Entanglement generation in uniformly accelerating atoms: Reexamination of the Unruh effect

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The master equation describing the completely positive time evolution of a uniformly accelerated two-level system in weak interaction with a scalar field in the Minkowski vacuum is derived and explicitly solved. The moving system is found to be subjected to dissipation that drives its density matrix to a purely thermal equilibrium state, exhibiting a nonvanishing probability of spontaneous excitation, this phenomenon being usually referred to as the Unruh effect. Remarkably, when the uniformly accelerating system is composed by two, independent two-level atoms, the corresponding asymptotic, equilibrium state turns out to be entangled.

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I. INTRODUCTION

When a particle detector moves with a uniform acceleration through an external field in its vacuum state, spontaneous excitation can occur; indeed, the detector behaves as if it were in a thermal bath, with temperature (the "Unruh temperature") proportional to its proper acceleration. This phenomenon is known as the "Unruh effect" [1–6]. Although originally presented for detectors in interaction with massless, free scalar fields, the effect has been later confirmed in the case of massive and higher spin fields, in arbitrary spacetime dimensions (with appropriate gray-body corrections to the thermal bath spectrum) [5], and even for arbitrary interacting fields, using the algebraic formulation of quantum field theory [7,8]. Experimental verification of the phenomena is hard to realize in practice, since measurable thermal effects require very high accelerations; nevertheless, clever proposals involving circular trajectories have been discussed and might realized in the future [9].

Most treatments of the Unruh effect focus on the study of the spontaneous excitation of an accelerated "DeWitt detector" [10], a nonrelativistic n-level system, linearly coupled to the external relativistic field. What is then studied is the excitation rate, i.e., the probability per unit time of a spontaneous transition from the ground state to one of its excited states. This probability clearly vanishes for a detector at rest, but turns out to be nonzero, and proportional to a Planckian factor, for a uniformly accelerated one. The same physical result can be obtained by examining the vacuum state of the external field as seen by the accelerating detector: it turns out to be thermal and described by the same Planckian factor.

This result, known as the "thermalization theorem" [5], implies that the accelerating detector behaves like an open system, i.e., a system immersed in an external heat bath [11–15]. In this respect, the study of its full dynamics and not only of its asymptotic excitation rate is of great physical interest. As for any open system, it can be obtained from the complete time evolution describing the total system (detector + external field) by integrating over the field degrees of freedom, which in fact are never observed.

In the following, we shall explicitly derive and study such reduced dynamics, restricting our attention to the case of a two-level detector in interaction with free, massless, scalar fields. As in the standard analysis of the Unruh effect, this simplified situation is not really restrictive since it is able to capture all the main features of the thermalization phenomena, without the algebraic complications needed to treat the case of higher-spin or massive fields.

In addition, we shall assume the interaction between the accelerating two-level system and the scalar fields to be weak [12–14]; while encompassing the most common physical situations, this hypothesis allows a rigorous, mathematically sound, derivation of the evolution equation describing the dynamics of the moving detector [16,17]. The corresponding finite-time evolution takes the form of a one-parameter semigroup of completely positive maps [11–14], whose action on the state of the system can be explicitly obtained in closed form. An explicit expression for the probability transitions among the states of the moving detector can then be obtained: they contain appropriate Planckian factors, so that the known forms of the spontaneous excitation rates are reproduced. Indeed, one finds that the equilibrium configuration of the detector is exactly thermal, with temperature equal to the Unruh temperature. This is just another manifestation of the "thermalization theorem:" a single, two-level system accelerating through a vacuum field is asymptotically driven to a thermal equilibrium with the bath.

A physically much more interesting asymptotic situation can be obtained when the uniformly accelerating system is formed by two independent two-level systems, in weak interaction with the same scalar field. The corresponding master equation that describes the reduced dynamics of the two detectors can be explicitly written generalizing the results obtained in the case of a single detector. It again generates a

semigroup of completely positive maps, whose asymptotic equilibrium state can be easily determined. In general, it turns out to be an entangled state, even in the case of a separable initial state.

It is known that, in certain circumstances, heat baths belonging to a specific class can enhance entanglement rather than destroying it [18–23]: two, mutually noninteracting systems immersed in one of these baths can then become quantum correlated; this can happen both at finite time and in the asymptotic regime. It is remarkable that the thermal bath seen by uniformly accelerating systems precisely belongs to the mentioned class. In turn, this result may suggest a new possibility for an experimental test of the Unruh effect: use of appropriate quantum optics devices to detect the asymptotic entanglement generated by the uniform acceleration.

The next section will be devoted to the discussion of the weak coupling limit and the derivation of the master equation describing the dynamics of the accelerating system in its comoving frame. The finite time evolution it generates will be analyzed in detail in Sec. III for a single two-level system. The case of a detector composed by two independent two-level systems will be instead treated in Sec. IV, and the possibility of generation of mutual quantum correlations analyzed; in particular, the entanglement content of the asymptotic equilibrium state will be determined by computing its corresponding concurrence. Finally, Sec. V contains further physical considerations on the open-system approach to the Unruh effect, while the Appendix illustrates some technical issues connected with the derivation of the master equation.

II. MASTER EQUATION

As explained in the introductory remarks, we shall use well-known techniques developed in the study of open quantum systems to analyze the behavior of an idealized detector immersed in a scalar vacuum field, moving along a prescribed, uniformly accelerating trajectory.

A frame-independent description of the dynamics of such a system would require a covariant formulation of the evolution equation of quantum mechanics.[1] Although rather cumbersome in practice, such a description exists; it has been introduced in the early days of quantum field theory, and is based on the so-called Schwinger-Tomonaga equation [26]. It extends the standard, Hamiltonian evolution to a local functional dynamics, where the time variable is replaced by a collection of three dimensional spacelike hypersurfaces, upon which spacetime has been sliced [15,27].

As an alternative to this rather complicated approach, one can fix *ab initio* a reference frame and study the dynamics of the system with respect to the natural (albeit in general only locally valid) time variable defined there. This choice is particularly convenient when a global time variable is available, as in an inertial or in an uniformly accelerated frame [3], since it allows drastic simplifications in the analysis and interpretation of the system dynamics. The price to pay is the lack of manifest covariance and thus of a simple way of translating a physical description obtained in one reference frame to that derived in a different one.

Both approaches are in line with principles always available, although, for practical reasons, one usually works in the "laboratory" frame. However, in certain physical situations, the use of the covariant formulation may, in practice, be precluded. This is indeed the case of a system in interaction with an external environment: in fact, a consistent, reduced dynamics can be derived only within reference frames for which environment correlations decay sufficiently fast [12,15]. Note that, in most cases, this is not really a serious drawback, since these preferred reference frames are precisely the ones used in the experiments.

For the system under study, since one is interested in the behavior of the accelerating detector, the natural reference frame to be adopted is that of comoving with it: its dynamics will then be described as an evolution in the proper time t. In this reference frame, the detector is always at rest, and without loss of generality, it can be positioned at the origin of the comoving spatial coordinates.

In the inertial reference frame, with Minkowski coordinates (x^0,x^1,x^2,x^3), the detector is instead seen following a hyperbolic trajectory. More specifically, in order to better conform with the standard open-system paradigm adopted below, we shall assume that the composed system (detector + external fields) be initially prepared in a factorized state, with the detector at rest and the fields in their vacuum state. The detector starts moving at time $t\equiv x^0=0$, after which it follows the path

$$x^0(t)=\frac{1}{a}\sinh at,$$

$$x^1(t)=\frac{1}{a}\cosh at,$$

$$x^2(t)=x^3(t)=0, \tag{2.1}$$

with a the constant proper acceleration.[2] Our aim is to derive and study the appropriate reduced "master equation" that describes the quantum evolution of the detector following this trajectory for positive t.

To this aim, the details of the detector internal dynamics result are irrelevant: one can then choose to model it as a simple two-level atom, i.e., as a nonrelativistic, quantum mechanical system, which can be fully described in terms of a two-dimensional Hilbert space. With respect to a fixed, arbitrary basis in this space, states of the system will be represented by a 2×2 density matrix ρ, i.e., a Hermitian, $\rho^\dagger=\rho$, normalized, $\mathrm{Tr}[\rho]=1$, operator, with non-negative eigenvalues, $\det[\rho]\geqslant 0$.

