Quantum Fields near Black Holes[*]

Andreas Wipf

Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität

Max Wien Platz 1, 07743 Jena

Abstract

This review gives an introduction into problems, concepts and techniques when quantizing matter fields near black holes. The first part focusses on quantum fields in general curved space-times. The second part is devoted to a detailed treatment of the Unruh effect in uniformly accelerated frames and the Hawking radiation of black holes. Particular emphasis is put on the induced energy momentum tensor near black holes.

1 Introduction

In the theory of quantum fields on curved space-times one considers gravity as a classical background and investigates quantum fields propagating on this background. The structure of spacetime is described by a manifold $\mathcal{M}$ with metric $g_{\mu\nu}$. Because of the large difference between the Planck scale (10^{-33}cm) and scales relevant for the present standard model ($\geq 10^{-17}$cm) the range of validity of this approximation should include a wide variety of interesting phenomena, such as particle creation near a black hole with Schwarzschild radius much greater than the Planck length.

The difficulties in the transition from flat to curved spacetime lie in the absence of the notion of global inertial observers or of Poincaré transformations which underlie the concept of particles in Minkowski spacetime. In flat spacetime, Poincaré symmetry is used to pick

---

[*]to appear in *Black Holes: Theory and Observation*, edited by F.W. Hehl, C. Kiefer and R. Metzler (Springer, Berlin, 1998).

1

out a preferred irreducible representation of the canonical commutation relations. This is achieved by selecting an invariant vacuum state and hence a particle notion. In a general curved spacetime there does not appear to be any preferred concept of particles. If one accepts that quantum field theory on general curved spacetime is a quantum theory of *fields*, not particles, then the existence of global inertial observers is irrelevant for the formulation of the theory. For linear fields a satisfactory theory can be constructed. Recently Brunelli and Fredenhagen [1] extended the Epstein-Glaser scheme to curved space-times (generalising an earlier attempt by Bunch [2]) and proved perturbative renormalizability of $\lambda\phi^4$.

The framework and structure of Quantum field theory in curved space-times emerged from Parker's analysis of particle creation in the very early universe [3]. The theory received enormous impetus from Hawking's discovery that black holes radiate as black bodies due to particle creation [4]. A comprehensive summary of the work can be found in the books [5].

2   Quantum Fields in Curved Spacetime

In a general spacetime no analogue of a 'positive frequency subspace' is available and as a consequence the states of the quantum field will not possess a physically meaningful particle interpretation. In addition, there are spacetimes, e.g. those with time-like singularities, in which solutions of the wave equation cannot be characterised by their initial values. The conditions of *global hyperbolocity* of $(\mathcal{M}, g_{\mu\nu})$ excludes such 'pathological' spacetimes and ensures that the field equations have a well posed initial value formulation. Let $\Sigma \subset \mathcal{M}$ be a hypersurface whose points cannot be joined by time-like curves. We define the *domain of dependence of* Σ by

$$\mathrm{D}(\Sigma) = \{p \in \mathcal{M} | \text{every inextendible causal curve through } p \text{ intersects } \Sigma\}.$$

If $\mathrm{D}(\Sigma) = \mathcal{M}$, Σ is called a *Cauchy surface* for the spacetime and $\mathcal{M}$ is called *globally hyperbolic*. Globally hyperbolic spacetimes can be *foliated* by a one-parameter family of smooth Cauchy surfaces Σ_t, i.e. a smooth 'time coordinate' t can be chosen on $\mathcal{M}$ such that each surface of constant t is a Cauchy surface [6]. There is a *well posed initial value problem* for linear wave equations [7]. For example, given smooth initial data $\phi_0, \dot{\phi}_0$, then there exists a unique solution ϕ of the *Klein-Gordon equation*

$$\Box_g \phi + m^2 \phi = 0, \qquad \Box_g = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \tag{1}$$

which is smooth on all of $\mathcal{M}$, such that on Σ we have $\phi = \phi_0$ and $n^\mu \nabla_\mu \phi = \dot{\phi}_0$, where

n^μ is the unit future-directed normal to Σ. In addition, ϕ varies continuously with the initial data.

For the phase-space formulation we slice $\mathcal{M}$ by space-like Cauchy surfaces Σ_t and introduce unit normal vector fields n^μ to Σ_t. The spacetime metric $g_{\mu\nu}$ induces a spatial metric $h_{\mu\nu}$ on each Σ_t by the formula

$$g_{\mu\nu} = n_\mu n_\nu - h_{\mu\nu}.$$

Let t^μ be a 'time evolution' vector field on $\mathcal{M}$ satisfying $t^\mu \nabla_\mu t = 1$. We decompose it into its parts normal and tangential to Σ_t,

$$t^\mu = N n^\mu + N^\mu,$$

where we have defined the *lapse function* N and the *shift vector* N^μ tangential to the Σ_t. Now we introduce adapted coordinates $x^\mu = (t, x^i), i = 1, 2, 3$ with $t^\mu \nabla_\mu x^i = 0$, so that $t^\mu \nabla_\mu = \partial_t$ and $N^\mu \partial_\mu = N^i \partial_i$. The metric coefficients in this coordinate system are

$$g_{00} = g(\partial_t, \partial_t) = N^2 - N^i N_i \quad \text{and} \quad g_{0i} = g(\partial_t, \partial_i) = -N_i,$$

where $N_i = h_{ij} N^j$, so that

$$
\begin{aligned}
ds^2 &= (N dt)^2 - h_{ij}(N^i dt + dx^i)(N^j dt + dx^j) \\
(\partial \phi)^2 &= \frac{1}{N^2}(\partial_0 \phi - N^i \partial_i \phi)^2 - h^{ij} \partial_i \phi \partial_j \phi.
\end{aligned}
$$

The determinant g of the 4-metric is related to the determinant h of the 3-metric as $g = -N^2 h$. Inserting these results into the Klein-Gordon action

$$S = \int L dt = \frac{1}{2} \int \eta \Big(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \Big), \qquad \eta = \sqrt{|g|} d^4 x,$$

one obtains for the momentum density, π, conjugate to the configuration variable ϕ on Σ_t

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \frac{\sqrt{h}}{N}(\dot{\phi} - N^i \partial_i \phi) = \sqrt{h}(n^\mu \partial_\mu \phi).$$

A point in classical phase space consists of the specification of functions (ϕ, π) on a Cauchy surface. By the result of Hawking and Ellis, smooth (ϕ, π) give rise to a unique solution to (1). The space of solutions is independent on the choice of the Cauchy surface.

For two (complex) solutions of the Klein-Gordon equation the inner product

$$(u_1, u_2) \equiv i \int_\Sigma \Big(\bar{u}_1 n^\mu \nabla_\mu u_2 - (n^\mu \nabla_\mu \bar{u}_1) u_2 \Big) \sqrt{h} \, d^3 x = i \int (\bar{u}_1 \pi_2 - \bar{\pi}_1 u_2) d^3 x$$

defines a natural symplectic structure. Natural means, that (u_1, u_2) is independent of the choice of Σ. This inner product is not positive definite. Let us introduce a complete set of conjugate pairs of solutions $(u_k, \bar{u}_k)$ of the Klein-Gordon equation[1] satisfying the following

---

[1] the k are any labels, not necessarily the momentum

ortho-normality conditions

$$(u_k, u_{k'}) = \delta(k, k') \Rightarrow (\bar{u}_k, \bar{u}_{k'}) = -\delta(k, k') \quad \text{and} \quad (u_k, \bar{u}_{k'}) = 0.$$

There will be an infinity of such sets. Now we expand the field operator in terms of these modes:

$$\phi = \int d\mu(k) \Big(a_k u_k + a_k^\dagger \bar{u}_k \Big) \quad \text{and} \quad \pi = \int d\mu(k) \Big(a_k \pi_k + a_k^\dagger \bar{\pi}_k \Big),$$

so that

$$(u_k, \phi) = a_k \quad \text{and} \quad (\bar{u}_k, \phi) = -a_k^\dagger.$$

By using the completeness of the u_k and the canonical commutation relations one can show that the operator-valued coefficients $(a_k, a_k^\dagger)$ satisfy the usual commutation relations

$$[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0 \quad \text{and} \quad [a_k, a_{k'}^\dagger] = \delta(k, k'). \tag{2}$$

We choose the Hilbert space $\mathcal{H}$ to be the Fock space built from a 'vacuum' state Ω_u satisfying

$$a_k \Omega_u = 0 \quad \text{for all} \quad k, \qquad (\Omega_u, \Omega_u)_{\mathcal{H}} = 1. \tag{3}$$

The 'vectors' $\Omega_u, a_k^\dagger \Omega_u, \ldots$ comprise a basis of $\mathcal{H}$. The scalar product given by (2,3) is positive-definite.

If $(v_p, \bar{v}_p)$ is a second set of basis functions, we may as well expand the field operator in terms of this set

$$\phi = \int d\mu(p) \Big(b_p v_p + b_p^\dagger \bar{v}_p \Big).$$

The second set will be linearly related to the first one by

$$v_p = \int d\mu(k) \Big((u_k, v_p) u_k - (\bar{u}_k, v_p) \bar{u}_k \Big) \equiv \int d\mu(k) \Big(\alpha(p, k) u_k + \beta(p, k) \bar{u}_k \Big).$$

The inverse transformation reads

$$u_k = \int d\mu(p) \Big(v_p \bar{\alpha}(p, k) - \bar{v}_p \beta(p, k) \Big).$$

As a consequence, the Bogolubov-coefficients are related by

$$\alpha\alpha^\dagger - \beta\beta^\dagger = 1 \quad \text{and} \quad \alpha\beta^t - \beta\alpha^t = 0. \tag{4}$$

If the $\beta(k,p)$ vanish, then the 'vacuum' is left unchanged, but if they do not, we have a nontrivial *Bogolubov transformation*

$$(a \quad a^\dagger) = (b \quad b^\dagger) \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b \\ b^\dagger \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \tag{5}$$

which mixes the annihilation and creations operators. If one defines a Fock space and a 'vacuum' corresponding to the first mode expansion, $a_k \Omega_u = 0$, then the expectation of the number operator $b_p^\dagger b_p$ defined with respect to the second mode expansion is

$$(\Omega_u, b_p^\dagger b_p \Omega_u) = \int d\mu(k) |\beta(p,k)|^2.$$

That is, the old vacuum contains new particles. It may even contain an infinite number of new particles, in which case the two Fock spaces cannot be related by a unitary transformation.

Stationary and static spacetimes. A spacetime is *stationary* if there exist coordinates for which the metric is time-independent. This property holds iff spacetime admits a time-like Killing field $K = K^\mu \partial_\mu$ and hence a natural choice for the mode functions u_k: We may scale K such that the Killing time t is the proper time measured by at least one comoving clock. Now we may choose as basis functions u_k the eigenfunctions of the Lie derivative,

$$iL_K u_k = \omega(k) u_k \quad \text{and} \quad iL_K \bar{u}_k = -\omega(k) \bar{u}_k,$$

where the $\omega(k) > 0$ are constant. The $\omega(k)$ are the frequencies relative to the particular comoving clock and the u_k and $\bar{u}_k$ are the positive and negative frequency solutions, respectively. Now the construction of the vacuum and Fock space is done as described above.

