Fermi-normal coordinates for the Newtonian approximation of gravity

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In this work, we compute the metric corresponding to a static mass distribution in the general relativistic weak field approximation to quadratic order in Fermi-normal coordinates surrounding a radial geodesic. To construct a geodesic and a convenient tetrad transported along it, we first introduce a general metric, use the Cartan formalism of differential forms, and then specialize the space-time by considering the nearly Newtonian metric. This procedure simplifies the calculations significantly, and the expression for the radial geodesic admits a simple form. We conclude that in quadratic order, the effects of a Schwarzschild gravitational field measured locally by a freely falling observer equals the measured by an observer in similar conditions in the presence of a Newtonian approximation of gravitation.

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I. INTRODUCTION

The local flatness theorem ensures that finding a local Lorentz frame at a point P in a given space-time is always possible. That means that, for a point space-time P, it is always possible to find a coordinate system in which the metric tensor corresponds to the Minkowski tensor with zero Christoffel symbols. Therefore, free-falling observers do not see any effect of gravity in their immediate vicinity [1, 2]. Pioneering ideas about this theorem probably came from the definitions of the Riemann-normal coordinates of a point P in space-time. In such coordinates of the point P, the metric tensor corresponds to the Minkowski metric, with the first derivative vanishing and the second derivatives expressed only in terms of the Riemann tensor. More general coordinate systems exist where a single point admits extension to an arbitrary curve, leading to the metric orthogonal and few nonzero Christoffel symbols. These coordinate systems are essential in numerous physical situations [3].

Under Fermi transport, the fourth member of an orthonormal tetrad remains tangent on a general curve. The other members of the tetrad provide a spatial triad forming a reference frame for an observer whose word line is precisely the general curve in question. On that orthonormal tetrad, it is possible to construct a reference frame at which the metric is orthogonal and with nonzero Christoffel symbols along the observer word line. It is common to refer to this tetrad as the Fermi reference frame or simply as the Fermi coordinates. This frame gives us the correct relativistic generalization of the Newtonian concept of a non-rotating frame.

The generalization of the Fermi reference frame to the case in which the frame rotation is significant remains an exciting area of research. Rotating Fermi reference frames help describe the Thomas precession and the Sagnac effect in general relativity [4–6], which is nowadays employed to clarify the basic ideas for global navigation satellite systems [7].

An exciting and particular case of the Fermi coordinates occurs when the curve in question is a geodesic. In this case, the Fermi transport of the orthonormal tetrad coincides with its parallel transport. If the operating coordinates satisfy the additional condition that the Christoffel symbols vanish along the geodesic, the resulting coordinates are called Fermi-normal coordinates. Manasse and Misner [8, 9] primarily used these coordinates to determine the lowest-order effects of a Schwarzschild gravitational field which can be measured locally by a freely falling observer.

In this work, we ask ourselves how a freely falling observer determines the quadratic order effects of a Newtonian approximation of gravitation. To answer, we construct a geodesic curve and transport a Fermi-normal frame along it. Following the ideas in the early paper by Manasse and Misner [8], Section II presents the geometric construction of the Fermi-normal coordinates. Usually, the calculations in resolving the Fermi-normal coordinates imply a non-easy procedure in determining the orthonormal basis to be parallel transported on the reference geodesic. To simplify the calculations, in Section III, we initially introduce a general metric, express it in an orthonormal tetrad, and use the Cartan formalism of differential forms. In Section IV, we specialize the space-time by considering the nearly Newtonian metric. Section V presents some concluding remarks and areas for further research.

II. GEOMETRIC CONSTRUCTION OF FERMI-NORMAL COORDINATES

In this section, we briefly illustrate the procedure for constructing Fermi-normal coordinates. To do so, we first consider a given space-time with a metric expressed in arbitrary coordinates $x^{\tilde{\alpha}}$ [1] . Then we choose a point O as the origin and assume a time-like geodesic γ, which starts at this point. To describe the geodesic, we let t be a proper time along it so that the point O is given by taking $t = 0$, that is, $O = \gamma(0)$.

We erect an orthonormal basis of vectors $\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}$ and $\mathbf{e_3}$ to fix the coordinates axis at O with tangent $\mathbf{e_0} = \mathbf{e_0}(0)$. Because γ is a geodesic, its tangent at two arbitrary points will relate by parallel displacement along γ. Therefore, the vector $\mathbf{e_0}(t)$ will be tangent to the geodesic in any point $P = \gamma(t)$ on it. Similarly, the other vectors $\mathbf{e_i}(0)$ of the tetrad are parallel displaced to get $\mathbf{e_i}(t)$ at any arbitrary point P. We can assume for simplicity that $\mathbf{e_0}(t)$ is a time-like vector and $\mathbf{e_i}(t)$ are space-like vectors.

