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Sources and Gravitons

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Gravitational theory is reconstructed from the source description of gravitons. As indirect evidence for this starting point, the theoretical basis for the four tests of Einsteinian gravitational theory is produced by elementary arguments. Following the electromagnetic example, the source description is recast as a numerical field theory characterized by an action principle. After recognizing that the physically restricted theory possesses invariance with respect to infinitesimal coordinate transformations, it is generalized to exhibit invariance under arbitrary coordinate transformations, which supplies the primitive interactions for multigraviton processes. There is a discussion of the necessary dependence of graviton sources upon the gravitational field, and a simple model is constructed. The qualitative structure of the modification that multiparticle exchange introduces in the graviton-propagation function is exhibited, with the corresponding modification in the Newtonian potential. There are some speculative remarks about Mach's principle and the accompanying interpretation of the gravitational constant. The paper concludes by pointing out empirical scaling laws that interconnect the cosmos, the laboratory, and the atom.

INTRODUCTION

THE present open-ended situation in high-energy physics has produced a new attitude toward particle theory,[1] one that is highly pragmatic and nonspeculative in its foundations, and yet does not bar the road to a conceivable deeper level of understanding. The method is not mere phenomenology. Its constructive principles, based on space-time uniformity and causality, have enabled the practical results of quantum electrodynamics to be regained, freed from unnecessary physical hypotheses and attendant mathematical difficulties, as an induction from the basic physical fact that accelerated charges radiate. Armed with this new general approach, one naturally seeks to encompass the portion of physical experience that is concerned with gravitational phenomena. But here again the special nature of gravitation is underlined, for the relevant particle—the graviton—is experimentally unknown. The systematic method that we seek to apply must find its justification in the more accessible domain of quasistatic macroscopic phenomena. It is as though the zero mass and unit spin of the photon could be verified only by reference to the laws of Coulomb and Ampère. This is the path we follow with the graviton, a massless particle of spin 2.

THE PHOTON REVIEWED

The theory of sources defines particles by reference to the collision processes that create or annihilate them. The abstraction of these realistic events, in the form of source functions, supplies the mathematical framework upon which the dynamical theory is erected. The experimenter's principal tool, a beam of physically noninteracting particles, is described completely, if ideally, by representing both the creation and the eventual detection of the particles. The quantum-mechanical characterization of the complete process, which begins and ends with the vacuum state, has the typical form

$$\langle 0_+|0_-\rangle^S=\exp(iW)\,,\qquad W=\frac{1}{2}\int(dx)(dx')S(x)G(x-x')S(x')\,. \tag{1}$$

The exponential structure is derived from the physical independence of sufficiently remote individual acts of emission, propagation, and absorption, which are embodied in the form of W. For the specific example of the photon, with real source function $J^\mu(x)$, the structure of the relativistically invariant W is (units in which $\hbar=c=1$ are used)

$$W=\frac{1}{2}\int(dx)(dx')J^\mu(x)D_+(x-x')J_\mu(x')\,, \tag{2}$$

where

$$D_+(x-x')=D_+(x'-x) \tag{3}$$

and

$$x^0>x^{0\prime}:\quad D_+(x-x')=i\int d\omega_k e^{ik(x-x')}\,,\qquad d\omega_k=\frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2k^0}\,,\quad k^0=|\mathbf{k}|\,. \tag{4}$$

In addition, it is necessary that

$$\partial_\mu J^\mu(x)=0 \tag{5}$$

if one is not to violate the probability interpretation of the formalism, as expressed by

$$|\langle 0_+|0_-\rangle^J|^2\leq 1\,. \tag{6}$$

The source restriction is a specific consequence of the masslessness of the photon. The probability interpreta-

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[1] The initial papers are J. Schwinger, Phys. Rev. 152, 1219 (1966); 158, 1391 (1967). A general survey will be found in the 1967 Brandeis Summer Institute Lectures (to be published).

tion of the formalism is also involved in defining $D_+(x-x')$ everywhere in the simplest possible manner, as given in (4), rather than by introducing an independent definition in the spacelike regions that are not explored (macroscopically) by propagating photons. This occurs since the vacuum amplitude has two independent functions, which must be consistent with the quantum laws. One of these is its direct significance in giving the vacuum persistence probability, which involves the structure of $D_+(x-x')$ throughout space-time; the other is its implicit reference to individual photon emission and absorption acts, being the origin of the plane-wave factors from which $D_+(x-x')$ is constructed.

The exhibition of W in a space-time form that makes no overt reference to the initial causal arrangement of emission and absorption sources leads inevitably to extrapolations that connect photon propagation with other physical phenomena. Thus, consider sources that are incapable of emitting or absorbing photons, owing to a very slow time variation. The structure of W then reduces to an account of the instantaneous coupling of sources, involving the spatial function

$$\int_{-\infty}^{\infty} dx^0 D_+(x-x') = \int \frac{(d\mathbf{k})}{(2\pi)^3} \frac{1}{|\mathbf{k}|^2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = 1/(4\pi|\mathbf{x}-\mathbf{x}'|). \quad (7)$$

The vacuum amplitude is now of the form

$$\langle 0_+|0_-\rangle^J = \exp\left[-i\int dx^0 E(x^0)\right], \quad (8)$$

where, ignoring all time variation ($\boldsymbol{\nabla}\cdot\mathbf{J}=0$),

$$E = \frac{1}{2}\int (d\mathbf{x})(d\mathbf{x}') \frac{J^0(\mathbf{x})J^0(\mathbf{x}') - \mathbf{J}(\mathbf{x})\cdot\mathbf{J}(\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}. \quad (9)$$

The exponential phase factor identifies the energy of a quasistatic source distribution in the familiar form that embodies Coulomb's and Ampère's laws of charge and current interactions. This is the simplest example of a characteristic pattern of extending the source concept to wider physical circumstances in order to include more phenomena within the same dynamical framework.

