat{a}_l\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right]$$
$$\frac{1}{4}\sum_{kj}\sum_{nl}\left[-\tilde{\beta}_{ln}\tilde{\beta}_{kj}^*\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{\rho}(0)\hat{a}_k\hat{a}_j\right]$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{kj}\sum_{nl}\Bigg[-\tilde{\beta}_{kj}\tilde{\beta}^*_{ln}\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\hat{a}_l\hat{a}_n \\
&+ \tilde{\beta}^*_{kj}\tilde{\beta}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_l\hat{a}^\dagger_n - \tilde{\beta}^*_{kj}\tilde{\beta}^*_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n + \tilde{\beta}^*_{kj}\tilde{\alpha}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_l\hat{a}_n - \tilde{\beta}^*_{kj}\tilde{\alpha}^*_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_n\hat{a}_l \\
&- \tilde{\beta}_{ln}\tilde{\beta}_{kj}\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0) + \tilde{\beta}^*_{ln}\tilde{\beta}_{kj}\hat{a}_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0) - \tilde{\alpha}_{ln}\tilde{\beta}_{kj}\hat{a}^\dagger_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0) + \tilde{\alpha}^*_{ln}\tilde{\beta}_{kj}\hat{a}^\dagger_n\hat{a}_l\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0) \\
&- \tilde{\beta}_{ln}\tilde{\beta}^*_{kj}\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{\rho}(0)\hat{a}_k\hat{a}_j \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{4}\sum_{kj}\sum_{nl}\Bigg[&\tilde{\beta}^*_{ln}\tilde{\beta}_{kj}\left(\hat{a}_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) + \tilde{\beta}^*_{kj}\tilde{\beta}_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_l\hat{a}^\dagger_n\right) \\
&+ \tilde{\alpha}^*_{ln}\tilde{\beta}_{kj}\left(\hat{a}^\dagger_n\hat{a}_l\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) + \tilde{\beta}^*_{kj}\tilde{\alpha}_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_l\hat{a}_n\right) \\
&- \tilde{\beta}_{ln}\tilde{\beta}_{kj}\left(\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) - \tilde{\beta}^*_{kj}\tilde{\beta}^*_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n\right) \\
&- \tilde{\alpha}_{ln}\tilde{\beta}_{kj}\left(\hat{a}^\dagger_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) - \tilde{\beta}^*_{kj}\tilde{\alpha}^*_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_n\hat{a}_l\right) \\
&- \tilde{\beta}_{kj}\tilde{\beta}^*_{ln}\left(\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\hat{a}_l\hat{a}_n\right) - \tilde{\beta}_{ln}\tilde{\beta}^*_{kj}\left(\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{\rho}(0)\hat{a}_k\hat{a}_j\right) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{4}\sum_{kj}\sum_{nl}\Bigg[&\tilde{\beta}^*_{ln}\tilde{\beta}_{kj}\left(\hat{a}_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) - \tilde{\beta}_{kj}\tilde{\beta}^*_{ln}\left(\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\hat{a}_l\hat{a}_n\right) + \tilde{\alpha}^*_{ln}\tilde{\beta}_{kj}\left(\hat{a}^\dagger_n\hat{a}_l\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) \\
&- \tilde{\beta}_{ln}\tilde{\beta}_{kj}\left(\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) - \tilde{\alpha}_{ln}\tilde{\beta}_{kj}\left(\hat{a}^\dagger_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) + \text{h.c.}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho}(t) = \hat{\rho}(0) - \frac{1}{2}\sum_{kj}&\left[\tilde{\beta}_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) + \tilde{\beta}_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\right] \\
-\frac{1}{8}\sum_{kj}\sum_{nl}&\left[\tilde{\beta}_{ln}^*\tilde{\beta}_{kj}\left(\hat{a}_l\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) + \tilde{\beta}_{kj}^*\tilde{\beta}_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n^\dagger\right)\right. \\
&- \tilde{\beta}_{kj}\tilde{\beta}_{ln}^*\left(\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l\hat{a}_n\right) - \tilde{\beta}_{ln}\tilde{\beta}_{kj}^*\left(\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{\rho}(0)\hat{a}_k\hat{a}_j\right) \\
&+ \tilde{\alpha}_{ln}^*\tilde{\beta}_{kj}\left(\hat{a}_n^\dagger\hat{a}_l\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) + \tilde{\beta}_{kj}^*\tilde{\alpha}_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n\right) \\
&- \tilde{\beta}_{ln}\tilde{\beta}_{kj}\left(\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) - \tilde{\beta}_{kj}^*\tilde{\beta}_{ln}^*\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n\right) \\
&\left.- \tilde{\alpha}_{ln}\tilde{\beta}_{kj}\left(\hat{a}_l^\dagger\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) - \tilde{\beta}_{kj}^*\tilde{\alpha}_{ln}^*\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_n^\dagger\hat{a}_l\right)\right]
\end{aligned}
$$

Chapter 3

Effective Hamiltonian

3.1 The theory

Let us consider a one-dimensional cavity composed by two mirrors, one fixed at positions $x = 0$, while the other can move in a prescribed trajectroy $x = L(t)$. Confined inside the cavity we consider a quantum scalar field $\hat{\phi}(x, t)$ satisfying the wave equation

$$\partial_x^2 \hat{\phi}(x, t) = \partial_t \hat{\pi}(x, t) \tag{3.1}$$

and the standard equal-time commutation relations

$$\begin{aligned} \left[\hat{\phi}(x, t), \hat{\pi}(x', t)\right] &= i\delta(x - x'), \\ \left[\hat{\phi}(x, t), \hat{\phi}(x', t)\right] &= [\hat{\pi}(x, t), \hat{\pi}(x', t)] = 0, \end{aligned} \tag{3.2}$$

where $\hat{\pi} = \partial_t \hat{\phi}$ is the conjugated momenta of the field. If the mirrors are ideal, the boundary conditions we impose on the field take the form $\hat{\phi}(0, t) = \hat{\phi}(L(t), t) = 0$.

Taking the mirrors to be initially at rest (with static boundary conditions $L(t < 0) = L_0$), for $t < 0$ the field can be decomposed as

$$\hat{\phi}^{\text{in}}(x, t) = \sum_n \frac{1}{\sqrt{2\omega_n}} \left[f_n^{\text{in}}(x, t) \hat{a}_n^{\text{in}} + f_n^{\text{in}*}(x, t) \hat{a}_n^{\text{in}\dagger} \right], \tag{3.3}$$

where the steady-state mode functions

$$f_n^{\text{in}}(x, t) = \sqrt{\frac{2}{L_0}} \sin\left(\frac{n\pi}{L_0} x\right) e^{-i\omega_n t} \tag{3.4}$$

are complex valued solutions of the wave equation with frequencies

$$\omega_n = \frac{\pi n}{L_0}, \tag{3.5}$$

where n is a positive integer. The annihilation (creation) operator $\hat{b}_n$ ($\hat{b}_n^\dagger$) satisfies the standard commutation relations

$$\left[\hat{a}_n^{\text{in}}, \hat{a}_m^{\text{in}\dagger}\right] = \delta_{nm}, \quad \left[\hat{a}_n^{\text{in}}, \hat{a}_m^{\text{in}}\right] = \left[\hat{a}_n^{\text{in}\dagger}, \hat{a}_m^{\text{in}\dagger}\right] = 0.$$

The initial vacuum state $|0; \text{in}\rangle$ is defined as the state annihilated by all $\hat{b}_n$.

For $t > 0$, one of the z-mirrors starts moving and the quantum field is subjected to non-stationary boundary conditions. The changes in the vacuum mode structure then translate, into the existence of new set of mode functions $f_n^{\text{out}}(x,t)$ such that the decomposition of the field reads

$$\hat{\phi}(x,t) = \sum_n \frac{1}{\sqrt{2\omega_n}} \left[f_n^{\text{out}}(x,t)\hat{a}_n^{\text{in}} + f_n^{\text{out}*}(x,t)\hat{a}_n^{\text{in}\dagger}\right], \tag{3.6}$$

with f_n^{out} standing for the instantaneous modes to be defined in the next section. Supposing that after some interval of time T, the moving mirror returns to the the static configuration in its initial position L_0. For $t > T$ we have a new set of physical ladder operators, $\hat{a}_m$ and $\hat{a}_m^\dagger$, with the following expansion for the field

$$\hat{\phi}^{\text{out}}(x,t) = \sum_n \frac{1}{\sqrt{2\omega_n}} \left[f_n^{\text{in}}(x,t)\hat{a}_n^{\text{out}} + f_n^{\text{in}*}(x,t)(\hat{a}_n^{\text{out}})^\dagger\right].$$

According with the Bogoliubov transformations both set of creation and annihilation operators (3.3) and (3.1) are related by

$$\hat{a}_m^{\text{out}} = \sum_n \left(\alpha_{nm}\hat{a}_n^{\text{in}} + \beta_{nm}^*\hat{a}_n^{\text{in}\dagger}\right). \tag{3.7}$$

From this is immediate that the initial vacuum state (as seen after the motion of the mirror) now contains a non-vanishing number of particles

$$N = \sum_m \langle 0; \text{in}|(\hat{a}_m^{\text{out}})^\dagger \hat{a}_m^{\text{out}}|0; \text{in}\rangle = \sum_{n,m} |\beta_{nm}|^2, \tag{3.8}$$

which characterizes the DCE.

In the next section we describe and effective Hamiltonian approach for the DCE. This formalism, introduced by C. K. Law formalism [**law**] is suitable for the study of entropy production due to the perturbations imposed on the field.

3.2 Effective Hamiltonian

We start by reviewing the effective Hamiltonian formalism developed in Ref. [**law**]. This will be the main tool we employ in order to investigate the entropy production due to the dynamical Casimir effect.

Inside the cavity, the field can be described in terms of instantaneous mode functions.

$$f_n^{\text{out}}(x,t) = \varphi_{n,L(t)}(x)e^{-i\omega_{n,L(t)}t},$$

which satisfies the differential equation

$$\partial_x^2 \varphi_{n,L(t)}(x) + \omega_{n,L(t)}^2 \varphi_{n,L(t)}(x) = 0, \tag{3.9}$$

subjected to the boundary conditions $\varphi_{n,L(t)}(0) = \varphi_{n,L(t)}(L(t)) = 0$ and the normalization

$$\int_0^{L(t)} d^3x \; \varphi_{n,L(t)}(x)\varphi_{m,L(t)}(x) = \delta_{nm}.$$

The subscript notation means that the frequency ω_n and the mode function φ_k depend implicitly on time through the mirror motion $L(t)$.

With these expressions it is straightforward to write down an effective Hamiltonian as [**law**]

$$\hat{H}_{\text{eff}} = \sum_n \omega_{n,L(t)} \hat{a}_n^\dagger \hat{a}_n + \frac{i}{2} \sum_{m,n} \left[\mu_{n,m}^+(t) \left(\hat{a}_n^\dagger \hat{a}_m^\dagger - \hat{a}_n a_m \right) + \mu_{n,m}^-(t) \left(\hat{a}_n^\dagger a_m - \hat{a}_m^\dagger \hat{a}_n \right) \right], \tag{3.10}$$

where beyond the field mode structure with instantaneous frequencies $\omega_{n,L(t)}$, there are additional terms representing non-adiabatic processes such as the creation of pairs of particles (even from the vacuum state) as well as their scattering to different modes. The coefficients $\mu_{n,m}^+(t)$ and $\mu_{n,m}^-(t)$ which govern the time-scales of these processes are defined as

$$\mu_{n,m}^\pm(t) = \frac{\chi_{n,m}^\pm}{\omega_{n,L(t)} \pm \omega_{m,L(t)}} \frac{\dot{L}(t)}{L(t)}$$

where

$$\chi_{n,m}^\pm = \frac{1}{2} \left[\frac{\omega_{m,L(t)}^2 - \omega_{n,L(t)}^2}{\sqrt{\omega_{n,L(t)}\omega_{m,L(t)}}} g_{n,m} - L(t) \frac{\omega_{n,L(t)} \pm \omega_{m,L(t)}}{\omega_{n,L(t)}} \frac{\partial \omega_{n,L(t)}}{\partial L_z} \delta_{n,m} \right],$$

and

$$g_{n,m} = -g_{m,n} = -L(t) \int_0^{L(t)} dz \; \varphi_{n,L(t)} \frac{\partial \varphi_{m,L(t)}}{\partial L_z}. \tag{3.11}$$

The time evolution of the system is governed by the Schrodinger equation

$$i\partial_t |\Psi(t)\rangle = \hat{H}_{\text{eff}} |\Psi(t)\rangle. \tag{3.12}$$

As pointed out by Law [**law**], as the formalism is based on a set of instantaneous basis functions $\{\varphi_{n,L(t)}(x)\}$, the vacuum states changes accordingly with the change in the time parameter, so the bosons associated with the creation and annihilation operators may not be regarded as real particles while the cavity is still in motion. Only in the static situation such operators become unique, thus acquiring the usual physical meaning.

Moving now to the interaction picture, the effective Hamiltonian reads

$$\hat{H}_I = \frac{i}{2}\sum_{m,n}\left[\mu^{+}_{n,m}(t)\left(\hat{\mathcal{A}}^{\dagger}_n\hat{\mathcal{A}}^{\dagger}_m - \hat{\mathcal{A}}_n\hat{\mathcal{A}}_m\right) + \mu^{-}_{n,m}(t)\left(\hat{\mathcal{A}}^{\dagger}_n\hat{\mathcal{A}}_m - \hat{\mathcal{A}}^{\dagger}_m\hat{\mathcal{A}}_n\right)\right],$$

where $\hat{\mathcal{A}}_n(t) = \hat{a}_n \exp\{-i\bar{\omega}_n(t)\}$ and $\bar{\omega}_{n,L(t)}(t) = \int^t dt'\omega_{n,L_z(t')}$. The time evolution of the system's density operator $\hat{\rho}(t)$ can be determined with the help of the dynamical equation $\dot{\hat{\rho}}(t) = -i\left[\hat{H}_I, \hat{\rho}(t)\right]$. Considering $\dot{L}_z(t)$ as a perturbation parameter, a formal solution for $\hat{\rho}(t)$ in the second order of $\hat{H}_I(t)$ can be putted as

$$\hat{\rho}(t) = \hat{\rho}(0) - i\int_0^t dt'\left[\hat{H}_I(t'), \hat{\rho}(0)\right] - \int_0^t dt'\int_0^{t'} dt''\left[\hat{H}_I(t'), \left[\hat{H}_I(t''), \hat{\rho}(0)\right]\right].$$

We are interested in the case where the cavity is initially prepared in the vacuum state $\hat{\rho}(0) = |0\rangle\langle 0|$ (from here on we use the shorthand notation $|0\rangle \equiv |0;\mathrm{in}\rangle$). Under such initial condition, the number of particles created inside the cavity due do the DCE can be written as

$$N(t) \;=\; \mathrm{Tr}\left\{\sum_n \hat{\rho}(t)\hat{a}^{\dagger}_n\hat{a}_n\right\} = \sum_{n,m}\mathcal{N}_{n,m}, \tag{3.13}$$

with

$$\mathcal{N}_{n,m}(t) := 2\,\mathrm{Re}\int_0^t dt'\mu^{+}_{n,m}(t')e^{i\{[\bar{\omega}_n(t')+\bar{\omega}_m(t')]\}}\int_0^{t'} dt''\,\mu^{+}_{n,m}(t'')e^{-i\{[\bar{\omega}_n(t'')+\bar{\omega}_m(t'')]\}},$$

which can be identified, from equation (3.8), as the Bogoliubov coefficient $|\beta_{n,m}|^2$.

Será que esta transição não está brusca? We are now ready to study irreversibility induced by the DCE. In order to quantity irreversibility, we consider the diagonal entropy [1]

$$S_d(\rho_{\mathrm{diag}}) = -\sum_{\eta}\rho^{(\eta)}_{\mathrm{diag}}\ln\rho^{(\eta)}_{\mathrm{diag}}, \tag{3.14}$$

as the main figure of merit, where $\rho^{(\eta)}_{\mathrm{diag}} = \langle\boldsymbol{\eta}|\rho|\boldsymbol{\eta}\rangle$ are the diagonal elements of the system's density operator in the time-independent energy eigenstates, defined as

$$|\boldsymbol{\eta}\rangle = |\eta_{n_1}, \eta_{n_2}, \ldots\rangle = \prod_{\eta_{n_i}}\frac{1}{\sqrt{\eta_{n_i}!}}\left(a^{\dagger}_{n_i}\right)^{\eta_{n_i}}|0\rangle\,,$$

for the number state populated with η_{n_i} photons in the n_i-th mode..

From Eq. (3.14), the diagonal entropy can be directly computed, resulting in

$$S_d(t) = -\rho^{(0)}_{\mathrm{diag}}(t)\ln\rho^{(0)}_{\mathrm{diag}}(t) - \sum_{n,m}\rho^{(n,m)}_{\mathrm{diag}}(t)\ln\rho^{(n,m)}_{\mathrm{diag}}(t)$$

with

$$\rho_{\text{diag}}^{(0)}(t) := \langle 0|\hat{\rho}(t)|0\rangle = 1 - \frac{1}{2}N(t)$$
$$\rho_{\text{diag}}^{(n,m)}(t) := \begin{cases} \langle 2_n|\hat{\rho}(t)|2_n\rangle, & \text{if } n = m \\ \langle 1_n, 1_m|\hat{\rho}(t)|1_n, 1_m\rangle, & \text{if } n \neq m \end{cases}$$
$$= \frac{1}{2}\mathcal{N}_{n,m}(t).$$

In order to illustrate the behavior of the entropy production, we concentrate in the case when the field inside the cavity is weakly perturbed by the mirrors in a periodically manner. In this moving-mirror setup the field mode function can be defined with the analogous form of Eq. (3.4)

$$f_n^{\text{out}}(x,t) = \sqrt{\frac{2}{L(t)}} \sin\left(\frac{n\pi}{L(t)}x\right) e^{-i\omega_{n,L(t)}t},$$

with an implicit time-dependence in the mode frequency $\omega_{n,L(t)}$, allowing us to directly obtain the parameter,

$$g_{n,m} \quad = \quad \begin{cases} (-1)^{m+n}\frac{2nm}{m^2-n^2}, & n \neq m \\ 0, & n = m \end{cases} \tag{3.15}$$

A convenient class of mirror trajectory to be chosen is when the second mirror satisfies the parametric equation

$$L(t) = L_0\left[1 + \epsilon\xi(t)\right],$$

where $\xi(t)$ is a periodic function and $\epsilon \ll 1$ is a small dimensionless number, needed to keep the field with well defined frequencies $\bar{\omega}_{n,L(t)} = \omega_n t$ as defined in (3.5).

In this regime, we directly obtain

$$\mathcal{N}_{n,m}(t) = \left\| \epsilon\chi_{n,m}^{+} \int_0^t dt' \xi(t') e^{-i(\omega_n+\omega_m)t'} \right\|^2.$$

We calculate the last expression for the case in which the mirror performs harmonic oscillations

$$\xi(t) = \sin\Omega_\mu t,$$

with an arbitrary frequency $\Omega_\mu = \mu\omega_1$, which is a real multiple μ of the first unperturbed field frequency ω_1. Therefore

$$\mathcal{N}_{n,m}(t) = \begin{cases} \frac{\epsilon^2\chi^{+2}_{n,m}}{8\omega^2}\left[2\omega^2t^2 + 2\omega t\sin(2\omega t) + \cos(2\omega t) - 1\right] & \text{for} \quad \Omega = \omega \\ \frac{\epsilon^2\chi^{+2}_{n,m}}{2\omega^2}\frac{\Omega^2}{\Omega^2-\omega^2}\left[1 + \frac{4\omega^2}{\Omega^2-\omega^2} + \frac{2\omega}{\Omega+\omega}\cos(\Omega+\omega)t - \frac{2\omega}{\Omega-\omega}\cos(\Omega-\omega)t - \cos 2\Omega t\right] & \text{for} \quad \Omega \neq \omega. \end{cases} \tag{3.16}$$

where $\Omega \equiv \Omega_\mu$ and $\omega \equiv \omega_n + \omega_m$.

As the diagonal terms of ρ are proportional to the last expression, we can interpret them as quantifying the different probabilities of finding the system in one of its energy eigenstates for different cavity frequencies Ω_μ. This makes clear that only at resonance, when the cavity oscillates at some unperturbed cavity field frequency, is when the amplification of the field energy happens. Out of the resonance frequencies, the probability of finding the system in different energy excitation oscillates very weakly in magnitude.

To show the adequacy of the latter formalism, we compute the number of particles and the diagonal entropy creation when the cavity oscillates in resonance with some unperturbed field frequency $\Omega_p = p\omega_1$ for $p = 1, 2, \ldots,$. Under the rotating wave approximation and up to second order, Eq. (3.16) becomes

$$N^p(t) = \frac{1}{6}p(p^2-1)\tau^2, \tag{3.17}$$

which is in accordance with literature [2]. Note that the above expression is valid under perturbation theory involving time and, therefore, it is a good approximation only when $\tau \ll 1$.

The diagonal entropy, our focus of interest here, takes the form

$$S^p_d(t) = \frac{1}{2}N^p(t)\left[1 - \ln\frac{1}{2}N^p(t) + \ln\frac{p(p^2-1)}{6} - \frac{6f(p)}{p(p^2-1)}\right].$$

where $f(p) = \sum_{m=1}^{p-1}(p-m)m\ln(p-m)m$. The first thing we note here is that it is proportional to the number of created particles. Therefore, it is a directly consequence of the DCE. If there are no particles being produced in the cavity, the entropy vanishes. It is important to observe that this holds when considering resonance and initial vacuum state. A more general discussion will be presented in the end of the article.

3.2.1 Discussion

The coherence generated in the energy basis can be measured with the help of the relative entropy of coherence [3]

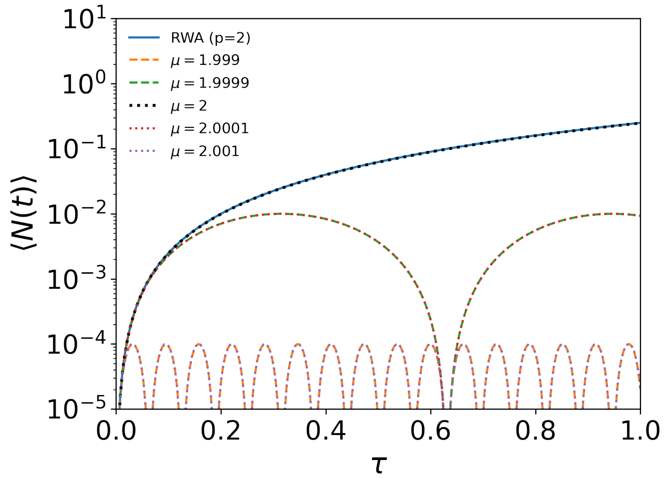

FIGURE 3.1: The average number of particles created inside the cavity as a function of time for different oscillating frequencies $\omega_\mu = \mu\omega_1$. The dotted lines represents numerical calculations of $\langle N(t) \rangle$ using equation (**??**) without approximations, whereas the continuous blue line uses the RWA approximation for resonance $p = 2$ in equation (3.17).

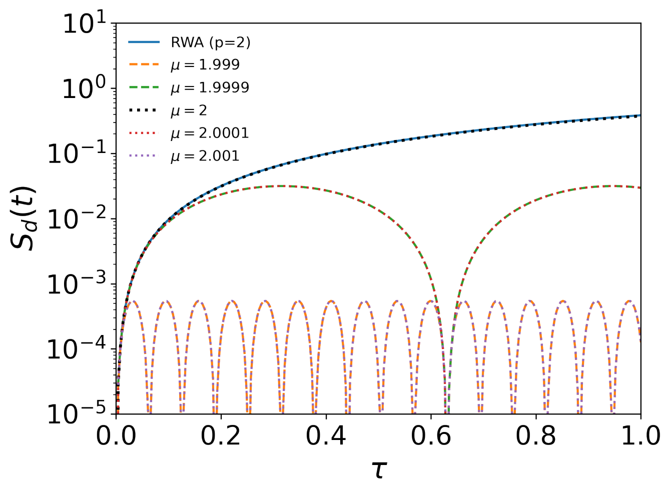

FIGURE 3.2: The diagonal entropy production inside the cavity as a function of time for different oscillating frequencies $\omega_p = p\pi/L_0$.

$$C(\rho) = S_d(\rho_{\text{diag}}) - S(\rho),$$

which in our case is identical to the diagonal entropy since $S(\rho) = 0$ for a closed system.

Another interesting concept to be analyzed is that of the non-adiabatic work performed by the system due to the unwanted energy levels transitions (typically associated with heat), the so called inner friction [4]

$$\langle w_{\text{fric}} \rangle = \frac{1}{\beta_A} D\left(\rho_\tau || \rho_A\right),$$

which is the quantum relative entropy between the actual process (unitary evolution to ρ_τ) to the ideal one (adiabatic reversible transformation to ρ_A).

Chapter 4

Instantaneous basis decomposition

Another approach to study the DCE in the one-dimensional case by expanding the modes defined in equation (H.1) in a series with respect to the instantaneous basis

$$f_n^{\text{out}}(x,t) = \sqrt{\frac{2}{L(t)}} \sum_{k=1}^{\infty} \sin\left(\frac{\pi k}{L(t)} x\right) Q_k^{(n)}(t),$$

with the initial conditions

$$Q_k^{(n)}(0) = \delta_{kn}, \qquad \dot{Q}_k^{(n)}(t) = -i\omega_n \delta_{kn}, \qquad k, n = 1, 2, \ldots.$$

From the restrictions imposed by the wave equation on the instantaneous coefficient we obtain the set of coupled differential equations

$$\ddot{Q}_k^{(n)} + \omega_k^2(t) Q_k^{(n)} = 2\sum_{j=1}^{\infty} G_{kj}(t)\dot{Q}_j^{(n)} + \sum_{j=1}^{\infty} \dot{G}_{kj} Q_k^{(n)} + \mathcal{O}(G_{kj}^2) \tag{4.1}$$

where $\omega_k(t) = k\pi/L(t)$ and the coefficients $G_{jk}(t)$ (for $j \neq k$)

$$G_{jk} = -G_{kj} = (-1)^{k-j} \frac{2kj}{(j^2 - k^2)} \frac{\dot{L}(t)}{L(t)}.$$

If the wall returns to its initial position $x = L_0$ after some interval of time T then

$$Q_k^{(n)}(t) = \alpha_{nk} e^{-\omega_k t} + \beta_{nk} e^{\omega_k t}, \qquad k, n = 1, 2, \ldots. \tag{4.2}$$

A simplification for the problem occurs for the case in which the mirror perform small oscillations at the frequency of some unperturbed field eigenfrequency $\omega_p = p\omega_1$

$$L(t) = L_0 \left[1 + \epsilon \cos\left(p\omega_1 t\right)\right], \qquad (p = 1, 2, \ldots).$$

Inserting expression (H.4) into (H.2), we obtain a set of differential equations from which is possible to extract the Bogoluibov coefficients α_{nm} and β_{nm} by assuming that they vary slowly in time. Despite their complicated dependence in terms of hypergeometrical function, we can write the first two Bogoliubov coefficients in terms of the elliptical integral as

$$\alpha_{11} = \frac{2}{\pi}\mathbf{E}(\kappa)$$
$$\beta_{11} = \frac{2}{\pi\kappa}\left[(1-\kappa^2)\mathbf{K}(\kappa) - \mathbf{E}(\kappa)\right],$$

where the principal mode index p appears implicitly in $\kappa = K(p\tau)/\sqrt{1-K^2(p\tau)}$ with $K(\tau) = \sinh(\tau)$.