[1] Combining quantum mecahnics and relativity in a fully consistent way is notoriously problematic; for recent reviews on different aspects of this question, see Refs. [24,25].

[2] This situation looks more physical than the usually adopted one for which the detector follows the hyperbolic motion for all times.

In the absence of any interaction with the external scalar fields, the atom internal dynamics will be driven by a 2×2 Hamiltonian matrix H_S, which in the chosen basis can be taken to assume the most general form

$$H_S = \frac{\omega}{2}\sum_{i=1}^{3} n_i\sigma_i \equiv \frac{\omega}{2}\vec{n}\cdot\vec{\sigma}, \tag{2.2}$$

where σ_i, $i=1, 2, 3$ are the Pauli matrices, n_i, $i=1, 2, 3$ are the components of a unit vector, while ω represents the gap between the two energy eigenvalues.

As mentioned in the Introduction, the interaction of the atom with the external scalar fields is assumed to be weak; it can then be described by an Hamiltonian H' that is linear in both atom and field variables:

$$H' = \sum_{\mu=0}^{3} \sigma_\mu \Phi_\mu(x(t)), \tag{2.3}$$

with σ_0 the 2×2 unit matrix. As explicitly indicated, the interaction is effective only along the trajectory (2.1), as the atom is assumed to be an idealized point, without size.[3] The operators (or better, operator-valued distributions) $\Phi_\mu(x)$ represent the external fields and satisfy the massless Klein-Gordon equation. They can be expanded as

$$\Phi_\mu(x) = \sum_{a=1}^{N} [\chi_\mu^a \phi_a^{(-)}(x) + (\chi_\mu^a)^* \phi_a^{(+)}(x)], \tag{2.4}$$

in terms of positive- $\phi_a^{(+)}(x)$ and negative-energy $\phi_a^{(-)}(x)$ field operators relative to a set of N independent, massless, free scalar fields, with total Hamiltonian H_Φ; the complex coefficients χ_μ^a "embed" the field modes into the two-dimensional detector Hilbert space and play the role of (generalized) coupling constants. The explicit form of the field Hamiltonian H_Φ need not be specified; it suffices to know that once transformed in the inertial reference frame it leads to the standard expansion of $\phi_a^{(+)}$, $\phi_a^{(-)}$ in terms of Minkowski creation and annihilation operators, respectively.

In the comoving frame, the total Hamiltonian H for the complete system (atom+external fields) can thus be written as

$$H = H_S + H' + H_\Phi. \tag{2.5}$$

It generates the evolution in t of the corresponding total density matrix $\rho_{\rm tot}$,

$$\frac{\partial\rho_{\rm tot}(t)}{\partial t} = -iL_H[\rho_{\rm tot}(t)], \tag{2.6}$$

starting at $t=0$ from the initial configuration: $\rho_{\rm tot}(0)=\rho(0)\otimes|0\rangle\langle0|$, where $|0\rangle$ is the Minkowski field vacuum state, as seen in the comoving frame; the symbol L_H in (2.6) represents the Liouville operator corresponding to H,

$$L_H[\cdot] \equiv [H, \cdot]. \tag{2.7}$$

The dynamics of the atom is then obtained by summing over the field Φ degrees of freedom, i.e., by applying to $\rho_{\rm tot}(t)$ the trace projection operator P:

$$\rho(t) = P[\rho_{\rm tot}(t)] \equiv {\rm Tr}_\Phi[\rho_{\rm tot}(t)] = {\rm Tr}_\Phi[e^{-itH}(\rho(0)\otimes|0\rangle\langle0|)e^{itH}]; \tag{2.8}$$

it is generated by an equation that is the result of the action of P, and its complement operator $Q=1-P$, on both sides of (2.6). It can be conveniently written as an integro-differential equation [12]:

$$\frac{\partial\rho(t)}{\partial t} = -iL_{H_S}[\rho(t)] + \int_0^t ds\mathcal{K}_s[\rho(t-s)], \tag{2.9}$$

where the time-dependent kernel $\mathcal{K}_s$ is defined as a formal power series expansion in the interaction Hamiltonian H', and it is explicitly given by the expression

$$\mathcal{K}_s[\rho] = -{\rm Tr}_\Phi\{L_{H'}Qe^{-isQL_HQ}QL_{H'}[\rho\otimes|0\rangle\langle0|]\}. \tag{2.10}$$

The resulting finite evolution map $\rho(0)\mapsto\rho(t)$ is in general very complicated, developing irreversibility and memory effects; indeed, the atom state $\rho(t)$ at time t as given by (2.9) depends not only on the initial state $\rho(0)$, but also on all states $\rho(s)$, with $s<t$. This is a general result, valid for any reduced dynamics: nevertheless, the form of the master equation (2.9) can be further simplified on the basis of additional physical considerations, which in the present case amounts to the requirement of a weak coupling between the moving atom and the external fields [11–15].

On general grounds, one expects the memory effects in (2.9) to be negligible when the ratio τ/τ_Φ between the typical variation time τ of $\rho(t)$ and the decay time τ_Φ of the field correlations is large. This is precisely the result of the "weak coupling limit" procedure [13]: it first amounts to a rescaling of the interaction Hamiltonian H' by a dimensionless coupling constant g, so that the first contribution in the right-hand side of Eq. (2.9), representing the unperturbed motion of the atom, is of order 1, while the remaining piece, taking care of the interaction with the external field, becomes of order g^2. This implies that the evolution of the state $\rho(t)$ develops on time scales of order $1/g^2$; in order to obtain a consistent physical description, one then needs to appropriately rescale the time variable, $t\to t/g^2$. After this, g can be safely taken to be vanishingly small, and the evolution equation attains a well-defined limit.

The procedure that we have briefly outlined can be given a precise mathematical meaning [16,17]: provided the field correlations decay sufficiently fast at large time separations, one can rigorously prove that in the weak coupling limit the formal expression in Eqs. (2.9) and (2.10) converges to a differential equation for the reduced density matrix $\rho(t)$, which is local in time. It takes the so-called Kossakowski-Lindblad form (see the Appendix for details) [28,29]:

[3] For a detailed discussion on the physical motivations justifying this simplifying assumption, see Ref. [5].

$$\frac{\partial\rho(t)}{\partial t} = -i[H_{\rm eff},\rho(t)] + \mathcal{L}[\rho(t)], \tag{2.11}$$

with

$$\mathcal{L}[\rho] = \frac{1}{2}\sum_{i,j=1}^{3} a_{ij}[2\sigma_j\rho\sigma_i - \sigma_i\sigma_j\rho - \rho\sigma_i\sigma_j]. \tag{2.12}$$

The corresponding finite-time evolution maps γ_t generated by this equation, $\rho(0)\mapsto\rho(t)=\gamma_t[\rho(0)]$, form a one-parameter semigroup of trace-preserving transformations: $\gamma_t\circ\gamma_s=\gamma_{t+s}$, $t, s\geqslant 0$.

The effective Hamiltonian $H_{\rm eff}$ and the coefficients of the 3×3 Kossakowski matrix a_{ij} depend on the Fourier transform of the field vacuum correlations (Wightman functions), evaluated along the trajectory (2.1):

$$\alpha_{\mu\nu}(\lambda) = \int_{-\infty}^{\infty} dt e^{i\lambda t}\langle 0|\Phi_\mu(x(t))\Phi_\nu(0)|0\rangle. \tag{2.13}$$

Being invariant, these correlation functions can be computed in any reference frame. Recalling that the field variables $\phi_a^{(\pm)}$ in (2.4) are all independent, one finds

$$\langle 0|\Phi_\mu(x)\Phi_\nu(y)|0\rangle = \sum_{a=1}^{N}\chi_\mu^a(\chi_\nu^a) * G(x-y), \tag{2.14}$$

where $G(x-y)$ is the standard four-dimensional Wightman function for a single scalar field, which, with the proper $i\varepsilon$ prescription, can be written as[4]

$$G(x) = \int \frac{d^4k}{(2\pi)^3}\theta(k^0)\,\delta(k^2)e^{ik\cdot x-\varepsilon k^0} = -\frac{1}{4\pi^2}\frac{1}{(x^0-i\varepsilon)^2-(\vec{x})^2}. \tag{2.15}$$