In a *static spacetime*, K is everywhere orthogonal to a family of hyper-surfaces and hence satisfies the Frobenius condition $\tilde{K} \wedge d\tilde{K} = 0, \quad \tilde{K} = K_\mu dx^\mu$. We may introduce adapted coordinates: t along the congruence ($K = \partial_t$) and x^i in one hypersurface such that the metric is time-independent and the shift vector N_i vanishes,

$$(g_{\mu\nu}) = \begin{pmatrix} N^2(x^i) & 0 \\ 0 & -h_{ij}(x^i) \end{pmatrix}.$$

As modes we use

$$u_k = \frac{1}{\sqrt{2\omega(k)}} e^{-i\omega(k)t} \phi_k(x^i)$$

which diagonalise L_K and for which the Klein-Gordon equation simplifies to

$$\mathcal{K}\phi_k \equiv \Big(-\frac{N}{\sqrt{h}}\partial_i(N\sqrt{h}h^{ij}\partial_j) + N^2m^2 \Big)\phi_k = \omega_k^2\phi_k.$$

Since $n^\mu\partial_\mu = N^{-1}\partial_t$, the inner product of two mode functions is

$$(u_1, u_2) = \frac{\omega_1 + \omega_2}{2\sqrt{\omega_1\omega_2}}\, e^{i(\omega_1-\omega_2)t}\underbrace{\int \bar{\phi}_1\phi_2\, N^{-1}\sqrt{h}\, d^3x}_{(\phi_1,\phi_2)_2}.$$

The elliptic operator $\mathcal{K}$ is symmetric with respect to the L_2 scalar product $(.,.)_2$ and may be diagonalised. Its positive eigenvalues are the $\omega^2(k)$ and its eigenfunctions form a complete 'orthonormal' set on Σ, $(\phi_k, \phi_{k'})_2 = \delta(k, k')$. It follows then that the u_k form a complete set with the properties discussed earlier.

Ashtekar and Magnon [8] and Kay [9] gave a rigorous construction of the Hilbert space and Hamiltonian in a stationary spacetime. They started with a *conserved positive scalar product* $(.,.)_E$

$$(\phi_1, \phi_2)_E = \int\limits_\Sigma T_{\mu\nu}(\phi_1, \phi_2)K^\nu n^\mu\sqrt{h}d^3x,$$

where the bilinear-form on the space of complex solutions is defined by the metric 'stress tensor':

$$T_{\mu\nu}(\phi, \psi) = \frac{1}{2}\Big(\phi^\dagger{}_{,\mu}\,\psi_{,\nu} + \phi^\dagger{}_{,\nu}\,\psi_{,\mu} - g_{\mu\nu}(\nabla\phi^\dagger\nabla\psi - m^2\phi^\dagger\psi)\Big).$$

This 'stress tensor' is symmetric and conserved and hence $\nabla_\mu(T^{\mu\nu}K_\nu) = 0$. It follows that the norm is invariant under the time-translation map

$$\alpha_t^*(\phi) = \phi \circ \alpha_t \quad \text{or} \quad (\alpha_t^*(\phi))(x) = \phi(\alpha_t(x)),$$

generated by the Killing field K. When completing the space of complex solutions in the 'energy-norm' one gets a complex (auxiliary) Hilbert space $\tilde{\mathcal{H}}$. The time translation map extends to $\tilde{\mathcal{H}}$ and defines a one-parameter unitary group

$$\alpha_t^* = e^{i\tilde{h}t}, \qquad \tilde{h} \quad \text{self-adjoint.}$$

Note, that from the definition of the Lie derivative,

$$\frac{d}{dt}(\alpha_t^*\phi)|_{t=0} = -L_K\phi = i\tilde{h}\phi.$$

The conserved inner product (ϕ_1, ϕ_2) can be bounded by the energy norm and hence extends to a quadratic form on $\tilde{\mathcal{H}}$. Let $\tilde{\mathcal{H}}^+ \subset \tilde{\mathcal{H}}$ be the positive spectral subspace in the spectral decomposition of $\tilde{h}$ and let P be the projection map $P : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}^+$. For all real solutions we may now define the *scalar product* as the inner product of the projected solutions, which are complex. The one-particle Hilbert space $\mathcal{H}$ is just the completion of the space $\tilde{\mathcal{H}}^+$ of 'positive frequency solutions' in the Klein-Gordon inner product.

Hadamard states. For a black hole the global Killing field is not everywhere time-like. One may exclude the non-time-like region from space time which corresponds to the imposition of boundary conditions. One may also try to retain this region but attempt to define a meaningful vacuum by invoking physical arguments. In general spacetimes there is no Killing vector at all. One probably has to give up the particle picture in this generic situation.

In (globally hyperbolic) spacetimes without any symmetry one can still construct a well-defined Fock space over a quasifree vacuum state, provided that the two-point functions satisfies the so-called Hadamard condition. Hadamard states are states, for which the two-point function has the following singularity structure

$$\omega(\phi(x)\phi(y)) \equiv \omega_2(x, y) = \frac{u}{\sigma} + v \log \sigma + w, \tag{6}$$

where $\sigma(x, y)$ is the square of the geodesic distance of x and y and u, v, w are smooth functions on $\mathcal{M}$. It has been shown that if ω_2 has the Hadamard singularity structure in a neighbourhood of a Cauchy-surface, then it has this form everywhere [11]. To show that, one uses that ω_2 satisfies the wave equation. This result can then be used to show that on a globally hyperbolic spacetime there is a wide class of states whose two-point functions have the Hadamard singularity structure.

The two-point function ω_2 must be positive,

$$\omega(\phi(f)^\dagger \phi(f)) = \int d\mu(x) d\mu(y) \ \bar{f}(x) \omega_2(x, y) f(y) \geq 0,$$

and must obey the Klein-Gordon equation. These requirements determine u and v uniquely and put stringent conditions on the form of w. In a globally hyperbolic spacetime there are unique retarded and advanced Green functions

$$\Delta_{ret}(x, y) \quad , \quad \Delta_{adv}(x, y) \quad \text{with} \quad \text{supp}(\Delta_{ret}) = \{(x, y); x \in J_+(y)\},$$

where $J_+(y)$ is the causal future of y. The *Feynman Green function* is related to ω_2 and the advanced Green function as

$$i\Delta_F(x, y) = \omega_2(x, y) + \Delta_{adv}(x, y).$$

Since Δ_{adv} is unique, the ambiguities of Δ_F are the same as those of ω_2. The *propagator function*

$$i\Delta(x,y) = [\phi(x), \phi(y)] = \Delta_{ret}(x,y) - \Delta_{adv}(x,y)$$

determines the antisymmetric part of ω_2,

$$\omega_2(x,y) - \omega_2(y,x) = i\Delta(x,y),$$

so that this part is without ambiguities. For a scalar field without self-interaction we expect that

$$\omega(\phi(x_1)\ldots\phi(x_{2n-1})) = 0, \qquad \omega(\phi(x_1)\ldots\phi(x_{2n})) = \sum_{\substack{i_1 < i_2 \ldots < i_n \\ j_1 < j_2 \ldots < j_n}} \prod_{k=1}^{n} \omega(\phi(x_{i_k})\phi(x_{j_k})).$$

A state ω fulfilling these conditions is called *quasifree*. Now one can show that any choice of $\omega_2(x,y)$ fulfilling the properties listed above gives rise to a well-defined Fock-space $\mathcal{F} = \oplus \mathcal{F}_n$ over a quasifree vacuum state. The scalar-product on the 'n-particle subspace' $\mathcal{F}_n$ in

$$\mathcal{F}_n = \{\psi \in \mathcal{D}(\mathcal{M}^n)_{symm}/\mathcal{N}\}^{completion}, \qquad n = 0,1,2,\ldots, \tag{7}$$

where $\mathcal{D}(\mathcal{M}^n)$ denotes the smooth symmetric functions on $\mathcal{M} \times \cdots \times \mathcal{M}$ (n factors) with compact support, is

$$(\psi_1, \psi_2) = \int d\mu(x_1,..,x_n,y_1,..,y_n) \prod_{i=1}^{n} \omega_2(x_i,y_i) \bar{\psi}_1(x_1,..,x_n) \psi_2(y_1,..,y_n),$$

where $d\mu(x_1,x_2,..) = d\mu(x_1)d\mu(x_2)\ldots$. Since ω_2 satisfies the wave equation, the functions in the image of $\Box + m^2$ have zero norm. The set of zero-norm states $\mathcal{N}$ has been divided out in order to end up with a positive definite Hilbert space.

The smeared field operator is now defined in the usual way: $\phi(f) = a(f)^\dagger + a(\bar{f})$, where

$$(a(\bar{f})\psi)_n(x_1,..,x_n) = \sqrt{n+1} \int d\mu(x,y)\omega_2(x,y)f(x)\psi_{n+1}(y,x_1,..,x_n)$$

$$(a(f)^\dagger\psi)_n(x_1,..,x_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(x_k)\psi_{n-1}(x_1,..,x_{k-1},x_{k+1}..,x_n), \quad n > 0$$

and $(a(f)^\dagger\psi)_0 = 0$. It is now easy to see that ω_2 is just the Wightman function of ϕ in the vacuum state ψ_0: $\omega_2(x,y) = (\psi_0, \phi(x)\phi(y)\psi_0)$.

3 The Unruh Effect

We may ask the question how quantum fluctuations appear to an accelerating observer? In particular, if the observer was carrying with him a robust detector, what would this detector register? If the motion of the observer undergoing constant (proper) acceleration is confined to the x^3 axis, then the world line is a hyperbola in the x^0, x^3 plane with asymptotics $x^3 = \pm x^0$. These asymptotics are *event horizons* for the accelerated observer. To find a natural comoving frame we consider a family of accelerating observers, one for each hyperbola with asymptotics $x^3 = \pm x^0$. The coordinate system is then the comoving one in which along each hyperbola the space coordinate is constant while the time coordinate τ is proportional to the proper time as measured from an initial instant $x^0 = 0$ in some inertial frame. The world lines of the uniformly accelerated particles are the orbits of one-parameter group of Lorentz boost isometries in the 3-direction:

$$\begin{pmatrix} x^0 \\ x^3 \end{pmatrix} = \rho \begin{pmatrix} \sinh \kappa t \\ \cosh \kappa t \end{pmatrix} = e^{\kappa \omega t} \begin{pmatrix} 0 \\ \rho \end{pmatrix}, \qquad (\omega^\mu{}_\nu) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the comoving coordinates (t, x^1, x^2, ρ)

$$ds^2 = \kappa^2 \rho^2 dt^2 - d\rho^2 - (dx^1)^2 - (dx^2)^2.$$

so that the proper time along a hyperbola $\rho =$const is $\kappa \rho t$. The orbits are tangential to the *Killing field*

$$K = \partial_t = \kappa(x^3 \partial_0 + x^0 \partial_3) \quad \text{with} \quad (K, K) = (\kappa \rho)^2 = g_{00}. \tag{8}$$

Some typical orbits are depicted in figure (1). Since the proper acceleration on the orbit with $(K, K) = 1$ or $\rho = 1/\kappa$ is κ, it is conventional to view the orbits of K as corresponding to a family of observers associated with an observer who accelerates uniformly with acceleration $a = \kappa$.