In order to study the quantum correlations between the particles created inside the cavity at a given field mode m (subsystem A) and the rest of the system (complement B), we shall compute the diagonal entropy S_d in terms of the reduced state

$$\hat{\rho}_A^{(m)} = \mathrm{Tr}_B\{\hat{\rho}\}, \tag{4.3}$$

where we trace out the system's density operator with respect to the subsystem B.

To compute the last expression, we can attend to two key points. The first refers to the fact that, as seen in the first part, the time evolution of moving cavity systems can be described in the Schrodinger picture with the help of a quadratic multidimensional time-dependent Hamiltonian. The second key point is the fact that the time evolution governed by quadratic Hamiltonians transforms any Gaussian state to another Gaussian state.

As the vacuum state belongs to this wider class of Gaussian states, it is in fact possible to describe our initial state in terms of the Wigner-Weyl formulation of quantum mechanics, with a simple distribution function for the m-th mode

$$W_m(\mathbf{q}) = \frac{1}{\sqrt{2\pi \det \mathbf{\Sigma}_m}} e^{-\frac{1}{2}(\mathbf{q}-\langle\mathbf{q}\rangle)\mathbf{\Sigma}_m^{-1}(\mathbf{q}-\langle\mathbf{q}\rangle)},$$

which depends only on the average values of the quadrature operators $\mathbf{q} = (\hat{q}_m, \hat{p}_m)$ and their variances which defines a symmetric matrix

$$\mathbf{\Sigma}_m \equiv \begin{pmatrix} \sigma_m^q & \sigma_m^{qp} \\ \sigma_m^{qp} & \sigma_m^p \end{pmatrix}$$

with elements

$$\sigma_m^q = \langle \hat{q}_m^2 \rangle - \langle \hat{q}_m \rangle^2, \qquad \sigma_m^p = \langle \hat{p}_m^2 \rangle - \langle \hat{p}_m \rangle^2,$$
$$\sigma_m^{qp} = \frac{1}{2}\langle \hat{p}_m\hat{q}_m + \hat{q}_m\hat{p}_m \rangle - \langle \hat{q}_m\hat{p}_m \rangle.$$

For the initial vacuum state where all average values $\langle \hat{q}_m \rangle = \langle \hat{p}_m \rangle = 0$, the diagonal components of the density operator can be calculated in terms of the quadrature variances with the help of the formula

[2]

$$\rho_m^{(nn)} = \frac{2\left[\left(2\sigma_m^q - 1\right)\left(2\sigma_m^p - 1\right)\right]^{n/2}}{\left[\left(2\sigma_m^q + 1\right)\left(2\sigma_m^p + 1\right)\right]^{(n+1)/2}} \times \mathrm{P}_n\left(\frac{4\sigma_m^q\sigma_m^p - 1}{\sqrt{(4(\sigma_m^q)^2 - 1)(4(\sigma_m^p)^2 - 1)}}\right),$$

where P_n is the Legendre polynomial of order n and $\rho_m^{(nn)} = \langle n | \hat{\rho}_A^{(m)} | n \rangle$ are the diagonal terms of the density operator (4.3) in the initial energy eigenstate basis. For the special case of parametric oscillation, its possible to relate the quadrature variances with the Bogoliubov coefficients as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_m^q = -m\left[\alpha_{1m} + \beta_{1m}\right]^2$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_m^p = +m\left[\alpha_{1m} - \beta_{1m}\right]^2.$$

where $m = 2\mu + 1$ with $\mu = 1, 2, \ldots$. From the above expression its possible to find their Taylor expansions

$$\left.\begin{array}{l}\sigma_{2m+1}^q\,|_{\tau\to 0}\\ \sigma_{2m+1}^p\,|_{\tau\to 0}\end{array}\right\} = \frac{1}{2} \mp \tau^{2m+1}\left[\frac{(2m-1)!!}{m!}\right]^2 \times\left[1 \mp \frac{2m+1}{(m+1)^2}\tau + \mathcal{O}(\tau^2)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_{2m+1}^q\,|_{\tau\to\infty} = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_{2m+1}^p\,|_{\tau\to\infty} = \frac{16}{\pi^2(2m+1)}$$

For the first mode of the field ($m = 1$), in the short time limit $\tau \ll 1$, the expression for the variances are $\sigma_1^q = \frac{1}{2}\left(1 - 2\tau + 2\tau^2 + \mathcal{O}(\tau^3)\right)$ and $\sigma_1^p = \frac{1}{2}\left(1 + 2\tau + 2\tau^2 + \mathcal{O}(\tau^3)\right)$, so the density matrix takes the form of

$$\rho_1^{(nn)}(\tau \ll 1) = (-1)^{n/2}\tau^n\left(1 - \frac{2n+2}{2}\tau^2 + \mathcal{O}(\tau^3)\right)P_n(0)$$
$$= \frac{(2k)!}{(2^k k!)^2}\tau^{2k}\left(1 - \frac{k+1}{2}\tau^2\right)$$

where $P_{2k+1}(0) = 0$ and $P_{2k} = (-1)^k(2k)!/2^{2k}(k!)^2$ for $k = 0, 1, 2, \ldots$.

In this case the diagonal entropy for the first mode can be computed as

$$S_{d,1}(\tau \ll 1) = -\sum_n \rho_1^{(nn)} \ln \rho_1^{(nn)} = \\ = \frac{1}{2}\tau^2\left(1 - \ln\frac{1}{2}\tau^2\right) + \mathcal{O}(\tau^4).$$

For the long-time limit $\tau \gg 1$ the system's quadrature variances take the asymptotic behaviour $\sigma_1^q \to 2/\pi^2$ and $\sigma_1^p \to 16\tau/\pi^2$ [2] so our density operator takes the form of

$$\rho_1^{(nn)}(\tau \gg 1) = eC_n[\Sigma_1(\tau)]^{-1/2} + \mathcal{O}(1/\tau) \tag{4.4}$$

with $\Sigma_1(\tau) \equiv \det\{\mathbf{\Sigma}_1\}$ at $\tau \gg 1$, e is Euler's number and

$$C_n = \frac{1}{e\sqrt{1+a}}\left(\frac{1-a}{\sqrt{1-a^2}}\right)^n \mathrm{P}_n\left(\frac{1}{\sqrt{1-a^2}}\right) \text{ with } a = \frac{1}{2\sigma_q^{(1)}}.$$

is a positive real coefficient. Using the last definition, the correspondent diagonal entropy can now be written as

$$S_d^{(1)}(\tau \gg 1) \approx S_A^{(1)}(\tau) + \mathcal{S}(C_n)[\Sigma_1(\tau)]^{-1/2},$$

where $\mathcal{S}(C_n) := -\sum_n C_n \ln(C_n)$ and $S_A^{(1)}(\tau) = 1 + \frac{1}{2}\ln\frac{32}{\pi^4}\tau$ is the entanglement entropy at long times between the subsystem A for $m = 1$ and the rest of the field modes, as obtained in Ref. [**romualdo**]. Using the integral test for convergence, we find the inequality $\mathcal{S}(C_n)\Sigma_1^{-1/2} \leq \frac{1}{2}\ln \pi N - 1$ for $N \to \infty$, meaning that the last term diverges logarithmically.

Chapter 5

Let us put gravity

Considering

$$\hat{H}_{\text{eff}} = -\frac{i}{2}\sum_{j,k}\left[\mathcal{B}^*_{kj}(t)\hat{b}^\dagger_k(t)\hat{b}^\dagger_j(t) - \mathcal{B}_{kj}(t)\hat{b}_k(t)\hat{b}_j(t) + \mathcal{A}^*_{kj}(t)\hat{b}^\dagger_k(t)\hat{b}_j(t) - \mathcal{A}_{kj}(t)\hat{b}^\dagger_j(t)\hat{b}_k(t)\right]. \tag{5.1}$$

so

$$\int_0^t dt'\hat{H}_I(t')\hat{\rho}_0 = -\frac{i}{2}\sum_{kj}\beta^*_{kj}\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}_0 \tag{5.2}$$

$$\hat{\rho}_0\int_0^t dt'\hat{H}_I(t') = \frac{i}{2}\sum_{kj}\beta_{kj}\hat{\rho}_0\hat{b}_k\hat{b}_j \tag{5.3}$$

thus

$$\int_0^t dt'\left[\hat{H}_I(t'),\hat{\rho}(0)\right] = -\frac{i}{2}\sum_{kj}\left(\beta^*_{kj}\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}_0 + \beta_{kj}\hat{\rho}_0\hat{b}_k\hat{b}_j\right) \tag{5.4}$$

Now to calculate the second term in (A.1) we consider the first order term in (2.5) and expression (5.4)

$$\begin{aligned}
&\int_0^t dt''\int_0^t dt'\left[H(t''),\left[\hat{H}_I(t'),\rho(0)\right]\right] \\
&= \int_0^t dt''H(t'')\int_0^t dt'\left[\hat{H}_I(t'),\rho(0)\right] - \int_0^t dt'\left[\hat{H}_I(t'),\rho(0)\right]\int_0^t dt''H(t'')
\end{aligned}$$

which can be expressed as

$$\begin{aligned}
&= -\frac{i}{2}\sum_{nl}\left[\beta^*_{ln}\hat{b}^\dagger_l\hat{b}^\dagger_n - \beta_{ln}\hat{b}_l\hat{b}_n + \alpha^*_{ln}\hat{b}^\dagger_l\hat{b}_n - \alpha_{ln}\hat{b}^\dagger_n\hat{b}_l\right]\int_0^{t''} dt'\left[\hat{H}_I(t'),\hat{\rho}(0)\right] \\
&+ \frac{i}{2}\sum_{nl}\int_0^t dt'\left[\hat{H}_I(t'),\hat{\rho}(0)\right]\left[\beta^*_{ln}\hat{b}^\dagger_l\hat{b}^\dagger_n - \beta_{ln}\hat{b}_l\hat{b}_n + \alpha^*_{ln}\hat{b}^\dagger_l\hat{b}_n - \alpha_{ln}\hat{b}^\dagger_n\hat{b}_l\right]
\end{aligned}$$

$$= \frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{kj}\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)+\beta_{kj}^*\hat{\rho}(0)\hat{b}_k\hat{b}_j\right]\left[\beta_{ln}\hat{b}_l^\dagger\hat{b}_n^\dagger-\beta_{ln}^*\hat{b}_l\hat{b}_n+\alpha_{ln}\hat{b}_l^\dagger\hat{b}_n-\alpha_{ln}^*\hat{b}_n^\dagger\hat{b}_l\right]$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{ln}\hat{b}_l^\dagger\hat{b}_n^\dagger-\beta_{ln}^*\hat{b}_l\hat{b}_n+\alpha_{ln}\hat{b}_l^\dagger\hat{b}_n-\alpha_{ln}^*\hat{b}_n^\dagger\hat{b}_l\right]\left[\beta_{kj}\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)+\beta_{kj}^*\hat{\rho}(0)\hat{b}_k\hat{b}_j\right]$$

$$= \frac{1}{4}\sum_{kj}\sum_{nl}\beta_{kj}\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\left[\beta_{ln}\hat{b}_l^\dagger\hat{b}_n^\dagger-\beta_{ln}^*\hat{b}_l\hat{b}_n+\alpha_{ln}\hat{b}_l^\dagger\hat{b}_n-\alpha_{ln}^*\hat{b}_n^\dagger\hat{b}_l\right]$$
$$+\frac{1}{4}\sum_{kj}\sum_{nl}\beta_{kj}^*\hat{\rho}(0)\hat{b}_k\hat{b}_j\left[\beta_{ln}\hat{b}_l^\dagger\hat{b}_n^\dagger-\beta_{ln}^*\hat{b}_l\hat{b}_n+\alpha_{ln}\hat{b}_l^\dagger\hat{b}_n-\alpha_{ln}^*\hat{b}_n^\dagger\hat{b}_l\right]$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{ln}\hat{b}_l^\dagger\hat{b}_n^\dagger-\beta_{ln}^*\hat{b}_l\hat{b}_n+\alpha_{ln}\hat{b}_l^\dagger\hat{b}_n-\alpha_{ln}^*\hat{b}_n^\dagger\hat{b}_l\right]\beta_{kj}\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{ln}\hat{b}_l^\dagger\hat{b}_n^\dagger-\beta_{ln}^*\hat{b}_l\hat{b}_n+\alpha_{ln}\hat{b}_l^\dagger\hat{b}_n-\alpha_{ln}^*\hat{b}_n^\dagger\hat{b}_l\right]\beta_{kj}^*\hat{\rho}(0)\hat{b}_k\hat{b}_j$$

$$= \frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{kj}\beta_{ln}\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\hat{b}_l^\dagger\hat{b}_n^\dagger-\beta_{kj}\beta_{ln}^*\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\hat{b}_l\hat{b}_n+\beta_{kj}\alpha_{ln}\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\hat{b}_l^\dagger\hat{b}_n-\beta_{kj}\alpha_{ln}^*\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\hat{b}_n^\dagger\hat{b}_l\right]$$
$$+\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{kj}^*\beta_{ln}\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_l^\dagger\hat{b}_n^\dagger-\beta_{kj}^*\beta_{ln}^*\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_l\hat{b}_n+\beta_{kj}^*\alpha_{ln}\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_l^\dagger\hat{b}_n-\beta_{kj}^*\alpha_{ln}^*\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_n^\dagger\hat{b}_l\right]$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{ln}\beta_{kj}\hat{b}_l^\dagger\hat{b}_n^\dagger\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)-\beta_{ln}^*\beta_{kj}\hat{b}_l\hat{b}_n\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)+\alpha_{ln}\beta_{kj}\hat{b}_l^\dagger\hat{b}_n\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)-\alpha_{ln}^*\beta_{kj}\hat{b}_n^\dagger\hat{b}_l\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\right]$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{ln}\beta_{kj}^*\hat{b}_l^\dagger\hat{b}_n^\dagger\hat{\rho}(0)\hat{b}_k\hat{b}_j-\beta_{ln}^*\beta_{kj}^*\hat{b}_l\hat{b}_n\hat{\rho}(0)\hat{b}_k\hat{b}_j+\alpha_{ln}\beta_{kj}^*\hat{b}_l^\dagger\hat{b}_n\hat{\rho}(0)\hat{b}_k\hat{b}_j-\alpha_{ln}^*\beta_{kj}^*\hat{b}_n^\dagger\hat{b}_l\hat{\rho}(0)\hat{b}_k\hat{b}_j\right]$$

$$= \frac{1}{4}\sum_{kj}\sum_{nl}\left[-\beta_{kj}\beta_{ln}^*\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\hat{b}_l\hat{b}_n\right]$$
$$+\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{kj}^*\beta_{ln}\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_l^\dagger\hat{b}_n^\dagger-\beta_{kj}^*\beta_{ln}^*\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_l\hat{b}_n+\beta_{kj}^*\alpha_{ln}\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_l^\dagger\hat{b}_n-\beta_{kj}^*\alpha_{ln}^*\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_n^\dagger\hat{b}_l\right]$$
$$\frac{1}{4}\sum_{kj}\sum_{nl}\left[-\beta_{ln}\beta_{kj}\hat{b}_l^\dagger\hat{b}_n^\dagger\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)+\beta_{ln}^*\beta_{kj}\hat{b}_l\hat{b}_n\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)-\alpha_{ln}\beta_{kj}\hat{b}_l^\dagger\hat{b}_n\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)+\alpha_{ln}^*\beta_{kj}\hat{b}_n^\dagger\hat{b}_l\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\right]$$
$$\frac{1}{4}\sum_{kj}\sum_{nl}\left[-\beta_{ln}\beta_{kj}^*\hat{b}_l^\dagger\hat{b}_n^\dagger\hat{\rho}(0)\hat{b}_k\hat{b}_j\right]$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{kj}\sum_{nl}\Bigg[-\beta_{kj}\beta^*_{ln}\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0)\hat{b}_l\hat{b}_n \\
&+ \beta^*_{kj}\beta_{ln}\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}^\dagger_l\hat{b}^\dagger_n - \beta^*_{kj}\beta^*_{ln}\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_l\hat{b}_n + \beta^*_{kj}\alpha_{ln}\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}^\dagger_l\hat{b}_n - \beta^*_{kj}\alpha^*_{ln}\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}^\dagger_n\hat{b}_l \\
&- \beta_{ln}\beta_{kj}\hat{b}^\dagger_l\hat{b}^\dagger_n\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0) + \beta^*_{ln}\beta_{kj}\hat{b}_l\hat{b}_n\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0) - \alpha_{ln}\beta_{kj}\hat{b}^\dagger_l\hat{b}_n\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0) + \alpha^*_{ln}\beta_{kj}\hat{b}^\dagger_n\hat{b}_l\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0) \\
&- \beta_{ln}\beta^*_{kj}\hat{b}^\dagger_l\hat{b}^\dagger_n\hat{\rho}(0)\hat{b}_k\hat{b}_j\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{4}\sum_{kj}\sum_{nl}\Bigg[&\beta^*_{ln}\beta_{kj}\left(\hat{b}_l\hat{b}_n\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0)\right) + \beta^*_{kj}\beta_{ln}\left(\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}^\dagger_l\hat{b}^\dagger_n\right) \\
&+ \alpha^*_{ln}\beta_{kj}\left(\hat{b}^\dagger_n\hat{b}_l\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0)\right) + \beta^*_{kj}\alpha_{ln}\left(\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}^\dagger_l\hat{b}_n\right) \\
&- \beta_{ln}\beta_{kj}\left(\hat{b}^\dagger_l\hat{b}^\dagger_n\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0)\right) - \beta^*_{kj}\beta^*_{ln}\left(\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_l\hat{b}_n\right) \\
&- \alpha_{ln}\beta_{kj}\left(\hat{b}^\dagger_l\hat{b}_n\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0)\right) - \beta^*_{kj}\alpha^*_{ln}\left(\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}^\dagger_n\hat{b}_l\right) \\
&- \beta_{kj}\beta^*_{ln}\left(\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0)\hat{b}_l\hat{b}_n\right) - \beta_{ln}\beta^*_{kj}\left(\hat{b}^\dagger_l\hat{b}^\dagger_n\hat{\rho}(0)\hat{b}_k\hat{b}_j\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{4}\sum_{kj}\sum_{nl}\Bigg[&\beta^*_{ln}\beta_{kj}\left(\hat{b}_l\hat{b}_n\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0)\right) - \beta_{kj}\beta^*_{ln}\left(\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0)\hat{b}_l\hat{b}_n\right) + \alpha^*_{ln}\beta_{kj}\left(\hat{b}^\dagger_n\hat{b}_l\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0)\right) \\
&- \beta_{ln}\beta_{kj}\left(\hat{b}^\dagger_l\hat{b}^\dagger_n\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0)\right) - \alpha_{ln}\beta_{kj}\left(\hat{b}^\dagger_l\hat{b}_n\hat{b}^\dagger_k\hat{b}^\dagger_j\hat{\rho}(0)\right) + \text{h.c.}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho}(t) = \hat{\rho}(0) - \frac{1}{2}\sum_{kj}&\left[\beta_{kj}\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0) + \beta_{kj}^*\hat{\rho}(0)\hat{b}_k\hat{b}_j\right] \\
-\frac{1}{8}\sum_{kj}\sum_{nl}&\Bigg[\beta_{ln}^*\beta_{kj}\left(\hat{b}_l\hat{b}_n\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\right) + \beta_{kj}^*\beta_{ln}\left(\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_l^\dagger\hat{b}_n^\dagger\right) \\
&- \beta_{kj}\beta_{ln}^*\left(\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\hat{b}_l\hat{b}_n\right) - \beta_{ln}\beta_{kj}^*\left(\hat{b}_l^\dagger\hat{b}_n^\dagger\hat{\rho}(0)\hat{b}_k\hat{b}_j\right) \\
&+ \alpha_{ln}^*\beta_{kj}\left(\hat{b}_n^\dagger\hat{b}_l\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\right) + \beta_{kj}^*\alpha_{ln}\left(\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_l^\dagger\hat{b}_n\right) \\
&- \beta_{ln}\beta_{kj}\left(\hat{b}_l^\dagger\hat{b}_n^\dagger\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\right) - \beta_{kj}^*\beta_{ln}^*\left(\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_l\hat{b}_n\right) \\
&- \alpha_{ln}\beta_{kj}\left(\hat{b}_l^\dagger\hat{b}_n\hat{b}_k^\dagger\hat{b}_j^\dagger\hat{\rho}(0)\right) - \beta_{kj}^*\alpha_{ln}^*\left(\hat{\rho}(0)\hat{b}_k\hat{b}_j\hat{b}_n^\dagger\hat{b}_l\right)\Bigg]
\end{aligned}
$$

5.1 The theory

Let us consider a real massless scalar field $\phi(x^\mu)$ in a fixed curved background whose spacetime structure is described by a $(1+1)$-dimensional manifold $\mathcal{M}$, with a Lorentzian metric $g_{\mu\nu}$. The correspondent field action

$$S = -\frac{1}{2}\int d^2x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{5.5}$$

when taken to be stationary, it gives rise to the Klein-Gordon equation in curved spacetime

$$\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\right)\phi(x^\mu) = 0, \tag{5.6}$$

where we can define

$$\pi(x^\mu) = -\sqrt{-g}g^{00}\partial_0\hat{\phi}(x^\mu).$$

as the field conjugated momentum.

Introducing a reference frame along the radial coordinate r with origin on the fixed mirror at $r = r_0$, the second mirror is then allow to move in the prescribed trajectory $r(t) = r_0 + L(t)$. As a next step, we expand the scalar field in terms of a complete and orthonormal set of instantaneous mode function $\{\varphi_{k;t}(r)\}$ with eigenfrequencies $\omega_{k;t}$ such that

$$
\begin{aligned}
\hat{\phi}(r,t) &= \sum_k \frac{1}{\sqrt{2\omega_{k;t}}}\left[\hat{b}_{k;t} + \hat{b}_{k;t}^\dagger\right]\varphi_{k;t}(r) \\
\hat{\pi}(r,t) &= i\sqrt{-g}g^{00}\sum_k\sqrt{\frac{\omega_{k;t}}{2}}\left[\hat{b}_{k;t} - \hat{b}_{k;t}^\dagger\right]\varphi_{k;t}(r)
\end{aligned}
$$

The mode function obeys the differential equation

$$\left[-\sqrt{-g}g^{tt}\omega_{k;t}^2 + \partial_r\sqrt{-g}g^{rr}\partial_r\right]\varphi_{k;t}(r) = 0 \tag{5.7}$$

normalized by the inner product

$$-\int_{r_0}^{r_0+L(t)} dr\sqrt{-g}g^{tt}\varphi_{j;t}(r)\varphi_{k;t}(r) = \delta_{k,j}.$$

We can obtain an analogous Hamiltonian, with coefficients

$$\mu_{kj}(t) = \sqrt{\frac{\omega_{k;t}}{\omega_{j;t}}}G_{kj} - \frac{1}{2}\frac{\dot{\omega}_{j;t}}{\omega_{j;t}}\delta_{kj}$$

and

$$G_{kj} = \int_{r_0}^{r_0+L(t)} dr\sqrt{-g}g^{tt}\varphi_{k;t}(r)\partial_t\varphi_{j;t}(r).$$

Here we will study the case in which the pair of mirrors oscillates with a proper length

$$L_p(t) = L_{p,0} + \tilde{A}\sin\mu\omega_1 t$$

while is embedded in the spacetime curvature of the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{r_S}{r}\right)dt^2 + \left(1 - \frac{r_S}{r}\right)^{-1}dr^2$$

with $r_S = 2GM$.

To calculate the expression for G_{kj} we switch to the tortoise coordinates $r^* = r + r_s\ln\left|\frac{r}{r_s} - 1\right|$, where the metric reads

$$ds^2 = \left(1 - \frac{r_S}{r}\right)\left(dt^2 - dr^{*2}\right),$$

and by being conformally symmetric with the Minkowski metric, the differential equation (5.7) for the mode function $\varphi_{k;t}$ in terms of r^* is simply

$$\left(\partial_{r^*}^2 - \omega_{k;t}^2\right)\varphi_{k;t}(r^*) = 0.$$

with solutions

$$\varphi_{k;t}(r^*) = \sqrt{\frac{2}{L(t)}}\sin\left[\frac{k\pi}{L(t)}(r^* - r_0^*)\right]$$

where $r_0^* = r_0 + r_S\ln|\frac{r_0}{r_S} - 1|$.

With this at hands we can now calculate G_{kj} as

$$G_{kj} = \int_{r_0^*}^{r_1^*} dr^* \varphi_{k;t}(r^*) \partial_t \varphi_{j;t}(r^*)$$

with $r_1^* = r_0 + L(t) + r_S \ln |\frac{r_0+L(t)}{r_S} - 1|$. Therefore

$$\begin{aligned}
G_{kj} &= -\frac{\dot{L}}{L} \int_{r_0^*}^{r_1^*} dr^* \varphi_k \left\{ \frac{1}{2}\varphi_j + k\pi(r^* - r_0^*)\sqrt{\frac{2}{L}} \cos\left[\frac{k\pi}{L}(r^* - r_0^*)\right] \right\} \\
&= -\frac{\dot{L}}{2L} \int_{r_0^*}^{r_1^*} dr^* \varphi_k \varphi_j + k\pi - \frac{2k\pi}{L^2}\frac{\dot{L}}{L} \int_0^{r_1^* - r_0^*} du\, u \sin\left[\frac{k\pi}{L}u\right] \cos\left[\frac{k\pi}{L}u\right] \\
&= -\frac{n\dot{L}}{\pi L^2} \left[\frac{\pi F}{j-k} \cos\left(\frac{\pi F(j-k)}{L}\right) - \frac{\pi F}{j+k} \cos\left(\frac{\pi F(j+k)}{L}\right) \right. \\
&\left. - \frac{L}{(j-k)^2} \sin\left(\frac{\pi F(j-k)}{L}\right) + \frac{L}{(j+k)^2} \sin\left(\frac{\pi F(j+k)}{L}\right) \right]
\end{aligned}$$

with $F = r^* - r_0^* = L + \Lambda$ and $\Lambda = r_s \ln \frac{L+r_0}{r_0} \frac{f(L+r_0)}{f(r_0)}$.