We can choose a time-like vector $\mathbf{u}(0) = \mathbf{e}_0(0)$ and a vector $\mathbf{v}(0) = v^i \mathbf{e}_i(0)$ orthonormal to it, i. e. $\mathbf{u}(0) \cdot \mathbf{v}(0) = 0$. In terms of the arbitrary coordinates, these vectors admit the relation

$$u^{\tilde{\mu}} = (\mathbf{e}_\alpha)^{\tilde{\mu}} u^\alpha = (\mathbf{e_0})^{\tilde{\mu}} \ , \tag{1}$$

$$v^{\tilde{\mu}} = (\mathbf{e}_\alpha)^{\tilde{\mu}} v^\alpha = (\mathbf{e_i})^{\tilde{\mu}} v^i \ , \tag{2}$$

[1] Fermi-normal coordinates will labelled as x^α. We shall use the conventions according to which Greek indices $\alpha, \beta, \ldots$, and first Latin indices $a, b, \ldots$, run over $0, \ldots, 3$, while mid-Latin indices as $i, j, \ldots$, run over 1,. . . , 3.

where we chose, conveniently, $v^0 = 0$, and $u^\alpha = \delta^\alpha_0$, as well as

$$\eta_{\mu\nu} = (\mathbf{e}_\mu)^{\tilde{\alpha}} (\mathbf{e}_\nu)^{\tilde{\beta}} g_{\tilde{\alpha}\tilde{\beta}} \ , \tag{3}$$

everywhere on γ. Here $g_{\tilde{\alpha}\tilde{\beta}}$ is the metric in arbitrary coordinates, and $\eta_{\mu\nu} = \mathrm{d}iag(-1,1,1,1)$ is the Minkowski metric.

Let us now consider a space-like geodesic β originating at a point P on γ, at which $t = t_P$. Then, given x^α, we construct the vector $\mathbf{v}(t) = v^i \mathbf{e}_i(t)$ tangent to β at point P. Naturally, we do

$$v^i = \frac{x^i}{s} \ , \quad s^2 \equiv (x^1)^2 + (x^2)^2 + (x^3)^2 \ , \tag{4}$$

to normalize $\mathbf{v}$ properly. We can construct a family of space-like geodesics $\beta(t, v^i)$ orthogonal to γ at P by choosing different v^i. This family of geodesics has proper distance s along β; we set $s = 0$ at P. Then, a unique geodesic intersects γ orthogonally at P, is tangent to the vector $\mathbf{v}$, and passes through a point Q. To specify the point Q in Fermi-normal coordinates, we then write

$$Q = \beta(t, v^i, s) \ , \tag{5}$$

with t denoting proper time at the intersection point, v^i the components of the vector $\mathbf{v}(t)$ at that point, and s the proper distance from P to Q.

Equation (5) determines a point of a geodesic in Fermi-normal coordinates; we can rewrite it in some arbitrary coordinates $x^{\tilde{\alpha}} = (Q)^{\tilde{\alpha}}$ as

$$x^{\tilde{\alpha}} = x^{\tilde{\alpha}}(t, v^i, s) \ . \tag{6}$$

It is not difficult to demonstrate that, given an arbitrary parameter κ, $x^{\tilde{\alpha}}(t, sv^i, \kappa)$ and $x^{\tilde{\alpha}}(t, v^i, s\kappa)$ satisfy the same geodesic equation and initial condition. Hence, the uniqueness theorem for solutions of differential equations gives us

$$x^{\tilde{\alpha}}(t, v^i, s\kappa) = x^{\tilde{\alpha}}(t, sv^i, \kappa) \ , \tag{7}$$

and, as a special case, we find

$$x^{\tilde{\alpha}}(t, v^i, s) = x^{\tilde{\alpha}}(t, sv^i, 1) \equiv x^{\tilde{\alpha}}(x^\alpha) \ , \tag{8}$$

which gives a transformation rule between $x^{\tilde{\alpha}}$ and the Fermi-normal coordinates.

Finally, the tangent vector to the geodesics $\beta(t, v^i, s)$ in the coordinates $x^{\tilde{\alpha}}$ is

$$v^{\tilde{\alpha}} = \frac{\partial x^{\tilde{\alpha}}}{\partial s}\bigg|_{t,v^i} \equiv \frac{dx^{\tilde{\alpha}}}{ds} \ . \tag{9}$$

The notation indicates that we take the derivative with respect to s while keeping t and v^i fixed. Accordingly, this vector calculated in the geodesic γ reveals that

$$v^{\tilde{\alpha}}\Big|_\gamma = \frac{dx^{\tilde{\alpha}}}{ds}\bigg|_\gamma = \frac{\partial x^{\tilde{\alpha}}}{\partial x^i}\bigg|_{s=0} \frac{dx^i}{ds} = \frac{\partial x^{\tilde{\alpha}}}{\partial x^i}\bigg|_{s=0} v^i \ , \tag{10}$$

where we used the relation $x^i = sv^i$ given by Equation (4). Thus, a direct comparison between (2) and (10) gives

$$(\mathbf{e_i})^{\tilde{\alpha}} = \frac{\partial x^{\tilde{\alpha}}}{\partial x^i}\bigg|_\gamma \ . \tag{11}$$

Additionally, from (1) we know that

$$(\mathbf{e_0})^{\tilde{\alpha}} = \frac{\partial x^{\tilde{\alpha}}}{\partial t}\bigg|_\gamma \ . \tag{12}$$

Therefore, from (11) and (12) we conclude that along the geodesic γ,

$$(\mathbf{e}_\mu)^{\tilde{\alpha}} = \frac{\partial x^{\tilde{\alpha}}}{\partial x^\mu}\bigg|_\gamma \ , \tag{13}$$

or, equivalently,

$$\mathbf{e}_\mu(t) = \frac{\partial}{\partial x^\mu}\bigg|_\gamma \ , \tag{14}$$

is the orthonormal tetrad on the geodesic γ in the Fermi-normal coordinates.