The coordinate system t, ρ covers the Rindler wedge R on which K is time-like future directed. The boundary H^+ and H^- of the wedge is given by $\rho = 0$ and appears as a *Killing horizon*, on which K becomes null. Beyond this event horizon the Killing vector field becomes space-like in the regions F, P and time-like past directed in L. The parameter κ plays the role of the *surface gravity*. To see that, we set $r - 2M = \rho^2/8M$ in the Schwarzschild solution and linearise the metric near the horizon $r \sim 2M$. One finds that

$$ds^2 \sim \underbrace{(\kappa \rho)^2 dt^2 - d\rho^2}_{\substack{\text{2-dim Rindler} \\ \text{spacetime}}} - \underbrace{\frac{1}{4\kappa^2} d\Omega^2}_{\substack{\text{2-sphere of} \\ \text{radius } 1/2\kappa}}$$
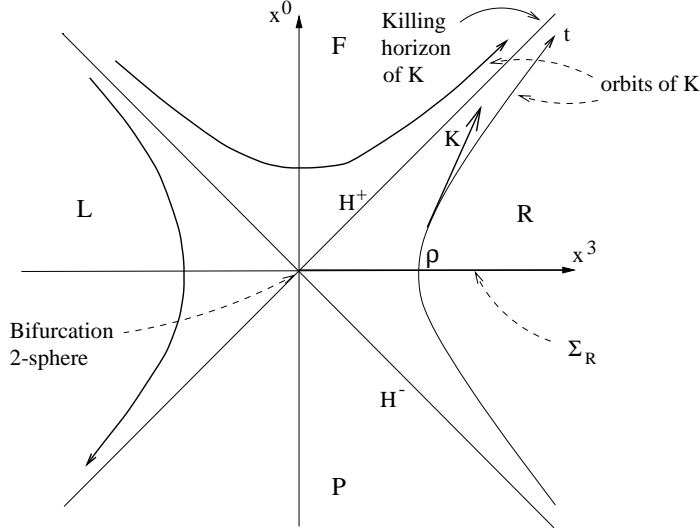
Figure 1: *A Rindler-observer sees only a quarter of Minkowski space*

contains the line element of two-dimensional Rindler spacetime, where $\kappa = 1/4M$ is indeed the surface gravity of the Schwarzschild black hole.

Killing horizons and surface gravity. The notion of Killing horizons is relevant for the Hawking radiation and the thermodynamics of black holes and can already be illustrated in Rindler spacetime. Let $S(x)$ be a smooth function and consider a family of hyper-surfaces $S(x) = \text{const}$. The vector fields normal to the hyper-surfaces are

$$l = g(x)(\partial^\mu S)\partial_\mu,$$

with arbitrary non-zero function g. If l is null, $l^2 = 0$, for a particular hypersurface $\mathcal{N}$ in the family, $\mathcal{N}$ is said to be a *null hypersurface*. For example, the normal vectors to the surfaces $S = r - 2M = \text{const}$ in Schwarzschild spacetime have norm

$$l^2 = g^2 g^{\mu\nu}\partial_\mu S \partial_\nu S = g^2\Big(1 - \frac{2M}{r}\Big),$$

and the horizon at $r = 2M$ is a null hypersurface.

Let $\mathcal{N}$ be a null hypersurface with normal l. A vector t tangent to $\mathcal{N}$ is characterised by $(t,l) = 0$. But since $l^2 = 0$, the vector l is itself a tangent vector, i.e.

$$l^\mu = \frac{dx^\mu}{d\lambda}, \quad \text{where} \quad x^\mu(\lambda) \quad \text{is a null curve on} \quad \mathcal{N}.$$

Now one can show, that $\nabla_l l^\mu|_{\mathcal{N}} \sim l^\mu$, which means that $x^\mu(\lambda)$ is a geodesic with tangent l. The function g can be chosen such that $\nabla_l l = 0$, i.e. so that λ is an affine parameter. A null hypersurface $\mathcal{N}$ is a *Killing horizon* of a Killing field K if K is normal to $\mathcal{N}$.

Let l be normal to $\mathcal{N}$ such that $\nabla_l l = 0$. Then, since on the Killing horizon $K = fl$ for some function f, it follows that

$$\nabla_K K^\mu = fl^\nu \nabla_\nu (fl^\mu) = fl^\mu l^\nu \partial_\nu f = (\nabla_K \log|f|) K^\mu \equiv \kappa K^\mu \quad \text{on} \quad \mathcal{N}. \tag{9}$$

One can show, that the *surface gravity* $\kappa = \frac{1}{2}\nabla_K \log f^2$ is constant on orbits of K. If $\kappa \neq 0$, then $\mathcal{N}$ is a bifurcate Killing horizon of K with bifurcation 2-sphere B. In this non-degenerate case κ^2 is constant on $\mathcal{N}$. For example, for the Killing field in Rindler spacetime (8) $\nabla_K K = \pm \kappa K$ on the Killing horizon and the bifurcation 'sphere' is at $\rho = 0$. If $\mathcal{N}$ is a Killing horizon of K with surface gravity κ, then it is also a Killing horizon of cK with surface gravity $c^2\kappa$. Thus the surface gravity depends on the normalisation of K. For asymptotically flat spacetimes there is the natural normalisation $K^2 \to 1$ and K future directed as $r \to \infty$. With this normalisation the surface gravity is the acceleration of a static particle near the horizon as measured at spatial infinity.

A Killing field is uniquely determined by its value and the value of its derivative $F_{\mu\nu} = \nabla_{[\mu} K_{\nu]}$ at any point $p \in M$. At the bifurcation point p of a bifurcate Killing horizon K vanishes and hence is determined by $F_{\mu\nu}(p)$. In two dimensions $F_{\mu\nu}(p)$ is unique up to scaling. The infinitesimal action of the isometries α_t generated by K takes a vector v^μ at p into

$$L_K v^\mu = F^\mu_\nu v^\nu. \tag{10}$$

The nature of this map on T_p depends upon the signature of the metric. For Riemannian signature it is an infinitesimal rotation and the orbits of α_t are closed with a certain period. For Lorentz signature (10) is an infinitesimal Lorentz boost and the orbits of α_t have the same structure as in the Rindler case. A similar analysis applies to higher dimensions.

The Rindler wedge R is globally hyperbolic with Cauchy hypersurface Σ_R (see fig. (1)). Thus it may be viewed as a spacetime in its own right, and we may construct a quantum field theory on it. When we do that, we obtain a remarkable conclusion, namely that the standard Minkowski vacuum Ω_M corresponds to a thermal state in the new construction. This means, that an accelerated observer will feel himself to be immersed in a thermal bath of particles with temperature proportional to his acceleration a [10],

$$kT = \hbar a / 2\pi c.$$

The noise along a hyperbola is greater than that along a geodesic, and this excess noise

excites the Rindler detector: A uniformly accelerated detector in its ground state may jump spontaneously to an excited state. Note that the temperature tends to zero when $\hbar$ tends to zero. Such a radiation has non-zero entropy. Since the use of an accelerated frame seems to be unrelated to any statistical average, the appearance of a non-vanishing entropy is rather puzzling. The Unruh effect shows, that at the quantum level there is a deep relation between the theory of relativity and the theory of fluctuations associated with states of thermal equilibrium, two major aspects of Einstein's work: The distinction between quantum zero-point and thermal fluctuations is not an invariant one, but depends on the motion of the observer. Note that the temperature is proportional to the acceleration a of the observer. Since $a = 1/\rho$ this means that $T\rho = $ const $\Longleftrightarrow T\sqrt{g_{00}} = $ const. This is just the *Tolman-Ehrenfest relation* [12] for the temperature in a fluid in hydrostatic equilibrium in a gravitational field. The factor $\sqrt{g_{00}}$ guarantees that no work can be gained by transferring radiation between two regions at different gravitational potentials.

Let us calculate the number of 'Rindler-particles' in Minkowski vacuum. To simplify the analysis, we consider a zero-mass scalar field in two-dimensional Minkowski space. In the Heisenberg picture, the expansions in terms of annihilation and creation operators are

$$\phi = \int dk \Big(a_k u_k + h.c. \Big), \quad \text{where} \quad u_k = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega x^0 + ikx^3}, \quad \omega = |k|$$

and

$$\phi = \int dp \Big(b_p v_p + h.c. \Big), \quad \text{where} \quad v_p = \frac{1}{\sqrt{4\pi\epsilon}} \rho^{ip/\kappa} e^{-i\epsilon t}, \quad \epsilon = |p|.$$

The β-coefficients are found to be

$$\beta(p,k) = -(\bar{u}_k, v_p) = \frac{1}{4\pi} \int\limits_0^\infty \Big(\sqrt{\frac{\omega}{\epsilon}} - \sqrt{\frac{\epsilon}{\omega}} \frac{1}{\kappa\rho} \Big) e^{ik\rho} \rho^{ip} d\rho,$$

where we have evaluated the time-independent 'scalar-product' at $t = 0$ for which $x^0 = 0$. Using the formula

$$\int\limits_0^\infty dx \, x^{\nu-1} e^{-(\alpha+i\beta)x} = \Gamma(\nu)(\alpha^2 + \beta^2)^{-\nu/2} e^{-i\nu \arctan(\beta/\alpha)} \tag{11}$$

we arrive at

$$\beta(p,k) = -\frac{\Gamma(ip/\kappa)}{4\pi\kappa} \omega^{-ip/\kappa} \Big(\sqrt{\frac{\epsilon}{\omega}} \pm \frac{p}{\sqrt{\epsilon\omega}} \Big) e^{\mp\pi p/2\kappa} \quad \text{for} \quad \frac{k}{\omega} = \pm 1,$$

or at

$$|\beta(p,k)|^2 = \frac{1}{2\pi\kappa\omega} \frac{1}{e^{2\pi\epsilon/\kappa} - 1}.$$

The Minkowski spacetime vacuum is characterised by $a_k\Omega_M = 0$ for all k. Assuming that this is the state of the system, the expectation value of the occupation number as defined by the Rindler observer, $n_p \equiv b_p^\dagger b_p$, is found to be

$$(\Omega_M, n_p\Omega_M) = \int dk |\beta(p,k)|^2 = \text{volume} \times \frac{1}{e^{2\pi\epsilon/\kappa} - 1}, \tag{12}$$

Thus for an accelerated observer the quantum field seems to be in an equilibrium state with temperature proportional to $T = \kappa/2\pi = a/2\pi$. An observer with $a = 10^{21}\text{cm/sec}^2$ feels a temperature $T \sim 1^0 K$. Since T tends to zero as $\rho \to \infty$ the Hawking temperature (i.e. temperature as measured at spatial ∞) is actually zero. This is expected, since there is nothing inside which could radiate. But for a black hole $T_{local} \to T_H$ at infinity and the black hole must radiate at this temperature.

