Diagonal entropy

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1 Effective Hamiltonian

Defining the quantity $\mathcal{A}_k(t) = a_k e^{-i\omega_k(t)}$ (with $\omega_k = k\pi/L$), the effective Hamiltonian in the interaction picture can be written as

$$\hat{H}_I = i\sum_k \left\{ \xi_k(t)\mathcal{A}_k^{\dagger 2}(t) + \sum_{j(\neq k)} \mu_{kj}(t)\mathcal{A}_k^\dagger(t)\left(\mathcal{A}_j^\dagger(t) + \mathcal{A}_j(t)\right) - \text{h.c.} \right\}, \tag{1}$$

Let the dynamical equation for the density matrix $\rho(t)$

$$\frac{d}{dt}\rho(t) = -i\left[\hat{H}_I(t), \rho(t)\right] \tag{2}$$

or in second order

$$\rho(t) = \rho(0) - i\int_0^t d\tau \left[\hat{H}_I(\tau), \rho(0)\right] - \int_0^t dt' \int_0^{t'} d\tau \left[\hat{H}_I(t'), \left[\hat{H}_I(\tau), \rho(0)\right]\right] + \mathcal{O}(\hat{H}_I^3(t)). \tag{3}$$

Considering the initial state as the vacuum state $\rho(0) = |0\rangle\langle 0|$, from the relations $\mathcal{A}_k|0\rangle = \langle 0|\mathcal{A}^\dagger = 0$ we obtain

$$\begin{aligned}\left[\hat{H}_I(\tau), \rho(0)\right] &= \hat{H}_I(\tau)|0\rangle\langle 0| - |0\rangle\langle 0|\hat{H}_I(\tau) \\ &= i\sum_k \left\{ \left(\mathcal{A}_k^{\dagger 2}\rho(0) + \rho(0)\mathcal{A}_k^2\right) + \sum_{j(\neq k)} \mu_{kj}\left(\mathcal{A}_k^\dagger\mathcal{A}_j^\dagger\rho(0) + \rho(0)\mathcal{A}_j\mathcal{A}_k\right) \right\},\end{aligned}$$

for the first order term, as well as the expressions for the second order terms

$$\begin{aligned}
&\left[H(t'), \left[\hat{H}_I(\tau), \rho(0)\right]\right] = H(t')\left[\hat{H}_I(\tau), \rho(0)\right] - \left[\hat{H}_I(\tau), \rho(0)\right]H(t') \\
&= i\sum_{k'} \left\{ \xi_{k'}\left(\mathcal{A}_{k'}^{\dagger 2} - \mathcal{A}_{k'}^2\right) + \sum_{j'(\neq k')} \mu_{k'j'}\left(\mathcal{A}_{k'}^\dagger\mathcal{A}_{j'}^\dagger + \mathcal{A}_{k'}^\dagger\mathcal{A}_{j'} - \mathcal{A}_{k'}\mathcal{A}_{j'} - \mathcal{A}_{j'}^\dagger\mathcal{A}_{k'}\right) \right\}\left[\hat{H}_I(\tau), \rho(0)\right] \\
&- i\sum_{k'} \left[\hat{H}_I(\tau), \rho(0)\right]\left\{ \xi_{k'}\left(\mathcal{A}_{k'}^{\dagger 2} - \mathcal{A}_{k'}^2\right) + \sum_{j'(\neq k')} \mu_{k'j'}\left(\mathcal{A}_{k'}^\dagger\mathcal{A}_{j'}^\dagger + \mathcal{A}_{k'}^\dagger\mathcal{A}_{j'} - \mathcal{A}_{k'}\mathcal{A}_{j'} - \mathcal{A}_{j'}^\dagger\mathcal{A}_{k'}\right) \right\} \\
&= i\sum_{k'} \xi_{k'}\mathcal{A}_{k'}^{\dagger 2}\left[\hat{H}_I(\tau), \rho(0)\right] - i\sum_{k'} \xi_{k'}\left[\hat{H}_I(\tau), \rho(0)\right]\mathcal{A}_{k'}^{\dagger 2} - i\sum_{k'} \xi_{k'}\mathcal{A}_{k'}^2\left[\hat{H}_I(\tau), \rho(0)\right] + i\sum_{k'} \xi_{k'}\left[\hat{H}_I(\tau), \rho(0)\right]\mathcal{A}_{k'}^2 \\
&+ i\sum_{k'}\sum_{j'(\neq k')} \mu_{k'j'}\mathcal{A}_{k'}^\dagger\mathcal{A}_{j'}^\dagger\left[\hat{H}_I(\tau), \rho(0)\right] - i\sum_{k'}\sum_{j'(\neq k')} \mu_{k'j'}\left[\hat{H}_I(\tau), \rho(0)\right]\mathcal{A}_{k'}^\dagger\mathcal{A}_{j'}^\dagger + i\sum_{k'}\sum_{j'(\neq k')} \mu_{k'j'}\mathcal{A}_{k'}^\dagger\mathcal{A}_{j'}\left[\hat{H}_I(\tau), \rho(0)\right] \\
&- i\sum_{k'}\sum_{j'(\neq k')} \mu_{k'j'}\left[\hat{H}_I(\tau), \rho(0)\right]\mathcal{A}_{k'}^\dagger\mathcal{A}_{j'} - i\sum_{k'}\sum_{j'(\neq k')} \mu_{k'j'}\mathcal{A}_{k'}\mathcal{A}_{j'}\left[\hat{H}_I(\tau), \rho(0)\right] + i\sum_{k'}\sum_{j'(\neq k')} \mu_{k'j'}\left[\hat{H}_I(\tau), \rho(0)\right]\mathcal{A}_{k'}\mathcal{A}_{j'} \\
&- i\sum_{k'}\sum_{j'(\neq k')} \mu_{k'j'}\mathcal{A}_{j'}^\dagger\mathcal{A}_{k'}\left[\hat{H}_I(\tau), \rho(0)\right] + i\sum_{k'}\sum_{j'(\neq k')} \mu_{k'j'}\left[\hat{H}_I(\tau), \rho(0)\right]\mathcal{A}_{j'}^\dagger\mathcal{A}_{k'}
\end{aligned}$$

