Supplemental Material: Quantifying Coherence

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I. THE MAXIMALLY COHERENT STATE

In the first section of the Supplemental Material, we show that every $d \times d$ state $\hat{\varrho}$ may be prepared from $|\Psi_d\rangle$ by using only incoherent operations. We do this by an explicit construction of Kraus operators $\hat{K}_n$, $n = 1, \ldots, d$. Let

$$\hat{K}_n = \sum_{i=1}^{d} c_i |i\rangle\langle m_{i+n-1}| \tag{I.1}$$

with $c_i \in \mathbb{C}$ such that $\sum_{i=1}^d |c_i|^2 = 1$ and $m_x = \mathrm{mod}(x-1, d) + 1 = x - \lfloor \frac{x-1}{d} \rfloor d$. Then $\sum_{n=1}^d \hat{K}_n^\dagger \hat{K}_n = \mathbb{1}$ as $\sum_n |m_{i+n-1}\rangle\langle m_{i+n-1}| = \sum_{n=1}^d |n\rangle\langle n| = \mathbb{1}$ for all i. Further, for any diagonal density matrix $\hat{\sigma} = \sum_i \sigma_i |i\rangle\langle i| \in \mathcal{I}$, we find

$$\begin{aligned}\hat{K}_n \hat{\sigma} \hat{K}_n^\dagger &= \sum_{i,j,k=1}^{d} \sigma_i c_j c_k^* \delta_{i,m_{j+n-1}} \delta_{i,m_{k+n-1}} |j\rangle\langle k| \\ &= \sum_k \sigma_{m_{k+n-1}} |c_k|^2 |k\rangle\langle k| \in \mathcal{I},\end{aligned} \tag{I.2}$$

i.e., these Kraus operators define an incoherent operation in terms of the two classes of quantum operations (A) and (B), see the main text for further details. Now, let $|\Psi_d\rangle$ be the maximally coherent state. Then

$$\begin{aligned}\hat{K}_n |\Psi_d\rangle &= \frac{1}{\sqrt{d}} \sum_{i=1}^{d} c_i |i\rangle \sum_{j=1}^{d} \delta_{j,m_{i+n-1}} \\ &= \frac{1}{\sqrt{d}} \sum_{i=1}^{d} c_i |i\rangle.\end{aligned} \tag{I.3}$$

Hence, for each outcome n, we have

$$\hat{\varrho}_n = \frac{\hat{K}_n |\Psi_d\rangle\langle\Psi_d| \hat{K}_n^\dagger}{p_n} = |\phi\rangle\langle\phi| \tag{I.4}$$

with probability $p_n = 1/d$ and where $|\phi\rangle = \sum_{i=1}^d c_i |i\rangle$. Thus, with certainty, every pure state $|\phi\rangle$ may be prepared by incoherent operations from the maximally coherent state. Now let $\hat{\varrho} = \sum_l q_l |\phi_l\rangle\langle\phi_l|$, $\sum_l q_l = 1$, $|\phi_l\rangle = \sum_i c_i^{(l)} |i\rangle$, be an arbitrary mixed state and define the Kraus operators

$$\hat{K}_n^{(l)} = \sqrt{q_l} \sum_{i=1}^{d} c_i^{(l)} |i\rangle\langle m_{i+n-1}|. \tag{I.5}$$

As above, they sum to unity and are incoherent. Further

$$\sum_{n,l} \hat{K}_n^{(l)} |\Psi_d\rangle\langle\Psi_d| \big(\hat{K}_n^{(l)}\big)^\dagger = \sum_l q_l |\phi_l\rangle\langle\phi_l| = \hat{\varrho}, \tag{I.6}$$

i.e., performing generalized measurements according to the $\hat{K}_n^{(l)}$ (and actively erasing the information about measurement outcomes in the case of (B)) prepares the state $\hat{\varrho}$ with certainty.

