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Brownian Motion of a Quantum Oscillator

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An action principle technique for the direct computation of expectation values is described and illustrated in detail by a special physical example, the effect on an oscillator of another physical system. This simple problem has the advantage of combining immediate physical applicability (e.g., resistive damping or maser amplification of a single electromagnetic cavity mode) with a significant idealization of the complex problems encountered in many-particle and relativistic field theory. Successive sections contain discussions of the oscillator subjected to external forces, the oscillator loosely coupled to the external system, an improved treatment of this problem and, finally, there is a brief account of a general formulation.

INTRODUCTION

THE title of this paper refers to an elementary physical example that we shall use to illustrate, at some length, a solution of the following methodological problem. The quantum action principle[1] is a differential characterization of transformation functions, $\langle a't_1|b't_2\rangle$, and thus is ideally suited to the practical computation of transition probabilities (which includes the determination of stationary states). Many physical questions do not pertain to individual transition probabilities, however, but rather to expectation values of a physical property for a specified initial state,

$$\langle X(t_1)\rangle_{b't_2}=\sum_{a'a''}\langle b't_2|a't_1\rangle\langle a't_1|X(t_1)|a''t_1\rangle\langle a''t_1|b't_2\rangle,$$

or, more generally, a mixture of states. Can one devise an action principle technique that is adapted to the direct computation of such expectation values, without requiring knowledge of the individual transformation functions?

The action principle asserts that ($\hbar=1$),

$$\delta\langle a't_1|b't_2\rangle=i\left\langle a't_1\left|\delta\left[\int_{t_2}^{t_1}dtL\right]\right|b't_2\right\rangle,$$

and

$$\delta\langle b't_2|a't_1\rangle=-i\left\langle b't_2\left|\delta\left[\int_{t_2}^{t_1}dtL\right]\right|a't_1\right\rangle,$$

in which we shall take $t_1>t_2$. These mutually complex-conjugate forms correspond to the two viewpoints whereby states at different times can be compared, either by progressing forward from the earlier time, or backward from the later time. The relation between the pair of transformation functions is such that

$$\delta\Big[\sum_{a'}\langle b't_2|a't_1\rangle\langle a't_1|b''t_2\rangle\Big]=0,$$

which expresses the fixed numerical value of

$$\langle b't_2|b''t_2\rangle=\delta(b',b'').$$

But now, imagine that the positive and negative senses of time development are governed by different dynamics. Then the transformation function for the closed circuit will be described by the action principle

$$\delta\langle t_2|t_2\rangle=\delta[\langle t_2|t_1\rangle\times\langle t_1|t_2\rangle]$$
$$=i\left\langle t_2\left|\delta\left[\int_{t_2}^{t_1}dtL_+-\int_{t_2}^{t_1}dtL_-\right]\right|t_2\right\rangle,$$

in which abbreviated notation the multiplication sign symbolizes the composition of transformation functions by summation over a complete set of states. If, in particular, the Lagrangian operators $L_\pm$ contain the dynamical term $\lambda_\pm(t)X(t)$, we have

$$\delta_\lambda\langle t_2|t_2\rangle=i\left\langle t_2\left|\int_{t_2}^{t_1}dt(\delta\lambda_+-\delta\lambda_-)X(t)\right|t_2\right\rangle,$$

and, therefore,

$$-i\frac{\delta}{\delta\lambda_+(t_1)}\langle t_2|t_2\rangle=i\frac{\delta}{\delta\lambda_-(t_1)}\langle t_2|t_2\rangle$$
$$=\langle t_2|X(t_1)|t_2\rangle,$$

where $\lambda_\pm$ can now be identified. Accordingly, if a system is suitably perturbed[2] in a manner that depends upon the time sense, a knowledge of the transformation function referring to a closed time path determines the expectation value of any desired physical quantity for a specified initial state or state mixture.

OSCILLATOR

To illustrate this remark we first consider an oscillator subjected to an arbitrary external force, as described by the Lagrangian operator

$$L=iy^\dagger(dy/dt)-\omega y^\dagger y-y^\dagger K(t)-yK^*(t),$$

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[1] Some references are: Julian Schwinger, Phys. Rev. 82, 914 (1951); 91, 713 (1953); Phil. Mag. 44, 1171 (1953). The first two papers also appear in *Selected Papers on Quantum Electrodynamics* (Dover Publications, New York, 1958). A recent discussion is contained in Julian Schwinger, Proc. Natl. Acad. Sci. U. S. 46, 883 (1960).

[2] Despite this dynamical language, a change in the Hamiltonian operator of a system can be kinematical in character, arising from the consideration of another transformation along with the dynamical one generated by the Hamiltonian. See the last paper quoted in footnote 1, and Julian Schwinger, Proc. Natl. Acad. Sci. U. S. 46, 1401 (1960).

in which the complementary pair of non-Hermitian operators y, $iy^\dagger$, are constructed from Hermitian operators q, p by

$$y=2^{-\frac{1}{2}}(q+ip)$$
$$iy^\dagger=2^{-\frac{1}{2}}(p+iq).$$

The equations of motion implied by the action principle are

$$i(dy/dt)-\omega y=K$$
$$-i(dy^\dagger/dt)-\omega y^\dagger=K^*,$$

and solutions are given by

$$y(t)=e^{-i\omega(t-t_2)}y(t_2)-i\int_{t_2}^{t}dt'e^{-i\omega(t-t')}K(t'),$$

together with the adjoint equation. Since we now distinguish between the forces encountered in the positive time sense, $K_+(t)$, $K_+^*(t)$, and in the reverse time direction, $K_-(t)$, $K_-^*(t)$, the integral must be taken along the appropriate path. Thus, when t is reached first in the time evolution from t_2, we have

$$y_+(t)=e^{-i\omega(t-t_2)}y_+(t_2)-i\int_{t_2}^{t}dt'e^{-i\omega(t-t')}K_+(t'),$$

while on the subsequent return to time t,

$$y_-(t)=e^{-i\omega(t-t_2)}y_+(t_2)-i\int_{t_2}^{t_1}dt'e^{-i\omega(t-t')}K_+(t')+i\int_{t}^{t_1}dt'e^{-i\omega(t-t')}K_-(t').$$

Note that

$$y_-(t_1)-y_+(t_1)=0,$$
$$y_-(t_2)-y_+(t_2)=i\int_{t_2}^{t_1}dte^{i\omega(t-t_2)}(K_--K_+)(t).$$

We shall begin by constructing the transformation function referring to the lowest energy state of the unperturbed oscillator, $\langle 0t_2|0t_2\rangle^{K\pm}$. This state can be characterized by

$$\langle 0t_2|y^\dagger y(t_2)|0t_2\rangle=0$$

or, equivalently, by the eigenvector equations

$$y(t_2)|0t_2\rangle=0,\quad \langle 0t_2|y^\dagger(t_2)=0.$$

Since the transformation function simply equals unity if $K_+=K_-$ and $K_+^*=K_-^*$, we must examine the effect of independent changes in K_+ and K_-, and of K_+^* and K_-^*, as described by the action principle

$$\delta_K\langle 0t_2|0t_2\rangle^{K\pm}=-i\left\langle 0t_2\left|\left[\int_{t_2}^{t_1}dt(\delta K_+^*y_+-\delta K_-^*y_-)+\int_{t_2}^{t_1}dt(y_+^\dagger\delta K_+-y_-^\dagger\delta K_-)\right]\right|0t_2\right\rangle^{K\pm}$$

The choice of initial state implies effective boundary conditions that supplement the equations of motion,

$$y_+(t_2)\rightarrow 0,\quad y_-^\dagger(t_2)\rightarrow 0.$$

Hence, in effect we have

$$y_+(t)=-i\int_{t_2}^{t_1}dt'e^{-i\omega(t-t')}\eta_+(t-t')K_+(t')$$

and

$$y_-(t)=-i\int_{t_2}^{t_1}dt'e^{-i\omega(t-t')}K_+(t')+i\int_{t_2}^{t_1}dt'e^{-i\omega(t-t')}\eta_-(t-t')K_-(t'),$$

together with the similar adjoint equations obtained by interchanging the $\pm$ labels. For convenience, step functions have been introduced:

$$\eta_+(t-t')=\begin{cases}1, & t-t'>0\\ 0, & t-t'<0\end{cases},$$

$$\eta_-(t-t')=\begin{cases}1, & t-t'<0\\ 0, & t-t'>0\end{cases},$$

$$\eta_+(t-t')+\eta_-(t-t')=1,\quad \eta_+(0)=\eta_-(0)=\tfrac{1}{2}.$$

We shall also have occasion to use the odd function

$$\epsilon(t-t')=\eta_+(t-t')-\eta_-(t-t').$$

The solution of the resulting integrable differential expression for $\log\langle 0t_2|0t_2\rangle^{K\pm}$ is given by

$$\langle 0t_2|0t_2\rangle^{K\pm}=\exp\left[-i\int_{t_2}^{t_1}dtdt'K^*(t)G_0(t-t')K(t')\right],$$

in a matrix notation, with

$$K(t)=\begin{pmatrix}K_+(t)\\ K_-(t)\end{pmatrix}$$

and

$$iG_0(t-t')=e^{-i\omega(t-t')}\begin{pmatrix}\eta_+(t-t') & 0\\ -1 & \eta_-(t-t')\end{pmatrix}.$$

The requirement that the transformation function reduce to unity on identifying K_+ with K_-, K_+^* with K_-^*, is satisfied by the null sum of all elements of G_0, as assured by the property $\eta_++\eta_-=1$.

