# Quantum Principle of Relativity and The Renormalizable Gravity

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We develop a purely quantum theory based on the novel principle of relativity, termed the quantum principle of relativity, without introducing general relativity. We demonstrate that the essence of the principle of relativity can be naturally extended into the quantum realm, maintaining the identical structures of active and passive transformations. By employing this principle, we show that gravitational effects are naturally incorporated into the renormalizable theory, with general relativity emerging in the classical regime. We derive graviton propagators and provide several examples grounded in this novel theory.

# I. INTRODUCTION

to be used, at least apparently.

During the early 20-th century, two revolutionary theories arose in the physics society, which are general relativity (GR) [1–3] and quantum mechanics (QM) [4–10]. Both theories explain physical phenomena very accurately in the regime of which each theory aimed for. GR can describe the gravitational phenomena in a very large scale, while QM describe the world in a very small scale. Because there were attempts to unify two theories, one of the most successful theory was formulated in 1920s, called quantum field theory (QFT) [11–14], which is a quantum theory with special relativity [15].

However, it was found out to be extremely difficult to combine OFT and GR due to the lack of renormalizability [16, 17], and there is no standard theory up to these days. There were many attempts to solve this problem, and new theories were introduced such as loop gravity and supergravity [18], teleparallel gravity [19], etc. But most of the theories focused on introducing the gravity effect to QFT, not the philosophical part of GR. Only a few theories, such as loop quantum gravity, has a equivalent principle in the background by possessing (general) principle of relativity (PR), geometrically realized by the diffeomorphism invariance [16]. This approach was fruitful and it gave a new insight of understanding quantum gravity. In this paper, however, we introduce a slightly different approach to the diffeomorphism invariance for achieving the PR, where only a few papers focus on the PR itself in quantum theories [20, 21].

PR is one of the postulates mentioned when Einstein formulated special relativity [15]. The PR asserts that no special frame of reference exists, and it allows us to set frames for arbitrary observers. In the cited paper, only the special principle of relativity is considered. However, it was later developed into the general principle of relativity for GR. On the other hand, diffeomorphism invariance, or general invariance, is the idea that coordinates do not exist a priori in nature, framed in the language of Riemannian geometry [22]. In classical physics, PR and diffeomorphism invariance are almost equivalent, as setting the frame of reference means choosing the coordinate

The aim of this paper is to construct a pure quantum theory with PR, assuming physical reality underlying the quantum objects. We begin by distinguishing the position space in quantum theories from the Riemannian manifold, highlighting that the position is merely an eigenvalue of a position operator. We then explore the natural generalization of PR in the quantum realm by determining the correct approach to describe quantum version of active and passive transformations. Subsequently, we identify a constraint on the Lagrangian that adheres to the quantum principle of relativity. Ultimately, we frame the theory within the path integral formalism. Our investigation yields a renormalizable quantum theory with PR, providing a natural explanation for quantum gravity.

The structure of this paper is as follows. In Sec. II. we present a thought experiment about changing the frame in a quantum manner and argue that only a specifically constrained subspace of the Hilbert space is physical. The novel constraint, termed the condition for quantum relativity, is revealed to be crucial in maintaining the identical property of applying both active and passive quantum transformations. By demonstrating that such transformations in general maps a manifold to a manifold that is not isomorphic to the original one, we show that underlying Riemannian manifold does not exist prior to the construction of a Hilbert space, resulting in the inadequacy of diffeomorphism invariance. In Sec. III, we apply the quantum principle of relativity to QFT, examining its physical implications and defining graviton as the quantum fluctuation of the metric field, distinct from the metric itself. We elucidate the form of the partition function based on the path integral formalism. In Sec. IV, graviton propagators are derived from the partition function established in Sec. III. Renormalizability is discussed in Sec. V, followed by illustrative examples in Sec. VI. We conclude with a summary of our findings, and discuss the emerging concepts for future framework development in Sec. VII.

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## II. QUANTUM PRINCIPLE OF RELATIVITY

## A. Thought experiment

In the framework of general relativity, a local coordinate, denoted as x, is defined on spacetime to represent a specific point on a manifold. Given that spacetime possesses a physical reality in general relativity due to its interaction with matter, a point in spacetime is often regarded as a physical entity that is universally shared, even though its value is dependent on the local coordinate. Conversely, from the quantum perspective, the position x of a particle is discerned as a measured value after applying a position operator  $\hat{X}$ , showing one of the eigenvalues of  $\hat{X}$ . Given that the "measured outcome" represents what an observer perceives post-experiment, a position in quantum theory is more of a readout than an intrinsic physical entity.

Importantly, the construction of the quantum operator  $\hat{X}$  is contingent upon the measurement device utilized. In this paper, we refer to the device used to measure position as a "ruler". A captivating thought experiment arises when one considers promoting the ruler itself as a state vector in Hilbert space. This proposition seems natural, especially when considering that the rulers used in all physical experiments are made of an ordinary matter.

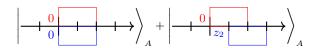
Let's consider three identical bar rulers, denoted as ruler A, ruler B, and ruler C, that can be used to measure the position of an object. Each ruler is marked with ticks to indicate position, and the spacings between these ticks are consistent. Within the framework of a Hilbert space, a single tick from a ruler can be selected and represented as a state vector as follows:

$$|\psi\rangle = |x\rangle_A \tag{1}$$

Here, x denotes the position value of the tick, while  $|\cdots\rangle_A$  indicates that the reading is taken from ruler A. It's crucial to note that  $|\psi\rangle_A$  must always be  $|x\rangle_A$ , since ruler A is interpreting its own ticks. Now, let's consider a entangled state of the rulers as described in Fig. 1, which is expressed as below.

$$|\psi_A, \psi_B, \psi_C\rangle_1 = |x\rangle_A(|y_1\rangle|z_1\rangle + |y_2\rangle|z_2\rangle)_A \tag{2}$$

If there is no specific, representative ruler in this universe, a condition that must hold true when the PR is valid, one should be able to switch the readout reference from ruler A to ruler B or ruler C, since the rulers are identical. Let's consider a case where the reference is switched to ruler C, and examine how the states of ruler A (the one being switched), ruler B (the one not involved in the switching), and ruler C (the one that is to be switched to) are affected. This switch is achieved by simply reading the values from the ticks of ruler C. Therefore, if we have  $|\psi_C\rangle = |z_3\rangle_C$  by interpreting  $|\psi_C\rangle$  with ruler C itself, it becomes possible to express the full quantum state of all rulers using ruler C instead of using



(a) Ruler A and ruler C in frame A



(b) Ruler A and ruler C in frame C

FIG. 1: Graphical representation of Eq. 2 (above) and Eq. 3 (below), assuming  $x=z_1=0$  for simplicity. Ruler B is not included in the figure, as it is not involved in the ruler swapping process. Both cases represent the same physical situation; only the coordinate values differ, while the relative distances are preserved.

ruler A, by keeping in mind that the spacing between the ticks are equidistant.

$$|\psi_A, \psi_B, \psi_C\rangle_2 = (|x + \Delta_1\rangle |y_1 + \Delta_1\rangle + |x + \Delta_2\rangle |y_2 + \Delta_2\rangle)_C |z_3\rangle_C$$
(3)

We have  $\Delta_1 = z_3 - z_1$  and  $\Delta_2 = z_3 - z_2$ . When comparing Eq. 2 and Eq. 3, it's evident that both expressions describe the same physical situation. The sole distinction lies in the ruler being utilized, suggesting that Eq. 2 and Eq. 3 convey equivalent informational quality. From this observation, we introduce a quantum principle of relativity.

Quantum principle of relativity: The unified Hilbert space, denoted as  $\mathcal{H}$ , exists independent of the chosen ruler. A physical state is represented as a vector within this Hilbert space, while the particular ruler in use defines a position operator, not  $\mathcal{H}$ , by assigning appropriate eigenvalues (or readouts) to each corresponding basis state.

Using the aforementioned principle, one can introduce the unified Hilbert space and equate Eq. 2 with Eq. 3, because they indicate the same vector in  $\mathcal{H}$ :

$$|\psi_A, \psi_B, \psi_C\rangle_1 = |\psi_A, \psi_B, \psi_C\rangle_2 \tag{4}$$

For the above equation to hold true across all values of  $\Delta_1$ ,  $\Delta_2$ , x,  $y_1$ , and  $y_2$ , we consider a new linear operator,  $\hat{O}_{AC}$ .

$$\hat{O}_{AC} = |z_3; C\rangle \int dz \langle z; C| e^{-i\sum_{j \in A, B} \hat{P}_j(z_3 - z)}$$
 (5)

$$\equiv |z_3; C\rangle \int dz \langle z; C| e^{-i\hat{Q}_C(z_3 - z)}$$
 (6)

Here,  $\hat{P}_j$  represents the momentum operators, acting as Lie generators [10]. A newly defined operator  $\hat{Q}_C$  is a translator to every rulers except ruler C. Although the subscript C like  $|\cdots\rangle_C$  is missing in Eq. 6, values used in

 $\hat{O}_{AC}$  reference ruler C. The operator  $\hat{O}_{AC}$  incorporates the value  $z_3$  by the definition of ruler C, which remains constant.

Implementing  $\hat{O}_{AC}$ , the relationship presented in Eq. 4 enables one to deduce the natural transformation between rulers:

$$|\psi_A, \psi_B, \psi_C\rangle_1 = |x\rangle_A(|y_1\rangle|z_1\rangle + |y_2\rangle|z_2\rangle)_A \tag{7}$$

$$= |\psi_A, \psi_B, \psi_C\rangle_2 \tag{8}$$

$$= \hat{O}_{AC}|x\rangle_C(|y_1\rangle|z_1\rangle + |y_2\rangle|z_2\rangle)_C \quad (9)$$

Given that the form of the ket vectors in Eq. 7 and Eq. 9 is identical, and defining  $\hat{O}_{CA}$  similarly to  $\hat{O}_{AC}$ ,

$$\therefore |\Psi\rangle_A = \hat{O}_{AC}|\Psi\rangle_C, \ |\Psi\rangle_C = \hat{O}_{CA}|\Psi\rangle_A$$
 (10)

Several properties of the operator deserve attention. Firstly, while  $O_{AC}$  is not unitary, it exhibits idempotent behavior  $(\hat{O}_{AC}^2 = \hat{O}_{AC})$ . This implies that  $\hat{O}_{AC}$  acts as a projector on  $\mathcal{H}$ , given that idempotent operators are always diagonalizable and their eigenvalues are restricted to either 0 or 1. Secondly, it is shown from Eq. 10 that  $|\Psi\rangle_A$  and  $|\Psi\rangle_C$  are inter-transformable. Such behavior is feasible only if the rulers span the same subspace within  $\mathcal{H}$ , which is interesting since the operators are revealed as projectors. These results allow us to define  $\mathcal{H}_{phy}$ , a subspace projected by the transformation operator. Since projecting onto  $\mathcal{H}_{phy}$  is necessary for defining a position operator, we introduce an additional postulate: an observer can identify the complete physical state, similar to the approach in traditional quantum mechanics. Then it becomes clear that  $\mathcal{H}_{\rm phy}$  represents the physical world on

Finally, but most importantly,  $\hat{O}_{AC}$  absorbs right-acting  $\hat{U}(a)$ , where  $\hat{U}(a) \equiv \exp\left(-ia\sum_j \hat{P}_j\right)$  is the total translator.

$$\hat{O}_{AC}\hat{U}$$

$$= |z_3; C\rangle \int dz \langle z; C| e^{-i\hat{Q}_C(z_3 - z + a) - i\hat{P}_C a}$$
(11)

$$=|z_3;C\rangle \int dz \langle z-a;C|e^{-i\hat{Q}_C(z_3-z+a)}$$
 (12)

$$=\hat{O}_{AC} \tag{13}$$

Given our findings, we have  $|\Psi\rangle_A = \hat{O}_{AC}|\Psi\rangle_C = \hat{O}_{AC}\hat{U}|\Psi\rangle_C$ . This indicates that  $\hat{O}_{AC}$  maps both  $|\Psi\rangle_C$  and  $\hat{U}|\Psi\rangle_C$  to the identical physical state.

To investigate more deeply, let's introduce a subspace of  $\mathcal{H}$ , termed  $\mathcal{H}_{QR}$ , where  $\hat{U}|\Psi\rangle_{QR} = |\Psi\rangle_{QR}$  for any state  $|\Psi\rangle_{QR} \in \mathcal{H}_{QR}$ . Given the significance of the condition to define  $\mathcal{H}_{QR}$ , we term it the "condition for quantum relativity" or just the "QR condition":

$$\hat{P}|\Psi\rangle_{\rm QR} \equiv \sum_{j} \hat{P}_{j}|\Psi\rangle_{\rm QR} = 0 \tag{14}$$

Within  $\mathcal{H}_{QR}$ , the operator  $\hat{U}(a)$  is essentially the same as the identity operator  $\hat{I}$  for any value of a. Furthermore,

under the QR condition,  $\hat{O}_{AC}$  becomes simpler.

$$\hat{O}_{AC} = |z_3; C\rangle \int dz \langle z; C| e^{-i\hat{Q}_C(z_3 - z)} \hat{I}$$
(15)

$$=|z_3;C\rangle \int dz\langle z;C|e^{-i\hat{Q}_C(z_3-z)}\hat{U}(z-z_3) \quad (16)$$

$$= \int dz (\hat{I}|z_3; C) \langle z_3; C|\hat{I})$$
 (17)

$$= \int dz \left( \hat{U}(z) | z_3; C \rangle \langle z_3; C | \hat{U}^{\dagger}(z) \right) \tag{18}$$

$$=\hat{I} \tag{19}$$

A projector can only be equivalent to the identity operator if  $\mathcal{H}_{QR} \subseteq \mathcal{H}_{phy}$ . Also, using this newly found property, one may re-express the definition of  $\mathcal{H}_{QR}$  to:  $\hat{O}_{AC}\hat{U}|\Psi\rangle_{QR} = \hat{O}_{AC}|\Psi\rangle_{QR}$ . This expression leads us to the opposite relationship,  $\mathcal{H}_{phy} \subseteq \mathcal{H}_{QR}$ , according to Eq. 13. As a result, the QR condition is both a necessary and sufficient condition for  $\mathcal{H}_{phy}$ . Additionally, in the space of  $\mathcal{H}_{QR}$ , not only  $\hat{O}_{AC}$  but any transformation operator  $\hat{O}$  becomes the identity operator, which is trivial due to the symmetries between the rulers. This offers an universal viewpoint that is independent of the specific ruler in use.