II. REALIZATION OF COHERENT GATES BY INCOHERENT OPERATIONS AND COHERENT STATES AS RESOURCE

We set out to show how to implement any unitary operation $\hat{U} = \sum_{i,j=0}^1 U_{ij} |i\rangle\langle j|$ only by means of incoherent operations $\{\hat{K}_i\}$ and the maximally coherent state

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}} \sum_{l=0}^{1} |l\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{II.1}$$

To this end, we let

$$\begin{aligned}\hat{K}_0 = {}& U_{00}|00\rangle\langle 00| + U_{10}|10\rangle\langle 01| \\ &+ U_{01}|00\rangle\langle 10| + U_{11}|10\rangle\langle 11|\end{aligned} \tag{II.2}$$

and

$$\begin{aligned}\hat{K}_1 = {}& U_{00}|01\rangle\langle 01| + U_{10}|11\rangle\langle 00| \\ &+ U_{01}|01\rangle\langle 11| + U_{11}|11\rangle\langle 10|\end{aligned} \tag{II.3}$$

be two Kraus operators. Note that (i) $\sum_{i=0}^1 \hat{K}_i^\dagger \hat{K}_i = \mathbb{1}$ and (ii) $\hat{K}_i \mathcal{I} \hat{K}_i^\dagger \subset \mathcal{I}$ for all $i = 0, 1$. This can be verified straightforwardly by inspection such that these operators form an incoherent operation as defined in the main text (of type (A) and (B)). Now, let $|\phi\rangle = \sum_{k=0}^1 \phi_k |k\rangle$, and

$$|\xi\rangle = |\phi\rangle \otimes |\Psi_2\rangle = \frac{1}{\sqrt{2}} \sum_{k,l=0}^{1} \phi_k |kl\rangle, \tag{II.4}$$

then $\hat{K}_0|\xi\rangle = \hat{U}|\phi\rangle \otimes |0\rangle/\sqrt{2}$ and $\hat{K}_1|\xi\rangle = \hat{U}|\phi\rangle \otimes |1\rangle/\sqrt{2}$. Thus, under type (A) and (B) operations, the system will be in the desired state $\hat{U}|\phi\rangle\langle\phi|\hat{U}^\dagger$ with certainty. Recall that this is achieved only by the incoherent operators $\hat{K}_i$, $i = 0, 1$, and the consumption of one maximally coherent state $|\Psi_2\rangle$.

III. FINITE COPY TRANSFORMATIONS

In this section, we provide a specific set of Kraus operators that allow – with finite probability – to transform a pure state into another. For this, let $|\psi\rangle = \sum_{l=1}^d \psi_l |l\rangle \in \mathbb{C}^d$ and $|\phi\rangle = \sum_{l=1}^d \phi_l |l\rangle \in \mathbb{C}^d$ be two pure quantum states. Denote as M_ψ

and M_ϕ the number of non-zero coefficients for the respective states, i.e.,

$$M_\psi = \Big|\{\psi_l|\psi_l \neq 0 \text{ for } l = 1,\dots,d\}\Big| \tag{III.1}$$

and M_ϕ equivalently. If $M_\psi \geq M_\phi$, then one can construct a set of incoherent Kraus operators $\{\hat{K}_n\}$ such that

$$\hat{\varrho}_1 = \frac{\hat{K}_1|\psi\rangle\langle\psi|\hat{K}_1^\dagger}{p_1} = |\phi\rangle\langle\phi| \tag{III.2}$$

with probability $p_1 = 1/\sum_{l,\psi_l\neq 0}\big|\frac{\phi_l}{\psi_l}\big|^2$.