$$
\begin{aligned}
&\Big[H(t'),\Big[\hat{H}_I(\tau),\rho(0)\Big]\Big] = \\
&-\sum_{k'}\sum_{k}\left\{\xi_k\xi_{k'}\left(\mathcal{A}_{k'}^{\dagger 2}\mathcal{A}_k^{\dagger 2}\rho(0)+\mathcal{A}_{k'}^{\dagger 2}\rho(0)\mathcal{A}_k^2\right)+\sum_{j(\neq k)}\xi_{k'}\mu_{kj}\left(\mathcal{A}_{k'}^{\dagger 2}\mathcal{A}_k^{\dagger}\mathcal{A}_j^{\dagger}\rho(0)+\mathcal{A}_{k'}^{\dagger 2}\rho(0)\mathcal{A}_j\mathcal{A}_k\right)\right\} \\
&+\sum_{k'}\sum_{k}\left\{\xi_k\xi_{k'}\left(\mathcal{A}_k^{\dagger 2}\rho(0)\mathcal{A}_{k'}^{\dagger 2}+\rho(0)\mathcal{A}_k^2\mathcal{A}_{k'}^{\dagger 2}\right)+\sum_{j(\neq k)}\mu_{kj}\xi_{k'}\left(\mathcal{A}_k^{\dagger}\mathcal{A}_j^{\dagger}\rho(0)\mathcal{A}_{k'}^{\dagger 2}+\rho(0)\mathcal{A}_j\mathcal{A}_k\mathcal{A}_{k'}^{\dagger 2}\right)\right\} \\
&+\sum_{k'}\sum_{k}\left\{\xi_k\xi_{k'}\left(\mathcal{A}_{k'}^{2}\mathcal{A}_k^{\dagger 2}\rho(0)+\mathcal{A}_{k'}^{2}\rho(0)\mathcal{A}_k^2\right)+\sum_{j(\neq k)}\xi_{k'}\mu_{kj}\left(\mathcal{A}_{k'}^{2}\mathcal{A}_k^{\dagger}\mathcal{A}_j^{\dagger}\rho(0)+\mathcal{A}_{k'}^{2}\rho(0)\mathcal{A}_j\mathcal{A}_k\right)\right\} \\
&-\sum_{k'}\sum_{k}\left\{\xi_k\xi_{k'}\left(\mathcal{A}_k^{\dagger 2}\rho(0)\mathcal{A}_{k'}^{2}+\rho(0)\mathcal{A}_k^2\mathcal{A}_{k'}^{2}\right)+\sum_{j(\neq k)}\mu_{kj}\xi_{k'}\left(\mathcal{A}_k^{\dagger}\mathcal{A}_j^{\dagger}\rho(0)\mathcal{A}_{k'}^{2}+\rho(0)\mathcal{A}_j\mathcal{A}_k\mathcal{A}_{k'}^{2}\right)\right\} \\
&-\sum_{k'}\sum_{j'(\neq k')}\sum_{k}\left\{\xi_k\mu_{k'j'}\left(\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}^{\dagger}\mathcal{A}_k^{\dagger 2}\rho(0)+\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}^{\dagger}\rho(0)\mathcal{A}_k^2\right)+\sum_{j(\neq k)}\mu_{k'j'}\mu_{kj}\left(\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}^{\dagger}\mathcal{A}_k^{\dagger}\mathcal{A}_j^{\dagger}\rho(0)+\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}^{\dagger}\rho(0)\mathcal{A}_j\mathcal{A}_k\right)\right\} \\
&+\sum_{k'}\sum_{j'(\neq k')}\sum_{k}\left\{\xi_k\mu_{k'j'}\left(\mathcal{A}_k^{\dagger 2}\rho(0)\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}^{\dagger}+\rho(0)\mathcal{A}_k^2\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}^{\dagger}\right)+\sum_{j(\neq k)}\mu_{kj}\mu_{k'j'}\left(\mathcal{A}_k^{\dagger}\mathcal{A}_j^{\dagger}\rho(0)\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}^{\dagger}+\rho(0)\mathcal{A}_j\mathcal{A}_k\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}^{\dagger}\right)\right\} \\
&-\sum_{k'}\sum_{j'(\neq k')}\sum_{k}\left\{\xi_k\mu_{k'j'}\left(\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}\mathcal{A}_k^{\dagger 2}\rho(0)+\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}\rho(0)\mathcal{A}_k^2\right)+\sum_{j(\neq k)}\mu_{kj}\mu_{k'j'}\left(\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}\mathcal{A}_k^{\dagger}\mathcal{A}_j^{\dagger}\rho(0)+\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}\rho(0)\mathcal{A}_j\mathcal{A}_k\right)\right\} \\
&+\sum_{k'}\sum_{j'(\neq k')}\sum_{k}\left\{\xi_k\mu_{k'j'}\left(\mathcal{A}_k^{\dagger 2}\rho(0)\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}+\rho(0)\mathcal{A}_k^2\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}\right)+\sum_{j(\neq k)}\mu_{kj}\mu_{k'j'}\left(\mathcal{A}_k^{\dagger}\mathcal{A}_j^{\dagger}\rho(0)\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}+\rho(0)\mathcal{A}_j\mathcal{A}_k\mathcal{A}_{k'}^{\dagger}\mathcal{A}_{j'}\right)\right\} \\
&+\sum_{k'}\sum_{j'(\neq k')}\sum_{k}\left\{\xi_k\mu_{k'j'}\left(\mathcal{A}_{k'}\mathcal{A}_{j'}\mathcal{A}_k^{\dagger 2}\rho(0)+\mathcal{A}_{k'}\mathcal{A}_{j'}\rho(0)\mathcal{A}_k^2\right)+\sum_{j(\neq k)}\mu_{kj}\mu_{k'j'}\left(\mathcal{A}_{k'}\mathcal{A}_{j'}\mathcal{A}_k^{\dagger}\mathcal{A}_j^{\dagger}\rho(0)+\mathcal{A}_{k'}\mathcal{A}_{j'}\rho(0)\mathcal{A}_j\mathcal{A}_k\right)\right\} \\
&-\sum_{k'}\sum_{j'(\neq k')}\sum_{k}\left\{\xi_k\mu_{k'j'}\left(\mathcal{A}_k^{\dagger 2}\rho(0)\mathcal{A}_{k'}\mathcal{A}_{j'}+\rho(0)\mathcal{A}_k^2\mathcal{A}_{k'}\mathcal{A}_{j'}\right)+\sum_{j(\neq k)}\mu_{kj}\mu_{k'j'}\left(\mathcal{A}_k^{\dagger}\mathcal{A}_j^{\dagger}\rho(0)\mathcal{A}_{k'}\mathcal{A}_{j'}+\rho(0)\mathcal{A}_j\mathcal{A}_k\mathcal{A}_{k'}\mathcal{A}_{j'}\right)\right\} \\
&+\sum_{k'}\sum_{j'(\neq k')}\sum_{k}\left\{\xi_k\mu_{k'j'}\left(\mathcal{A}_{j'}^{\dagger}\mathcal{A}_{k'}\mathcal{A}_k^{\dagger 2}\rho(0)+\mathcal{A}_{j'}^{\dagger}\mathcal{A}_{k'}\rho(0)\mathcal{A}_k^2\right)+\sum_{j(\neq k)}\mu_{kj}\mu_{k'j'}\left(\mathcal{A}_{j'}^{\dagger}\mathcal{A}_{k'}\mathcal{A}_k^{\dagger}\mathcal{A}_j^{\dagger}\rho(0)+\mathcal{A}_{j'}^{\dagger}\mathcal{A}_{k'}\rho(0)\mathcal{A}_j\mathcal{A}_k\right)\right\} \\
&-\sum_{k'}\sum_{j'(\neq k')}\sum_{k}\left\{\xi_k\mu_{k'j'}\left(\mathcal{A}_k^{\dagger 2}\rho(0)\mathcal{A}_{j'}^{\dagger}\mathcal{A}_{k'}+\rho(0)\mathcal{A}_k^2\mathcal{A}_{j'}^{\dagger}\mathcal{A}_{k'}\right)+\sum_{j(\neq k)}\mu_{kj}\mu_{k'j'}\left(\mathcal{A}_k^{\dagger}\mathcal{A}_j^{\dagger}\rho(0)\mathcal{A}_{j'}^{\dagger}\mathcal{A}_{k'}+\rho(0)\mathcal{A}_j\mathcal{A}_k\mathcal{A}_{j'}^{\dagger}\mathcal{A}_{k'}\right)\right\}.
\end{aligned}
$$

