Behavior of information flow near criticality

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Recent experiments have indicated that many biological systems self-organise near their critical point, which hints at a common design principle. While it has been suggested that information transmission is optimized near the critical point, it remains unclear how information transmission depends on the dynamics of the input signal, the distance over which the information needs to be transmitted, and the distance to the critical point. Here we employ stochastic simulations of a driven 2D Ising system and study the instantaneous mutual information and the information transmission rate between a driven input spin and an output spin. The instantaneous mutual information varies non-monotonically with the temperature, but increases monotonically with the correlation time of the input signal. n contrast, there exists not only an optimal temperature but also an optimal finite input correlation time that maximizes the information transmission rate. This global optimum arises from a fundamental trade-off between the need to maximize the frequency of independent input messages, the necessity to respond fast to changes in the input, and the need to respond reliably to these changes. The optimal temperature lies above the critical point, but moves towards it as the distance between the input and output spin is increased.

Most, if not all, living organisms need to respond to changes in their environment. Examples include bacteria searching for food, animals trying to catch prey, or birds in flocks trying to coordinate their motion. In all these cases, the flow of information, be it via an intracellular biochemical network, an intercellular neural network, or between the individuals within the group, is vital to function. Moreover, in all these examples not only the reliability of information transmission is important, but also the speed and the distance over which the information needs to be transmitted.

Recent experiments indicate that many biological systems self-organise at their critical point. Examples are the flocking behaviour of starlings [1], signal percolation within a bacterial community [2], or neural networks [3]. A hallmark of all these systems is that they process information that is in the dynamics of an input signal—a raptor [1], an electrochemical signal [2], a neuronal stimulus [3]—by mapping that input onto an output signal—a direction of the birds' course, halting cell growth, or a neuronal response, respectively. To characterise this information transmission capacity [4], we need to understand how these systems process time-varying signals. Moreover, as an information measure we need the information transmission rate [5–7] or the transfer entropy [11], which takes into account both the reliability and the speed of information transmission.

The observation that very different systems self-organize near the critical point hints at a common design principle [12]. The Ising system is, because of its generic properties, ideally suited to address this question. In fact, the role of criticality in information transmission in the Ising system has been studied extensively [13–17]. It has been shown that the mutual information between pairs of neighboring spins exhibits a sharp peak near the critical point [13, 14]. Additionally, the transfer entropy [11] between pairs of neighboring spins peaks at the critical point while a global transfer entropy measure peaks above it [15]. However, all these studies consider information transfer between spins in Ising systems that are in thermodynamic equilibrium. They do not address the question how the system processes time-varying signals, which, moreover, can drive the system out of thermodynamic equilibrium.

Here, we consider a 2D Ising system that needs to process a time-varying input signal. The input signal $\mathcal{S}$ is a spin, the *input spin*, which is flipped according to a stationary random telegraph process with a time scale $\tau_{\rm s}$. Its dynamics can thus be regulated independently of the remaining spins, which is controlled by the temperature. The input signal will drive the system out of equilibrium, and because this signal is stationary, our system is in a stationary non-equilibrium steady-state. The output signal $\mathcal{X}$ is another spin, *the output spin*, which is located at a distance d away from the input spin, see Fig. 1. The input and output signals produce the random variables $S, X \in \{+1, -1\}$ at each point in time, respectively. Because the information is propagated from the input to the output spin via the other spins, the dynamics of the output are distinctly non-Markovian [7]. Consequently, we need to recognize the history of the input and output signal in characterizing the information flow between them.

To characterize information flow, not only the mutual information [13, 14] and the transfer entropoy [15] have been employed, but also various decompositions thereof [16–21]. Yet, none of these studies have considered the information transmission rate or the multi-step transfer entropy [11], which take the full history of the input into account. This is critical, because correlations in the input and output signal will affect information transmission.

To characterize information transmission, we study two measures: the *instantaneous* mutual information I_{inst} and the information transmission rate I_{R}. The measure $I_{\text{inst}}(S; X)$ is the mutual information between the stationary input and output signal at a single point in time:

$$I_{\text{inst}}(S; X) = H(S) - H(S|X), \qquad (1)$$

where $H(S)$ is the Shannon entropy of the input signal and $H(S|X)$ is the Shannon entropy conditional on the output signal. The instantaneous mutual information has been used to study information transmission in intracellular signaling networks [22–25]. In contrast to the mutual information studied by Matsuda et al. [13], which characterizes the statistical mutual dependence between pairs of spins in an equilibrium system, $I_{\text{inst}}(S; X)$ quantifies the dependence between the input spin S and the output spin X of our driven system; $2^{I_{\text{inst}}(S;X)}$ can be interpreted as the number of distinct mappings between input and output that can be distinguished reliably [26]. The quantity $I_{\text{inst}}(S; X)$ depends on the input time scale τ_{s} and the response time τ_{r} of the system, which is determined by the temperature.