Its Fourier transform along the trajectory (2.1) can be easily evaluated through a contour integral [5]:

$$\mathcal{G}(\lambda) = \int_{-\infty}^{\infty} dt e^{i\lambda t}G(x(t)) = \frac{1}{2\pi}\frac{\lambda}{1-e^{-\beta_U\lambda}}, \tag{2.16}$$

where $\beta_U=1/T_U$, with $T_U=a/2\pi$, the so-called Unruh temperature. With the help of the 3×3 Hermitian matrices (see Appendix),

$$\psi_{ij}^{(0)} = n_in_j,\ \psi_{ij}^{(\pm)} = \frac{1}{2}(\delta_{ij}-n_in_j \pm i\epsilon_{ijk}n_k), \tag{2.17}$$

it proves convenient to define the transformed coupling coefficients $\chi_i^{(\xi)a}=\Sigma_j\chi_j^a\psi_{ji}^{(\xi)}$, with $\xi=0,+,-$, together with their corresponding complex conjugate ones $\overline{\chi}_i^{(\xi)a}=\Sigma_j(\chi_j^a)*\psi_{ji}^{(-\xi)}$; by means of them, the Kossakowski matrix a_{ij} in (2.12) explicitly reads

$$a_{ij} = \sum_{a=1}^{N}[\mathcal{G}(0)\chi_i^{(0)a}\overline{\chi}_j^{(0)a} + \mathcal{G}(\omega)\chi_i^{(+)a}\overline{\chi}_j^{(+)a} + \mathcal{G}(-\omega)\chi_i^{(-)a}\overline{\chi}_j^{(-)a}]. \tag{2.18}$$

Being the sum of three manifestly positive terms, the Hermitian matrix a_{ij} turns out to be positive. As a consequence, the one parameter family of transformations γ_t generated by Eqs. (2.11) and (2.12) is composed by completely positive maps [12–14,30]. As is well known, this property assures the positivity of the evolved density matrix $\rho(t)$ in any physical situation, thus guaranteeing the correct interpretation of its eigenvalues as probabilities. (For discussions on this relevant point, see Refs. [31–34].)

This result is nontrivial and shows the importance of adopting a physically consistent and mathematically precise procedure, the Davies weak–coupling limit, in deriving the reduced evolution equation [35]. Indeed, note that direct use of the standard second-order perturbative approximation in the original master equation (2.9) (as adopted in Ref. [36]) produces a finite time evolution for $\rho(t)$ that in general does not preserve the positivity of probabilities.

Besides producing the nonunitary evolution term (2.12), the coupling with the external fields Φ_μ induces also a correction to the system Hamiltonian (2.2), the so-called Lamb shift H_L: the complete Hamiltonian is now $H_{\rm eff}=H_S+H_L$. As for the Kossakowski matrix, this additional shift H_L can be expressed in terms of the field correlations along the accelerating trajectory. Introducing together with the Fourier transforms (2.13) also their corresponding Hilbert transforms [12,14]:

$$\beta_{\mu\nu}(\lambda) = \frac{P}{i\pi}\int_{-\infty}^{\infty} dz\frac{\alpha_{\mu\nu}(z)}{z-\lambda}, \tag{2.19}$$

where P denotes principal value, one finds

$$H_L = \frac{1}{2}\sum_{i=1}^{3} b_i\sigma_i, \tag{2.20}$$

with

$$b_i = i\sum_{j=1}^{3}[\alpha_{0j}(0)-\alpha_{j0}(0)-\beta_{0j}(0)-\beta_{j0}(0)]\psi_{ji}^{(0)} + \sum_{k,l,r,s=1}^{3}\epsilon_{ikl}[\beta_{rs}(0)\psi_{rk}^{(0)}\psi_{sl}^{(0)} + \beta_{rs}(\omega)\psi_{rk}^{(+)}\psi_{sl}^{(-)} + \beta_{rs}(-\omega)\psi_{rk}^{(-)}\psi_{sl}^{(+)}]. \tag{2.21}$$

This expression is however formal and requires renormalization. It involves the following integral transform of the scalar Wightman function [compare with (2.19)]:

$$\mathcal{K}(\lambda) = \frac{P}{i\pi}\int_{-\infty}^{\infty} dz\frac{\mathcal{G}(z)}{z-\lambda}, \tag{2.22}$$

which, recalling (2.16), can be split as

[4]The existence of the weak-coupling limit is guaranteed by the convergence of the integral $\int_0^\infty dt|G(x(t))|(1+t)^\delta$, for some $\delta>0$ (for details, see Ref. [16]); this is assured at infinity by the exponential fall off of $G(x(t))$, and at zero by the $i\varepsilon$ prescription [5].

$$\mathcal{K}(\lambda)=\frac{P}{2\pi^2 i}\int_0^\infty dz\frac{z}{z-\lambda}+\frac{P}{2\pi^2 i}\int_0^\infty dz\frac{z}{1-e^{\beta_U z}}\left[\frac{1}{z+\lambda}-\frac{1}{z-\lambda}\right] \quad (2.23)$$

into an inertial and an acceleration dependent piece. Although not expressible in terms of elementary functions [37], the acceleration-dependent second term is a finite, odd function of λ, vanishing as β_U becomes large, i.e., for a vanishing acceleration. The first contribution in (2.23) is, however, linearly divergent. As a consequence, despite some cancellations that occur in (2.21) (see below), the Lamb contribution H_L turns out to be infinite, and its definition requires the introduction of a suitable cutoff and a renormalization procedure.

This is a well-known fact and has nothing to do with the weak-coupling assumptions used in deriving the evolution equation (2.11), nor with the specific situation of an accelerating atom: as the splitting in (2.23) shows, the Lamb shift would be infinite even for an atom at rest. Rather, the appearance of the divergences is due to the nonrelativistic treatment of the moving two-level atom, while any sensible calculation of energy shifts would have required the use of quantum field theory techniques [38,39].

In our quantum mechanical setting, the procedure needed to make the Lamb contribution H_L well defined is therefore clear: perform a suitable acceleration independent subtraction so that the expression in (2.21) reproduces the correct quantum field theory result when the atom is at rest. However, since we are interested in analyzing the effects due to the uniformly accelerated motion of the atom, we do not need to do this explicitly. In the following we shall therefore ignore standard, acceleration-independent Hamiltonian contributions in the evolution equation (2.11) and concentrate on the phenomena induced by the motion of the atom.

III. SINGLE ACCELERATING ATOM AND DECOHERENCE

We shall now explicitly discuss the dynamics of the accelerated atom as described by the evolution equations (2.11) and (2.12). In order to simplify the treatment a bit and be able to write explicit expressions for the evolved state $\rho(t)$, we shall assume that the coupling coefficients χ_μ^a introduced in (2.4) satisfy the further condition

$$\sum_{a=1}^{N}\chi_\mu^a(\chi_\nu^a)^*\propto\delta_{\mu\nu}; \quad (3.1)$$

in this way, the field correlations in (2.14) become diagonal. In the following, for the sake of simplicity, we shall set to 1 the proportionality coupling coefficient. Then, the sum in (2.18) can be explicitly performed and the Kossakowski matrix a_{ij} takes the general form

$$a_{ij}=A\delta_{ij}-iB\varepsilon_{ijk}n_k+Cn_in_j, \quad (3.2)$$

where

$$A=\frac{1}{2}[\mathcal{G}(\omega)+\mathcal{G}(-\omega)]=\frac{\omega}{4\pi}\left[\frac{1+e^{-\beta_U\omega}}{1-e^{-\beta_U\omega}}\right],$$

$$B=\frac{1}{2}[\mathcal{G}(\omega)-\mathcal{G}(-\omega)]=\frac{\omega}{4\pi},$$

$$C=\frac{1}{2}[2\mathcal{G}(0)-\mathcal{G}(\omega)-\mathcal{G}(-\omega)]=\frac{\omega}{4\pi}\left[\frac{2}{\beta_U\omega}-\frac{1+e^{-\beta_U\omega}}{1-e^{-\beta_U\omega}}\right]. \quad (3.3)$$

Similarly, also the Lamb shift contribution b_i in (2.21) simplifies, and becomes directed along the unit vector $\vec{n}$; the effective Hamiltonian in (2.11) can then be written as in (2.2),

$$H_{\rm eff}=\frac{\Omega}{2}\vec{n}\cdot\vec{\sigma}, \quad (3.4)$$

in terms of a renormalized frequency

$$\Omega=\omega+i[\mathcal{K}(-\omega)-\mathcal{K}(\omega)]. \quad (3.5)$$

As explained at the end of Sec. II, a suitable acceleration-independent subtraction has been implicitly included in the definition of the combination $\mathcal{K}(-\omega)-\mathcal{K}(\omega)$, which otherwise would have been logarithmically divergent [compare with the result (2.23)].