An operator interpretation of G_0 is given by the second variation

$$-\delta_{K^*}\delta_K\langle 0t_2|0t_2\rangle^{K\pm}|_{K=K^*=0}=i\int dtdt'\delta K^*(t)G_0(t-t')\delta K(t').$$

Generally, on performing two distinct variations in the structure of L that refer to parameters upon which the dynamical variables at a given time are not explicitly dependent, we have

$$-\delta_1\delta_2\langle t_2|t_2\rangle=\left\langle t_2\left|\int_{t_2}^{t_1}dtdt'\{(\delta_1L_+(t)\delta_2L_+(t'))_+ -\delta_1L_-(t)\delta_2L_+(t')-\delta_2L_-(t)\delta_1L_+(t') +(\delta_1L_-(t)\delta_2L_-(t'))_-\}\right|t_2\right\rangle,$$

in which the multiplication order follows the sense of time development. Accordingly,

$$iG_0(t-t')\quad\begin{pmatrix}\langle(y(t)y^\dagger(t'))_+\rangle_0 & -\langle y^\dagger(t')y(t)\rangle_0\\ -\langle y(t)y^\dagger(t')\rangle_0 & \langle(y(t)y^\dagger(t'))_-\rangle_0\end{pmatrix},$$

where the expectation values and operators refer to the lowest state and the dynamical variables of the unperturbed oscillator. The property of G_0 that the sum of rows and columns vanishes is here a consequence of the algebraic property

$$(y(t)y^\dagger(t'))_++(y(t)y^\dagger(t'))_-=\{y(t),y^\dagger(t')\}.$$

The choice of oscillator ground state is no essential restriction since we can now derive the analogous results for any initial oscillator state. Consider, for this purpose, the impulse forces

$$K_+(t)=iy''\delta(t-t_2),$$
$$K_-^*(t)=-iy^{\dagger\prime}\delta(t-t_2),$$

the effects of which are described by

$$y_+(t_2+0)-y_+(t_2)=y'',$$
$$y_-^\dagger(t_2+0)-y_-^\dagger(t_2)=y^{\dagger\prime}.$$

Thus, under the influence of these forces, the states $|0t_2\rangle$ and $\langle 0t_2|$ become, at the time t_2+0, the states $|y''t_2\rangle$ and $\langle y^{\dagger\prime}t_2|$, which are right and left eigenvectors, respectively, of the operators $y(t_2)$ and $y^\dagger(t_2)$. On taking into account arbitrary additional forces, the transformation function for the closed time path can be expressed as

$$\langle y^{\dagger\prime}t_1|y''t_2\rangle^{K\pm} =\exp\left[y^{\dagger\prime}y''-y^{\dagger\prime}\left(\int_{t_2}^{t_1}dtG_0(t_2-t)K(t)\right) +\left(\int_{t_2}^{t_1}dtK^*(t)G_0(t-t_2)\right)_+y'' -i\int_{t_2}^{t_1}dtdt'K^*(t)G_0(t-t')K(t')\right],$$

in which

$$\left(\int_{t_2}^{t_1}dtG_0(t_2-t)K(t)\right)=-i\int_{t_2}^{t_1}dte^{i\omega(t-t_2)}(K_--K_+)(t)$$

and

$$\left(\int dtK^*(t)G_0(t-t_2)\right)_+ =-i\int_{t_2}^{t_1}dte^{-i\omega(t-t_2)}(K_+^*-K_-^*)(t).$$

The eigenvectors of the non-Hermitian canonical variables are complete and have an intrinsic physical interpretation in terms of q and p measurements of optimum compatibility.[3] For our immediate purposes, however, we are more interested in the unperturbed oscillator energy states. The connection between the two descriptions can be obtained by considering the unperturbed oscillator transformation function

$$\langle y^{\dagger\prime}t_1|y''t_2\rangle=\langle y^{\dagger\prime}|\exp[-i(t_1-t_2)\omega y^\dagger y]|y''\rangle.$$

Now

$$i(\partial/\partial t_1)\langle y^{\dagger\prime}t_1|y''t_2\rangle=\langle y^{\dagger\prime}t_1|\omega y^\dagger(t_1)y(t_1)|y''t_2\rangle =\omega y^{\dagger\prime}e^{-i\omega(t_1-t_2)}y''\langle y^{\dagger\prime}t_1|y''t_2\rangle,$$

since

$$y(t_1)=e^{-i\omega(t_1-t_2)}y(t_2),$$

which yields

$$\langle y^{\dagger\prime}t_1|y''t_2\rangle=\exp[y^{\dagger\prime}e^{-i\omega(t_1-t_2)}y''] =\sum_{n=0}^\infty\frac{(y^{\dagger\prime})^n}{(n!)^{\frac12}}e^{-in\omega(t_1-t_2)}\frac{(y'')^n}{(n!)^{\frac12}}.$$

We infer the nonnegative integer spectrum of $y^\dagger y$, and the corresponding wave functions

$$\langle y^{\dagger\prime}|n\rangle=(y^{\dagger\prime})^n/(n!)^{\frac12},\quad \langle n|y'\rangle=(y')^n/(n!)^{\frac12}.$$

Accordingly, a non-Hermitian canonical variable transformation function can serve as a generator for the transformation function referring to unperturbed oscillator energy states,

$$\langle y^{\dagger\prime}t_2|y''t_2\rangle^{K\pm}=\sum_{n,n'=0}^\infty\frac{(y^{\dagger\prime})^n}{(n!)^{\frac12}}\langle nt_2|n't_2\rangle^{K\pm}\frac{(y'')^{n'}}{(n'!)^{\frac12}}.$$

If we are specifically interested in $\langle nt_2|nt_2\rangle^{K\pm}$, which supplies all expectation values referring to the initial state n, we must extract the coefficient of $(y^{\dagger\prime}y'')^n/n!$ from an exponential of the form

$$\exp[y^{\dagger\prime}y''+y^{\dagger\prime}\alpha+\beta y''+\gamma] =\sum_{kl}\frac{(y^{\dagger\prime})^k}{k!}\frac{(y'')^l}{l!}\alpha^k\beta^l\exp[y^{\dagger\prime}y''+\gamma].$$

All the terms that contribute to the required coefficient

[3] A discussion of non-Hermitian representations is given in *Lectures on Quantum Mechanics* (Les Houches, 1955), unpublished.

are contained in

$$\sum_{k=0}^{\infty}\frac{(y^{\dagger\prime}y'')^k}{(k!)^2}(\alpha\beta)^k\exp[y^{\dagger\prime}y''+\gamma]$$
$$=\frac{1}{2\pi i}\oint\frac{d\lambda}{\lambda}e^{\lambda}\exp[y^{\dagger\prime}y''(1+\lambda^{-1}\alpha\beta)+\gamma],$$

where the latter version is obtained from

$$\frac{1}{2\pi i}\oint\frac{d\lambda}{\lambda^{k+1}}e^{\lambda}=\frac{1}{k!},$$

and

$$\langle nt_2|nt_2\rangle^{K\pm}=\exp\left[-i\int dtdt'K^*(t)G_0(t-t')K(t')\right]$$
$$\times L_n\left[\left(\int dtK^*(t)G_0(t-t_2)\right)_+\times\left(\int dtG_0(t_2-t)K(t)\right)_-\right],$$

in which the nth Laguerre polynomial has been introduced on observing that

$$\frac{1}{2\pi i}\oint\frac{d\lambda}{\lambda}e^{\lambda}(1-\lambda^{-1}x)^n=\frac{1}{n!}e^x\left(\frac{d}{dx}\right)^n x^ne^{-x}=L_n(x).$$

One obtains a much neater form, however, from which these results can be recovered, on considering an initial mixture of oscillator energy states for which the nth state is assigned the probability

$$(1-e^{-\beta\omega})e^{-n\beta\omega},$$

and

$$\beta^{-1}=\vartheta$$

can be interpreted as a temperature. Then, since

$$(1-e^{-\beta\omega})\sum_{n=0}^{\infty}e^{-n\beta\omega}L_n(x)=(1-e^{-\beta\omega})\frac{1}{2\pi i}\oint\frac{d\lambda}{\lambda}$$
$$\times e^{\lambda}[1-e^{-\beta\omega}+\lambda^{-1}e^{-\beta\omega}x]^{-1}=\exp\left[-\frac{x}{e^{\beta\omega}-1}\right],$$

we obtain

$$\langle t_2|t_2\rangle_\vartheta{}^{K\pm}=\exp\left[-i\int_{t_2}^{t_1}dtdt'K^*(t)G_\vartheta(t-t')K(t')\right],$$

with

$$iG_\vartheta(t-t')=iG_0(t-t')+(e^{\beta\omega}-1)^{-1}G_0(t-t_2)_+{}_-G_0(t_2-t'),$$

and in which

$$iG_0(t-t_2)_+=e^{-i\omega(t-t_2)}\begin{pmatrix}1\\-1\end{pmatrix},$$

$$i_-G_0(t_2-t)=e^{i\omega(t-t_2)}\begin{pmatrix}-1 & 1\end{pmatrix}.$$

Thus,

$$iG_\vartheta(t-t')=e^{-i\omega(t-t')}\begin{pmatrix}\eta_+(t-t')+\langle n\rangle_\vartheta, & -\langle n\rangle_\vartheta\\ -1-\langle n\rangle_\vartheta, & \eta_-(t-t')+\langle n\rangle_\vartheta\end{pmatrix},$$

where we have written

$$\langle n\rangle_\vartheta=(e^{\beta\omega}-1)^{-1},$$

and since the elements of G_ϑ are also given by unperturbed oscillator thermal expectation values

$$iG_\vartheta(t-t')=\begin{pmatrix}\langle(y(t)y^\dagger(t'))_+\rangle_\vartheta & -\langle y^\dagger(t')y(t)\rangle_\vartheta\\ -\langle y(t)y^\dagger(t')\rangle_\vartheta & \langle(y(t)y^\dagger(t'))_-\rangle_\vartheta\end{pmatrix},$$

the designation $\langle n\rangle_\vartheta$ is consistent with its identification as $\langle y^\dagger y\rangle_\vartheta$.