While the instantaneous mutual information $I_{\text{inst}}(S; X)$ quantifies how accurately the input spin is mapped onto the output spin, it does not quantify the rate of information transmission. The latter is not only determined by the accuracy of the input-output mapping, but also by the rate at which independent "messages" are transmitted through the system. Autocorrelations within the input and the output signal lower the information transmission rate. To take these correlations into account, we study the information transmission rate, which is defined as the rate at which the mutual information $I(\mathbf{S}_L; \mathbf{X}_L)$ between the trajectories of the input and output signal increases [6]:

$$I_{\text{R}} = \lim_{L \to \infty} \frac{I(\mathbf{S}_L; \mathbf{X}_L)}{L} = \lim_{L \to \infty} \frac{H(\mathbf{S}_L) - H(\mathbf{S}_L|\mathbf{X}_L)}{L}, \qquad (2)$$

where $\mathbf{S}_L = [S(t_1), S(t_2), \ldots, S(t_n)]$ and $\mathbf{X}_L = [X(t_1), X(t_2), \ldots, X(t_n)]$ are spin trajectories of duration $L = (n-1)\delta t$, containing n subsequent spin states S (X) at successive time points $t_i = (i-1)\delta t$, with δt the elementary timestep of the dynamics [7]. To capture the autocorrelations in the input and output signal, the trajectory lengths have to be longer than the longest timescale in the problem, $L > \tau_{\text{s}}, \tau_{\text{r}}$; I_{R} then properly takes into account the history of the input and output spin, in contrast to the one-step transfer entropy. Indeed, in general, I_R differs from $I_{\text{inst}}/\delta t$ precisely because of the correlations in input and output. Since in our system the output signal does not feed back on the input, I_R is equivalent the multi-step transfer entropy [5, 7].

In order to evaluate the effects of the dynamics and criticality on information flow, we will study both measures as a function of the input correlation time τ_{s}, the
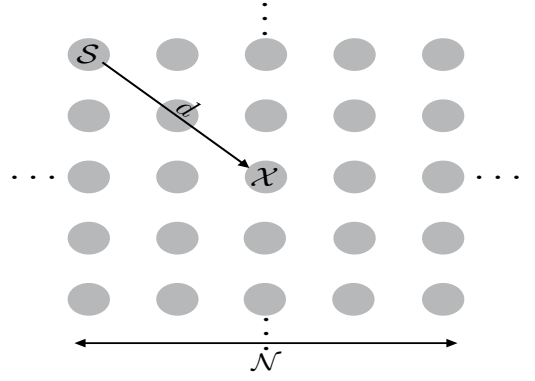

FIG. 1: We consider an Ising system containing $\mathcal{N} \times \mathcal{N}$ spins with periodic boundary conditions. One spin is chosen to be the input spin and is flipped according to a stationary random-telegraph progress. We measure the information transmission from the input to the output spin, positioned a distance d along the diagonal from the input; d in units of the distance between two neighboring spins along the diagonal.

temperature T, the distance d between the input and output spin, and the system size. We are mainly interested in temperatures higher than the critical temperature T_{c}, since for lower temperatures the system freezes down in the ferromagnetic phase, drastically slowing down information transmission. We will show that the non-trivial interaction between the diverging correlation length and the diverging response time near the critical point causes the information flow to be optimal close to, but not at, the critical point. The optimal temperature is determined by the distance over which the information needs to be transmitted and the size of the system.

Consider a 2D Ising system of $\mathcal{N} \times \mathcal{N}$ spins with periodic boundary conditions and no external magnetic field. The Hamiltonian of the system, with spin configuration $\boldsymbol{\sigma} = \sigma_1, \ldots, \sigma_{\mathcal{N}^2}$ and $\sigma_i \in \{+1, -1\}$, is $H(\boldsymbol{\sigma}) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$, where J is the coupling parameter and the sum is taken over all nearest neighbours. For isotropic coupling, the critical temperature is $k_{\text{B}} T_{\text{c}}/J = 2.269$ [27]. Following Barnett *et al.* [15], we use discrete-time Glauber spin-flip dynamics [28]. We define the response time τ_{r} as the time scale over which spontaneous fluctuations in the undriven system relax to equilibrium, as measured via the two-point time correlation function of the input and output spin [29]. Entropies are measured in nats.