Analysing the last expression, the only diagonal terms can be summarized as

$$
\begin{aligned}
\rho(t) = \rho(0) - \sum_k \int_0^t dt' \int_0^{t'} d\tau \Bigg\{ & \xi_k(\tau)\xi_k(t')\left[\mathcal{A}_k^2(t')\mathcal{A}_k^{\dagger 2}(\tau)\rho(0) - \mathcal{A}_k^{\dagger 2}(t')\rho(0)\mathcal{A}_k^2(\tau) - \mathcal{A}_k^{\dagger 2}(\tau)\rho(0)\mathcal{A}_k^2(t') + \rho(0)\mathcal{A}_k^2(\tau)\mathcal{A}_k^{\dagger 2}(t')\right] \\
& - \sum_{j(\neq k)} \mu_{kj}(\tau)\mu_{kj}(t')\left[\mathcal{A}_k^{\dagger}(t')\mathcal{A}_j^{\dagger}(t')\rho(0)\mathcal{A}_j(\tau)\mathcal{A}_k(\tau) + \mathcal{A}_k^{\dagger}(\tau)\mathcal{A}_j^{\dagger}(\tau)\rho(0)\mathcal{A}_j(t')\mathcal{A}_k(t')\right] \Bigg\}.
\end{aligned}
$$

where the diagonal entropy then takes the form of

$$
S_d(t) = -\rho_{00}\ln\rho_{00} - \sum_k \left(\rho_{kk}\ln\rho_{kk} + \sum_{j(\neq k)} \rho_{kj}\ln\rho_{kj}\right)
$$

with

$$\rho_{00} = \langle 0|\rho(t)|0\rangle = 1 - 2\sum_k \int_0^t dt' \int_0^{t'} d\tau \xi_k(\tau)\xi_k(t') \left(e^{2i\omega_k\left(\tau - t'\right)} + e^{-2i\omega_k\left(\tau - t'\right)}\right)$$

$$\rho_{kk} = \langle 2_k|\rho(t)|2_k\rangle = 2\int_0^t dt' \int_0^{t'} d\tau \xi_k(\tau)\xi_k(t') \left(e^{2i\omega_k\left(\tau - t'\right)} + e^{-2i\omega_k\left(\tau - t'\right)}\right)$$

$$\rho_{kj} = \langle 1_k, 1_j|\rho(t)|1_k, 1_j\rangle = \int_0^t dt' \int_0^{t'} d\tau \mu_{kj}(\tau)\mu_{kj}(t') \left(e^{i(\omega_k+\omega_j)\left(\tau - t'\right)} + e^{-i(\omega_k+\omega_j)\left(\tau - t'\right)}\right).$$