Furthermore, we note that the relation between metric components in the Fermi-normal coordinate x^α and t he components of the metric in the arbitrary coordinates $x^{\tilde{\alpha}}$ is given by

$$g_{\mu\nu} = \frac{\partial x^{\tilde{\alpha}}}{\partial x^\mu}\frac{\partial x^{\tilde{\beta}}}{\partial x^\nu}\, g_{\tilde{\alpha}\tilde{\beta}} \ . \tag{15}$$

Thus, by evaluating all the quantities on the geodesic and using (13), we get

$$g_{\mu\nu}\bigg|_\gamma = (\mathbf{e}_\mu)^{\tilde{\alpha}}(\mathbf{e}_\nu)^{\tilde{\beta}}\, g_{\tilde{\alpha}\tilde{\beta}} \ . \tag{16}$$

Hence, a direct comparison between (3) and (16) gives

$$g_{\mu\nu}\bigg|_\gamma = \mathbf{e}_\mu \cdot \mathbf{e}_\nu = \eta_{\mu\nu} \ . \tag{17}$$

The last result shows that, in the Fermi-normal coordinates, the metric is Minkowski everywhere on the geodesic γ. This result is consistent with the local-flatness theorem and the specialization of the Fermi original ideas. So, given a geodesic, it is possible to introduce Fermi-normal coordinates $x^\alpha = (t, x^i)$ near this so that the Christoffel symbols vanish along. Furthermore, to express the metric near the geodesic as [8]

$$g_{00} = -1 \ + \ R_{0l0m}\Big|_\gamma \ x^l x^m \ , \tag{18}$$

$$g_{0i} = 0 \ + \ \frac{2}{3} R_{0lim}\Big|_\gamma \ x^l x^m \ , \tag{19}$$

$$g_{ij} = \delta_{ij} \ + \ \frac{1}{3} R_{iljm}\Big|_\gamma \ x^l x^m \ . \tag{20}$$

Here, t is proper time along the geodesic γ, on which the spatial coordinates x^i vanish, and the curvature tensor is evaluated. The dependence of the metric on t is contained entirely in the curvature tensor components. Naturally, the metric corresponds to the Minkowski metric on the geodesic $x^i = 0$.

III. THE NEARLY NEWTONIAN SPACE-TIME

In the next section, we will calculate the metric corresponding to the nearly Newtonian space-time to quadratic order in Fermi-normal coordinates surrounding a radial geodesic. To do it, in the current section, we first must calculate the curvature tensor of the corresponding space-time in the Fermi-normal coordinates. This procedure requires previously computing the curvature of the given space-time in arbitrary coordinates. We initially introduce a general metric, express it in an orthonormal tetrad, and use the Cartan formalism of differential forms to emphasize the independence from the coordinates. Finally, specialize the space-time by considering the nearly Newtonian metric.

Consider a metric $\mathcal{G}$ in coordinates y^a and a set of differential forms $\vartheta^{\hat{\alpha}} \equiv dy^{\hat{\alpha}}$, such that

$$\mathcal{G} = g_{ab} dy^a \otimes dy^b = \eta_{\hat{\alpha}\hat{\beta}} \vartheta^{\hat{\alpha}} \otimes \vartheta^{\hat{\beta}} \ , \tag{21}$$

with $\eta_{\hat{\alpha}\hat{\beta}} = \mathrm{diag}(-1, 1, 1, 1)$, and $\vartheta^{\hat{\alpha}} = (\mathbf{e}^{\hat{\alpha}})_a dy^a$. The first and second Cartan equations (See [1, 10]) ,

$$d\vartheta^{\hat{\alpha}} = -\omega^{\hat{\alpha}}{}_{\hat{\beta}} \wedge \vartheta^{\hat{\beta}} \ , \tag{22}$$

$$\Omega^{\hat{\alpha}}{}_{\hat{\beta}} = d\omega^{\hat{\alpha}}{}_{\hat{\beta}} + \omega^{\hat{\alpha}}{}_{\hat{\mu}} \wedge \omega^{\hat{\mu}}{}_{\hat{\beta}} = \frac{1}{2} R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\mu}\hat{\nu}} \vartheta^{\hat{\mu}} \wedge \vartheta^{\hat{\nu}} \tag{23}$$

allow us to compute the Riemann curvature tensor components $R_{\hat{\alpha}\hat{\beta}\hat{\mu}\hat{\nu}}$ in the local orthonormal frame $\vartheta^{\hat{\alpha}}$. As an additional point, if we introduce the bivector representation that defines the curvature components $R_{\hat{\alpha}\hat{\beta}\hat{\mu}\hat{\nu}}$ as the components of a 6×6 matrix $\mathbf{R}_{AB}$ according to the procedure discussed in [11, 12], it is possible to calculate the corresponding eigenvalues of the curvature tensor.

To specialize the metric (21) so that it corresponds to the nearly Newtonian space-time, let us consider the following metric in spherical coordinates $y^a = (T, R, \Theta, \Phi)$ (see [1], page 470) in the form

$$\mathcal{G} = -(1+2\phi)dT \otimes dT + (1-2\phi)(dR \otimes dR + R^2 d\Theta \otimes d\Theta + R^2 \sin^2\Theta d\Phi \otimes d\Phi) \ . \tag{24}$$