Let us finally see, how the (massless) Feynman-Green function in Minkowski spacetime,

$$i\Delta_F(x,x') = \langle 0|T(\phi(x)\phi(x'))|0\rangle = \frac{i}{4\pi^2} \frac{1}{(x-x')^2 - i\epsilon},$$

appears to an accelerated observer. Let $x = (t,\rho)$ and $x' = (t',\rho)$ be two events on the world line of an accelerated observer. Since the invariant distance of these two events is $2\rho\sinh\frac{\kappa}{2}(t-t')$, one arrives at the following spectral representation of the Feynman-propagator as seen by this observer

$$\Delta_F(x,x') = \frac{1}{(2\pi)^4}\left(\frac{\kappa}{\rho}\right)^2 \int d^4p \, e^{-iE(t-t')}\left(\frac{1}{p^2 + i\epsilon} - 2\pi i\frac{\delta(p^2)}{e^{\beta|E|} - 1}\right). \tag{13}$$

This is the finite temperature propagator. It follows, that atoms dragged along the world line find their excited levels populated as predicted by a temperature $\beta^{-1} = a/2\pi$.

4   The Stress-Energy Tensor

Semiclassically one would expect that back-reaction is described by the 'semiclassical Einstein equation'

$$G_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle,$$

where the right-hand side contains the expectation value of the energy-momentum tensor of the relevant quantised field in the chosen state. If the characteristic curvature radius L in a region of spacetime is much greater then the Planck length l_{pl}, then in the calculation of $\langle T_{\mu\nu} \rangle$ one can expand in the small parameter $\epsilon = (l_{pl}/L)^2$ and retain only the terms up to first order in ϵ (one-loop approximation). The term of order ϵ, containing a factor $\hbar$, represents the main quantum correction to the classical result. In the one-loop approximation or free fields the contributions of all fields to $\langle T_{\mu\nu} \rangle$ are additive and thus can be studied independently.

The difficulties with defining $\langle T_{\mu\nu} \rangle = \omega(T_{\mu\nu})$ are present already in Minkowski spacetime. The divergences are due to the vacuum zero-fluctuations. The methods of extracting a finite, physically meaningful part, known as renormalisation procedures, were extensively discussed in the literature [14]. A simple cure for this difficulty is (for free fields) the *normal ordering* prescription. We first consider the ill-defined object $\phi^2(x)$, which is part of the stress-energy tensor. We may split the points and consider first the object $\omega(\phi(x)\phi(y))$ which solves the Klein-Gordon equation. This bi-distribution makes perfectly good sense. For physically reasonable states ω in the Fock space (e.g. states with a finite number of particles) the singular behaviour of this bi-distribution is the same as that belonging to the vacuum state, $\omega_0(\phi(x)\phi(y))$. For such states the difference

$$F(x,y) = \omega(\phi(x)\phi(y)) - \omega_0(\phi(x)\phi(y))$$

is a smooth function of its arguments. Hence, after performing this 'vacuum subtraction' the coincidence limit may be taken. We then define

$$\omega(\phi^2(x)) = \lim_{x \to y} F(x,y).$$

The same prescription can be used for the stress-energy tensor. We define

$$\omega(T_{\mu\nu}(x)) = \lim_{x \to x'} D_{\mu\nu'} F(x,x'), \quad D_{\mu\nu'} = \partial_\mu \partial_{\nu'} - \frac{1}{2} g_{\mu\nu} [\partial_\alpha \partial^{\alpha'} - m^2]). \tag{14}$$

In curved spacetime some restrictions should be expected on the class of states on which $\langle T_{\mu\nu} \rangle$ can be defined this way. The *Hadamard condition* provides a restriction of exactly this sort of states.

Although (14) is not a physical definition of expectation values of the stress-energy tensor itself (no preferred vacuum state, vacuum polarisation), it sensibly defines the *differences* of the expected stress energy between two states. In the absence of an obvious prescription it is useful to take an axiomatic approach. Wald showed that a renormalised stress tensor satisfying certain reasonable physical requirements is essentially unique [13]. Its ambiguity can be absorbed into redefinitions of the coupling constants in the (generalised) gravitational field equation. Wald's requirements are:

14

Consistency: Whenever $\omega_1(\phi(x)\phi(y)) - \omega_2(\phi(x)\phi(y))$ is a smooth function, then $\omega_1(T_{\mu\nu}) - \omega_2(T_{\mu\nu})$ is well-defined and should be given by the above 'point-splitting' prescription.

Conservation: There is a regularisation which respects the diffeomorphism invariance, so that $\nabla_\nu T^{\mu\nu} = 0$ holds. This property is needed for consistency of Einstein's gravitational field equation.

Normalisation: In Minkowski spacetime, we have $(\Omega_M, T_{\mu\nu}\Omega_M) = 0$.

Causality: For a fixed in-state in an asymptotically static spacetime $\omega_{in}(T_{\mu\nu}(x))$ is independent of variations of $g_{\mu\nu}$ outside the past light cone of x. For a fixed out-state, $\omega_{out}(T_{\mu\nu})$ is independent of metric variations outside the future light cone of x.

The Causality axiom can be replaced by a locality property, which does not assume an asymptotically static spacetime. The first and last properties are the key ones, since they uniquely determine the expected stress-energy tensor up to the addition of local curvature terms:

Uniqueness theorem: Let $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$ be operators on globally hyperbolic spacetime satisfying the axioms of Wald. Then the difference $U_{\mu\nu} = T_{\mu\nu} - \tilde{T}_{\mu\nu}$ is a multiple of the identity operator, is conserved, $\nabla_\nu U^{\mu\nu} = 0$ and is a local tensor of the metric. That is, it depends only on the metric and its derivatives, via the curvature tensor, at the same point x. As a consequence $\omega(T_{\mu\nu}) - \omega(\tilde{T}_{\mu\nu})$ is independent of the state ω and depends only locally on curvature invariants. The proofs of these properties are rather simple and can be found in the standard textbooks.

Calculating the stress-energy tensor. A 'point-splitting' prescription where one subtracts from $\omega(\phi(x)\phi(y))$ the expectation value $\omega_0(\phi(x)\phi(y))$ in some fixed state ω_0 fulfils the consistency requirement, but cannot fulfil the first and third axiom at the same time. However, if one subtracts a locally constructed bi-distribution $H(x, y)$ which satisfies the wave equation, has a suitable singularity structure and is equal to $(\Omega_M, \phi(x)\phi(y)\Omega_M)$ in Minkowski spacetime, then all four properties will be satisfied.

To find a suitable bi-distribution one recalls the singularity structure (6) of $\omega_2(x, y)$. In Minkowski spacetime and for massless fields $w = 0$ and this suggests that we take the bi-distribution

$$H(x, y) = \frac{u(x, y)}{\sigma} + v(x, y) \log \sigma$$

For massless fields the resulting stress-energy obeys all properties listed above (for massive fields a slight modification is needed).

Effective action. The classical metric energy momentum tensor

$$^{cl}T_{\mu\nu}(x) = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}(x)}$$

is symmetric and conserved (for solutions of the field equation) for a diffeomorphism-invariant classical action S. If we could construct a diffeomorphism-invariant *effective action* Γ, whose variation with respect to the metric yields an expectation value of the energy momentum tensor,

$$\langle T_{\mu\nu}(x)\rangle = \frac{2}{\sqrt{|g|}}\frac{\delta\Gamma}{\delta g^{\mu\nu}(x)},$$

then $\langle T_{\mu\nu}\rangle$ would be conserved by construction. There exists a number of procedures for regularising $\langle T_{\mu\nu}\rangle$, i.e. dimensional, point-splitting or zeta-function regularisation, to mention the most popular ones. Unfortunately the 'divergent' part of $T_{\mu\nu}$ cannot be completely absorbed into the parameters already present in the theory, i.e. gravitational and cosmological constant and parameters of the field theory under investigation. One finds that one must introduce new, dimensionless parameters.

The regularisation and renormalisation of the effective action is more transparent. The divergent geometric parts of the effective action, $\Gamma = \int \eta\gamma_{div} + \Gamma_{finite}$ have in the one-loop approximation the form

$$\gamma_{div} = A + BR + C(\text{Weyl})^2 + D[(\text{Ricci})^2 - R^2] + E\nabla^2 R + FR^2.$$

Only the part containing A and B can be absorbed into the classical action of gravity. The remaining terms with dimensionless parameters $C - F$ lead, upon variation with respect to the metric, to a 2-parameter ambiguity in the expression for $T_{\mu\nu}$.

Effective actions and $\langle T_{\mu\nu}\rangle$ in two dimensions. In two dimensions there are less divergent terms in the effective action. They have the form $\gamma_{div} = A + BR$. The last topological term does not contribute to $T_{\mu\nu}$ and the first one leads to an ambiguous term $\sim Ag_{\mu\nu}$ in the energy momentum tensor.

The symmetric stress-energy tensor has 3 components, two of which are (almost) determined by $T^{\mu\nu}_{\;\;;\nu} = 0$. As independent component we choose the trace $T = T^{\mu}_{\mu}$ which is a scalar of dimension L^{-2}. The ambiguities in the reconstruction of $T^{\mu\nu}$ from its trace is most transparent if we choose isothermal coordinates for which

$$ds^2 = e^{2\sigma}\left((dx^0)^2 - (dx^1)^2\right).$$

This is possible in two dimensions. Introducing null-coordinates

$$u = \frac{1}{2}(x^0 - x^1) \quad \text{and} \quad v = \frac{1}{2}(x^0 + x^1) \Rightarrow ds^2 = 4e^{2\sigma}dudv,$$

the non-vanishing Christoffel symbols are $\Gamma^u_{uu} = 2\partial_u\sigma$, $\Gamma^v_{vv} = 2\partial_v\sigma$ and the Ricci scalar reads $R = -2e^{-2\sigma}\partial_u\partial_v\sigma$. Rewriting the conservation in null-coordinates we obtain

$$\partial_u \langle T_{vv} \rangle + e^{2\sigma} \partial_v \langle T \rangle = 0 \quad , \quad \partial_v \langle T_{uu} \rangle + e^{2\sigma} \partial_u \langle T \rangle = 0, \tag{15}$$

where $T = T^\mu_\mu = e^{-2\sigma} T_{uv}$. The trace $\langle T \rangle$ determines $\langle T_{vv} \rangle$ up to a function $t_v(v)$ and $\langle T_{uu} \rangle$ up to a function $t_u(u)$. These free functions contain information about the state of the quantum system.

