Federal University of Goiás

Institute of Physics

Quantum Pequi Group

The static and the dynamical Casimir effects

Monography

Gustavo de Oliveira

Goiânia - 2022

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Federal University of Goiás
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The static and the dynamical Casimir effects

Gustavo de Oliveira

Monograph presented to the Institute of Physics of the Federal University of Goiás as partial requirement to obtain the Bachelor's degree in Physics.

Advisor: *Prof. Lucas Chibebe Céleri*

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Aos 04 dias do mês de Abril de 2022, a partir das 10h00min realizou-se a sessão pública de Defesa de Trabalho de Conclusão de Curso do estudante do curso de Física, Bacharelado, Gustavo de Oliveira**,** matrícula 201800321, para apresentar sua monografia intitulada: "**The static and the dynamical Casimir effects**". A banca examinadora foi composta pelos professores **Lucas Chibebe Céleri** (IF/UFG), **Norton Gomes de Almeida** (IF/UFG) e **Alexandre Dodonov** (IF/UnB). A sessão pública de Defesa de TCC foi aberta pelo Presidente da Banca Examinadora, Professor Lucas Chibebe Céleri (Orientador), que na sequência passou a palavra para o estudante apresentar sua monografia. Após a exposição, a Banca Examinadora realizou a arguição do estudante. Ao finalizar a arguição, a Banca reuniu-se em sessão secreta a fim de concluir o julgamento da monografia. A Banca atribuiu ao estudante a nota **9,80**, este foi **APROVADO** na disciplina de TCC. Proclamados os resultados pelo Professor Lucas Chibebe Céleri (Presidente), foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos membros da Banca Examinadora. Devido ao fato de o membro externo, o Prof. Alexandre Dodonov, não possuir cadastro no sistema SEI, o presidente da Banca Examinadora assinou assina a ata também em seu nome.

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ABSTRACT

Modifications of the structure of the vacuum state of quantum fields due to imposed boundaries conditions lie at the heart of two different but interconnected phenomena: the emergence of measurable forces on the vicinity of boundaries as described by the static Casimir effect and the process of creation of particles by non-stationary boundary conditions as expressed by the dynamical Casimir effect. In this sense, the present work aims to introduce and discuss these two fundamental quantum field theory phenomena by a theoretical reconstruction of their most important features.

Keywords: Casimir effect; dynamical Casimir effect; quantum field theory; zero point energy, vacuum fluctuations.

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List of Abbreviations

DCE	Dynamical Casimir Effect.
EMF	Electromagnetic Field.
QFT	Quantum Field Theory.
SCE	Static Casimir Effect.
ZPE	Zero Point Energy.

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Chapter 1

Introduction

Quantum Field Theory (QFT), the unified description of quantum mechanics, classical field theory and special relativity, is perhaps one of the most successful scientific paradigms ever developed, in the sense that it provides the framework for theoretical predictions with the best agreement with experiments in all of science [1]. The theory basically operates by associating each fundamental particle to an excitation of a correspondent quantum field which permeates space-time. Because of the intrinsic quantum aspect, not only the field persists in the absence of particles with an infinity of different modes, but more shocking, the field energy of each mode still fluctuates around a non-zero value. As we will see later, the modification of those fluctuations patterns imposed by boundary conditions are intrinsically related to two different but interconnected quantum phenomena, the emergence of forces in the edges of cavities, as described by the static Casimir effect (SCE) [2] and the creation of real particles by moving boundaries, as expressed by the dynamical Casimir effect (DCE) [3–6].

But let us start from the beginning, more exactly, in the turn of the 19th century, when classical physics is struggling to theoretically determine the correct spectrum available from experiments for the power emitted by an empty cavity at finite temperature. This anomaly, called black-body problem, would just meet a solution in 1900, when Planck in an "act of desperation", postulated that energy exchange between atoms and the electromagnetic radiation modes should only occur in an integral multiple of the mode oscillating frequency ν [7]. It was in this environment of disruptive ideas that later, in 1912, Planck introduced, as a correction to his former theory, the concept of zero point energy (ZPE): the lowest possible energy value that each oscillator would possess. Initially of only theoretical interest, this very unusual concept could be experimentally confirmed for the first time in 1924, when comparisons in molecular spectroscopy between the vibrational spectra of two molecule isotopes suggested a minimum vibrational energy for atomic systems [8].

With the formulation of non-relativistic quantum mechanics around 1925, it became clear that ZPE was a consequence of the intrinsic fluctuating character of the quantum harmonic oscillator, where Heisenberg uncertainty relations impose a lower limit for the simultaneous deviation of its canonical variables and, therefore, a non-vanishing energy value in the ground state. Despite very esoteric, for finite systems the notion was perfectly fine with the physics of that time, but everything changed with the 1927 Dirac's seminal paper "*The Quantum Theory of the Emission and Absorption of Radiation*" [9], where the basis to quantum mechanically treat infinite continuous systems like the radiation field were drawn[1]: QFT was born and with it a serious problem concerning ZPE. The issue revolved around the known result that the electromagnetic field (EMF) can be expanded in an infinite series of orthogonal mode functions with a harmonic oscillator character. Since the natural route founded to quantize the theory was by treating each oscillator in a quantum mechanical manner [10], as a side effect of the procedure, each one of the infinite oscillators adds to the total radiation energy a ZPE contribution, meaning that vacuum should have an infinite energy density. The issue configured the very first divergence encountered on the theory but was mostly ignored on the grounds that only energy differences have physical significance.

The attitude of not attributing physical meaning to ZPE of the radiation field was the zeitgeist of the time, but it was about to change after the dutch physicist Hendrick Casimir showed how to make measurable predictions from considerations of the vacuum energy. It begins in 1930 after London correctly explaining the dispersive van der Walls forces between neutral but polarizable atoms with quantum mechanics [11]. With experiments in colloidal suspensions showing disagreement for large separations, Casimir and Polder, in 1948, were taken to dive right into the lengthy calculations of perturbative quantum electrodynamics. By considering retardation effects they obtained the correct expression for the interaction, now known as Casimir-Polder forces, a retarded version of the van der Wall counterpart. After a brief conversation with Niels Bohr about the unexpected simplicity of the obtained expression, Casimir was taken to consider the possible role played by ZPE of the EMF in those interactions. It was then, in 1947, that Casimir himself would predict the emergence of an attractive force in vacuum between two neutral and parallel ideal conductor plates as a macroscopic manifestation of the ZPE of the radiation field: the so called *static Casimir effect*, or simply Casimir effect [2].

The idea behind the chosen configuration resides in the known fact by Casimir at the time, that contrary to the measurement of ZPE of atomic systems, there is no isotope analogue for vacuum *i.e.*, you can not duplicate the vacuum state and compare energy differences. The alternative method founded by Casimir to emulate such energy comparisons turns out to rely in the introduction of physical elements able to perturb the properties of the vacuum. In the EMF case (the most propitious quantum field for detection) ideal conductors fit like a glove for this purpose since they behave like perfect mirrors, killing parallel (perpendicular) components of electric (magnetic) fields at the plate vicinity [12]. The perturbation occurs because, despite vacuum mode structure before the plate introduction remaining unmodified with a continuous frequency spectra, the presence of the mirror in the form of a cavity ends up changing the field into a standing wave pattern of discrete allowed frequencies. By comparing the energy differences is the two conditions with an adequate regularization of the divergent terms, he was able to predict theoretically the emergence of a force of attraction per unit area of $F(a)/L^2 \approx 1{,}3\;(a/\mu m)^{-4}10^{-3}\;Ncm^{-2}$. To have an idea, for a $1\;cm^2$ plates separated by $1\;\mu m$, the expected pressure is of the order of $P_{\text{Cas}} \approx 10^{-8}\;atm$. Despite very small, such force should still be measurable by adequate equipment. Indeed, after an inconclusive experimental test made

[1] Born, Heisenberg and Jordan in 1926 already had applied the matrix mechanics to the description of electromagnetism, although without the photon creation and absorption framework of QFT.

by Sparnaay [13] in 1958, where difficulties in maintaining plate parallelism prevented conclusive results, Casimir force was widely confirmed experimentally after the modern stage of experiments which followed the work of Lamoreaux [14] and Mohideen and Roy [15] at the end of the 90's.

As a final remark, it is important to notice that Casimir effect is not an exclusivity of the EMF, but instead, a fundamental property of any QFT confined by adequate boundary conditions. In particular, this latter does not need to be induced by material objects, but just any potential able to modify the structure of the vacuum state. The geometry of the phenomena also does not need to be defined by parallel plates, nor the force needs to be attractive. For instance, the case of the two neutral spherical shells calculated firstly by Boyer [16], which defies common sense, it is repulsive.

The lesson we could learn from SCE is that since variations of boundaries configurations induce changes in the vacuum energy structure, the latter cannot be considered just as an inert actor but as an active physical entity reacting against distortions [17]. Indeed, because of the quantum fluctuations associated with ZPE, modern arguments by invoking the fluctuation-dissipation theorem, anticipates that a non-uniform accelerated mirror should see vacuum as a viscous medium by inducing reaction forces of dissipative character [18]. As friction requires constant energy insertion to maintain accelerated motion, if this is true, the question that immediately follows is: where does the energy pumped into the system goes if we are in the vacuum? The short answer, which holds the key for a new rich physics is that energy is dissipated through the creation of photons from an initial field in the vacuum state, as we will see with more details now.

It was in 1969, fruit of a PhD thesis, that the American physicist G. T. Moore published the surprising prediction of how non-stationary boundaries like a set of moving mirrors, could lead to the generation of real photons from the vacuum state of the radiation field: the so called *dynamical Casimir effect* [3]. To do this, he used a simplified model of a linear polarized EMF (scalar electrodynamics) confined by a one-dimensional cavity composed by two ideal mirrors: one at rest and the other describing an arbitrary trajectory in time. Using the conformal symmetry of the field equation in the (1+1)-dimensional plane, Moore could bypass the initial difficulty associated with time-dependent boundary conditions by mapping the problem into the static mirror model with the help of an auxiliary function associated with each mirror trajectory.

Moore's strategy basically summarized in a scattering-like problem, where the interval of time of the system is divided in three stages: an initial one where the mirror is at rest with standing wave solutions; an intermediate interval when the mirror can move arbitrarily; a third one, again, with fixed mirrors but now with a more general solution. By writing the final solution in terms of the initial standing wave patterns, Moore was able to compute the number of particles in the final stage in terms of the initial state by means of a special linear map called Bogoliubov transformation. As a consequence of the coupling between both solutions and the correspondent mixture of positive and negative frequencies terms, a Bogoliubov coefficient survives with a non-zero value even in the vacuum state, demonstrating the process of particles creation.

Again, as the above description does not make reference to the mirror properties, it is immediate that the DCE can occur without material boundaries. Instead, any coupling between a quantum field and an external potential capable of changing the vacuum state structure can do the job. Actually, it was studying QFT in the presence of gravitation potentials that DeWitt [4] demonstrated that the generation of particle from the vacuum could occur even in the case of a single moving mirror. The work of Fulling and Davies [5, 6] extended this analysis in a more general local framework and in the case of the single-mirror set-up, they showed that the radiation power emitted is proportional to the rate of change of the mirror acceleration (for the $(1 + 1)$-dimensional case) which is, as expected, consistent with Lorentz invariance.

Both SCE and DCE are best described in terms of the most intuitive QFT: the quantum theory of the electromagnetic radiation. So in order to properly introduce the present work, which aims to give a theoretical reconstruction of the most important results and features of these two interconnected effects, it will be imperative to understand more deeply the quantum aspects of the electromagnetic field. We will begin the second chapter, therefore, by introducing the EMF in the framework of a classical field theory and after this, by establishing the details of the radiation field and its mode decomposition. The analysis will be concluded by the canonical quantization procedure to obtain the quantum description of the field.

The third chapter properly introduces and contextualizes the SCE by presenting the mathematical derivations of Casimir forces either in the case of the Casimir original approach and in the more modern procedure which involves local quantities and Green's functions. To complement the discussion, the historical endeavour of its experimental confirmation is discussed along with the connections with Van der Wall interactions.

Dynamical Casimir Effect is the theme of the fourth chapter, which revolves around the conceptualization and introduction of Moore's approach based on an auxiliary function. Some exemplars of Moore's function will be presented with a quick connection with Casimir forces. We left the best for the end where we discuss the process of particle creation from the vacuum, together with an application for the special case of parametric resonance motion.

In the last chapter, as the name says, we will finalize the monograph giving the final words about what could be learned from the two quantum effects studied until here.

Chapter 2

Quantization of the radiation field

In order to understand the physics behind Casimir phenomena (static and dynamical) we need to familiarize ourselves with the quantum aspects of the electromagnetic field. Therefore, after describing electromagnetism as a classical field theory, our next step will consist in the quantization program for the radiation field.

2.1 Electromagnetism as a classical field theory

Continuous systems have infinite degrees of freedom and are best described by fields, *i.e.*, physical quantities ϕ_α ($\alpha = 1, ..., m$ indexing the field component) defined at every position of space $\mathbf{r}$ and instant of time t. Their dynamical description usually involves a formulation in the lagrangian language through the introduction of an action functional

$$S = \int_{t_1}^{t_2} dt \int_{\mathcal{V}} d^3r \mathcal{L}\left(\phi_\alpha, \frac{\partial \phi_\alpha}{\partial t}, \boldsymbol{\nabla}\phi_\alpha\right), \tag{2.1}$$

expressed in terms of the lagrangian density $\mathcal{L}$ of the field. The correct field equations can be found by taking the dynamical equations that turn the action stationary, which is equivalent to the Hamilton's principle for continuous systems

$$\begin{aligned} 0 = \delta S &= \int_{t_1}^{t_2} dt \int_{\mathcal{V}} d^3r \sum_{\alpha=1}^{m} \left\{ \frac{\partial \mathcal{L}}{\partial \phi_\alpha}\delta\phi_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\delta\dot{\phi}_\alpha + \frac{\partial \mathcal{L}}{\partial\left(\boldsymbol{\nabla}\phi_\alpha\right)} \cdot \delta\left(\boldsymbol{\nabla}\phi_\alpha\right)\right\} \\ &= \int_{t_1}^{t_2} dt \int_{\mathcal{V}} d^3r \sum_{\alpha=1}^{m} \left\{ \frac{\partial \mathcal{L}}{\partial \phi_\alpha} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\right) - \boldsymbol{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial\left(\boldsymbol{\nabla}\phi_\alpha\right)}\right)\right\} \delta\phi_\alpha. \end{aligned}$$

where we considered $\delta\phi_\alpha$ to vanish at t_1, t_2 and on the surface that encloses the volume $\mathcal{V}$ where the field is defined [19]. As the field variations are mutually independent and arbitrary, from Eq. (2.1) we can extract the Euler-Lagrange equations

$$\boldsymbol{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial\left(\boldsymbol{\nabla}\phi_\alpha\right)}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0 \quad \text{with} \quad \alpha = 1, ..., m \tag{2.2}$$

which gives the set of equations that govern the dynamical behavior of the field.

A very important field archetype, the classical electromagnetic field, is generally described in the framework of classical field theory by treating the field strengths $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, respectively called electric and magnetic field, as derived quantities of the scalar, V, and vector, $\mathbf{A}$, potentials. Consequently, V and the components A_1, A_2 and A_3 of the vector potential are promoted to the status of fundamental dynamical variables ϕ_α as in Eq. (2.1) and are related with the field strength through the relations

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{2.3}$$

Finally, from purely phenomenological considerations, the EMF in the presence of charges and current densities ρ and $\mathbf{J}$, respectively, can be described with the help of lagrangian density [12]

$$\mathcal{L}(V, \mathbf{A}) = \frac{1}{2}\left[\left|\boldsymbol{\nabla} V + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right|^2 - |\boldsymbol{\nabla} \times \mathbf{A}|^2\right] - \rho V + \mathbf{J} \cdot \mathbf{A}, \tag{2.4}$$

in the Heaviside-Lorentz units. By substituting (2.4) in Eq. (2.2), with help of the identity $|\boldsymbol{\nabla} \times \mathbf{A}|^2 = \sum_k \left[|\boldsymbol{\nabla} \mathrm{A}_k|^2 - \frac{\partial \mathbf{A}}{\partial x_k} \cdot \boldsymbol{\nabla} \mathrm{A}_k\right]$ $(k = 1, 2, 3)$, we can calculate

$$\boldsymbol{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial (\boldsymbol{\nabla} V)}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{V}}\right) - \frac{\partial \mathcal{L}}{\partial V} = \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} V + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) + \rho = 0$$
$$\boldsymbol{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial (\boldsymbol{\nabla} A_k)}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{A}_k}\right) - \frac{\partial \mathcal{L}}{\partial A_k} = -\nabla^2 A_k + \frac{\partial}{\partial x_k}\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c}\frac{\partial}{\partial t}\left(\boldsymbol{\nabla} V + \frac{1}{c}\frac{\partial A_k}{\partial t}\right) - \mathbf{J} = 0$$

obtaining as a result the field equations of electromagnetism

$$-\nabla^2 V - \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A}) = \rho \tag{2.5a}$$
$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = \mathbf{J} \tag{2.5b}$$

that, together with Eq. (2.3), form Maxwell's equations in the potential formulation. In general, the potential V and $\mathbf{A}$ are not measurable quantities, since electromagnetism exhibit gauge symmetry which allows to arbitrarily redefine it under a transformation,

$$V' \to V - \frac{1}{c}\frac{\partial \Lambda}{\partial t}, \qquad \text{and} \qquad \mathbf{A}' \to \mathbf{A} + \boldsymbol{\nabla}\Lambda, \tag{2.6}$$

with Λ being an arbitrary function os the space-time coordinates, without changing the physical and directly observable electric and magnetic fields. In general, Maxwell's equations (2.5a), (2.5b) and (2.3) are usually expressed in the more recognized field strength formulation version [12]

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \rho; \qquad & \boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0; \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0; \qquad & \boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{\mathbf{J}}{c}; \end{aligned} \tag{2.7}$$

2.2 The radiation field

Setting the problem

As both SCE and DCE can be described in terms of neutral and demagnetized boundary configurations in vacuum, it is sufficient for our analysis to consider the EMF in regions absent of sources, that is, $\rho = \mathbf{J} = 0$. A stronger statement of such condition is that we must contemplate only EMF contributions that survives far away from their original sources and that can transport energy out to infinity. This is called the radiation field, which we will describe now.

This picture can be further simplified by identifying the existence of a redundancy in our description. Since electromagnetic radiation only have two polarization states and the potential formulation is given in terms of 4 components, we proceed by eliminating the first superfluous degree of freedom associated with the gauge symmetry (2.6) by abandoning the relativistic covariance of the theory in favor of fixing the *Coulomb gauge* condition

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0, \tag{2.8}$$

which characterizes the transversality condition [10]. With this choice, Eq. (2.5a) simplifies to the Poisson equation characterizing an instantaneous Coulombian potential (this is the reason for the gauge name)

$$V = \int \mathrm{d}^3 r' \frac{\rho\left(\boldsymbol{r}', t\right)}{|\boldsymbol{r} - \boldsymbol{r}'|} = 0 \quad \text{as} \quad \rho = 0,$$

which is identically zero, signifying that we successfully eliminated all the spurious longitudinal degrees of freedom of our description.

Our dynamical problem then reduces to the homogeneous wave equation (2.5b) in $\mathbf{A}$ with the field strengths specified under the conditions [10]

$$\nabla^2 \mathbf{A} = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}, \tag{2.9}$$

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}; \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{2.10}$$

As a natural extension of this description we expand the solutions of Eq. (2.9) with help of Fourier analysis in a plane wave eigenfunction basis

$$\mathbf{A}\left(\mathbf{r}, t\right) = \frac{1}{\sqrt{\mathcal{V}}} \sum_{\boldsymbol{k}} \left[\mathcal{A}_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\mathbf{r}} + \mathcal{A}^*_{\boldsymbol{k}}(t) e^{-i\boldsymbol{k}\cdot\mathbf{r}}\right], \tag{2.11}$$

where $\mathcal{V}$ is the volume in which the field is contained and we guarantee $\mathbf{A}$ to be a real vector with help of the condition $\mathcal{A}_{-\boldsymbol{k}} = \mathcal{A}^*_{\boldsymbol{k}}$. Additionally, in order to satisfy the orthonormality condition, the basis choice in which we expanded the vector potential needs to fulfill the conditions

$$\int d^3 r \left(e^{i\boldsymbol{k}\cdot\mathbf{r}} e^{-i\boldsymbol{k}'\cdot\mathbf{r}}\right) = \mathcal{V}\delta_{\mathbf{k}\mathbf{k}'}; \qquad \int d^3 r \left\{ \begin{array}{c} e^{i\boldsymbol{k}\cdot\mathbf{r}}\; e^{i\boldsymbol{k}'\cdot\mathbf{r}} \\ e^{-i\boldsymbol{k}\cdot\mathbf{r}}\; e^{-i\boldsymbol{k}'\cdot\mathbf{r}} \end{array} \right\} = \mathcal{V}\delta_{\boldsymbol{k},-\boldsymbol{k}'}. \tag{2.12}$$

To ensure that expression (2.11) represents a physical solution in which momentum is conserved, we need to make sure that each point of space is equivalent. We do so by enclosing the field in a cubic box with sides L (volume $\mathcal{V} = L^3$) and surface points $\mathbf{L}$ so that by imposing periodic boundary conditions $\mathbf{A}(\mathbf{r}, t) = \mathbf{A}(\mathbf{r} + \mathbf{L}, t)$ we guarantee that spatial translations doesn't change the physics of the problem (as we allow the box to be arbitrarily re-scaled). As a result of the referred boundary condition, the components of the vector $\boldsymbol{k}$ can only assume integer values in the form of $\boldsymbol{k} = (k_1, k_2, k_3) = \frac{2\pi}{L}(n_1, n_2, n_3)$, with $n_1, n_2, n_3 = \pm 1, \pm 2, ...,$.

From the transversality condition (2.8), it can be shown that $\boldsymbol{\nabla} \cdot \mathbf{A} = i\boldsymbol{k} \cdot \mathbf{A} = 0$. This means that $\mathbf{A}$ must live in a plane perpendicular to the propagation direction $\boldsymbol{k}$, which basically translates in an ambiguity in its orientation. The vectorial nature of the field together with the last fact implies the existence of two polarization states $\boldsymbol{\epsilon}_{\boldsymbol{k}}^{(\mathbf{1})}$ and $\boldsymbol{\epsilon}_{\boldsymbol{k}}^{(\mathbf{2})}$, where $(\boldsymbol{\epsilon}_{\boldsymbol{k}}^{(\mathbf{1})}, \boldsymbol{\epsilon}_{\boldsymbol{k}}^{(\mathbf{2})}, \boldsymbol{k}/|\boldsymbol{k}|)$ constitute a set of mutually perpendicular unit vectors. This can be taken into account by simply writing the coefficients in Eq. (2.11) as

$$\mathcal{A}_{\boldsymbol{k}}(t) = \sum_{\alpha=1,2} a_{\boldsymbol{k},\alpha}(t)\boldsymbol{\epsilon}_{\boldsymbol{k}}^{(\boldsymbol{\alpha})} \qquad \text{with} \qquad \begin{cases} \boldsymbol{\epsilon}_{\boldsymbol{k}}^{(\alpha)} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k}}^{(\alpha')} = \delta_{\alpha\alpha'} \\ \boldsymbol{\epsilon}_{\boldsymbol{k}}^{(\alpha)} \times \boldsymbol{\epsilon}_{\boldsymbol{k}}^{(\alpha')} = \breve{\boldsymbol{k}} \end{cases} \tag{2.13}$$

where $\breve{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ and the vector potential $\mathbf{A}$ is then written as

$$\mathbf{A}\left(\mathbf{r}, t\right) = \frac{1}{\sqrt{\mathcal{V}}} \sum_{\alpha=1,2} \sum_{\boldsymbol{k}} \left[a_{\boldsymbol{k},\alpha}(t)e^{i\boldsymbol{k}\cdot\mathbf{r}} + a^*_{k,\alpha}(t)e^{-i\boldsymbol{k}\cdot\mathbf{r}}\right] \boldsymbol{\epsilon}_{\boldsymbol{k}}^{(\alpha)}. \tag{2.14}$$

Field modes as harmonic oscillators

Substituting our solution for the vector potential given in Eq. (2.14) in the wave equation (2.9), we obtain the differential equations for the coefficients $\ddot{a}_{\boldsymbol{k},\alpha} = -\omega^2 a_{\boldsymbol{k},\alpha}$, where $\omega = |\boldsymbol{k}|c$ is the dispersion relation for this system. This last relation means that the coefficients of each mode behave as harmonic oscillators, whose amplitudes evolve in time as

$$a_{\boldsymbol{k},\alpha}(t) = a_{\boldsymbol{k},\alpha}(0)e^{-i\omega t}, \qquad a^*_{\boldsymbol{k},\alpha}(t) = a^*_{\boldsymbol{k},\alpha}(0)e^{i\omega t}. \tag{2.15}$$

Such relations represent the *modes of oscillations* of the field.

