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UNCERTAINTY PRINCIPLE AND THE QUANTUM FLUCTUATIONS OF THE LIGHT CONES IN THE STATIC SPACE-TIMES

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The application of the uncertainty relation to position and velocity of a source point represented by a general asymptotically flat static metric leads to fluctuations of the metric and the light cone structure. The fluctations in the coordinate photon velocity are given by the formula $\Delta V_{\mathrm{ph}}(\mathrm{hor}) = 2\sqrt{\frac{\hbar k}{m}}$ where k is the surface gravity and m is the mass of the source point.

1. Introduction

During the last decade there has been remarkable progress in the construction of a unified field theory of strong, electromagnetic and weak interactions. But putting gravity into quantum framework is still an open problem. The components of the metric tensor cannot be measured with arbitrary accuracy if the measuring apparatus obeys laws of quantum mechanics. The natural way of incorporating this stochastic nature of the metric components is to quantize gravity.

Recently Padmanabhan, Seshadri and Singh[1] have considered this question in an interesting way. They have taken the metric due to a point mass as a toy model and have applied uncertainty relation to the position and velocity of point mass (i.e. the source of gravitation). This uncertainty in the position of source induces uncertainty in the metric tensor and that in turn induces fluctuations in the light cone structure. They have proved that fluctuations in the coordinate velocity of photon at the event horizon $\Delta V_{\mathrm{ph}}(\mathrm{hor})$ is directly proportional to the Hawking temperature,

$$\Delta V_{\mathrm{ph}}(\mathrm{hor}) = \frac{8\pi L_P}{\hbar} K_B T_H, \tag{1.1}$$

where T_H is the Hawking temperature, L_P is the planck length and K_B is the Boltzmann constant.

The quantity $\Delta V_{\mathrm{ph}}(\mathrm{hor})$ is too small for the stellar mass blackholes. Its significance lies in the fact that fluctuations in the light cones are related to the Hawking temperature. One is interested in such types of relations because sometimes they signal coming break through.

In this paper we examine the above question in a general static asymptotically flat space-time and obtain the formula

$$\Delta V_{\text{ph}}(\text{hor}) = 2\sqrt{\frac{\hbar k}{m}}, \tag{1.2}$$

where k is the surface gravity and m is the mass of the hole. Since $k \propto T_H$,

$$\Delta V_{\text{ph}}(\text{hor}) \propto \sqrt{T_H/m}, \tag{1.3}$$

where T_H is the Hawking temperature. For the Schwarzschild metric it gives rather the misleading impression that $\Delta V_{\text{ph}}(\text{hor})$ is directly proportional to the Hawking temperature. However in the case of a charged blackhole

$$\Delta V_{\text{ph}}(\text{hor}) = \frac{4\pi L_P K_B}{\hbar}\left(\frac{1+\sqrt{1-Q^2/m^2}}{(1-Q^2/m^2)^{1/4}}\right)T_H, \tag{1.4}$$

where Q is the charge on the hole. Clearly $\Delta V_{\text{ph}}(\text{hor})$ is not directly related to the temperature of the hole but it also depends upon the parameter Q/m of the hole. In general this would be the case. However in the Schwarzschild case there is only one parameter m to which all the physical properties are related. This results into a deceptively simple proportionality $\Delta V_{\text{ph}}(\text{hor}) \propto T_H$. But this is a special case and does not hold in general.

In Sec. 2 we calculate the coordinate velocity of photon for a Lorentz boosted static asymptotically flat metric. In Sec. 3 we compute the lower bound on the product $\Delta\gamma_{11}\Delta\dot{\gamma}^{11}$ where γ_{11} are physical components of metric tensor. In Sec. 4 we compute fluctuations in the light cone structure and finally get the general relation for the variation in photon velocity at the horizon (see Eq. 4.7).