In order to discuss the properties of the solutions of (2.11) and (2.12), it is convenient to express the density matrix ρ in terms of the Pauli matrices; recalling the normalization condition $\mathrm{Tr}[\rho]=1$, one has the standard expansion

$$\rho=\frac{1}{2}\left(1+\sum_{i=1}^{3}\rho_i\sigma_i\right). \quad (3.6)$$

Then, the evolution equation (2.11) can be conveniently rewritten as a Schrödinger-like equation for the coherence (Bloch) vector $|\rho(t)\rangle$ of components $\{\rho_1(t),\rho_2(t),\rho_3(t)\}$ [14]:

$$\frac{\partial}{\partial t}|\rho(t)\rangle=-2\mathcal{H}|\rho(t)\rangle+|\eta\rangle. \quad (3.7)$$

The constant vector $|\eta\rangle$, with components $\eta_i=-4B\,n_i$, $i=1$, 2, 3, comes from the imaginary part of the Kossakowski matrix (3.2), while the 3×3 matrix $\mathcal{H}$ includes contributions both from $H_{\rm eff}$ and the real part of a_{ij}:

$$\mathcal{H}_{ij}=\begin{bmatrix} a & b+\Omega_3 & c-\Omega_2 \\ b-\Omega_3 & \alpha & \beta+\Omega_1 \\ c+\Omega_2 & \beta-\Omega_1 & \gamma \end{bmatrix}, \quad (3.8)$$

where $\Omega_i=(\Omega/2)n_i$, $i=1$, 2, 3, and

$$a=2A+C(n_2^2+n_3^2),\quad b=-Cn_1n_2,$$

$$\alpha=2A+C(n_1^2+n_3^2),\quad c=-Cn_1n_3,$$

$$\gamma=2A+C(n_1^2+n_2^2),\quad \beta=-Cn_2n_3. \quad (3.9)$$

The matrix $\mathcal{H}$ is nonsingular; indeed, its eigenvalues can be explicitly determined: $\lambda_1=2A$, $\lambda_\pm=(2A+C)\pm i\Omega/2$. Fur-

ther, their real parts are positive, so that for large times $|\rho(t)\rangle$ reaches an equilibrium state $|\rho_\infty\rangle$ [40]. This asymptotic state can be easily determined by inverting $\mathcal{H}$,

$$|\rho_\infty\rangle = \frac{1}{2}\mathcal{H}^{-1}|\eta\rangle, \tag{3.10}$$

and turns out to be directed along the unit vector $\vec{n}$:

$$|\rho_\infty\rangle = \left[\frac{1-e^{\beta_U\omega}}{1+e^{\beta_U\omega}}\right]|n\rangle. \tag{3.11}$$

Inserting these components in the expansion (3.6), one finds that the asymptotic density matrix ρ_∞ is purely thermal, with a temperature given by the Unruh temperature:

$$\rho_\infty = \frac{e^{-\beta_U H_S}}{\mathrm{Tr}[e^{-\beta_U H_S}]}. \tag{3.12}$$

Therefore, a two-level system, which is uniformly accelerating in a vacuum scalar field, is driven to a thermal state with temperature T_U, irrespectively of its initial state. This thermalization phenomenon is the most obvious manifestation of the Unruh effect in the framework of open-system dynamics.

Nevertheless, further aspects of this phenomenon can be analyzed by studying the behavior of the solution of (3.7) for finite times, which can be formally written as

$$|\rho(t)\rangle = \mathcal{M}(t)|\rho(0)\rangle + [1-\mathcal{M}(t)]|\rho_\infty\rangle,\ \mathcal{M}(t) = e^{-2\mathcal{H}t}. \tag{3.13}$$

The matrix $\mathcal{M}(t)$ is defined through the series expansion of the exponential function and therefore seems to involve arbitrary powers of $\mathcal{H}$. However, by definition, this 3×3 matrix obeys its cubic eigenvalue equation, so that powers of $\mathcal{H}$ higher then 2 can always be reduced to combinations of $\mathcal{H}^2$, $\mathcal{H}$, and **1**, the unit 3×3 matrix. Then, a systematic use of this substitution allows one to write

$$\mathcal{M}(t) = \frac{4}{\Omega^2+4C^2}\left\{e^{-4At}\Lambda_1 + 2e^{-2(2A+C)t}\left[\Lambda_2\cos\Omega t + \Lambda_3\frac{\sin\Omega t}{\Omega}\right]\right\}, \tag{3.14}$$

where the three constant 3×3 matrices Λ_i are explicitly given by

$$\Lambda_1 = \left[(2A+C)^2+\frac{\Omega^2}{4}\right]\mathbf{1} - 2(2A+C)\mathcal{H} + \mathcal{H}^2,$$

$$\Lambda_2 = -2A(A+C)\mathbf{1} + (2A+C)\mathcal{H} - \frac{1}{2}\mathcal{H}^2,$$

$$\Lambda_3 = 2A\left[\frac{\Omega^2}{4} - C(2A+C)\right]\mathbf{1} + \left[C(4A+C) - \frac{\Omega^2}{4}\right]\mathcal{H} - C\mathcal{H}^2. \tag{3.15}$$

As expected, $\mathcal{M}(t)$ contains exponentially decaying factors involving the real parts of the eigenvalues of $\mathcal{H}$, modulated by oscillating terms in the effective frequency Ω. In other terms, an accelerating atom immersed in a vacuum scalar field is subjected to phenomena of decoherence and dissipation, all regulated by the Planckian factors appearing in the Kossakowski matrix (3.2) and (3.3).

These nonunitary effects can be studied by analyzing the time behavior of suitable atom observables. Indeed, any physical property of the moving atom can be represented by a Hermitian matrix $\mathcal{O}$, which can be conveniently decomposed as

$$\mathcal{O} = \sum_{\mu=0}^{3}\mathcal{O}_\mu\sigma_\mu. \tag{3.16}$$

The time behavior of its corresponding mean value is then determined by that of the density matrix $\rho(t)$:

$$\langle\mathcal{O}(t)\rangle = \mathrm{Tr}[\mathcal{O}\rho(t)] = \mathcal{O}_0 + \sum_{i=1}^{3}\mathcal{O}_i\rho_i(t). \tag{3.17}$$

When the observable $\mathcal{O}$ represents itself as an admissible atom state ρ_f, the mean value (3.17) coincides with the probability $\mathcal{P}_{\mathrm{i}\to\mathrm{f}}(t)$ that the evolved atom density matrix $\rho(t)$, initially in $\rho(0)\equiv\rho_\mathrm{i}$, will be found in such a state at time t. Using (3.14), this probability can be computed in general:

$$\mathcal{P}_{\mathrm{i}\to\mathrm{f}}(t) = \frac{1}{2}\left\{1 - (\vec{\rho}_\mathrm{f}\cdot\vec{n})(1-e^{-4At})\left[\frac{1-e^{-\beta_U\omega}}{1+e^{-\beta_U\omega}}\right] + e^{-4At}(\vec{\rho}_\mathrm{i}\cdot\vec{n})(\vec{\rho}_\mathrm{f}\cdot\vec{n}) + e^{-2(2A+C)t}([(\vec{\rho}_\mathrm{i}\cdot\vec{\rho}_\mathrm{f}) - (\vec{\rho}_\mathrm{i}\cdot\vec{n})(\vec{\rho}_\mathrm{f}\cdot\vec{n})]\cos\Omega t - \vec{n}\cdot(\vec{\rho}_\mathrm{i}\times\vec{\rho}_\mathrm{f})\sin\Omega t)\right\}, \tag{3.18}$$

where, expanding the density matrices ρ_i, ρ_f as in (3.6), the notations $(\vec{\rho}_\mathrm{i}\cdot\vec{\rho}_\mathrm{f})$ and $(\vec{\rho}_\mathrm{i}\times\vec{\rho}_\mathrm{f})$ (and similarly with $\vec{n}$) represent scalar and vector products of their corresponding coherence vectors.