Using

$$\begin{aligned}
\cos\frac{\pi(j \pm k)}{L} F &= \cos\left[\pi(j \pm k) + \frac{\pi(j \pm k)}{L}\Lambda\right] \\
&= (-1)^{j \pm k} \cos\frac{\pi(j \pm k)}{L}\Lambda
\end{aligned}$$

$$\begin{aligned}
\sin\frac{\pi(j \pm k)}{L} F &= \sin\left[\pi(j \pm k) + \frac{\pi(j \pm k)}{L}\Lambda\right] \\
&= -(-1)^{j \pm k} \sin\frac{\pi(j \pm k)}{L}\Lambda
\end{aligned}$$

Chapter 6

Instantaneous basis decomposition

Let us consider a real, massless and quantum scalar field $\hat{\phi}(x,t)$ respecting the wave equation

$$\nabla^2\hat{\phi}(x,t) - \partial_t^2\hat{\phi}(x,t) = 0, \tag{6.1}$$

and the boundary conditions $\hat{\phi}(0,t) = \hat{\phi}(L(t),t) = 0$. Expanding the field in terms of

$$\hat{\phi}(x,t) = \sum_n \left[\psi_n(x,t)\hat{b}_n + \psi_n^*(x,t)\hat{b}_n^\dagger\right]$$

with the mode functions $\psi_n(x,t)$ expressed in terms of a series with respect to the instantaneous basis

$$\psi_n(x,t) = \frac{1}{\sqrt{2\omega_n(t)}}\sum_k \varphi_k(x;t)Q_k^{(n)}(t) \tag{6.2}$$

with $\omega_n(t) = k\pi/L(t)$ and the basis function is defined such that respect the equation of motion

$$\nabla^2\varphi_k(x;t) + \omega_k^2(t)\varphi_k(x;t) = 0 \tag{6.3}$$

and the normalization condition $\int_0^{L(t)}\varphi_k(x;t)\varphi_j(x;t) = \delta_{k,j}$.

Considering that the cavity is static at $t<0$, such that the field have standing waves mode functions $\psi(x,t<0) = \frac{1}{\sqrt{\pi n}}\sin(\omega_n t)e^{-i\omega_n t}$, we expect the following initial conditions

$$Q_k^{(n)}(0) = \delta_{k,n} \qquad\qquad \dot{Q}_k^{(n)}(0) = -i\omega_n\delta_{k,n}.$$

Putting (6.2) into the field dynamical equations (6.1) (with the help of expression (6.3)) we can recover

$$\begin{aligned}
&\ddot{\psi}_n + \omega_k^2\psi_n = 0\\
&\sum_k\left\{\left[\ddot{Q}_k^{(n)} + \omega_k^2(t)Q_k^{(n)}\right]\varphi_k + 2\dot{\varphi}_k\dot{Q}_k^{(n)} + Q_k^{(n)}\ddot{\varphi}_k\right\} = 0.
\end{aligned}$$

Multiplying the last expression by φ_j and integrating over 0 and $L_(t)$, we obtain

$$\ddot{Q}_k^{(n)} + \omega_k^2(t) Q_k^{(n)} = -2\sum_j G_{j,k}\dot{Q}_j^{(n)} - \sum_j H_{j,k} Q_j^{(n)}$$

with

$$\begin{aligned}
G_{j,k} &= -\int_0^{L(t)} dx\ \varphi_k \partial_t \varphi_j \\
H_{j,k} &= -\int_0^{L(t)} dx\ \varphi_k \partial_t^2 \varphi_j = \partial_t G_{j,k} + \sum_s G_{js} G_{ks}.
\end{aligned}$$

Given origin to the set of coupled differential equations

$$\ddot{Q}_k^{(n)} + \omega_k^2(t) Q_k^{(n)} = 2\sum_j G_{kj}\dot{Q}_j^{(n)} + \sum_j \dot{G}_{kj} Q_k^{(n)} + \mathcal{O}(G_{kj}^2) \tag{6.4}$$

Searching for solutions of the form

$$Q_k^{(n)}(t) = \sqrt{\frac{\omega_n(t)}{\omega_k(t)}}\left[\alpha_{nk}(t) e^{-i\Omega_k(t)} + \beta_{nk}(t) e^{i\Omega_k(t)}\right]$$

where $\Omega_k(t) = \int_0^t \omega_k(t')dt'$, we can write the (6.2) as

$$\begin{aligned}
\hat{\phi}(x,t) &= \sum_n \frac{1}{\sqrt{2\omega_n(t)}}\left[\sum_k \varphi_k(x;t)\sqrt{\frac{\omega_n(t)}{\omega_k(t)}}\left[\alpha_{nk}(t)e^{-i\Omega_k(t)} + \beta_{nk}(t)e^{i\Omega_k(t)}\right]\hat{b}_n + \text{h.c}\right], \\
&= \sum_k \frac{1}{\sqrt{2\omega_k(t)}}\left[\sum_n \left[\alpha_{nk}(t)\hat{b}_n + \beta_{nk}(t)\hat{b}_n^\dagger\right]\varphi_k(x;t)e^{-i\Omega_k(t)} + \text{h.c.}\right] \\
&= \sum_k \frac{1}{\sqrt{2\omega_k(t)}}\left[\varphi_k(x;t)e^{-i\Omega_k(t)}\hat{b}_k + \text{h.c.}\right]
\end{aligned}$$

where we identify the set of instantaneous creation and annihilation operators $\hat{b}_k^\dagger(t)$ and $\hat{b}_k(t)$ as connected with the operator $\hat{b}_k^\dagger\ \hat{b}_k$ (which correspond to the particle notion for $t < 0$) through a Bogoliubov transformations

$$\hat{b}_k(t) = \sum_n \left[\alpha_{nk}(t)\hat{b}_n + \beta_{nk}(t)\hat{b}_n^\dagger\right]$$

$$\alpha_{nk} = \sqrt{\frac{\omega_k}{\omega_n}} A_k^{(n)}$$
$$\beta_{nk} = \sqrt{\frac{\omega_k}{\omega_n}} B_k^{(n)}$$

The ladder operators $\hat{b}_n^{\text{in}}$ and $\hat{b}_n^{\text{in}\dagger}$ correspond to the particle notion defined in the "in" region ($t < 0$) whereas the operators $\hat{b}_n^{\text{out}}$ and $\hat{b}_n^{\text{out}\dagger}$ are defined for the "out" region ($t > T$). According with the Bogoliubov transformations both sets of creation and annihilation operators (3.3) and (3.1) are related by

$$\hat{b}_m^{\text{out}} = \sum_n \left(\alpha_{nm} \hat{b}_n^{\text{in}} + \beta_{nm}^* \hat{b}_n^{\text{in}\dagger} \right) . \tag{6.5}$$

Supposing the wall returns to its initial position $x = L_0$ after some interval of time T, the right hand size of the equation (H.2) vanishes and we must obtain solutions with the form

$$Q_k^{(n)}(t > T) = A_k^{(n)} e^{i\omega_k t} + B_k^{(n)} e^{-i\omega_k t}, \qquad k, n = 1, 2, \dots . \tag{6.6}$$

where $A_k^{(n)}$ and $B_k^{(n)}$ are constant coefficients to be later determined.

Let

$$\hat{\phi}^{\text{in}}(x,t) = \sum_n \frac{1}{\sqrt{2\omega_n}} \left[\varphi_n e^{-i\omega_n t} \hat{b}_n^{\text{in}} + \text{h.c.} \right] , \tag{6.7}$$

$$\begin{aligned}
\hat{\phi}(x,t) &= \sum_n \frac{1}{\sqrt{2\omega_n}} \left[\sum_k \varphi_k \left(A_k^{(n)} e^{i\omega_k t} + B_k^{(n)} e^{-i\omega_k t} \right) \hat{b}_n^{\text{in}} + \text{h.c} \right] , \\
&= \sum_k \frac{1}{\sqrt{2\omega_k}} \left[\sum_n \left(\sqrt{\frac{\omega_k}{\omega_n}} A_k^{(n)} \hat{b}_n^{\text{in}} + \sqrt{\frac{\omega_k}{\omega_n}} B_k^{(n)*} \hat{b}_n^{\text{in}\dagger} \right) \varphi_k e^{i\omega_k t} + \text{h.c.} \right] \\
&= \sum_k \frac{1}{\sqrt{2\omega_k}} \left[\varphi_k e^{i\omega_k t} \hat{b}_k^{\text{out}} + \text{h.c.} \right]
\end{aligned}$$

so we can identify

$$\alpha_{nk} = \sqrt{\frac{\omega_k}{\omega_n}} A_k^{(n)}$$
$$\beta_{nk} = \sqrt{\frac{\omega_k}{\omega_n}} B_k^{(n)}$$

Appendix A

Density operator

In this appendix we aim to calculate the expression for the density operator in second-order in $\dot{L}(t)$

$$\hat{\rho}(t) = \hat{\rho}(0) - i\int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right] - \frac{1}{2}\int_0^t dt'' \int_0^t dt' \left[\hat{H}_I(t''), \left[\hat{H}_I(t'), \hat{\rho}(0)\right]\right] + \mathcal{O}(\dot{L}^2(t)). \tag{A.1}$$

with the effective Hamiltonian

$$\begin{aligned}\hat{H}_{\text{eff}} &= \sum_k \omega_k(t)\hat{a}_k^\dagger(t)\hat{a}_k(t) \\ &+ \frac{i}{2}\sum_{j,k}\left[\mu_{[k,j]}(t)\left(\hat{a}_k^\dagger(t)\hat{a}_j^\dagger(t) - \hat{a}_k(t)\hat{a}_j(t)\right) + \mu_{(k,j)}(t)\left(\hat{a}_k^\dagger(t)\hat{a}_j(t) - \hat{a}_j^\dagger(t)\hat{a}_k(t)\right)\right].\end{aligned} \tag{A.2}$$

where in general $\mu_{[k,j]}(t)$ and $\mu_{(k,j)}(t)$ have a first order dependence in $\dot{L}(t)$ whereas the time dependent annihilation (and creation) operators $\hat{a}(t)$ can be expanded as

$$\hat{a}_k(t) = e^{-i\Omega_k(t)}\left[\hat{b}_k + \sum_j\left(\tilde{\alpha}_{kj}(t)b_j + \beta_{kj}(t)b_j^\dagger\right)\right] + \mathcal{O}(\dot{L}^2),$$

with $\tilde{\alpha}_{kj}(t) = e^{i\Omega_k(t)}\alpha_{kj}(t) - \delta_{kj}$ and $\beta_{kj}(t) = e^{i\Omega_k(t)}\beta_{kj}(t)$.

A.1 Computing the first term

To compute expression (A.1) we begin with the first term

$$\int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right] = \int_0^t dt' \hat{H}_I(t')\hat{\rho}(0) - \hat{\rho}(0)\int_0^t dt' \hat{H}_I(t'),$$

more specifically, by computing the expression

$$\int_0^t dt' \hat{H}_I(t').$$

To do so, we start expanding the quadratic terms involving $\hat{a}_k(t)$ and $\hat{a}_k^\dagger(t)$ in terms of $\dot{L}(t)$

A.1.1 Computing the quadradic terms

$\hat{a}_k^\dagger \hat{a}_j^\dagger$ term

$$
\begin{aligned}
\hat{a}_k^\dagger(t)\hat{a}_j^\dagger(t) &= e^{i[\Omega_k(t)+\Omega_j(t)]}\left[\hat{b}_k^\dagger + \sum_l \left(\tilde{\alpha}_{kl}^*(t)b_l^\dagger + \beta_{kl}^*(t)b_l\right)\right]\left[\hat{b}_j^\dagger + \sum_n \left(\tilde{\alpha}_{jn}^*(t)b_n^\dagger + \beta_{jn}^*(t)b_n\right)\right] \\
&= e^{i[\Omega_k(t)+\Omega_j(t)]}\left[\hat{b}_k^\dagger\hat{b}_j^\dagger + \sum_l \left(\tilde{\alpha}_{kl}^*(t)b_l^\dagger\hat{b}_j^\dagger + \beta_{kl}^*(t)b_l\hat{b}_j^\dagger\right) + \sum_n \left(\tilde{\alpha}_{jn}^*(t)\hat{b}_k^\dagger b_n^\dagger + \beta_{jn}^*(t)\hat{b}_k^\dagger b_n\right)\right] \\
&= e^{i[\Omega_k(t)+\Omega_j(t)]}\left[\hat{b}_k^\dagger\hat{b}_j^\dagger + \sum_l \left(\tilde{\alpha}_{kl}^*(t)b_l^\dagger\hat{b}_j^\dagger + \tilde{\alpha}_{jl}^*(t)\hat{b}_k^\dagger b_l^\dagger + \beta_{kl}^*(t)b_l\hat{b}_j^\dagger + \beta_{jl}^*(t)\hat{b}_k^\dagger b_l\right)\right]
\end{aligned}
$$

$\hat{a}_k \hat{a}_j$ term

$$
\begin{aligned}
\hat{a}_k(t)\hat{a}_j(t) &= e^{-i[\Omega_k(t)+\Omega_j(t)]}\left[\hat{b}_k + \sum_l \left(\tilde{\alpha}_{kl}(t)b_l + \beta_{kl}(t)b_l^\dagger\right)\right]\left[\hat{b}_j + \sum_n \left(\tilde{\alpha}_{jn}(t)b_n + \beta_{jn}(t)b_n^\dagger\right)\right] \\
&= e^{-i[\Omega_k(t)+\Omega_j(t)]}\left[\hat{b}_k\hat{b}_j + \sum_l \left(\tilde{\alpha}_{kl}(t)b_l\hat{b}_j + \beta_{kl}(t)b_l^\dagger\hat{b}_j\right) + \sum_n \left(\tilde{\alpha}_{jn}(t)\hat{b}_k b_n + \beta_{jn}(t)\hat{b}_k b_n^\dagger\right)\right] \\
&= e^{-i[\Omega_k(t)+\Omega_j(t)]}\left[\hat{b}_k\hat{b}_j + \sum_l \left(\tilde{\alpha}_{kl}(t)b_l\hat{b}_j + \tilde{\alpha}_{jl}(t)\hat{b}_k b_l + \beta_{kl}(t)b_l^\dagger\hat{b}_j + \beta_{jl}(t)\hat{b}_k b_l^\dagger\right)\right]
\end{aligned}
$$

$\hat{a}_k^\dagger \hat{a}_j$ term

$$
\begin{aligned}
\hat{a}_k^\dagger(t)\hat{a}_j(t) &= e^{i[\Omega_k(t)-\Omega_j(t)]}\left[\hat{b}_k^\dagger + \sum_l \left(\tilde{\alpha}_{kl}^*(t)b_l^\dagger + \beta_{kl}^*(t)b_l\right)\right]\left[\hat{b}_j + \sum_n \left(\tilde{\alpha}_{jn}(t)b_n + \beta_{jn}(t)b_n^\dagger\right)\right] \\
&= e^{i[\Omega_k(t)-\Omega_j(t)]}\left[\hat{b}_k^\dagger\hat{b}_j + \sum_l \left(\tilde{\alpha}_{kl}^*(t)b_l^\dagger\hat{b}_j + \beta_{kl}^*(t)b_l\hat{b}_j\right) + \sum_n \left(\tilde{\alpha}_{jn}(t)\hat{b}_k^\dagger b_n + \beta_{jn}(t)\hat{b}_k^\dagger b_n^\dagger\right)\right] \\
&= e^{i[\Omega_k(t)-\Omega_j(t)]}\left[\hat{b}_k^\dagger\hat{b}_j + \sum_l \left(\tilde{\alpha}_{kl}^*(t)b_l^\dagger\hat{b}_j + \tilde{\alpha}_{jl}(t)\hat{b}_k^\dagger b_l + \beta_{kl}^*(t)b_l\hat{b}_j + \beta_{jl}(t)\hat{b}_k^\dagger b_l^\dagger\right)\right]
\end{aligned}
$$

A.1.2 Getting back to the calculating

$$
\begin{aligned}
&\int_0^t dt' \Bigg[\mu_{[k,j]}(t) \left(\hat{a}^\dagger_k(t)\hat{a}^\dagger_j(t) - \hat{a}_k(t)\hat{a}_j(t) \right) + \mu_{(k,j)}(t) \left(\hat{a}^\dagger_k(t)\hat{a}_j(t) - \hat{a}^\dagger_j(t)\hat{a}_k(t) \right) \Bigg] \\
&\int_0^t dt' \Bigg[\mu_{[k,j]}(t) e^{i[\Omega_k(t)+\Omega_j(t)]} \left[\hat{b}^\dagger_k\hat{b}^\dagger_j + \sum_l \left(\tilde{\alpha}^*_{kl}(t) b^\dagger_l \hat{b}^\dagger_j + \tilde{\alpha}^*_{jl}(t)\hat{b}^\dagger_k b^\dagger_l + \beta^*_{kl}(t) b_l \hat{b}^\dagger_j + \beta^*_{jl}(t)\hat{b}^\dagger_k b_l \right) \right] \\
&\qquad - \mu_{[k,j]}(t) e^{-i[\Omega_k(t)+\Omega_j(t)]} \left[\hat{b}_k\hat{b}_j + \sum_l \left(\tilde{\alpha}_{kl}(t) b_l \hat{b}_j + \tilde{\alpha}_{jl}(t)\hat{b}_k b_l + \beta_{kl}(t) b^\dagger_l \hat{b}_j + \beta_{jl}(t)\hat{b}_k b^\dagger_l \right) \right] \\
&\qquad + \mu_{(k,j)}(t) e^{i[\Omega_k(t)-\Omega_j(t)]} \left[\hat{b}^\dagger_k\hat{b}_j + \sum_l \left(\tilde{\alpha}^*_{kl}(t) b^\dagger_l \hat{b}_j + \tilde{\alpha}_{jl}(t)\hat{b}^\dagger_k b_l + \beta^*_{kl}(t) b_l \hat{b}_j + \beta_{jl}(t)\hat{b}^\dagger_k b^\dagger_l \right) \right] \\
&\qquad - \mu_{(k,j)}(t) e^{i[\Omega_j(t)-\Omega_k(t)]} \left[\hat{b}^\dagger_j\hat{b}_k + \sum_l \left(\tilde{\alpha}^*_{jl}(t) b^\dagger_l \hat{b}_k + \tilde{\alpha}_{kl}(t)\hat{b}^\dagger_j b_l + \beta^*_{jl}(t) b_l \hat{b}_k + \beta_{kl}(t)\hat{b}^\dagger_j b^\dagger_l \right) \right] \Bigg]
\end{aligned}
$$

A.1.3 What?

$$
\begin{aligned}
&\left[\hat{a}^\dagger_k(t)\hat{a}^\dagger_j(t) - \hat{a}_k(t)\hat{a}_j(t) \right] \hat{\rho}(0) = \\
&= e^{i[\Omega_k(t)+\Omega_j(t)]} \left[\hat{b}^\dagger_k\hat{b}^\dagger_j + \sum_l \left(\tilde{\alpha}^*_{kl}(t) b^\dagger_l \hat{b}^\dagger_j + \tilde{\alpha}^*_{jl}(t)\hat{b}^\dagger_k b^\dagger_l + \beta^*_{kl}(t) b_l \hat{b}^\dagger_j + \beta^*_{jl}(t)\hat{b}^\dagger_k b_l \right) \right] \hat{\rho}(0) \\
&- e^{-i[\Omega_k(t)+\Omega_j(t)]} \left[\hat{b}_k\hat{b}_j + \sum_l \left(\tilde{\alpha}_{kl}(t) b_l \hat{b}_j + \tilde{\alpha}_{jl}(t)\hat{b}_k b_l + \beta_{kl}(t) b^\dagger_l \hat{b}_j + \beta_{jl}(t)\hat{b}_k b^\dagger_l \right) \right] \hat{\rho}(0) \\
&= e^{i[\Omega_k(t)+\Omega_j(t)]} \left[\hat{b}^\dagger_k\hat{b}^\dagger_j + \sum_l \left(\tilde{\alpha}^*_{kl}(t) b^\dagger_l \hat{b}^\dagger_j + \tilde{\alpha}^*_{jl}(t)\hat{b}^\dagger_k b^\dagger_l \right) + \beta^*_{kj}(t) b_j \hat{b}^\dagger_j \right] \hat{\rho}(0) - e^{-i[\Omega_k(t)+\Omega_j(t)]} \beta_{jk}(t)\hat{b}_k b^\dagger_k \hat{\rho}(0)
\end{aligned}
$$

$$
\begin{aligned}
&\hat{\rho}(0) \left[\hat{a}^\dagger_k(t)\hat{a}^\dagger_j(t) - \hat{a}_k(t)\hat{a}_j(t) \right] = \\
&= e^{i[\Omega_k(t)+\Omega_j(t)]} \hat{\rho}(0) \left[\hat{b}^\dagger_k\hat{b}^\dagger_j + \sum_l \left(\tilde{\alpha}^*_{kl}(t) b^\dagger_l \hat{b}^\dagger_j + \tilde{\alpha}^*_{jl}(t)\hat{b}^\dagger_k b^\dagger_l + \beta^*_{kl}(t) b_l \hat{b}^\dagger_j + \beta^*_{jl}(t)\hat{b}^\dagger_k b_l \right) \right] \\
&- e^{-i[\Omega_k(t)+\Omega_j(t)]} \hat{\rho}(0) \left[\hat{b}_k\hat{b}_j + \sum_l \left(\tilde{\alpha}_{kl}(t) b_l \hat{b}_j + \tilde{\alpha}_{jl}(t)\hat{b}_k b_l + \beta_{kl}(t) b^\dagger_l \hat{b}_j + \beta_{jl}(t)\hat{b}_k b^\dagger_l \right) \right] \\
&= e^{i[\Omega_k(t)+\Omega_j(t)]} \hat{\rho}(0) \beta^*_{kj}(t) b_j \hat{b}^\dagger_j - e^{-i[\Omega_k(t)+\Omega_j(t)]} \hat{\rho}(0) \left[\hat{b}_k\hat{b}_j + \sum_l \left(\tilde{\alpha}_{kl}(t) b_l \hat{b}_j + \tilde{\alpha}_{jl}(t)\hat{b}_k b_l \right) + \beta_{jk}(t)\hat{b}_k b^\dagger_k \right]
\end{aligned}
$$

$$\int_0^t dt' \mu_{[k,j]}(t') \left[\left(\hat{a}_k^\dagger(t')\hat{a}_j^\dagger(t') - \hat{a}_k(t')\hat{a}_j(t') \right) \hat{\rho}(0) - \hat{\rho}(0) \left(\hat{a}_k^\dagger(t')\hat{a}_j^\dagger(t') - \hat{a}_k(t')\hat{a}_j(t) \right) \right]$$
$$= \beta_{kj} \left(\hat{b}_k^\dagger \hat{b}_j^\dagger \hat{\rho}_0 \right) + \frac{1}{2} \sum_l \left[\beta_{kj} \tilde{\alpha}_{kl}^* \left(b_l^\dagger \hat{b}_j^\dagger \hat{\rho}_0 \right) + \beta_{kj} \tilde{\alpha}_{jl}^* \left(\hat{b}_k^\dagger b_l^\dagger \hat{\rho}_0 \right) \right] + \frac{1}{2} \beta_{kj} \beta_{kj}^* \left(b_j \hat{b}_j^\dagger \hat{\rho}_0 \right) - \frac{1}{2} \beta_{kj}^* \beta_{jk} \left(\hat{b}_k b_k^\dagger \hat{\rho}_0 \right)$$
$$= \beta_{kj}^* \left(\hat{\rho}_0 \hat{b}_k \hat{b}_j \right) + \frac{1}{2} \sum_l \left[\beta_{kj}^* \tilde{\alpha}_{kl} \left(\hat{\rho}_0 b_l \hat{b}_j \right) + \beta_{kj}^* \tilde{\alpha}_{jl} \left(\hat{\rho}_0 \hat{b}_k b_l \right) \right] + \frac{1}{2} \beta_{kj}^* \beta_{jk} \left(\hat{\rho}_0 \hat{b}_k b_k^\dagger \right) - \frac{1}{2} \beta_{kj} \beta_{kj}^* \left(\hat{\rho}_0 b_j \hat{b}_j^\dagger \right)$$

whereas

$$\int_0^t dt' \mu_{(k,j)}(t') \left[\left(\hat{a}_k^\dagger(t')\hat{a}_j^\dagger(t') - \hat{a}_k(t')\hat{a}_j(t') \right) \hat{\rho}(0) - \hat{\rho}(0) \left(\hat{a}_k^\dagger(t')\hat{a}_j^\dagger(t') - \hat{a}_k(t')\hat{a}_j(t) \right) \right]$$
$$= \beta_{kj} \left(\hat{b}_k^\dagger \hat{b}_j^\dagger \hat{\rho}_0 \right) + \frac{1}{2} \sum_l \left[\beta_{kj} \tilde{\alpha}_{kl}^* \left(b_l^\dagger \hat{b}_j^\dagger \hat{\rho}_0 \right) + \beta_{kj} \tilde{\alpha}_{jl}^* \left(\hat{b}_k^\dagger b_l^\dagger \hat{\rho}_0 \right) \right] + \frac{1}{2} \beta_{kj} \beta_{kj}^* \left(b_j \hat{b}_j^\dagger \hat{\rho}_0 \right) - \frac{1}{2} \beta_{kj}^* \beta_{jk} \left(\hat{b}_k b_k^\dagger \hat{\rho}_0 \right)$$
$$+ \beta_{kj}^* \left(\hat{\rho}_0 \hat{b}_k \hat{b}_j \right) + \frac{1}{2} \sum_l \left[\beta_{kj}^* \tilde{\alpha}_{kl} \left(\hat{\rho}_0 b_l \hat{b}_j \right) + \beta_{kj}^* \tilde{\alpha}_{jl} \left(\hat{\rho}_0 \hat{b}_k b_l \right) \right] + \frac{1}{2} \beta_{kj}^* \beta_{jk} \left(\hat{\rho}_0 \hat{b}_k b_k^\dagger \right) - \frac{1}{2} \beta_{kj} \beta_{kj}^* \left(\hat{\rho}_0 b_j \hat{b}_j^\dagger \right)$$

Considering the vacuum state $\hat{\rho}(0) = |0; \text{in}\rangle \langle \text{in}; 0|$ as the initial state and the correspondent relations $\hat{b}_k \hat{\rho}(0) = \hat{\rho}(0) b_k^\dagger = 0$, we can obtain the first order term as

$$\begin{aligned} \int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0) \right] &= \int_0^t dt' \hat{H}_I(t') |0\rangle\langle 0| - |0\rangle\langle 0| \int_0^t dt' \hat{H}_I(t') \\ &= \frac{i}{2} \sum_{kj} \int_0^t dt' \left[\mu_{[k,j]}(t') \left(\hat{a}_k^\dagger(t')\hat{a}_j^\dagger(t')\hat{\rho}(0) - \hat{\rho}(0)\hat{a}_k(t')\hat{a}_j(t') \right) \right] && \text{(A.3)} \\ &= \frac{i}{2} \sum_{kj} \left[\beta_{kj} \hat{a}_k^\dagger \hat{a}_j^\dagger \hat{\rho}(0) + \beta_{kj}^* \hat{\rho}(0) \hat{a}_k \hat{a}_j \right], && \text{(A.4)} \end{aligned}$$