2. Lorentz Transformations for Static Asymptotically Flat Metric

Since we are interested in exploring source related uncertainties and wish to apply the uncertainty relation to the position and velocity of the source, we need to find the metric due to a static source moving with coordinate velocity V. This can simply be obtained by boosting the general static asymptotically flat metric in the usual manner. Consider

$$ds^2 = A^2(d\bar{t})^2 - B^2(d\bar{x})^2 - C^2(d\bar{y})^2 - D^2(d\bar{z})^2, \tag{2.1}$$

where A, B, C, D are functions of position coordinate only. The event horizon of the space-time is given by $A = 0$.

Consider an observer moving with respect to the source with coordinate velocity V along the $\bar{x}$-axis. Let the coordinates in his frame be (t, x, y, z). The Lorentz transformations relating the two coordinate systems are

$$\bar{x} - x_0 = \gamma(x - x_0 + Vt), \quad \bar{t} = \gamma\big(t + V(x - x_0)\big),$$
$$\bar{y} = y, \quad \bar{z} = z, \quad \gamma = (1 - V^2)^{-1/2}, \quad G = c = 1. \tag{2.2}$$

The initial conditions are as follows:

$$\bar{x} = x = x_0 \quad \text{at} \quad \bar{t} = t = 0, \quad x_0 > (\text{larger of the roots of } A = 0).$$

Note that the source of gravitation is moving with coordinate velocity $-V$ relative to the observer. The transformed metric is

$$ds^2 = \gamma^2 A^2 (dt - V\,dx)^2 - B^2\gamma^2(dx - V\,dt)^2 - C^2\,dy^2 - D^2\,dz^2, \tag{2.3}$$

where A, B, C, D are functions of $\bar{x}$, $\bar{y}$, $\bar{z}$.

We are interested in computing light cone fluctuations (see Sec. 4). Hence we should compute the coordinate velocity of photon. For the sake of simplicity let us restrict ourselves to the $x - t$ plane. Consider a ray of light moving in the $x - t$ plane. The metric in the $x - t$ plane is

$$ds^2 = \gamma^2 A^2 (dt - V\,dx)^2 - B^2\gamma^2(dx - V\,dt)^2. \tag{2.4}$$

In what follows we shall only be concerned with the limit of small velocities. Therefore we expand (2.4) to lowest order in V, obtaining,

$$ds^2 = A^2\,dt^2 - 2V(A^2 - B^2)\,dt\,dx - B^2\,dx^2. \tag{2.5}$$

The coordinate velocity of photon V_{ph} is given by the condition $ds^2 = 0$. Hence for the metric $ds^2 = g_{00}(dt)^2 + g_{11}(dx)^2 + 2g_{01}\,dx\,dt = 0$,

$$(V_{\text{ph}})_\pm = \left(\frac{dx}{dt}\right)_\pm = \frac{-g_{01}}{g_{11}} \pm \left[\left(\frac{g_{01}}{g_{11}}\right)^2 - \frac{g_{00}}{g_{11}}\right]^{1/2}. \tag{2.6}$$

Here the two solutions refer to the two light rays—one travelling along the positive x-axis and the other along negative x-axis. For the metric given by (2.5)

$$(V_{\text{ph}})_\pm = \frac{-V(A^2 - B^2)}{B^2} \pm \left(\frac{V^2(A^2 - B^2)}{B^4} + \frac{A^2}{B^2}\right)^{1/2}. \tag{2.7}$$

Since $V^2 \approx 0$

$$(V_{\text{ph}})_\pm = \frac{-V(A^2 - B^2)}{B^2} \pm \frac{A}{B}. \tag{2.8}$$

3. Fluctuations in the Metric

It is important to investigate the following question. Will uncertainty in position and velocity always induce uncertainty in the metric coefficients? Because if the position and velocity appearing in the metric (Eq. 2.5) are actually expectation values, then uncertainties in the source parameters will not lead to uncertainties in the metric coefficients. But replacing the source parameters by their expectation values is equivalent to interpreting Einstein's equations as

$$G_{ik} = 8\pi G\langle\psi|T_{ik}|\psi\rangle. \tag{3.1}$$

Here ψ is matter wave function. But (3.1) is not a correct description of gravity.[2,3] If it were so, then one has to assume that the matter wave function never collapses. Because if $\psi = \sum_i C_i(x^\alpha)\psi_i$,

$$8\pi\langle\psi|T^{\mu\nu}|\psi\rangle_{;\nu} = 8\pi\sum_{i,j}(C_i^* C_j)_{;\nu}\langle\psi_i|T^{\mu\nu}|\psi_j\rangle. \tag{3.2}$$