The thermal forms can also be derived directly by solving the equations of motion, in the manner used to find $\langle 0t_2|0t_2\rangle^{K\pm}$. On replacing the single diagonal element

$$\langle 0t_2|0t_2\rangle^{K\pm}=\langle 0t_2|U|0t_2\rangle$$

by the statistical average

$$(1-e^{-\beta\omega})\sum_0^\infty e^{-n\beta\omega}\langle nt_2|nt_2\rangle^{K\pm}$$
$$=(1-e^{-\beta\omega})\,\mathrm{tr}[\exp(-\beta\omega y^\dagger y)U],$$

we find the following relation,

$$y_-(t_2)=e^{\beta\omega}y_+(t_2),$$

instead of the effective initial condition $y_+(t_2)=0$. This is obtained by combining

$$\exp(-\beta\omega y^\dagger y)y\exp(\beta\omega y^\dagger y)=\exp(\beta\omega)y$$

with the property of the trace

$$\mathrm{tr}[\exp(-\beta\omega y^\dagger y)yU]=\mathrm{tr}[\exp(\beta\omega)y\exp(-\beta\omega y^\dagger y)U]$$
$$=\mathrm{tr}[\exp(-\beta\omega y^\dagger y)U\exp(\beta\omega)y].$$

We also have

$$y_-(t_2)-y_+(t_2)=-i\int_{t_2}^{t_1}dte^{i\omega(t-t_2)}(K_+-K_-)(t),$$

and therefore, effectively,

$$y_+(t_2)=-i\frac{1}{e^{\beta\omega}-1}\int_{t_2}^{t_1}dte^{i\omega(t-t_2)}(K_+-K_-)(t).$$

Hence, to the previously determined $y_\pm(t)$ is to be added the term

$$-i\langle n\rangle_\vartheta\int_{t_2}^{t_1}dt'e^{-i\omega(t-t')}(K_+-K_-)(t'),$$

and correspondingly

$$\langle t_2|t_2\rangle_\vartheta{}^{K\pm}=\langle t_2|t_2\rangle_0{}^{K\pm}\exp\left[-\langle n\rangle_\vartheta\int_{t_2}^{t_1}dtdt'\right.$$
$$\left.\times(K_+{}^*-K_-{}^*)(t)e^{-i\omega(t-t')}(K_+-K_-)(t')\right],$$

which reproduces the earlier result.

As an elementary application let us evaluate the expectation value of the oscillator energy at time t_1 for a system that was in thermal equilibrium at time t_2 and is subsequently disturbed by an arbitrarily time-varying force. This can be computed as

$$\langle t_2|\omega y^\dagger y(t_1)|t_2\rangle_\vartheta{}^K = \omega\frac{\delta}{\delta K_-(t_1)}\frac{\delta}{\delta K_+{}^*(t_1)}\langle t_2|t_2\rangle_\vartheta{}^{K\pm}|_{K_+=K_-,\ K_+{}^*=K_-{}^*}.$$

The derivative $\delta/\delta K_+{}^*(t_1)$ supplies the factor

$$-i\left(\int_{t_2}^{t_1} dt G_\vartheta(t_1-t')K(t')\right)_+,$$

the subsequent variation with respect to $K_-(t_1)$ gives

$$-iG_\vartheta(0)_{+-}+\left(\int dtK^*(t)G_\vartheta(t-t_1)\right)_-\times\left(\int dt'G_\vartheta(t_1-t')K(t')\right)_+,$$

and the required energy expectation value equals

$$\omega\langle n\rangle_\vartheta+\omega\left|\int_{t_2}^{t_1}dte^{i\omega t}K(t)\right|^2.$$

More generally, the expectation values of all functions of $y(t_1)$ and $y^\dagger(t_1)$ are known from that of

$$\exp\{-i[\lambda y^\dagger(t_1)+\mu y(t_1)]\},$$

and this quantity is obtained on supplementing K_+ and $K_+{}^*$ by the impulsive forces (note that in this use of the formalism a literal complex-conjugate relationship is not required)

$$K_+(t)=\lambda\delta(t-t_1),$$
$$K_+{}^*(t)=\mu\delta(t-t_1).$$

Then

$$\langle t_2|\exp\{-i[\lambda y^\dagger(t_1)+\mu y(t_1)]\}|t_2\rangle_\vartheta{}^K = \exp\left[-\lambda\mu(\langle n\rangle_\vartheta+\tfrac{1}{2})+\lambda\int_{t_2}^{t_1}dte^{i\omega(t_1-t)}K^*(t) -\mu\int_{t_2}^{t_1}dte^{-i\omega(t_1-t)}K(t)\right],$$

which involves the special step-function value

$$\eta_+(0)=\tfrac{1}{2}.$$

Alternatively, if we choose

$$K_+(t)=\lambda\delta(t-t_1),$$
$$K_+{}^*(t)=\mu\delta(t-t_1+0),$$

there appears

$$\langle t_2|\exp[-i\lambda y^\dagger(t_1)]\exp[-i\mu y(t_1)]|t_2\rangle_\vartheta{}^K = \exp\left[-\lambda\mu\langle n\rangle_\vartheta+\lambda\int_{t_2}^{t_1}dte^{i\omega(t_1-t)}K^*(t) -\mu\int_{t_2}^{t_1}dte^{-i\omega(t_1-t)}K(t)\right].$$

It may be worth remarking, in connection with these results, that the attention to expectation values does not deprive us of the ability to compute individual probabilities. Indeed, if probabilities for specific oscillator energy states are of interest, we have only to exhibit, as functions of y and $y^\dagger$, the projection operators for these states, the expectation values of which are the required probabilities. Now

$$P_n=|n\rangle\langle n|$$

is represented by the matrix

$$\langle y^{\dagger\prime}|P_n|y''\rangle=(y^{\dagger\prime}y'')^n/n! = [(y^{\dagger\prime}y'')^n/n!]\exp(-y^{\dagger\prime}y'')\langle y^{\dagger\prime}|y''\rangle,$$

and, therefore,

$$P_n=\frac{1}{n!}(y^\dagger)^n\left[\sum_{k=0}^\infty\frac{(-1)^k}{k!}(y^\dagger)^ky^k\right]y^n =\frac{1}{n!}(y^\dagger)^n\exp(-y^\dagger;y)y^n,$$

in which we have introduced a notation to indicate this ordered multiplication of operators. A convenient generating function for these projection operators is

$$\sum_{n=0}^\infty\alpha^nP_n=\exp[-(1-\alpha)y^\dagger;y],$$

and we observe that

$$\sum_0^\infty\alpha^nP_n=\exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\times\exp(-i\lambda y^\dagger)\exp(-i\mu y)|_{\lambda=\mu=0}.$$

Accordingly,

$$\sum_0^\infty\alpha^np(n,\vartheta,K)=\exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\exp\left[-\lambda\mu\langle n\rangle_\vartheta +\lambda e^{i\omega t_1}\int dte^{-i\omega t}K^*(t) -\mu e^{-i\omega t_1}\int dte^{i\omega t}K(t)\right]\Bigg|_{\lambda=\mu=0}$$

gives the probability of finding the oscillator in the nth energy state after an arbitrary time-varying force has acted, if it was initially in a thermal mixture of states.

To evaluate

$$X=\exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\exp[-\lambda\mu\langle n\rangle+\lambda\gamma^*-\mu\gamma]|_{\lambda=\mu=0},$$

we first remark that

$$\frac{\partial}{\partial\gamma^*}X=\exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\lambda\exp[\;]|_{\lambda=\mu=0}$$

$$=(1-\alpha)\exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\frac{\partial}{\partial\mu}\exp[\;]|$$

$$=(1-\alpha)\exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right](-\lambda\langle n\rangle-\gamma)\exp[\;]|,$$

from which follows

$$\frac{\partial}{\partial\gamma^*}X=-\frac{\gamma(1-\alpha)}{1+\langle n\rangle(1-\alpha)}X$$

or

$$X=X_0\exp\left[-|\gamma|^2\frac{1-\alpha}{1+\langle n\rangle(1-\alpha)}\right].$$

Here

$$X_0=\exp\left[(1-\alpha)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\exp[-\lambda\mu\langle n\rangle]|_{\lambda=\mu=0}$$

$$=[1+\langle n\rangle(1-\alpha)]^{-1},$$

as one shows with a similar procedure, or by direct series expansion. Therefore,

$$\sum_0^\infty\alpha^n p(n,\vartheta,K)=\frac{1-e^{-\beta\omega}}{1-\alpha e^{-\beta\omega}}\exp\left[-|\gamma|^2\frac{1-e^{-\beta\omega}}{1-\alpha e^{-\beta\omega}}(1-\alpha)\right],$$

where

$$|\gamma|^2=\left|\int dte^{i\omega t}K(t)\right|^2,$$

and on referring to the previously used Laguerre polynomial sum formula, we obtain

$$p(n,\vartheta,K)=(1-e^{-\beta\omega})e^{-n\beta\omega}\exp[-|\gamma|^2(1-e^{-\beta\omega})]\times L_n[-4|\gamma|^2\sinh^2(\beta\omega/2)].$$

In addition to describing the physical situation of initial thermal equilibrium, this result provides a generating function for the individual transition probabilities between oscillator energy states,

$$\sum_{n'=0}^\infty p(n,n',K)e^{-(n'-n)\beta\omega}$$

$$=\exp[-|\gamma|^2(1-e^{-\beta\omega})]L_n[-(1-e^{-\beta\omega})(e^{\beta\omega}-1)|\gamma|^2].$$

This form, and the implied transition probabilities, have already been derived in another connection,[4] and we shall only state the general result here:

$$p(n,n',K)=\frac{n_<!}{n_>!}(|\gamma|^2)^{n_>-n_<}[L_{n_<}^{(n_>-n_<)}(|\gamma|^2)]^2\times\exp(-|\gamma|^2),$$

in which $n_>$ and $n_<$ represent the larger and smaller of the two integers n and n'.