The construction of W as an "action at a distance" coupling of sources can be replaced by an equivalent local-action description, which is the introduction of the field concept. We shall now *derive* Maxwell's equations. The insertion of a weak test source $\delta J^\mu(x)$, obeying $\partial_\mu \delta J^\mu(x)=0$, alters W by

$$\delta W = \int (dx)\delta J^\mu(x) A_\mu(x), \quad (10)$$

where $A_\mu(x)$ measures the effect of the preexisting sources. It is not uniquely determined since δJ^μ is restricted by the divergence condition. The corresponding arbitrariness in $A_\mu(x)$ has the form $\partial_\mu\lambda(x)$, a gauge transformation. Thus,

$$A_\mu(x) = \int (dx') D_+(x-x') J_\mu(x') + \partial_\mu\lambda(x), \quad (11)$$

where the relation

$$\partial_\mu A^\mu(x) = \partial^2\lambda(x) \quad (12)$$

can be deduced from $\partial_\mu J^\mu = 0$. We now recognize that the explicitly defined propagation function $D_+(x-x')$ obeys the differential equation

$$-\partial^2 D_+(x-x') = \delta(x-x'), \quad (13)$$

where the four-dimensional δ function expresses the discontinuity term

$$2\delta(x^0-x^{0\prime})\partial_0\left(i\int d\omega_k e^{ik(x-x')}\right) = \delta(x-x'). \quad (14)$$

Accordingly, we have

$$-\partial^2 A_\mu(x) = J_\mu(x) - \partial_\mu\partial_\nu A^\nu(x), \quad (15)$$

or

$$\partial_\nu F^{\mu\nu}(x) = J^\mu(x) \quad (16)$$

with

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \quad (17)$$

as promised.

The fundamental significance of W is recognized by writing it in various equivalent versions:

$$W = \frac{1}{2}\int (dx) J^\mu A_\mu = \frac{1}{4}\int (dx) F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) = \frac{1}{4}\int (dx) F^{\mu\nu}F_{\mu\nu} = \int (dx)[J^\mu A_\mu + \mathfrak{L}(A,F)], \quad (18)$$

where

$$\mathfrak{L}(A,F) = -\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}. \quad (19)$$

In the latter form, the response of W to independent infinitesimal changes of J^μ, of A_μ and $F_{\mu\nu}$, is given by

$$\delta W = \int (dx)[\delta J^\mu A_\mu + J^\mu \delta A_\mu + \delta\mathfrak{L}(A,F)]. \quad (20)$$

It follows from the initial definition of A_μ, Eq. (10), that the terms involving δA_μ and $\delta F_{\mu\nu}$ must vanish. In short,

$$W = \int (dx)[J^\mu A_\mu + \mathfrak{L}(A,F)] \quad (21)$$

is an action, from which the field differential equations [Eqs. (16) and (17)] are recovered through the stationary requirement with respect to independent field variations.

GRAVITONS

These considerations are transferred to a spin-2 massless particle by introducing a real source

$$T^{\mu\nu}(x)=T^{\nu\mu}(x)\,, \tag{22}$$

and choosing

$$\begin{aligned}W&=\frac{1}{2}\int(dx)(dx')[T^{\mu\nu}(x)D_+(x-x')T_{\mu\nu}(x')\\&\qquad-\tfrac{1}{2}T(x)D_+(x-x')T(x')]\\&=\frac{1}{2}\int(dx)(dx')T^{\mu\nu}(x)[g_{\mu\lambda}g_{\nu\kappa}-\tfrac{1}{2}g_{\mu\nu}g_{\lambda\kappa}]\\&\qquad\times D_+(x-x')T^{\lambda\kappa}(x')\,,\end{aligned} \tag{23}$$

where

$$\partial_\mu T^{\mu\nu}(x)=0\,. \tag{24}$$

The crucial test of these assertions is given by

$$|\langle 0_+|0_-\rangle^T|^2=\exp[-\mathrm{Re}(1/i)2W]\,, \tag{25}$$

for $\mathrm{Re}(1/i)2W$ must be positive and composed of contributions from the sources that are effective in emitting the various particles. These should be two in number for a given momentum, referring to the only spin (helicity) states accessible to a massless particle. The relevant structure is

$$\begin{aligned}\mathrm{Re}(1/i)2W&=\int d\omega_k T^{\mu\nu}(k)^*[g_{\mu\lambda}g_{\nu\kappa}-\tfrac{1}{2}g_{\mu\nu}g_{\lambda\kappa}]T^{\lambda\kappa}(k)\,,\\T^{\mu\nu}(k)&=\int(dx)e^{-ikx}T^{\mu\nu}(x)\,,\end{aligned} \tag{26}$$

where

$$k_\mu T^{\mu\nu}(k)=0\,. \tag{27}$$

Relative to a specific graviton of momentum k^μ, let the third spatial axis point along the direction of motion ($k_3=k^0$) so that

$$T^{3\nu}(k)=T^{0\nu}(k)\,. \tag{28}$$

Then all reference to the time axis and the third spatial axis cancel in the bilinear source structure (26), leaving

$$\mathrm{Re}(1/i)2W=\int d\omega_k\sum_{a,b=1,2}|T_{ab}'(k)|^2\geq 0\,, \tag{29}$$

where

$$T_{ab}'(k)=T_{ab}(k)-\tfrac{1}{2}\delta_{ab}\sum_{c=1,2}T_{cc}(k) \tag{30}$$

obeys

$$\sum_{a=1,2}T_{aa}'(k)=0 \tag{31}$$

and has only two independent components. All other aspects of the discussion follow previously charted courses.