For the second order terms

$$\begin{aligned} &\int_0^t dt'' \int_0^t dt' \left[H(t''), \left[\hat{H}_I(t'), \rho(0) \right] \right] \\ &= \int_0^t dt'' H(t'') \int_0^t dt' \left[\hat{H}_I(t'), \rho(0) \right] - \int_0^t dt' \left[\hat{H}_I(t'), \rho(0) \right] \int_0^t dt'' H(t'') \\ &= \frac{i}{2} \sum_{nl} \left[\beta_{ln} \hat{a}_l^\dagger \hat{a}_n^\dagger - \beta_{ln}^* \hat{a}_l \hat{a}_n + \tilde{\alpha}_{ln} \hat{a}_l^\dagger \hat{a}_n - \tilde{\alpha}_{ln}^* \hat{a}_n^\dagger \hat{a}_l \right] \int_0^{t''} dt' \left[\hat{H}_I(t'), \hat{\rho}(0) \right] \\ &- \frac{i}{2} \sum_{nl} \int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0) \right] \left[\beta_{ln} \hat{a}_l^\dagger \hat{a}_n^\dagger - \beta_{ln}^* \hat{a}_l \hat{a}_n + \tilde{\alpha}_{ln} \hat{a}_l^\dagger \hat{a}_n - \tilde{\alpha}_{ln}^* \hat{a}_n^\dagger \hat{a}_l \right] \end{aligned}$$

$$= \frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)+\beta_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\right]\left[\beta_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger-\beta_{ln}^*\hat{a}_l\hat{a}_n+\tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n-\tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger-\beta_{ln}^*\hat{a}_l\hat{a}_n+\tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n-\tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]\left[\beta_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)+\beta_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\right]$$

$$= \frac{1}{4}\sum_{kj}\sum_{nl}\beta_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\left[\beta_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger-\beta_{ln}^*\hat{a}_l\hat{a}_n+\tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n-\tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]$$
$$+\frac{1}{4}\sum_{kj}\sum_{nl}\beta_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\left[\beta_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger-\beta_{ln}^*\hat{a}_l\hat{a}_n+\tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n-\tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger-\beta_{ln}^*\hat{a}_l\hat{a}_n+\tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n-\tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]\beta_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{ln}\hat{a}_l^\dagger\hat{a}_n^\dagger-\beta_{ln}^*\hat{a}_l\hat{a}_n+\tilde{\alpha}_{ln}\hat{a}_l^\dagger\hat{a}_n-\tilde{\alpha}_{ln}^*\hat{a}_n^\dagger\hat{a}_l\right]\beta_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j$$

$$= \frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{kj}\beta_{ln}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l^\dagger\hat{a}_n^\dagger-\beta_{kj}\beta_{ln}^*\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l\hat{a}_n+\beta_{kj}\tilde{\alpha}_{ln}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l^\dagger\hat{a}_n-\beta_{kj}\tilde{\alpha}_{ln}^*\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_n^\dagger\hat{a}_l\right]$$
$$+\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{kj}^*\beta_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n^\dagger-\beta_{kj}^*\beta_{ln}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n+\beta_{kj}^*\tilde{\alpha}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n-\beta_{kj}^*\tilde{\alpha}_{ln}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_n^\dagger\hat{a}_l\right]$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{ln}\beta_{kj}\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)-\beta_{ln}^*\beta_{kj}\hat{a}_l\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)+\tilde{\alpha}_{ln}\beta_{kj}\hat{a}_l^\dagger\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)-\tilde{\alpha}_{ln}^*\beta_{kj}\hat{a}_n^\dagger\hat{a}_l\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right]$$
$$-\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{ln}\beta_{kj}^*\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{\rho}(0)\hat{a}_k\hat{a}_j-\beta_{ln}^*\beta_{kj}^*\hat{a}_l\hat{a}_n\hat{\rho}(0)\hat{a}_k\hat{a}_j+\tilde{\alpha}_{ln}\beta_{kj}^*\hat{a}_l^\dagger\hat{a}_n\hat{\rho}(0)\hat{a}_k\hat{a}_j-\tilde{\alpha}_{ln}^*\beta_{kj}^*\hat{a}_n^\dagger\hat{a}_l\hat{\rho}(0)\hat{a}_k\hat{a}_j\right]$$

$$= \frac{1}{4}\sum_{kj}\sum_{nl}\left[-\beta_{kj}\beta_{ln}^*\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l\hat{a}_n\right]$$
$$+\frac{1}{4}\sum_{kj}\sum_{nl}\left[\beta_{kj}^*\beta_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n^\dagger-\beta_{kj}^*\beta_{ln}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n+\beta_{kj}^*\tilde{\alpha}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n-\beta_{kj}^*\tilde{\alpha}_{ln}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_n^\dagger\hat{a}_l\right]$$
$$\frac{1}{4}\sum_{kj}\sum_{nl}\left[-\beta_{ln}\beta_{kj}\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)+\beta_{ln}^*\beta_{kj}\hat{a}_l\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)-\tilde{\alpha}_{ln}\beta_{kj}\hat{a}_l^\dagger\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)+\tilde{\alpha}_{ln}^*\beta_{kj}\hat{a}_n^\dagger\hat{a}_l\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right]$$
$$\frac{1}{4}\sum_{kj}\sum_{nl}\left[-\beta_{ln}\beta_{kj}^*\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{\rho}(0)\hat{a}_k\hat{a}_j\right]$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{kj}\sum_{nl}\Bigg[-\beta_{kj}\beta^*_{ln}\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\hat{a}_l\hat{a}_n \\
&+ \beta^*_{kj}\beta_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_l\hat{a}^\dagger_n - \beta^*_{kj}\beta^*_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n + \beta^*_{kj}\tilde{\alpha}_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_l\hat{a}_n - \beta^*_{kj}\tilde{\alpha}^*_{ln}\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_n\hat{a}_l \\
&- \beta_{ln}\beta_{kj}\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0) + \beta^*_{ln}\beta_{kj}\hat{a}_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0) - \tilde{\alpha}_{ln}\beta_{kj}\hat{a}^\dagger_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0) + \tilde{\alpha}^*_{ln}\beta_{kj}\hat{a}^\dagger_n\hat{a}_l\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0) \\
&- \beta_{ln}\beta^*_{kj}\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{\rho}(0)\hat{a}_k\hat{a}_j\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{4}\sum_{kj}\sum_{nl}\Bigg[&\beta^*_{ln}\beta_{kj}\left(\hat{a}_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) + \beta^*_{kj}\beta_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_l\hat{a}^\dagger_n\right) \\
&+ \tilde{\alpha}^*_{ln}\beta_{kj}\left(\hat{a}^\dagger_n\hat{a}_l\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) + \beta^*_{kj}\tilde{\alpha}_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_l\hat{a}_n\right) \\
&- \beta_{ln}\beta_{kj}\left(\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) - \beta^*_{kj}\beta^*_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n\right) \\
&- \tilde{\alpha}_{ln}\beta_{kj}\left(\hat{a}^\dagger_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) - \beta^*_{kj}\tilde{\alpha}^*_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_n\hat{a}_l\right) \\
&- \beta_{kj}\beta^*_{ln}\left(\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\hat{a}_l\hat{a}_n\right) - \beta_{ln}\beta^*_{kj}\left(\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{\rho}(0)\hat{a}_k\hat{a}_j\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{4}\sum_{kj}\sum_{nl}\Bigg[&\beta^*_{ln}\beta_{kj}\left(\hat{a}_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) - \beta_{kj}\beta^*_{ln}\left(\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\hat{a}_l\hat{a}_n\right) + \tilde{\alpha}^*_{ln}\beta_{kj}\left(\hat{a}^\dagger_n\hat{a}_l\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) \\
&- \beta_{ln}\beta_{kj}\left(\hat{a}^\dagger_l\hat{a}^\dagger_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) - \tilde{\alpha}_{ln}\beta_{kj}\left(\hat{a}^\dagger_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right) + \text{h.c.}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho}(t) = \hat{\rho}(0) - \frac{1}{2}\sum_{kj} & \left[\beta_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) + \beta_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\right] \\
-\frac{1}{8}\sum_{kj}\sum_{nl} & \Bigg[\beta_{ln}^*\beta_{kj}\left(\hat{a}_l\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) + \beta_{kj}^*\beta_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n^\dagger\right) \\
& - \beta_{kj}\beta_{ln}^*\left(\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l\hat{a}_n\right) - \beta_{ln}\beta_{kj}^*\left(\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{\rho}(0)\hat{a}_k\hat{a}_j\right) \\
& + \tilde{\alpha}_{ln}^*\beta_{kj}\left(\hat{a}_n^\dagger\hat{a}_l\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) + \beta_{kj}^*\tilde{\alpha}_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n\right) \\
& - \beta_{ln}\beta_{kj}\left(\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) - \beta_{kj}^*\beta_{ln}^*\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n\right) \\
& - \tilde{\alpha}_{ln}\beta_{kj}\left(\hat{a}_l^\dagger\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) - \beta_{kj}^*\tilde{\alpha}_{ln}^*\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_n^\dagger\hat{a}_l\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho}(t) = \hat{\rho}(0) - \frac{1}{2}\sum_{kj} & \left[\beta_{kj}\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0) + \beta_{kj}^*\hat{\rho}(0)\hat{a}_k\hat{a}_j\right] \\
-\frac{1}{8}\sum_{kj}\sum_{nl} & \Bigg[\beta_{ln}^*\beta_{kj}\left(\hat{a}_l\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) + \beta_{kj}^*\beta_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l^\dagger\hat{a}_n^\dagger\right) \\
& - \beta_{kj}\beta_{ln}^*\left(\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\hat{a}_l\hat{a}_n\right) - \beta_{ln}\beta_{kj}^*\left(\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{\rho}(0)\hat{a}_k\hat{a}_j\right) \\
& - \beta_{ln}\beta_{kj}\left(\hat{a}_l^\dagger\hat{a}_n^\dagger\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) - \beta_{kj}^*\beta_{ln}^*\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_l\hat{a}_n\right) \\
& - 2\tilde{\alpha}_{ln}\beta_{kj}\left(\hat{a}_l^\dagger\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right) - 2\beta_{kj}^*\tilde{\alpha}_{ln}^*\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_n^\dagger\hat{a}_l\right)\Bigg]
\end{aligned}
$$

therefore

A.2 2 particles terms

A.2.1 2 by 2 terms

$$
\begin{aligned}
\langle 2_a|\,\hat\rho\,|2_a\rangle &= \frac{1}{8}\sum_{kj}\sum_{nl}\left[2\beta_{kj}\beta^*_{ln}\delta_{ka}\delta_{ja}\delta_{la}\delta_{na} + 2\beta_{ln}\beta^*_{kj}\delta_{ka}\delta_{ja}\delta_{la}\delta_{na}\right] \\
&= \frac{1}{2}|\beta_{aa}|^2
\end{aligned}
$$

$$
\begin{aligned}
\langle 1_a, 1_b|\,\hat\rho\,|1_a, 1_b\rangle &= \frac{1}{8}\sum_{kj}\sum_{nl}\left[\beta_{kj}\beta^*_{ln}\left(\delta_{ka}\delta_{la}\delta_{jb}\delta_{nb} + \delta_{ka}\delta_{na}\delta_{jb}\delta_{lb}\right) + \beta_{ln}\beta^*_{kj}\left(\delta_{la}\delta_{ka}\delta_{nb}\delta_{jb} + \delta_{la}\delta_{ja}\delta_{nb}\delta_{kb}\right)\right] \\
&= \frac{1}{2}|\beta_{ab}|^2
\end{aligned}
$$

So $\rho(2^{ab}|2_{ab}) = \frac{1}{2}|\beta_{ab}|^2$

A.3 4 particles terms

Here we consider the terms

$$
\frac{1}{8}\sum_{jknl}\left[\beta_{ln}\beta_{kj}\left(\hat a^\dagger_l \hat a^\dagger_n \hat a^\dagger_k \hat a^\dagger_j \hat\rho(0)\right) + \beta^*_{kj}\beta^*_{ln}\left(\hat\rho(0)\hat a_k \hat a_j \hat a_l \hat a_n\right)\right]
$$

$$
\begin{aligned}
\langle 4_a|\,\hat\rho\,|0\rangle &= \frac{\sqrt{4!}}{8}\beta_{aa}\beta_{aa} \\
\langle 3_a, 1_b|\,\hat\rho\,|0\rangle &= \frac{\sqrt{3!}}{8}4\beta_{aa}\beta_{ab} \\
\langle 2_a, 2_b|\,\hat\rho\,|0\rangle &= \frac{2!}{8}\left(4\beta_{ab}\beta_{ab} + 2\beta_{aa}\beta_{bb}\right) \\
\langle 1_a, 1_b, 1_c, 1_d|\,\hat\rho\,|0\rangle &= \frac{24}{8}\beta_{cd}\beta_{ab} \\
\langle 0|\,\hat\rho\,|1_a, 1_b, 1_c, 1_d\rangle &= \frac{24}{8}\beta^*_{cd}\beta^*_{ab} \\
\langle 0|\,\hat\rho\,|2_a, 2_b\rangle &= \frac{2!}{8}\left(4\beta^*_{ab}\beta^*_{ab} + 2\beta^*_{aa}\beta^*_{bb}\right) \\
\langle 0|\,\hat\rho\,|3_a, 1_b\rangle &= \frac{\sqrt{3!}}{8}4\beta^*_{aa}\beta^*_{ab} \\
\langle 0|\,\hat\rho\,|4_a\rangle &= \frac{\sqrt{4!}}{8}\beta^*_{aa}\beta^*_{aa}
\end{aligned}
$$

A.4 2 particles terms

Here we consider the terms

$$\frac{1}{4}\sum_{jknl}\left[\tilde{\alpha}_{ln}\beta_{kj}\left(\hat{a}_l^\dagger\hat{a}_n\hat{a}_k^\dagger\hat{a}_j^\dagger\hat{\rho}(0)\right)+\beta_{kj}^*\tilde{\alpha}_{ln}^*\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}_n^\dagger\hat{a}_l\right)\right]$$

$$\begin{aligned}\langle 2_a|\,\hat{\rho}\,|0\rangle &= \frac{\sqrt{2!}}{4}2\sum_k\tilde{\alpha}_{ak}\beta_{ka}\\ \langle 1_a,1_b|\,\hat{\rho}\,|0\rangle &= \frac{1}{4}\sum_k\left(\tilde{\alpha}_{ak}\beta_{kb}+\tilde{\alpha}_{ak}\beta_{bk}\right)\\ \langle 0|\,\hat{\rho}\,|1_a,1_b\rangle &= \frac{1}{4}\sum_k\left(\tilde{\alpha}_{ak}^*\beta_{kb}^*+\tilde{\alpha}_{ak}^*\beta_{bk}^*\right)\\ \langle 0|\,\hat{\rho}\,|2_a\rangle &= \frac{\sqrt{2!}}{4}2\sum_k\tilde{\alpha}_{ak}^*\beta_{ka}^*\end{aligned}$$

$\rho(2^{abcd}|0) = \frac{\sqrt{4!}}{8}\beta_{ab}\beta_{cd}$

A.5 2 particles terms - last

$$\langle 2_a| \hat{\rho} |0\rangle = \frac{1}{8}\sum_{kj}\sum_{nl}\left[\tilde{\alpha}_{ln}\beta_{kj}\left(\delta_{nk}\delta_{la}\delta_{ja}+\delta_{nj}\delta_{la}\delta_{ka}\right)+\beta^*_{kj}\tilde{\alpha}^*_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_n\hat{a}_l\right)\right] = \frac{\sqrt{4}}{8}\beta_{aa}\beta_{aa}$$

$$\langle 3_a, 1_b| \hat{\rho} |0\rangle = \frac{1}{8}\sum_{kj}\sum_{nl}\left[\tilde{\alpha}_{ln}\beta_{kj}\left(\hat{a}^\dagger_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right)+\beta^*_{kj}\tilde{\alpha}^*_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_n\hat{a}_l\right)\right] = \frac{\sqrt{3!}}{8}\beta_{aa}\beta_{ab}$$

$$\langle 2_a, 2_b| \hat{\rho} |0\rangle = \frac{1}{8}\sum_{kj}\sum_{nl}2!\left[\tilde{\alpha}_{ln}\beta_{kj}\left(\hat{a}^\dagger_l\hat{a}_n\hat{a}^\dagger_k\hat{a}^\dagger_j\hat{\rho}(0)\right)+\beta^*_{kj}\tilde{\alpha}^*_{ln}\left(\hat{\rho}(0)\hat{a}_k\hat{a}_j\hat{a}^\dagger_n\hat{a}_l\right)\right] = \frac{2!}{8}\left(\beta_{ab}\beta_{ab}+\beta_{aa}\beta_{bb}\right)$$

$$\langle 1_a, 1_b, 1_c, 1_d| \hat{\rho} |0\rangle = \frac{1}{8}\sum_{kj}\sum_{nl}\left[\beta_{ln}\beta_{kj}\delta_{ka}\delta_{jb}\delta_{lc}\delta_{nd}\right] = \frac{1}{8}\beta_{cd}\beta_{ab}$$

$$\langle 0| \hat{\rho} |1_a, 1_b, 1_c, 1_d\rangle = \frac{1}{8}\sum_{kj}\sum_{nl}\left[\beta^*_{kj}\beta^*_{ln}\delta_{ka}\delta_{jb}\delta_{lc}\delta_{nd}\right] = \frac{1}{8}\beta^*_{cd}\beta^*_{ab}$$

$$\langle 0| \hat{\rho} |2_a, 2_b\rangle = \frac{1}{8}\sum_{kj}\sum_{nl}\left[\beta^*_{kj}\beta^*_{ln}\delta_{ka}\delta_{ja}\delta_{la}\delta_{nb}\right]$$

$$\langle 0| \hat{\rho} |3_a, 1_b\rangle = \frac{1}{8}\sum_{kj}\sum_{nl}\left[\beta^*_{kj}\beta^*_{ln}\beta^*_{kj}\beta^*_{ln}\delta_{ka}\delta_{ja}\delta_{la}\delta_{nb}\right] = \frac{\sqrt{3!}}{8}\beta^*_{aa}\beta^*_{ab}$$

$$\langle 0| \hat{\rho} |4_a\rangle = \frac{1}{8}\sum_{kj}\sum_{nl}\left[\beta^*_{kj}\beta^*_{ln}\sqrt{4!}\delta_{la}\delta_{na}\delta_{ka}\delta_{ja}\right] = \frac{\sqrt{4!}}{8}\beta^*_{aa}\beta^*_{aa}$$

Appendix B

Derivation of the effective Hamiltonian

B.1 Bogoliubov coefficients

Let the equations of motion

$$\partial_t \hat{\phi}(x,t) = \hat{\pi}(x,t)$$
$$\partial_t \hat{\pi}(x,t) = -\omega_j^2(t)\hat{\phi}(x,t),$$

with

$$\hat{\phi}(x,t) = \sum_j \frac{1}{\sqrt{2\omega_j(t)}}\left[\hat{a}_j(t) + \hat{a}_j^\dagger(t)\right]\varphi_j(x,t) \tag{B.1}$$

$$\hat{\pi}(x,t) = i\sum_j \sqrt{\frac{\omega_j(t)}{2}}\left[\hat{a}_j^\dagger(t) - \hat{a}_j(t)\right]\varphi_j(x,t) \tag{B.2}$$

$$\partial_t\hat{\phi}(x,t) = \sum_j \frac{1}{\sqrt{2\omega_j(t)}}\left[\hat{a}_j(t) + \hat{a}_j^\dagger(t)\right]\left[\partial_t\varphi_j(x,t) - \frac{1}{2\omega_j}\frac{d\omega_j}{dt}\varphi_j(x,t)\right] \tag{B.3}$$

$$+\sum_j \frac{1}{\sqrt{2\omega_j(t)}}\left[\partial_t\hat{a}_j(t) + \partial_t\hat{a}_j^\dagger(t)\right]\varphi_j(x,t) \tag{B.4}$$

$$\partial_t\hat{\pi}(x,t) = i\sum_j \sqrt{\frac{\omega_j(t)}{2}}\left[\hat{a}_j^\dagger(t) - \hat{a}_j(t)\right]\left[\partial_t\varphi_j(t) + \frac{1}{2\omega_j}\frac{d\omega_j}{dt}\varphi_j(x,t)\right] \tag{B.5}$$

$$+ i\sum_j \sqrt{\frac{\omega_j(t)}{2}}\left[\partial_t\hat{a}_j^\dagger(t) - \partial_t\hat{a}_j(t)\right]\varphi_j(x,t). \tag{B.6}$$

therefore, since $\partial_t\hat{\phi}(x,t) = \hat{\pi}(x,t)$, the following quantities

$$\int d^3x\ \varphi_k\partial_t\hat{\phi} = \sum_j \frac{1}{\sqrt{2\omega_j}}\left(\hat{a}_j + \hat{a}_j^\dagger\right)\left(G_{kj} - \frac{1}{2\omega_j}\frac{d\omega_j}{dt}\delta_{k,j}\right) + \sum_j \frac{1}{\sqrt{2\omega_j}}\left(\partial_t\hat{a}_j + \partial_t\hat{a}_j^\dagger\right)\delta_{k,j}$$
$$\int d^3x\ \varphi_k\hat{\pi} = i\sum_j \sqrt{\frac{\omega_j}{2}}\left(\hat{a}_j^\dagger - \hat{a}_j\right)\delta_{k,j},$$

must be equal. This means that

$$\frac{\mathrm{d}\hat{a}_k^\dagger}{\mathrm{d}t} + \frac{\mathrm{d}\hat{a}_k}{\mathrm{d}t} = i\omega_k \left(\hat{a}_k^\dagger - \hat{a}_k\right) + \sum_j \mu_{k,j} \left(\hat{a}_j + \hat{a}_j^\dagger\right)$$

with $\mu_{k,j} = \sqrt{\frac{\omega_j}{\omega_k}} \left(G_{k,j} - \frac{1}{2\omega_k}\frac{d\omega_k}{dt}\delta_{k,j}\right)$. On the other hand, we also have that $\partial_t\hat{\pi} = -\omega_j^2\hat{\phi}$, meaning that

$$\int d^3x\ \varphi_k\partial_t\hat{\pi} = -i\sum_j \sqrt{\frac{\omega_j}{2}}\left(\hat{a}_j^\dagger - \hat{a}_j\right)\left(G_{j,k} - \frac{1}{2\omega_k}\frac{d\omega_k}{dt}\delta_{k,j}\right) + i\sum_j \sqrt{\frac{\omega_j}{2}}\left(\partial_t\hat{a}_j^\dagger - \partial_t\hat{a}_j\right)\delta_{k,j}$$

$$-\omega_j^2\int d^3x\ \varphi_k\hat{\phi} = -\sum_j \omega_j\sqrt{\frac{\omega_j}{2}}\left(\hat{a}_j^\dagger + \hat{a}_j\right)\delta_{k,j},$$

meaning

$$\frac{\mathrm{d}\hat{a}_k^\dagger}{\mathrm{d}t} - \frac{\mathrm{d}\hat{a}_k}{\mathrm{d}t} = i\omega_k \left(\hat{a}_k^\dagger + \hat{a}_k\right) + \sum_j \mu_{j,k} \left(\hat{a}_j^\dagger - \hat{a}_j\right),$$

therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_k(t) = -i\omega_k\hat{a}_k(t) + \sum_k \left[\mu_{(k,j)}(t)\hat{a}_k^\dagger(t) + \mu_{[k,j]}(t)\hat{a}_k(t)\right], \tag{B.7}$$

$$\frac{\mathrm{d}\hat{a}_k^\dagger}{\mathrm{d}t} = i\omega_k\hat{a}_k^\dagger + \sum_k \mu_{(k,j)}\hat{a}_k + \sum_k \mu_{[k,j]}\hat{a}_k^\dagger, \tag{B.8}$$

with

$$\mu_{[k,j]} = \frac{1}{2}\left(\mu_{k,j} - \mu_{j,k}\right)$$
$$\mu_{(k,j)} = \frac{1}{2}\left(\mu_{k,j} + \mu_{j,k}\right)$$

B.1.1 Finding the effective Hamiltonian

Considering a general quadratic Hamiltonian

$$\hat{H} = \sum_{kl}\left(A_{kl}\hat{a}_k^\dagger(0)\hat{a}_l^\dagger(0) + B_{kl}\hat{a}_k^\dagger(0)\hat{a}_l(0) + C_{kj}\hat{a}_l^\dagger(0)\hat{a}_k(0) + D_{kl}\hat{a}_k(0)\hat{a}_l(0)\right),$$

and supposing

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}a_k(t) &= i\left[\hat{H}, \hat{a}_k(t)\right] \qquad \text{(B.9)}\\
&= i\sum_{jl}\left(A_{jl}\left[\hat{a}_j^\dagger(0)\hat{a}_l^\dagger(0), a_k(t)\right] + B_{jl}\left[\hat{a}_j^\dagger(0)\hat{a}_l(0), a_k(t)\right] + C_{jl}\left[\hat{a}_l(0)^\dagger\hat{a}_j(0), a_k(t)\right] + D_{jl}[\hat{a}_j(0)\hat{a}_l(0), a_k(t)\right.\\
&= -i\left(\sum_{jl}A_{jl}\delta_{lk}\hat{a}_j^\dagger + \sum_{jl}A_{jl}\delta_{jk}\hat{a}_l^\dagger + \sum_{jl}B_{jl}\delta_{jk}\hat{a}_l + \sum_{jl}C_{jl}\delta_{lk}\hat{a}_j\right)\\
&= -i\left(\sum_{j}A_{jk}\hat{a}_j^\dagger + \sum_{l}A_{kl}\hat{a}_l^\dagger + \sum_{l}B_{kl}\hat{a}_l + \sum_{j}C_{jk}\hat{a}_j\right)\\
&- i\sum_j\left[\left(A_{jk}+A_{kj}\right)\hat{a}_j^\dagger + \left(B_{kj}+C_{jk}\right)\hat{a}_j\right] \qquad \text{(B.10)}
\end{aligned}
$$

where it was used

$$
\begin{aligned}
\left[\hat{a}_j^\dagger\hat{a}_l^\dagger, a_k\right] &= \hat{a}_j^\dagger\left[\hat{a}_l^\dagger, a_k\right] + \left[\hat{a}_j^\dagger, a_k\right]\hat{a}_l^\dagger = -\delta_{lk}\hat{a}_j^\dagger - \delta_{jk}\hat{a}_l^\dagger\\
\left[\hat{a}_j^\dagger\hat{a}_l, a_k\right] &= \left[\hat{a}_j^\dagger, a_k\right]\hat{a}_l = -\delta_{jk}\hat{a}_l\\
\left[\hat{a}_l^\dagger\hat{a}_j, a_k\right] &= \left[\hat{a}_l^\dagger, a_k\right]\hat{a}_j = -i\delta_{lk}\hat{a}_j\\
\left[\hat{a}_j\hat{a}_l, a_k\right] &= 0
\end{aligned}
$$

in terms of the complex conjugated

$$
\begin{aligned}
\frac{\mathrm{d}a_k^\dagger}{\mathrm{d}t} &= i\left[\hat{H}, \hat{a}_k^\dagger\right] = i\sum_{jl}\left(A_{jl}\left[\hat{a}_j^\dagger\hat{a}_l^\dagger, a_k^\dagger\right] + B_{jl}\left[\hat{a}_j^\dagger\hat{a}_l, a_k^\dagger\right] + C_{jl}\left[\hat{a}_l^\dagger\hat{a}_j, a_k^\dagger\right] + D_{jl}\left[\hat{a}_j\hat{a}_l, a_k^\dagger\right]\right)\\
&= i\left(\sum_{jl}D_{jl}\hat{a}_j\delta_{lk} + \sum_{jl}D_{jl}\delta_{jk}\hat{a}_l + \sum_{jl}B_{jl}\delta_{lk}\hat{a}_j^\dagger + \sum_{jl}C_{jl}\delta_{jk}\hat{a}_l^\dagger\right)\\
&= i\left(\sum_{j}D_{jk}\hat{a}_j + \sum_{l}D_{kl}\hat{a}_l + \sum_{j}B_{jk}\hat{a}_j^\dagger + \sum_{l}C_{kl}\hat{a}_l^\dagger\right)\\
& i\sum_j\left[\left(D_{jk}+D_{kj}\right)\hat{a}_j + \left(B_{jk}+C_{kj}\right)\hat{a}_j^\dagger\right] \qquad \text{(B.11)}
\end{aligned}
$$

where it was used

$$
\begin{aligned}
\left[\hat{a}_j\hat{a}_l, a_k^\dagger\right] &= \hat{a}_j\left[\hat{a}_l, a_k^\dagger\right] + \left[\hat{a}_j, a_k^\dagger\right]\hat{a}_l = \hat{a}_j\delta_{lk} + \delta_{jk}\hat{a}_l \\
\left[\hat{a}_j^\dagger\hat{a}_l, a_k^\dagger\right] &= \hat{a}_j^\dagger\left[\hat{a}_l, a_k^\dagger\right] = \delta_{lk}\hat{a}_j^\dagger \\
\left[\hat{a}_l^\dagger\hat{a}_j, a_k^\dagger\right] &= \hat{a}_l^\dagger\left[\hat{a}_j, a_k^\dagger\right] = \delta_{jk}\hat{a}_l^\dagger \\
\left[\hat{a}_j^\dagger\hat{a}_l^\dagger, a_k^\dagger\right] &= 0
\end{aligned}
$$