First, assume that $M_\psi = d$, i.e., $\psi_l \neq 0$ for all $l = 1,\dots,d$. Let

$$\hat{K}_n = \sum_{l=1}^{d}\frac{c_l}{\psi_l}|l\rangle\langle m_{l+n-1}|, \tag{III.3}$$

where m_x is defined as above, i.e., $m_x = \text{mod}(x-1,d)+1 = x - \lfloor\frac{x-1}{d}\rfloor d$ and $c_l = \phi_l\sqrt{p_1}$ with $p_1 = 1/\sum_l\big|\frac{\phi_l}{\psi_l}\big|^2$. Note that these operators define incoherent operations as $\hat{K}_n\hat{\delta}\hat{K}_n^\dagger \in \mathcal{I}$ for all $\hat{\delta} = \sum_i \delta_i|i\rangle\langle i| \in \mathcal{I}$. Further, the Kraus operators satisfy the normalization condition, i.e.,

$$\sum_n \hat{K}_n^\dagger\hat{K}_n = \sum_n\sum_l\left|\frac{\phi_l}{\psi_l}\right|^2 p_1|n\rangle\langle n| = \mathbb{1}, \tag{III.4}$$

where, as before, $\sum_n|m(l+n-1)\rangle\langle m(l+n-1)| = \sum_n|n\rangle\langle n| = \mathbb{1}$ for all l. Moreover, we have

$$\hat{K}_1|\psi\rangle = \sum_{k,l=1}^{d}\frac{c_l}{\psi_l}\psi_k|l\rangle\langle l|k\rangle = \sum_{l=1}^{d}c_l|l\rangle = \sqrt{p_1}|\phi\rangle \tag{III.5}$$

such that we find

$$\hat{\varrho}_1 = \frac{\hat{K}_1|\psi\rangle\langle\psi|\hat{K}_1^\dagger}{p_1} = |\phi\rangle\langle\phi| \tag{III.6}$$

with probability $p_1 = \mathbf{tr}\big[\hat{K}_1|\psi\rangle\langle\psi|\hat{K}_1^\dagger\big] = 1/\sum_l\big|\frac{\phi_l}{\psi_l}\big|^2$. Now, if $M_\psi \neq d$ use the permutations P_ψ and P_ϕ to rearrange the entries of the states such that $P_\psi|\psi\rangle = \sum_l \psi_{P_\psi(l)}|l\rangle$ with $\psi_{P_\psi(1)} \geq \ldots \geq \psi_{P_\psi(M_\psi)} > 0$, and similarly for $P_\phi|\phi\rangle$. Note that a permutation matrix maps $\mathcal{I}$ onto itself and hence is an incoherent operation. Now, separate the total Hilbert space $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1$ is of dimension M_ψ and $\mathcal{H}_2$ is of dimension $d - M_\psi$ (note that $P_\psi|\psi\rangle$ and $P_\phi|\phi\rangle$ are only supported in $\mathcal{H}_1$). Apply the Kraus operators $\hat{L}_n = \hat{K}_n \oplus \mathbb{O}$ for $n = 1,\dots,M_\psi$ and $\hat{L}_n = \mathbb{O}\oplus|l_n\rangle\langle l_n|$ with $l_n = 1,\dots,d-M_\psi$ for $n = M_\psi+1,\dots,d$ to the state $P_\psi|\psi\rangle$. Here, the $\hat{K}_n$ are as in equation (III.3) but restricted to the subspace $\mathcal{H}_1$ (i.e., the sum is over all $\psi_l \neq 0$). Note that the system $\{\hat{L}_n\}$ defines a valid incoherent operation as every element maps diagonal matrices onto diagonal matrices and further satisfies the normalization condition. As before, the application of the operator $\hat{K}_1$ to the subsystem $\mathcal{H}_1$ will produce the state $P_\phi|\phi\rangle$ in $\mathcal{H}$ with probability $p_1 = 1/\sum_{l,\psi_l\neq 0}\big|\frac{\phi_l}{\psi_l}\big|^2$ where the sum is over all $\psi_l \neq 0$ (i.e., in the subspace $\mathcal{H}_1$). Applying the inverse of the permutation P_ϕ (which is an incoherent operation) produces the desired state $|\phi\rangle$ with an overall probability of $p_1 = 1/\sum_{l,\psi_l\neq 0}\big|\frac{\phi_l}{\psi_l}\big|^2$. Note that this protocol may not be optimal, i.e., $P\big(|\psi\rangle \mapsto |\phi\rangle\big) \geq p_1 = 1/\sum_{l,\psi_l\neq 0}\big|\frac{\phi_l}{\psi_l}\big|^2$, and sub-selection is required (type (B) operations).