Another kind of probability is also easily identified, that referring to the continuous spectrum of the Hermitian operator

$$q=2^{-\frac{1}{2}}(y+y^\dagger)$$

[or $p=-2^{-\frac{1}{2}}i(y-y^\dagger)$]. For this purpose, we place $\lambda=\mu=-2^{-\frac{1}{2}}p'$ and obtain

$$\langle t_2|e^{ip'q(t_1)}|t_2\rangle_\vartheta{}^K=\exp[-\tfrac{1}{2}p'^2(\langle n\rangle_\vartheta+\tfrac{1}{2})+ip'\langle q(t_1)\rangle^K],$$

with

$$\langle q(t_1)\rangle^K=2^{-\frac{1}{2}}i\left[e^{i\omega t_1}\int_{t_2}^{t_1}dte^{-i\omega t}K^*(t)-e^{-i\omega t_1}\int_{t_2}^{t_1}dte^{i\omega t}K(t)\right].$$

If we multiply this result by $\exp(-ip'q')$ and integrate with respect to $p'/2\pi$ from $-\infty$ to ∞, we obtain the expectation value of $\delta[q(t_1)-q']$ which is the probability of realizing a value of $q(t_1)$ in a unit interval about q':

$$p(q't_1,\beta,K)=(\pi^{-1}\tanh\tfrac{1}{2}\beta\omega)^{\frac{1}{2}}\times\exp[-(\tanh\tfrac{1}{2}\beta\omega)(q'-\langle q(t_1)\rangle^K)^2].$$

Still another derivation of the formula giving thermal expectation values merits attention. Now we let the return path terminate at a different time $t_2'=t_2-T$, and on regarding the resulting transformation function as a matrix, compute the trace, or rather the trace ratio

$$\mathrm{tr}\langle t_2'|t_2\rangle^{K\pm}/\mathrm{tr}\langle t_2'|t_2\rangle,$$

which reduces to unity in the absence of external forces. The action principle again describes the dependence upon $K_\pm^*(t)$, $K_\pm(t)$ through the operators $y_\pm(t)$, $y_\pm^\dagger(t)$ which are related to the forces by the solutions of the equations of motion, and, in particular,

$$y_-(t_2')=e^{-i\omega(t_2'-t_2)}y_+(t_2)-i\int_{t_2}^{t_1}dte^{i\omega(t-t_2')}K_+(t)+i\int_{t_2'}^{t_1}dte^{i\omega(t-t_2')}K_-(t).$$

Next we recognize that the structure of the trace implies the effective boundary condition

$$y_-(t_2')=y_+(t_2).$$

[4] Julian Schwinger, Phys. Rev. 91, 728 (1953).

Let us consider

$$\mathrm{tr}\langle t_2'|y_-(t_2')|t_2\rangle=\sum_{a'}\langle a't_2'|y_-(t_2')|a't_2\rangle,$$

where we require of the a representation only that it have no explicit time dependence. Then

$$\langle a't_2'|y_-(t_2')=\sum_{a''}\langle a'|y|a''\rangle\langle a''t_2'|$$

and

$$\mathrm{tr}\langle t_2'|y_-(t_2')|t_2\rangle=\sum_{a'a''}\langle a''t_2'|a't_2\rangle\langle a'|y|a''\rangle =\mathrm{tr}\langle t_2'|y_+(t_2)|t_2\rangle,$$

which is the stated result.

The effective initial condition now appears as

$$y_+(t_2)=-\frac{1}{e^{-i\omega T}-1}i\left[\int_{t_2}^{t_1}dte^{i\omega(t-t_2)}K_+(t) -\int_{t_2'}^{t_1}dte^{i\omega(t-t_2)}K_-(t)\right],$$

and the action principle supplies the following evaluation of the trace ratio:

$$\exp\left[-i\int dtdt'K^*(t)G_0(t-t')K(t')\right] \times\exp\left[-(e^{-i\omega T}-1)^{-1}\left|\int dte^{i\omega t}(K_+-K_-)(t)\right|^2\right],$$

where the time variable in K_+ and K_- ranges from t_2 to t_1 and from t_2' to t_1, respectively. To solve the given physical problem we require that $K_-(t)$ vanish in the interval between t_2' and t_2 so that all time integrations are extended between t_2 and t_1. Then, since

$$\langle t_2'|=\langle t_2|e^{-i\omega(t_2'-t_2)n},\quad n=y^\dagger y(t_2),$$

what has been evaluated equals

$$\mathrm{tr}\langle t_2|e^{i\omega Tn}|t_2\rangle^{K\pm}/\mathrm{tr}\langle t_2|e^{i\omega Tn}|t_2\rangle,$$

and by adding the remark that this ratio continues to exist on making the complex substitution

$$-iT\rightarrow\beta>0,$$

the desired formula emerges as

$$\mathrm{tr}\langle t_2|e^{-\beta\omega n}|t_2\rangle^{K\pm}/\mathrm{tr}\langle t_2|e^{-\beta\omega n}|t_2\rangle =\exp\left[-i\int dtdt'K^*(t)G_\vartheta(t-t')K(t')\right].$$

EXTERNAL SYSTEM

This concludes our preliminary survey of the oscillator and we turn to the specific physical problem of interest: An oscillator subjected to prescribed external forces and loosely coupled to an essentially macroscopic external system. All oscillator interactions are linear in the oscillator variables, as described by the Lagrangian operator

$$L=iy^\dagger(dy/dt)-\omega_0y^\dagger y-y^\dagger K(t)-yK^*(t)-2^{\frac{1}{2}}qQ+L_{\mathrm{ext}},$$

in which L_{ext} characterizes the external system and $Q(t)$ is a Hermitian operator of that system.

We begin our treatment with a discussion of the transformation function $\langle t_2|t_2\rangle_{\vartheta_0\vartheta}{}^{K\pm}$ that refers initially to a thermal mixture at temperature ϑ for the external system, and to an independent thermal mixture at temperature ϑ_0 for the oscillator. The latter temperature can be interpreted literally, or as a convenient parametric device for obtaining expectation values referring to oscillator energy states. To study the effect of the coupling between the oscillator and the external system we supply the coupling term with a variable parameter λ, and compute

$$\frac{\partial}{\partial\lambda}\langle t_1|t_2\rangle^{K\pm} =-i\left\langle t_2\left|\int_{t_2}^{t_1}dt[2^{\frac{1}{2}}q_+(t)Q_+(t)-2^{\frac{1}{2}}q_-(t)Q_-(t)]\right|t_2\right\rangle^{K\pm}$$

where the distinction between the forward and return paths arises only from the application of different external forces $K_\pm(t)$ on the two segments of the closed time contour. The characterization of the external system as essentially macroscopic now enters through the assumption that this large system is only slightly affected by the coupling to the oscillator. In a corresponding first approximation, we would replace the operators $Q_\pm(t)$ by the effective numerical quantity $\langle Q(t)\rangle_\vartheta$. The phenomena that appear in this order of accuracy are comparatively trivial, however, and we shall suppose that

$$\langle Q(t)\rangle_\vartheta=0,$$

which forces us to proceed to the next approximation.

A second differentiation with respect to λ gives

$$-\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\langle t_2|t_2\rangle^{K\pm}=\left\langle t_2\left|\int_{t_2}^{t_1}dtdt'[(qQ(t)qQ(t'))_+ -2q_-Q_-(t)q_+Q_+(t')+(qQ(t)qQ(t'))_-]\right|t_2\right\rangle^{K\pm}.$$

The introduction of an approximation based upon the slight disturbance of the macroscopic system converts this into

$$-\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\langle t_2|t_2\rangle^{K\pm} =\left\langle t_2\left|\int_{t_2}^{t_1}dtdt'[(y(t')y^\dagger(t))_+A_{++}(t-t') -y_-(t')y_+^\dagger(t)A_{+-}(t-t')-y_-^\dagger(t)y_+(t')A_{-+}(t-t') +(y(t')y^\dagger(t))_-A_{--}(t-t')]\right|t_2\right\rangle^{K\pm}$$

where

$$A(t-t')=\begin{pmatrix}\langle(Q(t)Q(t'))_+\rangle_\vartheta & \langle Q(t')Q(t)\rangle_\vartheta\\ \langle Q(t)Q(t')\rangle_\vartheta & \langle(Q(t)Q(t'))_-\rangle_\vartheta\end{pmatrix},$$

and we have also discarded all terms containing $y(t)y(t')$ and $y^\dagger(t)y^\dagger(t')$. The latter approximation refers to the assumed weakness of the coupling of the oscillator to the external system, for, during the many periods that are needed for the effect of the coupling to accumulate, quantities with the time dependence $e^{\pm i\omega_0(t+t')}$ will become suppressed in comparison with those varying as $e^{\pm i\omega_0(t-t')}$. At this point we ask what effective term in an action operator that refers to the closed time path of the oscillator would reproduce this approximate value of $(\partial/\partial\lambda)^2\langle t_2|t_2\rangle$ at $\lambda=0$. The complete action that satisfies this requirement, with λ^2 set equal to unity, is given by

$$W=\int_{t_2}^{t_1}dt\left[iy^\dagger\frac{dy}{dt}-\omega_0y^\dagger y-y^\dagger K-yK^*\Big|_+-\Big|_-\right]$$
$$+i\int_{t_2}^{t_1}dtdt'[(y^\dagger(t)y(t'))_+A_{++}(t-t')$$
$$-y_-{}^\dagger(t)y_+(t')A_{-+}(t-t')-y_-(t')y_+{}^\dagger(t)A_{+-}(t-t')$$
$$+(y^\dagger(t)y(t'))_-A_{--}(t-t')].$$

The application of the principle of stationary action to this action operator yields equations of motion that are nonlocal in time, namely,

$$i\frac{dy_+}{dt}-\omega_0y_++i\int_{t_2}^{t_1}dt'[A_{++}(t-t')y_+(t')-A_{+-}(t-t')y_-(t')]=K_+(t)$$
$$i\frac{dy_-}{dt}-\omega_0y_--i\int_{t_2}^{t_1}dt'[A_{--}(t-t')y_-(t')-A_{-+}(t-t')y_+(t')]=K_-(t),$$

together with

$$-i\frac{dy_+{}^\dagger}{dt}-\omega_0y_+{}^\dagger+i\int_{t_2}^{t_1}dt'[y_+{}^\dagger(t')A_{++}(t'-t)-y_-{}^\dagger(t')A_{-+}(t'-t)]=K_+{}^*(t)$$
$$-i\frac{dy_-{}^\dagger}{dt}-\omega_0y_-{}^\dagger-i\int_{t_2}^{t_1}dt'[y_-{}^\dagger(t')A_{--}(t'-t)-y_+{}^\dagger(t')A_{+-}(t'-t)]=K_-{}^*(t).$$