Since in the Heisenberg picture ψ_i's are constants but $C_i(x^\alpha)$ change during a measurement. But obviously from (3.2) $8\pi\langle\psi|T^{\mu\nu}|\psi\rangle_{;\nu} \neq 0$ whereas $G^{\mu\nu}_{;\nu} = 0$, i.e., the right-hand side of (3.1) will not be conserved whereas the left-hand side of (3.1) will be conserved. So in order to retain (3.1) as a semi-classical theory of gravity, we must assume that the matter wave function ψ never collapses. But experimental results show that (3.1) is inconsistent with nature if the wave function does not collapse.[2]

So one cannot replace source parameters by their expectation values. Hence it follows that uncertainties in position and velocity of source will induce uncertainty in the metric tensor. In this paper we will investigate the quantum uncertainties in the metric of asymptotically flat space-time given by the equation (2.5).

The metric components g_{ij}'s do not represent true dynamical degrees of freedom and can be gauged away by coordinate transformations. So it is more appropriate to calculate the uncertainty relations between *the physical components of the metric tensor.* For that let us consider *induced 3-metric* $\gamma_{\alpha\beta}$ which is given by the relation[4] $\gamma_{\alpha\beta} = -g_{\alpha\beta} + \dfrac{g_{0\alpha}g_{0\beta}}{g_{00}}$. Calculating these quantities for (2.5), $\gamma_{11} = B^2$ and $\gamma^{11} = -B^{-2}$. Note that though B is not a function of $\bar{t}$ but it is a function of t. Hence we get $\dot{\gamma}^{11} = \dfrac{2}{B^3}\dot{B}$, $\Delta\dot{\gamma}^{11} = \dfrac{2}{B^3}B'\Delta V$, $\Delta\gamma_{11} = 2BB'\Delta r$ (where a dot represents differentiation with respect to t and a prime represents differentiation with respect to r). So

$$\Delta\gamma_{11}\Delta\dot{\gamma}^{11} \geq 4\left(\left\langle\frac{B'}{B}\right\rangle\right)^2 \Delta r\Delta V, \tag{3.3}$$

which will imply

$$\Delta\gamma_{11}\Delta\dot{\gamma}^{11} \geq 4\left\langle\frac{B'}{B}\right\rangle^2 \frac{\hbar}{m}. \tag{3.4}$$

For the Schwarzschild metric[1] $B = \left(1 + \dfrac{m}{2r}\right)^2$ and the above relation reads as

$$\Delta\gamma_{11}\Delta\dot{\gamma}^{11} \geq \left(\frac{2m}{\langle r\rangle^2}\right)^2 \frac{1}{\left(1 + \left\langle\dfrac{m}{2r}\right\rangle\right)^2} \frac{\hbar}{m}. \tag{3.5}$$

Note these fluctuations are everywhere finite and have maximum value at the horizon

$$\Delta\gamma_{11}\Delta\dot{\gamma}^{11}(\text{hor}) = \frac{16\hbar}{m^3}. \tag{3.6}$$

For the Reissner-Nordstrom solution $B = \left(\left(1 + \dfrac{m}{2r}\right)^2 - \left(\dfrac{Q}{2r}\right)^2\right)$ and so we have

$$\Delta\gamma_{11}\Delta\dot{\gamma}^{11} \geq \frac{4}{\langle r\rangle^4} \frac{\left\{m\left(1 + \left\langle\dfrac{m}{2r}\right\rangle\right) - \left\langle\dfrac{Q^2}{2r}\right\rangle\right\}^2}{\left[\left(1 + \left\langle\dfrac{m}{2r}\right\rangle\right)^2 - \left\langle\dfrac{Q}{2r}\right\rangle^2\right]^2} \frac{\hbar}{m}, \tag{3.7}$$

which at the horizon takes the form

$$\Delta\gamma_{11}\Delta\dot{\gamma}^{11}(\text{hor}) \geq \frac{16\hbar}{m(m^2 - Q^2)}. \tag{3.8}$$

In this case, it should however be noted that $\Delta\gamma_{11}\Delta\dot{\gamma}^{11}$ is not maximum at the horizon.