IV. A THIRD MONOTONICITY CRITERION

Besides the operations of type (A) and (B) that are discussed in the main text, one may argue that sub-selection based on measurement outcomes is described by adding a classical flag to the relevant quantum states $\hat{\varrho}_i$, i.e., that one obtains a state of the form $\sum_i p_i|i\rangle\langle i|\otimes\hat{\varrho}_i$, with p_i, $\hat{\varrho}_i$ as in (B) and $|i\rangle\langle i| \in \mathcal{I}$. Note that, here, (A) follows by erasing the classical flag, i.e., tracing out the auxiliary system, and (B) may be obtained by projective measurements $\hat{P}_i = |i\rangle\langle i|\otimes\mathbb{1}$ (which are incoherent operators with respect to the basis $\{|i\rangle\}$) and tracing over the ancilla. Monotonicity under these incoherent operations would then require:

(C2c) $C(\hat{\varrho}) \geq C\big(\sum_i p_i|i\rangle\langle i|\otimes\hat{\varrho}_i\big)$ for all $|i\rangle\langle i| \in \mathcal{I}$ and all $\{\hat{K}_i\}$ with $\sum_i \hat{K}_i^\dagger\hat{K}_i = \mathbb{1}$ and $\hat{K}_i\mathcal{I}\hat{K}_i^\dagger \subset \mathcal{I}$.

We find that the relative entropy of coherence and the l_1-norm of coherence straightforwardly fulfil this additional constraint as they satisfy (C2b), (C3) and $C(|i\rangle\langle i|\otimes\hat{\varrho}_i) \leq C(\hat{\varrho}_i)$:

For the relative entropy, one has

$$S(\hat{\varrho}\|\hat{\delta}) = S(|\alpha\rangle\langle\alpha|\otimes\hat{\varrho}\||\alpha\rangle\langle\alpha|\otimes\hat{\delta}) \tag{IV.1}$$

for all state vectors $|\alpha\rangle$ and all states $\hat{\varrho}$ and $\hat{\delta}$. Now, if $|i\rangle\langle i| \in \mathcal{I}$ and $\hat{\delta}\in\mathcal{I}$ then $|i\rangle\langle i|\otimes\hat{\delta}\in\mathcal{I}$, i.e., whenever $|i\rangle\langle i|\in\mathcal{I}$, we have

$$\begin{aligned} C_{\text{rel. ent.}}&\Big(\sum_i p_i|i\rangle\langle i|\otimes\hat{\varrho}_i\Big) \\ &\overset{\text{(C3)}}{\leq} \sum_i p_i C_{\text{rel. ent.}}\big(|i\rangle\langle i|\otimes\hat{\varrho}_i\big) \\ &\leq \sum_i p_i S\big(|i\rangle\langle i|\otimes\hat{\varrho}_i\||i\rangle\langle i|\otimes\hat{\delta}_i^*\big) \\ &\overset{\text{(IV.1)}}{=} \sum_i p_i S(\hat{\varrho}_i\|\hat{\delta}_i^*) = \sum_i p_i C_{\text{rel. ent.}}(\hat{\varrho}_i) \\ &\overset{\text{(C2b)}}{\leq} C_{\text{rel. ent.}}(\hat{\varrho}), \end{aligned} \tag{IV.2}$$

which is (C2c).