The latter set is also obtained by combining the formal adjoint operation with the interchange of the + and − labels attached to the operators and $K(t)$. Another significant form is conveyed by the pair of equations

$$\left(i\frac{d}{dt}-\omega_0\right)(y_--y_+)-i\int_{t_2}^{t_1}dt'(A_{--}-A_{+-})(t-t')\times(y_--y_+)(t')=K_--K_+$$

and

$$\left(i\frac{d}{dt}-\omega_0\right)(y_++y_-)+i\int_{t_2}^{t_1}dt'(A_{++}-A_{+-})(t-t')\times(y_++y_-)(t')-i\int_{t_2}^{t_1}dt'(A_{+-}+A_{-+})(t-t')\times(y_--y_+)(t')=K_++K_-,$$

where

$$(A_{--}-A_{+-})(t-t')=-(A_{++}-A_{-+})(t-t')=\langle[Q(t),Q(t')]\rangle_\vartheta\eta_-(t-t'),$$
$$(A_{++}-A_{+-})(t-t')=-(A_{--}-A_{-+})(t-t')=\langle[Q(t),Q(t')]\rangle_\vartheta\eta_+(t-t'),$$

and

$$(A_{+-}+A_{-+})(t-t')=\langle\{Q(t),Q(t')\}\rangle_\vartheta.$$

The nonlocal character of these equations is not very marked if, for example, the correlation between $Q(t)$ and $Q(t')$ in the macroscopic system disappears when $|t-t'|$ is still small compared with the period of the oscillator. Then, since the behavior of $y(t)$ over a short time interval is given approximately by $e^{-i\omega t}$, the matrix $A(t-t')$ is effectively replaced by

$$\int_{-\infty}^{\infty}d(t-t')e^{i\omega(t-t')}A(t-t')=A(\omega),$$

and the equations of motion read

$$[i(d/dt)-\omega_-](y_--y_+)=K_--K_+,$$
$$[i(d/dt)-\omega_+](y_++y_-)-ia(y_--y_+)=K_++K_-.$$

Here we have defined

$$\omega_-=\omega_0+i(A_{--}-A_{+-})(\omega)=\omega+\tfrac{1}{2}i\gamma,$$
$$\omega_+=\omega_0-i(A_{++}-A_{+-})(\omega)=\omega-\tfrac{1}{2}i\gamma,$$

and

$$a(\omega)=(A_{+-}+A_{-+})(\omega).$$

It should be noted that $A_{+-}(\omega)$ and $A_{-+}(\omega)$ are real positive quantities since

$$A_{-+}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle\left(\int_{-\frac{1}{2}T}^{\frac{1}{2}T}dte^{-i\omega t}Q(t)\right)^\dagger\times\left(\int_{-\frac{1}{2}T}^{\frac{1}{2}T}dte^{-i\omega t}Q(t)\right)\right\rangle$$

and

$$A_{+-}(\omega)=A_{-+}(-\omega).$$

One consequence is

$$a(\omega)=a(-\omega)\geq0.$$

It also follows from

$$\omega_--\omega_+=i(A_{--}+A_{++}-2A_{+-})(\omega)=i(A_{-+}-A_{+-})(\omega)$$

that

$$\gamma(\omega)=A_{-+}(\omega)-A_{+-}(\omega)=-\gamma(-\omega)$$

is real. Furthermore

$$\omega=\omega_0-\tfrac{1}{2}i(A_{++}-A_{--})(\omega),$$

where

$$(A_{++}-A_{--})(t-t')=\langle[Q(t),Q(t')]\rangle_\vartheta\epsilon(t-t')=(A_{-+}-A_{+-})(t-t')\epsilon(t-t'),$$

so that

$$-i(A_{++}-A_{--})(\omega)=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{d\omega'}{\omega-\omega'}\gamma(\omega'),$$

and ω emerges as the real quantity

$$\omega=\omega_0-\frac{1}{\pi}P\int_0^{\infty}\frac{\omega' d\omega'}{\omega'^2-\omega^2}\gamma(\omega').$$

We have not yet made direct reference to the nature of the expectation value for the macroscopic system, which is now taken as the thermal average:

$$\langle X\rangle_\vartheta=C\,\mathrm{tr}e^{-\beta H}X$$
$$C^{-1}=\mathrm{tr}e^{-\beta H},$$

where H is the energy operator of the external system. The implication for the structure of the expectation values is contained in

$$\begin{aligned}\langle Q(t)Q(t')\rangle_\vartheta&=C\,\mathrm{tr}e^{-\beta H}Q(t)Q(t')\\&=\langle Q(t')Q(t+i\beta)\rangle_\vartheta,\end{aligned}$$

which employs the formal property

$$e^{-\beta H}Q(t)e^{\beta H}=Q(t+i\beta).$$

On introducing the time Fourier transforms, however, this becomes the explicit relation

$$A_{-+}(\omega)=e^{\beta\omega}A_{+-}(\omega),$$

and we conclude that

$$\begin{aligned}e^{-\frac12\beta\omega}A_{-+}(\omega)&=e^{\frac12\beta\omega}A_{+-}(\omega)\\&=a(\omega)/2\cosh\tfrac12\beta\omega,\end{aligned}$$

which is a positive even function of ω. As a consequence, we have

$$\begin{aligned}\gamma(\omega)&=a(\omega)\tanh\tfrac12\beta\omega,\\&\geq 0,\quad \beta\omega>0,\end{aligned}$$

which can also be written as

$$a(\omega)=2\gamma(\omega)[(e^{\beta\omega}-1)^{-1}+\tfrac12].$$

The net result of this part of the discussion is to remove all explicit reference to the external system as a dynamical entity. We are given effective equations of motion for y_+ and y_- that contain the prescribed external forces and three parameters, the angular frequency $\omega(\simeq\omega_0)$, γ, and a, the latter pair being related by the temperature of the macroscopic system. The accompanying boundary conditions are

$$(y_--y_+)(t_1)=0$$

and, for the choice of an initial thermal mixture,

$$y_-(t_2)=e^{\beta_0\omega}y_+(t_2).$$

We now find that

$$(y_--y_+)(t)=i\int_{t_2}^{t_1}dt'e^{i\omega_-(t'-t)}\eta_-(t-t')(K_--K_+)(t'),$$

which supplies the initial condition for the second equation of motion,

$$(y_++y_-)(t_2)=\coth(\tfrac12\beta_0\omega)i\int_{t_2}^{t_1}dte^{i\omega_-(t-t_2)}(K_--K_+)(t),$$

and the required solution is given by

$$\begin{aligned}&i(y_++y_-)(t)\\&=\int_{t_2}^{t_1}dt'e^{-i\omega_+(t-t')}\eta_+(t-t')(K_++K_-)(t')\\&\quad-\coth(\tfrac12\beta\omega)\int_{t_2}^{t_1}dt'[e^{-i\omega_+(t-t')}\eta_+(t-t')\\&\quad+e^{i\omega_-(t'-t)}\eta_-(t-t')](K_--K_+)(t')\\&\quad+(\coth\tfrac12\beta\omega-\coth\tfrac12\beta_0\omega)\int_{t_2}^{t_1}dt'e^{-i\omega_+(t-t_2)}\\&\qquad\times e^{i\omega_-(t'-t_2)}(K_--K_+)(t').\end{aligned}$$

The corresponding solutions for $y_\pm^\dagger(t)$ are obtained by interchanging the $\pm$ labels in the formal adjoint equation.

The differential dependence of the transformation function $\langle t_2|t_2\rangle_{\vartheta_0\vartheta}^{K\pm}$ upon the external forces is described by these results, and the explicit formula obtained on integration is

$$\begin{aligned}&\langle t_2|t_2\rangle_{\vartheta_0\vartheta}^{K\pm}\\&\quad=\exp\left[-i\int dtdt'K^*(t)G_{\vartheta_0\vartheta}(t-t_2,t'-t_2)K(t')\right],\end{aligned}$$

where $[n_0=\langle n\rangle_{\vartheta_0},\ n=\langle n\rangle_\vartheta]$

$$\begin{aligned}&iG_{\vartheta_0\vartheta}(t-t_2,t'-t_2)\\&=e^{-i\omega_+(t-t')}\eta_+(t-t')\begin{pmatrix}n+1, & -n\\ -n-1, & n\end{pmatrix}\\&\quad+e^{-i\omega_-(t-t')}\eta_-(t-t')\begin{pmatrix}n, & -n\\ -n-1, & n+1\end{pmatrix}\\&\quad+e^{-i\omega_+(t-t_2)}e^{i\omega_-(t'-t_2)}(n_0-n)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}.\end{aligned}$$

Another way of presenting this result is

$$\begin{aligned}&iG_{\vartheta_0\vartheta}(t-t_2,t'-t_2)\\&=e^{-i\omega(t-t')}e^{-\frac12\gamma|t-t'|}\begin{pmatrix}\eta_+(t-t')+n, & -n\\ -n-1, & \eta_-(t-t')+n\end{pmatrix}\\&\quad+e^{-i\omega(t-t')}e^{-\gamma[\frac12(t+t')-t_2]}(n_0-n)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix},\end{aligned}$$

although the simplest description of G is supplied by the differential equation

$$\left[\left(i\frac{d}{dt}-\omega_+\right)G-\delta(t-t')\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\left(-i\frac{d^T}{dt'}-\omega_-\right)$$
$$=-i\delta(t-t')\gamma\begin{pmatrix}n, & -n\\ -n-1, & n+1\end{pmatrix},$$

(where d^T indicates differentiation to the left) in conjunction with the initial value

$$iG_{\vartheta_0\vartheta}(0,0)=\begin{pmatrix}n_0+\frac{1}{2}, & -n_0\\ -n_0-1, & n_0+\frac{1}{2}\end{pmatrix}$$

and the boundary conditions

$$[i(d/dt)-\omega_+]G=0,\quad t>t'$$
$$[i(d/dt)-\omega_-]G=0,\quad t<t'.$$

A more symmetrical version of this differential equation is given by

$$\left(i\frac{d}{dt}-\omega_+\right)\left(-i\frac{d}{dt'}-\omega_-\right)G$$
$$-\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(i\frac{d}{dt}-\omega\right)\delta(t-t')$$
$$=-i\delta(t-t')\gamma\begin{pmatrix}n+\frac{1}{2}, & -n\\ -n-1, & n+\frac{1}{2}\end{pmatrix}.$$