Here $\phi << 1$ is the Newtonian potential of a static mass distribution in a not rotating local frame of reference. In this work, we limit ourselves to studying spherically symmetric gravitational configurations, assuming that ϕ depends on R. The components of the orthonormal tetrad are then

$$\vartheta^{\hat{0}} = \sqrt{1+2\phi}\, dT \ \equiv d\mathcal{T} \ , \quad \vartheta^{\hat{1}} = \sqrt{1-2\phi}\, dR \ \equiv d\mathcal{R} \ , \tag{25}$$

$$\vartheta^{\hat{2}} = \sqrt{1-2\phi}\, R\, d\Theta \equiv d\theta \ , \quad \vartheta^{\hat{3}} = \sqrt{1-2\phi}\, R \sin\Theta\, d\Phi \equiv d\varphi \ , \tag{26}$$

which, in the first-order approximation, leads to the connection 1-form

$$\omega^{\hat{1}}{}_{\hat{0}} = -\phi_R \vartheta^{\hat{0}} \ , \quad \omega^{\hat{2}}{}_{\hat{3}} = -\frac{1}{R}(1+\phi)\cot\Theta \vartheta^{\hat{3}} \ ,$$

$$\omega^{\hat{1}}{}_{\hat{2}} = -\frac{1}{R}(1+\phi - R\phi_R)\vartheta^{\hat{2}} \ , \quad \omega^{\hat{1}}{}_{\hat{3}} = -\frac{1}{R}(1+\phi - R\phi_R)\vartheta^{\hat{3}} \ . \tag{27}$$

Moreover, the only non-vanishing components of the curvature 2-form can be expressed up to the first order in Φ as

$$\Omega^{\hat{0}}{}_{\hat{1}} = -\phi_{RR}\, \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \ , \quad \Omega^{\hat{0}}{}_{\hat{2}} = -\frac{1}{R}\phi_R\, \vartheta^{\hat{0}} \wedge \vartheta^{\hat{2}}, \quad \Omega^{\hat{0}}{}_{\hat{3}} = -\frac{1}{R}\phi_R\, \vartheta^{\hat{0}} \wedge \vartheta^{\hat{3}} \ , \tag{28}$$

$$\Omega^{\hat{2}}{}_{\hat{3}} = \frac{2}{R}\phi_R \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} \ , \quad \Omega^{\hat{3}}{}_{\hat{1}} = (\phi_{RR} + \frac{1}{R}\phi_R)\vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} \ , \quad \Omega^{\hat{1}}{}_{\hat{2}} = (\phi_{RR} + \frac{1}{R}\phi_R)\vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} \ . \tag{29}$$

It then follows that the only non-zero components of the curvature tensor are

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}} = \mathbf{R}_{\hat{1}\hat{1}} = \phi_{RR} \, , \quad R_{\hat{0}\hat{2}\hat{0}\hat{2}} = \mathbf{R}_{\hat{2}\hat{2}} = R_{\hat{0}\hat{3}\hat{0}\hat{3}} = \mathbf{R}_{\hat{3}\hat{3}} = \frac{1}{R}\phi_R \, , \tag{30}$$

$$R_{\hat{2}\hat{3}\hat{2}\hat{3}} = \mathbf{R}_{\hat{4}\hat{4}} = \frac{2}{R}\phi_R \, , \quad R_{\hat{3}\hat{1}\hat{3}\hat{1}} = \mathbf{R}_{\hat{5}\hat{5}} = R_{\hat{1}\hat{2}\hat{1}\hat{2}} = \mathbf{R}_{\hat{6}\hat{6}} = \phi_{RR} + \frac{1}{R}\phi_R \, . \tag{31}$$

Consequently, the curvature matrix $\mathbf{R_{AB}}$ is diagonal with eigenvalues

$$\lambda_1 = \phi_{RR} \ , \quad \lambda_2 = \lambda_3 = \frac{1}{R}\phi_R \ , \tag{32}$$

$$\lambda_4 = \frac{2}{R}\phi_R \ , \quad \lambda_5 = \lambda_6 = \phi_{RR} + \frac{1}{R}\phi_R \ , \tag{33}$$

which satisfies the relationship

$$\sum_{i=1}^{6} \lambda_i = 3\left(\phi_{RR} + \frac{2}{R}\phi_R\right) = 3\ \nabla^2\phi \ , \tag{34}$$

where ∇^2 denotes the usual Laplace operator in the spherical coordinates. Furthermore, in Newtonian gravity, the eigenvalue problems of the Riemann curvature give us

$$\sum_{i=1}^{6} \lambda_i = \frac{3\kappa}{2}\rho \ \Leftrightarrow\ \nabla^2\phi = \frac{\kappa}{2}\rho \ . \tag{35}$$

Here, ρ represents the mass density of the source corresponding to the gravitational. This last result indicates that the approach for determining the eigenvalues presented in [11] is compatible with the Poisson equation. It then follows that in a region containing no mass, i.e., $\rho = 0$, the gravitational potential of a source of mass M is written as $\phi = -M/R$. Hence, the only non-zero components of the curvature tensor are

$$\begin{aligned} R_{\hat{0}\hat{1}\hat{0}\hat{1}} = \mathbf{R}_{\hat{1}\hat{1}} = -\frac{2M}{R} \ , &\quad R_{\hat{0}\hat{2}\hat{0}\hat{2}} = \mathbf{R}_{\hat{2}\hat{2}} = R_{\hat{0}\hat{3}\hat{0}\hat{3}} = \mathbf{R}_{\hat{3}\hat{3}} = \frac{M}{R} \ , \\ R_{\hat{2}\hat{3}\hat{2}\hat{3}} = \mathbf{R}_{\hat{4}\hat{4}} = \frac{2M}{R} \ , &\quad R_{\hat{3}\hat{1}\hat{3}\hat{1}} = \mathbf{R}_{\hat{5}\hat{5}} = R_{\hat{1}\hat{2}\hat{1}\hat{2}} = \mathbf{R}_{\hat{6}\hat{6}} = -\frac{M}{R} \ . \end{aligned} \tag{36}$$

Consequently, the curvature matrix $\mathbf{R_{AB}}$ is diagonal with eigenvalues

$$\lambda_1 = -\lambda_4 = -\frac{2M}{R} \ , \quad \lambda_2 = \lambda_3 = -\lambda_5 = -\lambda_6 = \frac{M}{R} \ , \tag{37}$$

satisfying the relationship

$$\sum_{i=1}^{6} \lambda_i = 0 \ , \tag{38}$$

which agrees with equation (35) .