The information transmission rate is notoriously difficult to compute, because the state space of the input and output trajectories rapidly diverges with their length. We have therefore considered not only relatively small systems, but also developed the following scheme: To limit the size of the state space, we introduce a sampling interval Δt such that the trajectory length $L = (n-1)\Delta t$. We ensure that L is longer than the input and output correlation time such that $I(\mathbf{S}_L; \mathbf{X}_L)$ increases linearly
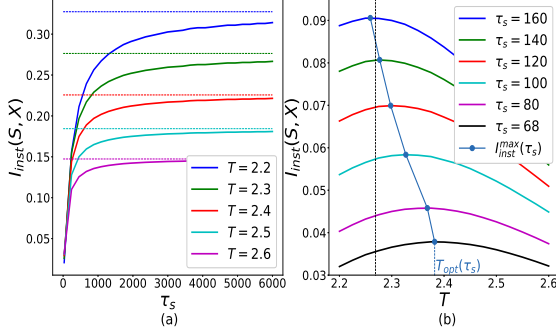
FIG. 2: The instantaneous mutual information $I_{\text{inst}}(S;X)$ as a function of the input correlation time τ_{s} and temperature T. (a) $I_{\text{inst}}(S;X)$ increases monotonically with τ_{s} until it reaches a plateau, $I_{\text{inst},\infty}(S;X)$, which is equal to the static mutual information (dashed line). The plateau value $I_{\text{inst},\infty}(S;X)$ increases as T decreases. However, for small τ_{s}, when τ_{s} is on the order of the response time τ_{r}, $I_{\text{inst}}(S;X)$ does not rise monotonically with decreasing temperature. This is more clearly seen in panel b: there exist an optimal temperature T_{opt} that maximises $I_{\text{inst}}(S;X)$ for a given τ_{s}; moreover, T_{opt} decreases when τ_{s} increases. The optimal temperature arises from a trade-off between responding rapidly and reliably. Vertical dashed line denotes critical temperature. The size of the system is 5×5, and the distance between input and output spin is $d = 2$. Fig. S8 of [7] shows that the results do not qualitatively change for a 10 x 10 times larger system.

with L and the information transmission rate I_{R} is independent of L (see [7]). We then compute for long enough L, $I_{\text{R}} = I_{\text{R}}(\Delta t)$ for a range of Δt values, where we verify that the entropy histograms are sampled accurately, using the Bayesian entropy estimator of Nemenman *et al.* [30]. We then extrapolate $I_{\text{R}}(\Delta t)$ to the quantity of interest, $I_{\text{R}}(\Delta t \to \delta t)$, where δt is the elementary time step of the Glauber dynamics; to verify this extrapolation procedure, we have also recomputed $I_{\text{R}}(\Delta t)$ for a number of extrapolated Δt values (see [7]).

Figure 2 shows the instantaneous mutual information $I_{\text{inst}}(S;X)$ between the input and output signal separated by a distance $d = 2$ as a function of the input correlation time τ_{s} and temperature T in an Ising system of 5×5 spins. The instantaneous mutual information rises with the input correlation time τ_{s} (Fig. 2a), because this gives the system more time to respond to changes in the input signal and hence more time to correlate the output with the input signal. For large $\tau_{\text{s}} \to \infty$, the instantaneous mutual information reaches a plateau value $I_{\text{inst},\infty}(S;X)$ that corresponds to the static mutual information, which is the mutual information between the output spin and the input spin when the latter is held fixed indefinitely for each realization $S = 1, -1$. The static mutual information increases as the temperature is decreased, because decreasing the temperature lowers the thermal noise in the transmitted signal.
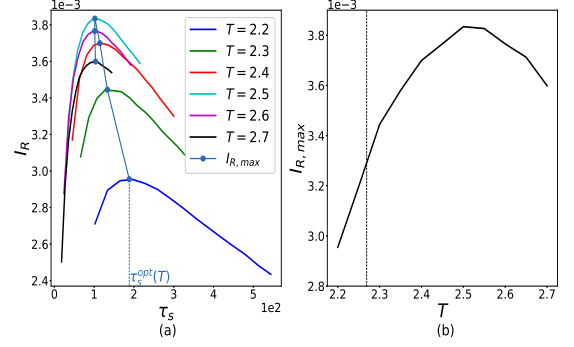