We note the vanishing sum of all G elements, and that the role of complex conjugation in exchanging the two segments of the closed time path is expressed by

$$-\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}G(t',t)^{T*}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=G(t,t'),$$

which is to say that

$$-G(t',t)_{+-}{}^*=G(t,t')_{+-},\quad -G(t',t)_{-+}{}^*=G(t,t')_{-+}$$
$$-G(t',t)_{--}{}^*=G(t,t')_{++}.$$

It will be observed that only when

$$\langle n\rangle_{\vartheta_0}=\langle n\rangle_\vartheta$$

is $G_{\vartheta_0\vartheta}(t-t_2,\, t'-t_2)$ independent of t_2 and a function of $t-t'$. This clearly refers to the initial physical situation of thermal equilibrium between the oscillator and the external system at the common temperature $\vartheta_0=\vartheta>0$, which equilibrium persists in the absence of external forces. If the initial circumstances do not constitute thermal equilibrium, that will be established in the course of time at the macroscopic temperature $\vartheta>0$. Thus, all reference to the initial oscillator temperature disappears from $G_{\vartheta_0\vartheta}(t-t_2,\, t'-t_2)$ when, for fixed $t-t'$,

$$\gamma[\tfrac{1}{2}(t+t')-t_2]\gg 1.$$

The thermal relaxation of the oscillator energy is derived from

$$\langle t_2|y^\dagger y(t_1)|t_2\rangle_{\vartheta_0\vartheta}=\frac{\delta}{\delta K_-(t_1)}\frac{\delta}{\delta K_+{}^*(t_1)}\langle t_2|t_2\rangle_{\vartheta_0\vartheta}{}^{K_\pm}\Big|_{K_\pm=0}$$
$$=-iG_{\vartheta_0\vartheta}(t_1-t_2,\, t_1-t_2)_{+-},$$

and is expressed by

$$\langle n(t_1)\rangle=\langle n\rangle_\vartheta+(\langle n\rangle_{\vartheta_0}-\langle n\rangle_\vartheta)e^{-\gamma(t_1-t_2)}.$$

The previously employed technique of impulsive forces applied at the time t_1 gives the more general result

$$\langle t_2|\exp[-i(\lambda y^\dagger(t_1)+\mu y(t_1))]|t_2\rangle_{\vartheta_0\vartheta}{}^K$$
$$=\exp\left[-\lambda\mu(\langle n(t_1)\rangle+\tfrac{1}{2})+\lambda\int_{t_2}^{t_1}dt\,e^{i\omega_-(t_1-t)}K^*(t)\right.$$
$$\left.-\mu\int_{t_2}^{t_1}dt\,e^{-i\omega_+(t_1-t)}K(t)\right],$$

from which a variety of probability distributions and expectation values can be obtained.

The latter calculation illustrates a general characteristic of the matrix $G(t,t')$, which is implied by the lack of dependence on the time t_1. Indeed, such a terminal time need not appear explicitly in the structure of the transformation function $\langle t_2|t_2\rangle^{K_\pm}$ and all time integrations can range from t_2 to $+\infty$. Then t_1 is implicit as the time beyond which K_+ and K_- are identified, and the structure of G must be such as to remove any reference to a time greater than t_1. In the present situation, the use of an impulsive force at t_1 produces, for example, the term

$$\int_{t_2}^{\infty}dt\,G(t_1-t_2,\, t-t_2)K(t),$$

in which K_+ and K_- are set equal. Hence it is necessary that

$$G(t,t')\begin{pmatrix}1\\ 1\end{pmatrix}=0,\quad t<t'$$

and similarly that

$$\begin{pmatrix}1 & 1\end{pmatrix}G(t,t')=0,\quad t>t',$$

which says that adding the columns of $G(t,t')$ gives retarded functions of $t-t'$, while the sum of rows supplies a vector that is an advanced function of $t-t'$. In each instance, the two components must have a zero sum. These statements are immediately verified for the explicitly calculated $G_{\vartheta_0\vartheta}(t-t_2,\, t'-t_2)$ and follow more generally from the operator construction

$$iG=\begin{pmatrix}\langle(y(t)y^\dagger(t'))_+\rangle, & -\langle y^\dagger(t')y(t)\rangle\\ -\langle y(t)y^\dagger(t')\rangle, & \langle(y(t)y^\dagger(t'))_-\rangle\end{pmatrix},$$

for, as we have already noted in connection with Q products,

$$(y(t)y^\dagger(t'))_+-y^\dagger(t')y(t)=y(t)y^\dagger(t')-(y(t)y^\dagger(t'))_-$$
$$=\eta_+(t-t')[y(t),y^\dagger(t')]$$

and

$$(y(t)y^\dagger(t'))_+ - y(t)y^\dagger(t') = y^\dagger(t')y(t) - (y(t)y^\dagger(t'))$$
$$= -\eta_-(t-t')[y(t),y^\dagger(t')].$$

Our results show, incidentally, that

$$\langle[y(t),y^\dagger(t')]\rangle_{\vartheta_0\vartheta} = e^{-i\omega_+(t-t')}\eta_+(t-t') + e^{-i\omega_-(t-t')}\eta_-(t-t')$$
$$= e^{-i\omega(t-t')}e^{-\frac{1}{2}\gamma|t-t'|}.$$

Another general property can be illustrated by our calculation, the positiveness of $-iG(t,t')_{+-}$,

$$-\int dtdt' K(t)iG(t-t_2,\, t'-t_2)_{+-}K^*(t')$$
$$= \left\langle t_2\left|\left(\int dtK(t)y(t)\right)^\dagger\left(\int dtK(t)y(t)\right)\right|t_2\right\rangle > 0.$$

We have found that

$$-iG_{\vartheta_0\vartheta}(t-t_2,\, t'-t_2)_{+-}$$
$$= \exp\{-i\omega(t-t') - \gamma[\tfrac{1}{2}(t+t')-t_2]\}\langle n\rangle_\vartheta$$
$$+ e^{-i\omega(t-t')}[e^{-\frac{1}{2}\gamma|t-t'|} - e^{-\gamma[\frac{1}{2}(t+t')-t_2]}]\langle n\rangle_{\vartheta_0},$$

and it is clearly necessary that each term obey separately the positiveness requirement. The first term is trivial,

$$\int dtdt' K(t)\exp\{-i\omega(t-t') - \gamma[\tfrac{1}{2}(t+t')-t_2]\}K^*(t')$$
$$= \left|\int dt e^{-i\omega_+(t-t_2)}K(t)\right|^2 > 0,$$

and the required property of the second term follows from the formula

$$e^{-\frac{1}{2}\gamma|t-t'|} - e^{-\gamma[\frac{1}{2}(t+t')-t_2]}$$
$$= \frac{2\gamma}{\pi}\int^\infty d\omega' \frac{\sin\omega'(t-t_2)\sin\omega'(t'-t_2)}{\omega'^2 + (\frac{1}{2}\gamma)^2}.$$

All the information that has been obtained about the oscillator is displayed on considering the forces

$$K_\pm(t) = \lambda_\pm(t) + K(t), \quad K_\pm^*(t) = \mu_\pm(t) + K^*(t),$$

and making explicit the effects of $\lambda_\pm(t)$, $\mu_\pm(t)$ by equivalent time-ordered operators:

$$\left\langle t_2\left|\left(\exp\left[i\int_{t_2}^\infty dt(\lambda_- y^\dagger + \mu_- y)\right]\right)_-\right.\right.$$
$$\times\left.\left.\left(\exp\left[-i\int_{t_2}^\infty dt(\lambda_+ y^\dagger + \mu_+ y)\right]\right)_+\right|t_2\right\rangle^K_{\vartheta_0\vartheta}$$
$$= \exp\left[-i\int dtdt'\mu(t)G(t-t_2,\, t'-t_2)\lambda(t')\right.$$
$$+ \int dtdt' K^*(t)e^{-i\omega_-(t-t')}\eta_-(t-t')(\lambda_+ - \lambda_-)(t')$$
$$\left. - \int dtdt'(\mu_+ - \mu_-)(t)e^{-i\omega_+(t-t')}\eta_+(t-t')K(t')\right].$$

This is a formula for the direct computation of expectation values of general functions of $y(t)$ and $y^\dagger(t)$. A less explicit but simpler result can also be given by means of expectation values for functions of the operators

$$[i(d/dt) - \omega_+]y(t) - K(t) = K_f(t),$$
$$[-i(d/dt) - \omega_-]y^\dagger(t) - K^*(t) = K_f^\dagger(t).$$

Let us recognize at once that

$$\langle K_f(t)\rangle = 0, \quad \langle K_f^\dagger(t)\rangle = 0,$$

and therefore that the fluctuations of $y(t)$, $y^\dagger(t)$ can be ascribed to the effect of the forces K_f, $K_f^\dagger$, which appear as the quantum analogs of the random forces in the classical Langevin approach to the theory of the Brownian motion. The change in viewpoint is accomplished by introducing

$$\lambda_\pm(t) = [i(d/dt) - \omega_-]u_\pm(t)$$
$$\mu_\pm(t) = [-i(d/dt) - \omega_+]v_\pm(t),$$

where we assume, just for simplicity, that the functions $u(t)$, $v(t)$ vanish at the time boundaries. Then, partial time integrations will replace the operators y, $y^\dagger$ with K_f, $K_f^\dagger$.