IV. NEARLY NEWTONIAN SPACE-TIME TO QUADRATIC ORDER IN FERMI-NORMAL COORDINATES

This section aims to calculate the metric corresponding to the nearly Newtonian space-time to quadratic order in Fermi-normal coordinates surrounding a radial geodesic. We start by considering the nearly Newtonian metric in the orthonormal tetrad previously discussed in Equations (24) and, (25) and (26), i.e.,

$$\mathcal{G} = -d\mathcal{T} \otimes d\mathcal{T} + d\mathcal{R} \otimes d\mathcal{R} + d\theta \otimes d\theta + d\varphi \otimes d\varphi \ . \tag{39}$$

To find the equations of a radial geodesic $\mathcal{T}(t)$, $\mathcal{R}(t)$ with θ and φ constants, one may replace the geodesic equations with two first integrals:

$$-\mathcal{T}'^2 + \mathcal{R}'^2 = -1 \ , \tag{40}$$

$$-\mathcal{T}' = -k \ , \tag{41}$$

the normalization of proper time and a dimensionless energy parameter $(-k)$, respectively. Here, " $'$ " denotes derivate with respect to the parameter t. By combining these equations, we get

$$\mathcal{R}'^2 = k^2 - 1 \ , \tag{42}$$

and, consequently, after renaming the constant $\alpha^2 \equiv (k^2-1)^{-1}$, we receive the geodesic equation given by

$$dt^2 = \alpha^2 d\mathcal{R}^2 \ . \tag{43}$$

Finally, after using the relations given in (25) and (26), we reach

$$dt^2 = \alpha^2(1-2\phi)dR^2 \ . \tag{44}$$

This expression allows us to get a geodesic explicitly after specifying a Newtonian potential. Along with this work, we will consider only central Newtonian potentials; therefore, this result indicates that one may take R in place of proper time t to identify points on this geodesic.

So far, we have studied a nearly Newtonian approach corresponding to a static and spherical gravitational source and have chosen a geodesic in that space-time. Usually, the difficult part of the subsequent calculations in resolving the Fermi coordinates is the determination of the orthonormal basis parallel transported on the reference geodesic. However, the calculations are simple since we have expressed the space-time on the orthonormal basis as given in Equations (25) and (26). On this basis, the geodesic looks like a right line and the transformation between the Fermi-normal coordinates $x^\alpha \equiv (t, x, y, z)$ and the arbitrary coordinates are then $x^{\tilde{\alpha}} \equiv (\mathcal{T}, \mathcal{R}, \theta, \varphi)$

$$\eta_{\alpha\beta} = (\mathbf{e}_\alpha)^{\hat{\mu}} (\mathbf{e}_\beta)^{\hat{\nu}} \eta_{\hat{\mu}\hat{\nu}} \ , \tag{45}$$

with

$$(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \bigg|_\gamma \ . \tag{46}$$

Hence, the non-zero components of this basis in terms of the orthonormal tetrad previously discussed can be derived from

$$\mathbf{e}_0 \equiv \frac{\partial}{\partial t}\bigg|_\gamma = \mathcal{T}' \frac{\partial}{\partial \mathcal{T}} + \mathcal{R}' \frac{\partial}{\partial \mathcal{R}} \ , \tag{47}$$

$$\mathbf{e}_1 \equiv \frac{\partial}{\partial x}\bigg|_\gamma = R' \frac{\partial}{\partial \mathcal{T}} + \mathcal{T}' \frac{\partial}{\partial \mathcal{R}} \ , \tag{48}$$

$$\mathbf{e}_2 \equiv \frac{\partial}{\partial y}\bigg|_\gamma = \frac{\partial}{\partial \theta} \ , \tag{49}$$

$$\mathbf{e}_3 \equiv \frac{\partial}{\partial z}\bigg|_\gamma = \frac{\partial}{\partial \varphi} \ . \tag{50}$$

After determining the orthonormal basis, the next step in expressing the metric in the Fermi-normal coordinates is to project the curvature tensor components given by equation (30) in this basis. For this purpose, we use the transformation formula given by

$$R_{\alpha\beta\mu\nu} = (\mathbf{e}_\alpha)^{\hat{\alpha}} (\mathbf{e}_\beta)^{\hat{\beta}} (\mathbf{e}_\mu)^{\hat{\mu}} (\mathbf{e}_\nu)^{\hat{\nu}} R_{\hat{\alpha}\hat{\beta}\hat{\mu}\hat{\nu}} \ . \tag{51}$$

Hence, the non-zero components of the Riemann curvature tensor are:

$$R_{0101} = R_{\hat{0}\hat{1}\hat{0}\hat{1}} = \phi_{RR} \ , \tag{52}$$

$$R_{0303} = R_{0202} = \mathcal{T}'^2 R_{\hat{0}\hat{2}\hat{0}\hat{2}} + \mathcal{R}'^2 R_{\hat{1}\hat{2}\hat{1}\hat{2}} = k^2 \nabla^2 \phi - (\phi_{RR} + \frac{1}{R}\phi_R) \ , \tag{53}$$

$$R_{2323} = R_{\hat{2}\hat{3}\hat{2}\hat{3}} = \frac{2}{R}\phi_R \ , \tag{54}$$

$$R_{1313} = R_{1212} = \mathcal{R}'^2 R_{\hat{0}\hat{2}\hat{0}\hat{2}} + \mathcal{T}'^2 R_{\hat{1}\hat{2}\hat{1}\hat{2}} = k^2 \nabla^2 \phi - \frac{1}{R}\phi_R \ . \tag{55}$$