For the l_1-norm, we observe that for any $|i\rangle\langle i|\in\mathcal{I}$ and any matrix $\hat{M}$, one has

$$\begin{aligned} \big\||i\rangle\langle i|\otimes\hat{M}\big\|_{l_1} &= \sum_{j,k,l,m}\Big|\big(|i\rangle\langle i|\otimes\hat{M}\big)_{(j,k),(l,m)}\Big| \\ &= \sum_{j,k,l,m}\delta_{j,i}\delta_{l,i}|M_{k,m}| \\ &= \sum_{k,m}|M_{k,m}|, \end{aligned} \tag{IV.3}$$

i.e., $\big\||i\rangle\langle i|\otimes(\hat\varrho-\hat\varrho_{\text{diag}})\big\|_{l_1}=\big\|\hat\varrho-\hat\varrho_{\text{diag}}\big\|_{l_1}$, and therefore, as above,

$$\begin{aligned}
&C_{l_1}\Big(\sum_i p_i|i\rangle\langle i|\otimes\hat\varrho_i\Big)\\
&\quad\overset{\text{(C3)}}{\leq}\sum_i p_i C_{l_1}\big(|i\rangle\langle i|\otimes\hat\varrho_i\big)\\
&\quad\leq\sum_i p_i\big\||i\rangle\langle i|\otimes\hat\varrho_i-|i\rangle\langle i|\otimes\hat\varrho_i^{\text{diag}}\big\|_{l_1}\\
&\quad=\sum_i p_i\big\|\hat\varrho_i-\hat\varrho_i^{\text{diag}}\big\|_{l_1}=\sum_i p_i C_{l_1}(\hat\varrho_i)\\
&\quad\overset{\text{(C2b)}}{\leq} C_{l_1}(\hat\varrho),
\end{aligned}\tag{IV.4}$$

which is condition (C2c).

V. (C2b) FOR THE RELATIVE ENTROPY OF COHERENCE

We set out to establish the monotonicity criterion for the relative entropy of coherence $C_{\text{rel. ent.}}=\min_{\delta\in\mathcal{I}}S(\hat\varrho\|\hat\delta)$ for condition (C2b), i.e., we show that

$$C_{\text{rel. ent.}}(\hat\varrho)\geq\sum_n p_n C_{\text{rel. ent.}}(\hat\varrho_n)\tag{V.1}$$

for all $\{\hat K_n\}$ with $\sum_n\hat K_n^\dagger\hat K_n=\mathbb{1}$ and $\hat K_n\mathcal{I}\hat K_n^\dagger\subset\mathcal{I}$. Let $\hat\varrho_n=\hat K_n\hat\varrho\hat K_n^\dagger/p_n$ with $p_n=\text{tr}[\hat K_n\hat\varrho\hat K_n^\dagger]$, then

$$S(\hat\varrho\|\hat\delta)\geq\sum_n p_n S(\hat\varrho_n\|\hat K_n\hat\delta\hat K_n^\dagger/\text{tr}[\hat K_n\hat\delta\hat K_n^\dagger]).\tag{V.2}$$

This follows as the quantum relative entropy satisfies conditions (F1)–(F5) in [1]. With this, we have

$$\begin{aligned}
C_{\text{rel. ent.}}(\hat\varrho)&=S(\hat\varrho\|\hat\delta^*)\\
&\geq\sum_n p_n S(\hat\varrho_n\|\hat K_n\hat\delta^*\hat K_n^\dagger/\text{tr}[\hat K_n\hat\delta^*\hat K_n^\dagger])\\
&\geq\sum_n p_n\min_{\delta\in\mathcal{I}}S(\hat\varrho_n\|\hat\delta)\\
&=\sum_n p_n C_{\text{rel. ent.}}(\hat\varrho_n),
\end{aligned}\tag{V.3}$$

as $\hat K_n\hat\delta^*\hat K_n^\dagger\in\mathcal{I}$. This proves (C2b) for the quantum relative entropy.