FIG. 3: The information transmission rate I_{R} as a function of the input correlation time τ_s and temperature T. (a) For a given temperature T, there exists an optimal τ_s that maximises the information transmission rate I_{R}, $I_{\text{R}}^{\text{max}}$; it arises from a trade-off between maximizing the frequency of independent inputs and responding reliably. The figure also shows that $I_{\text{R}}^{\text{max}}$ initially rises with T, but then decreases, which is highlighted in panel b: there exists an optimal temperature T_{opt} that maximises $I_{\text{R}}^{\text{max}}$. Vertical dashed line denotes critical temperature. System size is 5×5, and the distance between input and output spin is $d = 2$. Lines are truncated at high τ_s for large T, because it becomes exceedingly difficult to get good statistics in this regime.

Panel b of Fig. 2 shows that for a given correlation time τ_{s} of the input signal, there exists an optimal temperature T_{opt} that maximises the instantaneous mutual information $I_{\text{inst}}(S;X)$. Increasing the temperature raises the thermal noise in the signal, which tends to lower the instantaneous mutual information. On the other hand, increasing the temperature also reduces the response time τ_{r}. This allows the system to more accurately track the input signal, which tends to raise the instantaneous mutual information between the input and output signal. The interplay between these two effects gives rise to an optimal temperature $T_{\text{opt}}(\tau_{\text{s}})$ that maximizes the instantaneous mutual information, $I_{\text{inst}}^{\text{max}}(\tau_{\text{s}})$. This optimal temperature decreases as the input correlation time τ_{s} is increased, because the latter gives the system more time to respond to the changes in the input. Moreover, the maximum instantaneous mutual information $I_{\text{inst}}^{\text{max}}(\tau_{\text{s}})$ rises with τ_{s}, not only because increasing τ_{s} raises I_{inst} by itself, but also because the lower optimal temperature $T_{\text{opt}}(\tau_{\text{s}})$ reduces the thermal noise in the signal.

Figure 3a shows the information transmission rate I_{R} as a function of the correlation time of the input signal τ_{s} for different temperatures T. While, for a given temperature, the instantaneous mutual information I_{inst} increases monotonically with the input correlation time τ_{s} (see Fig. 2a), the information transmission rate I_{R} exhibits an optimal τ_{s} that maximizes I_{R}. When τ_{s} is too short, the signal is changing faster than the output can respond to, which decreases I_{R} by increasing the conditional entropy $H(\mathbf{S}_L|\mathbf{X}_L)$ (see Eq. 2). On the other hand,
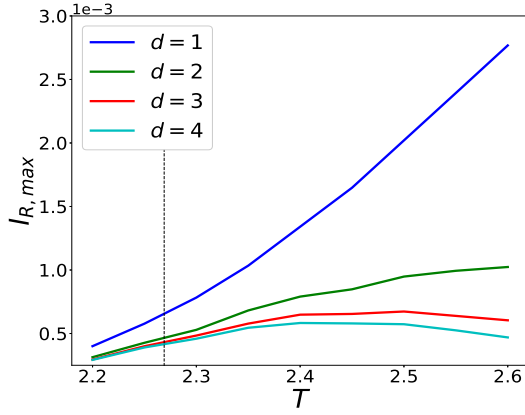
FIG. 4: The optimal temperature that maximises the information transmission rate I_R decreases as the distance d over which the information is transmitted increases. The figure shows for different values of d the maximum value of I_R, I_R^{max}, obtained by optimizing I_R over the input correlation time τ_s (see Fig. 3a), as a function of the temperature T. It is seen that I_R^{max} decreases as d is increased, while the optimal temperature moves closer to the critical temperature, denoted by vertical dashed line. The system size is 10×10.

for large τ_s time is wasted when the output has been correlated to the input yet is waiting for the signal to change again; indeed, the entropy of the input signal $H(\mathbf{S}_L)$ decreases as τ_s is increased, which tends to lower I_R (see Eq. 2). This interplay causes the information transmission rate to have a maximum at an optimal input time scale τ_s^{opt}. The value of τ_s^{opt} decreases with temperature, because at higher temperatures the system can respond more rapidly to changes in the input signal.