Comparing (B.12a) with (B.10) and (B.8) with (B.11), we obtain

$$
\begin{aligned}
-i\left(A_{jk} + A_{kj}\right) &= \frac{1}{2}\left(\mu_{kj} + \mu_{jk}\right) \\
-i\left(C_{jk} + B_{kj}\right) &= \frac{1}{2}\left(\mu_{kj} - \mu_{jk}\right) \\
i\left(D_{jk} + D_{kj}\right) &= \frac{1}{2}\left(\mu_{kj} + \mu_{jk}\right) \\
i\left(B_{jk} + C_{kj}\right) &= \frac{1}{2}\left(\mu_{kj} - \mu_{jk}\right).
\end{aligned}
$$

Naively someone would think the system's solution to be

$$
\begin{aligned}
A_{kj} &= \frac{i}{2}\mu_{kj}; \\
B_{kj} &= \frac{i}{2}\mu_{kj}; \\
C_{kj} &= -\frac{i}{2}\mu_{kj}; \\
D_{kj} &= -\frac{i}{2}\mu_{kj}.
\end{aligned}
$$

But since $\hat{H}$ is hermitian, in order for

$$
\begin{aligned}
\hat{H}^\dagger &= \sum_{kj}\left(A_{kj}^*\hat{a}_j\hat{a}_k + B_{kj}^*\hat{a}_j^\dagger\hat{a}_k + C_{jk}^*\hat{a}_k^\dagger\hat{a}_j + D_{jk}^*\hat{a}_j^\dagger\hat{a}_k^\dagger\right) \\
&= \sum_{kj}\left(D_{jk}^*\hat{a}_k^\dagger\hat{a}_j^\dagger + C_{jk}^*\hat{a}_k^\dagger\hat{a}_j + B_{kj}^*\hat{a}_j^\dagger\hat{a}_k + A_{kj}^*\hat{a}_j\hat{a}_k\right),
\end{aligned}
$$

to be equal to $\hat{H}$, it's necessary that

$$
\begin{aligned}
D_{kj} &= A_{kj}^* \\
C_{kj} &= B_{kj}^*.
\end{aligned}
$$

Meaning that

$$\hat{H} = \sum_{kj} \left(A_{kj} \hat{a}_k^\dagger \hat{a}_j^\dagger + A_{kj}^* \hat{a}_k \hat{a}_j + B_{kj} \hat{a}_k^\dagger \hat{a}_j + B_{kj}^* \hat{a}_j^\dagger \hat{a}_k \right).$$

We also expect the indices in $\hat{H}$ to muted, that is

$$\begin{aligned}
\hat{H} &= \sum_{kj} \left(A_{kj} \hat{a}_k^\dagger \hat{a}_j^\dagger + A_{kj}^* \hat{a}_k \hat{a}_j + B_{kj} \hat{a}_k^\dagger \hat{a}_j + B_{kj}^* \hat{a}_j^\dagger \hat{a}_k \right) \\
&= \sum_{jk} \left(A_{jk} \hat{a}_j^\dagger \hat{a}_k^\dagger + A_{jk}^* \hat{a}_j \hat{a}_k + B_{jk} \hat{a}_j^\dagger \hat{a}_k + B_{jk}^* \hat{a}_k^\dagger \hat{a}_j \right) \\
&= \sum_{kj} \left(A_{jk} \hat{a}_k^\dagger \hat{a}_j^\dagger + A_{jk}^* \hat{a}_k \hat{a}_j + B_{jk}^* \hat{a}_k^\dagger \hat{a}_j + B_{jk} \hat{a}_j^\dagger \hat{a}_k \right),
\end{aligned}$$

what only occurs when

$$\begin{aligned}
A_{kj} &= A_{jk} \\
A_{kj}^* &= A_{jk}^* \\
B_{kj} &= B_{jk}^* \\
B_{jk} &= B_{kj}^*
\end{aligned}$$

therefore we obtain instead the following system

$$\begin{aligned}
-i \left(A_{jk} + A_{kj} \right) = -2iA_{kj} &= \frac{1}{2} \left(\mu_{kj} + \mu_{jk} \right) \\
-i \left(B_{jk}^* + B_{kj} \right) = -2iB_{kj} &= \frac{1}{2} \left(\mu_{kj} - \mu_{jk} \right) \\
i \left(D_{jk} + D_{kj} \right) = 2iD_{kj} &= \frac{1}{2} \left(\mu_{kj} + \mu_{jk} \right) \\
i \left(B_{jk} + B_{kj}^* \right) = 2iB_{kj}^* &= \frac{1}{2} \left(\mu_{kj} - \mu_{jk} \right).
\end{aligned}$$

with the correct solutions

$$\begin{aligned}
A_{kj} &= \frac{i}{2} \mu_{(k,j)} \\
B_{kj} &= \frac{i}{2} \mu_{[k,j]} \\
C_{kj} &= -\frac{i}{2} \mu_{[k,j]} \\
D_{kj} &= -\frac{i}{2} \mu_{(k,j)}
\end{aligned}$$

B.2 Ladder operators

Let the dynamical equation for the annihilation operators

$$\frac{\mathrm{d}\hat{a}_k(t)}{\mathrm{d}t} = -i\omega_k \hat{a}_k(t) + \sum_j \left(\mu_{(k,j)}(t)\hat{a}_j^\dagger(t) + \mu_{[k,j]}(t)\hat{a}_j(t)\right) \tag{B.12a}$$

$$\frac{\mathrm{d}\hat{a}_k^\dagger(t)}{\mathrm{d}t} = i\omega_k \hat{a}_k^\dagger(t) + \sum_j \left(\mu_{(k,j)}(t)\hat{a}_j(t) + \mu_{[k,j]}(t)\hat{a}_j^\dagger(t)\right) \tag{B.12b}$$

If we consider the Bogoliubov coefficients that connect the ladder operators $\hat{a}_k(0)$ and $\hat{a}_k^\dagger(0)$ to $\hat{a}_k(t)$ and $\hat{a}_k^\dagger(t)$

$$\hat{a}_k(t) = \sum_j \left[\alpha_{kj}(t)\hat{a}_j(0) + \beta_{kj}(t)\hat{a}_j^\dagger(0)\right] \tag{B.13a}$$

$$\hat{a}_k^\dagger(t) = \sum_j \left[\alpha_{kj}^*(t)\hat{a}_j(0) + \beta_{kj}^*(t)\hat{a}_j^\dagger(0)\right] \tag{B.13b}$$

with the initial conditions $\alpha_{kj}(0) = \delta_{kj}$ and $\beta_{kj}(0) = 0$. Calculating the time derivatives for expressions (B.13) and substituting (B.12) into, we must obtain

Searching for solutions in order of $L(t)$

$$\hat{a}_k(t) = e^{-i\Omega_k(t)} \sum_\lambda \hat{a}_k^{(\lambda)}(t)$$

with $\hat{a}_k^{(0)} = \hat{a}_k(0)$, we obtain

$$\hat{a}_k^{(\lambda)}(t) = \int_0^t dt' e^{-i\Omega_k(t')} \sum_j \left[\mu_{(k,j)} e^{-i\Omega_j(t')} \hat{a}_j^{(\lambda-1)\dagger}(t') + \mu_{[k,j]} e^{-i\Omega_j(t')} \hat{a}_j^{(\lambda-1)}(t')\right].$$

In first order in $L(t)$ we obtain

$$\hat{a}_k(t) = e^{-\Omega_k(t)} \left[\hat{a}_k^{\text{in}} + \sum_j \left(\alpha_{kj}(t)\hat{a}_j^{\text{in}} + \beta_{kj}(t)\hat{a}_j^{\text{in}\dagger}\right) + \mathcal{O}(L^2)\right]$$

Appendix C

Interaction picture

Let's consider a Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$ where H_0 has a known dynamics and $\hat{V}$ is a time dependent perturbation as in the form

$$
\begin{aligned}
\hat{H}_0 &= \sum_k \omega_k(t) a_k^\dagger a_k \\
\hat{V}(t) &= i \sum_k \left[\xi_k(t) \left(a_k^{\dagger 2} - a_k^2 \right) + \sum_{j(\neq k)} \mu_{kj} \left(a_k^\dagger a_j^\dagger + a_k^\dagger a_j - a_j a_k - a_j^\dagger a_k \right) \right].
\end{aligned}
\tag{C.1}
$$

With $\xi_k(t) = \frac{1}{2} G_{k,k}(t) + \frac{1}{4\omega_k(t)} \frac{d\omega_k(t)}{dt}$ and $\mu_{kj}(t) = \frac{1}{2} \left[\frac{\omega_k(t)}{\omega_j(t)} \right]^{1/2} G_{k,j}(t)$. We can then define an interaction picture with ket $|\psi\rangle_I$ and an arbitrary time dependent observable $\hat{A}(t)$ such that

$$
\begin{aligned}
|\psi(t)\rangle_I &= \hat{U}_0^\dagger |\psi(t)\rangle_S \\
\hat{A}_I(t) &= \hat{U}_0^\dagger(t) \hat{A}_S \hat{U}_0(t)
\end{aligned}
\tag{C.2}
$$

where $\hat{A}$ is an arbitrary observable and $\hat{U}_0$ is an unitary operator for the time evolution of the Hamiltonian $\hat{H}_0$ that has the form

$$
\hat{U}_0(t) = \exp\left\{ -i \int_0^t d\tau \hat{H}_0(\tau) \right\}.
\tag{C.3}
$$

In this context the correct the Schrodinger equation is given by

$$
i\frac{d|\psi(t)\rangle_S}{dt} = i\frac{d}{dt}\left(\hat{U}_0 |\psi(t)\rangle_I \right) = i \left(\frac{d\hat{U}_0}{dt} |\psi(t)\rangle_I + \hat{U}_0 \frac{d|\psi(t)\rangle_I}{dt} \right) = \hat{H}|\psi(t)\rangle_S = \hat{H}\left(\hat{U}_0 |\psi(t)\rangle_I \right).
\tag{C.4}
$$

isolating the term $\hat{U}_0 \frac{d}{dt}|\psi(t)\rangle_I$ in the last equation (C.4) and multiplying both sides from the left by the expression $U_0^\dagger$ we have

$$
\begin{aligned}
i\hat{U}_0^\dagger \hat{U}_0 \frac{d|\psi(t)\rangle_I}{dt} &= \left(\hat{U}_0^\dagger \hat{H} \hat{U}_0 - U_0^\dagger \frac{dU_0}{dt} \right) |\psi(t)\rangle_I \\
i\frac{d|\psi(t)\rangle_I}{dt} &= \left[\hat{U}_0^\dagger \left(\hat{H}_0 + \hat{V} \right) \hat{U}_0 - iU_0^\dagger \frac{dU_0}{dt} \right] |\psi(t)\rangle_I
\end{aligned}
\tag{C.5}
$$

using the fact that $\left[\hat{U}_0, \hat{H}_0\right] = 0$ we have that $\hat{U}_0^\dagger \hat{H}_0 \hat{U}_0 = \hat{H}_0$ and $U_0^\dagger \frac{dU_0}{dt} = U_0^\dagger \left(-i\hat{H}_0 U_0\right) = -i\hat{H}_0$ what implies in

$$i\frac{d|\psi(t)\rangle_I}{dt} = \left[\hat{H}_0 + \hat{U}_0^\dagger \hat{V} \hat{U}_0 - \hat{H}_0\right] |\psi(t)\rangle_I = \hat{V}_I |\psi(t)\rangle_I, \tag{C.6}$$

which is the correct Schrodinger equation for the interaction picture. This is also described by the unitary evolution operator

$$\hat{U}(t) = \mathcal{T}\left\{\exp\{i\int d\tau \hat{V}_I(\tau)\}\right\}. \tag{C.7}$$

Theorem 1. *The effective Hamiltonian from Theorem* **??** *can be written in the interaction picture as*

$$H_I = i\sum_k \left\{e^{i\tilde{\omega}_k(t)}\left[\xi_k(t) a_k^{\dagger 2} e^{i\tilde{\omega}_k(t)} + \sum_{j(\neq k)} \mu_{kj}(t) a_k^\dagger \left(a_j^\dagger e^{i\tilde{\omega}_j(t)} + a_j e^{-i\tilde{\omega}_j(t)}\right)\right] - \textit{h.c.}\right\}, \tag{C.8}$$

Proof. In the interaction picture the effective hamiltonian (**??**) becomes

$$H_I(t) = \hat{U}_0^\dagger \hat{V} \hat{U}_0 = \exp\left\{i\sum_k \tilde{\omega}_k(t) a_k^\dagger a_k\right\} \hat{V} \exp\left\{-i\sum_k \tilde{\omega}_k(t) a_k^\dagger a_k\right\} \tag{C.9}$$

where $\tilde{\omega}_k(t) = \int_0^t d\tau \omega_k(\tau)$. Using the relation

$$e^{\alpha\hat{A}} \hat{B} e^{-\alpha\hat{A}} = \hat{B} + \alpha\left[\hat{A}, \hat{B}\right] + \frac{\alpha^2}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \dots \tag{C.10}$$

with the operators

$$\begin{aligned}
\hat{A} &= \sum_{k'} \tilde{\omega}_{k'}(t) a_{k'}^\dagger a_{k'} \\
\hat{B}_1 &= i\sum_k \xi_k(t) a_k^{\dagger 2} \\
\hat{B}_2 &= i\sum_k \sum_{j(\neq k)} \mu_{jk}(t) a_k^\dagger a_j^\dagger \\
\hat{B}_3 &= i\sum_k \sum_{j(\neq k)} \mu_{jk}(t) a_k^\dagger a_j
\end{aligned} \tag{C.11}$$

where $\hat{V} = \hat{B}_1 + \hat{B}_2 + \hat{B}_3 + \text{h.c.}$ and the commutation relations

$$\begin{aligned}
\left[\hat{A}, \hat{B}_1\right] &= i\sum_k \sum_{k'} \tilde{\omega}_{k'}(t) \xi_k(t) \left[a_{k'}^\dagger a_{k'}, a_k^{\dagger 2}\right] = i\sum_k 2\tilde{\omega}_k(t) \xi_k(t) a_k^{\dagger 2} \\
\left[\hat{A}, \hat{B}_2\right] &= i\sum_k \sum_{k'} \sum_{j(\neq k)} \tilde{\omega}_{k'}(t) \mu_{jk}(t) \left[a_{k'}^\dagger a_{k'}, a_k^\dagger a_j^\dagger\right] = i\sum_k \sum_{j(\neq k)} \mu_{jk}(t) \left(\tilde{\omega}_k(t) a_k^\dagger a_j^\dagger + a_k^\dagger a_j^\dagger \tilde{\omega}_j(t)\right) \\
\left[\hat{A}, \hat{B}_3\right] &= i\sum_k \sum_{k'} \sum_{j(\neq k)} \tilde{\omega}_{k'}(t) \mu_{jk}(t) \left[a_{k'}^\dagger a_{k'}, a_k^\dagger a_j\right] = i\sum_k \sum_{j(\neq k)} \mu_{jk}(t) \left(\tilde{\omega}_k(t) a_k^\dagger a_j - a_k^\dagger a_j \tilde{\omega}_j(t)\right)
\end{aligned} \tag{C.12}$$

$$
\begin{aligned}
\left[\hat{A},\left[\hat{A},\hat{B}_1\right]\right] &= i\sum_{k,k'} 2\tilde{\omega}_{k'}(t)\tilde{\omega}_k(t)\xi_k\left[a^\dagger_{k'}a_{k'},a^{\dagger 2}_k\right] = i\sum_k\left(2\tilde{\omega}_k(t)\right)^2\xi_k(t)a^{\dagger 2}_k\\
\left[\hat{A},\left[\hat{A},\hat{B}_2\right]\right] &= i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)\tilde{\omega}_{k'}(t)\left\{\tilde{\omega}_k(t)\left[a^\dagger_{k'}a_{k'},a^\dagger_k a^\dagger_j\right]+\left[a^\dagger_{k'}a_{k'},a^\dagger_k a^\dagger_j\right]\tilde{\omega}_j(t)\right\}\\
&= i\sum_k\sum_{j(\neq k)}\mu_{jk}(t)\left(\tilde{\omega}^2_k(t)a^\dagger_k a^\dagger_j+\tilde{\omega}_k(t)a^\dagger_k a^\dagger_j\tilde{\omega}_j(t)+\tilde{\omega}_k(t)a^\dagger_k a^\dagger_j\tilde{\omega}_j(t)+\tilde{\omega}^2_j a^\dagger_k a^\dagger_j\right)\\
\left[\hat{A},\left[\hat{A},\hat{B}_3\right]\right] &= i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)\tilde{\omega}_{k'}(t)\left\{\tilde{\omega}_k(t)\left[a^\dagger_{k'}a_{k'},a^\dagger_k a_j\right]-\left[a^\dagger_{k'}a_{k'},a^\dagger_k a_j\right]\tilde{\omega}_j(t)\right\}\\
&= i\sum_k\sum_{j(\neq k)}\mu_{jk}(t)\left(\tilde{\omega}^2_k(t)a^\dagger_k a_j-\tilde{\omega}_k(t)a^\dagger_k a_j\tilde{\omega}_j(t)-\tilde{\omega}_k(t)a^\dagger_k a_j\tilde{\omega}_j(t)+\tilde{\omega}^2_j a^\dagger_k a_j\right)
\end{aligned}
\tag{C.13}
$$

Using the relation above we find that

$$
\begin{aligned}
e^{i\hat{A}}\hat{B}_1e^{-i\hat{A}} &= \hat{B}_1+i\left[\hat{A},\hat{B}_1\right]+\frac{i^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}_1\right]\right]+\ldots\\
&= i\sum_k\xi_k(t)a^{\dagger 2}_k+i\sum_k 2\tilde{\omega}_k(t)\xi_k(t)a^{\dagger 2}_k+i\sum_k\left(2\tilde{\omega}_k(t)\right)^2\xi_k(t)a^{\dagger 2}_k\\
&= i\sum_k\xi_k(t)a^{\dagger 2}_k\left\{1+i2\tilde{\omega}_k(t)+\frac{i^2}{2!}\left(2\tilde{\omega}_k(t)\right)^2+\ldots\right\} = i\sum_k\xi_k(t)a^{\dagger 2}_k e^{2i\tilde{\omega}_k(t)}\\
e^{i\hat{A}}\hat{B}_2e^{-i\hat{A}} &= \hat{B}_2+i\left[\hat{A},\hat{B}_2\right]+\frac{i^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}_2\right]\right]+\cdots = i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)\left\{a^\dagger_k a^\dagger_j+i\left(\tilde{\omega}_k(t)a^\dagger_k a^\dagger_j+a^\dagger_k a^\dagger_j\tilde{\omega}_j(t)\right)\right.\\
&\left.+\frac{i^2}{2!}\left(\tilde{\omega}^2_k(t)a^\dagger_k a^\dagger_j+\tilde{\omega}_k(t)a^\dagger_k a^\dagger_j\tilde{\omega}_j(t)+\tilde{\omega}_k(t)a^\dagger_k a^\dagger_j\tilde{\omega}_j(t)+\tilde{\omega}^2_j a^\dagger_k a^\dagger_j\right)\right\}\\
&= i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)\left(1+i\tilde{\omega}_k(t)+\frac{i^2}{2!}\tilde{\omega}^2_k(t)+\ldots\right)a^\dagger_k a^\dagger_j\left(1+i\tilde{\omega}_j(t)+\frac{i^2}{2!}\tilde{\omega}^2_j(t)+\ldots\right)\\
&= i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)e^{i\tilde{\omega}_k(t)}a^\dagger_k a^\dagger_j e^{i\tilde{\omega}_j(t)}\\
e^{i\hat{A}}\hat{B}_3e^{-i\hat{A}} &= \hat{B}_3+i\left[\hat{A},\hat{B}_3\right]+\frac{i^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}_3\right]\right]+\cdots = i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)\left\{a^\dagger_k a_j+i\left(\tilde{\omega}_k(t)a^\dagger_k a_j-a^\dagger_k a_j\tilde{\omega}_j(t)\right)\right.\\
&\left.+\frac{i^2}{2!}\left(\tilde{\omega}^2_k(t)a^\dagger_k a_j-\tilde{\omega}_k(t)a^\dagger_k a_j\tilde{\omega}_j(t)-\tilde{\omega}_k(t)a^\dagger_k a_j\tilde{\omega}_j(t)+\tilde{\omega}^2_j a^\dagger_k a_j\right)\right\}\\
&= i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)\left(1+i\tilde{\omega}_k(t)+\frac{i^2}{2!}\tilde{\omega}^2_k(t)+\ldots\right)a^\dagger_k a_j\left(1-i\tilde{\omega}_j(t)+\frac{i^2}{2!}\tilde{\omega}^2_j(t)+\ldots\right)\\
&= i\sum_{k,k'}\sum_{j(\neq k)}\mu_{jk}(t)e^{i\tilde{\omega}_k(t)}a^\dagger_k a_j e^{-i\tilde{\omega}_j(t)}
\end{aligned}
\tag{C.14}
$$

Allowing us to obtain our effective Hamiltonian in the interaction picture as

$$\begin{aligned}\hat{H}_I &= \hat{U}_0^\dagger \hat{V} \hat{U}_0 = e^{i\hat{A}} \left(\hat{B}_1 + \hat{B}_2 + \hat{B}_3 + \text{h.c.} \right) e^{-i\hat{A}} \\ &= i \sum_k \left\{ \xi_k(t) e^{i\tilde{\omega}_k(t)} a_k^{\dagger 2} e^{i\tilde{\omega}_k(t)} + \sum_{j(\neq k)} \mu_{kj}(t) \left(e^{i\tilde{\omega}_k(t)} a_k^\dagger a_j^\dagger e^{i\tilde{\omega}_j(t)} + e^{i\tilde{\omega}_k(t)} a_k^\dagger a_j e^{-i\tilde{\omega}_j(t)} \right) \right\} + \text{h.c.} \\ &= i \sum_k \left\{ e^{i\tilde{\omega}_k(t)} \left[\xi_k(t) a_k^{\dagger 2} e^{i\tilde{\omega}_k(t)} + \sum_{j(\neq k)} \mu_{kj}(t) a_k^\dagger \left(a_j^\dagger e^{i\tilde{\omega}_j(t)} + a_j e^{-i\tilde{\omega}_j(t)} \right) \right] - \text{h.c.} \right\}\end{aligned} \tag{C.15}$$

proving the last theorem. □

Appendix D

Density operator

D.1 Density operator

In this section we calculate the expression for the diagonal terms of the density operator after the following expansion

$$\hat{\rho}(t) = \hat{\rho}(0) - i\int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right] - \int_0^t dt'' \int_0^{t''} dt' \left[\hat{H}_I(t''), \left[\hat{H}_I(t'), \hat{\rho}(0)\right]\right] + \mathcal{O}(\hat{H}_I^3(t)).$$

Considering the vacuum state $\hat{\rho}(0) = |0\rangle\langle 0|$ as the initial state and the correspondent relations $\hat{\mathcal{A}}_{\mathbf{n}}\hat{\rho}(0) = \hat{\rho}(0)\mathcal{A}^{\dagger}_{\mathbf{n}} = 0$, we can obtain

$$\begin{aligned}\left[\hat{H}_I(t'), \hat{\rho}(0)\right] &= \hat{H}_I(t')|0\rangle\langle 0| - |0\rangle\langle 0|\hat{H}_I(t') \\ &= \frac{i}{2}\sum_{\mathbf{n}',\mathbf{m}'} \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\left\{\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) + \hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right\},\end{aligned} \tag{D.1}$$

for the first order term. For the second order terms

$$\begin{aligned}&\left[H(t''), \left[\hat{H}_I(t'), \rho(0)\right]\right] = H(t'')\left[\hat{H}_I(t'), \rho(0)\right] - \left[\hat{H}_I(t'), \rho(0)\right] H(t'') \\ &= \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\left[\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} - \hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\right) + \chi^{(-)}_{\mathbf{n}'',\mathbf{m}''}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''}\right)\right]\left[\hat{H}_I(t'), \hat{\rho}(0)\right] \\ &- \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\left[\hat{H}_I(t'), \hat{\rho}(0)\right]\left[\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} - \hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\right) + \chi^{(-)}_{\mathbf{n}'',\mathbf{m}''}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] - \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \\
&- \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \chi^{(-)}_{\mathbf{n}'',\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] + \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \chi^{(-)}_{\mathbf{n}'',\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \\
&+ \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \chi^{(-)}_{\mathbf{n}'',\mathbf{m}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] - \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \chi^{(-)}_{\mathbf{n}'',\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \\
&- \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''} \left[\hat{H}_I(t'), \rho(0)\right] + \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''}.
\end{aligned}
$$

Substituting the first order commutator (G.2) in the last expression, we obtain

$$
\begin{aligned}
&\left[H(t''), \left[\hat{H}_I(t'), \rho(0)\right]\right] = \\
&- \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}''} \hat{\rho}(0) + \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \right\} \\
&+ \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} + \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \right\} \\
&+ \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) + \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \right\} \\
&- \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} + \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \right\} \\
&- \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(-)}_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) + \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \right\} \\
&+ \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(-)}_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} + \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \right\} \\
&+ \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(-)}_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) + \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \right\} \\
&- \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(-)}_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''} + \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''} \right\}.
\end{aligned}
$$

From those last expressions, the only candidates for the diagonal terms of $\hat{\rho}(t)$ can be identified as

$$\hat{\rho}(t)|_{\text{diag}} = \hat{\rho}(0) - \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}\sum_{\mathbf{n}'',\mathbf{m}''}\int_0^t dt''\int_0^{t''} dt'\chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}$$
$$\times\left\{\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'} - \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} + \hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\right\}.$$

The above terms are only diagonal if

$$\hat{\rho}(t)|_{\text{diag}} = \hat{\rho}(0) - \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\int_0^t dt''\int_0^{t''} dt'\chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}$$
$$\times\left\{\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'} - \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'} + \hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\right\},$$

where the factor $2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}$ comes from the fact that we have twice more diagonal terms when $\mathbf{n}' \neq \mathbf{m}'$.