VI. (C2b) FOR THE l_1-NORM OF COHERENCE

We show monotonicity for the l_1-norm of coherence according to (C2b). Recall the closed form of this coherence measure, that is,

$$C_{l_1}(\hat\varrho)=\sum_{\substack{i,j\\i\neq j}}|\varrho_{i,j}|.\tag{VI.1}$$

Now, for given $\hat\varrho$, consider

$$\begin{aligned}
\sum_n p_n C_{l_1}(\hat\varrho_n)&=\sum_n p_n\sum_{\substack{i,j\\i\neq j}}|[\hat\varrho_n]_{i,j}|\\
&=\sum_n\sum_{\substack{i,j\\i\neq j}}|[\hat K_n\hat\varrho\hat K_n^\dagger]_{i,j}|\\
&=\sum_n\sum_{\substack{i,j\\i\neq j}}\Big|\sum_{k,l}[\hat K_n]_{i,k}\varrho_{k,l}[\hat K_n^\dagger]_{l,j}\Big|,
\end{aligned}\tag{VI.2}$$

where, denoting by $\hat\varrho_{\text{diag}}$ the incoherent state $\hat\varrho_{\text{diag}}=\sum_k\varrho_{k,k}|k\rangle\langle k|$, we have

$$\begin{aligned}
\sum_k[\hat K_n]_{i,k}\varrho_{k,k}[\hat K_n^\dagger]_{k,j}&=\big(\hat K_n\hat\varrho_{\text{diag}}\hat K_n^\dagger\big)_{i,j}\\
&=\delta_{i,j}\big(\hat K_n\hat\varrho_{\text{diag}}\hat K_n^\dagger\big)_{i,i},
\end{aligned}\tag{VI.3}$$

i.e.,

$$\begin{aligned}
\sum_n p_n C_{l_1}(\hat\varrho_n)&=\sum_n\sum_{\substack{i,j\\i\neq j}}\Big|\sum_{\substack{k,l\\k\neq l}}[\hat K_n]_{i,k}\varrho_{k,l}[\hat K_n^\dagger]_{l,j}\Big|\\
&\leq\sum_{\substack{k,l\\k\neq l}}|\varrho_{k,l}|\sum_n\sum_{\substack{i,j\\i\neq j}}|[\hat K_n]_{i,k}[\hat K_n^\dagger]_{l,j}|.
\end{aligned}\tag{VI.4}$$

Further, we find

$$\begin{aligned}
&\sum_n\sum_{\substack{i,j\\i\neq j}}|[\hat K_n]_{i,k}[\hat K_n^\dagger]_{l,j}|\\
&\quad\leq\sum_n\sum_i|[\hat K_n]_{i,k}|\sum_j|[\hat K_n^\dagger]_{l,j}|\\
&\quad\leq\sqrt{\sum_n\Big(\sum_i|[\hat K_n]_{i,k}|\Big)^2\sum_m\Big(\sum_j|[\hat K_m^\dagger]_{l,j}|\Big)^2}
\end{aligned}\tag{VI.5}$$

and

$$\begin{aligned}
&\sum_n\Big(\sum_i|[\hat K_n]_{i,k}|\Big)^2\\
&\quad=\sum_n\sum_{i,j}|[\hat K_n]_{i,k}[\hat K_n^\dagger]_{k,j}|\\
&\quad=\sum_n\sum_{i,j}|\langle i|\hat K_n|k\rangle\langle k|\hat K_n^\dagger|j\rangle|\\
&\quad=\sum_n\sum_{i,j}\delta_{i,j}|\langle i|\hat K_n|k\rangle\langle k|\hat K_n^\dagger|j\rangle|\\
&\quad=\sum_n\sum_i|\langle i|\hat K_n|k\rangle\langle k|\hat K_n^\dagger|i\rangle|\\
&\quad=\sum_n\sum_i\langle k|\hat K_n^\dagger|i\rangle\langle i|\hat K_n|k\rangle=1,
\end{aligned}\tag{VI.6}$$

such that

$$\sum_n p_n C_{l_1}(\hat\varrho_n)\leq\sum_{\substack{k,l\\k\neq l}}|\varrho_{k,l}|=C_{l_1}(\hat\varrho).\tag{VI.7}$$

This proves (C2b) for the l_1-norm of coherence. Furthermore, this – together with the convexity of the measure, i.e., condition (C3) – implies monotonicity under (C2a) as discussed in the main text.