Panel b of Fig. 3 shows the maximum value of the information transmission rate I_R at the optimal input correlation time τ_s^{opt}, $I_R^{max} \equiv I_R(\tau_s = \tau_s^{opt})$, as a function of the temperature T. Clearly, there exists an optimal temperature that maximizes I_R^{max}. This is in marked contrast to the maximum value of the instantaneous mutual information, obtained for $\tau_s \to \infty$, $I_{inst,\infty}$, which increases monotonically with decreasing temperature (see Fig. 2a). The optimum in I_R^{max} arises from the trade-off between a faster response at higher temperatures, which allows for a more rapidly varying input that increases $H(\mathbf{S}_L)$, and less thermal noise in the transmitted signal at lower temperatures, which reduces $H(\mathbf{S}_L|\mathbf{X}_L)$ (Eq. 2). The reason I_R^{max} peaks above the critical temperature T_c is that the response time rapidly increases near T_c, thereby decreasing the amount of information that can be sent through the system per unit amount of time.

So far we have kept both the distance d between the input and output spin constant, as well as the system size. We now systematically vary these parameters. Figure 4 shows the maximum information transmission rate I_R^{max}, obtained by optimizing over τ_s (see Fig. 3), as a function of temperature T for different values of d in a 10×10 Ising system. The transmission rate decreases as d is increased, because the correlations between spins become weaker as the distance between them becomes larger. More interestingly, the optimal temperature that maximizes I_R^{max} moves closer to the critical temperature when d is increased. When the distance d between the input and output spin is increased, the correlation length must be increased in order to maintain the correlations between them. This can be achieved by bringing the system closer to the critical point.

Critical effects are stronger in systems of larger size. Close to T_c, the response time of our system increases up to six-fold when the system size is increased from 5×5 to 10×10 spins. This makes it beneficial for information transmission to move the system further away from the critical point when the system size is increased at constant d. Compare the case of $d = 2$ in the 5×5 system in Fig. 3 to that of $d = 2$ in the 10×10 system in Fig. 4: while I_R^{max} decreases because of the larger response time in the larger system, the optimal temperature that maximizes I_R^{max} increases from $T_{opt} \approx 2.5$ to $T_{opt} \approx 2.6$, to mitigate the effect of the larger response time, which tends to reduce the information transmisison rate.

The increase in the correlation length and the correlation time when the system is moved towards the critical temperature have opposite effects on information transmission. Moreover, these effects are more pronounced for larger systems, diverging in the thermodynamic limit. These two observations explain, as described above, why the optimal temperature T_{opt} that maximizes the information transmission rate moves towards the critical temperature T_c when the distance d between the input and output spin is increased in a system of constant size (Fig. 4), yet away from it when the system size is increased at constant d (cf. Figs. 3 and 4). This raises the question how T_{opt} changes as d is scaled *together* with the system size, which, as renormalization group theory indicates, is also the relevant finite-size scaling question for this problem. We have therefore also performed simulations for $d = 6$ and $\mathcal{N} = 15$. The optimal temperature that maximizes information transmission decreases from $T_{opt} \approx 2.53$ for $(d = 2, \mathcal{N} = 5)$, to $T_{opt} \approx 2.44$ for $(d = 4, \mathcal{N} = 10)$, and $T_{opt} \approx 2.38$ for $(d = 6, \mathcal{N} = 15)$ (see Fig. S10 [7]). We emphasise that these system sizes are small, but computing I_R^{max} becomes rapidly harder for larger systems. Nonetheless, our results do hint that T_{opt} moves towards T_c in the thermodynamic limit. In the SI, we discuss this question further; here we also study another quantity, the lagged mutual information [7].

In summary, the information transmission rate I_R is a dynamic quantity that depends on the input correlation time τ_c, the response time τ_r, and the reliability of the response. Increasing τ_c gives the system more time to respond to changes in the input, which tends to enhance the reliability of the response, yet also reduces the fre-

quency of independent input messages, which decreases I_R. Increasing the temperature decreases τ_r, which enhances the rate at which information can be transmitted, yet also increases the thermal noise, which reduces the reliability of the response. The information transmission rate thus exhibits an optimal input correlation time and an optimal temperature, which arise from a fundamental trade-off between the need to maximize the input frequency, the necessity to respond fast to changes in the input, and the need to respond reliably to these changes. The optimal temperature depends on the distance between the in- and output spin and the size of the system. The optimal temperature is close to yet above the critical point, although our results hint it moves towards the critical temperature in the thermodynamic limit.