## B. Discussions and the implications

There are key distinctions when compared to traditional quantum theories. Firstly, the information of the physical coordinate is fully contained in the state vector. This characteristic enables one to swap the ruler, which serves as the source of the coordinate, with the help of Eq. 6. From now on, we call this swap the "quantum coordinate transformation". Since the coordinate isn't grounded on a manifold but is solely constructed by the ticks on the ruler, we also introduce the term "quantum coordinate" to differentiate it from classical coordinates. When a quantum coordinate is established and the ruler isn't influenced by external physical entities, the ruler's state becomes stationary, and the ruler only acts as a manifold generator without having any physical interactions with others. For such cases, the ruler state can be taken out from the Hilbert space, apparently breaking the QR condition (instead the quantum coordinate transformation is blocked). Then the theory converges to the conventional quantum mechanics and the quantum coordinate aligns with the classical coordinate, by defining the manifold instead of explicitly expressing the ruler.

However, it is crucial to understand that this does not necessarily imply the existence of a classical coordinate transformation analogous to the quantum one. Some transformations, such as the one exemplified in Eq. 9, defy classical representation. As seen in Eq. 2, the tick on ruler A occupies a definite position denoted as x. Conversely, that very same tick, when referenced in Eq. 3,

disperses and superposes across two distinct positions. Such behavior violates the principles of one-to-one map between manifolds, thereby precluding a classical interpretation.

Secondly, the total momentum of the system must vanish, ensuring the ability to transform the quantum coordinate from one to another, without having logical inconsistencies. As a consequence, if one knows about the translation operator of the ruler in use, it essentially provides an ability to use the (ruler-in-use-excluded) system's total translator. This insight implies that the transformation of the quantum coordinate encompasses not just the rulers being swapped but all particles in the system. Given that the frame of a quantum system depends on a ruler that parametrizes the position values, this characteristic can be perceived as an effect of an active transformation. Viewing it from this angle, the QR condition is construed as a quantum counterpart of the active-passive transformation, wherein acting in both directions yields no difference.

The final distinguishing feature is the ability, through the QR condition, for a particular particle to be in rest to a position eigenstate,  $|x_0\rangle$ , by manipulating  $\hat{U}$  distinctly for each superposed state,  $|x\rangle$ .

$$|\Psi_{\text{particles}}\rangle_{\text{QR}} = \int dx A(x) \hat{I}|x\rangle |\Psi_{\text{else}}(x)\rangle$$

$$= \int dx A(x) \hat{U}(x_0 - x) |x\rangle |\Psi_{\text{else}}(x)\rangle$$

$$= |x_0\rangle |\Psi_{\text{else}}(x_0)\rangle \cdot \int dx A(x)$$
(22)

where A(x) is an arbitrary probability amplitude at a given position x. Using the above dedicated quantum coordinate, the notions of superposition and entanglement appear inapplicable to the state of the left particle, allowing one to treat it as a classical object. We coin the term "classical frame" to describe a collection of such particles. A typical example is the ruler that is in use, which lies in the classical frame by its definition.<sup>1</sup>

The existence of the classical frame resolves the fundamental question of how superposed objects perceive their own state: they can also sense themselves as existing at a single point, because such a frame of reference is always permissible to establish. In addition, this new perspective to classical frames sheds light on why measurement devices can be regarded as classical in quantum theories: we are used to employ our devices that align with a classical frame, at least approximately.

Conversely, the following situation is also interesting to discuss. When a ruler in use experiences an interaction with other particles, the quantum coordinate must be incessantly transformed to maintain the ruler within the classical frame. Such transformations cannot be addressed solely through classical coordinate transformations, if the superposition or the entanglement of the ruler state changes due to the interaction. This phenomenon can be analogized as a quantum variant of non-inertial frame.

### III. QUANTUM RELATIVITY IN QFT

# A. Canonical quantization

In this section we apply the QR condition to QFT with the requisite quantization. From Noether's theorem [23], we understand that the energy-stress (momentum) tensor in Minkowski spacetime is represented by:

$$(T_N)^{\mu}_{\nu} = \sum_{n} (T_N^{(n)})^{\mu}_{\nu} = \sum_{n} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi_n)} \partial_{\nu} \phi_n - \delta^{\mu}_{\nu} \mathcal{L}$$
 (23)

Here, n symbolizes the n-th quantum field. If all fields undergo canonical quantization [8], the below relation is followed:

$$\left[\phi_m(x), \frac{\partial \mathcal{L}(x^0, \vec{y})}{\partial (\partial_0 \phi_n)}\right] = i\delta_{mn}\delta^{(3)}(\vec{x} - \vec{y})$$
 (24)

 $[\cdot,\cdot]$  notation indicates super-commutator, which becomes anti-commutator when the both terms in the bracket are fermionic, otherwise it is just a commutator. It's important to note that this relationship holds true only when  $x^0=y^0$ , given that quantum operators are pinpointed on a specific time slice. Now:

$$[P_{\nu\neq0}(y^{0}),\phi_{m}(y)]$$

$$= \left[\sum_{n} \int d^{3}\vec{x} \frac{\partial \mathcal{L}(y^{0},\vec{x})}{\partial(\partial_{0}\phi_{n})} \partial_{\nu}\phi_{n},\phi_{m}(y)\right]$$

$$= \int d^{3}\vec{x} \sum_{n} [\Pi_{n}(y^{0},\vec{x})\partial_{\nu}\phi_{n}(y^{0},\vec{x}),\phi_{m}(y)]$$

$$= \int d^{3}\vec{x} \sum_{n} [\Pi_{n}(y^{0},\vec{x}),\phi_{m}(y)] \partial_{\nu}\phi_{n}(y^{0},\vec{x})$$

$$= -i\partial_{\nu}\phi_{m}(y)$$

$$(25)$$

$$= -i\partial_{\nu}\phi_{m}(y)$$

$$(26)$$

While  $P_{\nu}(y^0)$  is originally defined through the spatial integration of the net energy-stress tensor, it is also shown to act as a generator of the total translation. In situations where  $\nu=0$ , however, the uncertainty principle leads to a non-commutativity between  $\partial_0\phi_m$  and  $\phi_m$ , which challenges the above derivation. Fortunately, Schrödinger's equation [9] compensates for this, ensuring that Eq. 28 remains valid even for  $\nu=0$ .

$$[H(y^{0}), \phi_{m}(y)] = \int d^{3}\vec{x}[(T_{N})_{0}^{0}(y^{0}, \vec{x}), \phi_{m}(y)]$$
(29)  
=  $[E(y^{0}), \phi_{m}(y)] = -i\partial_{0}\phi_{m}(y)$  (30)

<sup>&</sup>lt;sup>1</sup> It's important to note that just because particles within a classical frame are in position eigenstates, it doesn't imply they remain stationary. Consider a particle with a state defined as  $|x_0 + vt\rangle$ , where v is a relative velocity in the frame.

It's worth to note that the energy operator E is also time-dependent to mark the chosen time slice.

Merging the above results, it becomes evident that  $\exp(-iP_{\nu}a^{\nu})$  functions as the total translator. With this background, the QR condition within the context of QFT can be expressed as:

$$P_{\nu}(x^{0}) = \int (T_{N})_{\nu}^{0}(x^{0}, \vec{x})d^{3}\vec{x} = 0$$
 (31)

The problem is that this universe is a boring place, as the conventional Hamiltonian in QFT is positive-definite and the only possible solution is a vacuum. Inspired by  $\delta S_{\rm QFT}/\delta g_{\mu\nu} = -T^{\mu\nu}$  in Minkowskian QFT (where  $S_{\rm QFT}$  is an action and  $T^{\mu\nu}$  is the Belinfante–Rosenfeld tensor [24–26]) and  $\delta S_{\rm GR}/\delta g_{\mu\nu}=0$  in general relativity (where  $S_{\rm GR}$  is a sum of matter action and Einstein–Hilbert action [27–29]), we overcome this problem by promoting a metric  $g_{\mu\nu}$  to a bosonic quantum field, and including it in Eq. 23 to add a negative energy part to the system. Although there exists a total derivative difference between  $(T_N)^{\mu\nu}$  and  $T^{\mu\nu}$  for fermions, we heuristically force  $(T_N)^{\mu\nu} \to T^{\mu\nu}$  to Eq. 28 and Eq. 31.

For theories with dynamic metrics, assuming a Minkowskian spacetime is no longer trivial. To effectively manage the theory, the conditions need to be generalized to the tensor forms.

$$P_{\nu}(x^{0}) = 0 \to P_{\nu}^{\mu}(x^{\perp}) \equiv \int_{V} T_{\nu}^{\mu}(x^{\perp}, \vec{x}) d^{3}\vec{x} = 0$$
 (32)

 $x^{\perp}$  represents time-like coordinate value, that is normal to a space-like hypersurface V (Eq. 32 must hold for any arbitrary V). Furthermore, to naturally generalize Eq. 28 to a tensor form, the following condition should be met for each component of  $P_{\mu}^{\mu}$ .

$$[P_{\nu}^{(n)\mu}(x^{\perp}), \phi_m(x)] = -i\delta_{mn}(\vec{e}^{\mu} \cdot \vec{E}_{\perp})\nabla_{\nu}\phi_n(x) \quad (33)$$

Here,  $\nabla_{\mu}$  denotes the external covariant derivative, and  $\vec{e}^{\,\mu}$  and  $\vec{E}_{\nu}$  are the contravariant and covariant basis vectors, respectively. n is used to distinguish the quantum fields (the same n used in  $T^{(n)}$ ). By using the metric compatibility condition,  $\nabla g_{\mu\nu} = 0$ , a particular case of Eq. 33 for  $g_{\alpha\beta}$  is expressed as below.

$$[P_{\nu}^{(n=g)\mu}(x^{\perp}), g_{\alpha\beta}(x)] = 0$$
 (34)

To ensure this holds for any arbitrary  $x^{\perp}$ , the spatial locality is introduced, further refining the condition of Eq. 34.

$$[T_{\nu}^{(n=g)\mu}(x), g_{\alpha\beta}(y)] = 0$$
 (35)

According to Eq. 24 and Eq. 28, it is also true that:

$$[T_{\nu\neq 0}^{(n\neq g)\mu=0}(x), g_{\alpha\beta}(y)] = 0$$
 (36)

By extending Eq. 36 to arbitrary  $\mu$  and  $\nu$  for  $T_{\nu}^{(n\neq g)\mu}$  to be a tensor, and combining this with Eq. 35, one arrives

at a result that the total energy-stress tensor,  $T^{\mu}_{\nu}$ , must also commute with  $g_{\alpha\beta}$ .

$$[T^{\mu}_{\nu}(x), g_{\alpha\beta}(y)] = 0 \tag{37}$$

Given that both  $T_{\mu\nu} = T^{\alpha}_{\nu} g_{\alpha\mu}$  and  $g_{\mu\nu}$  are rank 2 symmetric tensors and the operators commute, the eigenstates of the operators precisely align. This opens up a new possibility of configuring the fundamental field instead of  $g_{\alpha\beta}$ , which could be  $T_{\mu\nu}$ , or even  $T^{\mu}_{\nu}$ .