4. Fluctuations in the Light Cone

Quantum fluctuations in the metric tensor will result in fluctuations of the light cone structure of space-time. We shall compute the fluctuations just outside the horizon. We have computed in Sec. 2 the trajectory of a null ray travelling along the x-axis. A light cone is defined by the slope of this ray. The coordinate velocity of a light ray going in the positive x-direction is

$$V_{\text{ph}} = \frac{dx}{dt} = \frac{-V(A^2 - B^2)}{B^2} + \frac{A}{B}. \tag{4.1}$$

Let the uncertainty in position and velocity of the source about mean velocity $\langle V\rangle = 0$, be Δx and ΔV respectively, then at $t = 0$ we write

$$\Delta V_{\text{ph}} = \frac{-(A^2 - B^2)}{B^2}\Delta V + \frac{\Delta A}{B} - \frac{A\Delta B}{B^2}\,, \tag{4.2}$$

where $\Delta A = \dfrac{A'\hbar}{m\Delta V}$, $\Delta B = \dfrac{B'\hbar}{m\Delta V}$. Here $B' = \dfrac{\partial B}{\partial x}$, $A' = \dfrac{\partial A}{\partial x}$. Then

$$\Delta V_{\text{ph}} = \left(\frac{B^2 - A^2}{B^2}\right)\Delta V + \frac{\hbar A'}{mB\Delta V} - \frac{A}{B^2}\frac{B'\hbar}{\Delta V}\,. \tag{4.3}$$

The condition for minimum variation is $\dfrac{\partial \Delta V_{\text{ph}}}{\partial \Delta V} = 0$, so minimum ΔV is given by the following relation

$$\Delta V = \sqrt{\left(\frac{A'}{m} - \frac{AB'}{B}\right)\frac{B\hbar}{(B^2 - A^2)}}\,. \tag{4.4}$$

Putting this back into (4.3) and evaluating at horizon (i.e. $A = 0$)

$$\Delta V_{\text{ph}}(\text{hor}) = 2\sqrt{\frac{A'\hbar}{mB}}\,. \tag{4.5}$$

Now we compute the surface gravity k for the metric (2.5) which turns out to be

$$k = A'/B\,. \tag{4.6}$$

Then we get the general result

$$\Delta V_{\text{ph}}(\text{hor}) = 2\sqrt{\frac{\hbar k}{m}}\,, \tag{4.7}$$

which agrees with Padmanabhan *et al.*[1] for the Schwarzschild solution. It should be emphasized that theirs is a very special result because of the only one parameter (degenerate) specification of the hole. As pointed out above it is not so in the case of a charged hole. It can be easily verified by writing $k = \dfrac{\sqrt{m^2 - Q^2}}{(m + \sqrt{m^2 - Q^2})^2}$ for the Reissner-Nordstrom metric. Since $T_H = \dfrac{k\hbar}{k_B 2\pi}$,

$$\Delta V_{\text{ph}}(\text{hor}) = 2\sqrt{\frac{\hbar k}{m}} = \frac{4\pi L_P K_B}{\hbar} \times \left(\frac{1 + \sqrt{1 - \alpha^2}}{(1 - \alpha^2)^{1/4}}\right) \times T_H\,. \tag{4.8}$$

Here $\alpha = Q/m$. Evidently in the case of a charged blackhole ΔV_{ph}(hor) explicitly depends upon Q/m, i.e., on a parameter of the hole.

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