Our results may help to explain why a number of biological systems appear to be tuned close to a critical point [1–3, 31], and, more generally, will be relevant for understanding information transmission in these systems. For example, our results predict that increasing the interaction strength between the components of a system, be it between proteins in a biochemical network (between, e.g., the receptor and the kinase CheA in the bacterial chemotaxis system [32]), cells in a bacterial community, or birds in a flock, will on the one hand increase the reliability of information transmission, yet on the other hand also increase the response time of the system, giving rise to an optimal strength that maximises the information transmission rate. In fact, these principles likely pertain to systems outside the biological realm. Lastly, many systems, including biological systems, are high dimensional. Since the response time does not depend on the dimensionality of the system while correlations decay faster with distance in higher dimension, we conjecture that in higher dimensional systems the optimal temperature is closer to the critical point.

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[1] A. Cavagna, A. Cimarelli, I. Giardina, G. Parisi, R. Santagati, F. Stefanini, and M. Viale, Proceedings of the National Academy of Sciences of the United States of America **107**, 11865 (2010).

[2] J. W. Larkin, X. Zhai, K. Kikuchi, S. E. Redford, A. Prindle, J. Liu, S. Greenfield, A. M. Walczak, J. Garcia-Ojalvo, A. Mugler, et al., Cell Systems **7**, 137 (2018).

[3] G. Tkacik, T. Mora, O. Marre, D. Amodei, M. J. Berry, and W. Bialek, Proceedings of the National Academy of Sciences **112**, 11508 (2015).

[4] C. E. Shannon, The Bell System Technical Journal **27**, 379 (1948).

[5] J. L. Massey, Proc. 1990 Intl. Symp. on Info. Th. and its Applications (1990).

[6] F. Tostevin and P. R. Ten Wolde, Physical Review Letters **102**, 218101 (2009).

[7] See Supplemental Material at [URL] for an overview of different information theoretic measures, description of the inference procedure, and additional figures, which includes references [8–10].

[8] N. Becker and A. Mugler and P. R. Ten Wolde, Physical Review Letters **85**, 461 (2000).

[9] P. Grassberger, arXiv.org: 0307138v2, (2008).

[10] R.G. Miller, Biometrika **61**(1), (1974).

[11] T. Schreiber, Physical Review Letters **85**, 461 (2000).

[12] T. Mora and W. Bialek, Journal of Statistical Physics **144**, 268 (2011).

[13] H. Matsuda, K. Kudo, R. Nakamura, O. Yamakawa, and T. Murata, International Journal of Theoretical Physics **35**, 839 (1996).

[14] S.-J. Gu, C.-P. Sun, and H.-Q. Lin, J. Phys. A: Math. Theor. **41** (2008).

[15] L. Barnett, J. T. Lizier, M. Harré, A. K. Seth, and T. Bossomaier, Physical Review Letters **111**, 177203 (2013).

[16] V. S. Vijayaraghavan, R. G. James, and J. P. Crutchfield, Entropy **19**, 214 (2017).

[17] D. Marinazzo, L. Angelini, M. Pellicoro, and S. Stramaglia, Physical Review E **99**, 040101 (2019).

[18] J. T. Lizier, F. M. Atay, and J. Jost, Physical Review E **86**, 026110 (2012).

[19] M. Wibral, Frontiers in Neuroinformatics **8**, 1 (2014).

[20] L. Faes, A. Porta, G. Nollo, and M. Javorka, Entropy **19**, 5 (2017).

[21] A. Kolchinsky and B. Corominas-Murtra, J. R. Soc. Interface **17** (2020).

[22] F. Tostevin and P. R. Ten Wolde, Physical Review E **81**, 061917 (2010).

[23] R. Brittain, N. Jones, and T. Ouldridge, J. Stat. Mech. **2017**, 063502 (2017).

[24] S. Das, M. Rao, and G. Iyengar, Phys. Rev. E **95**, 062410 (2017).

[25] G. Malaguti and P. R. ten Wolde, arXiv.org: 1902.09332v1, (2019).

[26] G. Tkačik and A. M. Walczak, Journal of Physics: Condensed Matter **23**, 153102 (2011).

[27] L. Onsager, Physical Review **65**, 117 (1944).

[28] R. J. Glauber, Journal of Mathematical Physics **4**, 294 (1963).

[29] D. Chandler, *Introduction to Modern Statistical Mechanics* (Oxford University Press, 1987), 1st ed.

[30] I. Nemenman, W. Bialek, and R. de Ruyter van Steveninck, Physical Review E **69**, 056111 (2004).

[31] O. Feinerman, I. Pinkoviezky, A. Gelblum, E. Fonio, and N. S. Gov, Nature Communications **6**, 7729, (2018).

[32] D. Bray, M. D. Levin, and C. J. Morton-Firth, Nature **393**, 85 (1998).