#### B. Path integral formalism

By employing the path integral formalism [30–33], the QR condition can be straightforwardly introduced in the theory. To naturally utilize a projector  $|P^{\mu}_{\nu}=0\rangle\langle P^{\mu}_{\nu}=0|=\int DT^{\mu}_{\nu}\delta(P^{\mu}_{\nu})$  to partition function, we transfer all degrees of freedom from  $g_{\alpha\beta}$  to  $T^{\mu}_{\nu}$  and consider it as a fundamental field. Consequently, the QR condition transforms into a constraint for the new field  $T^{\mu}_{\nu}$ , and takes the form of the partition function like below:

$$Z[J] = \int D\phi_n DT^{\mu}_{\nu} \Big( \prod_i \delta \left( P^{\mu}_{\nu}(x_j^{\perp}) \right) \Big) e^{i\mathcal{S}}$$
 (38)

$$= \int D\phi_n DT^{\mu}_{\nu} \Big( \prod_j \delta \Big( \int_j T^{\mu}_{\nu} d^3 \vec{x} \Big) \Big) e^{i\mathcal{S}}$$
 (39)

where j is a time-slice index, while the source term is included in  $\mathcal{S}$ . Let's change the field variable from  $T^{\mu}_{\nu}$  to  $g_{\alpha\beta}$ , since  $g_{\alpha\beta}$  is more intuitive object.

$$Z[J] = \int D\phi_n Dg_{\alpha\beta} \Big( \prod_i \delta \Big( \int_j T^{\mu}_{\nu} d^3 \vec{x} \Big) \Big) \Big| \frac{\partial T^{\mu}_{\nu}}{\partial g_{\alpha\beta}} \Big| e^{i\mathcal{S}} \quad (40)$$

We limit our discussion to cases when the quantum system has a finite hypervolume: the system has well-known initial condition at the temporal boundary, and the spatial boundaries are fully analyzed at each time slices. We further constraint our discussion that the objects outside the system boundaries are entirely within a classical frame by a properly defined ruler. Under these conditions, which is actually the usual case of existing quantum experimental setups (see Fig. 2),  $g_{\alpha\beta}$  is completely determined across spacetime by the boundary conditions, given  $\nabla g_{\alpha\beta} = 0$ . For such cases, it is better to prepare various partition functions with differing metrics, and introduce a density function,  $\rho(g_{\mu\nu})$ , that integrates the individual partition functions.

$$Z_g[J] = \int \left(\prod D\phi_n\right) \det \left|\frac{\partial T^{\mu\nu}}{\partial g_{\alpha\beta}}\right| e^{iS}$$
 (41)

and,

$$Z = \int Dg_{\mu\nu} \,\rho(g_{\mu\nu}) Z_g \prod_i \delta\left(\int_j (\tilde{T}^{\mu}_{\nu} + T^{(g)\mu}_{\nu}) d^3\vec{x}\right) \quad (42)$$

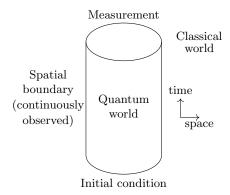


FIG. 2: Diagram of the quantum experiment in spacetime. The quantum system is assumed to be wellregulated within the experimental setup, ensuring a finite volume despite its vast size relative to the quantum scale. The system's spatial boundary, where the measurement devices are located, is constantly monitored to observe particles escaping from it. The experiment begins by identifying the initial conditions of the fields, and ends with the measurement of the fields across spacetime.

Here,  $\tilde{T}$  represents the total energy-stress tensor, excluding the contribution from the metric field. The determinant term in Eq. 41 can be evaluated by introducing fermionic ghost tensor field,  $\lambda_{\alpha\beta}$  and  $\bar{\lambda}_{\mu\nu}$ .

$$\det \left| \frac{\partial T^{\mu\nu}}{\partial g_{\alpha\beta}} \right| = \int D\lambda D\bar{\lambda} e^{-i\int d^4x \bar{\lambda}_{\mu\nu} \left( \frac{\delta^2 \mathcal{S}[J=0]}{\delta g_{\alpha\beta} \delta g_{\mu\nu}} \right) \lambda_{\alpha\beta}}$$
(43)

According to Eq. 42, the metric field is highly constrained by the delta function. Notably, when  $T_{\nu}^{(g)\mu}$  is identified with the Einstein tensor with a negative sign, the metric's response to matter matches precisely with the behavior of GR in the integrated form. Considering this together with the fact that various metrics can only be superposed under the control of  $\rho(g_{\mu\nu})$ , one can see that the metric serves more as a control variable in the classical world than as an independent variable in the quantum world. Yet, quantum fluctuations of spacetime still exist due to the Jacobian from the measure term, expressed as the interactions with the ghost field (see Eq. 43). In this context, instead of the metric itself, we propose naming the ghost field, the "graviton".

# C. Derivation of the free graviton Lagrangian

Although there appear to be 16 degrees of freedom for  $\lambda_{\mu\nu}$ , the true degrees of freedom are reduced to 10 due to the symmetric property of  $g_{\mu\nu}$ . To address this constraint, we identify  $\lambda_{\mu\nu}$  with  $\lambda_{\nu\mu}$ . Let's derive the free Lagrangian for the graviton when the Einstein-Hilbert

action is adopted.

$$-\mathcal{L}_{\lambda} = \bar{\lambda}_{\mu\nu} \left( \frac{\delta^2 \mathcal{S}_{\text{EH}}}{\delta g_{\alpha\beta} \delta g_{\mu\nu}} \right) \lambda_{\alpha\beta} \tag{44}$$

$$= \bar{\lambda}_{\mu\nu} \frac{\delta}{\delta g_{\alpha\beta}} \left( \frac{\sqrt{-g}}{2\kappa} \left( G^{\mu\nu} + g^{\mu\nu} \Lambda \right) \right) \lambda_{\alpha\beta}$$
 (45)

g is the determinant of  $g_{\mu\nu}$ ,  $\kappa=8\pi G$  is the Einstein gravitational constant,  $G^{\mu\nu}=R^{\mu\nu}-g^{\mu\nu}R/2$  is the Einstein tensor,  $R^{\mu\nu}$  is the Ricci tensor, and  $\Lambda$  is the cosmological constant. The above equation can be rearranged like below.

$$-\frac{2\kappa\mathcal{L}_{\lambda}}{\sqrt{-g}}$$

$$= \bar{\lambda}_{\mu\nu}\frac{\delta}{\delta g_{\alpha\beta}} \left(G^{\mu\nu} + g^{\mu\nu}\Lambda\right)\lambda_{\alpha\beta} + \frac{1}{2}G^{\mu\nu}\bar{\lambda}_{\mu\nu}\lambda$$

$$\equiv \bar{\lambda}_{\mu\nu}X^{\alpha\beta\mu\nu}\lambda_{\alpha\beta} - \frac{1}{2}\bar{\lambda}X^{\alpha\beta}\lambda_{\alpha\beta} + Y^{\alpha\beta\mu\nu}\bar{\lambda}_{\mu\nu}\lambda_{\alpha\beta}$$
(47)

Here,  $\bar{\lambda}$  and  $\lambda$  are the traces of the graviton field. Also,  $X^{\alpha\beta\mu\nu} = \delta R^{\mu\nu}/\delta g_{\alpha\beta}$ ,  $X^{\alpha\beta} = X^{\alpha\beta\mu\nu}g_{\mu\nu}$  are newly defined for simpler expression. The rest terms are collected to define  $Y^{\alpha\beta\mu\nu}$ .

$$Y^{\alpha\beta\mu\nu}\bar{\lambda}_{\mu\nu}\lambda_{\alpha\beta}$$

$$= \left(\frac{R}{2} - \Lambda\right)\bar{\lambda}_{\mu\nu}\lambda^{\mu\nu} - \frac{1}{2}R^{\alpha\beta}\bar{\lambda}\lambda_{\alpha\beta} + \frac{1}{2}G^{\mu\nu}\bar{\lambda}_{\mu\nu}\lambda$$

$$= \left(\frac{R}{2} - \Lambda\right)\left(\bar{\lambda}_{\mu\nu}\lambda^{\mu\nu} - \frac{\bar{\lambda}\lambda}{2}\right) + \frac{R^{\alpha\beta}}{2}(\bar{\lambda}_{\alpha\beta}\lambda - \bar{\lambda}\lambda_{\alpha\beta}) \tag{49}$$

Meanwhile,  $X^{\alpha\beta\mu\nu}$  terms are explicitly expressed as follows.

$$(X^{\alpha\beta\mu\nu} \text{ terms}) = (\bar{\lambda}_{\mu\nu} - \frac{1}{2}\bar{\lambda}g_{\mu\nu})\frac{\delta}{\delta g_{\alpha\beta}}(g^{\mu\rho}g^{\nu\sigma}R_{\rho\sigma})\lambda_{\alpha\beta}$$

$$(50)$$

$$=R^{\alpha\beta}(\bar{\lambda}\lambda_{\alpha\beta}-2\bar{\lambda}_{\mu\alpha}\lambda_{\beta}^{\mu})+(\bar{\lambda}^{\mu\nu}-\frac{\bar{\lambda}g^{\mu\nu}}{2})\frac{\delta R_{\mu\nu}}{\delta g_{\alpha\beta}}\lambda_{\alpha\beta} (51)$$

To find  $(\delta R_{\mu\nu}/\delta g_{\alpha\beta})\lambda_{\alpha\beta}$ ,  $(\delta \Gamma_{\mu\nu}^{\gamma}/\delta g_{\alpha\beta})\lambda_{\alpha\beta}$  is firstly explored as a building block  $(\Gamma_{\mu\nu}^{\gamma})$  are the Christoffel symbols).

$$\frac{\delta\Gamma_{\mu\nu}^{\gamma}}{\delta g_{\alpha\beta}} \lambda_{\alpha\beta} 
= \frac{\delta}{\delta g_{\alpha\beta}} \left( \frac{g^{\gamma\delta}}{2} (\partial_{\mu} g_{\nu\delta} + \partial_{\nu} g_{\mu\delta} - \partial_{\delta} g_{\mu\nu}) \right) \lambda_{\alpha\beta} 
= -\lambda_{\delta}^{\gamma} \Gamma_{\mu\nu}^{\delta} + \frac{g^{\gamma\delta}}{2} (\partial_{\mu} \lambda_{\nu\delta} + \partial_{\nu} \lambda_{\mu\delta} - \partial_{\delta} \lambda_{\mu\nu})$$
(53)  

$$= \frac{1}{2} (\nabla_{\mu} \lambda_{\nu}^{\gamma} + \nabla_{\nu} \lambda_{\mu}^{\gamma} - \nabla^{\gamma} \lambda_{\mu\nu})$$
(54)

Using the above it becomes obvious that:

$$2\frac{\delta}{\delta g_{\alpha\beta}}(\partial_{\gamma}\Gamma^{\gamma}_{\mu\nu})\lambda_{\alpha\beta}$$

$$=\nabla_{\gamma}(\nabla_{\mu}\lambda^{\gamma}_{\nu} + \nabla_{\nu}\lambda^{\gamma}_{\mu} - \nabla^{\gamma}\lambda_{\mu\nu})$$

$$+(\Gamma^{\delta}_{\gamma\mu}\nabla_{\delta}\lambda^{\gamma}_{\nu} + \Gamma^{\delta}_{\gamma\nu}\nabla_{\mu}\lambda^{\gamma}_{\delta} - \Gamma^{\gamma}_{\gamma\delta}\nabla_{\mu}\lambda^{\delta}_{\nu}) + (\mu \leftrightarrow \nu)$$

$$+(\Gamma^{\gamma}_{\gamma\delta}\nabla^{\delta}\lambda_{\mu\nu} - \Gamma^{\delta}_{\gamma\mu}\nabla^{\gamma}\lambda_{\delta\nu} - \Gamma^{\delta}_{\gamma\nu}\nabla^{\gamma}\lambda_{\delta\mu})$$
(55)

and,

$$-2\frac{\delta}{\delta q_{\alpha\beta}}(\partial_{\nu}\Gamma^{\gamma}_{\gamma\mu})\lambda_{\alpha\beta} = -\nabla_{\nu}\nabla_{\mu}\lambda - \Gamma^{\gamma}_{\mu\nu}\nabla_{\gamma}\lambda \qquad (56)$$

and also,

$$2\frac{\delta}{\delta g_{\alpha\beta}} (\Gamma_{\gamma\delta}^{\gamma} \Gamma_{\mu\nu}^{\delta} - \Gamma_{\mu\gamma}^{\delta} \Gamma_{\delta\nu}^{\gamma}) \lambda_{\alpha\beta}$$

$$= \Gamma_{\mu\nu}^{\gamma} \nabla_{\gamma} \lambda + \Gamma_{\gamma\delta}^{\gamma} (\nabla_{\mu} \lambda_{\nu}^{\delta} + \nabla_{\nu} \lambda_{\mu}^{\delta} - \nabla^{\delta} \lambda_{\mu\nu})$$

$$+ (-\Gamma_{\mu\gamma}^{\delta} (\nabla_{\delta} \lambda_{\nu}^{\gamma} + \nabla_{\nu} \lambda_{\delta}^{\gamma} - \nabla^{\gamma} \lambda_{\nu\delta})) + (\mu \leftrightarrow \nu)$$
(57)

Combining them all, one finds the desired form.

$$\frac{\delta R_{\mu\nu}}{\delta g_{\alpha\beta}} \lambda_{\alpha\beta} 
= \frac{\delta}{\delta g_{\alpha\beta}} (\partial_{\gamma} \Gamma^{\gamma}_{\mu\nu} - \partial_{\nu} \Gamma^{\gamma}_{\gamma\mu} + \Gamma^{\gamma}_{\gamma\delta} \Gamma^{\delta}_{\mu\nu} - \Gamma^{\delta}_{\mu\gamma} \Gamma^{\gamma}_{\delta\nu}) \lambda_{\alpha\beta} 
= \frac{1}{2} (\nabla_{\gamma} \nabla_{\mu} \lambda^{\gamma}_{\nu} + \nabla_{\gamma} \nabla_{\nu} \lambda^{\gamma}_{\mu} - \nabla_{\gamma} \nabla^{\gamma} \lambda_{\mu\nu} - \nabla_{\nu} \nabla_{\mu} \lambda) \quad (59)$$

From Eq. 59, the exact form for the kinetic terms in Eq. 51 is found.