VII. VIOLATION OF (C2b)

In this section of the Supplemental Material, we show that

$$C_{l_2}(\hat{\varrho}) := \min_{\hat{\sigma}\in\mathcal{I}} \|\hat{\varrho}-\hat{\sigma}\|_{l_2}^2 = \sum_{\substack{i,j \\ i\neq j}} |\varrho_{i,j}|^2 \tag{VII.1}$$

does not satisfy monotonicity under (C2b). We prove this fact by establishing a counter-example. For this, we construct a set of Kraus operators together with a specific quantum state and show that the functional C_{l_2} can increase on average under incoherent operations. Let

$$\hat{K}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \quad \text{and} \quad \hat{K}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix} \tag{VII.2}$$

with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. Note that the latter constraint guarantees that $\sum_n \hat{K}_n^\dagger \hat{K}_n = \mathbb{1}$. Moreover, we have $\hat{K}_n \hat{\delta} \hat{K}_n^\dagger \in \mathcal{I}$ for $n = 1, 2$ and all $\hat{\delta} \in \mathcal{I}$. Hence, in none of the outcomes the quantum operations $\{\hat{K}_n\}$ generate coherence from incoherent states. Now, consider the state

$$\hat{\varrho} = \frac{1}{2}\left[|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|\right] \tag{VII.3}$$

with $|\psi_1\rangle = [0\ 1\ 0]^{\mathrm{T}}$ and $|\psi_2\rangle = [1\ 0\ 1]^{\mathrm{T}}/\sqrt{2}$. We find $C_{l_2}(\hat{\varrho}) = 1/8$ and

$$\sum_{k=1}^{2} p_k C_{l_2}(\hat{\varrho}_k) = p_2 C_{l_2}(\hat{\varrho}_2) = \frac{|\beta|^2}{2(1+|\beta|^2)}. \tag{VII.4}$$

For $|\beta|^2 > 1/3$ monotonicity in formulation (C2b) is violated, that is, $\sum_{k=1}^{2} p_k C_{l_2}(\hat{\varrho}_k) > C(\hat{\varrho})$. This can be achieved with, e.g., $\alpha = 1/\sqrt{2} = \beta$. In words, allowing for sub-selection, the sum of the absolute values squared of the off-diagonal elements is not a proper measure to quantify coherence in a quantum system.

VIII. OTHER CANDIDATES FOR COHERENCE MEASURES

In the last section of the Supplemental Material, we briefly comment on other possible candidates for coherence measures. Consider the fidelity between quantum states [2]

$$F(\hat{\varrho}, \hat{\delta}) = \mathrm{tr}\left[\sqrt{\hat{\varrho}^{1/2}\hat{\delta}\hat{\varrho}^{1/2}}\right]^2 = \|\hat{\varrho}^{1/2}\hat{\delta}^{1/2}\|_{\mathrm{tr}}^2. \tag{VIII.1}$$

It is known that $\sqrt{F}$ is jointly concave, non-decreasing under CPTP maps, and $F(\hat{\varrho}, \hat{\delta}) = 1$ iff $\hat{\varrho} = \hat{\delta}$, see, e.g., Refs. [3, 4] and references therein. Hence, the coherence measure induced by

$$\mathcal{D}(\hat{\varrho}, \hat{\delta}) = 1 - \sqrt{F(\hat{\varrho}, \hat{\delta})} \tag{VIII.2}$$

fulfils (C1'), (C2a), and (C3).

The trace norm

$$\mathcal{D}_{\mathrm{tr}}(\hat{\varrho}, \hat{\delta}) = \|\hat{\varrho} - \hat{\delta}\|_{\mathrm{tr}} \tag{VIII.3}$$

is a matrix norm and contracting under CPTP maps [5], i.e., as discussed in the main text, the induced measure of coherence fulfils (C1'), (C2a), and (C3). Note, however, that in general $\min_{\hat{\delta}\in\mathcal{I}} \|\hat{\varrho} - \hat{\delta}\|_{\mathrm{tr}} \neq \|\hat{\varrho} - \hat{\varrho}_{\mathrm{diag}}\|_{\mathrm{tr}}$.

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