Finally, after introducing the last components of the Riemann curvature tensor in equations (18), (19) and (20), we get the Fermi-normal metric corresponding to the Newtonian approximation,

$$
\begin{aligned}
\mathcal{G} = &- \left[1 - \phi_{RR}x^2 + \left(\phi_{RR} + \frac{1}{R}\phi_R - k^2\nabla^2\phi\right)(y^2+z^2)\right] dt\otimes dt \\
&- \frac{1}{3R}\phi_R\big[(ydx - xdy)\otimes(ydx-xdy) + (xdz - zdx)\otimes(xdz-zdx) \\
&- 2(ydz - zdy)\otimes(ydz-zdy)\big] + dx\otimes dx + dy\otimes dy + dz\otimes dz \\
&+ \frac{k^2}{3R}\nabla^2\phi\big[(ydx - xdy)\otimes(ydx-xdy) + (xdz - zdx)\otimes(xdz-zdx)\big]
\end{aligned} \tag{56}
$$

By introducing spherical coordinates r, θ, φ related to the rectangular coordinates x, y, z by the standard formulas and taking the x direction as the polar axis, i.e.,

$$
x = r\cos\theta\ , \quad y = r\sin\theta\cos\varphi\ , \quad z = r\sin\theta\sin\varphi\ ,
$$

the Fermi-normal metric corresponding to the nearly Newtonian space-time reads

$$
\begin{aligned}
\mathcal{G} = &- \Big[1 - \big(2\phi_{RR} + \frac{1}{R}\phi_R - k^2\nabla^2\phi\big)r^2\cos^2\theta + \big(\phi_{RR} + \frac{1}{R}\phi_R - k^2\nabla^2\phi\big)r^2\Big]dt\otimes dt \\
&+ dr\otimes dr + \Big[1 - \frac{r^2}{3R}\phi_R + \frac{k^2}{3}r^2\nabla^2\phi\Big]r^2 d\theta\otimes d\theta + \Big[1 - \frac{r^2}{3R}\phi_R(3\cos^2\theta - 1) \\
&+ \frac{r^2}{3R^2}\phi_R + \frac{k^2}{3}r^2\cos^2\phi\nabla^2\phi\Big]r^2\sin^2\theta d\varphi\otimes d\varphi\ .
\end{aligned} \tag{57}
$$

To illustrate the Fermi-normal metric corresponding to the nearly Newtonian space-time, let us consider a spherically symmetric solution. Then, the exterior field should correspond to that of a sphere described by a solution of the Laplace equation with

$$
\rho = 0\ , \quad \phi = -\frac{M}{R}\ , \tag{58}
$$

where M is a constant. It is then straightforward to calculate the Fermi-normal metric, which turns out to be

$$
\begin{aligned}
\mathcal{G} = &- [1 + \mu(3\cos\theta - 1)]\, dt\otimes dt + dr\otimes dr + \left(1 - \frac{\mu}{3}\right) r^2 d\theta\otimes d\theta \\
&+ \left[1 - \mu(3\cos\theta - 1) + \frac{\mu}{3}\right] r^2\sin^2\theta d\varphi\otimes d\varphi\ , \qquad \mu \equiv \frac{Mr^2}{R^3}\ .
\end{aligned} \tag{59}
$$

Consistently with this metric, the geodesic (44) takes the particular form

$$
dt^2 = \alpha^2\left(1 + \frac{2M}{R}\right)dR^2\ . \tag{60}
$$

We finalize this section by calculating the shape of a sphere whose center of mass coincides with the coordinates of some point of the central geodesic. To do so, we define a sphere Σ

as the surface formed by all points a proper fixed distance r measured orthogonally from the such point. For the coordinates of Equation (59) this is the surface $t=$ constant, $r=$ constant, whose metric is,

$$\mathcal{G}\Big|_{\Sigma} = \left(1-\frac{\mu}{3}\right) r^2 d\theta\otimes d\theta + \left[1-\mu(3\cos\theta-1)+\frac{\mu}{3}\right] r^2\sin^2\theta d\varphi\otimes d\varphi \ . \tag{61}$$

The length of the great circle $\varphi=$ constant over the poles of this sphere is

$$L_{\text{Poles}} = \int_0^{2\pi}\left(1-\frac{\mu}{3}\right)^{1/2} r d\theta \approx 2\pi r\left(1-\frac{\mu}{6}\right) \ . \tag{62}$$

The circumference of the equator, $\theta=\pi/2$, is

$$L_{\text{Equator}} = \int_0^{2\pi}\left[1-\frac{\mu}{3}(3\cos\theta-1)+\frac{\mu}{3}\right]^{1/2} r\sin\theta d\varphi \mid_{\theta=\pi/2} \ \approx 2\pi r\left(1+\frac{\mu}{3}\right) \ . \tag{63}$$

Consequently, the distortion of the shape η of the sphere is given by

$$\eta \equiv \left|\frac{L_{\text{Poles}}-L_{\text{Equator}}}{L_{\text{Poles}}+L_{\text{Equator}}}\right| = \frac{\mu}{4} = \frac{Mr^2}{4R^3} \ . \tag{64}$$

This result indicates that a sphere r=constant is a surface shaped like a prolate spheroid.

V. CONCLUDING REMARKS

In this work, we find the metric corresponding to a static gravitational source for a weak-field approximation to quadratic order in fermi-normal coordinates surrounding a radial geodesic. We conclude that in the lowest order, the effects of a Schwarzschild gravitational field measured locally by a freely falling observer equals the measured by an observer in similar conditions in the presence of a Newtonian approximation of gravitation.