NEWTON AND EINSTEIN

It is now our intention to show that the theory of (hypothetical) gravitons is a correct starting point for a dynamical theory of gravitation by developing the analogs of the Coulomb-Ampère laws of electromagnetism, for comparison with the Newtonian mass-interaction law and the four experimental tests of the Einsteinian modification of gravitational theory. We do this by applying the gravitational-source picture to macroscopic bodies or electromagnetic waves under the hypothesis that the source function $T^{\mu\nu}(x)$ is identical, apart from a universal constant, with the stress tensor of the object as the only mechanical quantity that obeys the necessary restriction:

$$\partial_\mu T^{\mu\nu}(x)=0\,.$$

It is convenient to examine the interaction energy of two systems, one of which is effectively represented by the single stress tensor component $T^{00}(\mathbf{x})$, measuring energy or mass density. With $t^{\mu\nu}(\mathbf{x})$ referring to the other system, the interaction energy is

$$E=-\kappa\int(d\mathbf{x})(d\mathbf{x}')T^{00}(\mathbf{x})\times\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}[t^{00}(\mathbf{x}')+\tfrac{1}{2}t(\mathbf{x}')]\,, \tag{32}$$

where $\kappa^{1/2}>0$ converts mechanical measure to gravitational-source units. We first suppose that only t^{00} is significant for the second system. Then ($t=-t^{00}$)

$$E=-\frac{\kappa}{8\pi}\int(d\mathbf{x})(d\mathbf{x}')T^{00}(\mathbf{x})|\mathbf{x}-\mathbf{x}'|^{-1}t^{00}(\mathbf{x}')\,, \tag{33}$$

which is the Newtonian potential energy of attracting mass distributions. This supplies the identification ($c=1$)

$$\kappa=8\pi G\,. \tag{34}$$

We now consider four examples in which T^{00} describes the spherically symmetric mass distribution of an astronomical body with total mass M, while $t^{\mu\nu}(\mathbf{x})$ is sufficiently concentrated (there is an exception, to be discussed later) that one can introduce an effective distance R, as in

$$E=-\frac{GM}{R}\int(d\mathbf{x})(2t^{00}+t)\,. \tag{35}$$

The gravitational red shift is obtained in a familiar way by remarking that the total energy of a slowly moving atom with mass m becomes $m-(GM/R)m$ in the neighborhood of the massive body. The energy released in an atomic transition is thus modified by the red-shift factor: $1-(GM/R)$. Next, let $t_{\mu\nu}$ refer to a light beam, for which $t=0$. The interaction energy is now doubled relative to the Newtonian value, as is the deflection angle of light passing near the body, which is the Einsteinian result. The same effect alters the speed of light by the factor $1-2(GM/R)$, since the total energy of a photon is $|\mathbf{k}|[1-2(GM/R)]$. This gives a simple theory of the additional time delays observed in radar echoes from the inner planets. Thus, at superior conjunction we get[2]

$$\Delta t = 4GM \ln(4x_e x_p/d^2), \tag{36}$$

where d is the distance of closest approach to the sun of the beam and x_e, x_p are distances measured from the closest approach point to the earth and planet, respectively. Finally, we consider the perihelion precession of planetary orbits.

It is convenient to recall that the small relativistic modification of kinetic energy given by

$$\begin{aligned}(\mathbf{p}^2+m^2)^{1/2}-m &\simeq T-T^2/2m\\ &=T-(\epsilon-V)^2/2m,\end{aligned} \tag{37}$$

in which the symbols T, V, ϵ have standard nonrelativistic meanings, already produces a perihelion precession in the same sense and of $\frac{1}{6}$ the magnitude implied by the Einstein theory.[3] The significant perturbation term is $-V^2/2m$. We now approximate the interaction energy ($k=1, 2, 3$)

$$E=-\frac{GM}{R}\int(d\mathbf{x})(t^{00}+t_{kk}), \tag{38}$$

using a simple mechanical model ($t_{\mu\nu}=\sigma p_\mu p_\nu$) for which

$$t_{kk}\simeq(\mathbf{p}/m)^2 t^{00} \tag{39}$$

and

$$\int(d\mathbf{x})t_{kk}\simeq\left(\frac{2T}{m}\right)\int(d\mathbf{x})t^{00}. \tag{40}$$

The total energy of the planet is effectively reckoned as

$$\int(d\mathbf{x})t^{00}\simeq m+T+\tfrac{1}{2}V, \tag{41}$$

where the potential energy has been divided equally between the interacting partners. We return to the latter point shortly. The apparent interaction energy is therefore

$$V[1+(2T/m)][1+(T/m)+\tfrac{1}{2}(V/m)]-V^2/2m, \tag{42}$$

[2] Compare I. I. Shapiro, Phys. Rev. Letters **13**, 789 (1964).

[3] See for example, P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1942), p. 218.

where the last term records the significant kinetic-energy contribution. The net perturbation is $V(3T/m)$, of which the term relevant to perihelion precession is $-3V^2/m$, in agreement with the Einstein value.

The only delicate point in the above argument refers to the gravitational effect of gravitational energy. This is the phenomenon that causes the perihelion precession to be described as a higher-order or nonlinear effect, in contrast with the other tests of general relativity. In fact, only a small part of the total precession effect is attributed to this cause ($V^2/2m$), and the choice of $\frac{1}{2}V$ as the effective potential-energy contribution can be justified on purely Newtonian grounds by referring to the spatial distribution of the interaction energy. For two point masses separated by $\mathbf{R}$, this energy density is given by

$$-(G/4\pi)\nabla(M/|\mathbf{x}|)\cdot\nabla(m/|\mathbf{x}-\mathbf{R}|). \tag{43}$$

The interaction energy with the body of mass M is

$$\begin{aligned}(G^2/4\pi)M^2m\int(d\mathbf{x})|\mathbf{x}|^{-1}\nabla|\mathbf{x}|^{-1}\cdot\nabla|\mathbf{x}-\mathbf{R}|^{-1}\\ =\tfrac{1}{2}(G^2M^2m/R^2)=V^2/2m,\end{aligned} \tag{44}$$

which immediately follows from the observation that

$$|\mathbf{x}|^{-1}\nabla|\mathbf{x}|^{-1}=\tfrac{1}{2}\nabla|\mathbf{x}|^{-2}, \tag{45}$$

and a partial integration.

It is not the point of these remarks to dispense with a complete dynamical theory of gravitational interactions, but only to recognize that the experimentally successful predictions of the Newton-Einstein gravitational theory are easily reproduced by elementary arguments from the gravitational-source viewpoint, thus justifying the latter in the absence of actual graviton observations.