D.2 Number of particles

In this section we use the last expression for the density operator to compute the number of particles created inside the cavity. But before we shall compute the expression for the quantity $\hat{\rho}(t)\hat{N}$ using only the diagonal contribution of $\hat{\rho}$ (since we will take the trace of the expression anyway). Consider

$$\hat{\rho}(t)|_{\text{diag}}\hat{N} = \sum_{\mathbf{k}}\hat{\rho}(t)|_{\text{diag}}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}$$
$$= \frac{1}{4}\sum_{\mathbf{k},\mathbf{n}',\mathbf{m}'}2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\int_0^t dt''\int_0^{t''} dt'\chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'}\left\{\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\right\}$$
$$= \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}4^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\int_0^t dt''\int_0^{t''} dt'\chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'}\left\{\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}_{\mathbf{n}'} + \text{h.c.}\right\}.$$

Again, there will be twice more terms when $\mathbf{n}' \neq \mathbf{m}'$.

Term $2_{\mathbf{n}}$

The first diagonal coefficient of $\hat{\rho}_{diag}(t)\hat{N}$ can be computed as

$$\mathcal{N}_{\mathbf{n},\mathbf{n}}(t) = \langle 2_{\mathbf{n}} | \hat{\rho}(t)|_{\text{diag}} \hat{N} | 2_{\mathbf{n}} \rangle$$
$$\frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} 4^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} \left\{ \langle 2_{\mathbf{n}} | \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{a}^{\dagger}_{\mathbf{n}'} \hat{a}_{\mathbf{n}'} | 2_{\mathbf{n}} \rangle + \text{h.c.} \right\}.$$

As the coefficient $\langle 2_{\mathbf{n}} | \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{a}^{\dagger}_{\mathbf{n}'} \hat{a}_{\mathbf{n}'} | 2_{\mathbf{n}} \rangle$ are zero for $\mathbf{m}', \mathbf{n}' \neq \mathbf{n}$, the only surviving terms are

$$\begin{aligned}
\mathcal{N}_{\mathbf{n},\mathbf{n}}(t) &= \frac{1}{4} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{n}} \chi''^{(+)}_{\mathbf{n},\mathbf{n}} \left\{ \langle 2_{\mathbf{n}} | \hat{\mathcal{A}}''^{\dagger 2}_{\mathbf{n}} \hat{\rho}(0) \hat{\mathcal{A}}'^{2}_{\mathbf{n}} \hat{a}^{\dagger}_{\mathbf{n}} \hat{a}_{\mathbf{n}} | 2_{\mathbf{n}} \rangle + \text{h.c.} \right\} \\
&= \frac{1}{4} \langle 2_{\mathbf{n}} | \hat{a}''^{\dagger 2}_{\mathbf{n}} \hat{\rho}(0) \hat{a}'^{2}_{\mathbf{n}} \hat{a}^{\dagger}_{\mathbf{n}} \hat{a}_{\mathbf{n}} | 2_{\mathbf{n}} \rangle \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{n}} \chi''^{(+)}_{\mathbf{n},\mathbf{n}} \left\{ e^{-2i[\tilde{\omega}_{\mathbf{n}}(t'') - \tilde{\omega}_{\mathbf{n}}(t')]} + \text{h.c.} \right\} \\
&= 2\,\text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{n}} \chi''^{(+)}_{\mathbf{n},\mathbf{n}} e^{-2i[\tilde{\omega}_{\mathbf{n}}(t'') - \tilde{\omega}_{\mathbf{n}}(t')]}.
\end{aligned}$$

Term $1_{\mathbf{n}}, 1_{\mathbf{m}}$

The second type of diagonal coefficients of $\hat{\rho}_{diag}(t)\hat{N}$ can be computed as

$$\begin{aligned}
&\mathcal{N}_{\mathbf{n},\mathbf{m}}(t)|_{\mathbf{n}\neq\mathbf{m}} = \langle 1_{\mathbf{n}}, 1_{\mathbf{m}} | \rho_{\text{diag}}(t) N | 1_{\mathbf{n}}, 1_{\mathbf{m}} \rangle \\
&= \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} 4^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} \left\{ \langle 1_{\mathbf{n}}, 1_{\mathbf{m}} | \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{a}^{\dagger}_{\mathbf{n}'} \hat{a}_{\mathbf{n}'} | 1_{\mathbf{n}}, 1_{\mathbf{m}} \rangle + \text{h.c.} \right\} \\
&= \langle 1_{\mathbf{n}}, 1_{\mathbf{m}} | \hat{a}''^{\dagger}_{\mathbf{n}} \hat{a}^{\dagger}_{\mathbf{m}} \hat{\rho}(0) \hat{a}_{\mathbf{m}} \hat{a}'_{\mathbf{n}} \hat{a}^{\dagger}_{\mathbf{n}} \hat{a}_{\mathbf{n}} | 1_{\mathbf{n}}, 1_{\mathbf{m}} \rangle \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} \\
&\times \left\{ e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'') + \tilde{\omega}_{\mathbf{m}}(t'')] - [\tilde{\omega}_{\mathbf{n}}(t') + \tilde{\omega}_{\mathbf{m}}(t')]\}} + \text{h.c.} \right\} \\
&= 2\,\text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{m}} \chi''^{(+)}_{\mathbf{n},\mathbf{m}} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'') + \tilde{\omega}_{\mathbf{m}}(t'')] - [\tilde{\omega}_{\mathbf{n}}(t') + \tilde{\omega}_{\mathbf{m}}(t')]\}}.
\end{aligned}$$

Therefore, the total number of particles can be computed as

$$\begin{aligned}
\langle \hat{N}(t) \rangle &= \text{Tr} \left\{ \hat{\rho}(t)|_{\text{diag}} \hat{N} \right\} = \sum_{\mathbf{n}} \mathcal{N}_{\mathbf{n},\mathbf{n}} + \sum_{\mathbf{n}} \sum_{\mathbf{m}(\neq\mathbf{n})} \mathcal{N}_{\mathbf{n},\mathbf{m}} \\
&= \sum_{\mathbf{n},\mathbf{m}} \mathcal{N}_{\mathbf{n},\mathbf{m}} = 2\,\text{Re} \sum_{\mathbf{n},\mathbf{m}} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{m}} \chi''^{(+)}_{\mathbf{n},\mathbf{m}} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'') + \tilde{\omega}_{\mathbf{m}}(t'')] - [\tilde{\omega}_{\mathbf{n}}(t') + \tilde{\omega}_{\mathbf{m}}(t')]\}}
\end{aligned}$$

D.3 Diagonal Entropy

Let the expression for the diagonal terms of the density operator

$$\hat{\rho}(t)|_{\text{diag}} = \hat{\rho}(0) - \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''}$$
$$\times \left\{ \hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'} + \text{h.c.}\right\},$$

We can calculate the diagonal coefficients as

Term 0

$$\begin{aligned}
&\rho^{(0)}(t)|_{\text{diag}} = \langle 0|\, \hat{\rho}(t)|_{\text{diag}}\, |0\rangle \\
&= 1 - \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \left\{ \langle 0|\, \hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\, |0\rangle + \text{h.c.}\right\} \\
&= 1 - \frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'} \langle 0|\, \hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\, |0\rangle\, 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\, \text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} \\
&= 1 - \sum_{\mathbf{n}',\mathbf{m}'} \text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} \\
&= 1 - \frac{1}{2}\langle \hat{N}(t)\rangle.
\end{aligned}$$

Since $\langle 0|\, \hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\, |0\rangle = 2^{\delta_{\mathbf{n}',\mathbf{m}'}}$.

Term $2_{\mathbf{n}}$

$$\begin{aligned}
\rho^{(\mathbf{n},\mathbf{n})}(t)|_{\text{diag}} &= \langle 2_{\mathbf{n}}|\, \hat{\rho}_{\text{diag}}\, |2_{\mathbf{n}}\rangle = \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \\
&\times \left\{ \langle 2_{\mathbf{n}}|\, \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\, |2_{\mathbf{n}}\rangle + \text{h.c.}\right\} \\
&= \frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'} \langle 2_{\mathbf{n}}|\, \hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\, |2_{\mathbf{n}}\rangle\, 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\, \text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} \\
&= \text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{n}}\chi''^{(+)}_{\mathbf{n},\mathbf{n}} e^{-2i[\tilde{\omega}_{\mathbf{n}}(t'')-\tilde{\omega}_{\mathbf{n}}(t')]} \\
&= \frac{1}{2}\mathcal{N}_{\mathbf{n},\mathbf{n}}.
\end{aligned}$$

since $\langle 2_{\mathbf{n}}|\, \hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\, |2_{\mathbf{n}}\rangle = 2\delta_{\mathbf{n}',\mathbf{n}}\delta_{\mathbf{m}',\mathbf{n}}$.

Term $1_{\mathbf{n}}, 1_{\mathbf{m}}$

For the diagonal terms associated with the state $|1_{\mathbf{n}}, 1_{\mathbf{m}}\rangle$ we have

$$
\begin{aligned}
\rho^{(\mathbf{n},\mathbf{m})}(t)|_{\text{diag}} = \langle 1_{\mathbf{n}}, 1_{\mathbf{m}}| \hat{\rho}_{\text{diag}} |1_{\mathbf{n}}, 1_{\mathbf{m}}\rangle = \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \\
\times \left\{ \langle 1_{\mathbf{n}}, 1_{\mathbf{m}}| \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'} |1_{\mathbf{n}}, 1_{\mathbf{m}}\rangle + \text{h.c.}\right\} \\
= \frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'} \langle 1_{\mathbf{n}}, 1_{\mathbf{m}}| \hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'} |1_{\mathbf{n}}, 1_{\mathbf{m}}\rangle\, 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \,\text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]
\\
= \text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{m}}\chi''^{(+)}_{\mathbf{n},\mathbf{m}} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} \\
= \frac{1}{2}\mathcal{N}_{\mathbf{n},\mathbf{m}}.
\end{aligned}
$$

D.4 In the m-th mode

D.4.1 Density operator in the m-th mode

To compute the number of particles in the m-th mode, we begin computing the expression for the diagonal density operator after tracing it over the fields modes $\mathbf{m}, \mathbf{n} \neq \mathbf{k}$, such as

$$
\begin{aligned}
\hat{\rho}_{\mathbf{k}}(t)|_{\text{diag}} = \text{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}}\left(\hat{\rho}(0)\right) - \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \\
\times \left\{ \text{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}}\left(\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\right) - \text{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right) + \text{h.c.}\right\},
\end{aligned}
$$

To do so, we compute the trace operation for each one of the density operator.

First term

$$
\begin{aligned}
\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \text{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}}\left(\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\right) \\
= \sum_{\mathbf{n}'} \text{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}}\left(\hat{\mathcal{A}}''^{2}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger 2}_{\mathbf{n}'}\hat{\rho}(0)\right) + 2\sum_{\substack{\mathbf{n}',\mathbf{m}' \\ \mathbf{m}'\neq\mathbf{n}'}} \text{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}}\left(\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\right) \\
= 2\sum_{\mathbf{n}'} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} e^{-2i[\tilde{\omega}_{\mathbf{n}'}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]} |0_{\mathbf{k}}\rangle\langle 0_{\mathbf{k}}| \\
+ 2\sum_{\substack{\mathbf{n}',\mathbf{m}' \\ \mathbf{m}'\neq\mathbf{n}'}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{m}'}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{m}'}(t')]\}} |0_{\mathbf{k}}\rangle\langle 0_{\mathbf{k}}|\,;
\end{aligned}
$$

Second term

$$
\begin{aligned}
&\sum_{\mathbf{n}',\mathbf{m}'} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} \operatorname{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}} \left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right)\\
&= \sum_{\mathbf{n}'} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} \operatorname{Tr}_{\mathbf{n}'\neq\mathbf{k}} \left(\hat{\mathcal{A}}''^{\dagger 2}_{\mathbf{n}'}\hat{\rho}(0)\hat{\mathcal{A}}'^{2}_{\mathbf{n}'}\right) + \sum_{\substack{\mathbf{n}',\mathbf{m}'\\ \mathbf{m}'\neq\mathbf{n}'}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} \operatorname{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}} \left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right)\\
&= 2\chi'^{(+)}_{\mathbf{k},\mathbf{k}}\chi''^{(+)}_{\mathbf{k},\mathbf{k}} e^{-2i[\tilde{\omega}_{\mathbf{k}}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]} \left|2_{\mathbf{k}}\right\rangle\left\langle 2_{\mathbf{k}}\right| + \sum_{\mathbf{n}'(\neq\mathbf{k})} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} \operatorname{Tr}_{\mathbf{n}'\neq\mathbf{k}} \left(\hat{\mathcal{A}}''^{\dagger 2}_{\mathbf{n}'}\hat{\rho}(0)\hat{\mathcal{A}}'^{2}_{\mathbf{n}'}\right)\\
&+ 2\sum_{\mathbf{n}'\neq\mathbf{k}} \chi'^{(+)}_{\mathbf{n}',\mathbf{k}}\chi''^{(+)}_{\mathbf{n}',\mathbf{k}} \operatorname{Tr}_{\mathbf{n}'\neq\mathbf{k}} \left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{k}}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{k}}\hat{\mathcal{A}}'_{\mathbf{n}'}\right)\\
&+ \sum_{\substack{\mathbf{n}'(\neq\mathbf{k},\mathbf{m}')\\ \mathbf{m}'(\neq\mathbf{k},\mathbf{n}')}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} \operatorname{Tr}_{\mathbf{n}',\mathbf{m}'\neq\mathbf{k}} \left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\chi'^{(+)}_{\mathbf{k},\mathbf{k}}\chi''^{(+)}_{\mathbf{k},\mathbf{k}} e^{-2i[\tilde{\omega}_{\mathbf{k}}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]} \left|2_{\mathbf{k}}\right\rangle\left\langle 2_{\mathbf{k}}\right| + 2\sum_{\mathbf{n}'(\neq\mathbf{k})} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} e^{-2i[\tilde{\omega}_{\mathbf{n}'}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]} \left|0_{\mathbf{k}}\right\rangle\left\langle 0_{\mathbf{k}}\right|\\
&+ 2\sum_{\mathbf{n}'\neq\mathbf{k}} \chi'^{(+)}_{\mathbf{n}',\mathbf{k}}\chi''^{(+)}_{\mathbf{n}',\mathbf{k}} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{k}}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{k}}(t')]\}} \left|1_{\mathbf{k}}\right\rangle\left|1_{\mathbf{k}}\right\rangle\\
&+ \sum_{\substack{\mathbf{n}'(\neq\mathbf{k},\mathbf{m}')\\ \mathbf{m}'(\neq\mathbf{k},\mathbf{n}')}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{m}'}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{m}'}(t')]\}} \left|0_{\mathbf{k}}\right\rangle\left|0_{\mathbf{k}}\right\rangle\\
&= 2\chi'^{(+)}_{\mathbf{k},\mathbf{k}}\chi''^{(+)}_{\mathbf{k},\mathbf{k}} e^{-2i[\tilde{\omega}_{\mathbf{k}}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]} \left|2_{\mathbf{k}}\right\rangle\left\langle 2_{\mathbf{k}}\right| + 2\sum_{\mathbf{n}'(\neq\mathbf{k})} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} e^{-2i[\tilde{\omega}_{\mathbf{n}'}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]} \left|0_{\mathbf{k}}\right\rangle\left\langle 0_{\mathbf{k}}\right|\\
&+ 2\sum_{\mathbf{n}'\neq\mathbf{k}} \chi'^{(+)}_{\mathbf{n}',\mathbf{k}}\chi''^{(+)}_{\mathbf{n}',\mathbf{k}} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{k}}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{k}}(t')]\}} \left|1_{\mathbf{k}}\right\rangle\left|1_{\mathbf{k}}\right\rangle\\
&+ \sum_{\substack{\mathbf{n}'(\neq\mathbf{k},\mathbf{m}')\\ \mathbf{m}'(\neq\mathbf{k},\mathbf{n}')}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{m}'}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{m}'}(t')]\}} \left|0_{\mathbf{k}}\right\rangle\left|0_{\mathbf{k}}\right\rangle .
\end{aligned}
$$

If we trace out the modes different from $\mathbf{n}, \mathbf{m} = \mathbf{k}$ we must obtain

$$
\begin{aligned}
\hat{\rho}_{\mathbf{k}}(t)|_{\text{diag}} = & \left|0_{\mathbf{k}}\right\rangle \left\langle 0_{\mathbf{k}}\right| - \frac{1}{2} \operatorname{Re} \int_0^t dt'' \int_0^{t''} dt' \Bigg\{ 2 \sum_{\mathbf{n}'} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} e^{-2i[\tilde{\omega}_{\mathbf{n}}(t'') - \tilde{\omega}_{\mathbf{n}'}(t')]} \left|0_{\mathbf{k}}\right\rangle \left\langle 0_{\mathbf{k}}\right| \\
& + \sum_{\substack{\mathbf{n}',\mathbf{m}' \\ \mathbf{m}' \neq \mathbf{n}'}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'') + \tilde{\omega}_{\mathbf{m}}(t'')] - [\tilde{\omega}_{\mathbf{n}}(t') + \tilde{\omega}_{\mathbf{m}}(t')]\}} \left|0_{\mathbf{k}}\right\rangle \left\langle 0_{\mathbf{k}}\right| - 2 \chi'^{(+)}_{\mathbf{k},\mathbf{k}} \chi''^{(+)}_{\mathbf{k},\mathbf{k}} e^{-2i[\tilde{\omega}_{\mathbf{k}}(t'') - \tilde{\omega}_{\mathbf{n}'}(t')]} \left|2_{\mathbf{k}}\right\rangle \left\langle 2_{\mathbf{k}}\right| \\
& - 2 \sum_{\mathbf{n}'(\neq \mathbf{k})} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} e^{-2i[\tilde{\omega}_{\mathbf{n}'}(t'') - \tilde{\omega}_{\mathbf{n}'}(t')]} \left|0_{\mathbf{k}}\right\rangle \left\langle 0_{\mathbf{k}}\right| \\
& - 4 \sum_{\mathbf{n}' \neq \mathbf{k}} \chi'^{(+)}_{\mathbf{n}',\mathbf{k}} \chi''^{(+)}_{\mathbf{n}',\mathbf{k}} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'') + \tilde{\omega}_{\mathbf{k}}(t'')] - [\tilde{\omega}_{\mathbf{n}'}(t') + \tilde{\omega}_{\mathbf{k}}(t')]\}} \left|1_{\mathbf{k}}\right\rangle \left\langle 1_{\mathbf{k}}\right| \\
& - 2 \sum_{\substack{\mathbf{n}'(\neq \mathbf{k},\mathbf{m}') \\ \mathbf{m}'(\neq \mathbf{k},\mathbf{n}')}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'') + \tilde{\omega}_{\mathbf{m}'}(t'')] - [\tilde{\omega}_{\mathbf{n}'}(t') + \tilde{\omega}_{\mathbf{m}'}(t')]\}} \left|0_{\mathbf{k}}\right\rangle \left\langle 0_{\mathbf{k}}\right| \Bigg\}.
\end{aligned}
$$

D.4.2 Number of particles in the k-th mode

Let the expression for

$$
\begin{aligned}
\hat{\rho}_{\mathbf{k}}(t)|_{\text{diag}} \hat{N} = \operatorname{Re} \int_0^t dt'' \int_0^{t''} dt' \Bigg\{ & \chi'^{(+)}_{\mathbf{k},\mathbf{k}} \chi''^{(+)}_{\mathbf{k},\mathbf{k}} \left|2_{\mathbf{k}}\right\rangle \left\langle 2_{\mathbf{k}}\right| \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \\
& + 2 \sum_{\mathbf{n}' \neq \mathbf{k}} \chi'^{(+)}_{\mathbf{n}',\mathbf{k}} \chi''^{(+)}_{\mathbf{n}',\mathbf{k}} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'') + \tilde{\omega}_{\mathbf{k}}(t'')] - [\tilde{\omega}_{\mathbf{n}'}(t') + \tilde{\omega}_{\mathbf{k}}(t')]\}} \left|1_{\mathbf{k}}\right\rangle \left\langle 1_{\mathbf{k}}\right| \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \Bigg\},
\end{aligned}
$$

therefore

$$
\begin{aligned}
\langle N_{\mathbf{k}}(t) \rangle &= \operatorname{Tr} \left\{ \hat{\rho}_{\mathbf{k}}(t)|_{\text{diag}} \hat{N} \right\} \\
&= 2 \operatorname{Re} \sum_{\mathbf{n}'} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n}',\mathbf{k}} \chi''^{(+)}_{\mathbf{n}',\mathbf{k}} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'') + \tilde{\omega}_{\mathbf{k}}(t'')] - [\tilde{\omega}_{\mathbf{n}'}(t') + \tilde{\omega}_{\mathbf{k}}(t')]\}} \\
&= \sum_{\mathbf{n}'} \mathcal{N}_{\mathbf{n}',\mathbf{k}}
\end{aligned}
$$

D.4.3 Diagonal entropy in the $\mathbf{k}$-th mode

Term $0_\mathbf{k}$

$$
\begin{aligned}
&\rho_\mathbf{k}^{(0)}(t) = \langle 0_\mathbf{k}|\,\hat{\rho}_\mathbf{k}\,|0_\mathbf{k}\rangle = 1 - \mathrm{Re}\int_0^t dt'' \int_0^{t''} dt' \Bigg\{ \sum_{\mathbf{n}'} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} e^{-2i[\tilde{\omega}_\mathbf{n}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]} \\
&- \sum_{\mathbf{n}'(\neq\mathbf{k})} \chi'^{(+)}_{\mathbf{n}',\mathbf{n}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{n}'} e^{-2i[\tilde{\omega}_{\mathbf{n}'}(t'')-\tilde{\omega}_{\mathbf{n}'}(t')]} \\
&+ \sum_{\mathbf{n}',\mathbf{m}'} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} e^{-i\{[\tilde{\omega}_\mathbf{n}(t'')+\tilde{\omega}_\mathbf{m}(t'')]-[\tilde{\omega}_\mathbf{n}(t')+\tilde{\omega}_\mathbf{m}(t')]\}} \\
&- \sum_{\substack{\mathbf{n}'(\neq\mathbf{k},\mathbf{m}') \\ \mathbf{m}'(\neq\mathbf{k},\mathbf{n}')}} \chi'^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi''^{(+)}_{\mathbf{n}',\mathbf{m}'} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_{\mathbf{m}'}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_{\mathbf{m}'}(t')]\}} \Bigg\} \\
&= 1 - \mathrm{Re}\int_0^t dt'' \int_0^{t''} dt' \Bigg\{ \chi'^{(+)}_{\mathbf{k},\mathbf{k}} \chi''^{(+)}_{\mathbf{k},\mathbf{k}} e^{-2i[\tilde{\omega}_\mathbf{k}(t'')-\tilde{\omega}_\mathbf{k}(t')]} \\
&+ \sum_{\mathbf{n}'(\neq\mathbf{k})} \chi'^{(+)}_{\mathbf{n}',\mathbf{k}} \chi''^{(+)}_{\mathbf{n}',\mathbf{k}} e^{-i\{[\tilde{\omega}_\mathbf{n}(t'')+\tilde{\omega}_\mathbf{k}(t'')]-[\tilde{\omega}_\mathbf{n}(t')+\tilde{\omega}_\mathbf{k}(t')]\}} \Bigg\} \\
&1 - \frac{1}{2}\langle \hat{N}_\mathbf{k}(t)\rangle.
\end{aligned}
$$

Term $2_\mathbf{k}$

$$
\begin{aligned}
\rho_\mathbf{k}^{(2_\mathbf{k})} &= \langle 2_\mathbf{k}|\,\hat{\rho}_\mathbf{k}\,|2_\mathbf{k}\rangle = \mathrm{Re}\int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{k},\mathbf{k}} \chi''^{(+)}_{\mathbf{k},\mathbf{k}} e^{-2i[\tilde{\omega}_\mathbf{k}(t'')-\tilde{\omega}_\mathbf{k}(t')]} \\
&= \frac{1}{2}\mathcal{N}_{\mathbf{k},\mathbf{k}}
\end{aligned}
$$

Term $1_\mathbf{k}$

$$
\begin{aligned}
\rho_\mathbf{k}^{(1_\mathbf{k})} &= \langle 1_\mathbf{k}|\,\hat{\rho}_\mathbf{k}\,|1_\mathbf{k}\rangle = \mathrm{Re}\sum_{\mathbf{n}'\neq\mathbf{k}} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n}',\mathbf{k}} \chi''^{(+)}_{\mathbf{n}',\mathbf{k}} e^{-i\{[\tilde{\omega}_{\mathbf{n}'}(t'')+\tilde{\omega}_\mathbf{k}(t'')]-[\tilde{\omega}_{\mathbf{n}'}(t')+\tilde{\omega}_\mathbf{k}(t')]\}} \\
&= \frac{1}{2}\sum_{\mathbf{n}'\neq\mathbf{k}} \mathcal{N}_{\mathbf{n}',\mathbf{k}}
\end{aligned}
$$

Appendix E

Rotating Wave Approximation

E.1 Number of particles

In order to show the adequacy of our latter formalism we shall reproduce the literature results by computing our quantities when there is resonance between the cavity and some integral multiple of some unperturbed field frequency.