1.1 Vibrating cavity

For the situation in which the cavity vibrates at frequencies $\Omega_p = p\pi/L$ with small amplitude ϵL and trajectory

$$q(t) = Le^{\epsilon \cos \Omega_p t} \approx L\left(1 + \epsilon \cos \Omega_p t\right),$$

the no-adiabatic functions $\xi_k(t)$ and $\mu_{kj}(t)$ takes the form of

$$\xi_k(t) = \frac{\dot{q}(t)}{4q(t)}, \qquad \mu_{kj}(t) = \frac{1}{2}(-1)^{j+k}\frac{kj}{j^2 - k^2}\left(\frac{k}{j}\right)^{1/2}\frac{\dot{q}(t)}{q(t)}, \tag{4}$$

where

$$\frac{\dot{q}(t)}{q(t)} = \frac{L\left(-\Omega_p \epsilon \sin \Omega_p t\right) e^{\epsilon \cos \Omega_p t}}{Le^{\epsilon \cos \Omega_p t}} = -\Omega_p \epsilon \sin \Omega_p t = \frac{i\Omega_p \epsilon}{2}\left(e^{i\Omega_p t} - e^{-i\Omega_p t}\right). \tag{5}$$

Therefore, the coefficients for the matrix density can be written in terms of

$$\rho_{00} = \langle 0|\rho(t)|0\rangle = 1 - \frac{1}{8}\sum_k \Pi(t; 2k, p)$$

$$\rho_{kk} = \langle 2_k|\rho(t)|2_k\rangle = \frac{1}{8}\Pi(t; 2k, p)$$

$$\rho_{kj} = \langle 1_k, 1_j|\rho(t)|1_k, 1_j\rangle = \frac{k}{4j}\left(\frac{kj}{j^2 - k^2}\right)^2 \Pi(t; k + j, p).$$

where

$$\Pi(t; \kappa, p) = \Omega_p^2 \epsilon^2 \int_0^t dt' \int_0^{t'} d\tau \sin\left(\Omega_p \tau\right) \sin\left(\Omega_p t'\right) \left(e^{i\frac{\kappa\pi}{L}\left(\tau - t'\right)} + e^{-i\frac{\kappa\pi}{L}\left(\tau - t'\right)}\right).$$

Developing the last integrand

$$\begin{aligned}
&\sin\left(\Omega_p \tau\right) \sin\left(\Omega_p t'\right) \left(e^{i\frac{\kappa\pi}{L}\left(\tau - t'\right)} + e^{-i\frac{\kappa\pi}{L}\left(\tau - t'\right)}\right) \\
&= -\frac{1}{4}\left(e^{i\Omega_p \tau} - e^{-i\Omega_p \tau}\right)\left(e^{i\Omega_p t'} - e^{-i\Omega_p t'}\right)\left(e^{i\frac{\kappa\pi}{L}\left(\tau - t'\right)} + e^{-i\frac{\kappa\pi}{L}\left(\tau - t'\right)}\right) \\
&- \frac{1}{4}\left(e^{ip\frac{\pi}{L}t'} - e^{-ip\frac{\pi}{L}t'}\right)\left[\left(e^{i(p+\kappa)\frac{\pi}{L}\tau} - e^{-i(p-\kappa)\frac{\pi}{L}\tau}\right)e^{-i\kappa\frac{\pi}{L}t'} + \left(e^{i(p-\kappa)\frac{\pi}{L}\tau} - e^{-i(p+\kappa)\frac{\pi}{L}\tau}\right)e^{i\kappa\frac{\pi}{L}t'}\right] \\
&- \frac{1}{4}\left[\left(e^{i(p+\kappa)\frac{\pi}{L}\tau} - e^{-i(p-\kappa)\frac{\pi}{L}\tau}\right)\left(e^{i(p-\kappa)\frac{\pi}{L}t'} - e^{-i(p+\kappa)\frac{\pi}{L}t'}\right) + \left(e^{i(p-\kappa)\frac{\pi}{L}\tau} - e^{-i(p+\kappa)\frac{\pi}{L}\tau}\right)\left(e^{i(p+\kappa)\frac{\pi}{L}t'} - e^{-i(p-\kappa)\frac{\pi}{L}t'}\right)\right],
\end{aligned}$$