We can demonstrate more explicitly this last fact by analysing the structure of the radiation field energy. For the free electromagnetic field, Eq. (2.4) takes the simple form $\mathcal{L} = \frac{1}{2}\left(E^2 - B^2\right)$, so the Hamiltonian density for the system is given by,

$$\mathscr{H} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{A}}} \cdot \dot{\mathbf{A}} - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \mathbf{E}} \cdot \mathbf{E} - \mathcal{L} = \frac{1}{2}\left(E^2 + B^2\right). \tag{2.16}$$

Using this Hamiltonian density along with Eqs. (2.10) and (2.14) it is possible to express the Hamiltonian of the radiation field in terms of the normal modes coefficients as [10]

$$H = \int d^3r \mathscr{H} = \frac{1}{2}\int d^3r \left(\left|\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right|^2 + |\boldsymbol{\nabla} \times \mathbf{A}|^2\right) = \sum_{\mathbf{k},\alpha} 2\left(\frac{\omega}{c}\right)^2 a^*_{\boldsymbol{k},\alpha} a_{\boldsymbol{k},\alpha}. \tag{2.17}$$

At first sight, Eq. (2.17) does not give us any hint about the energy mode structure, so for this reason we define the following quantities

$$Q_{\boldsymbol{k},\alpha} = \frac{1}{c}\left(a_{\boldsymbol{k},\alpha} + a^*_{\boldsymbol{k},\alpha}\right) \qquad \text{and} \quad P_{\boldsymbol{k},\alpha} = -\frac{i\omega}{c}\left(a_{\boldsymbol{k},\alpha} - a^*_{\boldsymbol{k},\alpha}\right), \tag{2.18}$$

which are canonical variables that respect Hamilton's equation $\frac{\partial H}{\partial Q_{\boldsymbol{k},\alpha}} = \dot{P}_{\boldsymbol{k},\alpha}$ and $\frac{\partial H}{\partial P_{\boldsymbol{k},\alpha}} = -\dot{Q}_{\boldsymbol{k},\alpha}$, and the inverse transformations

$$a_{\boldsymbol{k},\alpha} = \frac{c}{2\omega}\left(\omega Q_{\boldsymbol{k},\alpha} + iP_{\boldsymbol{k},\alpha}\right) \qquad a^*_{\boldsymbol{k},\alpha} = \frac{c}{2\omega}\left(\omega Q_{\boldsymbol{k},\alpha} - iP_{\boldsymbol{k},\alpha}\right). \tag{2.19}$$

Putting Eq. (2.19) into Eq. (2.17) we obtain

$$H = \sum_{\boldsymbol{k},\alpha} \frac{1}{2} \left(P^2_{\boldsymbol{k},\alpha} + \omega^2 Q^2_{\boldsymbol{k},\alpha} \right).$$

This last equation makes clear that the radiation field can be regarded as a collection of infinite independent harmonic oscillators, each of which associated with a field mode $(\boldsymbol{k}, \alpha)$ [10]. This quadrature development was indeed central in the failed attempts at the end of the nineteenth century to explain the black body problem. As the classical argument goes, the equipartion of energy, by assigning an average value of energy $k_B T$ to each one of the infinite oscillators, mean that the cavity vacuum in which EMF was confined should posses an infinite heat capacity, absorbing all the thermal energy from the finite degrees of freedoms of the cavity walls and preventing thermal equilibrium ever to occur [20]. But this was about to change with the birth of quantum mechanics.

2.3 Electromagnetism as a quantum field theory

As history tell us, after Planck's *ad-hoc* solution for the black body problem, the work of Einstein on the photoelectric effect —the phenomena of electron ejection from metal by incident light of specific wavelength— was able to refine the latter hypothesis by quantizing not the energy exchange between the cavity atoms and the electromagnetic radiation modes, but the radiation energy itself by considering it as composed of energy packets of integral multiple of $\hbar\omega$. Einstein himself never proposed directly, but his bold claims were deeply suggesting that light of wavelength $\lambda = 2\pi c/\omega$ should behave as localized massless particles that latter would receive the name of photons. This somewhat elusive idea really gained traction with Compton demonstration in 1923 that experimental results of X-ray scattering on free electrons could only make sense under the assumption that light is made of particles of energy $\hbar\omega$ and momentum $\hbar|\boldsymbol{k}|$: the quantum character of light was also corpuscular after all.

After non-relativistic quantum mechanics was fully developed by Heisenberg, Born, Jordan, Schrodinger and others, P. A. M Dirac was taken to think in how this treatment could be extended to the theory of radiation [9]. The very first challenge was that, contrary to the latter theory which fundamentally treats dynamical systems with a finite and constant number of particles interacting instantaneously with each other, the interaction of a moving electron with the EMF, for example, should occur at the finite velocity of light and produce particles, *i.e.*, photons, as result of its motion. He clearly knew that someone trying to quantize the radiation field would automatically be dealing with a relativistic quantum field theory. By resolving the radiation field into an infinite Fourier series and treating its dynamical variables as non-compatible operators, Dirac could get around the relativistic problem by building up a theory for the emission and absorption of radiation in terms of the interaction of an atom with an EMF fixed in the non-covariant Coulomb gauge, as we will see in more details now.

Canonical quantization and Fock space

The route we choose to quantize the radiation field is the canonical quantization, which consists in the elevation of the canonical variables that describe each radiation oscillator $P_{\boldsymbol{k},\alpha}$ and $Q_{\boldsymbol{k},\alpha}$ to the status of non-commuting operators

$$\left[\hat{Q}_{\mathbf{k}',\alpha'}, \hat{P}_{\boldsymbol{k},\alpha}\right] = i\hbar\delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\alpha,\alpha'}; \qquad \left[\hat{P}_{\boldsymbol{k},\alpha}, \hat{P}_{\mathbf{k}',\alpha'}\right] = \left[\hat{Q}_{\boldsymbol{k},\alpha}, \hat{Q}_{\mathbf{k}',\alpha'}\right] = 0, \tag{2.20}$$

in the same way as the ordinary treatment of a one dimensional quantum harmonic oscillator. In analogy with the transformations shown in Eq. (2.19), we define the non-hermitian ladder operators

$$\hat{a}_{\boldsymbol{k},\alpha} = \frac{1}{\sqrt{2\hbar\omega}}\left(\omega\hat{Q}_{\boldsymbol{k},\alpha} + i\hat{P}_{\boldsymbol{k},\alpha}\right); \qquad \hat{a}^{\dagger}_{\boldsymbol{k},\alpha} = \frac{1}{\sqrt{2\hbar\omega}}\left(\omega\hat{Q}_{\boldsymbol{k},\alpha} - i\hat{P}_{\boldsymbol{k},\alpha}\right), \tag{2.21}$$

that respect the commutations relations

$$[\hat{a}_{\boldsymbol{k},\alpha}, \hat{a}^{\dagger}_{\boldsymbol{k}',\alpha'}] = \delta_{\boldsymbol{k}.\boldsymbol{k}'}\delta_{\alpha,\alpha'}; \qquad [\hat{a}_{\boldsymbol{k},\alpha}, \hat{a}_{\boldsymbol{k}',\alpha'}] = [\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}, \hat{a}^{\dagger}_{\boldsymbol{k}',\alpha'}] = 0. \tag{2.22}$$

After introducing the algebraic structure, to complete the quantization program we must define a Hilbert space in order to give the theory a particle interpretation. By remarking from (2.17) that in the classical theory the energy of each mode of oscillation was proportional to the square of their amplitude $|a_{\boldsymbol{k},\alpha}|^2 = a^{*}_{\boldsymbol{k},\alpha}a_{\boldsymbol{k},\alpha}$, we can analogously introduce the hermitian operator $\hat{N}_{\boldsymbol{k},\alpha} = \hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\hat{a}_{\boldsymbol{k},\alpha}$, called number operator for the mode of oscillation $(\boldsymbol{k}, \alpha)$ with the eigenvalue equation

$$\hat{N}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle = n_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle, \tag{2.23}$$

such that its basis elements —the set of eigenstates $|n_{\boldsymbol{k},\alpha}\rangle$— will span a special Hilbert space called Fock space which describes the quantum state of a variable number of particles.

From the commutation relations (2.22) is possible to obtain

$$\left[\hat{N}_{\boldsymbol{k}',\alpha'}, \hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\right] = \hat{a}^{\dagger}_{\boldsymbol{k}',\alpha'}\left[\hat{a}_{\boldsymbol{k}',\alpha'}, \hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\right] + \left[\hat{a}^{\dagger}_{\boldsymbol{k}',\alpha'}, \hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\right]\hat{a}_{\boldsymbol{k}',\alpha'} = \hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\delta_{\boldsymbol{k}.\boldsymbol{k}'}\delta_{\alpha,\alpha'}; \tag{2.24}$$

$$\left[\hat{N}_{\boldsymbol{k}',\alpha'}, \hat{a}_{\boldsymbol{k},\alpha}\right] = \hat{a}^{\dagger}_{\boldsymbol{k}',\alpha'}\left[\hat{a}_{\boldsymbol{k}',\alpha'}, \hat{a}_{\boldsymbol{k},\alpha}\right] + \left[\hat{a}^{\dagger}_{\boldsymbol{k}',\alpha'}, \hat{a}_{\boldsymbol{k},\alpha}\right]\hat{a}_{\boldsymbol{k}',\alpha'} = -\hat{a}_{\boldsymbol{k},\alpha}\delta_{\boldsymbol{k}.\boldsymbol{k}'}\delta_{\alpha,\alpha'}. \tag{2.25}$$

Allowing us to show that

$$\hat{N}_{\boldsymbol{k},\alpha}\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle = \left(\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\hat{N}_{\boldsymbol{k},\alpha} + \left[\hat{N}_{\boldsymbol{k},\alpha}, \hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\right]\right)\left|n_{\boldsymbol{k},\alpha}\right\rangle = (n_{\boldsymbol{k},\alpha}+1)\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle;$$
$$\hat{N}_{\boldsymbol{k},\alpha}\hat{a}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle = \left(\hat{a}_{\boldsymbol{k},\alpha}\hat{N}_{\boldsymbol{k},\alpha} + \left[\hat{N}_{\boldsymbol{k},\alpha}, \hat{a}_{\boldsymbol{k},\alpha}\right]\right)\left|n_{\boldsymbol{k},\alpha}\right\rangle = (n_{\boldsymbol{k},\alpha}-1)\hat{a}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle.$$

These relations demonstrate that both $\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle$ and $\hat{a}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle$ are also eigenstates of $\hat{N}_{\boldsymbol{k},\alpha}$, but with a eigenvalue increased (and decreased) by one unit. This also means they must be the same, up to a multiplicative constant as $|n_{\boldsymbol{k},\alpha}-1\rangle$ and $|n_{\boldsymbol{k},\alpha}+1\rangle$, respectively, which implies that

$$\hat{a}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle = c\left|n_{\boldsymbol{k},\alpha}-1\right\rangle \Longrightarrow \qquad \left\langle n_{\boldsymbol{k},\alpha}\right|\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\hat{a}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle = n_{\boldsymbol{k},\alpha} = |c|^2$$
$$\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle = d\left|n_{\boldsymbol{k},\alpha}+1\right\rangle \Longrightarrow \qquad \left\langle n_{\boldsymbol{k},\alpha}\right|\hat{a}_{\boldsymbol{k},\alpha}\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle = n_{\boldsymbol{k},\alpha}+1 = |d|^2.$$

Therefore, we obtain the correct result for the application of the ladder operators on $\hat{N}_{\boldsymbol{k},\alpha}$ eigenstates

$$\hat{a}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle = \sqrt{n_{\boldsymbol{k},\alpha}}\left|n_{\boldsymbol{k},\alpha}-1\right\rangle ; \qquad \hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\left|n_{\boldsymbol{k},\alpha}\right\rangle = \sqrt{n_{\boldsymbol{k},\alpha}+1}\left|n_{\boldsymbol{k},\alpha}+1\right\rangle ,$$

thus making possible the interpretation of $\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}$ and $\hat{a}_{\boldsymbol{k},\alpha}$, respectively, as the creation and annihilation operators, since they act in the eigenstates $\left|n_{\boldsymbol{k},\alpha}\right\rangle$ increasing and decreasing in one unit the occupational number $n_{\boldsymbol{k},\alpha}$.

Since $\hat{N}_{\boldsymbol{k},\alpha}$ is an observable, the successive applications of the annihilation operator $\hat{a}_{\boldsymbol{k},\alpha}$ on the eigenstates $\left|n_{\boldsymbol{k},\alpha}\right\rangle$ of the occupation number vector space must ensure $n_{\boldsymbol{k},\alpha}$ to be a positive integer number, allowing us to define the ground state for a mode of oscillation $(\boldsymbol{k},\alpha)$ as $\left|0_{\boldsymbol{k},\alpha}\right\rangle$, for which $\hat{a}_{\boldsymbol{k},\alpha}\left|0_{\boldsymbol{k},\alpha}\right\rangle = 0$. In this framework, we define the "vacuum state" as a composite system of every mode of oscillation with zero occupation number,

$$\left|0\right\rangle = \left|0_{\boldsymbol{k}_1,\alpha_1}\right\rangle \otimes \left|0_{\boldsymbol{k}_2,\alpha_2}\right\rangle \otimes \cdots \otimes \left|0_{\boldsymbol{k}_l,\alpha_l}\right\rangle \otimes \cdots . \tag{2.26}$$

The immediate consequence of this element is the possibility of the construction of a formal representation of a general state for every possible combination of the modes

$$\left|n\right\rangle = \left|n_{\boldsymbol{k}_1,\alpha_1}, n_{\boldsymbol{k}_2,\alpha_2}, \cdots\right\rangle = \prod_{\boldsymbol{k},\alpha} \frac{\left(\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\right)^{n_{\boldsymbol{k},\alpha}}}{\sqrt{n_{\boldsymbol{k},\alpha}!}}\left|0\right\rangle . \tag{2.27}$$

The development of Fock space forces us to reinterpret the vector potential, now written in terms of the ladder operators, no longer as a classical field function but instead as a field operator parameterized by position and time. It can be obtained with the substitutions $a_{\boldsymbol{k},\alpha} \to c\sqrt{\hbar/2\omega}\hat{a}_{\boldsymbol{k},\alpha}$ and $a^{*}_{\boldsymbol{k},\alpha} \to c\sqrt{\hbar/2\omega}\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}$ into Eq. (2.14), as

$$\hat{\mathbf{A}}\left(\mathbf{r},t\right) = c\sqrt{\frac{\hbar}{2\mathcal{V}}}\sum_{\alpha=1,2}\sum_{\boldsymbol{k}}\frac{1}{\sqrt{\omega_{\boldsymbol{k}}}}\left[\hat{a}_{\boldsymbol{k},\alpha}(t)e^{i\boldsymbol{k}\cdot\mathbf{r}} + \hat{a}^{\dagger}_{k,\alpha}(t)e^{-i\boldsymbol{k}\cdot\mathbf{r}}\right]\boldsymbol{\epsilon}^{(\alpha)}_{\boldsymbol{k}}, \tag{2.28}$$

resulting in the field strength (see Eq. (2.10))

$$\hat{\mathbf{E}} = -\frac{1}{c}\frac{\partial\hat{\mathbf{A}}}{\partial t} = i\sqrt{\frac{\hbar}{2\mathcal{V}}}\sum_{\alpha=1,2}\sum_{\boldsymbol{k}}\sqrt{\omega_{\boldsymbol{k}}}\left[\hat{a}_{\boldsymbol{k},\alpha}(t)e^{i\boldsymbol{k}\cdot\mathbf{r}} - \hat{a}^{\dagger}_{k,\alpha}(t)e^{-i\boldsymbol{k}\cdot\mathbf{r}}\right]\boldsymbol{\epsilon}^{(\alpha)}_{\boldsymbol{k}} \tag{2.29}$$

$$\hat{\mathbf{B}} = \boldsymbol{\nabla}\times\hat{\mathbf{A}} = ic\sqrt{\frac{\hbar}{2\mathcal{V}}}\sum_{\alpha=1,2}\sum_{\boldsymbol{k}}\frac{1}{\sqrt{\omega_{\boldsymbol{k}}}}\left[\hat{a}_{\boldsymbol{k},\alpha}(t)e^{i\boldsymbol{k}\cdot\mathbf{r}} - \hat{a}^{\dagger}_{k,\alpha}(t)e^{-i\boldsymbol{k}\cdot\mathbf{r}}\right]\boldsymbol{k}\times\boldsymbol{\epsilon}^{(\alpha)}_{\boldsymbol{k}} \tag{2.30}$$

The form of Eq. (2.28) also makes clear that just the temporal part of the field (coefficients of each normal mode) was quantized.

Photons as field excitations

To better understand the framework that we just created, it will be interesting to develop the properties of some of the observables acting on our Fock space that simultaneously commute with each other. As a consequence of the quantum character of the vector potential, the Hamiltonian operator $\hat{H}$ can be found (see appendix A) by following the same procedures as in Eq. (2.17), using instead Eq. (2.28) and taking care of the non-commutativity of the operators. Therefore

$$\hat{H} = \frac{1}{2}\int d^3r\left(\hat{\mathbf{E}}\cdot\hat{\mathbf{E}} + \hat{\mathbf{B}}\cdot\hat{\mathbf{B}}\right) = \frac{1}{2}\sum_{\boldsymbol{k},\alpha}\hbar\omega_{\boldsymbol{k}}\left(\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\hat{a}_{\boldsymbol{k},\alpha} + \hat{a}_{\boldsymbol{k},\alpha}\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\right) = \sum_{\boldsymbol{k},\alpha}\hbar\omega\left(\hat{N}_{\boldsymbol{k},\alpha} + \frac{1}{2}\right). \tag{2.31}$$

We can also define the electromagnetic momentum operator $\hat{\mathbf{P}}_{\boldsymbol{k},\alpha}$ with help of the quantum analog of Poynting vector $\frac{1}{c}\hat{\mathbf{E}}\times\hat{\mathbf{B}}$, as (see appendix A)

$$\hat{\mathbf{P}} = \frac{1}{c}\int d^3r\left(\hat{\mathbf{E}}\times\hat{\mathbf{B}}\right) = \sum_{\boldsymbol{k},\alpha}\hbar\boldsymbol{k}\left(\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\hat{a}_{\boldsymbol{k},\alpha} + \frac{1}{2}\right) = \sum_{\boldsymbol{k},\alpha}\hbar\boldsymbol{k}\hat{N}_{\boldsymbol{k},\alpha}, \tag{2.32}$$

where the term $\frac{1}{2}\hbar\boldsymbol{k}$ was cancelled out by the sum over $-\boldsymbol{k}$. As is known from electrodynamics [1] the density of total angular momentum is given in terms of the field components as

$$\hat{\boldsymbol{J}} = \frac{1}{c}\mathbf{r}\times\left(\hat{\mathbf{E}}\times\hat{\mathbf{B}}\right) = \sum_{i=1}^{3}\frac{1}{c}\hat{E}_i\left(\mathbf{r}\times\boldsymbol{\nabla}\right)\hat{A}_i + \frac{1}{c}\hat{\mathbf{E}}\times\hat{\mathbf{A}}$$

can be divided as the sum of an orbital (radius dependent) and the intrinsic term [21, 22], we introduce a set of circular polarization unit vectors $\boldsymbol{\epsilon}^{(\pm 1)} = \mp\frac{1}{\sqrt{2}}\left(\boldsymbol{\epsilon}_{\boldsymbol{k},1} \pm i\boldsymbol{\epsilon}_{\boldsymbol{k},2}\right)$ which allow us to define the intrinsic angular momentum of the electromagnetic field in the $\boldsymbol{k}$ direction [23] (see appendix A)

$$\hat{\mathbf{S}}_{\breve{\boldsymbol{k}}} = \frac{1}{c}\int d^3r\left(\hat{\mathbf{E}}\times\hat{\mathbf{A}}\right) = \sum_{\boldsymbol{k}}\sum_{\mu=\pm 1}\mu\hbar\breve{\boldsymbol{k}}\,\hat{a}^{\dagger}_{\boldsymbol{k},\mu}\hat{a}_{\boldsymbol{k},\mu} = \sum_{\boldsymbol{k}}\sum_{\mu=\pm 1}\mu\hbar\breve{\boldsymbol{k}}\,\hat{N}_{\boldsymbol{k},\mu}, \tag{2.33}$$

where $\breve{\boldsymbol{k}} = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}$ (do not confuse the symbol for unit vector with that for a quantum operator). From Eqs. (2.31), (2.32) and (2.33), with help of Eq. (2.23), its possible to obtain the eigenvalue equations

$$\hat{H}\left|n\right\rangle = \sum_{\boldsymbol{k},\alpha}\hbar\omega\left(n_{\boldsymbol{k},\alpha} + \frac{1}{2}\right)\left|n\right\rangle; \tag{2.34a}$$

$$\hat{\mathbf{P}}\left|n\right\rangle = \sum_{\boldsymbol{k},\alpha}\hbar\boldsymbol{k}\,n_{\boldsymbol{k},\alpha}\left|n\right\rangle; \tag{2.34b}$$

$$\hat{\mathbf{S}}_{\boldsymbol{k}}\left|n\right\rangle = \sum_{\boldsymbol{k},\mu}\mu\hbar\breve{\boldsymbol{k}}\,n_{\boldsymbol{k},\alpha}\left|n\right\rangle \tag{2.34c}$$

This shows that each mode of oscillation $(\boldsymbol{k},\alpha)$[1] can only contribute to the radiation field by adding an integer multiple of a quanta of energy $\hbar\omega_{\boldsymbol{k}}$, momenta $\hbar\boldsymbol{k}$ and helicity $\mu\hbar$ (to be interpreted as a spin 1 unit) to their respective total values. Another interesting heuristic result can be obtained if we associate for the quanta of the radiation field with the relativistic concept of mass $m^2c^4 = E^2 - c^2|\mathbf{p}|^2 = (\hbar\omega)^2 - c^2(\hbar|\boldsymbol{k}|)^2 = 0$, that is found to be zero.

We are then naturally compelled to interpret $\hat{H}$ as describing a collection of modes of oscillation $(\boldsymbol{k},\alpha)$ that, after being promoted to excited states in the context of Fock space, behave phenomenologically as a population of massless bosons with energy $\hbar\omega$ and momenta $\hbar\boldsymbol{k}$, which from our historical context can be identified simply as photons. This means that under the quantum description of

[1]Aside from the $\frac{1}{2}$ term in (2.34a).

electromagnetic field, photons must be understood as quantum mechanical excitation of the radiation field.

This also gives a physical interpretation for the elements of the Fock space. The eigenvalue $n_{\boldsymbol{k},\alpha}$ can be understood as the number of photons in a given state with wavevector $\boldsymbol{k}$ and polarization α, in the same time that the creation $\hat{a}^\dagger_{\boldsymbol{k},\alpha}$ and annihilation operators $\hat{a}_{\boldsymbol{k},\alpha}$ excite and de-excite the energy eigenstates of the set of commuting observables (2.34a), (2.34b) and (2.34c)[2].