GRAVITATIONAL FIELD

Following the electromagnetic model, we define a field $h_{\mu\nu}(x)$ by reference to a test source $\delta T^{\mu\nu}(x)$ obeying $\partial_\mu\delta T^{\mu\nu}(x)=0$:

$$\delta W=\int(dx)\delta T^{\mu\nu}(x)h_{\mu\nu}(x). \tag{46}$$

This field is undetermined to the extent expressed by $\partial_\mu\xi_\nu(x)+\partial_\nu\xi_\mu(x)$. Thus

$$\begin{aligned}h_{\mu\nu}(x)=\int(dx')D_+(x-x')\kappa[T_{\mu\nu}(x')-\tfrac{1}{2}g_{\mu\nu}T(x')]\\ +\partial_\mu\xi_\nu(x)+\partial_\nu\xi_\mu(x),\end{aligned} \tag{47}$$

where the mechanical measure of $T^{\mu\nu}$ is used, and we recall that

$$T(x)=g_{\mu\nu}T^{\mu\nu}(x). \tag{48}$$

With a similar definition for $h(x)$, we deduce that

$$h(x)=-\int(dx')D_+(x-x')\kappa T(x')+2\partial_\mu\xi^\mu(x)\,, \quad (49)$$

and therefore

$$h_{\mu\nu}(x)-\tfrac{1}{2}g_{\mu\nu}h(x)=\int(dx')D_+(x-x')\kappa T_{\mu\nu}(x')$$
$$+\partial_\mu\xi_\nu(x)+\partial_\nu\xi_\mu(x)-g_{\mu\nu}\partial_\lambda\xi^\lambda(x)\,. \quad (50)$$

The source restriction $\partial_\mu T^{\mu\nu}=0$ implies the relation

$$\partial_\mu(h^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}h)=\partial^2\xi^\nu\,, \quad (51)$$

with the aid of which we return to Eq. (47) and derive

$$-\partial^2 h_{\mu\nu}+\partial_\mu\partial^\lambda h_{\lambda\nu}+\partial_\nu\partial^\lambda h_{\lambda\mu}-\partial_\mu\partial_\nu h=\kappa(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T)\,. \quad (52)$$

The definition

$$\Gamma_{\mu\nu\lambda}=\partial_\mu h_{\nu\lambda}+\partial_\nu h_{\mu\lambda}-\partial_\lambda h_{\mu\nu}\,, \quad (53)$$

with its consequence

$$\Gamma_{\mu\lambda}{}^\lambda=\partial_\mu h\,, \quad (54)$$

then enables the gravitational field equations to be presented as

$$\partial^\lambda\Gamma_{\mu\nu\lambda}-\tfrac{1}{2}(\partial_\mu\Gamma_{\nu\lambda}{}^\lambda+\partial_\nu\Gamma_{\mu\lambda}{}^\lambda)=\kappa(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T)\,. \quad (55)$$

As the counterpart of the gauge invariance of the Maxwell equations, these field equations are invariant under the gravitational gauge transformation

$$h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu\,, \quad (56)$$

although

$$\Gamma_{\mu\nu\lambda}\to\Gamma_{\mu\nu\lambda}+2\partial_\mu\partial_\nu\xi_\lambda\,. \quad (57)$$

To construct the action form of W, we recognize that

$$W=\frac{1}{2}\int(dx)T^{\mu\nu}h_{\mu\nu}=\frac{1}{2}\int(dx)(T^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}T)(h_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}h)$$
$$=\frac{1}{2\kappa}\int(dx)(h^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}h)(\partial^\lambda\Gamma_{\mu\nu\lambda}-\partial_\mu\Gamma_{\nu\lambda}{}^\lambda)$$
$$=-\frac{1}{2\kappa}\int(dx)[\Gamma^{\lambda\mu\nu}\Gamma_{\mu\nu\lambda}-\Gamma^\mu{}_{\mu\lambda}\Gamma^{\lambda\nu}{}_\nu]. \quad (58)$$

The desired result is

$$W=\int(dx)[T^{\mu\nu}h_{\mu\nu}+\mathcal{L}(h,\Gamma)]\,, \quad (59)$$

with

$$\kappa\mathcal{L}=-(h^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}h)(\partial^\lambda\Gamma_{\mu\nu\lambda}-\partial_\mu\Gamma_{\nu\lambda}{}^\lambda)$$
$$-\tfrac{1}{2}(\Gamma^{\lambda\mu\nu}\Gamma_{\mu\nu\lambda}-\Gamma^\mu{}_{\mu\lambda}\Gamma^{\lambda\nu}{}_\nu)\,. \quad (60)$$

The response of the latter function to the gauge transformation given in Eqs. (56) and (57) is

$$\delta(\kappa\mathcal{L})=-\partial^\lambda[(\partial^\mu\xi^\nu+\partial^\nu\xi^\mu-g^{\mu\nu}\partial_\kappa\xi^\kappa)\Gamma_{\mu\nu\lambda}]$$
$$+\partial_\mu[(\partial^\mu\xi^\nu+\partial^\nu\xi^\mu-g^{\mu\nu}\partial_\kappa\xi^\kappa)\Gamma_\nu{}^\lambda{}_\lambda]\,, \quad (61)$$

which ensures the gauge invariance of W.

GENERAL COORDINATE INVARIANCE

As a first step toward the construction of a dynamical theory of gravitons in interaction with matter, comprising all other particles, we recall the definition of matter stress tensors through the effect of an infinitesimal arbitrary coordinate deformation,

$$\delta W_m=\int(dx)T_m{}^{\mu\nu}\partial_\mu\delta x_\nu\,. \quad (62)$$

Invariance of the matter action W_m under local Lorentz transformations demands the symmetry of the stress tensor, and the stationary action principle implies the local conservation law

$$\partial_\mu T_m{}^{\mu\nu}(x)=0\,. \quad (63)$$

In this first stage of developing a realistic dynamics, we express the idea that $T_m{}^{\mu\nu}$ is the gravitational source of matter by writing

$$W=W_m+\int(dx)[(T_s{}^{\mu\nu}+T_m{}^{\mu\nu})h_{\mu\nu}+\mathcal{L}(h,\Gamma)]\,, \quad (64)$$

where $T_s{}^{\mu\nu}$ now designates idealized sources. The response of W, to infinitesimal coordinate transformations in the matter term, and infinitesimal gauge transformations in the gravitational field part, is

$$\delta W=\int(dx)[T_m{}^{\mu\nu}\partial_\mu\delta x_\nu+(T_s{}^{\mu\nu}+T_m{}^{\mu\nu})2\partial_\mu\xi_\nu]. \quad (65)$$