we can find the coefficients $\Pi(t; \kappa, p)$ for $p \neq \kappa$ as

$$\begin{aligned}
\Pi(t; \kappa, p) = &-\frac{p^2\pi^2\epsilon^2}{4L^2}\int_0^t dt'\left[\left(e^{i(p-\kappa)\frac{\pi}{L}t'} - e^{-i(p+\kappa)\frac{\pi}{L}t'}\right)\int_0^{t'} d\tau\left(e^{i(p+\kappa)\frac{\pi}{L}\tau} - e^{-i(p-\kappa)\frac{\pi}{L}\tau}\right)\right. \\
&\left. + \left(e^{i(p+\kappa)\frac{\pi}{L}t'} - e^{-i(p-\kappa)\frac{\pi}{L}t'}\right)\int_0^{t'} d\tau\left(e^{i(p-\kappa)\frac{\pi}{L}\tau} - e^{-i(p+\kappa)\frac{\pi}{L}\tau}\right)\right] \\
= &\frac{ip^2\pi\epsilon^2}{4L}\int_0^t dt'\left[\left(e^{i(p-\kappa)\frac{\pi}{L}t'} - e^{-i(p+\kappa)\frac{\pi}{L}t'}\right)\left(\frac{e^{i(p+\kappa)\frac{\pi}{L}\tau}}{p+\kappa} + \frac{e^{-i(p-\kappa)\frac{\pi}{L}\tau}}{p-\kappa}\right)\Bigg|_0^{t'}\right. \\
&\left. + \left(e^{i(p+\kappa)\frac{\pi}{L}t'} - e^{-i(p-\kappa)\frac{\pi}{L}t'}\right)\left(\frac{e^{i(p-\kappa)\frac{\pi}{L}\tau}}{p-\kappa} + \frac{e^{-i(p+\kappa)\frac{\pi}{L}\tau}}{p+\kappa}\right)\Bigg|_0^{t'}\right] \\
= &\frac{ip^2\pi\epsilon^2}{4L}\int_0^t dt'\left[\left(e^{i(p-\kappa)\frac{\pi}{L}t'} - e^{-i(p+\kappa)\frac{\pi}{L}t'}\right)\left(\frac{e^{i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa} + \frac{e^{-i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa} - \frac{1}{p+\kappa} - \frac{1}{p-\kappa}\right)\right. \\
&\left. + \left(e^{i(p+\kappa)\frac{\pi}{L}t'} - e^{-i(p-\kappa)\frac{\pi}{L}t'}\right)\left(\frac{e^{i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa} + \frac{e^{-i(p+2k)\frac{\pi}{L}t'}}{p+\kappa} - \frac{1}{p+\kappa} - \frac{1}{p-\kappa}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{ip^2\pi\epsilon^2}{4L}\int_0^t dt'\Bigg[e^{i(p-\kappa)\frac{\pi}{L}t'}\left(\frac{e^{i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa}+\frac{e^{-i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa}-\frac{1}{p+\kappa}-\frac{1}{p-\kappa}\right)\\
&- e^{-i(p+\kappa)\frac{\pi}{L}t'}\left(\frac{e^{i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa}+\frac{e^{-i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa}-\frac{1}{p+\kappa}-\frac{1}{p-\kappa}\right)\\
&+ e^{i(p+\kappa)\frac{\pi}{L}t'}\left(\frac{e^{i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa}+\frac{e^{-i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa}-\frac{1}{p+\kappa}-\frac{1}{p-\kappa}\right)\\
&- e^{-i(p-\kappa)\frac{\pi}{L}t'}\left(\frac{e^{i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa}+\frac{e^{-i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa}-\frac{1}{p+\kappa}-\frac{1}{p-\kappa}\right)\Bigg]\\
&= \frac{ip^2\pi\epsilon^2}{4L}\int_0^t dt'\Bigg[\left(\frac{e^{i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa}e^{i(p-\kappa)\frac{\pi}{L}t'}+\frac{e^{-i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa}e^{i(p-\kappa)\frac{\pi}{L}t'}-\frac{e^{i(p-\kappa)\frac{\pi}{L}t'}}{p+\kappa}-\frac{e^{i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa}\right)\\
&-\left(\frac{e^{i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa}e^{-i(p+\kappa)\frac{\pi}{L}t'}+\frac{e^{-i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa}e^{-i(p+\kappa)\frac{\pi}{L}t'}-\frac{e^{-i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa}-\frac{e^{-i(p+\kappa)\frac{\pi}{L}t'}}{p-\kappa}\right)\\
&+\left(\frac{e^{i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa}e^{i(p+\kappa)\frac{\pi}{L}t'}+\frac{e^{-i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa}e^{i(p+\kappa)\frac{\pi}{L}t'}-\frac{e^{i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa}-\frac{e^{i(p+\kappa)\frac{\pi}{L}t'}}{p-\kappa}\right)\\
&-\left(\frac{e^{i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa}e^{-i(p-\kappa)\frac{\pi}{L}t'}+\frac{e^{-i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa}e^{-i(p-\kappa)\frac{\pi}{L}t'}-\frac{e^{-i(p-\kappa)\frac{\pi}{L}t'}}{p+\kappa}-\frac{e^{-i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa}\right)\Bigg]\\
&= \frac{ip^2\pi\epsilon^2}{4L}\int_0^t dt'\Bigg[\frac{e^{2ip\frac{\pi}{L}t'}}{p+\kappa}+\frac{1}{p-\kappa}-\frac{e^{i(p-\kappa)\frac{\pi}{L}t'}}{p+\kappa}-\frac{e^{i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa}-\frac{1}{p+\kappa}-\frac{e^{-2ip\frac{\pi}{L}t'}}{p-\kappa}+\frac{e^{-i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa}+\frac{e^{-i(p+\kappa)\frac{\pi}{L}t'}}{p-\kappa}\\
&+\frac{e^{2ip\frac{\pi}{L}t'}}{p-\kappa}+\frac{1}{p+\kappa}-\frac{e^{i(p+\kappa)\frac{\pi}{L}t'}}{p+\kappa}-\frac{e^{i(p+\kappa)\frac{\pi}{L}t'}}{p-\kappa}-\frac{1}{p-\kappa}-\frac{e^{-2ip\frac{\pi}{L}t'}}{p+\kappa}+\frac{e^{-i(p-\kappa)\frac{\pi}{L}t'}}{p+\kappa}+\frac{e^{-i(p-\kappa)\frac{\pi}{L}t'}}{p-\kappa}\Bigg]\\
&= \frac{ip^2\pi\epsilon^2}{4L}\int_0^t dt'\Bigg[\left(\frac{1}{p+\kappa}+\frac{1}{p-\kappa}\right)e^{2ip\frac{\pi}{L}t'}-\left(\frac{1}{p+\kappa}+\frac{1}{p-\kappa}\right)e^{-2ip\frac{\pi}{L}t'}-\left(\frac{1}{p+\kappa}+\frac{1}{p-\kappa}\right)e^{i(p-\kappa)\frac{\pi}{L}t'}\\
&+\left(\frac{1}{p+\kappa}+\frac{1}{p-\kappa}\right)e^{-i(p+\kappa)\frac{\pi}{L}t'}-\left(\frac{1}{p+\kappa}+\frac{1}{p-\kappa}\right)e^{i(p+\kappa)\frac{\pi}{L}t'}+\left(\frac{1}{p+\kappa}+\frac{1}{p-\kappa}\right)e^{-i(p-\kappa)\frac{\pi}{L}t'}\Bigg]\\
&= \frac{ip^2\pi\epsilon^2}{4L}\left(\frac{1}{p+\kappa}+\frac{1}{p-\kappa}\right)\int_0^t dt'\Bigg[\left(e^{2ip\frac{\pi}{L}t'}-e^{-2ip\frac{\pi}{L}t'}\right)-\left(e^{i(p-\kappa)\frac{\pi}{L}t'}-e^{-i(p-\kappa)\frac{\pi}{L}t'}\right)-\left(e^{i(p+\kappa)\frac{\pi}{L}t'}-e^{-i(p+\kappa)\frac{\pi}{L}t'}\right)\Bigg]\\
&= \frac{-p^2\pi\epsilon^2}{2L}\frac{2p}{p^2-\kappa^2}\int_0^t dt'\left\{\sin\left(\frac{2p\pi t'}{L}\right)-\sin\left[(p-\kappa)\frac{\pi t'}{L}\right]-\sin\left[(p+\kappa)\frac{\pi t'}{L}\right]\right\}\\
&= \frac{p^2\epsilon^2}{2}\frac{2p}{p^2-\kappa^2}\left\{\frac{1}{2p}\cos\left(\frac{2p\pi t'}{L}\right)-\frac{1}{p-\kappa}\cos\left[(p-\kappa)\frac{\pi t'}{L}\right]-\frac{1}{p+\kappa}\cos\left[(p+\kappa)\frac{\pi t'}{L}\right]\right\}\Bigg|_0^t\\
&= \frac{p^3\epsilon^2}{p^2-\kappa^2}\left\{\frac{1}{2p}\cos\left(\frac{2p\pi t}{L}\right)-\frac{1}{p-\kappa}\cos\left[(p-\kappa)\frac{\pi t}{L}\right]-\frac{1}{p+\kappa}\cos\left[(p+\kappa)\frac{\pi t}{L}\right]+\frac{2p}{p^2-\kappa^2}-\frac{1}{2p}\right\}
\end{aligned}
$$