When $\vec{\rho}_\mathrm{i}=-\vec{n}$ and $\vec{\rho}_\mathrm{f}=\vec{n}$, the density matrices ρ_i, ρ_f represent the ground and excited states of the system Hamiltonian H_S in (2.2) (see Appendix). In this case, the expression in (3.18) simplifies to

$$\mathcal{P}_{\mathrm{i}\to\mathrm{f}}(t) = \frac{1}{1+e^{\beta_U\omega}}(1-e^{-4At}), \tag{3.19}$$

giving the probability for a spontaneous transition of the atom from the ground state to its excited state. It is to this phenomenon of spontaneous excitation that one usually refers when discussing the Unruh effect; indeed, (3.19) vanishes as $\beta_U\to\infty$, i.e., for an atom at rest.

Although the behavior of $\mathcal{P}_{\mathrm{i}\to\mathrm{f}}(t)$ in (3.18) and (3.19) is in principle experimentally observable through the use of suitable interferometric devices, in the standard analysis of the Unruh effect one limits the discussion to the spontaneous excitation rate $\Gamma_{\mathrm{i}\to\mathrm{f}}$, the probability per unit time of the transition $\mathrm{i}\to\mathrm{f}$, in the limit of an infinitely slow switching on and off of the atom-field interaction. In our formalism, its expression can be easily obtained by taking the time derivative of $\mathcal{P}_{\mathrm{i}\to\mathrm{f}}(t)$ at $t=0$; in the case of (3.19), one then finds

$$\Gamma_{\mathrm{i}\to\mathrm{f}} = \frac{\omega}{\pi}\frac{1}{e^{\beta_U\omega}-1}, \tag{3.20}$$

which is the expected result for an interaction Hamiltonian of the form (2.3). One should nevertheless remark that the possibility of a nonvanishing $\Gamma_{\mathrm{i}\to\mathrm{f}}$ is just one of the many manifestations of the Unruh effect, which, as discussed above, actually involves phenomena of decoherence and dissipation; in this respect, the open-system approach to the description of a moving atom dynamics appears to be much more physically comprehensive than the most traditional treatments.

IV. TWO ACCELERATING ATOMS AND ENTANGLEMENT ENHANCEMENT

In Sec. III we have seen that a uniformly accelerating two-level atom immersed in a scalar field in its vacuum state can be consistently described as an open system in weak interaction with a heat bath: in the comoving frame, the atom is seen evolving in time according to a master equation in Kossakowski-Lindblad form, which through decoherence effects drives its state toward a purely thermal equilibrium state, characterized by the Unruh temperature. When the system that is subjected to the uniform acceleration along the trajectory (2.1) is formed by two, noninteracting two-level atoms, one thus expects similar mixing-enhancing phenomena to occur, leading in particular to loss of the mutual quantum correlation (entanglement) that might have been present at the beginning.

However, even though not directly coupled, the external vacuum field through which the two atoms move may provide an indirect interaction between them, and thus a means to entangle them. Indeed, entanglement generation through the action of an external heat bath has been shown to occur in certain circumstances [18–23]; it is therefore of physical interest to investigate the same issue in the case of accelerating atoms.

We shall therefore start by considering a system composed of two, equal two-level atoms that start moving along the trajectory (2.1) at proper time $t=0$. Being independent, without direct mutual interaction, in the common comoving frame, their internal dynamics can again be taken to be described by the generic Hamiltonian (2.2). Then, the total two-system Hamiltonian H_S is now the sum of the two terms:

$$H_S = H_S^{(1)} + H_S^{(2)}, \quad H_S^{(1)} = \frac{\omega}{2}\sum_{i=1}^{3} n_i(\sigma_i \otimes \sigma_0),$$

$$H_S^{(2)} = \frac{\omega}{2}\sum_{i=1}^{3} n_i(\sigma_0 \otimes \sigma_i). \tag{4.1}$$

Similarly, being immersed in the same field Φ_μ and within the weak-coupling hypothesis, the atom-field interaction Hamiltonian can be most simply assumed to be the generalization of that in (2.3):

$$H' = \sum_{\mu=0}^{3} [(\sigma_\mu \otimes \sigma_0) + (\sigma_0 \otimes \sigma_\mu)]\Phi_\mu(x(t)). \tag{4.2}$$

On the other hand, the field Hamiltonian H_Φ remains that of a collection of free, independent scalar fields.

The derivation of the appropriate master equation describing the dynamics of the two atoms in the comoving frame proceeds as in the case of a single moving atom, discussed in Sec. II. One starts from the Liouville–von Neumann equation (2.6) generating the time evolution of the state $\rho_{\mathrm{tot}}(0)$ of the total system (atoms+external fields), and then traces over the fields degrees of freedom, assuming a factorized initial state $\rho_{\mathrm{tot}}(0)=\rho(0)\otimes|0\rangle\langle 0|$. In the weak-coupling limit, the two-atom system density matrix $\rho(t)$ is seen evolving in time according to a quantum dynamical semigroup of completely positive maps, generated by an equation of Kossakowski-Lindblad form

$$\frac{\partial\rho(t)}{\partial t} = -i[H_{\mathrm{eff}},\rho(t)] + \mathcal{L}[\rho(t)]. \tag{4.3}$$

The unitary term depends on an effective Hamiltonian H_{eff} that is the sum of H_S in (4.1) and suitable Lamb contributions. In order to discuss them explicitly, we proceed as in the single–atom case and adopt the simplifying condition (3.1), which results in diagonal field correlations (2.14). In this case, the effective Hamiltonian consists of the sum of three pieces: $H_{\mathrm{eff}}=H_{\mathrm{eff}}^{(1)}+H_{\mathrm{eff}}^{(2)}+H_{\mathrm{eff}}^{(12)}$. The first two represent single system contributions; they can be written exactly as in (4.1), with the frequency ω replaced by the renormalized one Ω given in (3.5). The third term is a field-generated direct two-atom coupling term:

$$H_{\mathrm{eff}}^{(12)} = i\sum_{i,j=1}^{3}\{[\mathcal{K}(\omega)+\mathcal{K}(-\omega)]\delta_{ij} + [\mathcal{K}(0)-\mathcal{K}(\omega)-\mathcal{K}(-\omega)]n_i n_j\}\sigma_i\otimes\sigma_j, \tag{4.4}$$

where $\mathcal{K}(\lambda)$ is the function introduced in (2.22). As explained at the end of Sec. II, a suitable acceleration independent subtraction has implicitly been included in the definition (4.4) in order to make the contribution $H_{\mathrm{eff}}^{(12)}$ well defined. Further, recall that $\mathcal{K}(\lambda)$ can be split as in (2.23) into an acceleration dependent and an $a=0$ piece. Since, as observed there, the acceleration-dependent contribution to $\mathcal{K}(\lambda)$ is odd in λ, one deduces that $H_{\mathrm{eff}}^{(12)}$ does not actually involve a: it is the same Lamb term that would have been generated in the case of a two-atom system at rest. Being interested in acceleration-induced effects, and in particular, in those related to entanglement creation, we shall not consider any further this inertially generated term, nor the single-system contributions $H_{\mathrm{eff}}^{(1)}$, $H_{\mathrm{eff}}^{(2)}$ and move to analyze the effects produced by the dissipative term $\mathcal{L}$ in (4.3).