E.1.1 Total number of particles

To do so, we first consider the one-dimensional case in which the field frequency takes the form of

$$\omega_n = \frac{n\pi}{L},$$

such that

$$v^{(+)}_{n,m}(t) = \frac{1}{2}\frac{\omega_m^2 - \omega_n^2}{\sqrt{\omega_m \omega_n}} g_{m,n}(t) - L(t)\frac{\partial \omega_{n,L(t)}}{\partial L}\delta_{m,n}$$

for

$$g_{n,m} = -g_{m,n} = -L(t)\int_0^{L(t)} dz\ \varphi_n \frac{\partial \varphi_m}{\partial L}.$$

In the case of weak perturbed motion $L(t) = L_0\left[1 + \epsilon\xi(t)\right]$, we must obtain

$$g_{n,m}(t) = \begin{cases} (-1)^{m+n}\frac{2mn}{m^2-n^2} & n \neq m \\ 0 & n = m, \end{cases}$$

meaning

$$\chi^{(+)}_{n,m}(t) = -\epsilon\frac{v^{(+)}_{n,m}}{\omega_n + \omega_m}\dot{\xi}(t) \quad \text{with} \quad v^{(+)}_{n,m} = \frac{\pi}{L_0}\begin{cases} (-1)^{m+n}\sqrt{mn} & n \neq m \\ n & n = m \end{cases}.$$

meaning $v_{nm}^{(+)2} = \frac{\pi^2}{L_0^2} nm$. From this we can compute the total number of particles created in second order of ϵ as

$$N(t) = \sum_{n,m} \mathcal{N}_{n,m},$$

with

$$\begin{aligned}\mathcal{N}_{n,m} &= 2\,\mathrm{Re}\int_0^T dt'' \int_0^{t''} dt'\ \chi_{k',j'}(t')\chi_{k',j'}(t'')e^{-i[\bar{\omega}_n(t')+\bar{\omega}_m(t')-(\bar{\omega}_n(t'')+\bar{\omega}_m(t''))]} \\ &= \frac{2\epsilon^2 v_{m,n}^2}{(\omega_n+\omega_m)^2}\int_0^T dt'' \int_0^{t''} dt'\ \frac{d}{dt'}\xi(t')\frac{d}{dt''}\xi(t'')e^{-i(\omega_n+\omega_m)(t'-t'')}. \end{aligned} \tag{E.1}$$

To simplify this last integral, we can consider the following equality

$$\begin{aligned}&\int_0^T dt'' \int_0^{t''} dt'\ \frac{d}{dt'}\xi(t')\frac{d}{dt''}\xi(t'')e^{-i(\omega_n+\omega_m)(t'-t'')} \\ &\qquad = \frac{1}{2}\int_0^T dt'' \frac{d}{dt''}\xi(t'')e^{i(\omega_n+\omega_m)t''}\int_0^T dt'\ \frac{d}{dt'}\xi(t')e^{-i(\omega_n+\omega_m)t'}\end{aligned}$$

Remembering also that as the cavity remain at rest for instants of time $t < 0$ and $t > T$, the motion function $\xi(t)$ must respects $\xi(0) = \xi(T) = 0$. Using those last considerations by integration by parts the expression (E.1), we must obtain

$$\mathcal{N}_{n,m} = \epsilon^2 v_{n,m}^2\,\mathrm{Re}\int_0^T dt'' \int_0^T dt'\ \xi(t')\xi(t'')e^{-i(\omega_n+\omega_m)(t'-t'')}.$$

In this faction, we can interpret the coefficient as the square of the Bogoliubov coefficients

$$\mathcal{N}_{n,m} = |\beta_{n,m}|^2,$$

with

$$\beta_{n,m} = -i\epsilon v_{n,m}\int_0^T dt\ \xi(t)e^{-i(\omega_n+\omega_m)t}.$$

Considering the sinusoidal cavity motion $\xi(t) = \sin\Omega_p t$ with $\Omega = p\omega_1$ and $p = 1, 2, \ldots$, we can compute

$$\begin{aligned}\mathcal{N}_{m,n} &= \epsilon^2 v_{m,n}^2 \int_0^T dt'' \sin\Omega_p t'' e^{i(\omega_n+\omega_m)t''} \int_0^T dt' \sin\Omega_p t' e^{-i(\omega_n+\omega_m)t'} \\ &= -\frac{1}{4}\epsilon^2 v_{m,n}^2 \int_0^T dt'' \left[e^{i[\Omega_p+(\omega_n+\omega_m)]t''} - e^{-i[\Omega_p-(\omega_n+\omega_m)t'']}\right] \\ &\times \int_0^T dt' \left[e^{i[\Omega_p-(\omega_n+\omega_m)]t'} - e^{-i[\Omega_p+(\omega_n+\omega_m)]t'}\right].\end{aligned}$$

Considering the resonance condition $\Omega_p = \omega_n + \omega_m$, by applying the rotating wave approximation where we ignore exponentials with arguments greater than unity (as they oscillates so rapidly that in average its contributions can be neglected), we finally obtain

$$\begin{aligned}\mathcal{N}_{n,m} &= \frac{1}{4}\epsilon^2 v_{m,n}^2 T^2 \delta_{n,p-m} \\ &= \frac{\epsilon^2\pi^2}{4L_0^2} T^2 nm \delta_{n,p-m}\end{aligned}$$

Where as $m = p - n$ with $n = 1, 2, \ldots$, we have $m = 1, \ldots p - 1$. Therefore,

$$\begin{aligned}N(t) = \sum_{n,m} \mathcal{N}_{n,m} &= \frac{\epsilon^2\pi^2}{4L_0^2} T^2 \sum_{m=1}^{p-1} m(m-p) \\ &= \frac{\epsilon^2\pi^2}{4L_0^2} T^2 \left(\frac{p(p^2-1)}{6}\right) \\ &= \frac{\epsilon^2\pi^2}{24L_0^2} p(p^2-1)T^2.\end{aligned}$$

E.1.2 Number of particles in the k-th mode

The number of particles created in the k-mode in second order of time can be calculated through

$$\begin{aligned}\langle \hat{N}_k(t) \rangle &= \sum_n \mathcal{N}_{n,k} \\ &= \frac{\epsilon^2\pi^2}{4L_0^2} T^2 \sum_n nk\, \delta_{n,p-k} \\ &= \frac{\epsilon^2\pi^2}{4L_0^2} k(p-k)T^2,\end{aligned}$$

where $k = 1, \ldots, p - 1$.

E.2 Diagonal entropy

Considering

$$\mathcal{N}_{n,m} = \frac{6}{p(p^2-1)} N(t) nm \delta_{n,p-m},$$

the diagonal entropy can be written as

$$\begin{aligned}
S_d(t) &= -\rho^{(0)}_{\text{diag}}(t) \ln \rho^{(0)}_{\text{diag}}(t) - \sum_{n,m} \rho^{n,m}_{\text{diag}}(t) \ln \rho^{n,m}_{\text{diag}}(t) \\
&= -\left(1 - \frac{1}{2}\langle N(t)\right) \ln \left(1 - \frac{1}{2}\langle N(t)\right) - \sum_{n,m} \frac{1}{2}\mathcal{N}_{n,m} \ln \frac{1}{2}\mathcal{N}_{n,m} \\
&= \frac{1}{2}N(t) - \sum_{n,m} \frac{1}{2}\mathcal{N}_{n,m} \ln \frac{1}{2}\frac{6}{p(p^2-1)} N(t) nm \delta_{n,p-m} \\
&= \frac{1}{2}N(t) - \sum_{n,m} \frac{1}{2}\mathcal{N}_{n,m} \ln \frac{1}{2}\frac{6}{p(p^2-1)} N(t) - \sum_{n,m} \frac{1}{2}\mathcal{N}_{n,m} \ln nm\delta_{n,p-m} \\
&= \frac{1}{2}\langle N(t) - \frac{1}{2}N(t) \ln \frac{1}{2}N(t) - \frac{1}{2}N(t) \ln \frac{6}{p(p^2-1)} \\
&- \frac{1}{2}\frac{6}{p(p^2-1)} N(t) \sum_{n,m} nm\delta_{n,p-m} \ln nm\delta_{n,p-m} \\
&= \frac{1}{2}N(t)\left[1 - \ln\frac{1}{2}N(t) + \ln\frac{p(p^2-1)}{6} - \frac{6f(p)}{p(p^2-1)}\right]
\end{aligned}$$

where $f(p) = \sum_{m=1}^{p-1} (p-m)m \ln(p-m)m$

Appendix F

Computing $\mathcal{N}_{n,m}$

Let the function $\mathcal{N}_{n,m}$, with $\Omega = \Omega_\mu$ and $\omega = \omega_n + \omega_m$

$$\mathcal{N}_{n,m} = \frac{2\epsilon^2 v^2}{\omega^2} \int_0^t dt' \int_0^{t'} dt'' \frac{d}{dt''}\xi(t'') \frac{d}{dt'}\xi(t') \cos\omega(t'' - t').$$

Using $\xi(t) = \sin\Omega_\mu t$, we can develop the last integrant as

$$\begin{aligned}
&\frac{d}{dt''}\xi(t'') \frac{d}{dt'}\xi(t') \cos\omega(t'' - t') = \Omega^2 \cos\Omega t'' \cos\Omega t' \cos\omega(t'' - t') \\
&= \frac{\Omega^2}{8} \left(e^{i\Omega t''} + e^{-i\Omega_p t''}\right) \left(e^{i\Omega t'} + e^{-i\Omega t'}\right) \left(e^{i\omega(t''-t')} + e^{-i\omega(t''-t')}\right) \\
&= \frac{\Omega^2}{8} \left(e^{i\Omega t'} + e^{-i\Omega t'}\right) \left[\left(e^{i(\Omega+\omega)t''} + e^{-i(\Omega-\omega)t''}\right) e^{-i\omega t'} + \left(e^{i(\Omega-\omega)t''} + e^{-i(\Omega+\omega)t''}\right) e^{i\omega t'}\right] \\
&= \frac{\Omega^2}{8} \left[\left(e^{i(\Omega+\omega)t''} + e^{-i(\Omega-\omega)t''}\right) \left(e^{i(\Omega-\omega)t'} + e^{-i(\Omega+\omega)t'}\right) \right. \\
&\qquad \left. + \left(e^{i(\Omega-\omega)t''} + e^{-i(\Omega+\omega)t''}\right) \left(e^{i(\Omega+\omega)t'} + e^{-i(\Omega-\omega)t'}\right) \right],
\end{aligned}$$

F.1 $\Omega \neq \omega$

$$\begin{aligned}
\mathcal{N}_{n,m} &= \frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \int_0^t dt' \int_0^{t'} dt'' \left[\left(e^{i(\Omega+\omega)t''} + e^{-i(\Omega-\omega)t''}\right) \left(e^{i(\Omega-\omega)t'} + e^{-i(\Omega+\omega)t'}\right) \right. \\
&\quad \left. + \left(e^{i(\Omega-\omega)t''} + e^{-i(\Omega+\omega)t''}\right) \left(e^{i(\Omega+\omega)t'} + e^{-i(\Omega-\omega)t'}\right) \right] \\
&= \frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \int_0^t dt' \left[\left(e^{i(\Omega-\omega)t'} + e^{-i(\Omega+\omega)t'}\right) \int_0^{t'} dt'' \left(e^{i(\Omega+\omega)t''} + e^{-i(\Omega-\omega)t''}\right) \right. \\
&\quad \left. + \left(e^{i(\Omega+\omega)t'} + e^{-i(\Omega-\omega)t'}\right) \int_0^{t'} dt'' \left(e^{i(\Omega-\omega)t''} + e^{-i(\Omega+\omega)t''}\right) \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Omega^2}{4i\omega^2}\epsilon^2 v^2 \int_0^t dt' \Bigg[\left(e^{i(\Omega-\omega)t'} + e^{-i(\Omega+\omega)t'}\right) \left(\frac{e^{i(\Omega+\omega)t''}}{\Omega+\omega} - \frac{e^{-i(\Omega-\omega)t''}}{\Omega-\omega}\right)\Bigg|_0^{t'} \\
&\qquad + \left(e^{i(\Omega+\omega)t'} + e^{-i(\Omega-\omega)t'}\right) \left(\frac{e^{i(\Omega-\omega)t''}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t''}}{\Omega+\omega}\right)\Bigg|_0^{t'}\Bigg] \\
&= \frac{\Omega^2}{4i\omega^2}\epsilon^2 v^2 \int_0^t dt' \Bigg[\left(e^{i(\Omega-\omega)t'} + e^{-i(\Omega+\omega)t'}\right) \left(\frac{e^{i(\Omega+\omega)t'}}{\Omega+\omega} - \frac{e^{-i(\Omega-\omega)t'}}{\Omega-\omega} - \frac{1}{\Omega+\omega} + \frac{1}{\Omega-\omega}\right) \\
&\qquad + \left(e^{i(\Omega+\omega)t'} + e^{-i(\Omega-\omega)t'}\right) \left(\frac{e^{i(\Omega-\omega)t'}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t'}}{\Omega+\omega} - \frac{1}{\Omega-\omega} + \frac{1}{\Omega+\omega}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Omega^2}{4i\omega^2}\epsilon^2 v^2 \int_0^t dt' \Bigg[e^{i(\Omega-\omega)t'} \left(\frac{e^{i(\Omega+\omega)t'}}{\Omega+\omega} - \frac{e^{-i(\Omega-\omega)t'}}{\Omega-\omega} + \frac{2\omega}{\Omega^2-\omega^2}\right) \\
&\qquad + e^{-i(\Omega+\omega)t'} \left(\frac{e^{i(\Omega+\omega)t'}}{\Omega+\omega} - \frac{e^{-i(\Omega-\omega)t'}}{\Omega-\omega} + \frac{2\omega}{\Omega^2-\omega^2}\right) \\
&\qquad + e^{i(\Omega+\omega)t'} \left(\frac{e^{i(\Omega-\omega)t'}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t'}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\right) \\
&\qquad + e^{-i(\Omega-\omega)t'} \left(\frac{e^{i(\Omega-\omega)t'}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t'}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\right)\Bigg] \\
&= \frac{\Omega^2}{4i\omega^2}\epsilon^2 v^2 \int_0^t dt' \Bigg[\left(\frac{e^{2i\Omega t'}}{\Omega+\omega} - \frac{1}{\Omega-\omega} + \frac{2\omega}{\Omega^2-\omega^2} e^{i(\Omega-\omega)t'}\right) \\
&\qquad + \left(\frac{1}{\Omega+\omega} - \frac{e^{-2i\Omega t'}}{\Omega-\omega} + \frac{2\omega}{\Omega^2-\omega^2} e^{-i(\Omega+\omega)t'}\right) \\
&\qquad + \left(\frac{e^{2i\Omega t'}}{\Omega-\omega} - \frac{1}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2} e^{i(\Omega+\omega)t'}\right) \\
&\qquad + \left(\frac{1}{\Omega-\omega} - \frac{e^{-2i\Omega t'}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2} e^{-i(\Omega-\omega)t'}\right)\Bigg] \\
&= \frac{\Omega^2}{4i\omega^2}\epsilon^2 v^2 \int_0^t dt' \Bigg[\frac{2\Omega}{\Omega^2-\omega^2}\left(e^{2i\Omega t'} - e^{-2i\Omega t'}\right) + \frac{2\omega}{\Omega^2-\omega^2}\left(e^{i(\Omega-\omega)t'} - e^{-i(\Omega-\omega)t'}\right) \\
&\qquad - \frac{2\omega}{\Omega^2-\omega^2}\left(e^{i(\Omega+\omega)t'} - e^{-i(\Omega+\omega)t'}\right)\Bigg] \\
&= \frac{\Omega^2}{2\omega^2}\frac{\epsilon^2 v^2}{\Omega^2-\omega^2} \int_0^t dt' \Big[2\Omega \sin 2\Omega t' + 2\omega \sin(\Omega-\omega)t' - 2\omega \sin(\Omega+\omega)t' \Big] \\
&\qquad = \frac{\Omega^2}{2\omega^2}\frac{\epsilon^2 v^2}{\Omega^2-\omega^2} \left[\frac{2\omega}{\Omega+\omega}\cos(\Omega+\omega)t' - \frac{2\omega}{\Omega-\omega}\cos(\Omega-\omega)t' - \cos 2\Omega t'\right]\Bigg|_0^t
\end{aligned}
$$

$$= \frac{\Omega^2}{2\omega^2}\frac{\epsilon^2 v^2}{\Omega^2-\omega^2}\left[1+\frac{4\omega^2}{\Omega^2-\omega^2}+\frac{2\omega}{\Omega+\omega}\cos(\Omega+\omega)t-\frac{2\omega}{\Omega-\omega}\cos(\Omega-\omega)t-\cos 2\Omega t\right]$$

F.2 $\Omega = \omega$

$$\begin{aligned}
\frac{d}{dt''}\xi(t'')\frac{d}{dt'}\xi(t')\cos\omega(t''-t') &= \omega^2\cos\omega t''\cos\omega t'\cos\omega(t''-t') \\
&= \omega^2\cos\omega t''\cos\omega t'\left(\cos\omega t''\cos\omega t'+\sin\omega t''\sin\omega t'\right) \\
&= \frac{1}{4}\omega^2\left[\sin 2\omega t''\sin 2\omega t'+\left(1+\cos 2\omega t''\right)\left(1+\cos 2\omega t'\right)\right] \\
&= \frac{1}{4}\omega^2\left[\sin 2\omega t''\sin 2\omega t'+\cos 2\omega t'\cos 2\omega t''+\cos 2\omega t''+\cos 2\omega t'+1\right] \\
&= \frac{1}{4}\omega^2\left[\cos 2\omega(t''-t')+\cos 2\omega t''+\cos 2\omega t'+1\right]
\end{aligned}$$

integrating the last expression

$$\begin{aligned}
\mathcal{N}_{n,m} &= \frac{1}{2}\epsilon^2 v^2\int_0^t dt'\int_0^{t'} dt''\left[\cos 2\omega(t''-t')+\cos 2\omega t''+\cos 2\omega t'+1\right] \\
&= \frac{1}{2}\epsilon^2 v^2\int_0^t dt'\left[\frac{1}{2\omega}\sin 2\omega(t''-t')+\frac{1}{2\omega}\sin 2\omega t''+t''\cos 2\omega t'+t''\right]\Bigg|_0^{t'} \\
&= \frac{1}{2}\epsilon^2 v^2\int_0^t dt'\left[t'\cos 2\omega t'+t'\right] \\
&= \frac{1}{8}\frac{\epsilon^2 v^2}{\omega^2}\left[2\omega^2 t^2+2\omega t\sin(2\omega t)+\cos(2\omega t)-1\right]
\end{aligned}$$

F.3 the function

$$\mathcal{N}_{\mathbf{n},\mathbf{m}}(t) = \begin{cases} \frac{1}{2}\frac{\epsilon^2 v^2}{\omega^2}\left[\omega^2 t^2+2\omega t\sin(2\omega t)+\cos(2\omega t)-1\right] & \text{for} \quad \Omega=\omega \\ \frac{\Omega^2}{2\omega^2}\frac{\epsilon^2 v^2}{\Omega^2-\omega^2}\left[1+\frac{4\omega^2}{\Omega^2-\omega^2}+\frac{2\omega}{\Omega+\omega}\cos(\Omega+\omega)t-\frac{2\omega}{\Omega-\omega}\cos(\Omega-\omega)t-\cos 2\Omega t\right] & \text{for} \quad \Omega\neq\omega. \end{cases} \tag{F.1}$$

Using the simplification that in the end of the process $\Omega T = 2p\pi$ $(p = 1, 2, \dots)$

$$\mathcal{N}_{\mathbf{n},\mathbf{m}}(t) = nm\begin{cases} \tau^2 & \text{for} \quad \Omega=\omega \\ \frac{4\epsilon^2\omega_1^2\Omega^2}{(\Omega^2-\omega^2)^2}\sin^2\frac{\omega T}{2} & \text{for} \quad \Omega\neq\omega. \end{cases} \tag{F.2}$$

F.4 Other

Now consider

$$
\begin{aligned}
\mathcal{N}_{n,m} &= \frac{\epsilon^2 v^2}{\omega^2}\,\mathrm{Re}\int_0^t dt''\frac{d}{dt''}\xi(t'')e^{i\omega t''}\int_0^t dt'\;\frac{d}{dt'}\xi(t')e^{-i\omega t'} \\
&= \frac{\Omega^2}{\omega^2}\epsilon^2 v^2\,\mathrm{Re}\int_0^t dt''\cos\Omega t'' e^{i\omega t''}\int_0^t dt'\;\cos\Omega t' e^{-i\omega t'} \\
&= \frac{\Omega^2}{4\omega^2}\epsilon^2 v^2\,\mathrm{Re}\int_0^t dt''\left[e^{i\Omega t''}+e^{-i\Omega t''}\right]e^{i\omega t''}\int_0^t dt'\;\left[e^{i\Omega t'}+e^{-i\Omega t'}\right]e^{-i\omega t'} \\
&= \frac{\Omega^2}{4\omega^2}\epsilon^2 v^2\,\mathrm{Re}\int_0^t dt''\left[e^{i(\Omega+\omega)t''}+e^{-i(\Omega-\omega)t''}\right]\int_0^t dt'\;\left[e^{i(\Omega-\omega)t'}+e^{-i(\Omega+\omega)t'}\right] \\
&= -\frac{\Omega^2}{4\omega^2}\epsilon^2 v^2\,\mathrm{Re}\left[\frac{e^{i(\Omega+\omega)t''}}{\Omega+\omega}-\frac{e^{-i(\Omega-\omega)t''}}{\Omega-\omega}\right]\Bigg|_0^t\left[\frac{e^{i(\Omega-\omega)t'}}{\Omega-\omega}-\frac{e^{-i(\Omega+\omega)t'}}{\Omega+\omega}\right]\Bigg|_0^t
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\Omega^2}{4\omega^2}\epsilon^2 v^2\,\mathrm{Re}\left[\frac{e^{i(\Omega+\omega)t}}{\Omega+\omega}-\frac{e^{-i(\Omega-\omega)t}}{\Omega-\omega}-\frac{1}{\Omega+\omega}+\frac{1}{\Omega-\omega}\right]\left[\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega}-\frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega}-\frac{1}{\Omega-\omega}+\frac{1}{\Omega+\omega}\right] \\
&= -\frac{\Omega^2}{4\omega^2}\epsilon^2 v^2\,\mathrm{Re}\left[\frac{e^{i(\Omega+\omega)t}}{\Omega+\omega}-\frac{e^{-i(\Omega-\omega)t}}{\Omega-\omega}+\frac{2\omega}{\Omega^2-\omega^2}\right]\left[\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega}-\frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega}-\frac{2\omega}{\Omega^2-\omega^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \operatorname{Re}\Bigg\{ \frac{e^{i(\Omega+\omega)t}}{\Omega+\omega}\left[\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\right] \\
&\qquad\qquad - \frac{e^{-i(\Omega-\omega)t}}{\Omega-\omega}\left[\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\right] \\
&\qquad\qquad + \frac{2\omega}{\Omega^2-\omega^2}\left[\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega} - \frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\right]\Bigg\} \\
&= -\frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \operatorname{Re}\Bigg\{ \frac{e^{2i\Omega t}}{\Omega^2-\omega^2} - \frac{1}{(\Omega+\omega)^2} - \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{i(\Omega+\omega)t}}{\Omega+\omega} \\
&\qquad\qquad - \frac{1}{(\Omega-\omega)^2} + \frac{e^{-2i\Omega t}}{\Omega^2-\omega^2} + \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{-i(\Omega-\omega)t}}{\Omega-\omega} \\
&\qquad\qquad + \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega} - \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega} - \frac{4\omega^2}{(\Omega^2-\omega^2)^2}\Bigg\} \\
&= -\frac{\Omega^2}{4\omega^2}\epsilon^2 v^2 \operatorname{Re}\Bigg\{ \frac{e^{2i\Omega t}}{\Omega^2-\omega^2} + \frac{e^{-2i\Omega t}}{\Omega^2-\omega^2} - \frac{1}{(\Omega+\omega)^2} - \frac{1}{(\Omega-\omega)^2} \\
&\qquad\qquad + \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{i(\Omega-\omega)t}}{\Omega-\omega} + \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{-i(\Omega-\omega)t}}{\Omega-\omega} \\
&\qquad\qquad - \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{i(\Omega+\omega)t}}{\Omega+\omega} - \frac{2\omega}{\Omega^2-\omega^2}\frac{e^{-i(\Omega+\omega)t}}{\Omega+\omega} - \frac{4\omega^2}{(\Omega^2-\omega^2)^2}\Bigg\} \\
&= -\frac{\Omega^2}{2\omega^2}\epsilon^2 v^2 \operatorname{Re}\Bigg\{ \frac{\cos 2\Omega t}{\Omega^2-\omega^2} - \frac{\Omega^2+\omega^2}{(\Omega^2-\omega^2)^2} + \frac{2\omega}{\Omega^2-\omega^2}\frac{\cos(\Omega-\omega)t}{\Omega-\omega} \\
&\qquad\qquad - \frac{2\omega}{\Omega^2-\omega^2}\frac{\cos(\Omega+\omega)t}{\Omega+\omega} - \frac{2\omega^2}{(\Omega^2-\omega^2)^2}\Bigg\} \\
&= \frac{1}{2}\frac{\Omega^2}{\omega^2}\frac{\epsilon^2 v^2}{\Omega^2-\omega^2}\left\{1 + \frac{4\omega^2}{\Omega^2-\omega^2} - \frac{2\omega}{\Omega-\omega}\cos(\Omega-\omega)t + \frac{2\omega}{\Omega+\omega}\cos(\Omega+\omega)t - \cos 2\Omega t\right\}
\end{aligned}
$$

Appendix G

Density operator

G.1 Density operator

In this section we calculate the expression for the diagonal terms of the density operator after the following expansion

$$\hat{\rho}(t) = \hat{\rho}(0) - i\int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right] - \int_0^t dt'' \int_0^{t''} dt' \left[\hat{H}_I(t''), \left[\hat{H}_I(t'), \hat{\rho}(0)\right]\right] + \mathcal{O}(\hat{H}_I^3(t)).$$

Considering the vacuum state $\hat{\rho}(0) = |0\rangle\langle 0|$ as the initial state and the correspondent relations $\hat{\mathcal{A}}_{\mathbf{n}}\hat{\rho}(0) = \hat{\rho}(0)\mathcal{A}^{\dagger}_{\mathbf{n}} = 0$, we can obtain

$$\begin{aligned}\left[\hat{H}_I(t'), \hat{\rho}(0)\right] &= \hat{H}_I(t')|0\rangle\langle 0| - |0\rangle\langle 0|\hat{H}_I(t') \\ &= \frac{i}{2}\sum_{\mathbf{n}',\mathbf{m}'} \mu_{\mathbf{n}',\mathbf{m}'}\left\{\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) + \hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\right\},\end{aligned} \tag{G.1}$$

for the first order term. For the second order terms

$$\begin{aligned}&\left[H(t''), \left[\hat{H}_I(t'), \rho(0)\right]\right] = H(t'')\left[\hat{H}_I(t'), \rho(0)\right] - \left[\hat{H}_I(t'), \rho(0)\right] H(t'') \\ &= \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\left[\mu_{\mathbf{n}'',\mathbf{m}''}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} - \hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\right) + \mu_{\mathbf{n}'',\mathbf{m}''}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''}\right)\right]\left[\hat{H}_I(t'), \hat{\rho}(0)\right] \\ &- \frac{i}{2}\sum_{\mathbf{m}'',\mathbf{n}''}\left[\hat{H}_I(t'), \hat{\rho}(0)\right]\left[\mu_{\mathbf{n}'',\mathbf{m}''}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} - \hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\right) + \mu_{\mathbf{n}'',\mathbf{m}''}\left(\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\mathcal{A}}''_{\mathbf{n}''}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \mu_{\mathbf{n}'',\mathbf{m}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] - \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \mu_{\mathbf{n}'',\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \\
&- \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \mu_{\mathbf{n}'',\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] + \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \mu_{\mathbf{n}'',\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \\
&+ \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \mu_{\mathbf{n}'',\mathbf{m}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] - \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \mu_{\mathbf{n}'',\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \\
&- \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \mu_{\mathbf{n}'',\mathbf{m}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''} \left[\hat{H}_I(t'), \rho(0)\right] + \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \mu_{\mathbf{n}'',\mathbf{m}''} \left[\hat{H}_I(t'), \rho(0)\right] \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''}.
\end{aligned}
$$