To carry this out, however, we need the following lemma on time-ordered products:

$$\left(\exp\left[\int dt(A(t) + (d/dt)B(t))\right]\right)_+$$
$$= \left(\exp\left[\int dtA(t)\right]\right)_+ \exp\left(\int dt[A + \tfrac{1}{2}(dB/dt),\, B]\right),$$

which involves the unessential assumption that $B(t)$ vanishes at the time terminals, and the hypothesis that $[A(t), B(t)]$ and $[dB(t)/dt, B(t)]$ are commutative with all the other operators. The proof is obtained by replacing $B(t)$ with $\lambda B(t)$ and differentiating with respect to λ,

$$\frac{\partial}{\partial\lambda}\left(\exp\left[\int_{t_2}^{t_1} dt\left(A + \lambda\frac{d}{dt}B\right)\right]\right)_+$$
$$= \int_{t_2}^{t_1} dt\left(\exp\left[\int_t^{t_1}\right]\right)_+ \frac{d}{dt}B(t)\left(\exp\left[\int_{t_2}^{t}\right]\right)_+.$$

Then, a partial integration yields

$$\int_{t_2}^{t_1} dt\left(\exp\left[\int_t^{t_1}\right]\right)_+\left[A(t) + \lambda\frac{dB(t)}{dt},\, B(t)\right]$$
$$\times\left(\exp\left[\int_{t_2}^{t}\right]\right)_+$$
$$= \left(\exp\left[\int_{t_2}^{t_1}\right]\right)_+ \int_{t_2}^{t_1} dt\left[A + \lambda\frac{dB}{dt},\, B\right],$$

according to the hypothesis, and the stated result follows on integrating this differential equation.

The structure of the lemma is given by the rearrangement

$$-i(\lambda y^\dagger+\mu y)=-i[u(K^*+K_f^\dagger)+v(K+K_f)] +(d/dt)(uy^\dagger-vy),$$

and we immediately find a commutator that is a multiple of the unit operator,

$$[A+(d/dt)B,B]=-i[\lambda y^\dagger+\mu y,\, uy^\dagger-vy] =-i(\mu u+\lambda v)\to -2iv(i(d/dt)-\omega)u.$$

The last form involves discarding a total time derivative that will not contribute to the final result. To evaluate $[A,B]$ we must refer to the meaning of K_f and $K_f^\dagger$ that is supplied by the actual equations of motion,

$$K_f(t)=Q(t)+(\omega_0-\omega_+)y(t)$$
$$K_f^\dagger(t)=Q(t)+(\omega_0-\omega_-)y^\dagger(t),$$

for then

$$[A(t),B(t)]=-i[u(\omega_0-\omega_-)y^\dagger+v(\omega_0-\omega_+)y,\, uy^\dagger-vy] =2ivu(\omega-\omega_0),$$

which is also proportional to the unit operator. Accordingly,

$$\left(\exp\left[-i\int dt(\lambda y^\dagger+\mu y)\right]\right)_+$$
$$=\left(\exp\left[-i\int dt[u(K^*+K_f^\dagger)+v(K+K_f)]\right]\right)_+ \times\exp\left[i(\omega-\omega_0)\int dtvu-i\int dtv\left(i\frac{d}{dt}-\omega\right)u\right],$$

and complex conjugation yields the analogous result for negatively time-ordered products.

With the aid of the differential equation obeyed by G, we now get

$$\left\langle t_2\middle|\left(\exp\left[i\int dt(uK_f^\dagger+vK_f)\right]\right)_- \times\left(\exp\left[-i\int dt(uK_f^\dagger+vK_f)\right]\right)_+\middle|t_2\right\rangle^K_{\vartheta_0\vartheta} =\exp\left[-i\int_{t_2}^{t_1}dtv(t)\kappa u(t)\right],$$

where

$$\kappa=\gamma\begin{pmatrix} n+\frac{1}{2}, & -n\\ -n-1, & n+\frac{1}{2}\end{pmatrix}+i(\omega-\omega_0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The elements of this matrix are also expressed by

$$\kappa\delta(t-t')=\begin{pmatrix}\langle(K_f(t)K_f^\dagger(t'))_+\rangle, & -\langle K_f^\dagger(t')K_f(t)\rangle\\ -\langle K_f(t)K_f^\dagger(t')\rangle, & \langle(K_f(t)K_f^\dagger(t'))_-\rangle\end{pmatrix}.$$

Such expectation values are to be understood as effective evaluations that serve to describe the properties of the oscillator under the circumstances that validate the various approximations that have been used.

It will be observed that when n is sufficiently large to permit the neglect of all other terms,

$$\kappa\simeq\tfrac{1}{2}a\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\quad[\tfrac{1}{2}a=\gamma(n+\tfrac{1}{2})],$$

and the sense of operator multiplication is no longer significant. This is the classical limit, for which

$$\left\langle\exp\left[-i\int dt(uK_f^\dagger+vK_f)\right]\right\rangle_\vartheta =\exp\left[-\int dt\tfrac{1}{2}av(t)u(t)\right],$$

where we have placed $u_+-u_-=u$, $v_+-v_-=v$. On introducing real components of the random force

$$K_f=2^{-\frac{1}{2}}(K_1+iK_2),\quad K_f^\dagger=2^{-\frac{1}{2}}(K_1-iK_2),$$

the classical limiting result reads

$$\left\langle\exp\left[-i\int dt(u_1K_1+u_2K_2)\right]\right\rangle_\vartheta =\exp\left[-\int dt\tfrac{1}{4}a(u_1^2+u_2^2)\right].$$

The fluctuations at different times are independent. If we consider time-averaged forces,

$$\bar{K}=\frac{1}{\Delta t}\int_t^{t+\Delta t}dt'K(t'),$$

we find by Fourier transformation that

$$\langle\delta(\bar{K}_1-K_1')\delta(\bar{K}_2-K_2')\rangle_\vartheta=\frac{\Delta t}{\pi a}\exp\left[-\frac{\Delta t}{a}(K_1'^2+K_2'^2)\right],$$

which is the Gaussian distribution giving the probability that the force averaged over a time interval Δt will have a value within a small neighborhood of the point K'. In this classical limit the fluctuation constant a is related to the damping or dissipation constant γ and the macroscopic temperature ϑ by

$$a=(2\gamma/\omega)\vartheta.$$

Our simplified equations can also be applied to situations in which the external system is not at thermal equilibrium. To see this possibility let us return to the real positive functions $A_{-+}(\omega)$, $A_{+-}(\omega)$ that describe the external system and remark that, generally,

$$\frac{A_{-+}(-\omega)}{A_{+-}(-\omega)}=\left[\frac{A_{-+}(\omega)}{A_{+-}(\omega)}\right]^{-1}\geq 0.$$

These properties can be expressed by writing

$$A_{-+}(\omega)/A_{+-}(\omega)=e^{\omega\beta(\omega)},$$

where $\beta(\omega)$ is a real even function that can range from $-\infty$ to $+\infty$. When only one value of ω is of interest, all conceivable situations for the external system can be described by the single parameter β, the reciprocal of which appears as an effective temperature of the macroscopic system. A new physical domain that appears in this way is characterized by negative temperature, $\beta<0$. Since a is an intrinsically positive constant, it is γ that will reverse sign

$$-\gamma=a(1-e^{-|\beta|\omega})/(1+e^{-|\beta|\omega})>0,$$

and the effect of the external system on the oscillator changes from damping to amplification.

We shall discuss the following physical sequence. At time t_2 the oscillator, in a thermal mixture of states at temperature ϑ_0, is acted on by external forces which are present for a time, short in comparison with $1/|\gamma|$. After a sufficiently extended interval $\sim(t_1-t_2)$ such that the amplification factor or gain is very large,

$$k=e^{\frac{1}{2}|\gamma|(t_1-t_2)}\gg 1,$$

measurements are made in the neighborhood of time t_1. A prediction of all such measurements is contained in the general expectation value formula. Approximations that convey the physical situation under consideration are given by

$$\int dtdt'(\mu_+-\mu_-)(t)e^{-i\omega_+(t-t')}\eta_+(t-t')K(t')$$
$$\simeq k\int dt(\mu_+-\mu_-)(t)e^{-i\omega t}\int dt'e^{i\omega t'}K(t'),$$

$$\int dtdt'K^*(t)e^{-i\omega_-(t-t')}\eta_-(t-t')(\lambda_+-\lambda_-)(t')$$
$$\simeq k\int dtK^*(t)e^{-i\omega t}\int dt'e^{i\omega t'}(\lambda_+-\lambda_-)(t'),$$

and

$$i\int dtdt'\mu(t)G(t-t_2,t'-t_2)\lambda(t')$$
$$\simeq k^2(\langle n\rangle_{\vartheta_0}+(1-e^{-|\beta|\omega})^{-1})\int dt(\mu_+-\mu_-)(t)e^{-i\omega t}$$
$$\times\int dt'(\lambda_+-\lambda_-)(t')e^{i\omega t'}.$$

From the appearance of the combinations $\mu_+-\mu_-=\mu$, $\lambda_+-\lambda_-=\lambda$ only, we recognize that noncommutativity of operator multiplication is no longer significant, and thus the motion of the oscillator has been amplified to the classical level. To express the consequences most simply, we write

$$y(t)=ke^{-i\omega t}(y_s+y_n)$$
$$y^\dagger(t)=ke^{i\omega t}(y_s^*+y_n^*),$$

with

$$y_s=-i\int_{t_2}^{\infty}dt'e^{i\omega t'}K(t'),$$

and, on defining

$$u=k\int dte^{i\omega t}\lambda(t),\quad v=k\int dte^{-i\omega t}\mu(t),$$

we obtain the time-independent result

$$\langle\exp[-i(uy_n^*+vy_n)]\rangle$$
$$=\exp[-(\langle n\rangle_{\vartheta_0}+(1-e^{-|\beta|\omega})^{-1})vu],$$

which implies that

$$\langle y_n\rangle=\langle y_n^*\rangle=0$$
$$\langle|y_n|^2\rangle=\langle n\rangle_{\vartheta_0}+(1-e^{-|\beta|\omega})^{-1}\geq\langle n\rangle_{\vartheta_0}+1.$$

Thus, the oscillator coordinate $y(t)$ is the amplified superposition of two harmonic terms, one of definite amplitude and phase (signal), the other with random amplitude and phase (noise), governed by a two-dimensional Gaussian probability distribution.