In the case of a classical conformally invariant field, $^{cl}T^\mu_\mu = 0$. An important feature of $\langle T_{\mu\nu} \rangle$ is that its trace does not vanish any more. This trace-anomaly is a state-independent local scalar of dimension L^{-2} and hence must be proportional to the Ricci scalar,

$$\langle T \rangle = \frac{c}{24\pi} R = -\frac{c}{12\pi} e^{-2\sigma} \partial_u \partial_v \sigma,$$

where c is the *central charge*. Inserting this trace anomaly into (15) yields

$$\langle T_{uu,vv} \rangle = -\frac{c}{12\pi} e^\sigma \partial^2_{u,v} e^{-\sigma} + t_{u,v} \quad \text{and} \quad \langle T_{uv} \rangle = -\frac{c}{12\pi} \Box_0 \sigma. \tag{16}$$

Formally, the expectation value of the stress-energy tensor is given by the path integral

$$\langle T_{\mu\nu}(x) \rangle = -\frac{1}{Z[g]} \int \mathcal{D}\phi \, \frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} e^{-S[\phi]} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \Gamma[\phi],$$

where the effective action is given by

$$\Gamma[g] = -\log Z[g] = -\log \int \mathcal{D}\phi \, e^{-S[\phi]} = \frac{1}{2} \log \det(-\triangle_c)$$

and we made the transition to Euclidean spacetime (which is allowed for the $2d$ models under investigation). For arbitrary spacetimes the spectrum of $\triangle_c$ is not known. However, the variation of Γ with respect to σ in $g_{\mu\nu} = e^{2\sigma} \hat{g}_{\mu\nu}$ is proportional to the expectation value of the trace of the stress-energy tensor,

$$\frac{\delta\Gamma}{\delta\sigma(x)} = -2g^{\mu\nu}(x) \frac{\delta\Gamma}{\delta g^{\mu\nu}(x)} = -\sqrt{g} \langle T^\mu_\mu(x) \rangle$$

and can be calculated for conformally coupled particles in conformally flat spacetimes. From the conformal anomaly one can (almost) reconstruct the effective action. In particular, in two dimensions the result is the *Polyakov effective action*

$$\Gamma[g] - \Gamma[\delta] = \frac{c}{96\pi} \int \sqrt{g} R \frac{1}{\triangle} R,$$

where the central charge c is 1 for uncharged scalars and Dirac fermions [2]. The $\langle T_{\mu\nu} \rangle$ is found by differentiation with respect to the metric. The covariant expression is

---

[2] see [15] for modifications of this result, for a spacetime with nontrivial topology.

$$\langle T_{\mu\nu}\rangle = \frac{c}{48\pi}\Big(2g_{\mu\nu}R - 2\nabla_\mu\nabla_\nu S + \nabla_\mu S\cdot\nabla_\nu S - \frac{1}{2}g_{\mu\nu}\nabla^\alpha S\cdot\nabla_\alpha S\Big), \qquad S = \frac{1}{\triangle}R, \qquad (17)$$

and in isothermal coordinates this simplifies to (16), as it must be. This energy-momentum tensor is consistent, conserved and causality restricts the choice of the Green function $1/\triangle$. The ambiguities in inverting the wave operator in (17) shows up in the free functions $t_{u,v}$. A choice of these functions is equivalent to the choice of a state.

Let us now apply these results to the (t,r) part of the Schwarzschild black hole

$$ds^2 = \alpha(r)dt^2 - \frac{1}{\alpha(r)}\,dr^2, \qquad \alpha(r) = 1 - \frac{2M}{r}, \qquad (G = 1)$$

which we treat as two-dimensional black hole[3]. We use the 'Regge-Wheeler tortoise co-ordinate' $r_* = r + 2M\log\left(r/M - 2\right)$, such that the metric becomes conformally flat, $ds^2 = \alpha(dt^2 - dr_*^2)$. and introduce null-coordinates $2u = t - r_*$ and $2v = t + r_*$. Using $\partial_{r_*} = \alpha\partial_r$ we obtain for the light-cone components (16) of the energy momentum tensor

$$\langle T_{uu,vv}\rangle = -\frac{c}{12\pi}\Big(\frac{2M\alpha}{r^3} + \frac{M^2}{r^4}\Big) + t_{u,v}, \qquad \langle T_{uv}\rangle = -\frac{c}{12\pi}\frac{2M\alpha}{r^3}$$

or for $\langle T_{\mu\nu}\rangle$ in the $x^\mu = (t, r_*)$ coordinate system

$$\langle T_\mu{}^\nu\rangle = -\frac{cM}{24\pi r^4}\begin{pmatrix} 4r + \frac{M}{\alpha} & 0 \\ 0 & -\frac{M}{\alpha} \end{pmatrix} + \frac{1}{4\alpha}\begin{pmatrix} t_u + t_v & t_u - t_v \\ t_v - t_u & -t_u - t_v \end{pmatrix}. \qquad (18)$$

The *Boulware state* is the state appropriate to a vacuum around a static star and contains no radiation at spatial infinity $\mathcal{J}^\pm$. Hence t_u and t_v must vanish. This state is singular at the horizon. To see that, we use regular Kruskal coordinates:

$$U = -e^{-u/2M} \quad\text{and}\quad V = e^{v/2M} \quad\text{so that}\quad ds^2 = \frac{16M^3}{r}e^{-r/2M}dU\,dV. \qquad (19)$$

With respect to these coordinates the energy-momentum tensor takes the form

$$\langle T_{UU}\rangle = 4(\frac{M}{U})^2\langle T_{uu}\rangle, \qquad \langle T_{VV}\rangle = 4(\frac{M}{V})^2\langle T_{vv}\rangle \quad\text{and}\quad \langle T_{UV}\rangle = -4\frac{M^2}{UV}\langle T_{uv}\rangle.$$

For the Boulware vacuum $t_u = t_v = 0$ and $\langle\ldots\rangle$ is singular at the past horizon at $V = 0$ and future horizon at $U = 0$. The component $\langle T_{UU}\rangle$ is regular at the future horizon if

---

[3]The resulting energy-momentum tensor is not identical to the tensor that one gets when one quantises only the s-modes in the four-dimensional Schwarzschild metric [16].

$M^2 t_u = c/192\pi$ and $\langle T_{VV} \rangle$ is regular at the past horizon if $M^2 t_v = c/192\pi$. The state regular at both horizons is the *Israel-Hartle-Hawking state*. In this state the asymptotic form of the energy-momentum tensor is

$$\langle 0_{HH} | T^\mu_\nu | 0_{HH} \rangle \sim \frac{c}{384\pi M^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{c\pi}{6}(kT)^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{20}$$

with $T = 1/8\pi k M = \kappa/2\pi k$. This is the stress-tensor of a *bath* of thermal radiation at temperature T. Finally, demanding that energy-momentum is regular at the future horizon and that there is no incoming radiation, i.e. $M^2 t_u = c/192\pi$ and $t_v = 0$, results in

$$\langle 0_U | T^\mu_\nu | 0_U \rangle \sim \frac{c}{768\pi M^2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \frac{c\pi}{12}(kT)^2 \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \tag{21}$$

The *Unruh state* is regular on the future horizon and singular at the past horizon. It describes the Hawking evaporation process with only outward flux of thermal radiation.

Euclidean Black Holes. The most elegant and powerful derivation of the Hawking radiation involves an adaption of the techniques due to Kubo to show that the Feynman propagator for a spacetime with stationary black hole satisfies the KMS condition. Consider a system with time-independent Hamiltonian H. The time evolution of an observable A in the Heisenberg picture is $A(z) = e^{izH} A e^{-izH}$, where $z = t + i\tau$ is complex time. For $\tau = 0$ ($t = 0$) it is the time-evolution in a static spacetime with Lorentzian (Euclidean) signature. If $\exp(-\beta H), \beta > 0$ is trace class, one can define the equilibrium state of temperature $T = 1/\beta$:

$$\langle A \rangle_\beta = \frac{1}{Z} \mathrm{tr}\, e^{-\beta H} A, \qquad Z = \mathrm{tr}\, e^{-\beta H}. \tag{22}$$

Let us introduce the finite temperature correlation functions

$$G^\beta_+(z, \vec{x}, \vec{y}) = \langle \phi(z, \vec{x})\phi(0, \vec{y}) \rangle_\beta = \frac{1}{Z} \mathrm{tr}\left(e^{i(z+i\beta)H} \phi(0, \vec{x}) e^{-izH} \phi(0, \vec{y}) \right)$$

$$G^\beta_-(z, \vec{x}, \vec{y}) = \langle \phi(0, \vec{y})\phi(z, \vec{x}) \rangle_\beta = \frac{1}{Z} \mathrm{tr}\left(\phi(0, \vec{y}) e^{izH} \phi(0, \vec{x}) e^{-i(z-i\beta)H} \right)$$

We have used the cyclicity under the trace. Both exponents in G_+ have negative real parts if $-\beta < \tau < 0$; for G_- the condition reads $0 < \tau < \beta$. Therefore, these formulae define holomorphic functions in those respective strips with boundary values $G^\beta_\pm(t, \vec{x}, \vec{y})$. It follows immediately, that

$$G_-^\beta(z, \vec{x}, \vec{y}) = G_+^\beta(z - i\beta, \vec{x}, \vec{y}) \tag{23}$$

which is the KMS-condition (23). This condition is now accepted as a definition of 'thermal equilibrium at temperature $1/\beta$'.

So far the analytic functions $G_\pm$ have been defined in disjoint, adjacent strips in the complex time plane. The KMS-condition states that one of these is the translate of the other and this allows us to define a periodic function throughout the complex plane, with the possible exception of the lines $\tau = \Im(z) = n\beta$. Because of locality $\phi(x)$ and $\phi(y)$ commute for space-like separated events and

$$[\phi(t, \vec{x}), \phi(0, \vec{y})] = 0 \quad \text{for} \quad t \in I \subset R.$$

Then the boundary values of $G_\pm^\beta$ coincide on I and we conclude (by the edge-of-the-wedge theorem) that they are restrictions of a single holomorphic, periodic function, $\mathcal{G}^\beta(z, \vec{x}, \vec{y})$, defined in a connected region in the complex time plane except parts of the lines $\tau = n\beta$.

With these preparations we are now ready to show that the Green function in Schwarzschild spacetime satisfies the KMS-condition. Starting with the analytically continued Schwarzschild metric

$$ds^2 = \alpha dz^2 - \frac{1}{\alpha} dr^2 - r^2 d\Omega^2, \qquad \alpha = 1 - 2M/r, \quad z = t + i\tau,$$

we perform the same coordinate transformation to (complex) Kruskal coordinates as we did for the Lorentzian solution:

$$Z = V + U = 2e^{r_*/4M} \sinh \frac{z}{4M} \quad \text{and} \quad X = V - U = 2e^{r_*/4M} \cosh \frac{z}{4M}.$$

The line element reads

$$ds^2 = \frac{16M^3}{r} e^{-r/2M} \left(dZ^2 - dX^2 \right) - r^2 d\Omega^2$$

and the Killing field takes the form

$$K = \partial_z = \frac{1}{4M} \left(Z\partial_X + X\partial_Z \right) = \frac{1}{4M} \left(V\partial_V - U\partial_U \right).$$

Setting $Z = T + i\mathcal{T}$ the orbits of K are

$$\begin{pmatrix} T \\ X \end{pmatrix} = 2e^{r_*/4M} \begin{pmatrix} \sinh t/4M \\ \cosh t/4M \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathcal{T} \\ X \end{pmatrix} = 2e^{r_*/4M} \begin{pmatrix} \sin \tau/4M \\ \cos \tau/4M \end{pmatrix},$$

in the Lorentzian and Euclidean slices, respectively. As expected from the general properties of bifurcation spheres, these are Lorentz-boosts and rotations, respectively. Since the

Euclidean slice is periodic in τ, the analytic Green function $\mathcal{G}(z = t + i\tau, \vec{x}, \vec{y})$ is periodic in imaginary time τ with period $8\pi M$. This corresponds to a temperature $T = 1/8\pi M$, the Hawking temperature.

The vector field (with affine parametrisation) normal to the Killing horizon $\mathcal{N}$ (the past and future horizons) is $l = \partial_V$ on the future horizon and $l = \partial_U$ on the past horizon. It follows that the surface gravity κ (see (9)) is $1/4M$ on the future horizon and $-1/2M$ on the past horizon.