This shows the existence of a higher symmetry uniting matter and gravitation in which gravitational gauge transformations are linked to arbitrary coordinate transformations:

$$\delta\xi_\nu=-\tfrac{1}{2}\delta x_\nu\,, \quad (66)$$

as expressed by

$$\delta h_{\mu\nu}=-\tfrac{1}{2}(\partial_\mu\delta x_\nu+\partial_\nu\delta x_\mu) \quad (67)$$

and

$$\delta\Gamma_{\mu\nu\lambda}=-\partial_\mu\partial_\nu\delta x_\lambda\,. \quad (68)$$

The next stage is reached by remarking that the gravitational contribution to the action also responds to coordinate transformations, demanding the introduction of a gravitational stress tensor. Our discussion of perihelion precession is a simple application of that fact. We are now encountering higher-order effects, in the sense of a weak-field treatment. One might proceed in this way to construct a self-consistent theory.[4] But that theory could not be other than the self-consistent one produced by extending invariance under infinitesimal arbitrary coordinate transformations, in the weak-field situation, to invariance under finite arbitrary coordinate transformations, for strong fields. Judged by effectiveness in reaching the desired goal, the latter method is to be preferred, even had it not been sanctioned by historical precedent.

We define

$$g_{\mu\nu}(x)=g^{(0)}{}_{\mu\nu}+2h_{\mu\nu}(x)\,, \tag{69}$$

where $g^{(0)}{}_{\mu\nu}$ now refers to Minkowski metric, and seek to remove the weak-field limitation suggested by

$$|h_{\mu\nu}(x)|\ll 1\,. \tag{70}$$

A consistent generalization of the coordinate-induced gauge transformation of $h_{\mu\nu}$, Eq. (67), is given by

$$\delta' g_{\mu\nu}=-g_{\mu\lambda}\partial_\nu\delta x^\lambda-g_{\lambda\nu}\partial_\mu\delta x^\lambda\,, \tag{71}$$

which is the tensor transformation property

$$\bar{g}_{\mu\nu}(\bar{x})=g_{\lambda\kappa}(x)(\partial x^\lambda/\partial\bar{x}^\mu)(\partial x^\kappa/\partial\bar{x}^\nu)\,, \tag{72}$$

applied to the infinitesimal transformation

$$\bar{x}^\mu=x^\mu+\delta x^\mu(x)\,,\quad \bar{g}_{\mu\nu}(\bar{x})=g_{\mu\nu}(x)+\delta' g_{\mu\nu}(x)\,. \tag{73}$$

The expansions

$$\begin{aligned} g^{\mu\nu}(x)&=g^{(0)\mu\nu}-2h^{\mu\nu}(x)+\cdots\,,\\ -g(x)&=-\det g_{\mu\nu}(x)=1+2h(x)+\cdots\,, \end{aligned} \tag{74}$$

when combined as

$$(-g)^{1/2}g^{\mu\nu}=g^{(0)\mu\nu}-2(h^{\mu\nu}-\tfrac{1}{2}g^{(0)\mu\nu}h)+\cdots\,, \tag{75}$$

help to suggest the required generalization of the gravitational Lagrange function (60). It is

$$\mathfrak{L}_g(g,\Gamma)=(1/2\kappa)(-g)^{1/2}g^{\mu\nu}R_{\mu\nu}\,, \tag{76}$$

with

$$\begin{aligned} R_{\mu\nu}=\partial_\lambda\Gamma_{\mu\nu}{}^\lambda-\tfrac{1}{2}(\partial_\mu\Gamma_{\lambda\nu}{}^\lambda+\partial_\nu\Gamma_{\mu\lambda}{}^\lambda)\\ +\Gamma_{\lambda\kappa}{}^\kappa\Gamma_{\mu\nu}{}^\lambda-\Gamma_{\mu\kappa}{}^\lambda\Gamma_{\nu\lambda}{}^\kappa\,. \end{aligned} \tag{77}$$

It is familiar that the action principle based on this Lagrange function implies that the $\Gamma_{\mu\nu}{}^\lambda$ are the Christoffel symbols computed from $g_{\mu\nu}$, which is a consistent generalization of $\Gamma_{\mu\nu\lambda}$. We need hardly remark that general coordinate invariance has been attained, with $g^{\mu\nu}R_{\mu\nu}$ and $(-g)^{1/2}(dx)$ individually possessing that property.

The term in δW that introduces the total stress tensor $T^{\mu\nu}$ is easily generalized:

$$\int(dx)T^{\mu\nu}\delta h_{\mu\nu}\rightarrow\int(dx)(-g)^{1/2}\tfrac{1}{2}T^{\mu\nu}\delta g_{\mu\nu}\,. \tag{78}$$

This is the response to the variation $\delta g_{\mu\nu}$ of the source and matter components of the invariant action. Consideration of the coordinate-induced variation

$$\begin{aligned} \delta g_{\mu\nu}(x)=\bar{g}_{\mu\nu}(x)-g_{\mu\nu}(x)&=-g_{\mu\lambda}\partial_\nu\delta x^\lambda-g_{\lambda\nu}\partial_\mu\delta x^\lambda-\delta x^\lambda\partial_\lambda g^{\mu\nu}\\ &=-\partial_\mu(g_{\lambda\nu}\delta x^\lambda)-\partial_\nu(g_{\mu\lambda}\delta x^\lambda)+2\Gamma_{\mu\nu}{}^\lambda g_{\lambda\kappa}\delta x^\kappa \end{aligned} \tag{79}$$

implies

$$\partial_\mu[(-g)^{1/2}T^\mu{}_\nu]=\tfrac{1}{2}(-g)^{1/2}T^{\alpha\beta}\partial_\nu g_{\alpha\beta} \tag{80}$$

or

$$\partial_\mu[(-g)^{1/2}T^{\mu\nu}]=-(-g)^{1/2}\Gamma_{\alpha\beta}{}^\nu T^{\alpha\beta}\,. \tag{81}$$

It is well known that these relations are satisfied identically by the left-hand member of the gravitational field equations that are deduced from the action

$$W=W_s+\int(dx)[\mathfrak{L}_m+\mathfrak{L}_g]\,, \tag{82}$$

namely,

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=\kappa T_{\mu\nu}\,. \tag{83}$$