It is important to restate that throughout the quantization program we have only quantized the temporal part of the field by elevating the expansion coefficients (2.15) to non-hermitian operators $\hat{a}_{\boldsymbol{k},\alpha}(t)$ and $\hat{a}^\dagger_{\boldsymbol{k},\alpha}(t)$ in the Heisenberg picture. Therefore, the time development of the quantized field (2.28) can be done analysing their evolution with the Heisenberg equation of motion. From the commutation relations (2.24) and (2.25) its not difficult to prove that $[\hat{H}, \hat{a}_{\boldsymbol{k},\alpha}] = -\hbar\omega_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k},\alpha}$, which allow us to show that

$$\ddot{\hat{a}}_{\boldsymbol{k},\alpha} = \frac{i}{\hbar}\left[\hat{H}, \dot{\hat{a}}_{\boldsymbol{k},\alpha}\right] = \frac{i}{\hbar}\left[\hat{H}, \frac{i}{\hbar}\left[\hat{H}, \hat{a}_{\boldsymbol{k},\alpha}\right]\right] = -\omega_{\boldsymbol{k}}^2\hat{a}_{\boldsymbol{k},\alpha}, \tag{2.35}$$

Solving the differential equation (2.35) for $\hat{a}_{\mathbf{k},\alpha}$ we can obtain the explicit time dependence for the annihilation and the creation operators (by taking the correspondent hermitian conjugate)

$$\hat{a}_{\boldsymbol{k},\alpha}(t) = \hat{a}_{\boldsymbol{k},\alpha}(0)e^{-i\omega_{\boldsymbol{k}}t}, \qquad \hat{a}^\dagger_{\boldsymbol{k},\alpha}(t) = \hat{a}^\dagger_{\boldsymbol{k},\alpha}(0)e^{i\omega_{\boldsymbol{k}}t}, \tag{2.36}$$

which are called respectively as the coefficients operators with positive and negative frequencies. This last nomenclature is due to the fact that for their accompanying mode functions $\psi(t) = e^{\mp i\omega_{\boldsymbol{k}}t}$ (also called positive and negative modes) the application of the operator $i\hbar\partial/\partial t$ —such that $i\hbar\frac{\partial}{\partial t}\psi = \pm E\psi$— gives us the positive and negative energy eigenvalues $E = \hbar\omega_{\boldsymbol{k}}$, which can be understood as if the modes propagate forward and backward in time.

2.4 Zero point energy of the radiation field

At first sight, a very serious problem arises when we inspect the $\frac{1}{2}$ term more closely in Eq. (2.31): the so called zero point energy of the radiation field. Even in the vacuum state, where supposedly there should not be photons to contribute with energy for the field, the mean value of the total energy appears to be non vanishing, as

$$\langle 0|\,\hat{H}\,|0\rangle = \sum_{\boldsymbol{k},\alpha}\hbar\omega_{\boldsymbol{k}}\left(\langle 0|\,\hat{N}_{\boldsymbol{k},\alpha}\,|0\rangle + \frac{1}{2}\langle 0|0\rangle\right) = \sum_{\boldsymbol{k},\alpha}\frac{1}{2}\hbar\omega_{\boldsymbol{k}}. \tag{2.37}$$

But even worse, as we are summing over all the infinite field modes $(\boldsymbol{k}, \alpha)$ the expression (2.37) it is clearly infinity. To give a perspective of how troublesome the term (2.37) in free space is, even making use of a reasonable regularization where we truncate the frequencies to the maximum energy before

[2] It's also possible to construct the orbital angular momentum operator.

Planck scale ($\mathcal{E}_{max} \sim 10^{19}\ GeV$), the total energy of the vacuum per unit volume still would stand to incredible [24]

$$\frac{1}{\mathcal{V}}\sum_{\alpha,\boldsymbol{k}}\frac{1}{2}\hbar\omega_{\boldsymbol{k}} \rightarrow \frac{2}{\mathcal{V}}\int\limits_{0}^{\mathcal{E}_{\max}/\hbar c} 4\pi\boldsymbol{k}^2\frac{d|\boldsymbol{k}|}{(2\pi)^3}\left(\frac{1}{2}\hbar c|\boldsymbol{k}|\right) \sim 10^{115}\ GeV/cm^3.$$

The origin of the divergent ZPE can be heuristically understood if we go back to the decomposition of its Hamiltonian in oscillatory modes by canonical operators $\hat{Q}_{\boldsymbol{k},\alpha}$ and $\hat{P}_{\boldsymbol{k},\alpha}$. Due to their non-commutativity, the operators must respect standard uncertainty relations in the form of $\triangle\hat{P}_{\boldsymbol{k},\alpha}\triangle\hat{Q}_{\boldsymbol{k},\alpha} \geq \frac{\hbar}{2}$. Using the fact that $\langle\hat{P}_{\boldsymbol{k},\alpha}\rangle = \langle\hat{Q}_{\boldsymbol{k},\alpha}\rangle = 0$ and the definition of standard deviation $\triangle\hat{A} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$, we can write it as

$$\triangle\hat{P}_{\boldsymbol{k},\alpha}\triangle\hat{Q}_{\boldsymbol{k},\alpha} = \sqrt{\langle\hat{P}^2_{\boldsymbol{k},\alpha}\rangle}\sqrt{\langle\hat{Q}^2_{\boldsymbol{k},\alpha}\rangle} \geq \frac{\hbar}{2} \Longrightarrow \langle\hat{P}^2_{\boldsymbol{k},\alpha}\rangle\langle\omega^2_{\boldsymbol{k}}\hat{Q}^2_{\boldsymbol{k},\alpha}\rangle \geq \left(\frac{\hbar\omega_{\boldsymbol{k}}}{2}\right)^2. \tag{2.38}$$

Analogous to the classical result of the virial theorem [25], for a potential $\hat{U}_{\boldsymbol{k},\alpha} = U_0\hat{Q}^{\nu}_{\boldsymbol{k},\alpha}$ with the canonical position operator $\hat{Q}_{\boldsymbol{k},\alpha}$ raised to the ν-th power, the virial theorem for quantum mechanics guarantees that

$$2\langle\hat{T}_{\boldsymbol{k},\alpha}\rangle = \nu\langle\hat{U}_{\boldsymbol{k},\alpha}\rangle \Longrightarrow \langle\hat{P}^2_{\boldsymbol{k},\alpha}\rangle = \langle\omega^2\hat{Q}^2_{\boldsymbol{k},\alpha}\rangle,$$

where $\hat{T}_{\boldsymbol{k},\alpha} = \frac{1}{2}\hat{P}^2_{\boldsymbol{k},\alpha}$ and $\hat{U}_{\boldsymbol{k},\alpha} = \omega^2\hat{Q}^2_{\boldsymbol{k},\alpha}$ with $\nu = 2$ are respectively the kinetic and potential energies of each mode of oscillation. As a consequence, the inequality (2.38) takes the form $\langle\hat{P}^2_{\boldsymbol{k},\alpha}\rangle = \langle\omega^2_{\boldsymbol{k}}\hat{Q}^2_{\boldsymbol{k},\alpha}\rangle \geq \frac{\hbar\omega}{2}$. Therefore

$$\langle\hat{H}\rangle = \sum_{\boldsymbol{k},\alpha}\frac{1}{2}\left(\langle\hat{P}^2_{\boldsymbol{k},\alpha}\rangle + \langle\omega^2_{\boldsymbol{k}}\hat{Q}^2_{\boldsymbol{k},\alpha}\rangle\right) \geq \sum_{\boldsymbol{k},\alpha}\frac{1}{2}\left[2\left(\frac{\hbar\omega_{\boldsymbol{k}}}{2}\right)\right] = \sum_{\boldsymbol{k},\alpha}\frac{1}{2}\hbar\omega_{\boldsymbol{k}}. \tag{2.39}$$

This means that the ZPE can be interpreted as fluctuations of the normal modes of oscillation of the quantum field. As the field strengths (2.29) and (2.30) are written in terms of the last quadrature, it is expected that their magnitude should also fluctuate around a non-zero value. Indeed, although the vacuum expectation values of the electric field (and the magnetic field) can be show to vanish (because of $\hat{a}_{\boldsymbol{k},\alpha}|0\rangle = 0$), as

$$\langle 0|\hat{\mathbf{E}}|0\rangle = 0,$$

the same thing is not true for the square of the field (field magnitude), which can be show to satisfy

$$\langle 0|\hat{\mathbf{E}}\cdot\hat{\mathbf{E}}|0\rangle = \frac{1}{\mathcal{V}}\sum_{\boldsymbol{k}}\hbar\omega_{\boldsymbol{k}} \rightarrow \infty,$$

to be a divergent quantity. As a consequence, the mean square fluctuation $\left(\Delta\hat{\mathbf{E}}\right)^2 = \langle 0|\hat{\mathbf{E}}\cdot\hat{\mathbf{E}}|0\rangle - |\langle 0|\hat{\mathbf{E}}|0\rangle|^2 = \langle 0|\hat{\mathbf{E}}\cdot\hat{\mathbf{E}}|0\rangle$ should also diverge. This is telling us that if we fix the occupation number of particles then the field strength must be completely uncertain, even in vacuum.

We can circumvent this result by noticing that as we only measure the field strength averaged over some finite volume $\Delta\mathcal{V}$ in space as

$$\hat{\bar{\mathbf{E}}} = \frac{1}{\Delta V}\int_{\Delta\mathcal{V}} d^3r\hat{\mathbf{E}}$$

this more realistic consideration must give us the expression like [10]

$$\left(\Delta\hat{\vec{\mathbf{E}}}\right)^2 = \langle 0|\hat{\vec{\mathbf{E}}}\cdot\hat{\vec{\mathbf{E}}}|0\rangle \sim \frac{\hbar c}{(\Delta\mathcal{V})^{\frac{4}{3}}}.$$

Confirming the result that the electric and the magnetic field fluctuates around a fixed value even in the vacuum state where there is no photons present. The consequences of the latter implies that even removing all matter and isolating all electromagnetic radiation ($T \sim 0\ K$) we still would not be able to construct a perfect vacuum in the classical sense, since space would continue to be filled up with random quantum fluctuations.

Despite all of this, the usual reaction when first faced with this ground state with divergent energy is to interpret it as just a mathematical side effect of the theory with no physical consequences, since energy scale is arbitrary and for most practical applications is always possible to redefine $\hat{H}' \to \hat{H} - \langle 0|\hat{H}|0\rangle$, normal ordering it. But as the American physicist Bryce DeWitt once put it:

> "We should always push the mathematical formalism of physics and its internal logic to their ultimate conclusions."

And in fact, the vacuum energy after adequate regularization and if taken seriously, can lead to real and finite measurable physical consequences as contained in the explanations for the process of spontaneous emission of atoms, the Lamb shift (the difference in energy for the hydrogen levels ${}^2S_{1/2}$ and ${}^2\hat{P}_{1/2}$) and as we will see with more details in the next chapter, the Casimir effect.

Chapter 3

Static Casimir effect

The static Casimir effect is a physical phenomenon theoretically first proposed by Hendrik Casimir in 1948 [2] that, in the standard configuration consists in the emergence of a macroscopic and measurable force of attraction between two perfectly parallel and neutral conductor plates when placed close to each other in vacuum. The force expression (per unit area L^2) for the effect presents a surprising inverse proportionality with the fourth power of the plate separation a and take the form of

$$\frac{F}{L^2} = -\frac{\hbar c \pi^2}{240 a^4}.$$

Moreover, the presence of the Planck's constant $\hbar$ makes evident its fundamental quantum nature (the effect disappears as $\hbar \to 0$ in the classical limit) in the same time that the proportionality with the speed of light c reveals that retardation effects are relevant in the plate interactions. On the other hand, the absence of the elementary electric charge e suggests that the phenomena do not depend on the internal structure of the plate but just in the boundary condition effect.

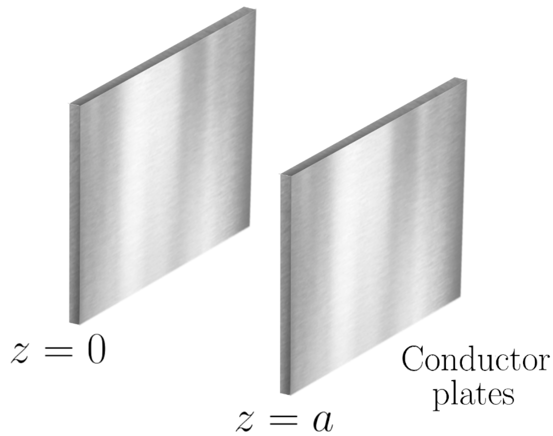


Figure 3.1: Geometry of the Casimir parallel plates scheme. Source: author.

Contrary to what one might think, the novelty of Casimir's result does not really lie on the prediction for the attractive force between neutral but polarizable materials such as conductor plates. In 1930, Fritz London already had applied perturbative methods in the recently developed quantum mechanics to show that the quantum fluctuations on the electronic distributions of neutral molecules could cause an attractive force by inducing instantaneous dipole moment. The derived interaction potential was shown to fall up to second-order as $V_{Lon}(r) \approx -(3/4)\left(\hbar\omega_0\alpha^2\right)/r^6$ with ω_0 characterizing the dominant transition frequency, α being the atom static polarizability while r represents the separation distance between the atoms [11]. The dispersive London force is indeed a special case of a more wide category of intermolecular interactions called Van der Waals forces.

In the 40's, experiments involving the stability of colloidal particles suspended in dispersive medium were used to study the nature of intermolecular interactions. It was found in this context, that London-Van der Waals interaction potential should fall much more rapidly ($V \sim r^7$) for large separations between the colloidal particles. The inconsistency could be solved only in 1947, when Casimir and Polder considered the retardation effects on the Van der Wall forces that appears when the light signal between the particles spent an interval of time of the order of the atomic transition frequencies to communicate the interaction between the atoms. So, after lengthy calculations using perturbative quantum electrodynamics they were successfull to obtain the correct expression for the interactions with a $V_{\mathrm{Ret}}(r) = -23\hbar c \alpha_A \alpha_B / r^7$ dependence, in agreement with experiments.

But the revealing aspect of Casimir's derivation just came to light during a brief conversation with Niels Bohr about the simplicity of the expression obtained for the retarded dispersive force [11], as in his own words

> In the summer or autumn 1947 (but I am not absolutely certain that it was not somewhat earlier or later) I mentioned my results to Niels Bohr, during a walk. "That is nice", he said, "That is something new." I told him that I was puzzled by the extremely simple form of the expressions for the interaction at very large distance and he mumbled something about zero-point energy. That was all, but it put me on a new track.

After this insight, Casimir was taken to consider the interactions to be the result of the change in the ZPE fluctuations of the radiation field due to the presence of the atoms. With this approach, he was able to re-derive very straightforwardly results obtained earlier with Polder at great cost, convincing himself, therefore, about the physical status played by the vacuum energy density in quantum field theory.

But how exactly do fluctuations enter this picture to explain the emergence of forces? The fact is that Casimir forces are just subset of a more general category of interactions called fluctuation-induced forces. The recipe for this last phenomenon has two ingredients [26]: (*i*) A dynamical system whose strength can fluctuate around some fixed value, *i.e.*, a fluctuating medium such an EMF or a heat bath; (*ii*) a physical element that interacts with the medium by suppressing its fluctuations such as a conductor or an adiabatic wall. Under theses conditions, whenever there is restriction on fluctuating medium, the increment in the free energy of the system induces an entropic effective interaction on the physical elements to restore equilibrium, explaining, therefore, how forces can be induced from fluctuations.

An important aspect of the last discussion is that fluctuation-induced forces can have any origin: from the quantum fluctuations of the EMF as manifested in the SCE with a force expression $F(a)/L^2 = -\hbar/a^4 \times \frac{\pi^2 c}{240}$ and strength proportional to $\hbar$ or in the case of thermal fluctuations such as the critical Casimir effect suggested by Fisher and deGennes in 1978, where a pair of walls immersed in a binary liquid mixture close to its critical point must experience an interacting force $F(a)/L^2 \sim k_B T/a^3$ proportional to the strength $k_B T$.

In this chapter, therefore, we must derive the Casimir force initially in the Casimir approach which emphasizes the regularization procedure followed by the re-derivation of the force expression with a more general local method where we use scalar functions and the Maxwell stress tensor. The experimental context and its relationship with Van der Waals forces are briefly discussed at the end.

3.1 Original derivation of the Casimir force

The direct measurement of vacuum energy has always been known to be untenable since in principle only differences of energy have a physical significance. But as Casimir realized [2], we can indirectly detect its effect by introducing a disturbance in the vacuum and comparing the energy after and before the perturbation by a suitable regularization of the divergent terms. We then define the Casimir energy $\mathcal{U}$ as [11]

$$\mathcal{U} = \lim_{s\to 0}\left[\left(\sum_{\boldsymbol{k},\alpha}\frac{1}{2}\hbar\omega_{\boldsymbol{k}}\right)_I - \left(\sum_{\boldsymbol{k},\alpha}\frac{1}{2}\hbar\omega_{\boldsymbol{k}}\right)_{II}\right], \tag{3.1}$$

where I and II labels, respectively, a regularized sum of the ZPE after and before the disturbance, while s represents a regularization parameter.

For the ZPE of the electromagnetic field, one way to accomplish this is by introducing two parallel and perfect neutral conductors plates separated by a small distance a in the z-direction. The choice is very intuitive since neutral ideal conductors cannot support parallel electric fields or normal magnetic fields on its surfaces and, therefore, work like perfect mirrors by reflecting electromagnetic radiation as a consequence of the boundary conditions

$$\hat{\mathbf{E}} \times \hat{\mathbf{n}}|_{\text{plates}} = 0 \qquad\qquad \hat{\mathbf{B}} \cdot \hat{\mathbf{n}}|_{\text{plates}} = 0, \tag{3.2}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the plates. This occurs because, otherwise, the existence of those field components would imply the emergence of electric currents on the surface, what is in full contradiction with the electrostatic equilibrium regime of the plates.

This arrangement, by imposing the boundary conditions (3.2) on the field (2.28) ends up disturbing the vacuum energy since it modifies the allowed frequency values and, therefore, the possible modes of excitation of the field operator inside the cavity. Considering the system to be contained in an usual box of volume L^3, we can divide it into three regions by positioning the two plates configuration in the form of a cavity with separation a, a box cover area of La, and a total volume of $L^2 a$ besides the Dirichlet boundary condition on the plates and periodic conditions in the remaining directions. In the intermediate region, inside the cavity we, therefore, expect the vector potential to satisfy [11]

$$\begin{aligned}\hat{\mathbf{A}} = \frac{L^2}{(2\pi)^2}\sum_{n=0}^{\infty}\int d^2k_{||}\left(\frac{c^2\hbar}{2\omega_{\boldsymbol{k}}\mathcal{V}}\right)^{1/2}\Big\{&\hat{a}_{(1)}(\breve{\boldsymbol{k}}_{||}\times\hat{\mathbf{z}})\sin(k_z z) + \\ &+\hat{a}_{(2)}\left[ik_z\frac{\boldsymbol{k}_{||}}{k}\sin(k_z z) - \hat{\mathbf{z}}\frac{k_{||}}{k}\cos(k_z z)\right]\Big\}e^{i(\boldsymbol{k}_{||}\cdot\mathbf{r}-\omega t)} + \text{ h.c.}\end{aligned} \tag{3.3}$$

where $\boldsymbol{k}_{||} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}$ is the projection of the wavevector parallel to the plates, and the components of $\boldsymbol{k}$ satisfy

$$k_x = \frac{2\pi l}{L}; \qquad k_y = \frac{2\pi m}{L}; \qquad k_z = \frac{\pi n}{a}, \tag{3.4}$$

with $l, m = \pm 1, \pm 2, \ldots$ and $n = 0, 1, 2, \ldots$.

In the limit that $a \ll L$, where the separation of the plates is much smaller than the other dimensions of the cavity, we can effectively consider with respect to a, that $L \to \infty$ in the same rate as the relative wavenumber separation $(\Delta k_x, \Delta k_y) = \frac{2\pi}{L} \to 0$[1]. In this circumstances the components $\boldsymbol{k}_{\|} = (k_x, k_y)$ parallel to the plates becomes continuous variables and the expression for the ZPE energy take the form

$$\begin{aligned}\sum_{\boldsymbol{k},\alpha} \frac{1}{2}\hbar\omega_{\boldsymbol{k}} = \frac{1}{2}\hbar\sum_{\alpha=1}^{2}\sum_{k_x,k_y,k_z} c|\mathbf{k}| \to \frac{c\hbar}{2}\frac{L^2}{(2\pi)^2}\sum_{\alpha=1}^{2}\lim_{\Delta\to 0}\sum_{k_x,k_y=-\infty}^{\infty}\Delta k_x\Delta k_y\sum_{n=0}^{\infty}\sqrt{k_x^2+k_y^2+k_z^2} \\ = \frac{\hbar c}{2}\sum_{\alpha=1}^{2}\frac{L^2}{(2\pi)^2}\sum_{n=0}^{\infty}\int d^2k_{\|}\sqrt{k_{\|}^2+\frac{n^2\pi^2}{a^2}},\end{aligned}$$

where it was used the substitution $\lim_{\Delta\to 0}\sum_{k_x,k_y}\Delta k_x\Delta k_y = \int d^2k_{||}$. We can, therefore define the ZPE of the field inside the cavity after being perturbed by the boundaries conditions as

$$\begin{aligned}\mathcal{U}_I &= \frac{\hbar c L^2}{2(2\pi)^2}\left[\int_0^\infty d^2k_{\|}\; k_{\|} + \sum_{\alpha}\sum_{n=1}^{\infty}\int_0^\infty d^2k_{\|}\sqrt{k_{\|}^2+\frac{n^2\pi^2}{a^2}}\right] \\ &= \frac{1}{2}\frac{\hbar c L^2}{2\pi}\left[\int_0^\infty k_{\|}^2 dk_{\|} + 2\sum_{n=1}^{\infty}\int_0^\infty k_{\|}\sqrt{k_{\|}^2+\frac{n^2\pi^2}{a^2}}dk_{\|}\right],\end{aligned} \tag{3.5}$$

where the term $n = 0$ was separated from the summation[2] and rewritten in the polar coordinates.

In contrast, the ZPE expression before the perturbation, when there was any conductor plates, takes the form of the continuous summation (integral)

$$\mathcal{U}_{II} = \frac{\sum \frac{1}{2}\hbar\omega}{L^2} = \frac{1}{2}\frac{\hbar c}{2\pi}\sum_{\alpha=1}^{2}\int_0^\infty\int_0^\infty k_{\|}\sqrt{k_{\|}^2+\frac{n^2\pi^2}{a^2}}dk_{\|}dn. \tag{3.6}$$

Making the substitution

$$\begin{cases} \lambda = \frac{a}{\pi}\sqrt{k_{\|}^2+\frac{n^2\pi^2}{a^2}} \\ n' = n \end{cases} \qquad \text{with Jacobian} \quad dk_{\|}dn = \frac{\pi}{a}\frac{\lambda}{\sqrt{\lambda^2-n'^2}}d\lambda dn'$$

we can write the difference between Eqs. (3.5) and (3.6) as

$$\begin{aligned}\mathcal{U}_I - \mathcal{U}_{II} &= \frac{\hbar c\pi^2 L^2}{2a^3}\left[\frac{1}{2}\int_0^\infty \lambda^2\big|_{n'=0}d\lambda + \sum_{n'=1}^{\infty}\int_{n'}^\infty\lambda^2 d\lambda - \int_0^\infty\int_{n'}^\infty\lambda^2 d\lambda dn'\right] \\ &= \frac{\hbar c\pi^2 L^2}{2a^3}\left[\sum_{n=(0)1}^{\infty}\int_n^\infty\lambda^2 d\lambda - \int_0^\infty\int_n^\infty\lambda^2 d\lambda dn\right],\end{aligned} \tag{3.7}$$

[1]As Δl and Δm are equal to 1, since $(\Delta k_x; \Delta k_y) = (k_x(l+1) - k_x(l); k_y(m+1) - k_y(m)) = 2\pi/L(1,1)$ from the relations (3.4).

[2]There is just one polarization state for the mode of oscillation in which $n = 0$.

where the index $(0)1$ means that the $n = 0$ term need to be corrected by a $\frac{1}{2}$ factor, and we changed $n' \to n$ to simplify notation. Equation (3.7) also makes explicit the effect of the boundary conditions by creating an abrupt energy difference in vacuum between the continuous frequency spectra of the free EMF that persists outside the plates (second term) and the set of discrete modes of oscillations in the form of a standing waves that fits inside the cavity (first term), as schematically represented in Fig. 3.2. This happens because the conductor plates act like perfect mirrors, reflecting propagating EMF in opposite directions and, by creating superposition patterns of constructive and destructive interference, culminates in the change of the possible modes of oscillation to a standing waves profile of discrete allowed frequencies.