Since the interaction of the two moving atoms with the external scalar field is mediated by the same field operator $\Phi(x)$, the Kossakowski matrix in $\mathcal{L}[\rho]$ involves the same field correlation functions (2.14) discussed in the case of a

single-atom system, and therefore can be expressed in terms of the single–atom Kossakowski matrix a_{ij} in (3.2) and (3.3). Explicitly, one finds [23]:

$$\mathcal{L}[\rho] = \sum_{i,j=1}^{3} a_{ij}\Bigg(\Bigg[(\sigma_j \otimes \sigma_0)\rho(\sigma_i \otimes \sigma_0) - \frac{1}{2}\{(\sigma_i\sigma_j \otimes \sigma_0), \rho\}\Bigg] + \Bigg[(\sigma_0 \otimes \sigma_j)\rho(\sigma_0 \otimes \sigma_i) - \frac{1}{2}\{(\sigma_0 \otimes \sigma_i\sigma_j), \rho\}\Bigg] + \Bigg[(\sigma_j \otimes \sigma_0)\rho(\sigma_0 \otimes \sigma_i) - \frac{1}{2}\{(\sigma_i \otimes \sigma_j), \rho\}\Bigg] + \Bigg[(\sigma_0 \otimes \sigma_j)\rho(\sigma_i \otimes \sigma_0) - \frac{1}{2}\{(\sigma_j \otimes \sigma_i), \rho\}\Bigg]\Bigg). \quad (4.5)$$

Describing the states of two two-level systems, the density matrix $\rho(t)$ is now a 4×4 matrix; in analogy with the decomposition (3.6) and recalling the normalization condition $\mathrm{Tr}[\rho(t)]=1$, we find convenient to decompose it as

$$\rho(t) = \frac{1}{4}[\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} \rho_{0i}(t)\sigma_0 \otimes \sigma_i + \sum_{i=1}^{3} \rho_{i0}(t)\sigma_i \otimes \sigma_0 + \sum_{i,j=1}^{3} \rho_{ij}(t)\sigma_i \otimes \sigma_j], \quad (4.6)$$

where the components $\rho_{0i}(t)$, $\rho_{i0}(t)$, $\rho_{ij}(t)$ are all real. Substitution of this expansion in the master equation (4.3) allows deriving the corresponding evolution equations for the above components of $\rho(t)$. As explained above, we shall ignore the Hamiltonian piece in (4.3) since it cannot give rise to acceleration-induced entanglement. Further, we shall work in the regime of large acceleration, i.e., in the limit of β_U small; while the conclusions concerning entanglement enhancement are actually independent of this choice, it will make many explicit formulas more readable. In fact, with this simplifying assumption, the Kossakowski matrix in (3.2) reduces to:[5]

$$a_{ij} = A\delta_{ij} - iB\epsilon_{ijk}n_k, \quad A = \frac{1}{2\pi\beta_U}, \quad B = \frac{\omega}{4\pi}. \quad (4.7)$$

A straightforward, but lengthy calculation allows one then to derive from (4.3) and (4.5)–(4.7) the following result:

$$\frac{\partial\rho_{0i}(t)}{\partial t} = -4A\rho_{0i}(t) + 2B(2+\tau)n_i - 2B\sum_{k=1}^{3} n_k\rho_{ik}(t), \quad (4.8a)$$

$$\frac{\partial\rho_{i0}(t)}{\partial t} = -4A\rho_{i0}(t) + 2B(2+\tau)n_i - 2B\sum_{k=1}^{3} n_k\rho_{ki}(t), \quad (4.8b)$$

$$\frac{\partial\rho_{ij}(t)}{\partial t} = -4A[2\rho_{ij}(t) + \rho_{ji}(t) - \tau\delta_{ij}] + 4B[n_i\rho_{0j}(t) + n_j\rho_{i0}(t)] + 2B[n_i\rho_{j0}(t) + n_j\rho_{0i}(t)] - 2B\delta_{ij}\sum_{k=1}^{3} n_k[\rho_{k0}(t) + \rho_{0k}(t)]. \quad (4.8c)$$

In these formulas, the quantity $\tau=\Sigma_{i=1}^{3}\rho_{ii}$ represents the trace of ρ_{ij}; it is a constant of motion, as is easily seen by taking the trace of both sides of (4.8c). Further, the value of τ can not be chosen arbitrarily, since it has to comply with the requirement of positivity of the initial density matrix $\rho(0)$; indeed, using the decomposition (4.6), one finds $-3\leqslant\tau\leqslant 1$.

The system of first-order differential equations in (4.8) naturally splits into two independent sets, involving the symmetric, $\rho_{(0i)}=\rho_{0i}+\rho_{i0}$, $\rho_{(ij)}=\rho_{ij}+\rho_{ji}$, and antisymmetric, $\rho_{[0i]}=\rho_{0i}-\rho_{i0}$, $\rho_{[ij]}=\rho_{ij}-\rho_{ji}$, variables. Although both sets of equations can be exactly integrated, the form of the explicit solutions looks cumbersome and is not very inspiring. Nevertheless, by just examining the structure of the two sets of differential equations, one can conclude that the antisymmetric variables involve exponentially decaying factors, so that they vanish for large times, while the symmetric variables approach in the same limit a nonvanishing asymptotic value. As a consequence, the evolution equations (4.8) admit an equilibrium state $\hat{\rho}$, whose explicit form can be obtained by inverting the appropriate coefficient tensors multiplying the variables $\rho_{0i}(t)$, $\rho_{i0}(t)$, $\rho_{ij}(t)$ in the rhs of (4.8).

The equilibrium density matrix $\hat{\rho}$ can be expanded as in (4.6) its components are given by

$$\hat{\rho}_{0i} = \hat{\rho}_{i0} = \frac{R}{3+R^2}(\tau+3)n_i,$$

$$\hat{\rho}_{ij} = \frac{1}{3+R^2}[(\tau-R^2)\delta_{ij} + R^2(\tau+3)n_in_j], \quad (4.9)$$

where $R=B/A$ is the ratio of the two constants appearing in the Kossakowski matrix in (4.7), whose positivity implies $0\leqslant R\leqslant 1$. As expected, the antisymmetric components $\hat{\rho}_{[0i]}$ and $\hat{\rho}_{[ij]}$ are zero, while the only dependence on the initial state is through the constant $\tau=\Sigma_i\rho_{ii}(0)$.[6] It is remarkable that in general this equilibrium state turns out to be entangled.

In the case of two two-level systems, a measure of the entanglement content of any state ρ is provided by the concurrence $\mathcal{C}[\rho]$ [41–43]. Indeed, it has been shown that $\mathcal{C}$ is a monotonically increasing function of the entanglement of formation [44]; its value ranges from zero, for separable states, to 1, for fully entangled states, like the Bell states. In order to compute the concurrence of any 4×4 density matrix ρ representing the state of two atoms, one starts from the

[5]Notice that the condition $B\leqslant A$, implicit in the original expressions for A and B in (3.3), needs now to be formally imposed in order to maintain the positivity of a_{ij}; it is physically justified by the assumption of a small β_U.

[6]One can check that the form (4.9) for the equilibrium state remains unchanged even considering the more general evolution generated by the Kossakowski matrix (3.2) instead of the simplified version in (4.7). The considerations about entanglement production below are therefore valid in general and not only in the case of large accelerations.

auxiliary matrix $\tilde{\rho}=(\sigma_2\otimes\sigma_2)\rho^T(\sigma_2\otimes\sigma_2)$, where T indicates transposition. Although not necessarily Hermitian, the matrix $\rho\tilde{\rho}$ has real, non-negative eigenvalues, whose square roots λ_μ, $\mu=1, 2, 3, 4$, can be ordered in decreasing order: $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$. The concurrence of ρ is then defined to be $\mathcal{C}[\rho]=\max\{\lambda_1-\lambda_2-\lambda_3-\lambda_4,0\}$.

In the case of the asymptotic state $\hat{\rho}$ in (4.9), one finds that the concurrence is indeed nonvanishing, provided

$$\tau < \frac{5R^2-3}{3-R^2}, \tag{4.10}$$

and that, in this case, its explicit expression is given by

$$\mathcal{C}[\hat{\rho}] = \frac{(3-R^2)}{2(3+R^2)}\left[\frac{5R^2-3}{3-R^2}-\tau\right]. \tag{4.11}$$

The concurrence is therefore a linearly decreasing function of τ, starting from its maximum $\mathcal{C}[\hat{\rho}]=1$ for $\tau=-3$ and reaching zero at $\tau=(5R^2-3)/(3-R^2)$; note that this ratio is an admissible value for τ, since it is always within the interval $[-1,1]$ for the allowed values of R.

This result is remarkable, since it implies that the dynamics in (4.8) can generate entanglement: one prepares the two atoms in a separable state at $t=0$; then, provided the condition (4.10) is satisfied, their long-time equilibrium state will turn out to be entangled. The simplest example of a separable state is provided by the direct product of pure states:

$$\rho(0)=\rho_n\otimes\rho_m, \quad \rho_n=\frac{1}{2}(1+\vec{n}\cdot\vec{\sigma}), \quad \rho_m=\frac{1}{2}(1+\vec{m}\cdot\vec{\sigma}), \tag{4.12}$$

where $\vec{n}$ and $\vec{m}$ are two unit vectors. In this case, one easily finds that $\tau=\vec{n}\cdot\vec{m}$, so that, recalling (4.11), the asymptotic entanglement is maximized when $\vec{n}$ and $\vec{m}$ are collinear and pointing in opposite directions. Explicitly, one finds

$$\mathcal{C}[\hat{\rho}] = \frac{2R^2}{3+R^2} \tag{4.13}$$

which reaches its maximum value, $\mathcal{C}[\hat{\rho}]=1/2$, for $R=1$.