for the case $p=\kappa$

$$
\begin{aligned}
&\sin(\Omega_\kappa\tau)\sin(\Omega_\kappa t')\left(e^{i\frac{\kappa\pi}{L}(\tau-t')}+e^{-i\frac{\kappa\pi}{L}(\tau-t')}\right)=\\
&=-\frac{p^2\pi^2\epsilon^2}{4L^2}\left[\left(e^{i(p+\kappa)\frac{\pi}{L}\tau}-e^{-i(p-\kappa)\frac{\pi}{L}\tau}\right)\left(e^{i(p-\kappa)\frac{\pi}{L}t'}-e^{-i(p+\kappa)\frac{\pi}{L}t'}\right)+\left(e^{i(p-\kappa)\frac{\pi}{L}\tau}-e^{-i(p+\kappa)\frac{\pi}{L}\tau}\right)\left(e^{i(p+\kappa)\frac{\pi}{L}t'}-e^{-i(p-\kappa)\frac{\pi}{L}t'}\right)\right]\\
&=-\frac{\pi^2\epsilon^2}{4L^2}\kappa^2\left[\left(e^{i\frac{2\kappa\pi}{L}\tau}-1\right)\left(1-e^{-i\frac{2\kappa\pi}{L}t'}\right)+\left(1-e^{-i\frac{2\kappa\pi}{L}\tau}\right)\left(e^{i\frac{2\kappa\pi}{L}t'}-1\right)\right]\\
&=-\frac{\pi^2\epsilon^2}{4L^2}\kappa^2\left[\left(e^{i\frac{2\kappa\pi}{L}\tau}-1\right)\left(1-e^{-i\frac{2\kappa\pi}{L}t'}\right)+\left(1-e^{-i\frac{2\kappa\pi}{L}\tau}\right)\left(e^{i\frac{2\kappa\pi}{L}t'}-1\right)\right]\\
&=-\frac{\pi^2\epsilon^2}{4L^2}\kappa^2\left[e^{i\frac{\kappa\pi}{L}\tau}\left(e^{i\frac{\kappa\pi}{L}\tau}-e^{-i\frac{\kappa\pi}{L}\tau}\right)e^{-i\frac{\kappa\pi}{L}t'}\left(e^{i\frac{\kappa\pi}{L}t'}-e^{-i\frac{\kappa\pi}{L}t'}\right)+e^{-i\frac{\kappa\pi}{L}\tau}\left(e^{i\frac{\kappa\pi}{L}\tau}-e^{-i\frac{\kappa\pi}{L}\tau}\right)e^{i\frac{\kappa\pi}{L}t'}\left(e^{i\frac{\kappa\pi}{L}t'}-e^{-i\frac{\kappa\pi}{L}t'}\right)\right]\\
&=\frac{\pi^2\epsilon^2}{L^2}\kappa^2\left[e^{i\frac{\kappa\pi}{L}(\tau-t')}\sin\left(\frac{\kappa\pi}{L}\tau\right)\sin\left(\frac{\kappa\pi}{L}t'\right)+e^{-i\frac{\kappa\pi}{L}(\tau-t')}\sin\left(\frac{\kappa\pi}{L}\tau\right)\sin\left(\frac{\kappa\pi}{L}t'\right)\right]\\
&=\frac{2\pi^2\epsilon^2}{L^2}\kappa^2\sin\left(\frac{\kappa\pi}{L}\tau\right)\sin\left(\frac{\kappa\pi}{L}t'\right)\cos\left[\frac{\kappa\pi}{L}(\tau-t')\right]\\
&=\frac{2\pi^2\epsilon^2}{L^2}\kappa^2\sin\left(\frac{\kappa\pi}{L}\tau\right)\sin\left(\frac{\kappa\pi}{L}t'\right)\left[\cos\left(\frac{\kappa\pi}{L}\tau\right)\cos\left(\frac{\kappa\pi}{L}t'\right)+\sin\left(\frac{\kappa\pi}{L}\tau\right)\sin\left(\frac{\kappa\pi}{L}t'\right)\right]\\
&=\frac{\pi^2\epsilon^2}{2L^2}\kappa^2\left\{\sin\left(\frac{2\kappa\pi}{L}\tau\right)\sin\left(\frac{2\kappa\pi}{L}t'\right)+\left[1-\cos\left(\frac{2\kappa\pi}{L}\tau\right)\right]\left[1-\cos\left(\frac{2\kappa\pi}{L}t'\right)\right]\right\}\\
&=\frac{\pi^2\epsilon^2}{2L^2}\kappa^2\left[\sin\left(\frac{2\kappa\pi}{L}\tau\right)\sin\left(\frac{2\kappa\pi}{L}t'\right)+\cos\left(\frac{2\kappa\pi}{L}t'\right)\cos\left(\frac{2\kappa\pi}{L}\tau\right)-\cos\left(\frac{2\kappa\pi}{L}\tau\right)-\cos\left(\frac{2\kappa\pi}{L}t'\right)+1\right]\\
&=\frac{\pi^2\epsilon^2}{2L^2}\kappa^2\left[\cos\frac{2\kappa\pi}{L}(\tau-t')-\cos\left(\frac{2\kappa\pi}{L}\tau\right)-\cos\left(\frac{2\kappa\pi}{L}t'\right)+1\right]
\end{aligned}
$$