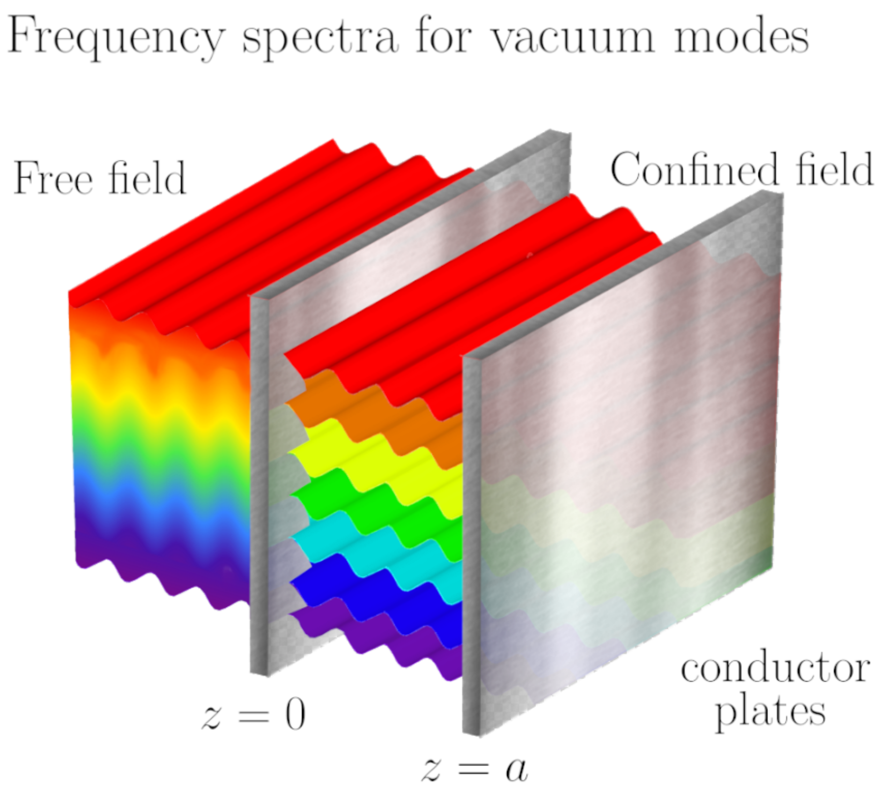


Figure 3.2: Casimir energy is the change in the ZPE after (confined field) and before (free field) the introduction of the conductor plates (boundary conditions). The scheme is merely illustrative to show the difference in the frequency spectra of the vacuum modes in both situations, where the region outside the cavity represent the continuous range of frequency when space is not confined in contrast to the discrete allowed values between the plates. Source: author.

Unfortunately, the difference shown in Eq. (3.7) cannot have physical significance since we are subtracting divergent quantities. Therefore, somehow we need a regularization procedure to extract finite results from that expression.

Casimir insight was to perceive that despite the idealization of his boundary condition, in real life, a conductor plate would not be able to constrain some very high energetic excited modes inside the cavity. So effectively, the mirror should be transparent to those ultra-violet divergent terms and, in order to regularize the sum in the form of Eq. (3.1), we need to introduce a cut-off function able to cancel those inopportune contributions.

For this purpose we use a cut-off function $f(s\lambda) \equiv f\left(\frac{\pi}{ak_c}\lambda\right)$ with k_c associated with a cut-off frequency and $s = \frac{\pi}{ak_c}$ a regularization parameter as described in Eq. (3.1) that we will take to be zero (or taken k_c to infinity) in the end of the process. In general we demand the following behavior

$$f\left(\frac{\pi}{ak_c}\lambda\right) \to \begin{cases} 1, & \text{for} \quad \lambda \ll \frac{a}{\pi}k_c \\ 0, & \text{for} \quad \lambda \gg \frac{a}{\pi}k_c. \end{cases} \tag{3.8}$$

To properly regularize the Casimir energy (3.1) we multiply it by the correspondent cut-off function (3.8)

$$\begin{aligned} \mathcal{U} = \frac{\hbar c\pi^2}{2a^3} \lim_{s\to 0} &\left[\sum_{n=(0)1}^{\infty} \int_n^{\infty} \lambda^2 f\left(\frac{\pi}{ak_c}\lambda\right) d\lambda - \int_0^{\infty}\int_n^{\infty} \lambda^2 f\left(\frac{\pi}{ak_c}\lambda\right) d\lambda dn \right] \\ &= \frac{\hbar c\pi^2}{2a^3} \lim_{s\to 0} \left[\sum_{n=(0)1}^{\infty} F(n) - \int_0^{\infty} F(n)\, dn \right], \end{aligned} \tag{3.9}$$

where we defined the auxiliary function

$$F(n) \equiv \int_n^{\infty} \lambda^2 f\left(\frac{\pi}{ak_c}\lambda\right) d\lambda. \tag{3.10}$$

Now, since $F(n)$ is a convergent integral, we can use the Euler-Maclaurin formula [2],

$$\sum_{n=(0)1}^{\infty} F(n) - \int_0^{\infty} F(n)\, dn = -\frac{1}{12}F^{'}(0) + \frac{1}{720}F^{'''}(0) + \cdots . \tag{3.11}$$

to rewrite Eq. (3.9) in terms of the infinite sum

$$\mathcal{U} = \frac{\hbar c\pi^2}{2a^3} \lim_{s\to 0} \left[-\frac{1}{12}F^{'}(0) + \frac{1}{720}F^{'''}(0) + \cdots \right].$$

To calculate the values for the derivatives of $F^{(p)}(0)$, we can use the Leibniz rule

$$\frac{d}{dn}\left(\int_{a(n)}^{\infty} g(n,\lambda) d\lambda\right) = \int_{a(n)}^{\infty} \frac{\partial}{\partial n} g(n,t) d\lambda - g(n,a(n))\frac{da}{dn}$$

in such a way that for the case of Eq. (3.10), considering $g(n,\lambda) = \lambda^2 f\left(\frac{\pi}{ak_c}\lambda\right)$ we obtain

$$F'(n) = -n^2 f\left(\frac{\pi}{ak_c}n\right) = -n^2 f(sn)$$

and, consequently

$$F'(0) = 0; \qquad F'''(0) = -2; \qquad F^{(p>3)} = \mathcal{O}(s^{(p-3)}). \tag{3.12}$$

with $p = 1, 3, 5, \ldots$. This regularization method is very general and demonstrates that the procedure is independent of the cut-off function choice. But to give a more explicit example, if we choose $F(s\lambda) = e^{-s\lambda}$ it would be immediately that, by integration by parts

$$F(n) = \int_n^{\infty} \lambda^2 e^{-s\lambda} d\lambda = \left[(sn+1)^2 + 1\right] \frac{e^{-sn}}{s^3}$$

$$F^{(p)}(n) = -s^{p-3}\left[(sn+p-1)^2 - (p-1)\right]e^{-s\lambda},$$

finding, therefore, the exact coefficients shown in Eq. (3.12), as expected.

After discussing about the regularization process, we can finally compute the Casimir energy by substituting the coefficients (3.12) into Eq. (3.9), finding

$$\mathcal{U}(a) = \lim_{s\to 0}\frac{\hbar c\pi^2 L^2}{2a^3}\left[0 - \frac{2}{720} + \mathcal{O}(s^2)\right] = -L^2\frac{\pi^2\hbar c}{720a^3},$$

and, as a consequence, the force per unit area acting on the plate is given by

$$\frac{F(a)}{L^2} = -\frac{1}{L^2}\frac{\partial\mathcal{U}}{\partial a} = -\frac{\pi^2\hbar c}{240a^4}. \tag{3.13}$$

Showing that from physical considerations of the ZPE of the electromagnetic field it can be predicted the emergence of an attractive force between the parallel conductor plates, which characterizes the static Casimir effect. In more adequate units for comparison —1 dyn is equal to 10^{-5} N— Casimir force has the following order of magnitude [11]

$$\frac{F(a)}{L^2} \approx 0,013\frac{1}{(a/\mu m)^4}\frac{dyn}{cm^2}$$

To have an idea of the magnitude of this force, for the idealized situation of two perfectly conducting plates with surface area $L^2 = 1\ \mathrm{cm}^2$ separated by $1\mu\mathrm{m}$ we would expect for the modulus of this attractive force to be $0,013\ \mathrm{dyn} \approx 10^{-7}N$ and giving, therefore, a corespondent pressure of $P_{\text{Cas}} \approx 10^{-8}\ atm.$

3.2 Local derivation of the Casimir force

As we will see in this section, SCE is not an exclusivity for the quantum fluctuations of the electromagnetic field. In fact, any relativistic quantum field under appropriate boundary conditions —whether caused by material objects (such as the mirrors) or external potentials able to distort the vacuum field mode structure [11]— is in principle susceptible to the emergence of Casimir forces. In order to show this more explicitly, we will derive Casimir results for a generic scalar field using, instead of the global approach followed in the previous section where we considered the mirrors perturbing the whole energy mode structure, here we will follow the work of K. A. Milton [27] and proceed by computing the local flux of momentum from the field to the boundaries using the Maxwell stress tensor .

Scalar field and Green's function solutions

A more formal approach for deriving Casimir's result involves the identification of the adequate component of the field operator $\hat{\mathbf{A}}$ in Eq. (2.28) as a massless scalar field $\hat{\phi}(\mathbf{r},t)$ where the complications reminiscent from the polarization states are initially ignored. The Lagrangian density associated with this description is given by

$$\mathcal{L} = -\frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial\hat{\phi}}{\partial t}\right)^2 - \left(\boldsymbol{\nabla}\hat{\phi}\right)^2\right]. \tag{3.14}$$

Not surprising is the fact that using Euler-Lagrange's equations (2.2), we can recover the wave equation for the field component in the form

$$\nabla^2\hat{\phi} - \frac{1}{c^2}\frac{\partial^2\hat{\phi}}{\partial t^2} = 0. \tag{3.15}$$

Again, this field can be expanded in plane wave basis (even though with $\boldsymbol{k}$ assuming continuous values)

$$\hat{\phi}(\boldsymbol{r},t) = c\sqrt{\frac{\hbar}{2}}\sum_{\alpha}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{\omega_{\boldsymbol{k}}}}\left(\hat{a}_{\boldsymbol{k}}e^{i(\boldsymbol{k}\cdot\mathbf{r}-\omega t)} + \hat{a}^{\dagger}_{\boldsymbol{k}}e^{-i(\boldsymbol{k}\cdot\mathbf{r}-\omega t)}\right). \tag{3.16}$$

Using relations $\hat{a}_{\boldsymbol{k}}|0\rangle = \langle 0|\hat{a}^{\dagger}_{\boldsymbol{k}} = 0$ and $\langle 0|\hat{a}_{\boldsymbol{k}}\hat{a}^{\dagger}_{\boldsymbol{k}}|0\rangle = (2\pi)^3\delta^3(\boldsymbol{k}-\boldsymbol{k}')$, it can be easily shown that the ground state expectation value of $\hat{\phi}(\boldsymbol{r},t)\hat{\phi}(\boldsymbol{r}',t)$ is

$$\langle 0|\hat{\phi}(\boldsymbol{r},t)\hat{\phi}(\boldsymbol{r}',t)|0\rangle = c^2\hbar\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{k}}}e^{i[\boldsymbol{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}. \tag{3.17}$$

Using the function $f(\omega) = 1/(\omega^2-\omega^2_{\boldsymbol{k}})$ with $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|$ in the expression for the residue theorem [28] over a closed curve C

$$\frac{1}{2\pi i}\oint_C f(\omega)d\omega = \mathrm{Res}\,f(\omega_{\boldsymbol{k}}) \implies -i\int\frac{d\omega}{2\pi}\frac{1}{\omega^2-\omega^2_{\boldsymbol{k}}} = \mathrm{Res}\,f(\omega_{\boldsymbol{k}}) = \frac{1}{2\omega_{\boldsymbol{k}}}, \tag{3.18}$$

its possible to identify the $1/2\omega_{\boldsymbol{k}}$ term in Eq. (3.17) as a the residue of the function $f(\omega)$ when integrated over the complex plane around the simple pole[3] $\omega = +ck$ for $\mathbf{r} - ct > \mathbf{r}' - ct'$. So with this in hands we can rewrite Eq. (3.17) using (3.18) as

$$\langle 0|\hat{\phi}(\boldsymbol{r},t)\hat{\phi}(\boldsymbol{r}',t)|0\rangle = -i\hbar\int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}\frac{e^{i[\boldsymbol{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]}}{\omega^2/c^2-k^2} \quad \text{for } \mathbf{r}-\mathbf{r}'-c(t-t')>0 \tag{3.19}$$

Expression (3.19) can be recognized as the Green's function $G(x,x') = \frac{i}{\hbar}\langle 0|\hat{\phi}(\boldsymbol{r},t)\hat{\phi}(\boldsymbol{r}',t)|0\rangle$ [4] for the non-homogeneous wave equation (3.15) since it satisfies the identity [12]

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(x,x') = \delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{3.20}$$

The geometry of the conductor plates, as shown in Fig. 3.1, justify the introduction of a convenient reduced Green's function $g(z,z')$ in the form as

$$G(x,x') = \int\frac{d^2k}{(2\pi)^2}e^{i\boldsymbol{k}\cdot(\mathbf{r}-\mathbf{r}')}\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}g(z,z'),$$

that after inserted into Eq. (3.20) must satisfy the equation

$$\left(-\frac{\partial^2}{\partial z^2} - \lambda^2\right)g(z,z') = \delta(z-z'), \tag{3.21}$$

subjected to Dirichlet conditions $g(0,z') = g(a,z') = 0$. In this equation, $\lambda^2 = \omega^2/c^2 - k^2$.

We can solve Eq. (3.21) by the standard discontinuity method [28]. For the region inside the boundary condition, let the two solutions for the homogeneous equation, each one respecting one of

[3] we used $\mathrm{Res}\,f(\omega_{\boldsymbol{k}}) = \lim_{\omega\to\omega_{\boldsymbol{k}}}(\omega-\omega_{\boldsymbol{k}})f(\omega)$ for the simple pole.

[4] More formally, the correspondence between Green's functions and the vacuum average value of the stress tensor can be done only after adequate time-ordering prescriptions through Feynman's propagator: see Ref. [1]

the Dirichlet conditions, to be $g_1(z) = \sin\lambda z$ and $g_2(z) = \frac{\sin\lambda(z-a)}{\cos\lambda a}$ together with $p(z) = 1$[5] and the Wronskian $W(g_1, g_2) = g_1 g_2' - g_1' g_2 = \frac{\lambda}{\cos\lambda a}\sin\lambda a$. We then obtain the solutions

$$g^{(+)}(z, z') = \begin{cases} \dfrac{g_1(z)g_2(z')}{p(z')W(z')} = \dfrac{\sin(\lambda z)\sin\lambda(z'-a)}{\lambda\sin\lambda a}, & 0 < z < z' < a \\ \dfrac{g_1(z')g_2(z)}{p(z')W(z')} = \dfrac{\sin(\lambda z')\sin\lambda(z-a)}{\lambda\sin\lambda a}, & a > z > z' > 0 \end{cases} \tag{3.22}$$

We also solve g' for the free region, outside the plates. For this, we use the same later form, but we modify Eq. (3.22) by considering $sin\lambda z \to e^{i\lambda(z-a)}$ as we force $z \to \infty$ on the right of the plate $z = a$, so that

$$g^{(-)}(z, z') = \begin{cases} \dfrac{\sin\lambda(z'-a)e^{i\lambda(z-a)}}{\lambda\sin\lambda a}, & 0 < z < z' < a \\ \dfrac{\sin\lambda(z-a)e^{i\lambda(z-a)}}{\lambda\sin\lambda a}, & a > z > z' > 0 \end{cases} \tag{3.23}$$

Casimir force as a stress tensor component

Contrary to the free EMF where it is possible to globally introduce a well defined operator for the linear momentum, as done in Eq. (2.32), when we impose kinematic constraints such as dividing space in regions delimited by Dirichlet boundary conditions, in general the total linear momentum is no longer conserved as the plates are fixed and do not exchange momentum with the field. Nevertheless, it is still tempting to analyse SCE in terms of how the momentum from the modes of the field induce a radiation pressure at the plates. With this intent we introduce a local quantity to describe the pressure and stress exerted by the EMF on a given surface: Maxwell stress tensor, a well known physical quantity from classical electromagnetism. More specifically, to compute the Casimir force on this geometry we make use of the z-diagonal component of the Maxwell stress tensor T_{zz}, which characterizes the local radiation pressure exerted in the normal direction of the plates. The expression for the scalar field in terms of its Lagrangian can be obtained by considerations of the Noether's theorem (whose details are not important here) and is given by [27][6]

$$\overleftrightarrow{T}_{zz} = \frac{\partial\mathcal{L}}{\partial(\partial\hat{\phi}/\partial z)}\frac{\partial\hat{\phi}}{\partial z} - \mathcal{L} = \frac{1}{2}\left[\left(\frac{1}{c}\frac{\partial\hat{\phi}}{\partial t}\right)^2 + \left(\frac{\partial\hat{\phi}}{\partial z}\right)^2 - \left(\frac{\partial\hat{\phi}}{\partial x}\right)^2 - \left(\frac{\partial\hat{\phi}}{\partial y}\right)^2\right] \tag{3.24}$$

It is useful to think of the stress tensor as an operator

$$\overleftrightarrow{\mathcal{D}}_{zz} = \frac{\partial}{\partial z}\frac{\partial}{\partial z'} + \frac{1}{c^2}\frac{\partial}{\partial t}\frac{\partial}{\partial t'} - \frac{\partial}{\partial x}\frac{\partial}{\partial x'} - \frac{\partial}{\partial y}\frac{\partial}{\partial y'}$$

acting on the product $\hat{\phi}(\boldsymbol{r}, t)\hat{\phi}(\boldsymbol{r}', t')$ of scalar field operators. In this context its vacuum expectation value can be computed by applying $\overleftrightarrow{\mathcal{D}}_{zz}$ in the vacuum expectation value of the same products, such as

$$\langle 0|\overleftrightarrow{T}_{zz}|0\rangle = \frac{1}{2}\lim_{(\boldsymbol{r},t)\to(\boldsymbol{r}',t')}\overleftrightarrow{\mathcal{D}}_{zz}\langle 0|\hat{\phi}(\boldsymbol{r}, t)\hat{\phi}(\boldsymbol{r}', t')|0\rangle = -\frac{i\hbar}{2}\lim_{(\boldsymbol{r},t)\to(\boldsymbol{r}',t')}\overleftrightarrow{\mathcal{D}}_{zz}G(x, x').$$

[5] $p(z)$ correspond to the function expression present in $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - s(x)y = f(x)$: the Sturm-Liouville ODE associated with the problem.

[6] For the sign in the stress tensor expression see page 24 of Milton's book Ref. [27]

By writing its mean value in terms of a reduced stress tensor t_{zz} we have

$$\langle 0|\overset{\leftrightarrow}{T}_{zz}|0\rangle = -\frac{i\hbar}{2}\int \frac{d^2k}{(2\pi)^2}\frac{d\omega}{2\pi}\overset{\leftrightarrow}{\mathcal{D}}_{zz}\left[e^{i\boldsymbol{k}\cdot(\mathbf{r}-\mathbf{r}')}e^{-i\omega(t-t')}g\left(z,z'\right)\right] = \int \frac{d^2k}{(2\pi)^2}\frac{d\omega}{2\pi}\langle 0|t_{zz}|0\rangle$$

where

$$\langle 0|t_{zz}(z)|0\rangle = -\frac{i\hbar}{2}\lim_{z'\to z}\left(\frac{\partial}{\partial z}\frac{\partial}{\partial z'}+\lambda^2\right)g\left(z,z'\right) \tag{3.25a}$$

$$= -\frac{i\hbar}{2}\lim_{(z'\to z)}\begin{cases} \dfrac{\lambda\cos(\lambda z_<)\cos\lambda(z_>-a)}{\sin\lambda a}+\dfrac{\lambda\sin(\lambda z_<)\sin\lambda(z_>-a)}{\sin\lambda a}, & \text{for } g^{(+)}(z,z') \\ \dfrac{\lambda\cos\lambda(z_<-a)e^{i\lambda(z_>-a)}}{\sin\lambda a}+\dfrac{\lambda\sin\lambda(z_<-a)e^{i\lambda(z_>-a)}}{\sin\lambda a}, & \text{for } g^{(-)}(z,z') \end{cases} \tag{3.25b}$$

and the notation $z_>$ ($z_<$) for greater (smaller) of z and z'.

On the other hand, the Casimir force per unit area can be calculated as the difference between the radiation pressure in the plate from the inside and the outside, that is

$$\mathcal{F} = \left\langle 0\left|\overset{\leftrightarrow}{T}{}^{(+)}_{zz}\right|0\right\rangle - \left\langle 0\left|\overset{\leftrightarrow}{T}{}^{(-)}_{zz}\right|0\right\rangle = \int\frac{d^2k}{(2\pi)^2}\frac{d\omega}{2\pi}\left[\left.\langle t_{zz}\rangle\right|_{z\to z'=a^-} - \left.\langle t_{zz}\rangle\right|_{z\to z'\to a^+}\right]. \tag{3.26}$$

The first term corresponds to the calculation of the average reduced stress tensor $\langle t_{zz}\rangle$ in the vicinity of the plate position from inside the cavity while the second corresponds to the outside region. The limits on $\langle t_{zz}\rangle$ instruct us to compute z and z' for the first term at the plate position a from the left using $g^{(+)}(z,z')$ and z and z' for the second term at a from the right using $g^{(-)}(z,z')$. Plugging expression (3.25b) into (3.26) results

$$\mathcal{F} = -\frac{i\hbar}{2}\int\frac{d^2k}{(2\pi)^2}\frac{d\omega}{2\pi}\left(\lambda\cot\lambda a-\lambda\right). \tag{3.27}$$

Performing a complex frequency rotation in the last integral by means of the transformation

$$\frac{\omega}{c}\to i\zeta, \quad \text{so} \quad \lambda=\sqrt{\omega^2/c^2-k^2}\to i\sqrt{\zeta^2+k^2}\equiv i\kappa, \tag{3.28}$$

that for $\cot\left(i\kappa a\right) = -\coth\left(\kappa a\right)$, we can rewrite the latter expression for the force per unit area as

$$\mathcal{F} = -\frac{\hbar c}{2}\int\frac{d^2k}{(2\pi)^2}\int\frac{d\zeta}{2\pi}\kappa\left(\coth\kappa a-1\right). \tag{3.29}$$

But as we are using the general framework of an arbitrary scalar field, it is easy at this stage to generalize the last expression to a $(d+1)$-dimensional space to obtain the force expression for smaller dimensions than 3, as described in Ref. [27]. We can do this simply rewriting (3.29) in the form of

$$\mathcal{F} = -\frac{\hbar c}{2}\int\frac{d^dk}{(2\pi)^d}\int\frac{d\zeta}{2\pi}\kappa\left(\coth\kappa a-1\right). \tag{3.30}$$

In this context it is possible to identify in Eq. (3.28) the interpretation of the set $\boldsymbol{k}$ and ζ as spanning a $(d+1)$-polar coordinate, where κ is the radial distance of the space with d components k_i defined in a hyper-plane and ζ identifiable as the zenithal component (the same role of z in the conventional polar coordinate). So, for $\coth(x) - 1 = 2/(e^{2x} - 1)$ and changing the expression with a $(d+1)$-polar coordinate transformation $\int d^d k \int d\zeta \equiv \int d^{d+1}\kappa = \int_0^\infty A_{d+1}\kappa^d d\kappa$, Eq. (3.30) can be shown to become

$$\mathcal{F} = -\hbar c \frac{A_{d+1}}{(2\pi)^{d+1}} \int_0^\infty \kappa^d d\kappa \frac{\kappa}{e^{2\kappa a} - 1}, \tag{3.31}$$

where

$$A_{d+1} = \frac{2\pi^{(d+1)/2}}{\Gamma\left[(d+1)/2\right]}$$

is the surface area of a $(d+1)$-dimensional sphere and $\Gamma(x)$ is the Gamma function.