(Kinetic terms)

$$= \frac{1}{2} \nabla^{\gamma} \bar{\lambda}^{\mu\nu} \nabla_{\gamma} \lambda_{\mu\nu} - \nabla^{\gamma} \bar{\lambda}^{\mu\nu} \nabla_{\mu} \lambda_{\gamma\nu}$$

$$+ \frac{1}{2} \nabla_{\nu} \bar{\lambda}^{\mu\nu} \nabla_{\mu} \lambda + \frac{1}{2} \nabla_{\nu} \bar{\lambda} \nabla_{\mu} \lambda^{\mu\nu} - \frac{1}{2} \nabla_{\gamma} \bar{\lambda} \nabla^{\gamma} \lambda$$

$$(60)$$

The second term can be rewritten as follows (vanishing total derivatives are used).

$$-\nabla^{\gamma}\bar{\lambda}^{\mu\nu}\nabla_{\mu}\lambda_{\gamma\nu} = (\nabla_{\mu}\nabla_{\gamma}\bar{\lambda}^{\mu\nu})\lambda_{\nu}^{\gamma} \tag{61}$$

$$= (\nabla_{\gamma} \nabla_{\mu} \bar{\lambda}^{\mu\nu}) \lambda_{\nu}^{\gamma} + (\nabla_{[\mu} \nabla_{\gamma]} \bar{\lambda}^{\mu\nu}) \lambda_{\nu}^{\gamma}$$
 (62)

$$= -\nabla_{\mu}\bar{\lambda}^{\mu\nu}\nabla_{\gamma}\lambda_{\nu}^{\gamma} + R_{\mu\nu}\bar{\lambda}^{\mu\gamma}\lambda_{\gamma}^{\nu} + R^{\nu}_{\ \alpha\mu\gamma}\bar{\lambda}^{\alpha\mu}\lambda_{\nu}^{\gamma} \quad (63)$$

 $R^{\nu}_{\alpha\mu\gamma}$  is the Riemann tensor. By applying Eq. 63 to the half of  $\nabla^{\gamma}\bar{\lambda}^{\mu\nu}\nabla_{\mu}\lambda_{\gamma\nu}$  in Eq. 60, the kinetic terms can be expressed with the more symmetric manner. Together with Eqs. 47, 49, 51, and 59, the full expression for the graviton's free Lagrangian,  $\mathcal{L}_{\lambda}$ , is derived.

$$4\kappa \frac{\mathcal{L}_{\lambda}}{\sqrt{-g}} = 4K_{\lambda} - (R - 2\Lambda)(\bar{\lambda}_{\mu\nu}\lambda^{\mu\nu} - \frac{1}{2}\bar{\lambda}\lambda)$$

$$- R^{\mu\nu}(\bar{\lambda}\lambda_{\mu\nu} + \bar{\lambda}_{\mu\nu}\lambda - 3\bar{\lambda}_{\mu}^{\gamma}\lambda_{\gamma\nu}) - R^{\nu}_{\alpha\mu\gamma}\bar{\lambda}^{\alpha\mu}\lambda_{\nu}^{\gamma}$$
(64)

where.

$$K_{\lambda} = -\frac{1}{4} \nabla^{[\gamma} \bar{\lambda}^{\mu]\nu} \nabla_{[\gamma} \lambda_{\nu]\mu} + \frac{1}{4} \nabla^{[\gamma} \bar{\lambda}^{\mu]}_{\mu} \nabla_{[\gamma} \lambda^{\nu]}_{\nu]}$$

$$= -\frac{1}{4} g_{\alpha\mu} \nabla^{[\gamma} \bar{\lambda}^{\mu][\nu} \nabla_{[\gamma} \lambda^{\alpha]}_{\nu]}$$

$$(66)$$

is the kinetic part of the graviton.

#### IV. GRAVITON PROPAGATORS

# A. Perturbation theory with a cut-off scale

In this section, we determine the graviton propagators for QR free scalar theory as a toy model. In this model, we introduce a free real scalar field,  $\phi$ , to an approximately flat spacetime  $(R^{\rho}_{\sigma\mu\nu} \approx 0)$ . Under such conditions, it's possible to define a local coordinate that is Minkowskian, denoted as  $g_{\mu\nu} \approx \eta_{\mu\nu}$ . Then the Lagrangian of the free scalar field is expressed like below (we're using the (-,+,+,+) metric convention here).

$$\frac{\mathcal{L}_{\phi}}{\sqrt{-g}} = -\frac{1}{2} (\partial^{\rho} \phi \partial_{\rho} \phi + m^2 \phi^2) \tag{67}$$

Due to the different sign convention, the sign of the kinetic term is inverted from the usual expression of QFT. Referring to Eqs. 41, 43, and 66, the free and self-interacting terms for the scalar field and the graviton are expressed as:

$$\mathcal{L}_{\text{free}} \approx -\frac{1}{4\kappa} \eta_{\alpha\mu} \partial^{[\gamma} \bar{\lambda}^{\mu][\nu} \partial_{[\gamma} \lambda_{\nu]}^{\alpha]} + \mathcal{L}_{\phi}$$
 (68)

In the above, we used the approximation  $\sqrt{-g} \approx \sqrt{-\eta} = 1$ . The next step is to obtain the graviton-scalar interaction terms described by Eq. 43.

$$2\frac{\mathcal{L}_{\text{int}}}{\sqrt{-g}} = \frac{(\bar{\lambda}_{\mu\nu}\lambda)\tilde{T}^{\mu\nu}}{2\sqrt{-g}} + 2(\bar{\lambda}_{\mu}^{\alpha}\lambda^{\mu\beta})\partial_{\alpha}\phi\partial_{\beta}\phi + \frac{\bar{\lambda}_{\mu\nu}\lambda^{\mu\nu}\mathcal{L}_{\phi}}{\sqrt{-g}} - \frac{1}{2}(\bar{\lambda}\lambda^{\alpha\beta})\partial_{\alpha}\phi\partial_{\beta}\phi$$

$$(69)$$

$$= (\bar{\lambda}_{\mu\nu}\lambda^{\mu\nu} - \frac{1}{2}\bar{\lambda}\lambda)\mathcal{L}_{\phi} - \frac{1}{2}(\bar{\lambda}\lambda^{\alpha\beta} + \bar{\lambda}^{\alpha\beta}\lambda - 4\bar{\lambda}^{\alpha}_{\mu}\lambda^{\mu\beta})\partial_{\alpha}\phi\partial_{\beta}\phi$$
 (70)

Again,  $g_{\mu\nu} \approx \eta_{\mu\nu}$  has been applied. After redefining the graviton field as  $\lambda_{\mu\nu} \to \sqrt{\kappa}\lambda_{\mu\nu}$  and imposing the condition  $O(\kappa m^2) \ll O(1)$ , the perturbation becomes valid, meaning that  $\mathcal{L}_{\text{free}} \gg \mathcal{L}_{\text{int}}$  by an order of magnitude of  $\kappa$ 

Concurrently, to handle the Dirac delta terms in Eq. 42, a limit is set on the measurement sensitivity of the curvature. We can define the precision scale of spacetime curvature with an order parameter  $\epsilon$ , where this parameter is expressed in energy units.

$$\delta R^{\rho}_{\ \sigma\mu\nu} \sim \epsilon^2$$
 (71)

To incorporate the condition  $R^{\rho}_{\sigma\mu\nu} \approx 0$  into the partition function, the density function in Eq. 42 is now parameterized by  $\epsilon$ .

$$Z \approx \int (DG_{\nu}^{\mu}) Z_{\eta} \rho_{\epsilon}(G_{\nu}^{\mu}) \prod_{j} \delta\left(\int_{j} \left(\tilde{T}_{\nu}^{\mu} - \frac{G_{\nu}^{\mu}}{2\kappa}\right) d^{3}\vec{x}\right)$$
(72)

In the above expression, the measure term is transformed to  $G^{\mu}_{\nu}$ , as  $\epsilon$  is the parameter for the curvature. Also,

all the irrelevant terms are absorbed to  $\rho_{\epsilon}$  for simplicity. Here, to have a better understandings to the partition function, we introduce a new parameter that is more intuitive compared to  $\epsilon$ . Given the form of the Lagrangian,  $R/2\kappa + \mathcal{L}_{\phi}$ , and specifying a spatial volume for the system, denoted as  $V_0$ , the new quantity  $\mu_G \equiv V_0 \epsilon^2/16\pi$  serves as a suitable measure for assessing the smallest gravitational constant  $(\kappa M/8\pi)$  that can be detected by an observer.

Utilizing the relation  $T_{\nu}^{(g)\mu} = -\sqrt{-g}G_{\nu}^{\mu}/2\kappa \approx -G_{\nu}^{\mu}/2\kappa$ , let's examine the simplest case with a uniform  $\rho_{\epsilon}$  that has a rectangular shape with its width equal to  $\epsilon$ . Its non-zero region begins at  $G_{\nu}^{\mu} = 0$ , suggesting  $\Lambda = 0$  to achieve  $R_{\nu}^{\mu} = 0$ . By integrating over  $G_{\nu}^{\mu}$  and applying normalization, one obtains the approximated partition function for an approximately flat spacetime:

$$Z[J] \approx Z_{\eta}[J] \Big|_{P_{\nu}^{\mu}(x^{\perp}) < O(8\pi\mu_G/\kappa)}$$
 (73)

 $P^{\mu}_{\nu}$  is referenced from Eq. 32. In order to retain the tensor property, it's clear that  $O(8\pi\mu_G/\kappa)$  must behave as a tensor, allowing us to denote it as  $M^{\mu}_{\nu}$ . Here, let's define  $\mu_G$  explicitly, not just by a scale:  $M^0_r = 8\pi\mu_G/\kappa$ . The radial coordinate is specially selected for the definition of  $\mu_G$ , as the radial UV cut-off emerges naturally in this coordinate, expressed as  $M^0_r \equiv \mu > P^0_r$  according to Eq. 73.

Although  $\mu$  and  $\mu_G$  are good parameters to understand the system, the direct usage of them is not recommended due to the non-Lorentz-invariance of the radial coordinate. An alternative is to bring into play  $M_{\rho}^0$ , where  $\rho = \sqrt{|r^2 - t^2|}$ , and express  $\mu$  via  $M_{\rho}^0$ :

$$\mu = \frac{\partial \rho}{\partial r} M_{\rho}^{0} = \left| \frac{r}{\rho} \right| M_{\rho}^{0} \tag{74}$$

The drawback with the above expression is the ill-definition of  $\mu$  at  $\rho=0$ . Considering that  $\mu$  is being utilized as a regulator and that  $\rho=0$  is a divergence point in Feynman diagrams (which is discussed in the later sections), this poses a challenge. To resolve it, a different Lorentz-invariant coordinate,  $s=r^2-t^2$ , is introduced. Leveraging this coordinate results in:

$$\mu = \frac{8\pi}{\kappa} \mu_G = \frac{\partial s}{\partial r} M_s^0 = 2r M_s^0 \tag{75}$$

The above indicates that  $\mu_G$  has a direct proportionality to r, if  $M_s^0$  is treated as a constant. By using it, a more suitable definition would be  $\phi_G \equiv \kappa M_s^0/4\pi = \mu_G/r$ . This value,  $\phi_G$ , is Lorentz-invariant and serves as a measure of the finest resolution attainable for the gravitational potential.

As a last check, let's find conditions for the parameter scales. First, for  $\mu$  to significantly surpass the momentum scale of the system, the condition  $O(|p_i|) \ll O(\mu)$  is required, where  $p_i$  denotes the initial momenta of the introduced particles in the system. Also, given that the system resolution scale of r aligns with  $O(1/|p_i|)$  from a

wave-perspective view, the associated condition relative to  $O(\phi_G)$  becomes,

$$O(\phi_G) \approx O(|p_i| \times (\mu_G)|_{\min}) \gg O(\kappa |p_i|^2)$$
 (76)

#### B. Derivation of the graviton propagators

This subsection is dedicated to derive the graviton propagators. As usual, the approach starts with rearranging the partition function:

$$Z_{\lambda} \propto \int D\lambda D\bar{\lambda} \exp\left(i \int d^4x (K_{\lambda} + \bar{J} \cdot \lambda + \bar{\lambda} \cdot J)\right)$$
(77)  
=  $Z_0 \exp\left(i \int d^4x \bar{J} \cdot \stackrel{\leftrightarrow}{\Delta} \cdot J\right)$  (78)

so that,

$$\left. \left( \frac{1}{Z_{\lambda}} \frac{\delta^2 Z_{\lambda}}{\delta J(x) \delta \bar{J}(y)} \right) \right|_{J=0} = \langle \lambda(y) \bar{\lambda}(x) \rangle = i \overset{\leftrightarrow}{\Delta}(x, y) \quad (79)$$

J and  $\bar{J}$  are graviton source terms and  $\stackrel{\leftrightarrow}{\Delta}$  is graviton propagator expressed in a tensor form. To find the right expression of  $Z_{\lambda}$  and to derive the graviton propagator, the physical part of the graviton Lagrangian is investigated. By additionally introducing the total derivatives that are unphysical, the free Lagrangian of graviton can be re-expressed in momentum basis:

$$\mathcal{L}_{\lambda} = \bar{\lambda}_{ij} A^{ijkl} \lambda_{kl} \tag{80}$$

where the tensor  $A^{ijkl}$  is obtained from Eq. 66.

$$A^{ijkl} = \frac{1}{4} \eta_{ab} \eta^{i[b} p^{c]} \eta^{l[a} \eta^{d]j} p_{[c} \delta^k_{d]}$$
 (81)

$$= \frac{1}{4} (\eta^{i[l} p^{c]} p_{[c} \delta^k_{d]} \eta^{dj} - \eta^{i[j} p^{c]} p_{[c} \delta^k_{d]} \eta^{dl})$$
 (82)