Energy-momentum tensor near a black hole. In any vacuum spacetime $R_{\mu\nu}$ vanishes and so do the two local curvature terms which enter the formula for $T_{\mu\nu}$ with undetermined coefficients. Hence $T_{\mu\nu}$ is well-defined in the Schwarzschild spacetime. The symmetry of $\langle T^\mu_\nu \rangle$ due to the $SO(3)$ symmetry of the spacetime of a non-rotating black hole and the conservation $\nabla_\nu \langle T^{\mu\nu} \rangle$ reduce the number of independent components of $\langle T^\mu_\nu \rangle$. Christensen and Fulling [18] showed that in the coordinates (t, r_*, θ, ϕ) the tensor is block diagonal. The (t, r_*) part admits the representation

$$\langle T^\mu_\nu \rangle = \begin{pmatrix} \frac{T}{2} - \frac{H+G}{\alpha r^2} - 2\Theta & 0 \\ 0 & \frac{H+G}{\alpha r^2} \end{pmatrix} + \frac{W}{4\pi\alpha r^2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \frac{N}{\alpha r^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{24}$$

and the (θ, ϕ)-part has the form

$$\langle T^\mu_\nu \rangle = (\frac{T}{4} + \Theta) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{25}$$

Here N and W are two constants and

$$\alpha(r) = \left(1 - \frac{2M}{r}\right), \quad T(r) = \langle T^\mu_\mu \rangle, \quad \Theta(r) = \langle T^\theta_\theta \rangle - \frac{1}{4}T(r)$$

$$H(r) = \frac{1}{2} \int_{2M}^r (r' - M)T(r')dr', \quad G(r) = 2 \int_{2M}^r (r' - 3M)\Theta(r')dr'.$$

The energy-momentum tensor is characterised unambiguously by fixing two functions $T(r), \Theta(r)$ and two constants N, W. The constant W gives the intensity of radiation of the black hole at infinity and N vanishes if the state is regular on the future horizon.

The radiation intensity W is non-vanishing only in the *Unruh vacuum*. It has been calculated for the massless scalar field ($s = 0$), two-components neutrino field ($s = 1/2$), electromagnetic field ($s = 1$) and gravitational field ($s = 2$) by Page and Elster [19]:

$M^2 W_0$	$M^2 W_{1/2}$	$M^2 W_1$	$M^2 W_2$
$7.4 \cdot 10^{-5}$	$8.2 \cdot 10^{-5}$	$3.3 \cdot 10^{-5}$	$0.4 \cdot 10^{-5}$

The coefficient N vanishes for the Unruh and Israel-Hartle-Hawking states.

The calculation of the functions in (24,25) meets technical difficulties connected with the fact that solutions of the radial mode equation (see below) are not expressed through known transcendental functions and, consequently, one needs to carry out renormalisation in divergent integrals within the framework of numerical methods. The results for $\langle T^t_t \rangle$ and $\langle T^r_r \rangle$ for the Israel-Hartle-Hawking and the Unruh states have been calculated by Howard/Candelas and Elster [20].

In the Hartle-Hawking state the Kruskal coordinate components of $\langle T_{\mu\nu} \rangle$ near the horizon are found to be of order $1/M^4$. The energy flux into the black hole is negative, as it must be since the 'Hartle-Hawking vacuum' is time independent and the energy flux at future infinity is positive. This is possible since $\langle T_{\mu\nu} \rangle$ need not satisfy the energy conditions.

s-wave contribution to $\langle T_{\mu\nu} \rangle$. The covariant perturbation theory for the $4d$ effective action Γ as developed in [23] is very involved for concrete calculations. Here we shall simplify the problem by considering s-modes of a minimally coupled massless scalar field propagating in an arbitrary (possibly time-dependent) spherically symmetric four-dimensional spacetime. The easiest way to perform this task is to compute the contribution of these modes to the effective action. We choose adapted coordinates for which the Euclidean metric takes the form

$$ds^2 = \gamma_{ab}(x^a)\,dx^a dx^b + \Omega^2(x^a)\omega_{ij}dx^i dx^j,$$

where the last term is the metric on S^2. Now one can expand the (scalar) matter field into spherical harmonics. For s-waves, $\phi = \phi(x^a)$, the action for the coupled gravitational and scalar field is

$$S = -\frac{1}{4}\int [\Omega^2\,{}^\gamma\mathcal{R} + {}^\omega\mathcal{R} + 2(\nabla\Omega)^2]\sqrt{\gamma}d^2x + 2\pi\int\Omega^2(\nabla\phi)^2\sqrt{\gamma}d^2x,$$

where ${}^\gamma\mathcal{R}$ is the scalar curvature of the $2d$ space metric γ_{ab}, ${}^\omega\mathcal{R} = 2$ is the scalar curvature of S^2 and $(\nabla\Omega)^2 = \gamma^{ab}\partial_a\Omega\partial_b\Omega$. The purely gravitational part of the action is almost the action belonging to $2d$ dilatonic gravity with two exceptions: first, the numerical coefficient in front of $(\nabla\Omega)^2$ is different and second, the action is not invariant under Weyl transformation due to the ${}^\omega\mathcal{R}$ term. The action is quite different from the actions usually considered in $2d$ (string-inspired) field theories, because of the unusual coupling of ϕ to the dilaton field Ω. Choosing isothermal coordinates, $\gamma_{ab} = e^{2\sigma}\gamma^f_{ab}$, where γ^f_{ab} is the metric of the flat $2d$ space, one arrives with ζ-function methods at the following exact result for the effective action for the s-modes [16]

$$\Gamma_s = {}^{(n)}\Gamma_s + {}^{(i)}\Gamma$$
$$^{(n)}\Gamma[\sigma,\Omega] = \frac{1}{8\pi}\int\Big(\frac{1}{12}{}^\gamma\mathcal{R}\frac{1}{\triangle_\gamma}{}^\gamma\mathcal{R} - \frac{\triangle_\gamma\Omega}{\Omega}\frac{1}{\triangle_\gamma}{}^\gamma\mathcal{R}\Big)\sqrt{\gamma}d^2x$$

$$^{(i)}\Gamma[\Omega] \;=\; \Gamma_s[\sigma=0,\Omega] = \frac{1}{2}\log\det\Big(-\triangle_f + \frac{\triangle_f\Omega}{\Omega}\Big).$$

The second contribution $^{(i)}\Gamma$ is invariant under $2d$ Weyl transformation, whereas the first one is not. Unfortunately, the determinant cannot be calculated exactly and one must resort to some perturbation expansion. For details I refer to [16]. Ignoring backscattering one finds

$$^{(1)}\Gamma = \frac{1}{8\pi}\int\Big(\frac{1}{12}{}^\gamma\mathcal{R}\frac{1}{\triangle_\gamma}{}^\gamma\mathcal{R} - \frac{\triangle_\gamma\Omega}{\Omega}\times\Big[1+\log\frac{\triangle_\gamma\Omega}{\mu^2\Omega}\Big]\Big)\sqrt{\gamma}d^2x.$$

Due to backscattering one needs to add the following term:

$$^{(2)}\Gamma = -\frac{\xi}{12\cdot 8\pi}\int\Big({}^\gamma\mathcal{R}\frac{1}{\triangle_\gamma}{}^\gamma\mathcal{R} + \text{local terms}\Big)\sqrt{\gamma}d^2x,$$

where $\xi\sim 0.9$. From the action $\Gamma_2 = {}^{(1)}\Gamma + {}^{(2)}\Gamma$ one obtains $\langle T_{\mu\nu}\rangle$ by variation with respect to the metric. To get the flux of the Hawking radiation we need to continue back to Lorentzian spacetime by changing the signs in the appropriate places. According to [23] we arrive at the in-vacuum energy-momentum tensor by replacing $-1/\triangle$ by the retarded Green function. Neglecting backscattering, the luminosity of the black hole is found to be

$$L = -\frac{\pi}{12}\frac{1}{(8\pi M)^2}.$$

This coincides with the total s-wave flux of the Hawking radiation obtained with other methods [5] without taking backscattering effects into account. With backscattering, the Hawking radiation is modified and compares well with that obtained by other means [24].

5 Wave equation in Schwarzschild spacetime

We study the classical wave propagation of a Klein-Gordon scalar field in fig.2. At late times, one expects that every solution will propagate into the black hole region II and/or propagate to $\mathcal{J}^+$.

In the spherically symmetric spacetime we may set

$$\phi = \frac{f(t,r)}{r}Y_{lm}e^{-i\omega t}$$

and the wave equation $(\Box + m^2)\phi = 0$ reduces to the radial equation
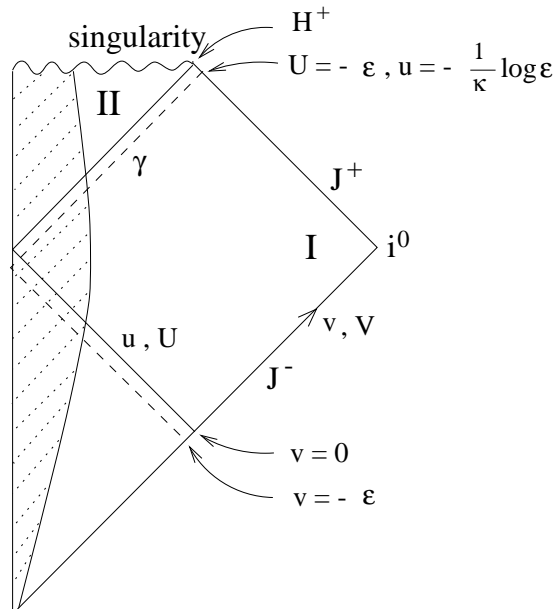
Figure 2: *The propagation of particles in the geometric optics approximation.*

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial r_*^2} - V(r_*)f = 0, \quad V(r_*) = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2} + m^2\right), \tag{26}$$

where M is the mass of the black hole and m that of the Klein-Gordon field. As $r_* \to -\infty$ (i.e. $r \to 2M$) the potential falls off exponentially, $V \sim \exp(r_*/2M)$, and as $r_* \to \infty$ the potential behaves as $\sim m^2 - 2Mm^2/r_*$ in the massive case and $\sim l(l+1)/r^2$ in the massless case. In the asymptotic region $r \to \infty$ this equation possesses outgoing solution $\sim e^{i\omega r_*}$ and ingoing solutions $\sim e^{-i\omega r_*}$. In terms of the null-coordinates the asymptotic solutions look like

$$f_\omega^{out} \sim e^{-2i\omega u} \quad \text{and} \quad f_\omega^{in} \sim e^{-2i\omega v}. \tag{27}$$