integrating the last expression

$$
\begin{aligned}
\Pi(t;\kappa,\kappa)&=\frac{\pi^2\epsilon^2}{2L^2}\kappa^2\int_0^t dt'\left\{\int_0^{t'}d\tau\cos\left[\frac{2\kappa\pi}{L}(\tau-t')\right]-\int_0^{t'}d\tau\cos\left(\frac{2\kappa\pi}{L}\tau\right)-\cos\left(\frac{2\kappa\pi}{L}t'\right)\int_0^{t'}d\tau+\int_0^{t'}d\tau\right\}\\
&=\frac{\pi\epsilon^2}{4L}\kappa\int_0^t dt'\left\{\sin\left[\frac{2\kappa\pi}{L}(\tau-t')\right]-\sin\left(\frac{2\kappa\pi}{L}\tau\right)-\frac{2\kappa\pi}{L}\tau\cos\left(\frac{2\kappa\pi}{L}t'\right)+\frac{2\kappa\pi}{L}\tau\right\}\Bigg|_0^{t'}\\
&=\frac{\pi\epsilon^2}{4L}\kappa\int_0^t dt'\left[\sin\left(\frac{2\kappa\pi}{L}t'\right)-\sin\left(\frac{2\kappa\pi}{L}t'\right)-\frac{2\kappa\pi}{L}t'\cos\left(\frac{2\kappa\pi}{L}t'\right)+\frac{2\kappa\pi}{L}t'\right]\\
&=\frac{\pi\epsilon^2}{4L}\kappa\left[\frac{2\kappa\pi}{L}\frac{t^2}{2}-\frac{L}{2\kappa\pi}\left[\frac{2\kappa\pi t'}{L}\sin\left(\frac{2\kappa\pi t'}{L}\right)+\cos\left(\frac{2\kappa\pi t'}{L}\right)\right]\Bigg|_0^t\right]\\
&=\frac{\epsilon^2}{8}\left[1+\frac{2\kappa^2\pi^2}{L^2}t^2-\cos\left(\frac{2\kappa\pi t'}{L}\right)-\frac{2\kappa\pi t'}{L}\sin\left(\frac{2\kappa\pi t'}{L}\right)\right].
\end{aligned}
$$

After those calculations we finally write

$$
\begin{aligned}
\rho_{00}&=\langle 0|\rho(t)|0\rangle=1-\frac{1}{8}\sum_k\Pi(t;2k,p)\\
\rho_{kk}&=\langle 2_k|\rho(t)|2_k\rangle=\frac{1}{8}\Pi(t;2k,p)\\
\rho_{kj}&=\langle 1_k,1_j|\rho(t)|1_k,1_j\rangle=\frac{k}{4j}\left(\frac{kj}{j^2-k^2}\right)^2\Pi(t;k+j,p).
\end{aligned}
$$

where

$$
\Pi(t;\kappa,p)=\begin{cases}\frac{\epsilon^2}{8}\left[1+\frac{2\kappa^2\pi^2}{L^2}t^2-\cos\left(\frac{2\kappa\pi t}{L}\right)-\frac{2\kappa\pi t}{L}\sin\left(\frac{2\kappa\pi t}{L}\right)\right] & \text{for}\quad p=\kappa\\ \frac{p^3\epsilon^2}{(p^2-\kappa^2)}\left[\frac{1}{2p}\cos\left(\frac{2p\pi t}{L}\right)-\frac{1}{p-\kappa}\cos(p-\kappa)\frac{\pi t}{L}-\frac{1}{p+\kappa}\cos(p+\kappa)\frac{\pi t}{L}+\frac{2p}{p^2-\kappa^2}-\frac{1}{2p}\right] & \text{for}\quad p\neq\kappa\end{cases}
$$