The graviton self-interactions in Eq. 64 vanishes as the metric is flat and  $\Lambda=0$ . The above equation shows that  $p_iA^{ijkl}=p_jA^{ijkl}=p_kA^{ijkl}=p_lA^{ijkl}=0$ , ensuring that only  $p^i\lambda_{ij}(p)=p^j\lambda_{ij}(p)=0$  terms survive in the action. Because gravitons are ghost particles that have no external lines,  $\lambda_{ij}(p)$  domain can be reduced without losing the physical degrees of freedom, if there is a region that does not affect to the action (if the external lines exist, on the other hand, there is a possibility that a field resides in the outer region, depending on its initial conditions). Applying the idea, the Lagrangian is modified by introducing divergenceless  $\lambda_{ij}$  condition, without altering the physical effects.

$$\mathcal{L}_{\lambda}(p) = \frac{p^{2}}{4} \bar{\lambda}_{ij} \eta^{i[l} \eta^{j]k} \lambda_{kl} + \bar{J}^{(ij)} \lambda_{(ij)} + \bar{\lambda}_{(ij)} J^{(ij)} + \bar{\chi}^{k} P_{k}^{(ij)} \lambda_{(ij)} + \bar{\lambda}_{(ij)} P_{k}^{(ij)} \chi^{k}$$
(83)

 $J^{(ij)}$  are the source terms.  $\chi^k$  and  $P_k^{(ij)}$  are Grassmanian variables and coefficients, respectively, introduced

to add Lagrange multipliers for giving constraints. A new notation (ij) indicates that the (ij) and (ji) terms are identified and counted only a single time during the contraction, to avoid double counts of  $\lambda_{ij}$  (as mentioned,  $\lambda_{ij}$  is identified with  $\lambda_{ii}$ ).

$$P_{k}^{(ij)} = \begin{pmatrix} 01 & 02 & 03 & 012 & 013 & 023 \\ 0 & p_{1} & p_{2} & p_{3} & 0 & 0 & 0 \\ -p_{0} & 0 & 0 & p_{2} & p_{3} & 0 \\ 0 & -p_{0} & 0 & p_{1} & 0 & p_{3} \\ 0 & 0 & -p_{0} & 0 & p_{1} & p_{2} \end{pmatrix}$$
(84)

and,

$$P_k^{(ii)} = \begin{pmatrix} 0 & 0 & (11) & (22) & (33) \\ -p_0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix}$$
(85)

The contractions of  $P_k^{(ij)}\lambda_{(ij)}$  yield nothing but  $p^i\lambda_{ik}$ . The Grassmanian variable  $\lambda_{ij}$  in  $Z_{\lambda}$  is integrated using  $\int D\lambda D\bar{\lambda} \exp(\bar{\lambda}\cdot A\cdot \lambda) = \det |A|$ . After normalizing the partition function,

$$Z_{\lambda} = \int D\chi D\bar{\chi} e^{i \int d^4 p \mathcal{L}_{\chi}(p)}$$
 (86)

remains with the leftover Lagrangian,  $\mathcal{L}_{\chi}(p)$ .

$$\mathcal{L}_{\chi}(p) = (\bar{J}^{(ij)} + \bar{\chi}^a P_a^{(ij)}) G_{ijkl} (J^{(kl)} + P_b^{(kl)} \chi^b)$$
 (87)

 $G_{ijkl}$  indicates the inverse of  $A^{ijkl}$ . For easier calculation, it is better to change the basis of A:  $A^{ijkl} \rightarrow A^{(ij)(kl)}$ .

$$\frac{4}{p^{2}}A^{(ij)(kl)} = \begin{pmatrix}
01 & 02 & 03 & (12) & (13) & (23) \\
02 & -2 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix} (88)$$

and,

$$\frac{4}{p^2}A^{(ii)(kk)} = \begin{pmatrix} 000 & (11) & (22) & (33) \\ 001 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$$
(89)

Other terms are zeros. The inverse of  $A^{(ij)(kl)}$ ,  $G_{(ij)(kl)}$ , is easily computed using block matrices.

$$G_{(ij)(kl)} = \frac{2}{p^2} \begin{pmatrix} \begin{pmatrix} -I_3 \\ & I_3 \end{pmatrix} & & & \\ & & \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & -1 & -1 \\ 1 & -1 & 2 & -1 \\ 1 & -1 & -1 & 2 \end{pmatrix} \end{pmatrix}$$
 with, 
$$\frac{\Delta_{(ij)(kk)}^3 =}{\begin{cases} \eta^{kk} p_i p_j (\eta^{kk} p_k^2 - p^2) & i = k \text{ or } j = k \\ \eta^{kk} p_i p_j (\eta^{kk} p_k^2 + p^2) & \text{else} \end{cases}$$

The empty places are zeros, and  $I_3$  indicates  $3 \times 3$  identity matrix. Now,

$$(\bar{J}^{(ij)} + \bar{\chi}^a P_a^{(ij)}) G_{ijkl} (J^{(kl)} + P_b^{(kl)} \chi^b)$$

$$= \bar{J}^{(ij)} G_{(ij)(kl)} J^{(kl)}$$

$$- \bar{J}^{(ij)} G_{(ij)(kl)} P_a^{(kl)} C^{ab} P_b^{(wx)} G_{(wx)(yz)} J^{(yz)}$$

$$+ \bar{D}^a B_{ab} D^b$$
(91)

where

$$B_{ab} = P_a^{(ij)} G_{(ij)(kl)} P_b^{(kl)}$$
(92)

$$= \frac{2}{3p^2} \begin{pmatrix} 4p_0^2 - 3|\vec{p}|^2 & p_0 \cdot p_1 & p_0 \cdot p_2 & p_0 \cdot p_3 \\ p_0 \cdot p_1 & p_{231} & p_1 \cdot p_2 & p_1 \cdot p_3 \\ p_0 \cdot p_2 & p_1 \cdot p_2 & p_{132} & p_2 \cdot p_3 \\ p_0 \cdot p_3 & p_1 \cdot p_3 & p_2 \cdot p_3 & p_{123} \end{pmatrix}$$
(93)

and  $C = B^{-1}$ , with

$$D^{b} = \chi^{b} + C^{bc} P_{c}^{(ij)} G_{(ij)(kl)} J^{(kl)}$$
(94)

Here,  $p_{ijk} \equiv 3(p_i^2 + p_j^2 - p_0^2) + 4p_k^2$  is defined and used. Taking the inverse of Eq. 93,  $C^{ab}$  is also obtained.

$$C^{ab} = \frac{9}{2}\eta^{ab} + \frac{3}{2p^2}|p^a p^b| \tag{95}$$

Using the fact that the last term in Eq. 91 is a constant after  $\chi$  integration, the normalized  $Z_{\lambda}$  becomes:

$$Z_{\lambda} = \exp\left(i \int d^4 p \bar{J}^{(ij)} \Delta_{(ij)(xy)} J^{(xy)}\right)$$
 (96)

where  $\Delta_{(ij)(xy)}$  are the graviton propagators.

$$\Delta \equiv G_{(ij)(kl)} \left( I_{(xy)}^{(kl)} - P_a^{(kl)} C^{ab} P_b^{(uv)} G_{(uv)(xy)} \right)$$
 (97)

$$= \frac{2}{p^6} \begin{pmatrix} \Delta^1 & \Delta^3 \\ (\Delta^3)^{\dagger} & \Delta^2 \end{pmatrix}_{\substack{(i\neq j)\\(i=j)}}$$
(98)

The values for  $\Delta^1$ ,  $\Delta^2$ , and  $\Delta^3$  are:

$$\Delta_{(ij)(xy)}^{1} = \begin{cases}
\eta^{ii} \eta^{jj} (p^{2} - \eta^{ii} p_{i}^{2}) (p^{2} - \eta^{jj} p_{j}^{2}) & (ij)(ij) \\
\eta^{ii} p_{j} p_{k} (\eta^{ii} p_{i}^{2} - p^{2}) & (ij)(ik), \ j \neq k \\
p_{0} p_{1} p_{2} p_{3} & \text{else}
\end{cases} (99)$$

$$\begin{split} \Delta_{(ii)(jj)}^2 &= \\ \left\{ (p^2 - \eta^{ii} p_i^2)^2 & i = j \\ p_i^2 p_j^2 - \eta^{ii} \eta^{jj} p^2 (p^2 - \eta^{ii} p_i^2 - \eta^{jj} p_j^2) & \text{else} \\ \end{array} \right. \end{split} \tag{100}$$

$$\Delta_{(ij)(kk)}^{3} = \begin{cases} \eta^{kk} p_i p_j (\eta^{kk} p_k^2 - p^2) & i = k \text{ or } j = k \\ \eta^{kk} p_i p_j (\eta^{kk} p_k^2 + p^2) & \text{else} \end{cases}$$
(101)

These terms can also be expressed in another way, by dividing  $\Delta_{(ij)(xy)}$  into 5 categories.

$$\begin{cases}
\Delta_{(ij)(ij)}/2 &= \eta^{ii}\eta^{jj}(p^2 - \eta^{ii}p_i^2)(p^2 - \eta^{jj}p_j^2)/p^6 \\
\Delta_{(ii)(jj)}^{i\neq j}/2 &= p_i^2 p_j^2/p^6 - \eta^{ii}\eta^{jj}/p^2 \\
\Delta_{(ij)(ik)}^{j\neq k}/2 &= p_i^2 p_j p_k/p^6 - \eta^{ii}p_j p_k/p^4 \\
\Delta_{(ij)(kk)}^{i\neq j}/2 &= p_k^2 p_i p_j/p^6 + \eta^{kk} p_i p_j/p^4 \\
\Delta_{(ij)(kl)}^{i\neq j\neq k\neq l}/2 &= p_0 p_1 p_2 p_3/p^6
\end{cases}$$
(102)

# C. Propagators in position basis

As it can be clearly seen in Eq. 70, the QR scalar theory is non-renormalizable with the employment of standard propagators. To resolve this, we modify the boundary condition for the graviton by introducing a new propagator, which we like to call the principal propagator. Contrary to Feynman propagators [34], principal propagators utilize principal values for the integration over  $p_0$  within the 4-dimensional Fourier transform. These principal propagators can also be defined via other propagators.

$$\Delta = \frac{1}{2} (\Delta_{\text{retarded}} + \Delta_{\text{advanced}})$$
(103)  
$$= \frac{1}{2} (\Delta_{\text{Feynman}} + \Delta_{\text{anti-Feynman}})$$
(104)

An important property of the principal propagators is that they are anti-symmetric with respect to t, since  $\Delta_{\text{retarded}}$  and  $\Delta_{\text{advanced}}$  satisfy the relationship  $\Delta_{\text{retarded}}(-t) = -\Delta_{\text{advanced}}(t)$ .<sup>2</sup> Due to their anti t-symmetry, principal propagators stand out for two main reasons. Firstly, they satisfy  $\Delta(t=0)=0$ . This condition is crucial to negate the diverging 1-point self-loops by assigning zero values, as gravitons have the potential to create self-loops according to Eq. 70. Secondly,  $|\Delta|$  is fully Lorentz-invariant, as both  $\Delta_{\text{retarded}}$  and  $\Delta_{\text{advanced}}$  maintain Lorentz invariance when the sign of t is preserved, with their sum violating the invariance only by a sign flip at t=0.

However, a notable side effect of the principal propagators is the presence of terms that propagate inversely in time. This stems from the relation  $\Delta_{\rm anti-Feynman}(x-y)=-i\langle \bar{T}(\phi(x)\phi(y))\rangle$ , where  $\bar{T}(\cdot,\cdot)$  indicates reverse time ordering. Nonetheless, we justify this anomaly by recognizing that gravitons emerge from the metric field, a physical object that defines the causality rather than obeying it.

Employing the relation  $\mathcal{F}^{-1}\{p_jF(p)\}=-i\partial_jf(x)$ , where  $\mathcal{F}\{\cdot\}$  represents the Fourier transform and  $\mathcal{F}\{f(x)\}=F(p)$ , all position basis propagators that are transformed from Eq. 102 can be generated using  $\mathcal{F}^{-1}\{1/p^6\}$  with proper derivatives. Because there is anti t-symmetry for the principal propagators, let's assume t>0 first and find  $\Delta(x)$  without loss of generality.

$$\mathcal{F}^{-1}\left\{\frac{(2\pi)^4}{p^6}\right\} = \int d^3\vec{p} \int dp_0 \frac{\exp(ip \cdot x)}{(p_r^2 - p_0^2)^3}$$
 (105)

 $f dp_0$  denotes Cauchy principal value integration. Decomposing the integration to  $(\Delta_{\text{retarded}} + \Delta_{\text{advanced}})/2$ ,  $\Delta_{\text{advanced}}$  gives zero contribution in t > 0 region, and the equation is further simplified using the residue theorem.

$$\mathcal{F}^{-1}\left\{\frac{(2\pi)^4}{p^6}\right\} = \frac{1}{2} \cdot 2\pi i \int d^3 \vec{p} \frac{e^{i\vec{p}\cdot\vec{x}}}{(2p_r)^3} \left(it^2 \sin(p_r t) - 3i\frac{\sin(p_r t)}{p_r^2} + 3it\frac{\cos(p_r t)}{p_r}\right)$$

$$= \frac{\pi^2}{4} \int_0^\infty dp_r \int_\pi^0 d\cos\theta e^{ip_r r\cos\theta} \times \left(\left(\frac{3}{p_r^3} - \frac{t^2}{p_r}\right)\sin(p_r t) - 3t\frac{\cos(p_r t)}{p_r^2}\right)$$
(106)

After the integration of  $\theta$ :

$$\mathcal{F}^{-1}\left\{\frac{(2\pi)^4}{p^6}\right\} = \frac{\pi^2}{2r} \int_0^\infty dp_r \left(\left(\frac{3}{p_r^4} - \frac{t^2}{p_r^2}\right) \sin(p_r t) \times \sin(p_r r) - \frac{3t}{p_r^3} \cos(p_r t) \sin(p_r r)\right)$$
(108)