[The electromagnetic analog is $\partial_\mu(\partial_\nu F^{\mu\nu})\equiv 0$.] A simple example of a matter Lagrange function is that of a zero-spin field appropriately generalized to ensure coordinate invariance:

$$\mathfrak{L}_m=(-g)^{1/2}[-\varphi^\mu\partial_\mu\varphi+\tfrac{1}{2}\varphi^\mu g_{\mu\nu}\varphi^\nu-\tfrac{1}{2}m^2\varphi^2]\,. \tag{84}$$

The implied matter stress tensor

$$T_m{}^{\mu\nu}=\varphi^\mu\varphi^\nu+g^{\mu\nu}(-g)^{-1/2}\mathfrak{L}_m \tag{85}$$

obeys the necessary divergence equations, (80) or (81), in consequence of the matter field equations. All this demands that the sources $T_s{}^{\mu\nu}$ separately obey

$$\partial_\mu[(-g)^{1/2}T_s{}^{\mu\nu}]=-(-g)^{1/2}\Gamma_{\alpha\beta}{}^\nu T_s{}^{\mu\nu}\,. \tag{86}$$

Thus, gravitational sources cannot be assigned independently of the gravitational field that they help to generate. In contrast with photons, which do not possess electric charge, gravitons do carry energy and momentum, which demands a corresponding reaction in the gravitational source strengths.

MULTIPARTICLE EXCHANGE

The problem is thus posed of exhibiting a model of a graviton source that incorporates the necessary dependence upon the gravitational field.[5] But, first, let us place this question in proper perspective. The source concept has already served its principal function in providing the model for matter sources in the dynamical theory that has been constructed. The idealized sources that appear in this formulation are used to represent individual gravitons, injected into or emitted by the physical system of interest, for which purpose the initial characterization of graviton sources is quite adequate, at least for terrestrial experiments. A more elaborate

[4] This route has been taken by Gupta, Feynman, and others.

[5] This question has already received some discussion from D. Boulware and S. Deser, Nuovo Cimento **30**, 1009 (1963). We must dissent from their pessimistic conclusion concerning the utility of graviton sources.

treatment is required only when one proceeds to the dynamical level in which single-particle propagation is extended to equivalent multiparticle propagation. In electrodynamics, this is the stage that introduces so-called vacuum polarization effects, electromagnetic form factors, and related phenomena. The graviton-source problem is the quest for an analytic expression of the fact that physical mechanisms which create a graviton are inherently capable of generating several gravitons in the same act. This is analogous to the photon emission that inherently accompanies the creation of a charged particle. We shall be content to discuss two-graviton emission, which refers to a linear dependence of $T_s^{\mu\nu}$ upon the gravitational field.

We seek an expression for $(-g)^{1/2}T_s^{\mu\nu}$, a function that obeys the differential equation (86), and is a functional of $g_{\mu\nu}$, responding as a tensor density to the coordinate induced transformation (79):

$$\begin{aligned}\delta[(-g)^{1/2}T_s^{\mu\nu}]\\ =\partial_\lambda[\delta x^\mu(-g)^{1/2}T_s^{\lambda\nu}+\delta x^\nu(-g)^{1/2}T_s^{\mu\lambda}-\delta x^\lambda(-g)^{1/2}T_s^{\mu\nu}]\\ +\delta x^\mu\Gamma_{\alpha\beta}{}^\nu(-g)^{1/2}T_s^{\alpha\beta}+\delta x^\nu\Gamma_{\alpha\beta}{}^\mu(-g)^{1/2}T_s^{\alpha\beta}.\end{aligned} \quad (87)$$

The latter terms involving $\Gamma_{\alpha\beta}{}^\mu$ can be ignored for our limited objectives. We first devise a functional of the gravitational field that responds to coordinate transformations in the manner

$$\delta X_\lambda(g)=\delta x_\lambda. \quad (88)$$

Let

$$f^\mu(x-x')=-f^\mu(x'-x) \quad (89)$$

be a class of functions that obey

$$\partial_\mu f^\mu(x-x')=\delta(x-x'). \quad (90)$$

We also define

$$f^\mu f^\nu(x-x')=f^\nu f^\mu(x-x')=\int(d\xi)f^\mu(x-\xi)f^\nu(\xi-x'), \quad (91)$$

which obeys

$$\partial_\mu f^\mu f^\nu(x-x')=f^\nu(x-x') \quad (92)$$

and

$$\partial_\mu\partial_\nu f^\mu f^\nu(x-x')=\delta(x-x'). \quad (93)$$

Using the kind of shorthand notation already illustrated in $f^\mu f^\nu$, we exhibit X_λ for our purposes as

$$\begin{aligned}X_\lambda&=-f^\mu f^\nu\Gamma_{\mu\nu\lambda}\\ &=-(f^\mu\delta_\lambda{}^\nu+f^\nu\delta_\lambda{}^\mu-f^\mu f^\nu\partial_\lambda)h_{\mu\nu},\end{aligned} \quad (94)$$

which refers to the coordinate-induced transformations (67) and (68). An initial step in the construction is, therefore,

$$(-g)^{1/2}T_s^{\mu\nu}=T_0^{\mu\nu}+\partial_\lambda(X^\mu T_0^{\lambda\nu}+X^\nu T_0^{\mu\lambda}-X^\lambda T_0^{\mu\nu})+\tau^{\mu\nu}, \quad (95)$$

where $T_0^{\mu\nu}$ is independent of the gravitational field and obeys

$$\partial_\mu T_0^{\mu\nu}=0, \quad (96)$$

while $\tau^{\mu\nu}$ is invariant under the coordinate-induced transformation of $g_{\mu\nu}$.