To simplify the calculations in this work, we introduced a general metric, expressed it in an orthonormal tetrad, and used the Cartan formalism of differential forms. Incidentally, we cast out the eigenvalues of the Riemann tensor corresponding to the nearly Newtonian gravitation. It is possible to directly compare these eigenvalues and the corresponding to the nearly Newtonian gravitation in its quadratic approximation. This comparison reveals that the metric corresponds to a non-static space-time in its quadratic approximation.

The results obtained in this work indicate the possibility of studying the nearly Newtonian gravitation in its quadratic approximation in other theories and scenarios, for instance, the problem of two gravitational sources, the interaction between gravitational sources in Einstein-Maxwell gravity, etcetera. We will analyze these problems in other works. We expect to consider this problem in future works.

Appendix A: Fermi normal coordinates for the Schwarzschild solution

In this section, we compute the metric corresponding to the Schwarzschild solution to quadratic order in Fermi-normal coordinates surrounding a radial geodesic. The Schwarzschild metric in a not rotating local frame of reference in spherical coordinates $y^a = (T, R, \Theta, \Phi)$ admits the form

$$\mathcal{G} = -X dT \otimes dT + \frac{1}{X} dR \otimes dR + R^2 \left(d\Theta \otimes d\Theta + \sin^2 \Theta d\Phi \otimes d\Phi \right) , \; X \equiv 1 - \frac{2M}{R} \; . \quad \text{(A1)}$$

To use the Cartan formalism, we introduce the orthonormal tetrad

$$\vartheta^{\hat{0}} = X^{1/2} dT \; \equiv d\mathcal{T}, \quad \vartheta^{\hat{1}} = X^{-1/2} dR \; \equiv d\mathcal{R}, \quad \text{(A2)}$$

$$\vartheta^{\hat{2}} = R\, d\Theta \equiv d\theta \; , \quad \vartheta^{\hat{3}} = R \sin\Theta \, d\Phi \equiv d\varphi \; , \quad \text{(A3)}$$

which leads to the metric in the rectangular form

$$\mathcal{G} = -d\mathcal{T} \otimes d\mathcal{T} + d\mathcal{R} \otimes d\mathcal{R} + d\theta \otimes d\theta + d\varphi \otimes d\varphi \; , \quad \text{(A4)}$$

and the connection 1-form

$$\omega^{\hat{1}}{}_{\hat{0}} = -\frac{M}{R^3} X^{-1/2} \vartheta^{\hat{0}}, \quad \omega^{\hat{2}}{}_{\hat{3}} = -\frac{1}{R} \cot\Theta \vartheta^{\hat{3}}, \quad \text{(A5)}$$

$$\omega^{\hat{1}}{}_{\hat{2}} = -\frac{1}{R} X^{1/2} \vartheta^{\hat{2}}, \quad \omega^{\hat{1}}{}_{\hat{3}} = -\frac{1}{R} X^{1/2} \vartheta^{\hat{3}} \; . \quad \text{(A6)}$$

It then follows that the only non-zero components of the curvature tensor are

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}} = \mathbf{R}_{\hat{1}\hat{1}} = -\frac{2M}{R} , \quad R_{\hat{0}\hat{2}\hat{0}\hat{2}} = \mathbf{R}_{\hat{2}\hat{2}} = R_{\hat{0}\hat{3}\hat{0}\hat{3}} = \mathbf{R}_{\hat{3}\hat{3}} = \frac{M}{R} , \quad \text{(A7)}$$

$$R_{\hat{2}\hat{3}\hat{2}\hat{3}} = \mathbf{R}_{\hat{4}\hat{4}} = \frac{2M}{R} , \quad R_{\hat{3}\hat{1}\hat{3}\hat{1}} = \mathbf{R}_{\hat{5}\hat{5}} = R_{\hat{1}\hat{2}\hat{1}\hat{2}} = \mathbf{R}_{\hat{6}\hat{6}} = -\frac{M}{R} . \quad \text{(A8)}$$

Consequently, the curvature matrix $\mathbf{R_{AB}}$ is diagonal with eigenvalues

$$\lambda_1 = -\frac{2M}{R} \; , \quad \lambda_2 = \lambda_3 = \frac{M}{R} \; , \quad \text{(A9)}$$

$$\lambda_4 = \frac{2M}{R} \; , \quad \lambda_5 = \lambda_6 = -\frac{M}{R} \; , \quad \text{(A10)}$$

which satisfies the relationship

$$\sum_{i=1}^{6} \lambda_i = 0 \; , \quad \text{(A11)}$$

in consistency with the fact that the Schwarzschild solution is a vacuum solution of the Einstein equations.