Consider a geometric optics approximation in which a particle's world line is a null ray, γ, of constant phase u and trace this ray backwards in time from $\mathcal{J}^+$. The later it reaches $\mathcal{J}^+$ the closer it must approach H^+. As $t \to \infty$ the ray γ becomes a null geodesic generator γ_H of H^+. We specify γ by its affine distance from γ_H along an ingoing null geodesic through H^+ (see fig.3a). The affine parameter on the ingoing null geodesic is U, so that
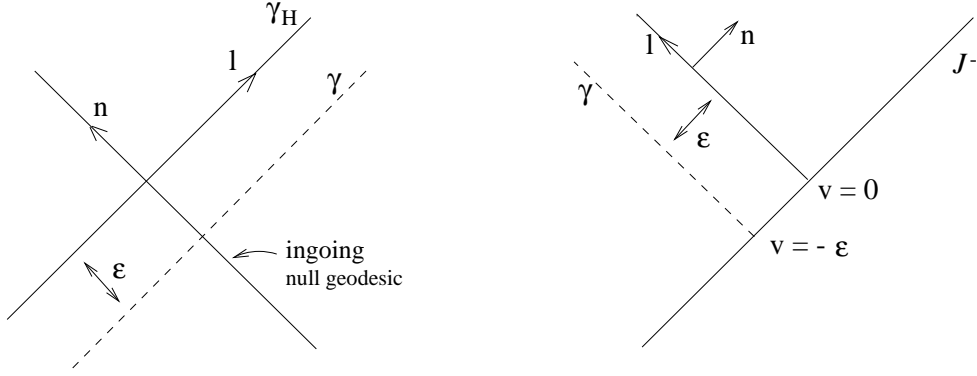
Figure 3:

according to (19)

$$U = -\epsilon \Rightarrow u = -\frac{1}{2\kappa}\log\epsilon, \quad f_\omega^{out} \sim \exp\left(\frac{i\omega}{\kappa}\log\epsilon\right).$$

This oscillates rapidly at later times t and this justifies the geometric optics approximation. Now we must match f_ω^{out} onto a solution near $\mathcal{J}^-$. In our approximation we just need to parallel-transport n and l along the continuation of γ_H back to $\mathcal{J}^-$. We choose v such that this continuation meets $\mathcal{J}^-$ at $v = 0$. The continuation of γ will meet $\mathcal{J}^-$ at an affine distance ϵ along an outgoing null geodesic on $\mathcal{J}^-$. Since $ds^2 = 4dudv + \dots$ on $\mathcal{J}^-$ the coordinate $2v$ is the affine parameter measuring this distance, so $2v = -\epsilon$ on γ and

$$f_\omega \sim \exp\left(\frac{i\omega}{\kappa}\log(-2v)\right)\theta(-v),$$

where we took into account, that null rays with $v > 0$ do not reach $\mathcal{J}^+$. Now we take the Fourier transform

$$\tilde{f}_\omega(\omega') = \int_{-\infty}^{0} e^{2i\omega'v} f_\omega(v)\, dv = \frac{1}{2}\int_0^\infty \tilde{v}^{i\omega/\kappa} e^{-i\omega'\tilde{v}} d\tilde{v}, \quad \omega' > 0.$$

Using (11) one sees, that

$$\tilde{f}_\omega(\omega') = -e^{\pi\omega/\kappa}\tilde{f}_\omega(-\omega') \quad \text{for} \quad \omega' > 0.$$

It follows, that a mode of positive frequency ω on $\mathcal{J}^+$ matches onto mixed positive and negative frequency modes on $\mathcal{J}^-$. We see, that the Bogolubov coefficients are related by $\beta_{ij} = -\exp(-\pi\omega_i/\kappa)\alpha_{ij}$. From the Bogolubov relations (4) one then gets

25

$$\left(\beta\beta^\dagger\right)_{ii} = \frac{1}{e^{2\pi\omega_i/\kappa} - 1}. \tag{28}$$

For calculating the late time particle flux through $\mathcal{J}^+$ we need the inverse β-coefficients, $\beta' = -\beta^t$. One easily finds, that $\langle N_i \rangle_{\mathcal{J}^+} = (\beta'^\dagger \beta')_{ii} = (\beta\beta^\dagger)_{ii}$. This is the Planck-distribution at the Hawking temperature $T_H = \hbar\kappa/2\pi$.

The detailed form of the potential in (26) is irrelevant in the geometric optics approximation. But the incoming waves will partially scatter off the gravitational field (on the l-dependent potential V in (26)) to become a superposition of incoming and outgoing waves. The backscattering is a function of ω and the spectrum is not precisely Planckian. The total luminosity of the hole is given by

$$L = \frac{1}{2\pi} \sum_{l=0}^{\infty} (2l+1) \int\limits_{0}^{\infty} d\omega \; \omega \frac{\Gamma_{\omega l}}{e^{8\pi M\omega} - 1}. \tag{29}$$

A black hole is actually grey, not black. The dependency on the angular momentum (and spin) of the particles resides in the grey-body factor $\Gamma_{\omega l}$.

6   Back-reaction

The main effect of the quantum field will be a decrease of M at the rate at which energy is radiated to infinity by particle creation. Since the spacetime is static outside the collapsing matter, the expected energy current $J_\mu = \langle T_{\mu\nu}\rangle K^\nu$ is conserved in that region. The calculation showed, that there will be a steady nonzero flux F. In [21] the contribution of the different particle species to this flux has been determined. The contribution of massive particles of rest mass m is exponentially small if $m > \kappa$. Black holes of mass $M > 10^{17}$g can only emit neutrinos, photons and gravitons. Black holes of mass $5 \cdot 10^{14}$g$\leq M \leq 10^{17}$g can also emit electrons and positrons. Black holes of smaller mass can emit heavier particles. A non-rotating black hole emits almost as a body heated to the temperature

$$T[^0\text{K}] = \frac{\hbar\kappa}{2\pi c} = \frac{\hbar c^3}{8\pi GkM} \sim 10^{26} \frac{1}{M[\text{g}]}.$$

The deviation from thermal radiation is due to the frequency dependence of the penetration coefficient $\Gamma_{s\omega l}$. This coefficient is also strongly spin-dependent, $\Gamma_{s\omega l} \sim \omega^{2s+1}$. As spin increases, the contribution of particles to the radiation of a non-rotating black hole diminishes. The distribution of the radiated particles in different mass-intervals is shown in the following table:

M [g]	L [$\frac{\text{erg}}{\text{sec}}$]	particles radiated
$M > 10^{17}$	$3.5 \times 10^{12} \left(\frac{10^{17} g}{M}\right)^2$	81.4% $\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu$ 16.7% γ 1.9% g
$10^{17} > M > 5 \times 10^{14}$	$6.3 \times 10^{16} \left(\frac{10^{15} g}{M}\right)^2$	45% $\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu$ 9% γ 1% g 45% e^-, e^+
$10^{14} > M > 10^{13.5}$	$10^{19} \left(\frac{10^{14} g}{M}\right)^2$	48% $\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu$ 28% e^-, e^+ 11% γ 1% g 12% $N, \bar{N}$

The following formula describes the rate of mass loss

$$-\frac{dM}{dt} \sim 4 \cdot 10^{-5} f \cdot \left(\frac{m_{pl}}{M}\right)^2 \frac{m_{pl}}{t_{pl}} = 7.7 \cdot 10^{24} f \cdot \left(\frac{1}{M[\text{g}]}\right)^2 \frac{\text{g}}{\text{sec}} = \frac{\alpha}{M^2}. \tag{30}$$

The contributions of the (massless) particle species are encoded in $f(M)$. From Page we take

$$f = 1.02 h(\frac{1}{2}) + 0.42 h(1) + 0.05 h(2),$$

where $h(s)$ is the number or distinct polarisations of spin-s particles. The rate equation (30) is easily integrated to yield

$$M(t) = (M_0^3 - 3\alpha t)^{1/3},$$

We see that a black hole radiates all of its mass in a finite time $\tau \sim M_0^3/3\alpha$. Inserting for α yields

$$\tau \sim 10^{71} (\frac{M}{M_\odot})^3 \text{sec}.$$

If primordial black holes of mass $\sim 5 \cdot 10^{14}$g were produced in the early universe, they would be in the final stages of evaporation now. Primordial black hole of smaller mass would have already evaporated and contributed to the γ-ray background. See the review of Carr [22] for the possibility of observing quantum explosions of small black holes.

The magnitude of the Kruskal coordinate components of $\langle T_{\mu\nu} \rangle_H$ near the black hole are found to be of order $1/M^4$ in Planck units, as expected on dimensional grounds. Since the background curvature is of order $1/M^2$ the quantum field should only make a small correction to the structure of the black hole for $M \gg 1$, or $M \gg 10^{-5}$g.

7 Generalisations and Discussion

In the previous section we have studied the Hawking effect in the case of the Schwarzschild black hole. Lets us consider now different generalisations of this effect and its possible consequences.

Hawking radiation of rotating and charged holes. The *Kerr solution* has null-hypersurfaces at

$$r = r_\pm = M \pm \sqrt{M^2 - a^2},$$

where $a = J/M$, which are Killing horizons of the Killing fields

$$K_\pm = k + \Omega m = k + \left(\frac{a}{r_\pm^2 + a^2}\right)m \qquad k = \partial_t, \quad m = \partial_\phi,$$

with surface gravities

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2(r_\pm^2 + a^2)}.$$

For the extreme Kerr solution with $a^2 = M^2$ the surface gravity vanishes.

For a Schwarzschild hole the number of particles per unit time in the frequency range ω to $\omega + d\omega$ passing out through a surface of the sphere is

$$\frac{1}{e^{8\pi M\omega} - 1}\frac{d\omega}{2\pi}.$$

For a Kerr Black hole ω is replaced by $\omega - m\Omega$ in this formula, where m is the azimuthal quantum number of the spheroidal harmonics, and Ω is the angular speed of the event horizon. Hence, the Planck factor at J^+ becomes

$$\frac{1}{e^{2\pi(\omega - m\Omega)/\kappa} \pm 1}, \qquad +\,\text{fermions}, \quad -\,\text{bosons}.$$

The emission is stronger for positive m than for negative m. In the boson case the Planck factor becomes negative when $\omega < m\Omega$ and super-radiance occurs: the effect of radiation amplifies the incoming classical wave with positive m. The result admits the following interpretation: Consider a rotating black hole enclosed in a mirror-walled cavity. A scattering of a 'particle' in a super-radiant mode by the black hole increases the number of quanta. After reflection by the mirror, these quanta are again scattered on the black hole and their number increases again, and so on. No stationary equilibrium distribution is possible for such modes. However, if the size of the cavity is not too large, $r < 1/\Omega$,

then the super-radiative modes are absent and equilibrium is possible. A related effect is that the rotation of the hole enhances the emission of particles with higher spins.

For a charged hole with *Reissner-Nordström metric*

$$ds^2 = \alpha(r)dt^2 - \frac{1}{\alpha(r)}dr^2 + r^2 d\Omega^2, \qquad \alpha(r) = 1 - \frac{2M}{r} + \frac{q^2}{r^2}$$

the event horizon is at $r = r_+ = M + (M^2 - q^2)^{1/2}$ and the surface gravity is found to be

$$\kappa = \frac{1 - 16\pi^2 q^4/A^2}{4M},$$

where $A = 4\pi r_+^2$ is the area of the horizon. If follows that the presence of the charge depresses the temperature $kT_H = \kappa/2\pi$ of the hole. For an extremal hole with charge $q = M$ or with $a^2 = M^2$ the Hawking temperature is zero, whereas the area is not ($A = 4\pi M^2$ for the extreme Reissner-Nordström hole). In the laws of black hole thermodynamics the entropy of a black hole is $S = A/4$ and hence non-vanishing for extreme black holes. The formulation of the third law, namely that $S \to 0$ as $T \to 0$, is not true for extremal holes[4]. The failure of the formulation of the third law may not be too disturbing. There other quantum systems with a degenerate ground state for which it fails as well.