Using an integral identity with the Riemann zeta function $\zeta(s)$,

$$\int_0^\infty dy \frac{y^{s-1}}{e^y - 1} = \Gamma(s)\zeta(s),$$

it is possible to finally find the expression for force per unit transverse area

$$\mathcal{F} = -\frac{\hbar c}{a^{d+2}} \frac{2^{-2(d+1)}\pi^{-\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} \int_0^\infty \frac{(2a\kappa)^{(d+2)-1}}{e^{2a\kappa} - 1} d(2a\kappa) = -\frac{\hbar c}{a^{d+2}} \frac{2^{-2(d+1)}\pi^{-\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} \Gamma\left(d+2\right)\zeta(d+2).$$

This result can be simplified through the identity [27]

$$\Gamma(2z) = (2\pi)^{-1/2} 2^{2z-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \tag{3.32}$$

to the final expression

$$\mathcal{F} = -\hbar c \frac{(d+1)}{2^{d+2}\pi^{\frac{d}{2}+1}} \frac{\Gamma\left(1 + \frac{d}{2}\right)\zeta(d+2)}{a^{d+2}}, \tag{3.33}$$

For $d = 2$ (3-dimensional original analysis), by using $\Gamma(2) = 1$, $\zeta(4) = \pi^4/90$ and considering the existence of 2 states of polarization, we can recover the usual expression shown in Eq. (3.13)

$$\mathcal{F} = -\frac{\pi^2 \hbar c}{240 a^4}$$

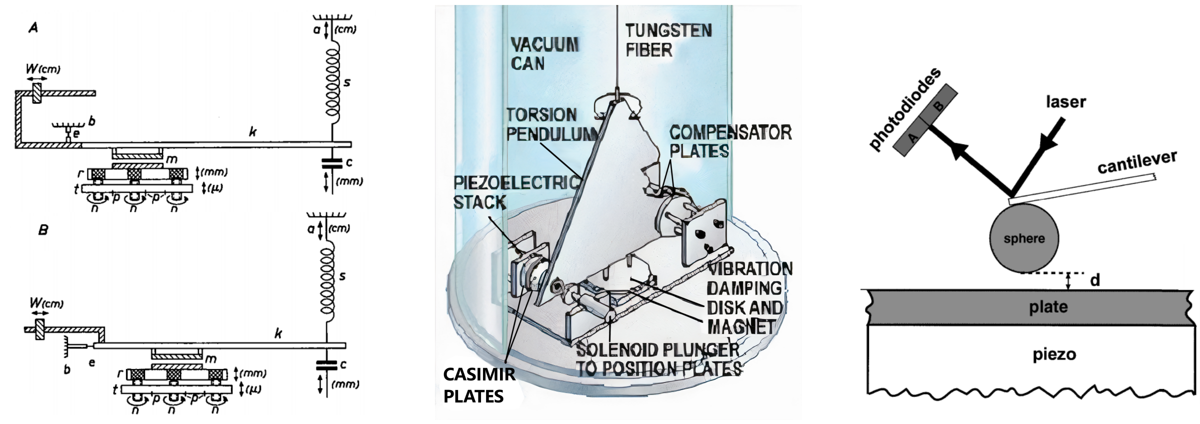
For $d = 0$ (one-dimensional cavity, where the mirrors are mere points), by using $\Gamma(1) = 1$ and $\zeta(2) = \pi^2/6$ we can obtain the expression for the Casimir force

$$\mathcal{F} = -\frac{\pi \hbar c}{24 a^2}. \tag{3.34}$$

3.3 Experimental verification of the Casimir effect

As the vacuum does not have an isotope analog to the ZPE of atomic systems, Casimir was taken to think in terms of changing its proprieties by altering the boundary conditions and ending choosing the parallel plane scheme as a direct way to give a testable theoretical prediction for the physical manifestation of the ZPE of the radiation field.

In 1958, by using this configuration, Sparnaay [13] was able to perform an experiment to verify Casimir's predictions. He used a balance system of springs attached to a pair of movable parallel metal plates and measured the effect in terms of the change of the capaciance of a flat plate condenser.



(a) (b) (c)

Figure 3.3: Schematic representation of: (a) the balance system used by Sparnaay's. Source: [13]; (b) the torsion pendulum system used Lamoreaux's. Source: (S. K. Lamoreaux) Scientific American; (c) the atomic force microscopic system used Mohideen and Roy. Source: [15].

However, and unfortunately, the results obtained with this method were inconclusive showing an effective 100% uncertainty, only not contradicting the theoretical predictions.

In spite of the myriad of theoretical literature in Casimir phenomena that followed Sparnaay's work, for the next 39 years, the empirical inquiry in the area was marked by the absence of significant improvement in the experimental accuracy. One of the major difficulties preventing precise measurements was the necessity of a high degree of parallelism between the plates, a direct consequence of the very nontrivial geometry dependence of the Casimir forces.

In 1997, bypassing the parallelism problem, Lamoreaux [14] initiate the modern era of precise experimental attempts by considering the attraction force between a flat plane and a spherical lens employing a proximity correction formula. The experimental apparatus was comprised of a torsion pendulum with a feedback circuit control and measured the effect of torsion angle in terms of voltage sign in a capacitor system. Despite great improvements in the sensibility and the reported 5% of agreement, latter reviews pointed out to a significant underestimation of errors (temperature corrections) in the analysis that could not be considered of a conclusive character for plate distances superior to $0{,}1\ \mu m$ [27, 29].

But just one year later, in 1998, Mohideen and Roy [15] were able perform an improved measurement of Casimir forces with help of an atomic force microscope and finite conductivity, roughness, and temperature corrections. Using the arrangement of a sphere and a plate supported by a cantilever, they measured the attraction force by the deflection of a laser beam on the flexing support and detected it by a photodiode. The achieved results where in agreement between theory with incredible 1% level of confidence. After that, many other experimental procedures were realized, proving conclusively the existence of Casimir effect with increasing precision [29].

3.4 Connections between Casimir and Van der Waals forces

As discussed in the first section, Casimir's prediction originated in colloidal chemistry from the study of dispersive forces between molecules. Such an approach, however, becomes problematic when extended to condensed matter regime, where intermolecular forces acquire a non additive character. This was until 1955, when Evgeny Lifshitz and collaborators developed a general theory for dispersive interactions between macroscopic materials by treating them as a continuum with a frequency dependent dielectric susceptibility $\epsilon(\omega)$ [30]. The main idea behind Lifshitz's approach was to consider the interactions as a result of the electromagnetic fluctuations that take place even in absolute zero through the ZPE. In particular, Lifshitz theory naturally unifies Van der Waals and Casimir forces between macroscopic materials as distinct manifestations of the same underlying physical principle. In this framework, Van der Waals forces are then thought to be a limiting case where mutual distances among objects are small compared to the relevant wavelengths of their absorption spectra, meanwhile Casimir forces emerge in the opposite circumstances where mutual distances are sufficiently large for relativistic effects (retardation) to take place.

Another important key aspect that distinguishes Casimir from Van der Waals forces is the very sensible geometry dependence —a direct consequence of the lack of pairwise additivity as found in the Lifshitz's description— and can be manifest in the form of repulsive forces for some particular configurations. One exemplar of these proprieties has a history that dates to the beginning of the nineteenth century where classical models for the structure of the electron were gaining large interest as a way to circumvent the infinite self-energy that electric particles had in classical electromagnetism. The most famous, the Abraham-Lorentz model pictured the electron as a spherical conductor with a surface layer electric charge. Despite its promises to ascribe the electron mass to electromagnetic properties the model always suffered from many inconsistencies and needed to ad-hoc postulate stresses to maintain stability. In 1956, Casimir was taken to think in the Abraham-Lorentz model with the lens of its successful derivation for force between two parallel plates, by imagining it as two opposites conductor spherical shells balancing the outward electrostatic repulsion pressure with the correspondent Casimir force. Casimir could in fact obtain the correct module expression for the force and it was providentially able to explain its stability, but unfortunately in 1968, Boyer [16], after lengthy calculation in a "nightmare in Bessel functions", was able to show that the sign of the force was in fact positive, and therefore, repulsive, destroying Casimir hopes and showing with this very not intuitive result that Casimir forces are much more complicated that someone would naively expect.

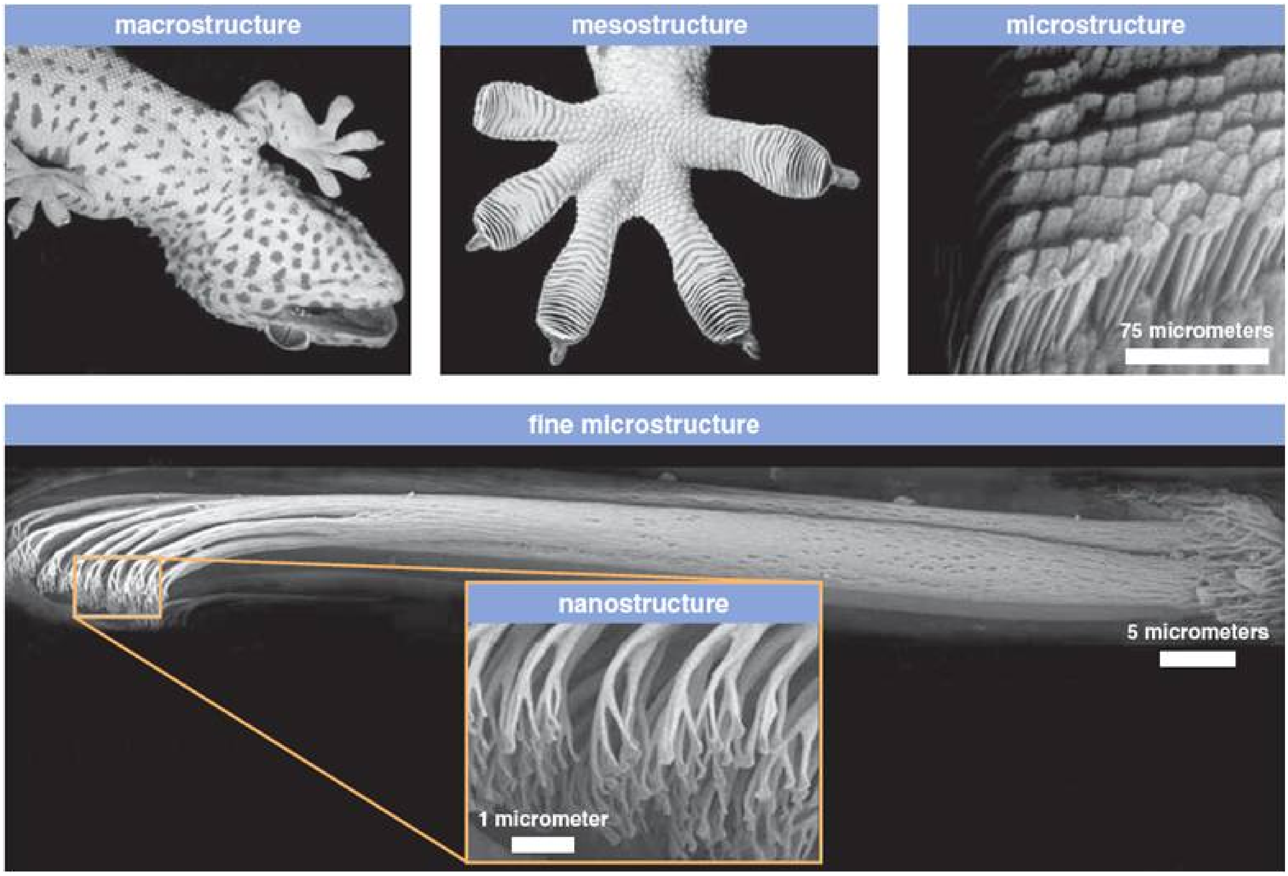


Figure 3.4: How gecko toes stick. Source: (Autumn, K.) American Scientist 94, 124-132.

Maybe one of the most surprising and perplexing applications of the Casimir force as an extension of Van der Waals forces is found in the explanation of the gecko ability to climb walls and suspend from ceilings. The adhesion force between the animal foot and surfaces, once thought to be of many origins —from friction to hydrophilic nature— turned out to resides in the gecko foot anatomy. With nearly half a million keratinous hair called setae, each of which with hundreds of nanostructures projections named spatulae, the elucidation for its sticking power was conclusively demonstrated, in 2002 [31], to be caused mainly from the Van der Waals attractive forces between the gecko spatulae structure and the closely spaced surfaces. In this context, the weak adhesion of mere $0{,}4\ \mu N$ between individual spatulae-surface, when combined over all the millions of structures, become responsible to produce an attraction force of around $10\ N$ in a unique foot, allowing thus the gecko trick.

Chapter 4

Dynamical Casimir effect

In the last chapter we could appreciate how the imposition of static boundaries conditions —such as the introduction of conductor plates fixed in the vacuum— can change the ZPE of the radiation field and induce forces in their vicinities. As the next logical step for this context, someone could ask what kind of physics would emerge if we allow one of the plates to start moving. Indeed, this was exactly what G. T. Moore did in his 1969 paper called *"Quantum Theory of the Electromagnetic Field in a Variable Length One Dimensional Cavity"* [3], where he considered a scalar field under a non-stationary boundary condition induced by moving mirrors. The found result is contained in the description of the *dynamical Casimir effect* (DCE) which refers to the macroscopic phenomena of particle creation due to changes in the mode structure of the quantum field induced by non-stationary boundary conditions.

Throughout the years that followed Moore's paper, the work of DeWitt [4], Fulling and Davies [5, 6] and many others extended the analysis and firmly establish the phenomena as a fundamental consequence of QFT. The DCE name itself was baptized by the great figure of Schwinger in his fascination with Sonoluminescence —the intriguing phenomena of intense light flashing by a bubble of air in an extreme acoustic field [27]— cultivated in his last years of life, as he conjectured (wrongly, unfortunately) that dynamical consideration of the Casimir effect could explain the phenomenon. The research program has received a great deal of attention in more than fifty years of existence and despite the theoretical fruitfulness the effect still lacks direct experimental verification, in contrast to his static counterpart.

In this respect, although DCE can occur even with a single mirror set in motion [4, 5], more feasible schemes for future practical implementations are presumed to take the form of vibrating cavities configurations with the special ingredient of parametric resonance to amplify the photon generation process. One of the main problems for direct experimental observation in the conventional motion resides that its detection in the visible part of the spectrum would necessitate the mirror to oscillate at very high speeds which is constrained by how rapidly one can deform it without breaking it apart (deformation velocities occurs at sound speed in the medium). For this reason is imagined that the only way to directly observe the phenomena must involve a still not developed mechanism to excite high frequency surface oscillations with a high quality factor. There is also some very interesting proposals as well as performed experiments in an indirect manner that aims to emulate DCE in other platforms. A broader discussion on the experimental proposals and their correspondent difficulties can be found in V. Dodonov compendium [32] .

In the following discussion of this chapter, we aim to introduce DCE under the umbrella of Moore's approach and, after defining and giving examples of the Moore's functions —an auxiliary functional relation associated with the mirror trajectory— we will proceed by demonstrating how the mirrors motion can induce the creation of particles from an initial vacuum state and conclude by computing, in the long-time limit, the specific number of particles created in the case of vibrating cavities in parametric resonance.

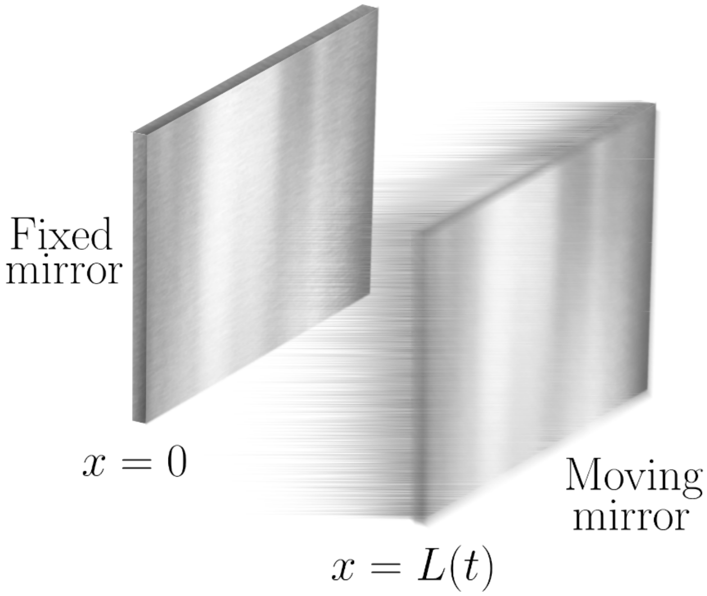


Figure 4.1: Schematic representation of the dynamical Casimir effect with a cavity scheme thought by Moore (although in three dimensions here). Source: author.

4.1 Moore's solutions for moving boundaries

Similar to the local analysis of the static case, in order to model the DCE we consider the simplified description of a one-dimensional cavity resonator confining a linearly polarized electric (and magnetic) field. This can be done with the identification of the z-component of the vector potential operator $\mathbf{A}$ in Eq. (2.28) as a massless scalar field $\hat{\phi}$ with Lagrangian density (3.14) (ignoring the y and z contributions), implying the Klein-Gordon equation of motion (or wave equation for the most intimate)

$$\frac{1}{c^2}\frac{\partial^2\hat{\phi}}{\partial t^2} - \frac{\partial^2\hat{\phi}}{\partial x^2} = 0. \tag{4.1}$$

Specifically in this case, the cavity is composed by two parallel ideal mirrors (perfect conductors): one fixed at $x = 0$ and the second moving in an externally prescribed trajectory $x = L(t) > 0$, both under Dirichlet boundary conditions in the form of $\hat{\phi}(0, t) = \hat{\phi}(L(t), t) = 0$.

In complete analogy with Eq. (2.28), the quantum field development of scalar electrodynamics is made in terms of the expansion in terms of the the annihilation/creation operators[1][2] in the form of

$$\hat{\phi}(x,t) = 2c\sqrt{\hbar}\sum_{n=1}^{\infty}\left[\hat{a}_n\psi_n(x,t) + \hat{a}_n^{\dagger}\psi_n(x,t)^*\right], \tag{4.2}$$

where the mode functions $\psi_n(x,t)$ —complex solutions of the wave equation (4.1)— form a complete and orthonormal set of functions that spams a vector space with an inner product [33]

$$(\psi,\Psi) = \frac{1}{i}\int_0^{L(t)} dx\left\{\psi\frac{\partial\Psi^*}{\partial t} - \frac{\partial\psi}{\partial t}\Psi^*\right\} = (\Psi,\psi) = -(\Psi^*,\psi^*) \quad \text{with} \left\{\begin{array}{l}(\psi_n,\psi_m^*) = 0\\ (\psi_n,\psi_m) = \delta_{nm}\end{array}\right. \tag{4.3}$$

In the special case in which both mirrors positions are fixed as in the static Casimir effect (which translates in the second mirror position $L(t) = L_0$), it is possible to represent a mode solution of Eq. (4.1) as a linear combination of the standing waves

$$\psi_n^{(0)}(x,t) = \frac{1}{\sqrt{\omega_n L_0}}\sin\left(\frac{\omega_n}{c}x\right)e^{-i\omega_n t} = \frac{1}{2i\sqrt{\omega_n L_0}}\left[e^{-i\frac{\omega_n}{c}(ct-x)} - e^{-i\frac{\omega_n}{c}(ct+x)}\right], \tag{4.4}$$

where $\omega_n = cn\pi/L_0$ with $n = 1,2,\ldots$.

In order to attack the time-dependent boundary condition problem, we exploit the conformal invariance of the wave equation (4.1), that under the coordinate transformation $(x,t)\to(\xi,\tau)$,

$$ct - x = f(c\tau-\xi); \qquad ct + x = g(c\tau+\xi), \tag{4.5}$$

preserves the wave equation form

$$\frac{1}{c^2}\frac{\partial^2\hat{\phi}}{\partial\tau^2} - \frac{\partial^2\hat{\phi}}{\partial\xi^2} = 0 \rightleftarrows \frac{1}{c^2}\frac{\partial^2\hat{\phi}}{\partial t^2} - \frac{\partial^2\hat{\phi}}{\partial x^2} = 0.$$

Choosing the mapping f and g in such a way the ξ-coordinate (running over $0<\xi<1$) coincides with the mirror trajectory $x = L(t)$ when $\xi = 1$, we can rewrite our time-dependent boundary problem into a time-independent form. Solving the conformal transformed scalar field equation for $\hat{\phi}(\xi,\tau)$, under the boundary conditions $\hat{\phi}(0,\tau) = \hat{\phi}(1,\tau) = 0$, we can analogously find solutions in the form

$$\psi_n(\xi,\tau) = \frac{1}{\sqrt{\pi cn}}\sin(\pi n\xi)e^{-i\pi cn\tau} = \frac{1}{2i\sqrt{\pi cn}}\left\{e^{-i\pi n(c\tau-\xi)} - e^{-i\pi n(c\tau+\xi)}\right\} \tag{4.6}$$

and return to (x,t) representation using the inverses of the conformal transformations (4.5) as

$$\psi_n(x,t) = \frac{i}{\sqrt{4\pi cn}}\left\{e^{-i\pi n g^{-1}(ct+x)} - e^{-i\pi n f^{-1}(ct-x)}\right\}. \tag{4.7}$$

Using the identity

$$g^{-1}(ct+x) - f^{-1}(ct-x) = (c\tau+\xi) - (c\tau-\xi) = 2\xi, \tag{4.8}$$

[1] As in Eq. (2.22), respecting the commutation relations $\left[\hat{a}_n,\hat{a}_n^{\dagger}\right] = \delta_{nn'}$ and $[\hat{a}_n\hat{a}'_n] = [\hat{a}_n,\hat{a}_{n'}^{\dagger}] = 0$.

[2] The transition from classical coefficients to annihilation/creation operators was done analogously by the transformation $\hat{a}_n \to c\sqrt{\hbar/\omega_n L_0}\hat{a}_n$ and $\hat{a}_n^* \to c\sqrt{\hbar/\omega_n L_0}\hat{a}_n^{\dagger}$ where the term $1/\sqrt{\omega_n L_0}$ was absorbed by the mode function ψ_n.

for $\xi = 1 \to x = L(t)$ we can then force $f^{-1}(ct-x) = R(ct-x)$ and $g^{-1}(ct+x) = R(ct+x)$ and search for solutions in the form

$$\psi_n(x,t) = \frac{i}{\sqrt{4\pi cn}}\left[e^{-i\pi nR(ct+x)} - e^{-i\pi nR(ct-x)}\right] \tag{4.9}$$

provided that we can find solutions for the functional relation derived from (4.8)

$$R\Big(ct + L(t)\Big) - R\Big(ct - L(t)\Big) = 2, \tag{4.10}$$

known as Moore's equation. This mode solution, in combination with Eq. (4.2), constitute Moore's solution for the non-stationary problem.

4.2 Moore function $R(\zeta)$

4.2.1 Examples of $R(\zeta)$ functions

As the self-evident "conservation of difficulty" states, there is no free lunch in mathematics. Whenever you convert a hard problem into a more amenable one, an intrinsic 'difficulty' appears to be preserved in the process. The price we paid by transferring the problem of time dependent boundary conditions into the one of solving Moore's equation (4.9), is that for every mirror trajectory $L(t)$ we analyse, turns out to be necessary to find the correspondent Moore's function $R(\zeta)$, which suffers from a severe scarcity in the market of analytical solutions.

As a consequence of this trade-off, we just have a few number of complete solutions at our disposal. One of these is in the case of the uniform moving mirror described by the trajectory $L(t) = L_0 + vt$ with velocity v. Its correspondent $R(\zeta)$-function for $L_0 = 0$ was firstly found in 1921 by Nicolai [34] in the context of string vibrations with variable length. Another 8 years passed by until Havelock [35] in the discussion of radiation pressure would manage to obtain the expression for the case

$$R(\zeta) = \frac{2\ln\left(1 + \frac{v}{cL_0}\zeta\right)}{\ln(c+v)/(c-v)}. \tag{4.11}$$

A very illustrative special case can be obtained if by using the low limit expansion $\ln(1 + v\zeta/cL_0) \approx v\zeta/cL_o$ and $\ln(c+v)/(c-v) \approx 2v/(c-v)$ we rewrite R in the form of a velocity expansion $R(\zeta) = 2(1-v/c)\frac{\zeta}{2L_0} + \mathcal{O}(v^2)$, becoming trivial that for the static case $v \to 0$, we obtain

$$R(\zeta) = \zeta/L_0 = \frac{1}{L_0}(ct \pm x) \tag{4.12}$$

recovering the R function for the stationary solution (4.4) in terms of standing wave mode functions.