To find the equations of a radial geodesic $\mathcal{T}(t)$, $\mathcal{R}(t)$ with θ and φ constants, one may replace the geodesic equations with two first integrals:

$$-\mathcal{T}'^2 + \mathcal{R}'^2 = -1 \ , \tag{A12}$$

$$-\mathcal{T}' = -k \ , \tag{A13}$$

the normalization of proper time and a dimensionless energy parameter $(-k)$, respectively. Here, " $'$ " denotes derivate with respect to the parameter t. After combining these equations, we get

$$\mathcal{R}'^2 = k^2 - 1 \ , \tag{A14}$$

and, consequently, after renaming the constant $\alpha^2 = (k^2 - 1)^{-1}$ we receive the geodesic equation given by

$$dt^2 = \alpha^2 d\mathcal{R}^2 \ . \tag{A15}$$

Finally, after using the relations given in (A2) and (A3), we reach

$$dt^2 = \frac{\alpha^2}{1 - 2M/R} dR^2 \ . \tag{A16}$$

This expression allows us to get a radial geodesic in the Schwarzschild background. As we can see, because of its functional dependence, one may take R in place of proper time t to identify points on this geodesic. As we can see, similarly to the early Newtonian approach, the calculations are simple since we have expressed the space-time on the orthonormal basis as given in Equations (A2) and (A3) . On this basis, the geodesic looks like a right line and the transformation between the Fermi-normal coordinates $x^\alpha \equiv (t, x, y, z)$ and the arbitrary coordinates are then $x^{\tilde{\alpha}} \equiv (\mathcal{T}, \mathcal{R}, \theta, \varphi)$

$$\eta_{\alpha\beta} = (\mathbf{e}_\alpha)^{\hat{\mu}} (\mathbf{e}_\beta)^{\hat{\nu}} \eta_{\hat{\mu}\hat{\nu}} \tag{A17}$$

with

$$(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \bigg|_\gamma \ . \tag{A18}$$

Hence, the non-zero components of this basis in terms of the orthonormal tetrad previously discussed can be derived from

$$\mathbf{e}_0 \equiv \left.\frac{\partial}{\partial t}\right|_\gamma = \mathcal{T}'\frac{\partial}{\partial \mathcal{T}} + \mathcal{R}'\frac{\partial}{\partial \mathcal{R}} \ , \tag{A19}$$

$$\mathbf{e}_1 \equiv \left.\frac{\partial}{\partial x}\right|_\gamma = R'\frac{\partial}{\partial \mathcal{T}} + \mathcal{T}'\frac{\partial}{\partial \mathcal{R}} \ , \tag{A20}$$

$$\mathbf{e}_2 \equiv \left.\frac{\partial}{\partial y}\right|_\gamma = \frac{\partial}{\partial \theta} \ , \tag{A21}$$

$$\mathbf{e}_3 \equiv \left.\frac{\partial}{\partial z}\right|_\gamma = \frac{\partial}{\partial \varphi} \ . \tag{A22}$$

After determining the orthonormal basis, the next step in expressing the metric in the Fermi-normal coordinates is to project the curvature tensor components given by equation (30) in this basis. For this purpose, we use the transformation formula given by

$$R_{\alpha\beta\mu\nu} = (\mathbf{e}_\alpha)^{\hat{\alpha}}(\mathbf{e}_\beta)^{\hat{\beta}}(\mathbf{e}_\mu)^{\hat{\mu}}(\mathbf{e}_\nu)^{\hat{\nu}} R_{\hat{\alpha}\hat{\beta}\hat{\mu}\hat{\nu}} \ . \tag{A23}$$

Hence, the non-zero components of the Riemann curvature tensor are:

$$R_{0101} = R_{\hat{0}\hat{1}\hat{0}\hat{1}} = -\frac{2M}{R^3} \ , \tag{A24}$$

$$R_{0303} = R_{0202} = \mathcal{T}'^2 R_{\hat{0}\hat{2}\hat{0}\hat{2}} + \mathcal{R}'^2 R_{\hat{1}\hat{2}\hat{1}\hat{2}} = \frac{M}{R^3} \ , \tag{A25}$$

$$R_{2323} = R_{\hat{2}\hat{3}\hat{2}\hat{3}} = \frac{M}{R^3} \ , \tag{A26}$$

$$R_{1313} = R_{1212} = \mathcal{R}'^2 R_{\hat{0}\hat{2}\hat{0}\hat{2}} + \mathcal{T}'^2 R_{\hat{1}\hat{2}\hat{1}\hat{2}} = -\frac{M}{R^3} \ . \tag{A27}$$

Finally, after introducing the last components of the Riemann curvature tensor in equations (18), (19) and (20), we get the Fermi-normal metric corresponding to the Schwarzschild space-time,

$$\begin{aligned}\mathcal{G} = &-\left[1 - \frac{M}{R^3}(y^2 + z^2 - 2x^2)\right] dt \otimes dt \\ &+ \frac{2M}{3R^3}\left[xydx \otimes dy + xzdx \otimes dz - 2yzdy \otimes dz\right] \\ &+ \left[1 - \frac{M}{3R^3}(y^2 + z^2)\right] dx \otimes dx \\ &+ \left[1 - \frac{M}{3R^3}(x^2 - 2z^2)\right] dy \otimes dy \\ &+ \left[1 - \frac{M}{3R^3}(x^2 - 2y^2)\right] dx \otimes dx \ .\end{aligned} \tag{A28}$$

By introducing spherical coordinates r, θ, φ related to the rectangular coordinates x, y, z by the standard formulas and taking the x direction as the polar axis, i.e. ,

$$x = r\cos\theta\ , \quad y = r\sin\theta\cos\varphi\ , \quad z = r\sin\theta\sin\varphi\ ,$$

the Fermi-normal metric corresponding to the Schwarzschild space-time reads

$$\begin{aligned}\mathcal{G} = &-\left[1 + \mu(3\cos\theta - 1)\right] dt \otimes dt + dr \otimes dr + \left(1 - \frac{\mu}{3}\right) r^2 d\theta \otimes d\theta \\ &+ \left[1 - \mu(3\cos\theta - 1) + \frac{\mu}{3}\right] r^2 \sin^2\theta d\varphi \otimes d\varphi\ , \qquad \mu \equiv \frac{Mr^2}{R^3}\ .\end{aligned} \tag{A29}$$

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