Loss of Quantum Coherence. Consider the behaviour of the quantum field in the spacetime of a collapse, fig.4 in which back-reaction effects are not taken into account. The state of the field at late times in region I, and in particular the flux of thermal particles reaching infinity, must be described by a density matrix. The particles which entered the black hole at early times are correlated with the particles in region I. There is always a loss of information whenever one performs an inclusive[5] measurement outside the horizon. Such entropy increase is common to all inclusive measurements in physics. Perhaps we can understand this situation better if we recall the resolution of the well-known question raised by Einstein, Podolsky and Rosen. A pure quantum state is defined globally; its coherence may extend over field variables located at well-separated points on a space-like surface.

Let us distinguish between the set of out-states corresponding to particles moving away from the black hole (the visible ones) and those falling into the hole (the invisible ones). When one calculates expectation values $\langle A \rangle = (\psi, A\psi)$ of operators A depending only on the creation and annihilation operators belonging to the visible modes, this expectation value can be written as $\langle A \rangle = \text{tr}\, \rho A$. In a Fock space construction one can derive an

---

[4]see the contribution of Claus Kiefer: the canonical theory of gravity predicts $S(T \to 0) = 0$, whereas superstring-theory predicts $S(T \to 0) = A/4$.
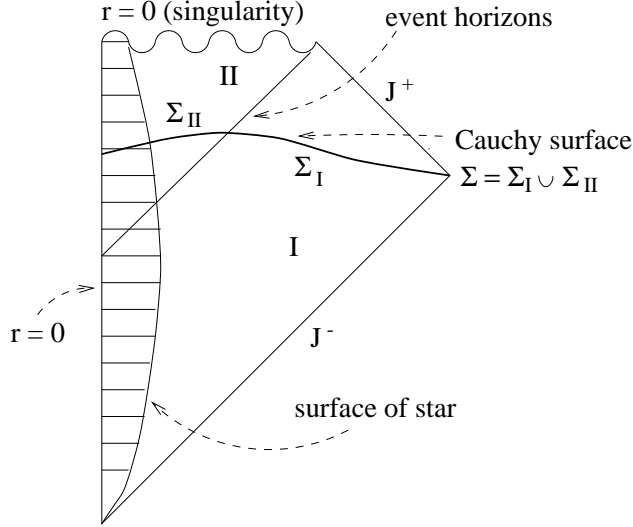
[5]not all commuting observables are measured

Figure 4: *A conformal diagram of the spacetime resulting from a complete collapse of a spherical body. The region II lies outside of the chronological past of J^+.*

explicit formula for the density matrix ρ in terms of the pure state ψ. Here it suffices to sketch the emergence of a mixed state from a pure one. let $\psi = \psi_i^I \otimes \psi_j^{II}$ be orthonormal pure states in the big Hilbert space $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$. Let us further assume that the observable A is the identity in $\mathcal{H}_{II}$. Then the expectation value

$$(\psi, A\psi) \quad \text{in the pure state} \quad \psi = \sum \alpha_i \psi_i^I \otimes \psi_i^{II}, \quad \sum |\alpha_i|^2 = 1$$

becomes

$$(\psi, A\psi) = \sum_{ij} \bar{\alpha}_i \alpha_j (\psi_i^I \otimes \psi_i^{II}, A\psi_j^I \otimes \psi_j^{II}) = \sum p_i (\psi_i^I, A\psi_i^I) = \text{tr}\,(\rho A),$$

where $p_i = |\alpha_i|^2$ and $\rho = \sum p_i P_i$. The P_i are the projectors on the states ψ_i^I. We have used, that the ψ_i^{II} are orthonormal. Thus, if we are only measuring observables in the region I outside of the black hole and ignore the information about the inside, then pure states become indeed mixed states. For a black hole $\alpha_i \sim \exp(-\pi\omega_i/\kappa)$ (see (28)) and ρ is the thermal state. As is also clear, for operators A which are not the identity in $\mathcal{H}_I$ the expectation values $(\psi, A\psi)$ cannot be written as $\text{tr}\,\rho A$.

Consider now the spacetime fig.5 in which back-reaction causes the black hole to 'evaporate'. The visible particles propagating to infinity can be described by a (thermal) density matrix. The particle creation and scattering will be described by a unitary S-matrix, provided that the invisible particles are represented in the 'out'-Hilbert space. What happens
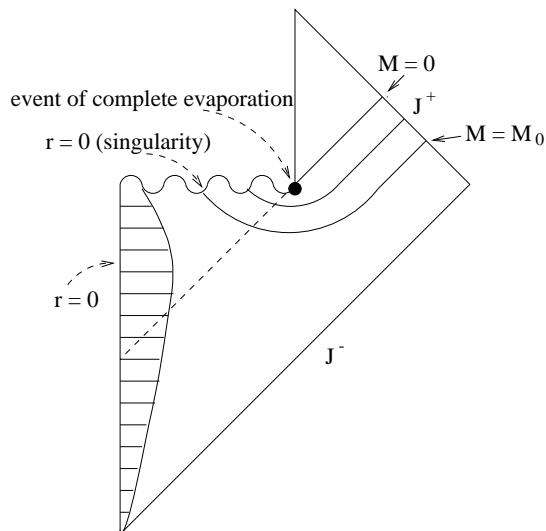
30

Figure 5: *A conformal diagram of a spacetime in which black hole formation and evaporation occurs. The contour labelled $M = 0$ lies at the (retarded) time corresponding to the final instant of evaporation.*

now when the black hole disappears from the spacetime? Apparently at late times, if one takes the 'out'-Hilbert space to be the Fock space associated with visible particles, the entire state of the field is mixed. Then one cannot describe particle creation and scattering by a unitary S-matrix, since an initial pure state evolved into a density matrix. This is the phenomenom of *loss of quantum coherence*. What are the possible ways out of this problem? A complete calculation including all back-reaction effects might resolve the issue, but even this is controversial, since the resolution very probably requires an understanding of the Planck scale physics. For example, QFT predicts that $T_{loc} \to \infty$ on the horizon of a black hole. This should not be believed when T reaches the Planck energy. The quantum aspects of gravity cannot be any longer ignored and this temperature is then of the order of the maximum (Hagedorn) temperature of string theory [6].

A natural approach to dealing with this situation is to consider 'toy models', for example in two spacetime dimensions, in which the semiclassical analysis could be done. In lower dimensions one adds a 'dilaton' field to render gravity non-trivial (this field naturally arises in low energy string theory). The resulting two-dimensional theories are dynamically nontrivial and mimic many features of four-dimensional general relativity: they possess black-hole solutions, Hawking radiation and there exist laws of black hole thermodynamics

---

[6]See the contribution of G. 't Hooft.

which are completely analogous to the laws in four dimensions. Callen et.al [25] studied the model

$$S = \frac{1}{2\pi} \int d^2x \sqrt{-g} \Big(e^{-2\sigma}[R + 4(\nabla\sigma)^2 + 4\lambda^2] + \frac{1}{2}(\nabla f)^2 \Big), \tag{31}$$

containing a metric field $g_{\mu\nu}$, a dilaton field σ and a matter field f. The Hawking radiation of the f-'particles' can be calculated the way we explained in our two-dimensional model calculations above. So far these model calculations have not resolved the problems with the final stage of the black hole evaporations (the problems are the same as those with the Liouville theory at strong-coupling). A further simplification of (31) has been discovered by Russo, Susskind and Thorlacius [26]. Rather recent calculations seem to indicate[7] that information is not destroyed, but slowly released as the black hole decays back to vacuum [27].

References

[1] Brunelli, R. and Fredenhagen, K. (1997), DESY-preprint 97-005.

[2] Bunch, T.S., (1981), Annals of Physics 131, 118.

[3] Parker, L. (1969), Phys. Rev. D 183, 1057

[4] Hawking, S. (1975), Commun. Math. Phys. 43, 199.

[5] Birrell, N.D. and Davies, P.C.W. (1982), 'Quantum Fields in Curved Space', CUP; Fulling, S.A. (1989), 'Aspects of Quantum Field Theory in Curves Spacetime', CUP; Wald, M.W. (1994), 'Quantum Theory in Curved Spacetime and Black Hole Thermodynamics', Chicago Lect. in Phys.; Novikov, I.D. and Frolov, V.P. (1989), 'Physics of Black Holes', Kluver Acad. Publ., London.; Audretsch, J. (1990), in 'Quantum Mechanics in Curved Spacetime', eds. Audretsch, and de Sabbata, Plenum, New York

[6] Geroch, R. (1979), J. Math. Phys. 11, 437.

[7] Hawking, S.W. and Ellis, G.F.R. (1973), ' The Large Scale Structure of Spacetime', CUP.; Townsend, P.K. (1997), gr-qc/9707012.

[8] Ashtekar, A. and Magnon, A. (1975), Proc. Roy. Soc. London A346, 375.

[9] Kay, B. (1978), Commun. Math. Phys. 62, 55.

---

[7] see the contribution of C. Kiefer

[10] Unruh, W.G. (1976), Phys. Rev. D14, 870.

[11] Fulling, S.A., Sweeny, M., and Wald, R.M (1978), Commun. Math. Phys.', 63, 257.

[12] Tolman, R.C. and Ehrenfest, P. (1930), Phys. Rev. 36, 1791.

[13] Wald, R.M. (1984), Phys. Rev. D17, 1477.

[14] De Witt, B.S. (1975), Phys. Rep. C19, 297; Grib, A.A., Mamayev, S.G. and Mostepa-nenko, V.M. (1994), 'Vacuum Quantum Effects in Strong Fields', Friedman Laboratory Publishing, St. Petersburg; Christensen, S.M. (1978), Phys. Rev. D17, 946.

[15] Wipf, A. and Sachs, I., (1995), Annals of Physics 249, 380.

[16] Mukhanov, V., Wipf, A. and Zelnikov, A. (1994), Phys. Lett. B 332, 283.

[17] Kubo, R. (1957), Phys. Soc. Japan 12, 570; Martin, P.C. and Schwinger, J. (1959), Phys. Rev. 115, 1342.

[18] Christensen, S.M. and Fulling S.A., (1977), Phys. Rev. D15, 2088.

[19] Page, D.N. (1982), Phys. Rev. D25, 1499; Elster, T. (1983), Phys. Lett. A94, 205.

[20] Howard, K.W. and Candelas, P. (1984), Phys. Rev. Lett. 53, 403; Elster, T. (1984), Class. Quant. Grav. 1, 43.

[21] Page, D.N. (1977), Phys. Rev. D14, 1360.

[22] Carr, B.J. (1983), in M.A. Markov and P.C. West (eds.), Quantum Gravity, Plenum Press, N.Y., p. 337.

[23] Barvinsky, A.O. and Vilkovisky, G.A. (1990), Nucl. Phys. B333, 471; Barvinsky, A.O. and Vilkovisky, G.A. (1987), Nucl. Phys. B282, 163.

[24] Simkins, R.D. (1986), Massive Particle Emission from Schwarzschild Black Hole, Thesis, Pennsylvania State University.

[25] Callen, C.G., Giddings, S.B., Harvey, J.A. and Strominger, A. (1992), Phys. Rev. D45, 1005.

[26] Russo, J.G., Susskind, L., and Thorlacius, R. (1993), Phys. Rev. D47, 533.

[27] Polchinsky. J. and Strominger, A. (1994), Phy. Rev. D50, 7403.