The differential equation (86) now requires that

$$\partial_\mu\tau^{\mu\nu}=-(\Gamma_{\alpha\beta}{}^\nu+\partial_\alpha\partial_\beta X^\nu)T_0^{\alpha\beta} \quad (97)$$

which is consistent with the invariance of $\tau^{\mu\nu}$. A solution of this equation is exhibited on writing the final form as

$$\begin{aligned}(-g)^{1/2}T_s^{\mu\nu}=T_0^{\mu\nu}+\partial_\lambda[X^\mu T_0^{\lambda\nu}+X^\nu T_0^{\mu\lambda}-X^\lambda T_0^{\mu\nu}]\\ -[f^\mu\delta_\lambda{}^\nu+f^\nu\delta_\lambda{}^\mu-f^\mu f^\nu\partial_\lambda]\\ \times(\Gamma_{\alpha\beta}{}^\lambda+\partial_\alpha\partial_\beta X^\lambda)T_0^{\alpha\beta}.\end{aligned} \quad (98)$$

The correctness of this choice can be confirmed by verifying the integrability of the differential expression

$$\delta W_s=\frac{1}{2}\int(dx)(-g)^{1/2}T_s^{\mu\nu}\delta g_{\mu\nu}. \quad (99)$$

The several terms become

$$\delta W_s=\int(dx)[T_0^{\mu\nu}\delta h_{\mu\nu}-T_0^{\mu\nu}X^\lambda\delta\Gamma_{\mu\nu\lambda}-T_0^{\mu\nu}(\Gamma_{\mu\nu\lambda}+\partial_\mu\partial_\nu X_\lambda)\delta X^\lambda] \quad (100)$$

and

$$W_s=\int(dx)T_0^{\mu\nu}[h_{\lambda\nu}-\Gamma_{\mu\nu\lambda}X^\lambda+\tfrac{1}{2}\partial_\mu X^\lambda\partial_\nu X_\lambda]. \quad (101)$$

The direct construction of W_s, an invariant under the coordinate-induced variation of $g_{\mu\nu}$, is an alternative procedure.[6] As in electrodynamics, simplifications can be introduced by adopting a particular gauge, relative to the function f^μ:

$$f^\mu h_{\mu\nu}=0. \quad (102)$$

Then $X_\lambda=0$ and only the linear term survives in W_s. In this gauge two-graviton emission depends entirely upon the cubic coupling terms in $\mathfrak{L}_g$.

The additional coupling between graviton sources that is mediated by a pair of gravitons, or photons, or neutrinos, and so on, is expressed by a modified propagation function. One function will suffice if the sources are restricted to be traceless. In the analogous electrodynamic situation this function is expressed by

$$\bar{D}_+(k)=\frac{1}{k^2-i\epsilon}+\alpha\int\frac{dM}{M}\frac{f(M)}{k^2+M^2-i\epsilon}, \quad (103)$$

where $f(M)$ approaches a positive constant when M becomes large compared with the threshold mass of

[6] For this and related approaches see A. Radkowski, Harvard Ph.D. thesis, 1968 (unpublished). He gave the first derivation of Eq. (101), and has also discussed the necessary gravitational modification of the coupling of matter fields to their sources, which we have not considered.

pairs of spin-0 or spin-$\frac{1}{2}$ charged particles. Let us note here that, if zero-mass charged particles were supposed to exist, the small-momentum form of the propagation function would be dominated by (A, and later, B and C, are constants)

$$\bar{D}_+(k)=\frac{1}{k^2-i\epsilon}+\alpha A\int_{\to 0}^{\infty}\frac{dM}{M}\frac{1}{k^2+M^2-i\epsilon}. \quad (104)$$

The implied static potential, at sufficiently large distances, is

$$4\pi\bar{\mathfrak{D}}(r)=\frac{1}{r}+\alpha A\int_{\to 0}^{\infty}\frac{dM}{M}\frac{e^{-Mr}}{r}$$
$$\sim\frac{1}{r}\left[1+\alpha A\ \ln\frac{1}{\mu r}\right]\sim\frac{B}{r^{(1+\alpha A)}}, \quad (105)$$

where μ is a small cutoff mass characteristic of the infrared problem. Thus the whole structure of the long-range interaction of static charges would be changed. We mention this, not as a proof that massless charged particles could not exist, but to indicate the serious complications that would have plagued the phenomenological theory. Now, particles of zero rest mass do carry "gravitational charge." The dimensionless coupling constant α is replaced by G, which demands an additional (mass)^2 factor on dimensional grounds. That cannot be M^2 since the spectral integral would cease to exist. It is therefore given by $-k^2$, and we infer the dominant part of the modified graviton propagation function, for sufficiently small momenta, to be

$$\bar{D}_{+g}(k)=\frac{1}{k^2-i\epsilon}-k^2GC\int_{\to 0}^{\infty}\frac{dM}{M}\frac{1}{k^2+M^2-i\epsilon}$$
$$=\frac{1}{k^2-i\epsilon}+GC$$
$$\times\int_{\to 0}^{\infty}MdM\left(\frac{1}{k^2+M^2-i\epsilon}-\frac{1}{M^2}\right). \quad (106)$$

The corresponding static potential at appropriately large distances is

$$4\pi\bar{\mathfrak{D}}_g(r)=\frac{1}{r}+GC\int_0^{\infty}MdM\frac{e^{-Mr}}{r}$$
$$=\frac{1}{r}+C\frac{G}{r^3}, \quad (107)$$

and the long-range character of the Newtonian potential is not modified, despite the absence of a mass gap between the graviton and the particle pairs to which it is coupled. The additional $1/r^3$ term is equivalent to a distance uncertainty of magnitude $\sim G^{1/2}$ where, restoring $\hbar$ and c,

$$(G\hbar/c^3)^{1/2}=1.6\times 10^{-33}\ \text{cm}. \quad (108)$$

The appearance of this tiny fundamental length is a gentle reminder that, with conceptual problems no longer barring the way to performing the calculations, the practical interest attached to such refinements of gravitational dynamics is, and for the foreseeable future will remain, nil.