Each integral in the above diverges, but integration of the total function does not. To handle this, small  $\varepsilon$  is introduced. Using the integral by parts to the first term,

$$\int_{\varepsilon}^{\infty} dq \frac{\sin(qt)\sin(qr)}{q^4} = \left[ -\frac{\sin(qt)\sin(qr)}{3q^3} \right]_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty} dq \frac{t\cos(qt)\sin(qr) + r\sin(qt)\cos(qr)}{3q^3}$$
(109)

Repeating the integral by parts two times more:

First term = 
$$\frac{tr}{\varepsilon} - \frac{t^2 + r^2}{6} \int_{\varepsilon}^{\infty} dq \frac{\sin(qt)}{q} \frac{\sin(qr)}{q}$$
  
 $-\frac{tr}{3} \int_{\varepsilon}^{\infty} \frac{dq}{q} (t\sin(qt)\cos(qr) + r\cos(qt)\sin(qr))$ 
(110)

The result of the first integral in Eq. 110 is,

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} dq \frac{\sin(qt)}{q} \frac{\sin(qr)}{q} = \frac{\pi}{4} (|r+t| - |r-t|) \quad (111)$$

<sup>&</sup>lt;sup>2</sup> One might question this result due to the t-symmetric property of  $\Delta_{\text{Feynman}}$  and  $\Delta_{\text{anti-Feynman}}$ . This issue arises from the ill-defined nature that occurs when  $\Delta_{\text{Feynman}}$  is added with  $\Delta_{\text{anti-Feynman}}$ , as they appear to cancel each other out. The crucial factor lies in the process of taking the limit  $\varepsilon \to 0$  around  $p_0 = \pm p_r$ , when integrating over  $p_0$  to derive the position space (anti-)Feynman propagator [34]. If one first computes the sum  $\Delta_{\text{Feynman}} + \Delta_{\text{anti-Feynman}}$  and then applies  $\varepsilon \to 0$ , it is found that the second leading term remains finite on behalf of the first leading term, and exhibits anti t-symmetry.

$$\begin{split} -\mathcal{F}^{-1}\{p_0^2/p^6\} - r\delta(t-r)/64\pi &= \mathcal{F}^{-1}\{p_3^2/p^6\} + \frac{z^2}{r}\delta(t-r)/64\pi = \mathcal{F}^{-1}\{1/p^4\}/4\\ \mathcal{F}^{-1}\{p_0p_1p_2p_3/p^6\} &= -\left(\delta'(t-r)/r + \delta''(t-r)\right) \cdot xyz/64\pi r^2\\ \mathcal{F}^{-1}\{p_3^2/p^4\} &= \left(\rho^2\delta(t-r)/r^3 - z^2\delta'(t-r)/r^2\right)/16\pi\\ \mathcal{F}^{-1}\{p_0^2/p^4\} &= -\delta'(t-r)/16\pi\\ \mathcal{F}^{-1}\{p_0p_3/p^4\} &= \delta'(t-r) \cdot z/16\pi r\\ \mathcal{F}^{-1}\{p_1p_3/p^4\} &= -\left(\delta(t-r)/r + \delta'(t-r)\right) \cdot xz/16\pi r^2\\ \mathcal{F}^{-1}\{1/p^2\} &= \delta(t-r)/8\pi r \end{split}$$

$$-\partial_{0}\partial_{3}(r\delta(t-r)) = -(z/r) \cdot \delta'(t-r) + z\delta''(t-r)$$

$$-\partial_{1}\partial_{3}(r\delta(t-r)) = (xz/r) \cdot \left(\delta(t-r)/r^{2} + \delta'(t-r)/r - \delta''(t-r)\right)$$

$$-\partial_{3}^{2}(r\delta(t-r)) = -(\rho^{2}/r^{3}) \cdot \delta(t-r) + (\rho^{2} + 2z^{2})/r^{2} \cdot \delta'(t-r) - (z^{2}/r) \cdot \delta''(t-r)$$

$$-\partial_{0}\partial_{1}((z^{2}/r) \cdot \delta(t-r)) = (xz^{2}/r^{2}) \cdot \left(\delta'(t-r)/r + \delta''(t-r)\right)$$

$$-\partial_{0}\partial_{3}((z^{2}/r) \cdot \delta(t-r)) = (z/r) \cdot \left(-(2\rho^{2} + z^{2})/r^{2} \cdot \delta'(t-r) + (z^{2}/r) \cdot \delta''(t-r)\right)$$

$$-\partial_{1}\partial_{3}((z^{2}/r) \cdot \delta(t-r)) = (xz/r^{2}) \cdot \left((2\rho^{2} - z^{2})/r^{2} \cdot \delta(t-r)/r + (2\rho^{2} + z^{2})/r^{2} \cdot \delta'(t-r) - (z^{2}/r) \cdot \delta''(t-r)\right)$$

$$-\partial_{1}\partial_{2}((z^{2}/r) \cdot \delta(t-r)) = -(xyz^{2}/r^{4}) \cdot \left(3\delta(t-r)/r + 3\delta'(t-r) + r\delta''(t-r)\right)$$

$$-\partial_{1}^{2}((z^{2}/r) \cdot \delta(t-r)) = (z^{2}/r^{2}) \cdot \left(1 - 3(x^{2}/r^{2})\right) \left(\delta(t-r)/r + \delta'(t-r)\right) - (x^{2}z^{2}/r^{3}) \cdot \delta''(t-r)$$

TABLE I: List of the useful relations to Fourier transform Eq. 102 and obtain  $\Delta(x)$ .

Similarly, the limit after the second integral is,

$$\frac{\pi}{4}(t\operatorname{sgn}(r+t) - t\operatorname{sgn}(r-t) + r\operatorname{sgn}(r+t) - r\operatorname{sgn}(t-r))$$
(112)

 $\operatorname{sgn}(x)$  is a signum function, giving 1 if x > 0, -1 if x < 0, and 0 if x = 0. Now substituting the results to Eq. 110 with multiplied factors from Eq. 108,

First term = 
$$\frac{3\pi^2 t}{2\varepsilon} - \frac{\pi^3}{8} \begin{cases} (t^2 + 3r^2) \cdot t/r & r > t \\ 3t^2 + r^2 & r \le t \end{cases}$$
 (113)

By applying the aforementioned integrals, both the second and third terms in Eq. 108 can be calculated as well.

Second term = 
$$-\pi^3 \frac{t^2}{8r} (|r+t| - |r-t|)$$
 (114)

Third term = 
$$-\frac{3\pi^2 t}{2\varepsilon} + \frac{3\pi^3}{8} \begin{cases} t^3/r + tr & r > t \\ 2t^2 & r \le t \end{cases}$$
(115)

Adding three terms, one finds that Eq. 108 converges, and the transform is completed after restoring anti t-symmetry.

$$\therefore \mathcal{F}^{-1}\left\{\frac{128\pi}{p^6}\right\} = -(r^2 - t^2)u(|t| - r)\operatorname{sgn}(t) \qquad (116)$$

u(x) is the Heaviside step function. Generated from Eq. 116, few handy relations (t>0) for obtaining the explicit shape of the graviton propagators are listed in Tab. I  $(\rho \equiv \sqrt{x^2 + y^2})$  is used for simpler expression).

### V. THEORY RENORMALIZATION

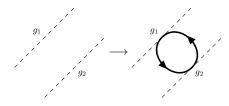
In this section we demonstrate a new way to renormalize the theory using the QR free scalar theory. The renormalization is achieved by following two procedures.

- 1. Classify diagrams without graviton lines.
- 2. Add graviton lines to the classified diagrams.

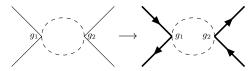
Diagrams derived from the QR free scalar theory, devoid of graviton lines, can be classified based on their connectivity to external sources. Some diagrams, called the elements of group A (see Fig. 3 (a)), are connected to the external sources. Given the free scalar's unique feature where interaction vertices have only two legs, the ends of such diagrams directly link to the external sources, with no intermediate branching. It's clear that these straightforward diagrams do not exhibit divergent terms.

Conversely, there are diagrams not following this structure, which are categorized to group **B** (see Fig. 3 (b)). In the QR free scalar theory, where tadpole type diagrams are disallowed, every one of these diagrams represents 'vacuum bubbles', which are entirely closed off and not exposed externally. Unlike the traditional QFTs that usually regard vacuum bubbles as unphysical, they gain prominence in QR due to gravitons' ability to serve as the mediator between the bubble and the external environment.

The theory's structure suggests that the vacuum bubbles will manifest as loops, but their size isn't fixed. Referring to Eq. 70, each interaction vertex can be described by two types of couplings: kinetic  $(\kappa \bar{\lambda} \lambda \partial \phi \partial \phi)$  type) and



(a) Freely propagating particles (group A)



(b) Scalar vacuum bubble (group **B**)

FIG. 3: Example diagrams for group **A** (above) and group **B** (below), respectively. The left figures show the diagrams when the graviton channels are turned off, while the right figures show the complete diagrams. Each interaction vertex has its coupling constant,  $g_n$ , which can be either kinetic ( $\propto \kappa p_{n\alpha} p_{n\beta}$ ) or massive ( $\propto \kappa m^2$ ).

massive  $(\kappa m^2 \bar{\lambda} \lambda \phi^2)$  type). If a loop has a number of kinetic interaction vertices and b number of mass interaction vertices, its Feynman amplitude can be represented as follows:

$$\mathcal{M} \propto \int d^4p \Big(\frac{\kappa}{p^2 + m^2}\Big)^{a+b} \Big(\prod_{n=1}^a p_{n\alpha} p_{n\beta}\Big) m^{2b}$$
 (117)

In the above, the momentum conservation is used.  $p_{n\alpha}$  and  $p_{n\beta}$  refer to kinetic couplings according to the shape of  $\partial_{\alpha}\phi\partial_{\beta}\phi$ , which possess indices to distinguish their directivity. As it is clearly shown in the above equation, vacuum bubbles may induce divergences because of their loop structure. The magnitude of these divergences can be measured by introducing the parameter  $\mu$  as a UV cut-off. To be more specific, if  $a \geq 2$  and b = 0, then  $\mathcal{M} \to O(\kappa^a \mu^4)$ ; if  $a \geq 1$  and b = 1,  $\mathcal{M} \to O(\kappa^{a+1}\mu^2)$ ; if b = 2,  $\mathcal{M} \to O(\kappa^{a+2}(\ln \mu))$ ; and the diagram converge when b > 2.

The conventional method for treating divergences involves introducing counter terms. However, this method does not work for vacuum bubbles, as there is an infinite number of one-particle-irreducible diagrams (1PIs). In QR, on the other hand, a new approach is possible due to the concrete relation between  $\kappa$  and  $\mu$ . By examining the strongest diverging term, where a=2 and b=0, and using Eq. 73, the divergent scale is determined.

$$\mathcal{M} \to O(\kappa^2 \mu^4) = O\left((8\pi)^4 \left(\frac{\phi_G^2}{\kappa}\right)^2\right)$$
 (118)

Therefore, even without canceling the divergence,  $\mathcal{M}$  can converge if  $\phi_G \ll \sqrt{\kappa} |p_i|$  holds, where  $|p_i|$  represents the

momentum scale of the system. For such cases, a new order parameter,  $\bar{\kappa} \equiv \phi_G^2/\kappa |p_i|^2$ , is introduced in place of  $\kappa$ , ensuring the perturbation theory remains valid. Conveniently, given that  $\kappa$  has a dimension of length squared while  $\bar{\kappa}$  is dimensionless, the same coupling  $\kappa$  can be employed, if the appropriate length unit is chosen such that  $\bar{\kappa} = \kappa = \phi_G/|p_i|$ .

After classifying the diagrams, graviton lines must be added to generate the true diagram according to certain rules. First, all interactions in the theory possess  $\bar{\lambda}\lambda$ couplings, meaning that each interaction vertex in the diagram should have two graviton legs. Like the case of group A diagrams, this implies that the chain of graviton propagators disallow branchings. Second, the chain of graviton propagators must form a closed loop. Considering that gravitons cannot form 1-point self-loops due to the use of the principal propagator and the fact that gravitons are ghost particles without external lines, this is inevitable. Lastly, only an even number of graviton propagators are allowed in the graviton-chain loop. This constraint arises from r-symmetric and the anti t-symmetric nature of the principal propagator. Graviton-chain loops exhibit fermionic directionality, and loops with opposing directions cancel out for odd loops.<sup>3</sup>

The important observation from Tab. I is that terms in the graviton propagators contain either  $\delta(t-r)$ ,  $\delta'(t-r)$ , or  $\delta''(t-r)$ . Also, given that  $\delta(t-r)/r = \delta(s)$  where  $s=r^2-t^2$ , all the denominators in the first section of Tab. I vanish in the newly defined coordinates,  $(t,s,\theta,\phi)$ . In this coordinate system, the addition of graviton propagators constrains the mediated points to be light-like separated, together with additional effects to the original diagram, such as multiplication of polynomial sine factors or application of differentiation.