To show that the formalism is adequate, let us calculate the coefficients for the case of parametric oscillations where the cavity oscillates with the double of the fundamental frequency ω_1, i.e., $p = 2$. For $\kappa = 2k$, the last quantity is

$$\begin{aligned}
\Pi(t; 2k, p) = {} & \frac{4k^2\pi^2}{L^2}\epsilon^2 \int_0^t dt' \int_0^{t'} d\tau \sin(\Omega_2 \tau) \sin(\Omega_2 t') \left(e^{i\frac{2k\pi}{L}(\tau - t')} + e^{-i\frac{2k\pi}{L}(\tau - t')}\right) \\
& - \frac{k^2\pi^2}{L^2}\epsilon^2 \int_0^t dt' \int_0^{t'} d\tau \Bigg[\left(e^{i(2+2k)\frac{\pi}{L}\tau} - e^{-i(2-2k)\frac{\pi}{L}\tau}\right)\left(e^{i(2-2k)\frac{\pi}{L}t'} - e^{-i(2+2k)\frac{\pi}{L}t'}\right) \\
& \qquad + \left(e^{i(2-2k)\frac{\pi}{L}\tau} - e^{-i(2+2k)\frac{\pi}{L}\tau}\right)\left(e^{i(2+2k)\frac{\pi}{L}t'} - e^{-i(2-2k)\frac{\pi}{L}t'}\right) \Bigg] \\
= {} & -\frac{k^2\pi^2}{L^2}\epsilon^2 \int_0^t dt' \int_0^{t'} d\tau \begin{cases} -2 & \text{for} \quad k = 1 \\ 0 & \text{otherwise} \end{cases} \\
= {} & \int_0^t dt' \int_0^{t'} d\tau \begin{cases} \frac{2\pi^2}{L^2}\epsilon^2 & \text{for} \quad k = 1 \\ 0 & \text{otherwise} \end{cases} \\
= {} & \frac{t^2}{2} \begin{cases} \frac{2\pi^2}{L^2}\epsilon^2 & \text{for} \quad k = 1 \\ 0 & \text{otherwise} \end{cases} \\
= {} & \begin{cases} \frac{\pi^2\epsilon^2}{L^2} t^2 & \text{for} \quad k = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and, therefore, we find the same results obtained previously with the effective Hamiltonian after a rotating wave approximation

$$\begin{aligned}
\rho_{00} &= 1 - \frac{1}{8}\sum_k \Pi(t; 2k, 2) = 1 - \frac{1}{8}\Pi(t; 2, 2) = 1 - \frac{\pi^2\epsilon^2}{8L^2}t^2 \\
\rho_{11} &= \frac{1}{8}\Pi(t; 2, 2) = \frac{\pi^2\epsilon^2}{8L^2}t^2 \\
\rho_{kj} &= 0.
\end{aligned}$$

1.2 Uniform velocity

For the situation in which the second mirror move uniformly with a small velocity ϵv and a prescribed trajectory

$$q(t) = Le^{\epsilon\frac{v}{L}t} \approx L + \epsilon v t$$

the no-adiabatic functions $\xi_k(t)$ and $\mu_{kj}(t)$ takes the form of

$$\xi_k(t) = \frac{\dot{q}(t)}{4q(t)}, \qquad \mu_{kj}(t) = \frac{1}{2}(-1)^{j+k}\frac{kj}{j^2 - k^2}\left(\frac{k}{j}\right)^{1/2}\frac{\dot{q}(t)}{q(t)}, \tag{6}$$

where

$$\frac{\dot{q}(t)}{q(t)} = \frac{L\left(\epsilon\frac{v}{L}\right)e^{\epsilon\frac{v}{L}t}}{Le^{\epsilon\frac{v}{L}t}} = \epsilon\frac{v}{L}. \tag{7}$$

Therefore, the coefficients for the matrix density can be written in terms of

$$\begin{aligned}
\rho_{00} &= \langle 0|\rho(t)|0\rangle = 1 - \frac{1}{8}\sum_k \Lambda(t; 2k) = 1 - \frac{\epsilon^2 v^2}{64}\left[1 - \mathrm{B}_2(t/L)\right] \\
\rho_{kk} &= \langle 2_k|\rho(t)|2_k\rangle = \frac{1}{8}\Lambda(t; 2k) \\
\rho_{kj} &= \langle 1_k, 1_j|\rho(t)|1_k, 1_j\rangle = \frac{k}{4j}\left(\frac{kj}{j^2 - k^2}\right)^2 \Lambda(t; k + j, p).
\end{aligned}$$

where

$$\Lambda(t; \kappa) = \frac{\epsilon^2 v^2}{L^2}\int_0^t dt' \int_0^{t'} d\tau \left(e^{i\kappa(\tau - t')} + e^{-i\kappa(\tau - t')}\right) = \frac{2\epsilon^2 v^2}{\kappa^2\pi^2}\left[1 - \cos\left(\frac{\kappa\pi}{L}t\right)\right]$$

with

$$\sum_k \Lambda(t; 2k) = \frac{\epsilon^2 v^2}{2\pi^2}\left[\sum_k \frac{1}{k^2} - \sum_k \frac{\cos 2\pi k\frac{t}{L}}{k^2}\right] = \frac{\epsilon^2 v^2}{8}\left[1 - \mathrm{B}_2(t/L)\right]$$

where $\mathrm{B}_2(x) = x^2 - x + 1/6$ is the Bernoulli polynomial of order 2.

Referências