Substituting the first order commutator (G.2) in the last expression, we obtain

$$
\begin{aligned}
&\left[H(t''), \left[\hat{H}_I(t'), \rho(0)\right]\right] = \\
&- \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \mu_{\mathbf{n}',\mathbf{m}'} \mu_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}''} \hat{\rho}(0) + \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \right\} \\
&+ \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \mu_{\mathbf{n}',\mathbf{m}'} \mu_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} + \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \right\} \\
&+ \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \mu_{\mathbf{n}',\mathbf{m}'} \mu_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) + \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \right\} \\
&- \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \mu_{\mathbf{n}',\mathbf{m}'} \mu_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} + \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \right\} \\
&- \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \mu_{\mathbf{n}',\mathbf{m}'} \mu_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) + \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \right\} \\
&+ \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \mu_{\mathbf{n}',\mathbf{m}'} \mu_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} + \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''} \hat{\mathcal{A}}''_{\mathbf{m}''} \right\} \\
&+ \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \mu_{\mathbf{n}',\mathbf{m}'} \mu_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) + \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \right\} \\
&- \frac{1}{4} \sum_{\mathbf{n}',\mathbf{m}'} \sum_{\mathbf{n}'',\mathbf{m}''} \mu_{\mathbf{n}',\mathbf{m}'} \mu_{\mathbf{n}'',\mathbf{m}''} \left\{ \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''} + \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''} \hat{\mathcal{A}}''_{\mathbf{n}''} \right\}.
\end{aligned}
$$

From those last expressions, the only candidates for the diagonal terms of $\hat{\rho}(t)$ can be identified as

$$\hat{\rho}(t)|_{\text{diag}} = \hat{\rho}(0) - \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'}\sum_{\mathbf{n}'',\mathbf{m}''}\int_0^t dt''\int_0^{t''} dt'\mu_{\mathbf{n}',\mathbf{m}'}\mu_{\mathbf{n}'',\mathbf{m}''}$$
$$\times\left\{\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'} - \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}''_{\mathbf{n}''}\hat{\mathcal{A}}''_{\mathbf{m}''} + \hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}''}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}''}\right\}.$$

The above terms are only diagonal if

$$\begin{aligned}
&\hat{\rho}(t)|_{\text{diag}} = \hat{\rho}(0)\\
&-\frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'}2^{-\delta_{\mathbf{n}',\mathbf{m}'}}\int_0^t dt''\int_0^t dt'\mu'_{(\mathbf{n}',\mathbf{m}')}\mu''_{(\mathbf{n}',\mathbf{m}')}\left\{\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'} + \text{h.c.}\right\}\\
&-\frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'}2^{-\delta_{\mathbf{n}',\mathbf{m}'}}\int_0^t dt''\int_0^t dt'\mu''_{(\mathbf{n}',\mathbf{m}')}\mu'_{[\mathbf{n}',\mathbf{m}']}\left\{\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) - \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'} + \text{h.c.}\right\},
\end{aligned}$$

$$\begin{aligned}
&\hat{\rho}(t)|_{\text{diag}} = \hat{\rho}(0)\\
&-\frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'}2^{-\delta_{\mathbf{n}',\mathbf{m}'}}|\beta_{\mathbf{n}',\mathbf{m}'}|^2\left\{\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) - \hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{m}'}\hat{a}_{\mathbf{n}'} + \text{h.c.}\right\}\\
&-\frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'}2^{-\delta_{\mathbf{n}',\mathbf{m}'}}\left\{\gamma_{\mathbf{n}',\mathbf{m}'}\beta^*_{\mathbf{n}',\mathbf{m}'}\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0) - \gamma^*_{\mathbf{n}',\mathbf{m}'}\beta_{\mathbf{n}',\mathbf{m}'}\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{m}'}\hat{a}_{\mathbf{n}'} + \text{h.c.}\right\},
\end{aligned}$$

with

$$\begin{aligned}
\gamma_{\mathbf{n}',\mathbf{m}'} &= \int_0^t dt'\mu_{[\mathbf{n}',\mathbf{m}']}e^{i(\Omega_{\mathbf{n}'}(t')+\Omega_{\mathbf{m}'}(t'))} = p\omega_1\epsilon\frac{\sqrt{nm}}{m-n}[1-\delta_{n,m}]\int_0^t dt'\dot{\xi}(t')e^{i(\Omega_{\mathbf{n}'}(t')+\Omega_{\mathbf{m}'}(t'))}\\
\beta_{\mathbf{n}',\mathbf{m}'} &= \int_0^t dt'\mu_{(\mathbf{n}',\mathbf{m}')}e^{i(\Omega_{\mathbf{n}'}(t')+\Omega_{\mathbf{m}'}(t'))} = p\omega_1\epsilon\frac{\sqrt{nm}}{m+n}\int_0^t dt'\dot{\xi}(t')e^{i(\Omega_{\mathbf{n}'}(t')+\Omega_{\mathbf{m}'}(t'))}
\end{aligned}$$

where the factor $2^{-\delta_{\mathbf{n}',\mathbf{m}'}}$ comes from the fact that we have twice more diagonal terms when $\mathbf{n}' \neq \mathbf{m}'$.

We can calculate the diagonal coefficients as

Term 0

$$
\begin{aligned}
&\rho^{(0)}(t)|_{\mathrm{diag}} = \langle 0|\,\hat{\rho}(t)|_{\mathrm{diag}}\,|0\rangle \\
&= 1 - \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \left\{ \langle 0|\,\hat{\mathcal{A}}''_{\mathbf{n}'}\hat{\mathcal{A}}''_{\mathbf{m}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\,|0\rangle + \text{h.c.} \right\} \\
&= 1 - \frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'} \langle 0|\,\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\,|0\rangle\, 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\, \mathrm{Re}\int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} \\
&= 1 - \sum_{\mathbf{n}',\mathbf{m}'} \mathrm{Re}\int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} \\
&= 1 - \frac{1}{2}\langle \hat{N}(t)\rangle.
\end{aligned}
$$

Since $\langle 0|\,\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\,|0\rangle = 2^{\delta_{\mathbf{n}',\mathbf{m}'}}$.

Term $2_{\mathbf{n}}$

$$
\begin{aligned}
\rho^{(\mathbf{n},\mathbf{n})}(t)|_{\mathrm{diag}} = \langle 2_{\mathbf{n}}|\,\hat{\rho}_{\mathrm{diag}}\,|2_{\mathbf{n}}\rangle &= \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \\
&\times \left\{ \langle 2_{\mathbf{n}}|\,\hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'}\hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{\mathcal{A}}'_{\mathbf{m}'}\hat{\mathcal{A}}'_{\mathbf{n}'}\,|2_{\mathbf{n}}\rangle + \text{h.c.} \right\} \\
&= \frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'} \langle 2_{\mathbf{n}}|\,\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\,|2_{\mathbf{n}}\rangle\, 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\, \mathrm{Re}\int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'}\chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} \\
&= \mathrm{Re}\int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{n}}\chi''^{(+)}_{\mathbf{n},\mathbf{n}} e^{-2i[\tilde{\omega}_{\mathbf{n}}(t'')-\tilde{\omega}_{\mathbf{n}}(t')]} \\
&= \frac{1}{2}\mathcal{N}_{\mathbf{n},\mathbf{n}}.
\end{aligned}
$$

since $\langle 2_{\mathbf{n}}|\,\hat{a}^{\dagger}_{\mathbf{n}'}\hat{a}^{\dagger}_{\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{m}'}\,|2_{\mathbf{n}}\rangle = 2\delta_{\mathbf{n}',\mathbf{n}}\delta_{\mathbf{m}',\mathbf{n}}$.

Term $1_{\mathbf{n}}, 1_{\mathbf{m}}$

For the diagonal terms associated with the state $|1_{\mathbf{n}}, 1_{\mathbf{m}}\rangle$ we have

$$
\begin{aligned}
\rho^{(\mathbf{n},\mathbf{m})}(t)|_{\text{diag}} &= \langle 1_{\mathbf{n}}, 1_{\mathbf{m}}| \hat{\rho}_{\text{diag}} |1_{\mathbf{n}}, 1_{\mathbf{m}}\rangle = \frac{1}{4}\sum_{\mathbf{n}',\mathbf{m}'} 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} \\
&\times \left\{ \langle 1_{\mathbf{n}}, 1_{\mathbf{m}}| \hat{\mathcal{A}}''^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}''^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} |1_{\mathbf{n}}, 1_{\mathbf{m}}\rangle + \text{h.c.} \right\} \\
&= \frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'} \langle 1_{\mathbf{n}}, 1_{\mathbf{m}}| \hat{a}^{\dagger}_{\mathbf{n}'} \hat{a}^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{n}'} \hat{a}_{\mathbf{m}'} |1_{\mathbf{n}}, 1_{\mathbf{m}}\rangle\, 2^{1-\delta_{\mathbf{n}',\mathbf{m}'}}\, \text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi^{(+)}_{\mathbf{n}',\mathbf{m}'} \chi^{(+)}_{\mathbf{n}'',\mathbf{m}''} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]} \\
&= \text{Re} \int_0^t dt'' \int_0^{t''} dt' \chi'^{(+)}_{\mathbf{n},\mathbf{m}} \chi''^{(+)}_{\mathbf{n},\mathbf{m}} e^{-i\{[\tilde{\omega}_{\mathbf{n}}(t'')+\tilde{\omega}_{\mathbf{m}}(t'')]-[\tilde{\omega}_{\mathbf{n}}(t')+\tilde{\omega}_{\mathbf{m}}(t')]\}} \\
&= \frac{1}{2}\mathcal{N}_{\mathbf{n},\mathbf{m}}.
\end{aligned}
$$

G.2 Density operator

In this section we calculate the expression for the diagonal terms of the density operator after the following expansion

$$
\hat{\rho}(t) = \hat{\rho}(0) - i\int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right] - \frac{1}{2}\int_0^t dt'' \int_0^t dt' \left[\hat{H}_I(t''), \left[\hat{H}_I(t'), \hat{\rho}(0)\right]\right] + \mathcal{O}(\hat{H}_I^3(t)).
$$

Considering the vacuum state $\hat{\rho}(0) = |0\rangle\langle 0|$ as the initial state and the correspondent relations $\hat{\mathcal{A}}_{\mathbf{n}}\hat{\rho}(0) = \hat{\rho}(0)\mathcal{A}^{\dagger}_{\mathbf{n}} = 0$, we can obtain

$$
\begin{aligned}
\int_0^t dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right] &= \int_0^t dt' \hat{H}_I(t')|0\rangle\langle 0| - |0\rangle\langle 0| \int_0^t dt' \hat{H}_I(t') \\
&= \frac{i}{2}\sum_{\mathbf{n}',\mathbf{m}'} \int_0^t dt' \mu_{(\mathbf{n}',\mathbf{m}')} \left\{ \hat{\mathcal{A}}'^{\dagger}_{\mathbf{n}'} \hat{\mathcal{A}}'^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) + \hat{\rho}(0) \hat{\mathcal{A}}'_{\mathbf{m}'} \hat{\mathcal{A}}'_{\mathbf{n}'} \right\} \qquad \text{(G.2)} \\
&= \frac{i}{2}\sum_{\mathbf{n}',\mathbf{m}'} \left\{ \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}^{\dagger}_{\mathbf{n}'} \hat{a}^{\dagger}_{\mathbf{m}'} \hat{\rho}(0) + \beta^{*}_{\mathbf{n}',\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \right\}, \qquad \text{(G.3)}
\end{aligned}
$$

for the first order term. For the second order terms

$$
\begin{aligned}
&\int_0^t dt'' \int_0^{t''} dt' \left[H(t''), \left[\hat{H}_I(t'), \rho(0)\right]\right] \\
&= \int_0^t dt'' H(t'') \int_0^{t''} dt' \left[\hat{H}_I(t'), \rho(0)\right] - \int_0^{t''} dt' \left[\hat{H}_I(t'), \rho(0)\right] \int_0^t dt'' H(t'') \\
&= \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \left[\beta_{\mathbf{n}'',\mathbf{m}''} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}^\dagger_{\mathbf{m}''} - \beta^*_{\mathbf{n}'',\mathbf{m}''} \hat{a}_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} + \alpha_{\mathbf{n}'',\mathbf{m}''} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} - \alpha^*_{\mathbf{n}'',\mathbf{m}''} \hat{a}^\dagger_{\mathbf{m}''} \hat{a}_{\mathbf{n}''}\right] \int_0^{t''} dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right] \\
&- \frac{i}{2} \sum_{\mathbf{m}'',\mathbf{n}''} \int_0^{t''} dt' \left[\hat{H}_I(t'), \hat{\rho}(0)\right] \left[\beta_{\mathbf{n}'',\mathbf{m}''} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}^\dagger_{\mathbf{m}''} - \beta^*_{\mathbf{n}'',\mathbf{m}''} \hat{a}_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} + \alpha_{\mathbf{n}'',\mathbf{m}''} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} - \alpha^*_{\mathbf{n}'',\mathbf{m}''} \hat{a}^\dagger_{\mathbf{m}''} \hat{a}_{\mathbf{n}''}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{4} \sum_{\mathbf{m}'',\mathbf{n}''} \sum_{\mathbf{n}',\mathbf{m}'} [&\beta_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) \hat{a}^\dagger_{\mathbf{n}''} \hat{a}^\dagger_{\mathbf{m}''} + \beta_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}^\dagger_{\mathbf{m}''} \\
&- \beta^*_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} - \beta^*_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \hat{a}_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} \\
&+ \alpha_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) \hat{a}^\dagger_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} + \alpha_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} \\
&- \alpha^*_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) \hat{a}^\dagger_{\mathbf{m}''} \hat{a}_{\mathbf{n}''} - \alpha^*_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}''} \hat{a}_{\mathbf{n}''} \\
&- \beta_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}^\dagger_{\mathbf{m}''} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) - \beta_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}^\dagger_{\mathbf{m}''} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \\
&+ \beta^*_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) + \beta^*_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{a}_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \\
&- \alpha_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) - \alpha_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \\
&+ \alpha^*_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{m}''} \hat{a}_{\mathbf{n}''} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) + \alpha^*_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{m}''} \hat{a}_{\mathbf{n}''} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'}]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{4} \sum_{\mathbf{m}'',\mathbf{n}''} \sum_{\mathbf{n}',\mathbf{m}'} [&\beta^*_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) + \beta_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}^\dagger_{\mathbf{m}''} \\
&- \beta^*_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} - \beta_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}^\dagger_{\mathbf{m}''} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \\
&- \beta_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}^\dagger_{\mathbf{m}''} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) - \beta^*_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \hat{a}_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} \\
&+ \alpha^*_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{m}''} \hat{a}_{\mathbf{n}''} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) + \alpha_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} \\
&- \alpha_{\mathbf{n}'',\mathbf{m}''} \beta_{\mathbf{n}',\mathbf{m}'} \hat{a}^\dagger_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{n}''} \hat{a}_{\mathbf{m}''} \hat{a}^\dagger_{\mathbf{m}'} \hat{\rho}(0) - \alpha^*_{\mathbf{n}'',\mathbf{m}''} \beta^*_{\mathbf{n}',\mathbf{m}'} \hat{\rho}(0) \hat{a}_{\mathbf{m}'} \hat{a}_{\mathbf{n}'} \hat{a}^\dagger_{\mathbf{m}''} \hat{a}_{\mathbf{n}''}]
\end{aligned}
$$

therefore

$$\hat{\rho}(t) = \hat{\rho}(0) + \frac{1}{2}\sum_{\mathbf{n}',\mathbf{m}'}\left(\beta_{\mathbf{n}',\mathbf{m}'}\hat{a}^\dagger_{\mathbf{n}'}\hat{a}^\dagger_{\mathbf{m}'}\hat{\rho}(0) + \beta^*_{\mathbf{n}',\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{m}'}\hat{a}_{\mathbf{n}'}\right)$$
$$-\frac{1}{8}\sum_{\mathbf{m}'',\mathbf{n}''}\sum_{\mathbf{n}',\mathbf{m}'}[\beta^*_{\mathbf{n}'',\mathbf{m}''}\beta_{\mathbf{n}',\mathbf{m}'}\hat{a}_{\mathbf{n}''}\hat{a}_{\mathbf{m}''}\hat{a}^\dagger_{\mathbf{n}'}\hat{a}^\dagger_{\mathbf{m}'}\hat{\rho}(0) + \beta_{\mathbf{n}'',\mathbf{m}''}\beta^*_{\mathbf{n}',\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{m}'}\hat{a}_{\mathbf{n}'}\hat{a}^\dagger_{\mathbf{n}''}\hat{a}^\dagger_{\mathbf{m}''}$$
$$- \beta^*_{\mathbf{n}'',\mathbf{m}''}\beta_{\mathbf{n}',\mathbf{m}'}\hat{a}^\dagger_{\mathbf{n}'}\hat{a}^\dagger_{\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{n}''}\hat{a}_{\mathbf{m}''} - \beta_{\mathbf{n}'',\mathbf{m}''}\beta^*_{\mathbf{n}',\mathbf{m}'}\hat{a}^\dagger_{\mathbf{n}''}\hat{a}^\dagger_{\mathbf{m}''}\hat{\rho}(0)\hat{a}_{\mathbf{m}'}\hat{a}_{\mathbf{n}'}$$
$$- \beta_{\mathbf{n}'',\mathbf{m}''}\beta_{\mathbf{n}',\mathbf{m}'}\hat{a}^\dagger_{\mathbf{n}''}\hat{a}^\dagger_{\mathbf{m}''}\hat{a}^\dagger_{\mathbf{n}'}\hat{a}^\dagger_{\mathbf{m}'}\hat{\rho}(0) - \beta^*_{\mathbf{n}'',\mathbf{m}''}\beta^*_{\mathbf{n}',\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{m}'}\hat{a}_{\mathbf{n}'}\hat{a}_{\mathbf{n}''}\hat{a}_{\mathbf{m}''}$$
$$+ \alpha^*_{\mathbf{n}'',\mathbf{m}''}\beta_{\mathbf{n}',\mathbf{m}'}\hat{a}^\dagger_{\mathbf{m}''}\hat{a}_{\mathbf{n}''}\hat{a}^\dagger_{\mathbf{n}'}\hat{a}^\dagger_{\mathbf{m}'}\hat{\rho}(0) + \alpha_{\mathbf{n}'',\mathbf{m}''}\beta^*_{\mathbf{n}',\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{m}'}\hat{a}_{\mathbf{n}'}\hat{a}^\dagger_{\mathbf{n}''}\hat{a}_{\mathbf{m}''}$$
$$- \alpha_{\mathbf{n}'',\mathbf{m}''}\beta_{\mathbf{n}',\mathbf{m}'}\hat{a}^\dagger_{\mathbf{n}'}\hat{a}^\dagger_{\mathbf{n}''}\hat{a}_{\mathbf{m}''}\hat{a}^\dagger_{\mathbf{m}'}\hat{\rho}(0) - \alpha^*_{\mathbf{n}'',\mathbf{m}''}\beta^*_{\mathbf{n}',\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{m}'}\hat{a}_{\mathbf{n}'}\hat{a}^\dagger_{\mathbf{m}''}\hat{a}_{\mathbf{n}''}]$$

$$\rho_{\text{ndiag}}^{1^i,1^j,1_k,1_l} = \langle 1_i, 1_j|\,\hat{\rho}(t)\,|1_k, 1_l\rangle = \beta^*_{\mathbf{n}'',\mathbf{m}''}\beta_{\mathbf{n}',\mathbf{m}'}\hat{a}^\dagger_{\mathbf{n}'}\hat{a}^\dagger_{\mathbf{m}'}\hat{\rho}(0)\hat{a}_{\mathbf{n}''}\hat{a}_{\mathbf{m}''} - \beta_{\mathbf{n}'',\mathbf{m}''}\beta^*_{\mathbf{n}',\mathbf{m}'}\hat{a}^\dagger_{\mathbf{n}''}\hat{a}^\dagger_{\mathbf{m}''}\hat{\rho}(0)\hat{a}_{\mathbf{m}'}\hat{a}_{\mathbf{n}'}$$

Appendix H

Instantaneous basis decomposition

Another approach to study the DCE in the one-dimensional case by expanding the modes defined in equation (H.1) in a series with respect to the instantaneous basis

$$f_n^{\text{out}}(x,t) = \sum_{k=1}^{\infty} \varphi_{k,L(t)}(x) Q_k^{(n)}(t),$$

with

$$\varphi_{k,L(t)}(x) = \sqrt{\frac{2}{L(t)}} \sin\left(\frac{\pi k}{L(t)} x\right)$$

the initial conditions

$$Q_k^{(n)}(0) = \delta_{kn}, \qquad \dot{Q}_k^{(n)}(0) = -i\omega_n \delta_{kn}, \qquad k, n = 1, 2, \ldots.$$

Introducing the field

$$\hat{\phi}(x,t) = \sum_n \frac{1}{\sqrt{2\omega_n}} \left[f_n^{\text{out}}(x,t)\hat{b}_n + f_n^{\text{out}*}(x,t)\hat{b}_n^\dagger \right], \tag{H.1}$$

into the field dynamical equation

$$\partial_t^2 \hat{\phi}(x,t) = \partial_x^2 \hat{\phi}(x,t) = -\omega_{k,L(t)}^2 \hat{\phi}(x,t)$$

we obtain

$$\begin{aligned}
&\ddot{f}_n^{\text{out}} + \omega_{k,L(t)}^2 f_n^{\text{out}} = 0 \\
&\sum_k \left\{ \left[\ddot{Q}_k^{(n)} + \omega_{k,L(t)}^2 Q_k^{(n)} \right] \varphi_{k,L(t)} + 2\dot{\varphi}_{k,L(t)} \dot{Q}_k^{(n)} + Q_k^{(n)} \ddot{\varphi}_{k,L(t)} \right\} = 0.
\end{aligned}$$

Multiplying the last expression by $\varphi_{j,L(t)}$ and integrating over 0 and $L_(t)$, we obtain

$$\ddot{Q}_k^{(n)} + \omega_{k,L(t)}^2 Q_k^{(n)} = -2 \sum_j G_{j,k} \dot{Q}_j^{(n)} - \sum_j H_{j,k} Q_j^{(n)}$$

with

$$
\begin{aligned}
G_{j,k} &= -\int_0^{L(t)} dx\ \varphi_{k,L(t)} \partial_t \varphi_{j,L(t)} \\
H_{j,k} &= -\int_0^{L(t)} dx\ \varphi_{k,L(t)} \partial_t^2 \varphi_{j,L(t)} = \partial_t G_{j,k} + \sum_s G_{js} G_{ks}.
\end{aligned}
$$

Given origin to the set of coupled differential equations

$$
\ddot{Q}_k^{(n)} + \omega_k^2(t) Q_k^{(n)} = 2 \sum_j G_{kj} \dot{Q}_j^{(n)} + \sum_j \dot{G}_{kj} Q_k^{(n)} + \mathcal{O}(G_{kj}^2) \tag{H.2}
$$

where $\omega_{k,L(t)}(t) = k\pi/L(t)$ and the coefficients G_{jk} (for $j \neq k$)

$$
G_{jk} = -G_{kj} = (-1)^{k-j} \frac{2kj}{(j^2 - k^2)} \frac{\dot{L}(t)}{L(t)}.
$$

The ladder operators $\hat{a}_n^{\text{in}}$ and $\hat{a}_n^{\text{in}\dagger}$ correspond to the particle notion defined in the "in" region ($t < 0$) whereas the operators $\hat{a}_n^{\text{out}}$ and $\hat{a}_n^{\text{out}\dagger}$ are defined for the "out" region ($t > T$). According with the Bogoliubov transformations both sets of creation and annihilation operators (3.3) and (3.1) are related by

$$
\hat{a}_m^{\text{out}} = \sum_n \left(\alpha_{nm} \hat{a}_n^{\text{in}} + \beta_{nm}^* \hat{a}_n^{\text{in}\dagger} \right). \tag{H.3}
$$

Supposing the wall returns to its initial position $x = L_0$ after some interval of time T, the right hand size of the equation (H.2) vanishes and we must obtain solutions with the form

$$
Q_k^{(n)}(t > T) = A_k^{(n)} e^{i\omega_k t} + B_k^{(n)} e^{-i\omega_k t}, \qquad k, n = 1, 2, \dots. \tag{H.4}
$$

where $A_k^{(n)}$ and $B_k^{(n)}$ are constant coefficients to be later determined.

Let

$$
\hat{\phi}^{\text{in}}(x,t) = \sum_n \frac{1}{\sqrt{2\omega_n}} \left[\varphi_n e^{-i\omega_n t} \hat{a}_n^{\text{in}} + \text{h.c.} \right], \tag{H.5}
$$

$$
\begin{aligned}
\hat{\phi}(x,t) &= \sum_n \frac{1}{\sqrt{2\omega_n}} \left[\sum_k \varphi_k \left(A_k^{(n)} e^{i\omega_k t} + B_k^{(n)} e^{-i\omega_k t} \right) \hat{a}_n^{\text{in}} + \text{h.c} \right], \\
&= \sum_k \frac{1}{\sqrt{2\omega_k}} \left[\sum_n \left(\sqrt{\frac{\omega_k}{\omega_n}} A_k^{(n)} \hat{a}_n^{\text{in}} + \sqrt{\frac{\omega_k}{\omega_n}} B_k^{(n)*} \hat{a}_n^{\text{in}\dagger} \right) \varphi_k e^{i\omega_k t} + \text{h.c.} \right] \\
&= \sum_k \frac{1}{\sqrt{2\omega_k}} \left[\varphi_k e^{i\omega_k t} \hat{a}_k^{\text{out}} + \text{h.c.} \right]
\end{aligned}
$$

so we can identify

$$\alpha_{nk} = \sqrt{\frac{\omega_k}{\omega_n}} A_k^{(n)}$$
$$\beta_{nk} = \sqrt{\frac{\omega_k}{\omega_n}} B_k^{(n)}$$

Bibliography

[1] A. Polkovnikov, “Microscopic diagonal entropy and its connection to basic thermodynamic relations”, Annals of Physics **326**, 486–499 (2011).

[2] V. V. Dodonov, “Nonstationary casimir effect and analytical solutions for quantum fields in cavities with moving boundaries”, 10.48550/ARXIV.QUANT-PH/0106081 (2001).

[3] T. Baumgratz, M. Cramer, and M. Plenio, “Quantifying coherence”, Physical Review Letters **113**, 10.1103/physrevlett.113.140401 (2014).

[4] F. Plastina, A. Alecce, T. Apollaro, G. Falcone, G. Francica, F. Galve, N. L. Gullo, and R. Zambrini, “Irreversible work and inner friction in quantum thermodynamic processes”, Physical Review Letters **113**, 10.1103/physrevlett.113.260601 (2014).