To see the renormalizability of the graviton-involved diagrams, let's separate a diagram into its singular part and regular part (graviton channels are still turned off at this point). The Feynman propagator of the scalar field, which is a building block of the diagrams in the QR free scalar theory, is expressed in the position basis as follows.

$$G(s) = \begin{cases} -\frac{1}{4\pi}\delta(s) + \frac{m}{8\pi\sqrt{|s|}}H_1^{(1)}(m\sqrt{|s|}) & s \le 0\\ -\frac{im}{4\pi^2\sqrt{|s|}}K_1(m\sqrt{|s|}) & s > 0 \end{cases}$$
(119)

Here,  $H_1^{(n)}$  is a Hankel function and  $K_1$  is a modified Bessel function. As it can be seen in the equation above, the singular part of the scalar propagator exists only around s=0, whereas the remaining parts comprise a regular function where the integration is well-defined.

For regular functions, the regions where integrability might fail due to the modification of the original function are typically near boundaries. Fortunately, around

 $<sup>^3</sup>$  Given that  $|\Delta|$  is Lorentz-invariant, it follows that the theory itself is also Lorentz-invariant, a consequence of this even number constraint on graviton propagators.

the  $|s| \to \infty$  boundary, which is the only boundary except s=0, the regular part of the scalar propagator converges to exponentially decaying continuous function in the asymptotic sense. Hence, the effects by introducing gravitons, such as multiplying by polynomials, differentiating functions, or constraining the function to light-like sections, does not break the integrability of the diagrams. Since there is no way for regular functions to induce divergences except at the boundaries, the diagrams remain non-divergent upon the addition of graviton propagators.

The issue arises in the singular part of the diagram, which appears when any of the propagators in the diagram mediates points separated by a light-like distance. Given that the singular part of the scalar propagator is also represented by a delta function, like graviton propagators, a diagram with a graviton propagator will only diverge if there are overlapping constraints set by the delta functions and their derivatives. The obvious divergences appear at 2-point loops, as they always contain terms like  $\delta^{(m)}(s)\delta^{(n)}(s) \to \infty$  ( $\delta^{(m)}(s)$  indicates m-th derivative of  $\delta(s)$ ). In the other cases, however, no divergences are introduced. To clarify this, let's consider npoints in 4-dimensional Minkowskian spacetime, labeling them with natural numbers (where n > 2). Here, even if light-like separation conditions are applied between the *i*-th and (i + 1)-th points for  $i = 1, 2, \ldots, n - 1$ , it does not imply that the i = n point and the i = 1 point are light-like separated. Therefore, there are no overlapping constraints for the chains of delta functions when n > 2.

Given that there are only a finite number of 2-point loops involving gravitons, renormalization is achievable. There are four types of 2-point loops with gravitons in the theory: graviton-graviton-scalar loop, graviton-scalar loop, graviton-scalar loop, graviton-scalar loop (see Fig. 4). For all these cases, the infinities can be represented as products of  $\delta^{(n)}(0)$ . Bearing in mind that the delta function arises from computing  $\mathcal{F}^{-1}\{1/p^2\}$ :

$$\delta(|t| - r) = \frac{2}{\pi} \int_0^\infty dp_r \sin(p_r|t|) \sin(p_r r)$$
 (120)

is achieved. Hence:

$$\lim_{t \to r} \delta(|t| - r) \to \frac{2}{\pi} \int_0^\infty dp_r \sin^2(p_r r)$$
 (121)

$$= \frac{1}{\pi} \int_0^{\mu} dp_r \left( 1 - \cos(2p_r r) \right)$$
 (122)

$$=\frac{1}{\pi}\left(\mu - \frac{\sin(2r\mu)}{2r}\right) \tag{123}$$

Because  $\mu$  is a large number,  $\delta(0) \to \mu/\pi = 2rM_s^0/\pi = 8r\phi_G/\kappa$  is derived. On the other hand,  $\delta''(0)$  is more tricky to handle, as the derivative can see the hidden r inside  $\mu$ . To properly resolve the issue, let's change the coordinate to s.

$$\delta(|t| - r) = 2r\delta(s) \tag{124}$$

Then, the derivatives of  $\delta(|t|-r)$  is computed as:

$$\frac{\partial^2}{\partial r^2} (2r\delta(s)) = 12r\delta'(s) + 8r^3 \delta''(s) \tag{125}$$

$$\frac{\partial^2}{\partial t^2} (2r\delta(s)) = 8rt^2 \delta''(s) \tag{126}$$

$$-\frac{\partial}{\partial r}\frac{\partial}{\partial t}(2r\delta(s)) = 4t\delta'(s) + 8r^2t\delta''(s)$$
 (127)

Because  $\delta(s)$  can be regarded as a square function around s=0 with its height  $1/\Delta s$ , where  $\Delta s \propto 1/M_s^0$ , one may find  $\Delta s$  that matches  $\delta(|t|-r=0)=2rM_s^0/\pi$ .

$$\delta(|t| - r = 0) = 2r\delta(s = 0) = \frac{2r}{\Delta s} = 2r\frac{M_s^0}{\pi}$$
 (128)

Using the results:

$$-\frac{\partial^2}{\partial r^2} \left( 2r\delta(s) \right) \Big|_{s=0} = 12r \left( \frac{4\phi_G}{\kappa} \right)^2 + 8r^3 \left( \frac{4\phi_G}{\kappa} \right)^3 \quad (130)$$

$$-\frac{\partial^2}{\partial t^2} \left( 2r\delta(s) \right) \Big|_{s=0} = 8r^3 \left( \frac{4\phi_G}{\kappa} \right)^3 \tag{131}$$

$$\left. \frac{\partial}{\partial r} \frac{\partial}{\partial t} \left( 2r \delta(s) \right) \right|_{s=0} = 4t \left( \frac{4\phi_G}{\kappa} \right)^2 + 8r^2 t \left( \frac{4\phi_G}{\kappa} \right)^3 \quad (132)$$

Due to the above relations,  $\delta''(|t|-r=0)$  have  $r\phi_G^2/\kappa^2(\propto \mu^2)$  and  $r^3\phi_G^3/\kappa^3(\propto \mu^3)$  terms. Interestingly, two terms show different responses when applying Lorentz transformation, and the actual surviving term may vary depending on the given contractions of the graviton indices. For this reason, the leading order is not just  $\mu^3$ , but it is required to see the full tensor expression.

Using  $\delta(0)$  and  $\delta''(0)$ , we can now determine the diverging order of the loops. For graviton-graviton-scalar loops, there are three delta functions multiplied, causing two of them to diverge. Noting that the highest diverging order of each graviton or kinematically coupled scalar propagator is  $\mu^3$  (massively coupled scalar propagator has less diverging order), the leading order term will be  $\kappa^2\mu^6 \propto (\phi_G^3/\kappa^2)^2$ . To address these terms perturbatively, the system momentum scale,  $|p_i|$ , must satisfy  $O(\phi_G^3/\kappa^2) \ll O(|p_i|^4)$ , requiring  $O(\bar{\kappa}^2) \ll O(\phi_G)$ . Alongside the previously mentioned conditions,  $O(\kappa|p_i|^2) \ll O(\phi_G)$  and  $O(\phi_G) \ll O(\sqrt{\kappa}|p_i|)$ , the following hierarchy is established for a valid perturbative theory:

$$O(\kappa |p_i|^2) \ll O(\phi_G) \ll O((\kappa |p_i|^2)^{2/3})$$
 (133)

or equivalently,

$$O(\bar{\kappa}^2) \ll O(\phi_G) \ll O(\bar{\kappa}) \ll O(1)$$
 (134)

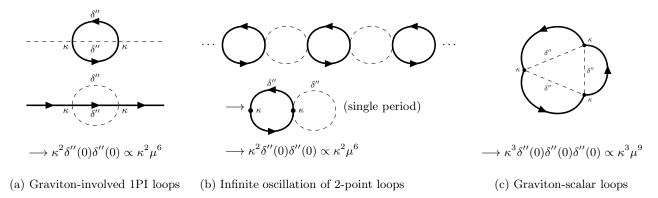


FIG. 4: Graviton-involved 2-point loops in QR free scalar theory. (a) Two possible 1PI loops, including graviton-graviton-scalar loop and graviton-scalar-scalar loop. Both loops have a diverging order of  $O(\kappa^2\mu^6)$ . (b) Particle oscillation between gravitons and scalar particles. As the number of oscillation periods increases, the highest diverging order of a single oscillation unit also increases and eventually converges to  $O(\kappa^2\mu^6)$ . (c) Graviton-scalar loops fully formed in a scalar vacuum bubble. In such cases, the highest diverging order is  $O((\kappa\mu^3)^n)$ , when the size of the vacuum bubble is denoted by n.

This condition implies that the gravitational potential of a single scalar particle should be immeasurable, ensuring the event horizon remains concealed within the theory. Simultaneously, the potential's resolution must be sufficient to constrain the kinetic fluctuation of the virtual particle.

The same logic can be applied to graviton-scalar-scalar loops, as it is shown in Fig. 4 (a). The highest order of  $\mu$  in graviton-scalar-scalar loops is also  $O(\kappa^2\mu^6) \to O((\phi_G^3/\kappa^2)^2) \ll O(|p_i|^8)$ , ensuring the survival of the perturbation theory (in the order estimation, r is omitted because it is free to set O(r) as the O(1) unit). Similarly, graviton-graviton loops can be addressed in the same way. The tricky case is when each vertex of graviton-graviton loops is connected to the scalar loop, like the ones illustrated in Fig. 4 (b). The highest diverging order for such cases arises when a large chain of oscillating graviton-graviton loop and kinetic scalar-scalar loop is formed. In this case, the limit of the highest diverging order of a single unit is  $O(\kappa^2\mu^6)$ , identical to the case of graviton-graviton-scalar loops.

Lastly, for graviton-scalar loops: if the loops form outside scalar loops,  $\mu$  can be handled straightforwardly. However, if they form within a scalar loop, complications arise as divergence from the scalar vacuum bubble is also involved, like an example of Fig. 4 (c). To solve this problem, let us consider the most divergent case, where a graviton connects all vertices of a vacuum bubble, maximizing the number of graviton-scalar loops. In such case, with a vacuum bubble size of n, there are n number of both  $\kappa$  and graviton-scalar loops. Therefore, the highest diverging order in this case is  $O((\kappa \mu^3)^n)$ , just like other types of 2-point loops. As a result, all infinities, including those from vacuum bubbles, graviton-involved 2-point loops, and hybrids of the two, are effectively regulated, completing the renormalization of the theory. Although we only showed the case of free scalar theory, the same

logic can be extended to any renormalizable theories.

#### VI. EXAMPLES

### A. Gravitational self-energy of the scalar field

As an initial example, we determine the gravitational self-energy of a scalar particle. The self-energy diagrams are computed by collecting all 1PIs that have single scalar lines at both the beginning and the end. Among the three order parameters  $(\kappa, \phi_G, \text{ and } \bar{\kappa})$  we focus on terms associated with the largest parameter for simplicity. It's notable that both the massive and kinetic couplings attain their largest parameters ( $\phi_G$  for massive couplings and  $\bar{\kappa}$  for kinetic couplings) exclusively in 2-point gravitongraviton-scalar loops. This implies that an infinite summation over n number of graviton-graviton-scalar loops will be the key, as illustrated in Fig. 5. To approach this systematically, we begin by disabling the kinetic coupling terms, and compute the infinite sum of the massively coupled loops. The corresponding Feynman amplitude, parameterized by momentum p, is expressed as follows.

$$\mathcal{M} = \frac{1}{-p^2 - m^2} \left( 1 - \frac{1}{-p^2 - m^2} \Sigma(p) + \dots \right)$$
 (135)

$$= \frac{1}{-p^2 - m^2 + \Sigma(p)} \tag{136}$$

Here,  $\Sigma(p)$  is a graviton-graviton-scalar single loop that can be computed in position space, using anti t-symmetry and x, y, z-symmetry. As the graviton constraints the 2-point to be separated in light-like manner, the scalar propagator becomes  $-\delta(|t|-r)/4\pi r$ . Introducing a new

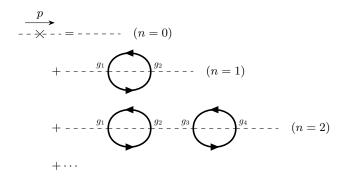


FIG. 5: Infinite summation of graviton-graviton-scalar loops. The index n denotes a number of involved loops. The interaction couplings,  $g_n$ , can be either massive or kinetic, resulting in each loop having four possible combinations: massive-massive, kinetic-kinetic, kinetic-massive, and massive-kinetic.

variable:  $z \equiv x - y$ , and using Eq. 70,

$$\Sigma = \frac{\kappa^2 m^4}{16} \int d^4 z e^{ip \cdot z} \frac{\delta(|t| - r)}{4\pi r} \times \left( \Delta_{ijkl} \Delta^{ijkl} - \Delta_{ijk}^{\ k} \Delta^{ijl}_{\ l} + \frac{1}{4} \Delta \Delta \right)$$
(137)

During the computation of  $\Sigma$  one faces various infinities:  $\delta(0)$ ,  $\delta'(0)$ , and  $\delta''(0)$ . Considering that the delta functions appeared by an approximate  $\mu \to \infty$ , we regard  $\delta'(x)$  as an odd function, thus  $\delta'(0) \to 0$ . Then the explicit expression of  $\Sigma$  becomes relatively short after the extensive and analytical simplification process.

$$\Sigma = \frac{\kappa^2 m^4}{32\pi} \int d^3 \vec{z} \frac{\cos(p_0 r) e^{ip_r r \cos \theta}}{r} \frac{226}{256} \frac{\delta^2(0)}{\pi^2 r^2}$$
(138)
$$= \frac{113}{1024} \frac{\kappa^2 m^4}{\pi^4 p_r} \left(\frac{8\pi \phi_G}{\kappa}\right)^2 \int_0^\infty dr \cos(p_0 r) \sin(p_r r)$$
(139)
$$= \frac{113}{4096} \frac{\kappa^2 m^4}{\pi^4 p_r} \left(\frac{8\pi \phi_G}{\kappa}\right)^2 \int_{-\infty}^\infty dr (-i \operatorname{sgn}(r)) \times$$
(140)
$$\left(e^{i(p_0 + p_r)r} - e^{i(p_0 - p_r)r}\right)$$

$$= \frac{113}{1024} \frac{\kappa^2 m^4}{\pi^4} \left(\frac{8\pi \phi_G}{\kappa}\right)^2 \frac{1}{p_r^2 - p_0^2}$$
(141)

Using the above results, one may renormalize the scalar mass as an effect of the self-energy diagrams. The equation for  $m_{\rm phy}$  is then expressed as follows.

$$m_{\rm phy}^2 = m^2 \left( 1 + \frac{113}{16\pi^2} \frac{(\phi_G m)^2}{-n^2} \right)$$
 (142)

One can see  $m_{\rm phy}$  affecting the scalar propagator, meaning that the mass of inertia is adjusted by the given gravitational parameter sensitivity,  $\phi_G$ . While the inertial mass of the scalar field runs, the gravitational mass remains to m.