But in terms of applicability, arrangements in the form of periodically moving mirrors (also called vibrating cavities) represents the best candidates for future practical implementation on DCE and, for this reason, constitutes the most important analytical cases for consideration. In special, the most simple periodically motion is the one in which the moving mirror oscillates back and forth between its equilibrium position L_0 with small amplitudes εL_0 ($\varepsilon \ll 1$) and angular frequencies $\Omega_q = \pi q c/L_0$ with $q = 1, 2, \ldots$ defining the trajectory

$$L(t) = L_0 \left[1 + \varepsilon \sin\left(\Omega_q t\right)\right], \tag{4.13}$$

The solution for the oscillation around L_0 in the long time limit ($\varepsilon ct/L_0 \gg 1$) can be found exactly as [33, 36]

$$R(ct \pm L_0) = \frac{ct \pm L_0}{L_0} - \frac{2}{\pi q} \operatorname{Im} \left\{ \ln \left[\frac{1+\xi(t)}{1-\xi(t)} + e^{i\frac{\pi q (ct \pm L_0)}{L_0}} \right] \right\} \quad \text{with} \quad \xi(t) = e^{(-1)^{q+1} \frac{\pi q c \varepsilon t}{L_0}}. \tag{4.14}$$

Another way to write expression (4.14) is with the substitution $ct \pm L_0 \to x'$ and identifying that for a complex number $z = x + iy$ we can represent $\ln z = \ln \sqrt{x^2 + y^2} + i \arctan\left(\frac{y}{x}\right)$, so that its correspondent imaginary part is equivalent to

$$R(x') = \frac{x'}{L_0} - \frac{2}{\pi q} \arctan \left[\frac{\sin\left(\frac{\pi q x'}{L_0}\right)}{\frac{1+\xi(t)}{1-\xi(t)} + \cos\left(\frac{\pi q x'}{L_0}\right)} \right] \tag{4.15}$$

For further analysis, it will be interesting to define the function $\Theta(x') = R(x') - x'/L_0$. Its form is better understood if we inspect the asymptotic case for even q and $c\varepsilon t/L_0 \to \infty$ in which $\xi \to 0$, where by noticing that $\frac{\sin(\alpha)}{1+\cos(\alpha)} = \tan(\alpha/2)$, we can rewrite it as

$$\Theta(x')_{\xi \to 0} = -\frac{2}{\pi q} \arctan \left[\tan\left(\frac{\pi q x'}{2L_0}\right) \right] = \begin{cases} -x'/L_0 & \text{for } 0 < x'/L_0 < 1/q \\ \left(\frac{1}{q} - \frac{x'}{L_0}\right) & \text{for } 1/q < x'/L_0 < 2/q \\ \text{with period } T = \frac{2L_0}{q} & \text{for } x' \end{cases} \tag{4.16}$$

In Dodonov compendium [32] there is an extensive list of bibliographic references towards systems with Moore's equation solutions. Nonetheless, from now on, our discussion will be centered around the case of oscillating mirror, as constitute the most simple and still interesting model with exact solution for DCE.

4.2.2 Connection with static Casimir effect

Even thought SCE and DCE are very different, in the DCE configuration of a moving cavity the confinement of the vacuum energy by the mirrors should analogously account for the emergence of Casimir forces in the vicinity of the non-stationary boundaries. So for consistency and as a final remark on Moore's function $R(\zeta)$, we must expect to recover from the mathematical framework developed so far, the standard Casimir force acting on the mirrors in the special condition where both mirrors are at rest. We calculate this "dynamical" version of Casimir force (in our old scalar field $\hat{\phi}_\alpha$) by, again, computing the vacuum expectation value of the appropriate component of the stress tensor shown in Eq. (3.24)

$$\langle 0 | \overleftrightarrow{T}_{xx}(\hat{\phi}_n, \hat{\phi}_{n'}) | 0 \rangle = \frac{1}{2} \left\langle \left[\left(\frac{1}{c} \frac{\partial \hat{\phi}_n}{\partial t} \right)^2 + \left(\frac{\partial \hat{\phi}_n}{\partial x} \right)^2 \right] \right\rangle \tag{4.17}$$

Since this expression is a well known divergent quantity (as a result of the infinity summation in Eq. (4.2)), we employ a normalization procedure called *point-splitting* [5] where, instead of evaluating ψ_n and ψ_n^* in the same space-time point, we compute them, respectively in $(x, t+\frac{\epsilon}{2})$ and $(x, t-\frac{\epsilon}{2})$, where ϵ is a small parameter that we are going to take to be zero at the end of the process. Using the notation $u = ct + x$ and $v = ct - x$ as well as the relations $\hat{a}_m|0\rangle = \langle 0|\hat{a}_m^\dagger = 0$ and $\langle 0|\hat{a}_n\hat{a}_{n'}^\dagger|0\rangle = \delta_{n,n'}$, it is possible to write [5]

$$\begin{aligned}\langle 0|\overleftrightarrow{T}_{xx}|0\rangle = \frac{c\hbar\pi}{4}\sum_{n=1}^{\infty} n\Big\{ & R'\left(v+\tfrac{\epsilon}{2}\right)R'\left(v-\tfrac{\epsilon}{2}\right)e^{in\pi\left[R(v+\frac{\epsilon}{2})-R(v-\frac{\epsilon}{2})\right]} \\ & + R'\left(u+\tfrac{\epsilon}{2}\right)R'\left(u-\tfrac{\epsilon}{2}\right)e^{in\pi\left[R\left(u+\frac{\epsilon}{2}\right)-R\left(u-\frac{\epsilon}{2}\right)\right]}\Big\}.\end{aligned} \tag{4.18}$$

No, by using $\Delta R(\zeta) = R(\zeta+\frac{\epsilon}{2}) - R(\zeta-\frac{\epsilon}{2})$, Eq. (4.18) can be computed through the identity $\sum_{n=1}^{\infty} ny^n = \frac{y}{(1-y)^2}$ where in our case $y = \exp\left(i\pi\Delta R(\zeta)\right)$, such that

$$\begin{aligned}\langle 0|\overleftrightarrow{T}_{xx}|0\rangle = \frac{c\hbar\pi}{4}\Big\{ & \frac{R'(v+\epsilon/2)R'(v-\epsilon/2)}{\left[1-\exp i\pi\Delta R(v)\right]^2}e^{i\pi\Delta R(v)} + \frac{1}{\epsilon^2} \\ & + \frac{R'(u+\epsilon/2)R'(u-\epsilon/2)}{\left[1-\exp i\pi\Delta R(u)\right]^2}e^{i\pi\Delta R(u)} + \frac{1}{\epsilon^2}\Big\} - \frac{\hbar c}{2\pi\epsilon^2}.\end{aligned} \tag{4.19}$$

Expanding this last expression in a power series in ϵ it is possible (see Appendix B for details) to show that

$$\mathcal{F} = \langle 0|\overleftrightarrow{T}_{xx}|0\rangle = -h\left(ct+x\right) - h\left(ct-x\right), \tag{4.20}$$

with

$$h(\zeta) = \frac{c\hbar}{24\pi}\left[\frac{R'''}{R'} - \frac{3}{2}\left(\frac{R''}{R'}\right)^2 + \frac{1}{2}\pi^2\left(R'\right)^2\right], \tag{4.21}$$

where we discarted the divergent term $-\hbar c/2\pi\epsilon^2$ of Eq. (4.19) by renormalizing the expression.

It is immediate from Eq. (4.21) that, for the case where Moore's function assumes the form given in Eq. (4.12), only the last term, proportional to its first derivative, should survive, obtaining therefore

$$\mathcal{F} = -\frac{\pi\hbar c}{24L_0^2},$$

thus recovering the Casimir force density for the static case in $(1+1)$-dimensional as latter derived in Eq. (3.34) [5].

4.3 Quantum vacuum and particle creation

4.3.1 Setting the stage

At this stage, after gathering all the necessary ingredients, we can begin the discussion about the central result (and most perplexing feature) of DCE: How the modification of the vacuum mode structure by a non-stationary cavity can induce the generation of real particles. To do so, we use the mathematical machinery so far developed to calculate the occupational number of excited modes (particles).

Again, following Moore's approach [3], we can state a 'scattering problem' route in which we divide our problem in three stages by an initial instant of time t_{in} and final one t_{out} as represented in Fig. 4.2. In the first stage, for $t < t_{\text{in}}$, we consider the second mirror fixed in $L(t) = L_0$ as described in Eqs. (4.4) and (4.2) with physical creation/annihilation operators $\hat{a}_n^{\text{in}}$ and $\hat{a}_n^{\text{in}\dagger}$, respectively. The steady-state standing waves mode solutions of the wave equation are written as

$$\begin{aligned} \hat{\phi}^{(\text{in})}(x, t < t_{\text{in}}) &= 2c\sqrt{\hbar}\sum_{n=1}^{\infty}\left[\hat{a}_n^{\text{in}}\psi_n^{(\text{in})}(x,t) + \hat{a}_n^{\text{in}\dagger}\psi_n^{(\text{in})}(x,t)^*\right] \\ &= 2c\sqrt{\hbar}\sum_{n=1}^{\infty}\left\{\frac{\hat{a}_n^{\text{in}}}{\sqrt{\pi n}}\sin\left(\frac{\pi n}{L_0}x\right)e^{-in\pi\frac{ct}{L_0}} + \text{h.c.}\right\}. \end{aligned} \tag{4.22}$$

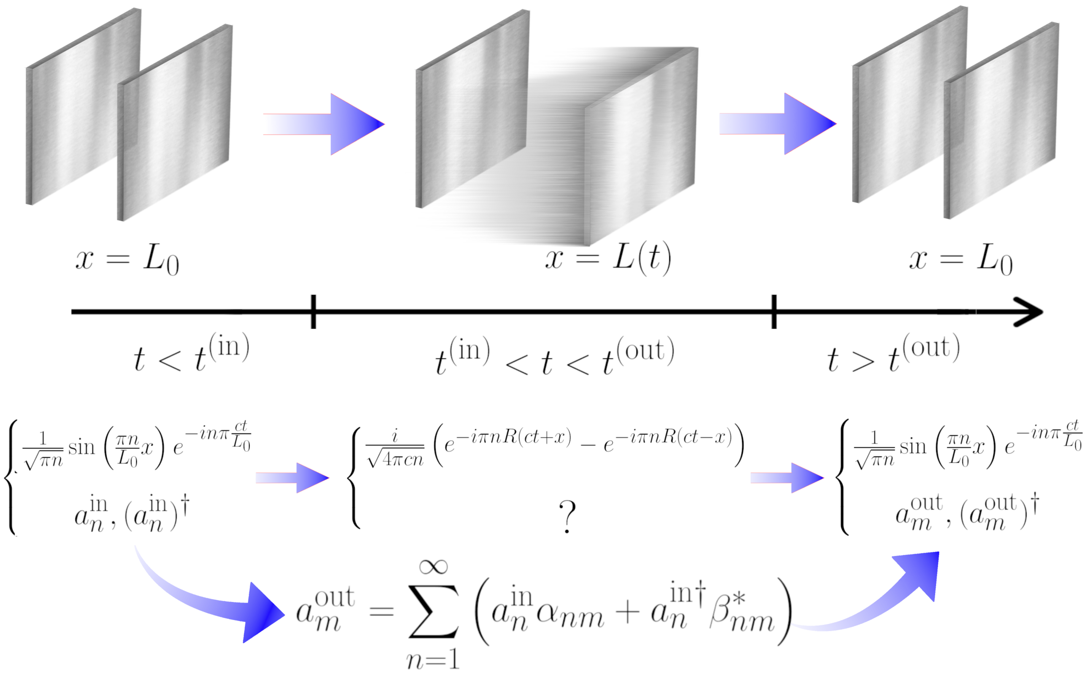


Figure 4.2: Scheme for the Moore's scattering problem. The scheme is divided from top to bottom as: (i) the mirror configuration (at rest or in motion); (ii) an arrow representing the instants in time; (iii) the mode functions and their correspondent set of creation/annihilation operators; (iv) the expression for the Bogoliubov transformations. Source: author.

For subsequent instants of time, in the second stage ($t_{\text{in}} < t < t_{\text{out}}$), the second mirror can execute its arbitrary motion thus imposing non-stationary boundaries conditions and scalar field dynamics is given by Eq. (4.9). At time t_{out} the mirror returns to its initial position and the field is characterized by the modes

$$\begin{aligned} \hat{\phi}^{(\text{out})}(x, t > t_{\text{out}}) &= 2c\sqrt{\hbar}\sum_{n=1}^{\infty}\left[\hat{a}_n^{\text{in}}\psi_n^{(\text{out})}(x,t) + \hat{a}_n^{\text{in}\dagger}\psi_n^{(\text{out})}(x,t)^*\right] \\ &= 2c\sqrt{\hbar}\sum_{n=1}^{\infty}\left\{\frac{i\hat{a}_n^{\text{in}}}{\sqrt{4\pi nc}}\left[e^{-i\pi nR(ct+x)} - e^{-i\pi nR(ct-x)}\right] + \text{h.c.}\right\}, \end{aligned} \tag{4.23}$$

for $t > t_{\text{out}}$. As commented by V. Dodonov [33, 36], it is plausible to suppose that a photon measuring device must react to a steady-state standing waves (4.22) which possess definite energy values. With this in mind, we exploit the fact that the mirror remains stationary after time t_{out} and expand Eq. (4.23) in standing modes using the linear combination

$$\psi_n^{(\text{out})}(x,t) = \sum_{m=1}^{\infty}\left[\alpha_{nm}\psi_m^{(\text{in})}(x,t) + \beta_{nm}\psi_m^{(\text{in})}(x,t)^*\right]. \tag{4.24}$$

Inserting Eq. (4.24) into Eq. (4.23) we obtain

$$\begin{aligned}\hat{\phi}^{(\text{out})} &= 2c\sqrt{\hbar}\sum_{m=1}^{\infty}\left\{\left[\sum_{n=1}^{\infty}\left(\hat{a}_n^{\text{in}}\alpha_{nm} + \hat{a}_n^{\text{in}\dagger}\beta_{nm}^*\right)\right]\psi_n^{(\text{in})}(x,t) + \left[\sum_{n=1}^{\infty}\left(\hat{a}_n^{\text{in}}\alpha_{nm}^* + \hat{a}_n^{\text{in}\dagger}\beta_{nm}\right)\right]\psi_n^{(\text{in})}(x,t)^*\right\}\\ &= 2c\sqrt{\hbar}\sum_{m=1}^{\infty}\left[\hat{a}_m^{\text{out}}\psi_n^{(\text{in})}(x,t) + \hat{a}_n^{\text{out}\dagger}\psi_n^{(\text{in})}(x,t)^*\right],\end{aligned}$$

where now we have an expansion in different creation/annihilation operators $(\hat{a}_n^{\text{out}})^\dagger$ and $\hat{a}_n^{\text{out}}$, which are connected with the initial operators $(\hat{a}_n^{\text{in}})^\dagger$ and $\hat{a}_n^{\text{in}}$ by means of the Bogoliubov transformations [32, 37]

$$\hat{a}_m^{\text{out}} = \sum_{n=1}^{\infty}\left(\hat{a}_n^{\text{in}}\alpha_{nm} + \hat{a}_n^{\text{in}\dagger}\beta_{nm}^*\right), \quad m = 1, 2, \ldots$$

The coefficients of such transformation must satisfy the unitary conditions

$$\sum_{m=1}^{\infty}(\alpha_{nm}^*\alpha_{km} - \beta_{nm}^*\beta_{km}) = \delta_{nk}, \quad \sum_{n=1}^{\infty}(\alpha_{nm}^*\alpha_{nj} - \beta_{nm}^*\beta_{nj}) = \delta_{mj}, \quad \sum_{n=1}^{\infty}(\beta_{nm}^*\alpha_{nk} - \beta_{nk}^*\alpha_{nm}) = 0.$$

Since the final position of the moving boundary coincides with initial position L_0, the Bogoliubov coefficients can be found using the definition of inner product given in Eq. (4.3) in the expression (4.24) as

$$\left.\begin{aligned}\alpha_{mn} &= \left(\psi_n^{(\text{out})}, \psi_m^{(\text{in})}\right)\\ \beta_{mn} &= -\left(\psi_n^{(\text{out})}, \psi_m^{(\text{in})*}\right)\end{aligned}\right\} = \frac{1}{2L_0}\sqrt{\frac{m}{n}}\int_{ct-L_0}^{ct+L_0} dx' e^{-i\frac{\pi}{L_0}\left[nL_0 R\left(x'\right) \mp mx'\right]}, \tag{4.25}$$

where we used a transformation of variable $ct \pm x \to x'$ throughout the derivation as can be seen in the Appendix C.

The mean number of particles in the m-th mode can be calculated as the average value of the operator $(\hat{a}_m^{\text{out}})^\dagger\hat{a}_m^{\text{out}}$ in the initial state $|\text{in}\rangle$ (the state basis associated with the set of ladder operators $a^{(\text{in})}$ and $a^{\dagger(\text{in})}$ in the stage $t < t^{(\text{in})}$), since this operator has a physical meaning for $t > t^{(out)}$. Therefore,

$$\begin{aligned}\mathcal{N}_m \equiv \left\langle \text{in}\left|(\hat{a}_m^{\text{out}})^\dagger\hat{a}_m^{\text{out}}\right|\text{in}\right\rangle &= \left\langle\left[\sum_{n=1}^{\infty}\left((\hat{a}_n^{\text{in}})^\dagger\alpha_{nm}^* + \hat{a}_n^{\text{in}}\beta_{nm}\right)\right]\left[\sum_{k=1}^{\infty}\left(\hat{a}_k^{\text{in}}\alpha_{km} + \hat{a}_k^{\text{in}\dagger}\beta_{km}^*\right)\right]\right\rangle\\ \sum_{n}|\beta_{nm}|^2 &+ \sum_{n,k=1}^{\infty}\left[(\alpha_{nm}^*\alpha_{km} + \beta_{nm}^*\beta_{km})\left\langle(\hat{a}_n^{\text{in}})^\dagger\hat{a}_k^{\text{in}}\right\rangle + 2\,\text{Re}\left(\beta_{nm}\alpha_{km}\left\langle\hat{a}_n^{\text{in}}\hat{a}_k^{\text{in}}\right\rangle\right)\right].\end{aligned} \tag{4.26}$$

From this analysis it is immediate that for an initial vacuum state $|\text{in}\rangle = |0\rangle$ only the first term survives such that

$$\mathcal{N}_m^{(0)} = \langle 0\left|(\hat{a}_m^{\text{out}})^\dagger\hat{a}_m^{\text{out}}\right|0\rangle = \sum_n |\beta_{nm}|^2, \tag{4.27}$$

indicating that the initial vacuum state now contains particles due to the mixture of positive and negative frequency modes (associated with the coefficient operators (2.36)) during the time in which the system were under non-stationary boundary conditions.

4.3.2 Creation of particles by parametric resonance

In order to exemplify the previous discussion, we follow the steps developed by V. Dodonov [36] and compute the expression for the occupation number of particles generated from the vacuum state for the case of vibrating cavity considering the mirror trajectory (4.13) and its correspondent Moore's function (4.15).

For $\varepsilon > 0$ and q even, the function $\xi(t) = \exp\left[(-1)^{q+1}\pi\varepsilon\frac{ct}{L_0}\right]$ is very small in the long run $c\varepsilon t/L_0 \gg 1$, which implies by the condition $\varepsilon \ll 1$ that the function $\xi(t)$ is practically constant in the interval of integration $ct - L_0 < x' < ct + L_0$ of the Bogoliubov coefficients (4.24). In this circumstance, we approach the asymptotic case (4.16) where $\Theta(x')$ can be considered as a periodic function

$$\Theta(x') = \Theta(x' + kT) \qquad \text{with} \quad T = 2L_0/q \quad \text{and} \quad k \in \mathbb{Z}. \tag{4.28}$$

So, to perform the calculation of the integral (4.25), we proceed by dividing the interval of integration with total size $\Delta I = (ct + L_0) - (ct - L_0) = 2L_0$ in N partitions each of which with size T (function period) so that $NT = \Delta I = 2L_0$. It is immediate from the periodicity condition (4.28) that $N = q$, so instead of evaluating the integral in the original interval, we can sum q integrals between 0 and T [3] obtaining

$$\left.\begin{matrix}\beta_{nm}\\ \alpha_{nm}\end{matrix}\right\} = \frac{1}{2L_0}\sqrt{\frac{m}{n}}\sum_{k=0}^{q-1}\int_0^T dx'\, e^{-\frac{i\pi}{L_0}[(n\pm m)(x'+kT)+nL_0\Theta(x')]}, \tag{4.29}$$

where

$$\Theta(x') = -\frac{2}{\pi q}\arctan\left[\frac{\sin\left(\frac{\pi q x'}{L_0}\right)}{\frac{1+\xi(t)}{1-\xi(t)} + \cos\left(\frac{\pi q x'}{L_0}\right)}\right]. \tag{4.30}$$

Before continuing the calculation, it is important to notice that the DCE can happen even in the case of a single moving mirror, the reason we choose a cavity, specially a vibrating cavity, is primarily to amplify the rate of creation of particles as we will discuss right now.

Parametric Resonance

To what frequency Ω_q we need to vibrate the cavity in order to maximize the particle creating rate? For our case of study, which corresponds to a resonant cavity enclosed by periodically moving boundaries $L(t) = L_0\left[1 + \varepsilon\cos(\Omega t)\right]$ with small oscillating frequency amplitudes ε, we can understand our system qualitatively as a harmonic oscillator modulated by a time dependent frequency $\omega(t) = c\pi n/L(t)$

$$\ddot{Q} + \omega(t)^2 Q = 0, \quad \text{with} \quad \omega(t)^2 = \omega_0^2\left(1 - 2\varepsilon\cos\Omega t\right),$$

where $\varepsilon \ll 1$, $\omega_0 = c\pi n/L_0$ and Q is a generalized coordinate. This is also equivalent to a driven harmonic oscillator with coordinate dependence

$$\ddot{Q} + \omega_0^2 Q = F(q,t) = 2\varepsilon\omega_0^2 Q\cos\Omega t.$$

[3] $\Theta(x)$ is invariant under such transformation, simplifying the calculation.

The very first noticeable aspect of the force expression is that despite very unstable, an initially static system ($Q(0) = 0$) should remain at rest indefinitely. Also, contrary to a simple driving oscillator in resonance frequency where its amplitude increases linearly in time, in optimal frequency conditions, we expect the increasing force with coordinate to pump additional energy in the form of increasing amplitude, which should increase correspondingly the force and, therefore, continue this process indefinitely in a feedback cycle of exponential growth.

But how could we achieve such an amplitude increase? Recapitulating, the natural frequency of vibration ω_0 (resonance) is the optimal one for a simple driving oscillator to increase its amplitude (linearly) and can be achieved by applying a driving force $G(t) \sim \cos\omega_0 t$. So, let us suppose that our system already starts oscillating at this frequency (ω_0), the correspondent driven force in this first cycle should be proportional to

$$F(t) \sim \cos(\omega_0 t)\cos(\Omega t) = 1/2\left[\cos(\Omega - \omega_0)t + \cos(\Omega + \omega_0)t\right], \tag{4.31}$$

showing us that the simplest way to obtain effective resonance ($F(t) \sim G(t)$) is by vibrating the system with the double of its natural frequency, or $\Omega = 2\omega_0$ (we discarded non-resonant terms proportional to $\cos(3\omega_0)$ in Eq. (4.31), characterizing the phenomena of *parametric resonance*, where for sufficient small perturbations σ of Ω, later cycles of increasing coordinate/amplitude result in an exponential growth of amplitude for $-\frac{1}{2}\varepsilon\omega_0 < \sigma < \frac{1}{2}\varepsilon\omega_0$ [38].