SPECULATIVE REMARKS

The weak-field structure of $g_{\mu\nu}$,

$$g_{\mu\nu}(x)=g^{(0)}{}_{\mu\nu}+2h_{\mu\nu}(x), \quad (109)$$

where

$$h_{\mu\nu}(x)=\frac{4\pi G}{c^4}\int(dx')D_+(x-x')$$
$$\times[2T_{\mu\nu}(x')-g^{(0)}{}_{\mu\nu}T(x')]+\text{gauge terms}, \quad (110)$$

makes it natural to assume that $g^{(0)}{}_{\mu\nu}$ is the local contribution of very distant masses. This is Mach's principle, as we interpret it.[7] It can be restated more tautologically as: Without matter ($T_{\mu\nu}=0$), there is no world ($g_{\mu\nu}=0$). The local nature of space-time is thereby attributed to the general matter distribution in the universe, which is one side of a self-consistent dynamical description. Any constant tensor $g^{(0)}{}_{\mu\nu}$ can be diagonalized by real linear coordinate transformations, leading to entries of unit magnitude, but its signature cannot be altered and is therefore an attribute of the world matter distribution. Thus the distinction between space and time is comprehensible, in principle. That the details of the distant matter distribution are not otherwise relevant, as conveyed by the diagonal form of $g^{(0)}{}_{\mu\nu}$ and the equivalent well-tested isotropy of space-time, is in the nature of physical theory, for physical systems are described only in relation to other physical systems, all being under the common influence of the external universe.

Perhaps the only immediate value of this viewpoint is the understanding it gives for the remarkable characteristics of the gravitational coupling constant G: that it is not dimensionless, and is so fantastically small in atomic units. We recall the qualitatively valid property of numbers representing the mass and radius of the observed universe[8]:

$$1\sim GM/Rc^2, \quad (111)$$

which indicates that a weak-field estimate of the

[7] Another interpretation of Mach's principle has lead to the introduction of hypothetical massless, spinless particles, represented by a scalar field that modifies the gravitational coupling constant. See R. H. Dicke, *The Theoretical Significance of Experimental Relativity* (Gordon and Breach Science Publishers, Inc., New York, 1964).

[8] See for example, H. Bondi, *Cosmology* (Cambridge University Press, New York, 1952), Chap. VII.

distant-mass contribution is not totally misleading. According to our version of Mach's principle, the effect of a nearby mass m, at distance r, is measured by comparing it with the contribution of the remote universe:

$$\sim\frac{m/r}{M/R}=\frac{Gm}{rc^2}. \tag{112}$$

Thus G emerges as a conversion factor between terrestrial standards and cosmical properties.

There are some interesting regularities that position the laboratory between the universe and the atom. With μ of the proton-mass order of magnitude, we note that

$$\frac{M}{1\text{ kg}}\sim\left(\frac{1\text{ kg}}{\mu}\right)^2,\quad \frac{R}{1\text{ cm}}\sim\left(\frac{1\text{ cm}}{\hbar/\mu c}\right)^2, \tag{113}$$

in the sense that the large powers of 10 being compared differ only by small integers. An additional aspect of these empirical scaling laws is expressed by

$$\frac{M}{R^2}\sim\frac{1\text{ kg}}{(1\text{ cm})^2}\sim\frac{\mu}{(\hbar/\mu c)^2}, \tag{114}$$

with a similar understanding concerning small powers of 10. A very suggestive consequence is the derived relation

$$\frac{G\mu^2}{R}\sim\frac{\hbar^2}{\mu R^2}, \tag{115}$$

for this can be read as a statement of dynamical equilibrium—the gravitational attraction of two atoms across the universe is balanced by the quantum kinetic energy demanded by localization within the universe. Does the quantum stabilize the cosmos?

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Determination of the Nucleon-Nucleon Scattering Matrix. IX. (n,p) Analysis from 7 to 750 MeV*

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Papers VII and VIII in this series contained phase-shift analyses of (p,p) data from 0–400 MeV and from 0–750 MeV, respectively. The present paper gives the corresponding (n,p) analyses. In the energy region below 400 MeV, six single-energy analyses were carried out, as well as an energy-dependent analysis. The combined (p,p) plus (n,p) energy-dependent analysis below 400 MeV includes 839 (p,p) data and 912 (n,p) data. With 45 phenomenological parameters representing 14 free isovector phases and 11 free isoscalar phases, we obtain a solution that yields an average χ^2 per datum of 1.08 for the 1751 data. This solution gives a precision fit to the data from 400 down to 4 MeV, and it extrapolates well above 400 MeV. At energies below 50 MeV, we find that the existing (n,p) data are not yet complete enough to permit a unique determination of the isoscalar phases. Single-energy analyses were also carried out at 425 and 630 MeV, as well as a combined (p,p) plus (n,p) energy-dependent analysis from 0 to 750 MeV. The energy-dependent solution, which includes 53 phenomenological parameters that represent 25 free phases, was obtained by fitting 1147 (p,p) data from 23 to 736 MeV and 901 (n,p) data from 14 to 730 MeV. It has an average χ^2 per datum of 1.34 for the 2048 data. However, at energies above 450 MeV, where, as shown in Paper VIII, the isovector amplitudes are not well known, we cannot uniquely define the isoscalar amplitudes. Nevertheless, the restriction on the phases imposed by fitting to experiments near 425 and 630 MeV enables us to sharpen our knowledge of the phase shifts at lower energies. We find that the $^1P_1$ and $^3D_3$ phases exhibit maxima in the magnitudes near 300 MeV, and that the $^1F_3$ phase is monotonic. Second-derivative and error matrices are tabulated for the single-energy solutions at 25, 50, 95, 142, 210, 330, and 425 MeV. These matrices, which represent our phase-shift solutions fitted to 683 (p,p) data and 572 (n,p) data, contain most of the physical content of the entire elastic nucleon-nucleon data collection. Fitting potential models to these matrices is essentially equivalent to fitting directly to the data. Computationally, using the matrices is vastly simpler.

I. INTRODUCTION

IN Paper VII of this series,[1] we published a phase-shift analysis of the (p,p) scattering data below 400 MeV. Paper VIII gave the corresponding analysis when (p,p) data from 400 to 750 MeV are included.[2] The present paper contains the results of our (n,p) analyses. As in the above papers, we first analyze the energy region below 400 MeV, and then we extend the analysis to include data up to 730 MeV.

Papers VII–IX form a self-contained set, and they supersede the results obtained in Papers I–VI. The pres-

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[1] M. H. MacGregor, R. A. Arndt, and R. M. Wright, Phys. Rev. **169**, 1128 (1968).

[2] M. H. MacGregor, R. A. Arndt, and R. M. Wright, Phys. Rev. **169**, 1149 (1968).