The infinite sum of the kinetic graviton-graviton-scalar loops without the massive couplings are also computed to complete the mass renormalization.

$$\mathcal{M} = \frac{1}{-p^2 - m^2} \left( 1 + h^{ij}(p) \frac{p_i p_j}{-p^2 - m^2} + \cdots \right)$$
(143)  
$$= \frac{1}{-(\eta^{ij} + h^{ij}) p_i p_j - m^2}$$
(144)

Unlike the massively coupled case, the above has plus sign in the summation, as  $p_i$  and  $p_j$  absorbed the imaginary numbers. To obtain the explicit form of  $h^{ij}(p)$  leading terms, below three integrals are computed. The integrals are:

$$I_1 = 2 \int d^3 \vec{r} \cos(p_0 r) \frac{e^{ip_r r \cos \theta}}{r} r^2$$
 (145)

$$=8\pi \int_0^\infty dr \frac{r^2}{p_r} \cos(p_0 r) \sin(p_r r) \tag{146}$$

$$= -\frac{8\pi}{p_r} \left( \frac{1}{(p_0 + p_r)^3} - \frac{1}{(p_0 - p_r)^3} \right)$$
 (147)

$$= -16\pi \frac{3p_0^2 + p_r^2}{p^6} \tag{148}$$

and,

$$I_2 = 2 \int d^3 \vec{r} \cos(p_0 r) \frac{z^2 e^{i p_r r \cos \theta}}{r^3} r^2$$
 (149)

$$=I_1 - 16\pi \frac{p_0^2 + p_r^2}{p^4 p_r^2} - 16\pi \frac{1}{p^2 p_r^2}$$
 (150)

with,

$$I_3 = -2i \int d^3 \vec{r} \sin(p_0 r) \frac{z e^{i p_r r \cos \theta}}{r^2} r^2$$
 (151)

$$= -8\pi \int_0^\infty dr \frac{r^2}{p_r} \sin(p_0 r) \cos(p_r r)$$

$$+ 8\pi \int_0^\infty dr \frac{r}{p_r^2} \sin(p_0 r) \sin(p_r r)$$
(152)

$$=16\pi \frac{p_0^2+3p_r^2}{p^6} \frac{p_0}{p_r} + 16\pi \frac{p_0}{p^4 p_r}$$
 (153)

By the convention, z direction is defined as a direction of  $\vec{p}$ , and  $h^{ij}$  can be reduced to  $2 \times 2$  matrix without loss of generality. Using  $I_1$ ,  $I_2$  and  $I_3$ :

$$h^{ij} = -\frac{5\kappa^2}{2^{14} \cdot 3^2 \pi^4} \left( 8 \left( \frac{4\phi_G}{\kappa} \right)^2 \right)^2 \begin{pmatrix} I_1 & I_3 \\ I_3 & I_2 \end{pmatrix}$$
 (154)

What one wants to compute is  $h^{ij}p_ip_i$ .

$$h^{ij}p_ip_j = \frac{80}{3\pi^3} \left(\frac{\phi_G^2}{\kappa}\right)^2 \frac{1}{p^2} = -\frac{80}{3\pi^3} \frac{\bar{\kappa}^2 |p_i|^4}{-p^2}$$
(155)

Absorbing the results to  $m_{\rm phy}$  again, the final form of renormalized scalar propagator in momentum space is obtained. Additionally incorporating the effects from the

kinetic-massive (and massive-kinetic) coupling loops, the result becomes:

$$\mathcal{M}_{\text{renorm.}} = \frac{1}{-p^2 - (m_{\text{phy}}^2 + m_{\text{add}}^2)}$$
 (156)

where.

$$\left(\frac{m_{\text{phy}}}{m}\right)^{2} - 1 = \frac{1}{-\pi^{2}p^{2}} \left(\frac{113}{16}(\phi_{G}m)^{2} - \frac{80}{3\pi} \left(\frac{\bar{\kappa}|p_{i}|^{2}}{m}\right)^{2}\right) \tag{157}$$

with a new parameter,  $m_{\rm add}$ ,

$$\left(\frac{m_{\text{add}}}{m}\right)^2 = \frac{2260}{3\pi^3} \frac{(\phi_G \cdot \bar{\kappa}|p_i|^2)^2}{p^4 + (p^2 \cdot m_{\text{phy}}^2)}$$
(158)

Interestingly, the mass parameter running becomes significant as  $p^2$  approaches zero. For this reason,  $O(|p|) \leq O(\phi_G m)$ ,  $O(\bar{\kappa}|p_i|)$  must be met to see the mass parameter running. If the data resolution is worse than the condition, the self-energies are hidden behind the noise, and the propagator converges to the traditional QFT.

### B. Orbiting particle in a circular motion

As a second example, a Feynman amplitude for circularly orbiting scalar field is computed in the tree level. To achieve this, two scalar fields,  $\varphi_1$  and  $\varphi_2$ , are introduced, with their mass  $m_1$  and  $m_2$ , respectively. There are two constraints given to the parameters and the fields. First,  $m_1$  is set to a large value, so that only  $\varphi_2$  is in motion. Next,  $\varphi_2$  is modeled to be localized to a plane with its thickness equal to  $\Delta z$ , together with forcing the propagating direction of  $\varphi_2$  to z-direction. This setup captures a specific moment of the orbit and simplifies the calculation, while not specifying the radius of the orbit. Then the initial momentum of  $\varphi_1$ , p, is described by  $(m_1, 0, 0, 0)$ , while the initial momentum of  $\varphi_2$ , q, is  $(q_t = \sqrt{m_2^2 + q_z^2}, 0, 0, q_z)$  for  $\varphi_2$  to have a circular motion. The corresponding situation is described in Fig. 6. Using Eq. 70 and intensively reducing the terms with a coordinate in which  $\varphi_2$  is localized to z=0, the Feynman amplitude with the given conditions is computed as below  $(\delta'(0) = 0 \text{ is applied}).$ 

$$\mathcal{M}_{\text{tree}} = \int d^4w \big( V_1(w) + V_2(w) \big) \big( \delta(z) \Delta z \big) e^{i\Delta p \cdot w} \quad (159)$$

Here, w stands for the 4-coordinate (t, x, y, z),  $\Delta p$  is a momentum transfer from  $\varphi_1$  to  $\varphi_2$ , and  $\Delta z$  is inserted to compensate  $\delta(z)$ . The explicit forms of  $V_1$  and  $V_2$  are:

$$V_{1} = \frac{m_{1}^{2}m_{2}^{2}}{256r^{2}} \cdot \kappa^{2}\delta(0)\delta(|t| - r) \times \left(1816 + 112\left(\frac{q_{z}}{m_{2}}\right)^{2} + 324\frac{x^{2} + y^{2}}{r^{2}}\left(\frac{q_{z}}{m_{2}}\right)^{2}\right)$$
(160)

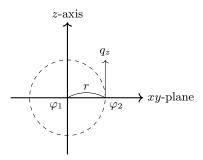


FIG. 6: Circular orbit of  $\varphi_2$  around  $\varphi_1$ . The figure depicts a moment when  $\varphi_2$  is positioned in the xy-plane, with its spatial momentum,  $\vec{q} = q_z \hat{z}$ .

and

$$V_{2} = 5 \cdot \frac{m_{1}^{2} m_{2}^{2}}{256r^{2}} \cdot \kappa^{2} \delta(0) \delta(|t| - r) \times \left(6 - (z\Delta p)^{2} + 6iz\Delta p\right) \left(\frac{q_{t}}{m_{2}} + \frac{zq_{z}}{m_{2}r}\right)^{2}$$
(161)

To obtain  $V_2$ ,  $\delta(x)\delta''(x) \to \delta(0)\delta''(x)$  is used, which has an ambiguity since it is not clear to use whether  $\delta(0)$  or  $\delta''(0)$ . In this paper,  $\delta(0)$  choice is adopted, because: 1)  $\delta(0)$  is a weaker renormalization than  $\delta''(0)$  while the diagram still converges with the choice, and 2)  $\delta(x)$  has a single peak with a single value while  $\delta''(x)$  has three peaks with different values, which is safer to regard  $\delta(0)$  as  $r \times (\text{constant})$  ( $\delta(s=0) \propto \delta(r=0)/r$  is a constant). Applying  $\delta(z)$  to  $V_1$  and  $V_2$ , the surviving terms are:

$$V_1 + V_2 \to V = V_r \cdot \delta(|t| - r) \tag{162}$$

where

$$V_r = (\kappa \phi_G) \frac{m_1^2 m_2^2}{16r} \left( 979 + 233 \frac{x^2 + y^2}{r^2} \left( \frac{q_z}{m_2} \right)^2 \right)$$
 (163)

From the well-known property of the scattering theory, it is possible to relate V with the perturbative potential energy around  $z \approx 0$  [35].

$$V_{\text{pot.}}(\vec{r}) = -\frac{1}{m_1 m_2} \int V dt$$
 (164)

 $m_1$  and  $m_2$  are the remnants after considering the scalar current flows. Defining  $G \equiv \kappa \phi_G \cdot 233/8$ ,  $l^2 \equiv 233(x^2 + y^2)q_z^2/979m_2^2$  (l corresponds to an angular momentum per mass), and adding the  $\varphi_2$  centrifugal term from the kinetic source, the Schwarzschild effective radial potential [22, 36] is recovered.

$$\frac{233}{979}V_{\text{eff}}(r,z=0) = \frac{m_2l^2}{2r^2} - \frac{Gm_1m_2}{r} - \frac{Gm_1m_2l^2}{r^3}$$
(165)

The dimensionless factor 233/979 can be absorbed to the remaining constants such as  $m_2$ , or the other parameters depending on the model.

#### VII. CONCLUSION

In this study, we presented a novel approach to integrating the principle of relativity into quantum theories. Instead of introducing general relativity, our theory extends the principle of relativity to the quantum domain. The application of this principle introduces a new constraint to the theory, which we term the QR condition. This condition was found essential for preserving identical structures in the application of active and passive transformations. Given that the theory is based on a newly introduced fundamental principle, the QR condition has led us to define clear concepts of both the quantum coordinate and the classical frame. As a consequence, it elucidated the role of the quantum observer: an individual who defines a quantum coordinate, aligning him to a classical frame.

Our results suggest that the most natural way to construct a dynamic universe is to promote the metric tensor to a quantum field, which makes the corresponding classical Lagrangian identical to the Einstein-Hilbert action in the integrated form. Using the path integral formalism, we applied the QR condition and derived the full expression of the partition function. Based on our findings, the quantum fluctuation of the metric field can be decoupled, while the classical part of the metric obeys strictly the Einstein field equations, constrained by the delta function. We further explored the mathematical expression of the quantum fluctuation, which we identify as a ghost particle called the graviton, by deriving the effective Lagrangian.

Introducing a free scalar field in an approximately flat spacetime, our results indicate that gravitational effects can be naturally incorporated into a quantum theory. Furthermore, the emergence of general relativity in the classical limit highlights the potential harmony between the macroscopic and microscopic realms. By assuming

gravion propagators as principal propagators with few acceptible scale conditions, we showed that the theory is even renormalizable. We also computed graviton propagators in position space, and the provided examples emphasize the practical implications and validity of our theory. Future studies can explore the applications of our theory, potentially bridging long-standing divides between quantum mechanics and general relativity.

Although several postulates and assumptions have been newly adopted, our approach offers new insights into the understanding of the quantum world and the renormalization strategy. There is also much more to explore on the conceptual side as well, which is not something we delve deeply into in its development. For instance, one could begin with the intriguing observation that a state vector of the full system in quantum relativity is universal and independent of the observer, given that the quantum coordinate transform is well-defined. Consequently, the phenomenon of wave function collapse, the unidentified irreversible process, is also universal, by applying the appropriate quantum coordinate transformation operator to the collapse projector.

What's particularly interesting is comparing two separate wave function collapses. Since the quantum coordinate transformation essentially is applying translations to each superposed state, there's no mechanism to alter the order of collapse events, although the collapsed time and the resulting field configuration may appear different to different observers. This universality of orders between the irreversible events leads us to think about the concept of the fundamental time. However, we leave such explorations of the framework for future investigations.

# ACKNOWLEDGMENTS

The authors would like to give special thanks to Dr. Junu Jeong, Dr. Danho Ahn, and Dr. Younggeun Kim for their valuable contributions, discussions, and support.

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