We can physically understand parametric resonance in the very intuitive case of a child swinging in a stand up swing. The amplification of the oscillating amplitude is made by propelling its center of mass twice a cycle in each furthest point from equilibrium. By forcing the body outward, the effective string length is decreased in a frequency doubled the natural oscillation periodicity, what acts as an energy pump into the system in an exponential rate.

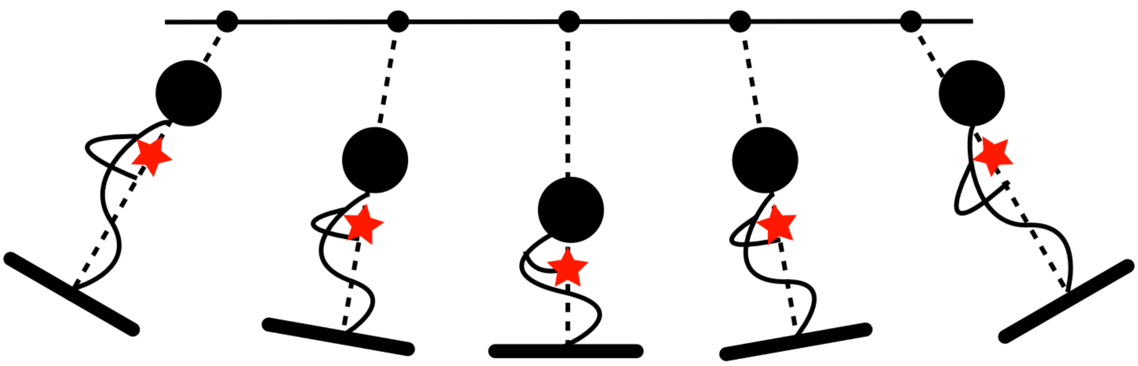


Figure 4.3: Parametric resonance and a swing: Contrary to the parent's method of swinging his child one time in each frequency cycle (amplifying linearly the amplitude), a more independent kid in a stand up swing can amplify the oscillation amplitude by propelling itself twice a cycle in each furthest point from equilibrium, changing thus the effective length of the string and consequently the frequency of oscillation to a parametric resonance (until physical limitations stops the exponential grow in amplitude). Source: [39].

Getting back to the calculation

With the latter discussion in mind, to amplify the DCE we must chose the parametric resonance mode $\Omega_q = 2\omega_n$ what implies $q = 2n$. Using the main resonance case of $n = 1$ so that $q = 2$ and after defining the small quantity $\delta = \sqrt{\xi}/2\pi = \exp\left(-\pi\varepsilon\frac{ct}{L_0}\right) \ll 1$, we can use it to approximate $\Theta(x')$ by a piecewise linear function

$$\Theta(x') \approx \begin{cases} -(1-2\delta)x'/L_0, & 0 < x'/L_0 < 1/2 - \delta \\ (1/2-2\delta)(x'/L_0 - 1/2)\delta^{-1}, & 1/2-\delta < x'/L_0 < 1/2+\delta \\ -(1-2\delta)(x'/L_0 - 1), & 1/2+\delta < x'/L_0 < 1, \end{cases} \tag{4.32}$$

in analogy with the asymptotic case (4.16). Inserting the last expression into Eq. (4.29) we can finally compute the Bogoliubov coefficients [36] (see Appendix D for details)

$$\left.\begin{matrix}\beta_{nm} \\ \alpha_{nm}\end{matrix}\right\} = \frac{2}{\pi}\sqrt{\frac{m}{n}}\frac{\sin\frac{\pi}{2}(2\delta n \pm m)}{(2\delta n \pm m)}\cos\frac{\pi}{2}(n \pm m)e^{i\frac{\pi}{2}(n\pm m)}. \tag{4.33}$$

It is also not difficult to show (see again Appendix D) that the expression for the moduli squared of the Bogoliubov coefficients[4] are given by,

$$\left.\begin{matrix}|\beta_{nm}|^2 \\ |\alpha_{nm}|^2\end{matrix}\right\} = \frac{m}{n\pi^2}\frac{[1-(-1)^m\cdot\cos(2n\pi\delta)]}{(2n\delta \pm m)^2}\left[1+(-1)^{m+n}\right]. \tag{4.34}$$

With the expression for the coefficients in hand, we can compute the total number of particles created in the m-th mode with help of the transformation shown in Eq. (4.26). But first we rewrite Eq. (4.34) by using the transformations $\bar{x} = 2\pi\delta \ll 1$ and $\bar{z} = m/2\delta \gg 1$ (such that $\bar{x}\bar{z} = \pi m$)

$$\begin{aligned}\left.\begin{matrix}|\beta_{nm}|^2 \\ |\alpha_{nm}|^2\end{matrix}\right\} &= \frac{m}{n\pi^2}\frac{[1-(-1)^m\cos(2n\pi\delta)]}{4\delta^2\left(n \pm \frac{m}{2\delta}\right)^2}[1+(-1)^m] \\ &= \frac{m}{4\delta^2\pi^2}\frac{[1-(-1)^m\cos(n\bar{x})]}{n\left(n \pm \bar{z}\right)^2}\left[1+(-1)^{m+n}\right]. \end{aligned} \tag{4.35}$$

Supposing an initial vacuum state, the number of particles created at time t_{out} can be computed by inserting the coefficient (4.35) into Eq. (4.27) and it is given by

$$\begin{aligned}\mathcal{N}_m^{(0)} &= \left\langle 0\left|\hat{\alpha}_m^\dagger\hat{\alpha}_m\right|0\right\rangle = \sum_n |\beta_{nm}|^2 \\ &= \frac{m}{4\pi^2\delta^2}\left\{\sum_{n=1}^{\infty}\frac{1}{n(\bar{z}+n)^2} + (-1)^m\sum_{n=1}^{\infty}\frac{(-1)^n}{n(\bar{z}+n)^2} - (-1)^m\sum_{n=1}^{\infty}\frac{\cos(\bar{x}n)}{n(\bar{z}+n)^2}\right. \\ &- \left.\sum_{n=1}^{\infty}\frac{(-1)^n\cos(\bar{x}n)}{n(\bar{z}+n)^2}\right\} \\ &= \frac{m}{4\pi^2\delta^2}\left\{S(0,\bar{z}) + (-1)^m S(\pi,\bar{z}) - (-1)^m S(\bar{x},\bar{z}) - S(\bar{x}+\pi,\bar{z})\right\} \end{aligned} \tag{4.36}$$

where we have used the auxiliary function

$$S(\bar{x},\bar{z}) = \sum_{n=1}^{\infty}\frac{\cos(\bar{x}n)}{n(\bar{z}+n)^2}. \tag{4.37}$$

[4]One can check that the normalization condition $\sum_m\left\{|\alpha_{nm}|^2 - |\beta_{nm}|^2\right\} = 1$ holds.

$\boldsymbol{S(\bar{x}, \bar{z})}$: The expression for the more general form of Eq. (4.37) can be computed with the help of the integral representation of the function $1/(\bar{z}+n) = \int_0^\infty d\bar{y}\ \bar{y}e^{-\bar{y}(\bar{z}+n)}$, such that

$$S(\bar{x},\bar{z}) = \int_0^\infty d\bar{y}\ \bar{y}e^{-\bar{y}\bar{z}} \left[\sum_{n=1}^{\infty} \frac{\cos(n\bar{x})}{n} e^{-\bar{y}n}\right] = \bar{z}^{-3} - \frac{1}{2}\int_0^\infty d\bar{y}\ \bar{y}e^{-\bar{z}\bar{y}} \ln[2\cosh(\bar{y}) - 2\cos(\bar{x})] \tag{4.38}$$

where

$$\sum_{n=1}^{\infty} \cos(n\bar{x}) \frac{e^{-n\bar{y}}}{n} = \frac{\bar{y}}{2} - \frac{1}{2}\ln\left[2\cosh(\bar{y}) - 2\cos(\bar{x})\right]. \tag{4.39}$$

Since by definition $\bar{x} \ll 1$, in the main region of integration ($\bar{y} < 1/\bar{z} < \bar{x} < 1$) we can write $\cosh(\bar{y}) = 1 + \bar{y}^2/2 + \mathcal{O}(\bar{y}^4)$ and $\cos(\bar{x}) = 1 - \bar{x}^2/2 + \mathcal{O}(\bar{x}^4)$, such that $2\cosh(\bar{y}) - 2\cos(\bar{x}) = (\bar{x}^2 + \bar{y}^2) + \mathcal{O}(\bar{x}^4, \bar{y}^4) \sim \bar{x}^2$ by neglecting terms containing $\bar{y}^2$ in comparison with $\bar{x}^2$. Then the integral becomes trivial, and we obtain

$$S(\bar{x},\bar{z}) = \bar{z}^{-3} - \ln(\bar{x}) \int_0^\infty d\bar{y}\ \bar{y}e^{-\bar{z}\bar{y}} = -\ln(\bar{x})/\bar{z}^2 + O\left(\bar{z}^{-3}\right). \tag{4.40}$$

$\boldsymbol{S(0, \bar{z})}$: For the first term in Eq. (4.36) written in $S(\bar{x},\bar{z})$ representation with $\bar{x} = 0$, by remembering that $\bar{y} < 1/\bar{z}$, we have that $\ln[2\cosh(\bar{y}) - 2\cos(0)] < \ln[2\cosh\frac{1}{\bar{z}} - 2] \sim -2\ln\bar{z}^2$, so

$$S(0,\bar{z}) = \bar{z}^{-3} + \ln(\bar{z}) \int_0^\infty d\bar{y}\ \bar{y}e^{-\bar{z}\bar{y}} = \ln(\bar{z})/\bar{z}^2 + O\left(\bar{z}^{-3}\right). \tag{4.41}$$

$\boldsymbol{S(\pi, \bar{z})}$ and $\boldsymbol{S(\bar{x}+\pi, \bar{z})}$: From the terms proportional to $(-1)^n$, expressed as a function of $S(\bar{x},\bar{z})$ in Eq. (4.36) and written considering the replacement $\bar{x} \to \bar{x} + \pi$

$$S(\bar{x}+\pi,\bar{z}) = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{\cos(\bar{x}n)}{n(\bar{z}+n)^2},$$

the correspondent sum in Eq. (4.39) must be proportional to $\ln\left[2\cosh(\bar{y}) + 2\cos(\bar{x})\right]$, which can be analogously expanded in terms of $2\cosh(\bar{y}) + 2\cos(\bar{x}) = 4 + \bar{y}^2 - \bar{x}^2 + \mathcal{O}(\bar{x}^2, \bar{y}^2) \sim 0$ so we can write $\ln\left(4 + \bar{y}^2 - \bar{x}^2\right) < |\ln\bar{x}^2|$ for $\bar{x} < 1$ and $\bar{y} \leq 1$ such that the correspondent terms can be omitted.

The final result is given, therefore, in terms of Eqs. (4.40) and (4.41) as

$$\begin{aligned} \mathcal{N}_m \approx & \frac{m}{4\pi^2\delta^2}\left[S(0,\bar{z}) - (-1)^m S(\bar{x},\bar{z})\right] = \frac{m}{4\pi^2\delta^2\bar{z}^2}\left[\ln(\bar{z}) - (-1)^m \ln\left(\tfrac{1}{\bar{x}}\right)\right] \\ = & \frac{1}{m\pi^2}\left[\ln\left(\frac{m}{2\delta}\right) - (-1)^m \ln\left(\frac{1}{2\pi\delta}\right)\right]. \end{aligned} \tag{4.42}$$

Remembering that $\delta(t) = \exp(-\pi c\varepsilon t/L_0)/\pi$ such that $\frac{d}{dt}\delta = -\frac{c\epsilon\pi}{L_0}\delta$, by deriving Eq. (4.42) in time we can finally obtain

$$\frac{d\mathcal{N}_m}{dt} = \frac{c\varepsilon}{L_0\pi m}\left[1 - (-1)^m\right], \tag{4.43}$$

showing that the rate of particles creation in the m-th mode for a vibrating cavity in parametric resonance in its first resonator eigenmode varies linearly in time in the limit $c\varepsilon t/L_0 \gg 1$. As a restriction due to the approximations made before the result is in fact only valid for values of m not that much large. Another characteristic that can be observed from expression (4.43) is that only odd modes m contributes with the creation of photons, a direct consequence of the imposed parametric resonance that by doubling the first cavity mode of oscillation enhanced the photon generation of theses last in detriment of the even ones.

Chapter 5

Conclusions

In order to develop this work we considered three main subjects: The very first consisted in the characterization and quantization of the radiation field. This program lead us to the result that the field can be expressed in terms of a structure of modes of oscillation which describes photons as excitations in the context of Fock space, as well as the the recognition that its fluctuations, even in the vacuum state, translates on the existence of a zero point energy for the radiation field.

The second topic involved the conceptualization of the static Casimir effect as a fundamental phenomenon of quantum fields in the vacuum due to imposed boundaries conditions. Using as a model a pair o parallel mirrors, we could illustrate how changes in the ZPE of the radiation field can be responsible for the emergence of forces in the vicinity of the cavity.

Finally, the last subject revolved around the dynamical counterpart of the last phenomenon, the so called dynamical Casimir effect. The analysis was done mainly through Moore's approach, where a scalar field confined by non-stationary boundary conditions takes us to consider how the non-trivial changes in the vacuum state throughout the movement translates into the creation of real particles due to the mixture of positive and negative frequencies modes.

Looking back to what we could learn from this monograph, those three ingredients here studied demonstrate together that modifications of the quantum vacuum structure by adequate boundary conditions can lead to macroscopic manifestations of the ZPE of a quantum field as delineated in the description of the two above quantum phenomena whether in the emergence of forces or in the particle creation from vacuum.

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Appendices

Appendix A

Explicit calculation of the EMF observables

Hamiltonian operator

Introducing $\mathbf{u} = c\sqrt{\frac{\hbar}{2\mathcal{V}}}\boldsymbol{\epsilon}_{\boldsymbol{k}}^{(\boldsymbol{\alpha})} e^{i\mathbf{k}\cdot\mathbf{r}}$ for conciseness (with de $(\boldsymbol{k}, \alpha)$ dependence implicit in the expression), we reframe expressions (2.28), (2.29) and (2.30) as

$$\hat{\mathbf{A}} = \sum_{\mathbf{k},\alpha} \frac{1}{\sqrt{\omega_{\boldsymbol{k}}}} \left[\hat{a}\mathbf{u} + \hat{a}^\dagger \mathbf{u}^*\right]; \tag{A.1}$$

$$\hat{\mathbf{E}} = \frac{i}{c} \sum_{\mathbf{k},\alpha} \sqrt{\omega_{\boldsymbol{k}}} \left[\hat{a}\mathbf{u} - \hat{a}^\dagger \mathbf{u}\right]; \qquad \hat{\mathbf{B}} = i \sum_{\mathbf{k},\alpha} \frac{1}{\sqrt{\omega_{\boldsymbol{k}}}} \boldsymbol{k} \times \left[\hat{a}\mathbf{u} - \hat{a}^\dagger \mathbf{u}^*\right]. \tag{A.2}$$

The normalization conditions (2.12) and (2.13) can be unified in this context as

$$\int d^3r \left(\mathbf{u}\cdot\mathbf{u}^{*\prime}\right) = \frac{c^2\hbar}{2}\delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,\alpha'}; \qquad \int d^3x \left\{ \begin{array}{c} \mathbf{u}\cdot\mathbf{u}' \\ \mathbf{u}^*\cdot\mathbf{u}^{*\prime} \end{array} \right\} = \frac{c^2\hbar}{2}\delta_{\mathbf{k},-\mathbf{k}'}\delta_{\alpha\alpha'}.$$

To compute (2.31) expression we need first to calculate the following expressions: For the electric field contribution

$$\frac{1}{2}\int d^3r \hat{\mathbf{E}}\cdot\hat{\mathbf{E}} = \sum_{\mathbf{k}\mathbf{k}'\alpha\alpha'} \frac{i^2\omega_{\boldsymbol{k}}}{2c^2} \int d^3r \left(\hat{a}\mathbf{u} - \hat{a}^\dagger\mathbf{u}^*\right)\cdot\left(\hat{a}'\mathbf{u}' - \hat{a}^{\dagger\prime}\mathbf{u}^{*\prime}\right)$$

$$= \sum_{\mathbf{k},\mathbf{k}',\alpha,\alpha'} \frac{\omega_{\boldsymbol{k}}}{2c^2} \left\{ \left(\hat{a}\hat{a}^{\dagger\prime} + \hat{a}^\dagger\hat{a}'\right) \int d^3r \left(\mathbf{u}\cdot\mathbf{u}^{*\prime}\right) - \hat{a}\hat{a}' \int d^3r (\mathbf{u}\cdot\mathbf{u}^{'}) - \hat{a}^\dagger\hat{a}^{\dagger\prime} \int d^3r \left(\mathbf{u}^*\cdot\mathbf{u}^{*\prime}\right) \right)$$

$$= \sum_{\mathbf{k},\alpha} \frac{\hbar\omega_{\boldsymbol{k}}}{4} \left(\hat{a}_{\boldsymbol{k},\alpha}\hat{a}^\dagger_{\boldsymbol{k},\alpha} + \hat{a}^\dagger_{\boldsymbol{k},\alpha}\hat{a}_{\boldsymbol{k},\alpha} - \hat{a}_{\boldsymbol{k},\alpha}\hat{a}_{-\boldsymbol{k},\alpha} - \hat{a}^\dagger_{\boldsymbol{k},\alpha}\hat{a}^\dagger_{-\boldsymbol{k},\alpha} \right). \tag{A.3}$$

We can also calculate the magnetic field contribution for the hamiltonian as

$$\frac{1}{2}\int d^3r \hat{\mathbf{B}}\cdot\hat{\mathbf{B}} = \sum_{\mathbf{k},\mathbf{k}',\alpha,\alpha'} \frac{i^2}{2\omega_{\boldsymbol{k}}} \int d^3r \left[\mathbf{k}\times\left(\hat{a}\mathbf{u} - a^\dagger_k\mathbf{u}\right)\right]\cdot\left[\mathbf{k}'\times\left(\hat{a}'\mathbf{u}' - \hat{a}^{\dagger\prime}\mathbf{u}'\right)\right]$$

$$= -\sum_{\mathbf{k},\mathbf{k}',\alpha,\alpha'} \frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{2\omega_{\boldsymbol{k}}} \int d^3r \left(\hat{a}\mathbf{u} - \hat{a}^\dagger\mathbf{u}^*\right)\cdot\left(\hat{a}'\mathbf{u}' - \hat{a}^{\dagger\prime}\mathbf{u}^{*\prime}\right)$$

$$= \sum_{\mathbf{k},\mathbf{k}',\alpha,\alpha'} \left\{ \frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{2\omega_{\boldsymbol{k}}} \left[\left(\hat{a}\hat{a}^{\dagger\prime} + \hat{a}^\dagger\hat{a}'\right) \int d^3r \left(\mathbf{u}\cdot\mathbf{u}^{*\prime}\right) \right] - \frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{2\omega_{\boldsymbol{k}}} \left[\hat{a}\hat{a}' \int d^3r (\mathbf{u}\cdot\mathbf{u}^{'}) + \hat{a}^\dagger\hat{a}^{\dagger\prime} \int d^3r \left(\mathbf{u}^*\cdot\mathbf{u}^{*\prime}\right) \right] \right\}$$

$$= \sum_{\mathbf{k},\alpha} \frac{k^2}{2\omega_{\boldsymbol{k}}} \left[\frac{c^2\hbar}{2} \left(\hat{a}_{\boldsymbol{k},\alpha}\hat{a}^\dagger_{\boldsymbol{k},\alpha} + \hat{a}^\dagger_{\boldsymbol{k},\alpha}\hat{a}_{\boldsymbol{k},\alpha} + \hat{a}_{\boldsymbol{k},\alpha}\hat{a}_{-\boldsymbol{k},\alpha} + \hat{a}^\dagger_{\boldsymbol{k},\alpha}\hat{a}^\dagger_{-\boldsymbol{k},\alpha} \right) \right] \tag{A.4}$$

where we used the result $\mathbf{u}\cdot\boldsymbol{k} = 0$ and the fact that $\boldsymbol{k}'$ turns into $-\boldsymbol{k}$ for sum over the delta Kronecker

$\delta_{\boldsymbol{k},-\boldsymbol{k}}$ together with the follow identity

$$\int d^3r\,(\mathbf{k}\times\mathbf{u})\cdot(\mathbf{k}\times\mathbf{v}) = \int d^3r(\boldsymbol{k}\cdot\boldsymbol{k})(\mathbf{u}\cdot\mathbf{v}) - \int d^3r(\boldsymbol{k}\cdot\mathbf{u})(\boldsymbol{k}\cdot\mathbf{v}) = |\boldsymbol{k}|^2\int d^3r\,(\mathbf{u}\cdot\mathbf{v}).$$

Therefore, by using (A.3) and (A.4) we can finally compute

$$H = \frac{1}{2}\int d^3r\left(\hat{\mathbf{E}}\cdot\hat{\mathbf{E}}+\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}\right) = \frac{1}{2}\sum_{\mathbf{k},\alpha}\hbar\omega_{\boldsymbol{k}}\left(\hat{a}_{\mathbf{k},\alpha}\hat{a}^{\dagger}_{\mathbf{k},\alpha}+\hat{a}^{\dagger}_{\mathbf{k},\alpha}\hat{a}_{\mathbf{k},\alpha}\right) = \sum_{\mathbf{k},\alpha}\hbar\omega_{\boldsymbol{k}}\left(\hat{a}^{\dagger}_{\mathbf{k},\alpha}\hat{a}_{\mathbf{k},\alpha}+\frac{1}{2}\right) \tag{A.5}$$

since $\hat{a}\hat{a}^{\dagger}+\hat{a}^{\dagger}\hat{a} = [\hat{a},\hat{a}^{\dagger}]+2\hat{a}^{\dagger}\hat{a} = 2(\hat{a}^{\dagger}\hat{a}+1/2)$.

Momentum operator

The momentum expression as given by Eq. (2.32) is given with help of the identities $\hat{\mathbf{A}}\times(\hat{\mathbf{B}}\times\mathbf{C}) = (\hat{\mathbf{A}}\cdot\mathbf{C})\hat{\mathbf{B}}-(\hat{\mathbf{A}}\cdot\hat{\mathbf{B}})\mathbf{C}$ together with $\mathbf{u}\cdot\boldsymbol{k}=0$ as

$$\begin{aligned}
\hat{\mathbf{P}} &= \frac{1}{c}\int d^3r\left(\hat{\mathbf{E}}\times\hat{\mathbf{B}}\right) = \frac{1}{c}\sum_{\mathbf{k},\mathbf{k}',\alpha,\alpha'}\frac{i^2}{c}\int d^3r\left(\hat{a}\mathbf{u}-\hat{a}^{\dagger}\mathbf{u}^*\right)\times\boldsymbol{k}'\times\left(\hat{a}'\mathbf{u}'-\hat{a}^{\dagger\prime}\mathbf{u}^{*\prime}\right)\\
&= \frac{i^2}{c^2}\sum_{\mathbf{k},\mathbf{k}',\alpha,\alpha'}\int d^3r\,\boldsymbol{k}'\left(\hat{a}\mathbf{u}-\hat{a}^{\dagger}\mathbf{u}^*\right)\cdot\left(\hat{a}\mathbf{u}-\hat{a}^{\dagger}\mathbf{u}^*\right)\\
&= \frac{1}{c^2}\sum_{\mathbf{k},\alpha}\boldsymbol{k}\left[\frac{c^2\hbar}{2}\left(\hat{a}_{\boldsymbol{k},\alpha}\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}+\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\hat{a}_{\boldsymbol{k},\alpha}+\hat{a}_{\boldsymbol{k},\alpha}\hat{a}_{-\boldsymbol{k},\alpha}+\hat{a}^{\dagger}_{\boldsymbol{k},\alpha}\hat{a}^{\dagger}_{-\boldsymbol{k},\alpha}\right)\right]
\end{aligned} \tag{A.6}$$

where the two last terms in (A.6) can be disregarded since as we sum over the negative values of $\boldsymbol{k}$ we will have expressions like $\boldsymbol{k}\hat{a}_{\mathbf{k},\alpha}\hat{a}_{-\mathbf{k},\alpha}+(-\boldsymbol{k}\hat{a}_{-\mathbf{k},\alpha}\hat{a}_{\mathbf{k},\alpha}) = 0$ as a result of the commutation relation $[\hat{a}_{\mathbf{k},\alpha},\hat{a}_{-\mathbf{k},\alpha}] = 0$, as well as with the terms $\hat{a}^{\dagger}_{\mathbf{k},\alpha}\hat{a}^{\dagger}_{\mathbf{k},\alpha}$. With this in mind we conclude that

$$\hat{\mathbf{P}} = \sum_{\mathbf{k},\alpha}\frac{\hbar\boldsymbol{k}}{2}\left(\hat{a}_{\mathbf{k},\alpha}\hat{a}^{\dagger}_{\mathbf{k},\alpha}+\hat{a}^{\dagger}_{\mathbf{k},\alpha}\hat{a}_{\mathbf{k},\alpha}\right) = \sum_{\mathbf{k},\alpha}\hbar\boldsymbol{k}\left(\hat{a}^{\dagger}_{\mathbf{k},\alpha}\hat{a}_{\mathbf{k},\alpha}+\frac{1}{2}\right) = \sum_{\mathbf{k},\alpha}\hbar\boldsymbol{k}\hat{a}^{\dagger}\hat{a} \tag{A.7}$$

where the $\frac{1}{2}\hbar\boldsymbol{k}$ term was cancelled over the summation on negative values of $\boldsymbol{k}$.