The phenomenon of entanglement production occurs also in cases when the initial state $\rho(0)$ already has a nonvanishing concurrence. As observed before, $\mathcal{C}[\hat{\rho}]$ reaches its maximum when $\tau=-3$; in this case, the equilibrium state $\hat{\rho}$ coincides with the totally entangled singlet state:

$$\rho_- = \frac{1}{4}\left[\sigma_0\otimes\sigma_0-\sum_{i=1}^{3}\sigma_i\otimes\sigma_i\right], \tag{4.14}$$

a fixed point of the dynamical equations (4.8), as easily seen by direct inspection. Then, let us consider the following initial state:

$$\rho(0)=(1-\varepsilon)\rho_- + \frac{\varepsilon}{4}\sigma_0\otimes\sigma_0, \tag{4.15}$$

which interpolates between ρ_- and the totally mixed (separable) state; for $\varepsilon<2/3$, it is entangled, with $\mathcal{C}[\rho(0)]=1-3\varepsilon/2$. In passing from this initial state to its corresponding equilibrium state $\hat{\rho}$ as $t\to\infty$, the corresponding increase in concurrence, and thus of entanglement, can be easily computed:

$$\mathcal{C}[\hat{\rho}]-\mathcal{C}[\rho(0)] = \frac{3R^2\varepsilon}{3+R^2}, \tag{4.16}$$

which is indeed nonvanishing. Further, notice that for the state (4.15), $\tau=-3(1-\varepsilon)$, so that it can be taken to be very close to its lower limit -3; unfortunately, the entanglement production (4.16) becomes vanishingly small as $\rho(0)$ approaches ρ_-. In other terms, the maximally entangled state ρ_- can never be asymptotically reached, and thus the maximum of concurrence obtained, unless one already starts with it at $t=0$.

V. DISCUSSION

Moving detectors, modeled as simple two-level atoms, immersed in external vacuum fields and following an uniformly accelerating trajectory are seen to possess a nonvanishing probability of spontaneous excitation, reproducing a thermal spectrum; this phenomenon is usually referred to as the Unruh effect.

It turns out that the dynamics of these accelerated atoms can be consistently assimilated with that of subsystems in interaction with an external environment, i.e., the so-called open-quantum systems. General techniques and results obtained in the analysis of the latter can then be fruitfully applied to the study of the former, enabling the discussion of physical aspects of the Unruh effect that might not be easily identified in the standard treatments. As shown in the previous sections, the open system paradigm is particularly suitable for analyzing, on the one hand, issues connected to the appearance of decoherence effects, and on the other hand, questions related to the phenomenon of entanglement enhancement.

The starting point of our analysis has been the derivation of the appropriate master equation that generates the time evolution of the states of an accelerating system, in its comoving frame. In the framework of a physically justified weak-coupling hypothesis, the system subdynamics takes the form of a semigroup of completely positive maps, transforming density matrices into density matrices, while preserving their normalization and positivity.

In the case of a single two-level atom, the master equation has been explicitly integrated; this has allowed discussing in detail the mixing-enhancing properties of the associated finite-time evolution through the analysis of the behavior of appropriate atom observables. In particular, one finds that the moving atom is subjected to dissipative effects that asymptotically drive its density matrix to an equilibrium state, with a purely thermal spectrum.[7]

When the accelerating system is composed by two, independent atoms, its physical characterization naturally in-

[7]Note that this result is different from (but complementary to) the "thermalization theorem" discussed in the usual treatments of the Unruh effect; there, it is the field "vacuum" state that appears to be purely thermal to an accelerating observer.

volves the analysis of their mutual quantum correlations. Because of the just mentioned decohering and mixing-enhancing phenomena, one would be led to regard the Unruh effect as counteracting entanglement enhancement.

Nevertheless, quite in general the presence of an external environment (in the specific case of the external fields) can provide an indirect interaction between the two otherwise totally decoupled two-level systems, thus a means to correlate them. This picture has indeed been confirmed by the analysis of specific models; in particular [23], entanglement can be created by the action of an external bath through a purely noisy mechanism during the memoryless, Markovian regime, when the corresponding subdynamics is generated by an equation of the form (2.11) and (2.12).

The master equation describing the dynamics of two accelerating atoms discussed in Sec. IV is precisely of the form identified in Ref. [23] as generating initial entanglement. It is remarkable that the unavoidable decoherence that subsequently builds up is not sufficient to counteract this quantum correlations enhancing effect. The fate of the entanglement production by acceleration can in fact be discussed by analyzing the entanglement content of the final two-atom equilibrium state, through the evaluation of its concurrence: quite in general, one finds an asymptotic nonvanishing concurrence even for totally separable initial states.

Although obtained in an idealized situation, this result offers new possibilities for an actual direct verification of the Unruh effect. So far all efforts have been devoted to the analysis of possible experimental settings that could allow the measure of the tiny spontaneous excitation thermal rate induced in single-accelerating systems. Alternatively, using suitable devices, one can instead try to detect the quantum correlation enhancement that is generated when the accelerating system is formed by two, initially unentangled atoms; in view of the high accuracy and sophistication reached by present quantum optics experiments, this possibility might actually be realized in the future.

As a concluding comment, let us remark that the presented open-system treatment of the Unruh effect is not limited to the analysis of the simple setting of atoms interacting with free, scalar fields. Extension to the case of higher spin fields is straightforward, while the analysis of situations involving self-interacting fields would require the use of algebraic quantum field theory. This might not be as formidable task as it looks, since the quantum theory of open systems is amenable to a rigorous, algebraic formulation. Finally, it is known that the Unruh effect has many similarities and analogies with the phenomenon of particle creation in curved space-times. In this regard, we expect the open-system paradigm to be applicable also to those cases, possibly providing new insight in the physical interpretation of the corresponding effects.

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APPENDIX

Using the rigorous results of Refs. [16,17], we shall present here the derivation of the master equation generating the reduced dynamics of a two-level system in interaction with a generic external environment, in the weak-coupling limit. We shall generalize the treatment of Ref. [45] and take the generic Hamiltonian (2.2),

$$H_S = \frac{\omega}{2}\vec{n}\cdot\vec{\sigma}, \quad |\vec{n}| = 1, \tag{A1}$$

as system Hamiltonian; instead, the most general term that is linear in both system and environment variables,

$$H' = \sum_{\mu=0}^{3} \sigma_\mu \otimes B_\mu, \tag{A2}$$

will be taken to represent the interaction Hamiltonian. The environmental dynamical variables B_μ are left unspecified (they coincide with the field Φ_μ in the case of an accelerating system), as the corresponding Hamiltonian H_B, which nevertheless is assumed to generate time translations:

$$e^{iH_B t} B_\mu e^{-iH_B t} = B_\mu(t). \tag{A3}$$

The time evolution of the density matrix $\rho_{\rm tot}$ representing the state of the complete system is then generated by the total Hamiltonian

$$H = H_0 + H', \quad H_0 = H_S \otimes \mathbf{1} + \mathbf{1} \otimes H_B, \tag{A4}$$

through the standard unitary evolution, starting with an initial state taken in factorized form: $\rho_{\rm tot}(0) = \rho(0) \otimes \rho_B$; $\rho(0)$ is the 2×2 density matrix describing the subsystem state, while ρ_B is the analogous one for the environment, assumed to be stationary: $[H_B, \rho_B] = 0$ (in the case of moving atoms, $\rho_B \equiv |0\rangle\langle 0|$, with $|0\rangle$ the Minkowski vacuum).