Intrinsic angular momentum operator (helicity)

We begin by writing the expressions (A.1) and (A.2) in terms of the circular polarization unit vectors

$$\boldsymbol{\epsilon}^{(\mu)} = -\frac{\operatorname{sign}\mu}{\sqrt{2}}\left(\boldsymbol{\epsilon}_{\boldsymbol{k},1}+\mu i\boldsymbol{\epsilon}_{\boldsymbol{k},2}\right)\quad\text{and}\quad\left\{\begin{array}{l}\boldsymbol{\epsilon}^{(\mu)}\times\boldsymbol{\epsilon}^{(\mu')} = -\frac{i}{2}\delta_{\mu,-\mu'}\left(\mu'+\mu\right)\breve{\boldsymbol{k}} = -\boldsymbol{\epsilon}^{(\mu)*}\times\boldsymbol{\epsilon}^{(\mu')*}\\ \boldsymbol{\epsilon}^{(\mu)}\times\boldsymbol{\epsilon}^{(\mu')*} = \frac{i}{2}\delta_{\mu,-\mu'}\left(\mu'-\mu\right)\breve{\boldsymbol{k}}\end{array}\right.$$

with $\mu=\pm1$, where we update the expression $\mathbf{u} = c\sqrt{\frac{\hbar}{2\mathcal{V}}}\boldsymbol{\epsilon}^{(\mu)}_{\boldsymbol{k}}e^{i\mathbf{k}\cdot\mathbf{r}}$ and the normalization conditions (2.12) and (2.13) to

$$\int d^3r\left\{\begin{array}{c}\mathbf{u}\times\mathbf{u}^{*\prime}\\ \mathbf{u}\times\mathbf{u}'\\ \mathbf{u}^*\times\mathbf{u}^{*\prime}\end{array}\right\} = \left\{\begin{array}{l}i\frac{c^2\hbar}{4}\left(\mu'-\mu\right)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mu,-\mu'}\breve{\boldsymbol{k}}\\ -i\frac{c^2\hbar}{4}\left(\mu'+\mu\right)\delta_{\mathbf{k},-\mathbf{k}'}\delta_{\mu,-\mu'}\breve{\boldsymbol{k}}\\ i\frac{c^2\hbar}{4}\left(\mu'+\mu\right)\delta_{\mathbf{k},-\mathbf{k}'}\delta_{\mu,-\mu'}\breve{\boldsymbol{k}}\end{array}\right..$$

With all the ingredients necessary, we can finally compute the expression for the helicity

$$\hat{\mathbf{S}}_{\boldsymbol{k}} = \frac{1}{c}\int d^3r \left(\hat{\mathbf{E}} \times \hat{\mathbf{A}}\right) = \frac{1}{c}\sum_{\mathbf{k},\mathbf{k}',\mu,\mu'} \frac{i}{c}\int d^3r \left(\hat{a}\mathbf{u} - \hat{a}^\dagger\mathbf{u}^*\right) \times \left(\hat{a}'\mathbf{u}' + \hat{a}^{\dagger\prime}\mathbf{u}^{*\prime}\right)$$

$$= \frac{i}{c^2}\sum_{\mathbf{k},\mathbf{k}',\alpha,\alpha'} \left\{\hat{a}\hat{a}'\int d^3r(\mathbf{u}\times\mathbf{u}^{'}) - \hat{a}^\dagger\hat{a}^{\dagger\prime}\int d^3r\,(\mathbf{u}^*\times\mathbf{u}^{*\prime}) + \left(\hat{a}^\dagger\hat{a}' + \hat{a}\hat{a}^{\dagger\prime}\right)\int d^3r\,(\mathbf{u}\times\mathbf{u}^{*\prime})\right)$$

$$= \frac{i}{c^2}\sum_{\mathbf{k},\alpha}\left[-\hat{a}\hat{a}^\dagger\,(0) + \hat{a}^\dagger\hat{a}\,(0) + \left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right)\left(-\frac{ic^2\hbar}{2}\mu\breve{\boldsymbol{k}}\right)\right]$$

$$= \sum_{\mathbf{k},\alpha}\frac{\hbar\mu\breve{\boldsymbol{k}}}{2}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right) = \sum_{\mathbf{k},\alpha}\hbar\mu\breve{\boldsymbol{k}}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \sum_{\mathbf{k},\alpha}\hbar\mu\breve{\boldsymbol{k}}\,\hat{a}^\dagger\hat{a}. \tag{A.8}$$

Appendix B

Vacuum Stress Tensor Calculation

From (4.18), the mean value of the vacuum state from the stress tensor can be computed as

$$\langle T_{11}\rangle = -h(ct+x) - h(ct-x) - \tfrac{\hbar c}{2\pi\epsilon^2}; \quad h(\zeta) = -\frac{c\hbar\pi}{4}\left[\frac{R'(\zeta+\frac{\epsilon}{2})R'(\zeta-\frac{\epsilon}{2})}{\left(1-e^{i\pi\Delta R(\zeta)}\right)^2}e^{i\pi\Delta R(\zeta)} + \frac{1}{\pi^2\epsilon^2}\right]. \tag{B.1}$$

Using the expansion $R(\zeta \pm \frac{\epsilon}{2}) = R \pm \frac{\epsilon}{2}R' + \frac{\epsilon^2}{8}R'' \pm \frac{\epsilon^3}{48}R''' + \mathcal{O}(\epsilon^4)$ it can easily be shown that

$$\Delta R(\zeta) = \epsilon R'\left[1 + \tfrac{\epsilon^2}{24}\tfrac{R'''}{R'} + \mathcal{O}(\epsilon^3)\right] \Longrightarrow \begin{cases} \Delta R(\zeta)^2 = \epsilon^2 R'^2\left[1 + \frac{\epsilon^2}{12}\frac{R'''}{R'} + \mathcal{O}(\epsilon^3)\right] \\ \Delta R(\zeta)^{-2} = \frac{1}{\epsilon^2 R'^2}\left[1 - \frac{\epsilon^2}{12}\frac{R'''}{R'} + \mathcal{O}(\epsilon^3)\right]. \end{cases} \tag{B.2}$$

So for,

$$\frac{e^{i\pi\Delta R(\zeta)}}{\left(1-e^{i\pi\Delta R(\zeta)}\right)^2} = \left[e^{-i\frac{\pi}{2}\Delta R} - e^{i\frac{\pi}{2}\Delta R}\right]^{-2} = -\frac{1}{4}\sin^{-2}\left(\frac{\pi\Delta R(\zeta)}{2}\right), \tag{B.3}$$

by using $\sin(x) = x(1 - x^2/6 + \mathcal{O}(x^3))$ we have that $\sin^{-2}(x) = \frac{1}{x^2}\left(1 + x^2/3 + \mathcal{O}(x^4)\right)$ so, for $x = \frac{\pi\Delta R(\zeta)}{2}$ we can rewrite (B.3) as

$$\frac{e^{i\pi\Delta R(\zeta)}}{\left(1-e^{i\pi\Delta R(\zeta)}\right)^2} = -\frac{1}{\pi^2\Delta R(\zeta)^2}\left[1 + \frac{\pi^2\Delta R(\zeta)^2}{12}\right] = -\frac{\Delta R(\zeta)^{-2}}{\pi^2}\left[1 + \frac{\pi^2\epsilon^2 R'^2}{12} + \mathcal{O}(\epsilon^4)\right]. \tag{B.4}$$

This means that $h(\zeta)$ in (B.1) can be written with help of (B.4) (ignoring $\mathcal{O}(\epsilon^4)$) as

$$h(\zeta) = \frac{\hbar c}{4\pi\epsilon^2}\Delta R^{-2}(\zeta)\left[\epsilon^2 R'(\zeta+\tfrac{\epsilon}{2})R'(\zeta-\tfrac{\epsilon}{2})\left(1 + \tfrac{\pi^2\epsilon^2 R'^2}{12}\right) - \Delta R(\zeta)^2\right] \tag{B.5}$$

Analogously,

$$\begin{aligned} R'(\zeta+\tfrac{\epsilon}{2})R'(\zeta-\tfrac{\epsilon}{2}) &= \left[R' + \tfrac{\epsilon}{2}R'' + \tfrac{\epsilon^2}{8}R''' + \mathcal{O}(\epsilon^3)\right]\left[R' - \tfrac{\epsilon}{2}R'' + \tfrac{\epsilon^2}{8}R''' - \mathcal{O}(\epsilon^3)\right] \\ &= R'^2\left[1 + \tfrac{\epsilon^2}{4}\tfrac{R'''}{R'} - \tfrac{\epsilon^2}{4}\left(\tfrac{R''}{R'}\right)^2 + \mathcal{O}(\epsilon^3)\right]. \end{aligned} \tag{B.6}$$

so ignoring $\mathcal{O}(\epsilon^3)$ terms we can calculate (B.5) using the expressions (B.2) and (B.6) in the form

$$\begin{aligned} h(\zeta) &= \frac{\hbar c}{4\pi\epsilon^4 R'^2}\left(1 - \tfrac{\epsilon^2}{12}\tfrac{R'''}{R'}\right)\epsilon^2 R'^2\left\{\left[1 + \tfrac{\epsilon^2}{4}\tfrac{R'''}{R'} - \tfrac{\epsilon^2}{4}\left(\tfrac{R''}{R'}\right)^2\right]\left(1 + \tfrac{\pi^2}{12}\epsilon^2 R'^2\right) - \left(1 + \tfrac{\epsilon^2}{12}\tfrac{R'''}{R'}\right)\right\} \\ &= \frac{\hbar c}{4\pi\epsilon^2}\left(1 - \tfrac{\epsilon^2}{12}\tfrac{R'''}{R'}\right)\left\{\tfrac{1}{6}\epsilon^2\left[\tfrac{R'''}{R'} - \tfrac{3}{2}\left(\tfrac{R''}{R'}\right)^2\right] + \tfrac{\pi^2}{12}\epsilon^2 R'^2 + \mathcal{O}(\epsilon^4)\right\} \\ &= \frac{\hbar c}{24\pi}\left[\tfrac{R'''}{R'} - \tfrac{3}{2}\left(\tfrac{R''}{R'}\right)^2 + \tfrac{1}{2}\pi^2 R'^2\right] + \mathcal{O}(\epsilon^2) \end{aligned} \tag{B.7}$$

proving the expression (4.21) in the limit in which $\epsilon \to 0$.

Appendix C

Bogoliubov Coefficients expression

Considering $\psi_n^{(\text{out})} = \frac{i}{2\sqrt{c\pi n}}\left[e^{-i\pi nR(u)} - e^{-i\pi nR(v)}\right]$ and $\psi_m^{(\text{in})} = \frac{1}{\sqrt{c\pi m}}\sin\left(\frac{m\pi}{L_0}x\right)e^{-im\pi\frac{ct}{L_0}}$ and using $p = n\pi$ and $\omega = m\pi/L_0$

$$\psi_n^{(\text{out})}\partial_t\psi_m^{(\text{in})*} = \frac{i^2}{2\pi\sqrt{mn}}\left[\omega e^{-ipR(u)} - \omega e^{-ipR(v)}\right]\sin(\omega x)\,e^{i\omega ct} \tag{C.1}$$

$$\partial_t\psi_n^{(\text{out})}\psi_m^{(\text{in})*} = -\frac{i^2}{2c\pi\sqrt{mn}}\left[p\,\partial_t R(u)e^{-ipR(u)} - p\,\partial_t R(v)e^{-ipR(v)}\right]\sin(\omega x)\,e^{i\omega ct} \tag{C.2}$$

Using the fact that $\partial_t R(u) = c\partial_x R(u)$ and $\partial_t R(v) = -c\partial_x R(v)$, we have that

$$\begin{aligned}&\psi_n^{(\text{out})}\partial_t\psi_m^{(\text{in})*} - \partial_t\psi_n^{(\text{out})}\psi_m^{(\text{in})*}\\ &\quad = -\frac{1}{2\pi\sqrt{mn}}\left\{\left[p\,\partial_x R(u) + \omega\right]e^{-ipR(u)} + \left[p\,\partial_x R(v) - \omega\right]e^{-ipR(v)}\right\}\sin(\omega x)\,e^{i\omega ct}.\end{aligned} \tag{C.3}$$

Using expression (C.3) we can finally compute the Boguliubov coefficient as

$$\begin{aligned}&\alpha_{mn} = (\psi_n^{(\text{out})},\psi_m^{(\text{in})}) = -i\int_0^{L_0} dx\left[\psi_n^{(\text{out})}\partial_t\psi_m^{(\text{in})*} - \partial_t\psi_n^{(\text{out})}\psi_m^{(\text{in})*}\right]\\ &= \frac{i}{2\pi\sqrt{mn}}\left\{\int_0^{L_0} dx\left[p\,\partial_x R(u) + \omega\right]\sin(\omega x)\,e^{-ipR(u)} + \int_0^{L_0} dx\left[p\,\partial_x R(v) - \omega\right]\sin(\omega x)\,e^{-ipR(v)}\right\}e^{i\omega ct}.\end{aligned} \tag{C.4}$$

Integrating by parts the first term in each integral with the identity $i\int \mathrm{d}x\left[pf'(x) \pm \omega\right]\mathrm{e}^{-ipf(x)}\sin\omega x$ $= \omega\int dx e^{-ipf(x)\pm i\omega x}$ [6] we are now able to obtain

$$\begin{aligned}&(\psi_n^{(\text{out})},\psi_m^{(\text{in})}) = \frac{1}{2L_0}\sqrt{\frac{m}{n}}\left\{\int_0^{L_0} dx e^{-ipR(u)-i\omega(ct+x)} + \int_0^{L_0} dx e^{-ipR(v)-i\omega(ct-x)}\right\}\\ &= \frac{1}{2L_0}\sqrt{\frac{m}{n}}\left\{\int_0^{L_0} dx e^{-\frac{i\pi}{L_0}[nL_0R(ct+x)-m(ct+x)]} + \int_0^{L_0} dx e^{-\frac{i\pi}{L_0}[nL_0R(ct-x)-m(ct-x)]}\right\}\end{aligned} \tag{C.5}$$

choosing a Cauchy surface of integration (t = cte), we can make a substitution $ct \pm x \to x'$ with differentials $dx = \pm dx'$ and limits $ct < x' < ct \pm L_0$

$$\begin{aligned}\alpha_{mn} &= \frac{1}{2L_0}\sqrt{\frac{m}{n}}\left\{\int_{ct}^{ct+L_0} dx' e^{-\frac{i\pi}{L_0}[nL_0R(x')-mx']} - \int_{ct}^{ct-L_0} dx' e^{-\frac{i\pi}{L_0}[nL_0R(x')-mx']}\right\}\\ &= \frac{1}{2L_0}\sqrt{\frac{m}{n}}\int_{ct-L_0}^{ct+L_0} dx' e^{-\frac{i\pi}{L_0}[nL_0R(x')-mx']}\end{aligned} \tag{C.6}$$

the same procedure can be done for β, differing just by a sign in the exponential.

Appendix D

Bogoliubov coefficients α_{mn} and β_{mn} for a vibrating cavity

For $q = 2$, we have that $T = 2L_0/2 = L_0$ and, therefore, the Bogoliubov coefficients (4.24) becomes

$$\left.\begin{array}{c}\beta_{nm}\\ \alpha_{nm}\end{array}\right\} = \frac{1}{2L_0}\sqrt{\frac{m}{n}}\sum_{k=0}^{1} e^{-i\pi(n\pm m)k}\int_0^{L_0} dx\, e^{-\frac{i\pi}{L_0}[(n\pm m)x+nL_0\Theta(x)]}. \tag{D.1}$$

with the approximation (4.32) for $\Theta(x)$ in a piecewise linear function

$$L_0\Theta(x') \approx \left\{\begin{array}{ll} -(1-2\delta)x', & 0 < x' < (1/2-\delta)L_0 \\ (1/2-2\delta)(x'-L_0/2)\delta^{-1}, & L_0/2-\delta < x' < (1/2+\delta)L_0 \\ -(1-2\delta)(x'-L_0), & (1/2+\delta)L_0 < x' < L_0, \end{array}\right. \tag{D.2}$$

We begin by calculating the expression for the sum

$$\sum_{k=0}^{1} e^{-i\pi(n\pm m)k} = 1 + e^{-i\pi(n\pm m)} = 2\cos\frac{\pi}{2}(n\pm m)e^{-i\frac{\pi}{2}(n\pm m)}, \tag{D.3}$$

and continue by attacking the integral of (D.1)

$$I = \int_0^{L_0} dx\, e^{-i\frac{\pi}{L_0}[(n\pm m)x+nL_0\Theta(x)]} = \int_0^{(1/2-\delta)L_0} e^{-i\pi a x}dx + \int_{(1/2+\delta)L_0}^{L_0} e^{-i\pi(ax+b)} \tag{D.4}$$

with $a = (2\delta n \pm m)/L_0$ and $b = n(1-2\delta)$ where we neglected the contribution of the middle term in the interval $(1/2-\delta)L_0 < x < L_0/2+\delta$ which is proportional to δ, we obtain (provided $m\delta \ll 1$)

$$\begin{aligned} I &= \frac{i}{\pi a}\left\{\left[e^{-i\pi a(1/2-\delta)L_0} - 1\right] + \left[e^{-i\pi(aL_0+b)} - e^{-i\pi(a(1/2+\delta)L_0+b)}\right]\right\} \\ &\sim \frac{i}{\pi a}\left\{e^{-i\frac{\pi a}{2}L_0} - 1 + e^{-i\pi(n\pm m)} - e^{i\frac{\pi a}{2}L_0}e^{-i\pi(n\pm m)}\right\} \end{aligned}$$

where we used $aL_0 + b = (n\pm m)$ and $a\delta \ll 1$. Rearranging the terms,

$$\begin{aligned} I &= \frac{i}{\pi a}\left\{e^{-i\frac{\pi a}{2}L_0}\left(1 - e^{i\frac{\pi a}{2}L_0}\right) + \left(1 - e^{i\frac{\pi a}{2}L_0}\right)e^{-i\pi(n\pm m)}\right\} \\ &= \frac{i}{\pi a}e^{-i\frac{\pi}{2}(n\pm m)}e^{-i\frac{\pi a}{4}}\left(1 - e^{i\frac{\pi a}{2}L_0}\right)\left\{e^{i\frac{\pi}{2}(n\pm m)}e^{-i\frac{\pi a}{4}L_0} + e^{-i\frac{\pi}{2}(n\pm m)}e^{i\frac{\pi a}{4}L_0}\right\} \\ &= \frac{i}{\pi a}e^{-i\frac{\pi}{2}(n\pm m)}\left(e^{-i\frac{\pi a}{4}L_0} - e^{i\frac{\pi a}{4}L_0}\right)\left\{e^{i\frac{\pi}{2}[(n\pm m)-\frac{a}{2}L_0]} + e^{-i\frac{\pi}{2}[(n\pm m)-\frac{a}{2}L_0]}\right\}, \end{aligned}$$

and using $2i\sin ax = e^{iax} - e^{-iax}$ and $2\cos ax = e^{iax} + e^{-iax}$, we can rewrite the the last expression as

$$I = \tfrac{-4i^2}{\pi a}\sin\left(\tfrac{\pi a L_0}{4}\right)\cos\tfrac{\pi}{2}\left[(n\pm m) - \tfrac{aL_0}{2}\right]e^{-i\frac{\pi}{2}(n\pm m)} \tag{D.5}$$

With the integral expression (D.4) calculated in (D.5) together withe sum expression (D.3), we can finally compute the Bogoluibov coefficients (D.1) as

$$\begin{aligned}
\left.\begin{matrix}\beta_{nm}\\ \alpha_{nm}\end{matrix}\right\} &= \tfrac{4}{\pi a L_0}\sqrt{\tfrac{m}{n}}\sin\left(\tfrac{\pi a L_0}{4}\right)\cos\tfrac{\pi}{2}\left[(n\pm m)-\tfrac{aL_0}{2}\right]\cos\tfrac{\pi}{2}\left(n\pm m\right)e^{-i\pi(n\pm m)}\\
&= \tfrac{4}{\pi a L_0}\sqrt{\tfrac{m}{n}}\sin\left(\tfrac{\pi a L_0}{4}\right)\cos\left(\tfrac{\pi a L_0}{4}\right)\cos^2\tfrac{\pi}{2}(n\pm m)e^{-i\pi(n\pm m)}\\
&= \tfrac{2}{\pi a L_0}\sqrt{\tfrac{m}{n}}\sin\left(\tfrac{\pi a L_0}{2}\right)\cos^2\tfrac{\pi}{2}(n\pm m)e^{-i\pi(n\pm m)},
\end{aligned} \tag{D.6}$$

so

$$\left.\begin{matrix}\beta_{nm}\\ \alpha_{nm}\end{matrix}\right\} = \frac{2}{\pi}\sqrt{\frac{m}{n}}\frac{\sin\frac{\pi}{2}(2\delta n\pm m)}{(2\delta n\pm m)}\cos^2\tfrac{\pi}{2}(n\pm m)e^{-i\pi(n\pm m)}. \tag{D.7}$$

It's very important to point out that expression (D.7) is different from the one obtained by V. Dodonov in [36] and expressed in Eq. (4.33) by a factor of $\cos\frac{\pi}{2}(n\pm m)e^{-\frac{3}{2}\pi(n\pm m)}$. I do not know exactly what went wrong in the derivation process but this terms seems to contribute in the physics as a phase, since as it will show next, the square moduli of the coefficients are somehow in exactly agreement with Dodonov expression. By squaring the last expression, we have

$$\left.\begin{matrix}|\beta_{nm}|^2\\ |\alpha_{nm}|^2\end{matrix}\right\} = \tfrac{4}{\pi^2}\tfrac{m}{n}\frac{\sin^2\frac{\pi}{2}(2\delta n\pm m)}{(2\delta n\pm m)^2}\cos^4\tfrac{\pi}{2}(n\pm m). \tag{D.8}$$

With the help of the following identities

$$\begin{aligned}
&\sin^2\tfrac{\pi}{2}(2\delta n\pm m) = \tfrac{1}{2}\left[1-\cos(2\pi\delta n\pm m)\right] = \tfrac{1}{2}\left[1-(-1)^m\cos(2\pi\delta n)\right]\\
&\cos^4\tfrac{\pi}{2}(n\pm m) = \left\{\tfrac{1}{2}\left[1+\cos\pi(n\pm m)\right]\right\}^2 = \tfrac{1}{4}\left[1+(-1)^{m+n}\right]^2 = \tfrac{1}{2}\left[1+(-1)^{m+n}\right],
\end{aligned}$$

we can finally find the most final expression for the moduli square of the Bogoliubov coefficients

$$\left.\begin{matrix}|\beta_{nm}|^2\\ |\alpha_{nm}|^2\end{matrix}\right\} = \frac{m}{n\pi^2}\frac{[1-(-1)^m\cos(2\pi\delta n)]}{(2\delta n\pm m)^2}[1+(-1)^{m+n}]. \tag{D.9}$$