Correspondingly, the subdynamics describing the evolution of the subsystem alone is obtained with a trace operation over the environment degrees of freedom. As briefly described in Sec. II, and rigorously proven in Refs. [16,17], in the limit of weak coupling between subsystem and environment, the reduced density matrix $\rho(t) = {\rm Tr}[\rho_{\rm tot}(t)]$ is found to obey the following evolution equation:

$$\frac{\partial \rho(t)}{\partial t} = -iL_{H_S}[\rho(t)] + \mathcal{K}^\sharp[\rho(t)], \quad L_{H_S}[\rho] \equiv [H_S, \rho], \tag{A5}$$

where

$$\mathcal{K}^\sharp[\cdot] = -\lim_{T\to\infty} \frac{1}{T}\int_0^T ds \mathcal{U}(-s)\mathcal{K}\mathcal{U}(s)[\cdot], \ \mathcal{U}(s) = e^{-isL_{H_S}} \tag{A6}$$

and

$$\mathcal{K}[\rho] = \int_0^\infty dt {\rm Tr}([e^{iH_0 t} H' e^{-iH_0 t}, [H', \rho \otimes \rho_B]]). \tag{A7}$$

The ergodic mean over the system dynamics in (A6) is necessary in order to obtain a completely positive subdynamics, and it is physically justified as an averaging over the fast microscopic system oscillations; indeed, the weak-coupling limit procedure involves a suitable rescaling of the time vari-

able, so that Eq. (A5) actually generates a "coarse grained" subdynamics with respect to the free system motion. With the definitions (A1) and (A2), one can more explicitly write

$$\mathcal{K}^{\sharp}[\rho] = \lim_{T\to\infty}\frac{1}{T}\int_0^T ds\int_0^\infty dt[\sigma_\mu(t+s)\rho\sigma_\nu(s)\langle B_\nu B_\mu(t)\rangle + \sigma_\nu(s)\rho\sigma_\mu(t+s)\langle B_\mu(t)B_\nu\rangle - \sigma_\mu(t+s)\sigma_\nu(s)\rho\langle B_\mu(t)B_\nu\rangle - \rho\sigma_\nu(s)\sigma_\mu(t+s)\langle B_\nu B_\mu(t)\rangle], \tag{A8}$$

where

$$\langle B_\mu(t)B_\nu\rangle = \mathrm{Tr}[B_\mu(t)B_\nu\rho_B] \tag{A9}$$

are the environment correlations.

In order to proceed further, it is convenient to introduce the two projector operators

$$P_\pm = \frac{1}{2}(1 \pm \vec{n}\cdot\vec{\sigma}); \tag{A10}$$

they represent the density matrices of the two eigenstates of the system Hamiltonian (A1), with eigenvalues $\pm\omega/2$. One can then use the auxiliary matrices $\sigma_\mu^{(\xi)}$, $\xi=0,+,-$, explicitly defined by

$$\sigma_\mu^{(0)} = P_+\sigma_\mu P_+ + P_-\sigma_\mu P_-,\ \sigma_\mu^{(\pm)} = P_\pm\sigma_\mu P_\mp, \tag{A11}$$

and represent the system free time evolution in terms of the following spectral decomposition:

$$\sigma_\mu(t) = e^{iH_St}\sigma_\mu e^{-iH_St} = \sum_{\xi=0,\pm} e^{i\xi\omega t}\sigma_\mu^{(\xi)}. \tag{A12}$$

This allows performing explicitly the limit in (A8) and expressing the result in terms of the following Fourier and Hilbert transform of the environment correlations:

$$\alpha_{\mu\nu}^{(\xi)}(\omega) = \int_{-\infty}^{\infty} dt e^{i\xi\omega t}\langle 0|B_\mu(t)B_\nu|0\rangle \tag{A13}$$

and

$$\beta_{\mu\nu}^{(\xi)}(\omega) = \int_0^\infty dt e^{i\xi\omega t}\langle 0|B_\mu(t)B_\nu|0\rangle - \int_0^\infty dt e^{-i\xi\omega t}\langle 0|B_\mu B_\nu(t)|0\rangle; \tag{A14}$$

notice that the first 4×4 matrix is Hermitian, $[\alpha_{\mu\nu}^{(\xi)}(\omega)]^\dagger = \alpha_{\mu\nu}^{(\xi)}(\omega)$, while the second is anti-Hermitian, $[\beta_{\mu\nu}^{(\xi)}(\omega)]^\dagger = -\beta_{\mu\nu}^{(\xi)}(\omega)$. Explicitly, one finds

$$\mathcal{K}^{\parallel}[\rho] = \frac{1}{2}\sum_{\xi=0,\pm}\sum_{\mu,\nu=0}^{3}\{\alpha_{\mu\nu}^{(\xi)}[2\sigma_\nu^{(-\xi)}\rho\sigma_\mu^{(\xi)} - \sigma_\mu^{(\xi)}\sigma_\nu^{(-\xi)}\rho - \rho\sigma_\mu^{(\xi)}\sigma_\nu^{(-\xi)}] + \beta_{\mu\nu}^{(\xi)}[\rho,\sigma_\mu^{(\xi)}\sigma_\nu^{(-\xi)}]\}. \tag{A15}$$

This expression can be further simplified by expanding the auxiliary matrices $\sigma_\mu^{(\xi)}$ in terms of Pauli matrices:

$$\sigma_0^{(\xi)} = \delta_{\xi 0}\sigma_0, \quad \sigma_i^{(\xi)} = \sum_{j=1}^3 \psi_{ij}^{(\xi)}\sigma_j, \tag{A16}$$

with

$$\psi_{ij}^{(0)} = n_i n_j, \quad \psi_{ij}^{(\pm)} = \frac{1}{2}(\delta_{ij} - n_i n_j \pm i\epsilon_{ijk}n_k). \tag{A17}$$

The entire master equation in (A5) can then be rewritten in standard Kossakowski-Lindblad form:

$$\frac{\partial\rho(t)}{\partial t} = -i[H_{\mathrm{eff}},\rho(t)] + \mathcal{L}[\rho(t)], \tag{A18}$$

where

$$\mathcal{L}[\rho] = \frac{1}{2}\sum_{i,j=1}^3 a_{ij}[2\sigma_j\rho\sigma_i - \sigma_i\sigma_j\rho - \rho\sigma_i\sigma_j], \tag{A19a}$$

$$H_{\mathrm{eff}} = \frac{1}{2}\sum_{i=1}^3[\omega n_i + b_i], \tag{A19b}$$

while the Kossakowski matrix a_{ij} and hamiltonian vector b_i can be expressed as

$$a_{ij} = \sum_{\xi=0,\pm}\sum_{k,l=1}^3 \alpha_{kl}^{(\xi)}\psi_{ki}^{(\xi)}\psi_{lj}^{(-\xi)}, \tag{A20a}$$

$$b_i = i\sum_{j=1}^3[\alpha_{0j}^{(0)} - \alpha_{j0}^{(0)} - \beta_{0j}^{(0)} - \beta_{j0}^{(0)}]n_j n_i + \sum_{j,k,l,m=1}^3 \epsilon_{ijk}\left[\sum_{\xi=0,\pm}\beta_{lm}^{(\xi)}\psi_{lj}^{(\xi)}\psi_{mk}^{(-\xi)}\right]. \tag{A20b}$$

Since $[\psi_{ij}^{(\xi)}]^* = \psi_{ij}^{(-\xi)}$, the matrix a_{ij} in (A20a) is manifestly Hermitian and positive, being the combination of Fourier transform of correlation functions [14]. Further, by choosing for the environment variables B_μ the fields Φ_μ, the master equation (A18)–(A20) reduces to that discussed in the text.

Notice that in general the dissipative piece in (A19a) is a function of nine real parameters, the independent entries of the matrix a_{ij}.[8] However, because of the structure given in (A20a), the Kossakowski matrix obtained through a weak-coupling procedure appears to depend on a lesser number of free parameters. This fact has been already noted in Ref. [45], where it is further observed that the most general master equation obtainable in the weak-coupling limit appears to coincide with the old Bloch equation, describing the dissipative motion of a spin in a constant magnetic field.

In the light of the previous derivation, this conclusion looks, however, to be too restrictive: by starting with the

[8]These are not actually all independent: since the matrix a_{ij} is positive, they need to satisfy certain inequalities; see Ref. [28] for details.

most general system Hamiltonian (A1), instead of the one with $\vec{n}=(0,0,1)$ as adopted in Ref. [45], one is able to obtain a master equation in the form (A18)–(A20), certainly more general than the Bloch equation. This observation might have interesting applications in the study of specific open system models.

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