These considerations with regard to amplification can be viewed as a primitive model of a maser device,[5] with the oscillator corresponding to a single mode of a resonant electromagnetic cavity, and the external system to an atomic ensemble wherein, for a selected pair of levels, the thermal population inequality is reversed by some means such as physical separation or electromagnetic pumping.

AN IMPROVED TREATMENT

In this section we seek to remove some of the limitations of the preceding discussion. To aid in dealing successfully with the nonlocal time behavior of the oscillator, it is convenient to replace the non-Hermitian operator description with one employing Hermitian operators. Accordingly, we begin the development again, now using the Lagrangian operator

$$L=p(dq/dt)-\tfrac{1}{2}(p^2+\omega_0^2q^2)+qF(t)+qQ+L_{\text{ext}},$$

where Q has altered its meaning by a constant factor. One could also include an external prescribed force that is coupled to p. We repeat the previous approximate construction of the transformation function $\langle t_2|t_2\rangle_{\vartheta_0\vartheta}{}^{F\pm}$ which proceeds by the introduction of an effective action operator that retains only the simplest correlation aspects of the external system, as comprised in

$$A(t-t')=\begin{pmatrix}\langle(Q(t)Q(t'))_+\rangle_\vartheta & \langle Q(t')Q(t)\rangle_\vartheta\\ \langle Q(t)Q(t')\rangle_\vartheta & \langle(Q(t)Q(t'))_-\rangle_\vartheta\end{pmatrix}.$$

[5] A similar model has been discussed recently by R. Serber and C. H. Townes, *Symposium on Quantum Electronics* (Columbia University Press, New York, 1960).

The action operator, with no other approximations, is

$$W=\int_{t_2}^{t_1}dt\left[p\frac{dq}{dt}-\tfrac{1}{2}(p^2+\omega_0{}^2q^2)+qF(t)|_+-|_-\right]$$
$$+\tfrac{1}{2}i\int_{t_2}^{t_1}dtdt'[(q(t)q(t'))_+A_{++}(t-t')$$
$$-2q_-(t)q_+(t')A_{-+}(t-t')+(q(t)q(t'))_-A_{--}(t-t')],$$

and the implied equations of motion, presented as second-order differential equations after eliminating

$$p=dq/dt,$$

are

$$\left(\frac{d^2}{dt^2}+\omega_0{}^2\right)q_+(t)$$
$$-i\int_{t_2}^{t_1}dt'[A_{++}(t-t')q_+(t')-A_{+-}(t-t')q_-(t')]=F_+(t)$$

and

$$\left(\frac{d^2}{dt'}+\omega_0{}^2\right)q_-(t)$$
$$+i\int_{t_2}^{t_1}dt[A_{--}(t-t')q_-(t')-A_{-+}(t-t')q_+(t')]=F_-(t).$$

It will be seen that the adjoint operation is equivalent to the interchange of the $\pm$labels.

We define

$$-iA_r(t-t')=\langle[Q(t),Q(t')]\rangle_\beta\eta_+(t-t')$$
$$=A_{++}-A_{+-}=A_{-+}-A_{--}$$

and

$$-iA_a(t-t')=-\langle[Q(t),Q(t')]\rangle_\beta\eta_-(t-t')$$
$$=A_{++}-A_{-+}=A_{+-}-A_{--},$$

together with

$$a(t-t')=\langle\{Q(t),Q(t')\}\rangle_\beta$$
$$=A_{+-}+A_{-+},$$

which enables us to present the integro-differential equations as

$$\left(\frac{d^2}{dt^2}+\omega_0{}^2\right)(q_--q_+)(t)-\int_{t_2}^{t_1}dt'A_a(t-t')(q_--q_+)(t')$$
$$=(F_--F_+)(t)$$

and

$$\left(\frac{d^2}{dt^2}+\omega_0{}^2\right)(q_++q_-)(t)-\int_{t_2}^{t_1}dt'A_r(t-t')(q_++q_-)(t')$$
$$+i\int_{t_2}^{t_1}dt'a(t-t')(q_--q_+)(t')=(F_++F_-)(t).$$

The accompanying boundary conditions are

$$(q_--q_+)(t_1)=0,\quad (d/dt)(q_--q_+)(t_1)=0$$

and

$$q_-(t_2)=q_+(t_2)\cosh\beta_0\omega_0+\frac{i}{\omega_0}\frac{d}{dt}q_+(t_2)\sinh\beta_0\omega_0$$
$$\frac{d}{dt}q_-(t_2)=-i\omega_0q_+(t_2)\sinh\beta_0\omega_0+\frac{d}{dt}q_+(t_2)\cosh\beta_0\omega_0,$$

or, more conveniently expressed,

$$(q_++q_-)(t_2)=\frac{i}{\omega_0}\coth(\tfrac{1}{2}\beta_0\omega_0)\frac{d}{dt}(q_--q_+)(t_2)$$
$$\frac{d}{dt}(q_++q_-)(t_2)=-i\omega_0\coth(\tfrac{1}{2}\beta_0\omega_0)(q_--q_+)(t_2),$$

which replace the non-Hermitian relations

$$y_-(t_2)=e^{\beta_0\omega_0}y_+(t_2),\quad y_-{}^\dagger(t_2)=e^{-\beta_0\omega_0}y_+{}^\dagger(t_2).$$

Note that it is the intrinsic oscillator frequency ω_0 that appears here since the initial condition refers to a thermal mixture of unperturbed oscillator states.

The required solution of the equation for q_--q_+ can be written as

$$(q_--q_+)(t)=\int_{-\infty}^{\infty}dt'G_a(t-t')(F_--F_+)(t'),$$

where $G_a(t-t')$ is the real Green's function defined by

$$\left(\frac{d^2}{dt^2}+\omega_0{}^2\right)G_a(t-t')-\int_{-\infty}^{\infty}d\tau A_a(t-\tau)G_a(\tau-t')=\delta(t-t')$$

and

$$G_a(t-t')=0,\quad t>t'.$$

Implicit is the time t_1 as one beyond which F_--F_+ equals zero. The initial conditions for the second equation, which this solution supplies, are

$$(q_++q_-)(t_2)=\frac{i}{\omega_0}\coth(\tfrac{1}{2}\beta_0\omega_0)\int_{t_2}^{\infty}dt'\frac{\partial}{\partial t_2}$$
$$\times G_a(t_2-t')(F_--F_+)(t')$$

and

$$\frac{d}{dt}(q_++q_-)(t_2)$$
$$=-i\omega_0\coth(\tfrac{1}{2}\beta_0\omega_0)\int_{t_2}^{\infty}dt'G_a(t_2-t')(F_--F_+)(t').$$

The Green's function that is appropriate for the equation obeyed by q_++q_- is defined by

$$\left(\frac{d^2}{dt^2}+\omega_0{}^2\right)G_r(t-t')-\int_{-\infty}^{\infty}d\tau A_r(t-\tau)G_r(\tau-t')=\delta(t-t'),$$
$$G_r(t-t')=0,\quad t<t',$$

and the two real functions are related by

$$G_a(t-t')=G_r(t'-t).$$

The desired solution of the second differential equation is

$$(q_++q_-)(t)=\int_{t_2}^{\infty}dt'G_r(t-t')(F_++F_-)(t')$$
$$-i\int_{t_2}^{\infty}dt'w(t-t_2,\,t'-t_2)(F_--F_+)(t'),$$

where

$$w(t-t_2,\,t'-t_2)$$
$$=\int_{t_2}^{\infty}d\tau d\tau'G_r(t-\tau)a(\tau-\tau')G_a(\tau'-t')$$
$$+\frac{1}{\omega_0}\coth(\tfrac{1}{2}\beta_0\omega_0)\left[\frac{\partial}{\partial t_2}G_r(t-t_2)\frac{\partial}{\partial t_2}G_a(t_2-t')\right.$$
$$\left.+\omega_0{}^2G_r(t-t_2)G_a(t_2-t')\right]$$

is a real symmetrical function of its two arguments.

The differential description of the transformation function that these solutions imply is indicated by

$$\delta_{F\pm}\langle t_2|t_2\rangle^{F\pm}=i\left\langle t_2\left|\int dt(\delta F_+q_+-\delta F_-q_-)\right|t_2\right\rangle$$
$$=-\tfrac{1}{2}i\left\langle t_2\left|\int dt[\delta(F_--F_+)(q_++q_-)\right.\right.$$
$$\left.\left.+\delta(F_++F_-)(q_--q_+)]\right|t_2\right\rangle,$$

and the result of integration is

$$\langle t_2|t_2\rangle_{\vartheta_0\vartheta}{}^{F\pm}$$
$$=\exp\left\{-\tfrac{1}{2}i\int dtdt'(F_--F_+)(t)G_r(t-t')(F_++F_-)(t')\right.$$
$$\left.-\tfrac{1}{4}\int dtdt'(F_--F_+)(t)w(t-t_2,\,t'-t_2)(F_--F_+)(t')\right\}.$$

This can also be displayed in the matrix form

$$\langle t_2|t_2\rangle_{\vartheta_0\vartheta}{}^{F\pm}=\exp\left\{\tfrac{1}{2}i\int dtdt'F(t)G_{\vartheta_0\vartheta}(t-t_2,\,t'-t_2)F(t')\right\},$$

with

$$G_{\vartheta_0\vartheta}(t-t_2,\,t'-t_2)$$
$$=\tfrac{1}{2}G_r(t-t')\begin{pmatrix}1&1\\-1&-1\end{pmatrix}+\tfrac{1}{2}G_a(t-t')\begin{pmatrix}1&-1\\1&-1\end{pmatrix}$$
$$+\tfrac{1}{2}iw(t-t_2,\,t'-t_2)\begin{pmatrix}1&-1\\-1&1\end{pmatrix}.$$

The latter obeys

$$G(t',t)^T=G(t,t')$$
$$-\begin{pmatrix}0&1\\1&0\end{pmatrix}G(t,t')^*\begin{pmatrix}0&1\\1&0\end{pmatrix}=G(t,t'),$$

and its elements are given by

$$G=i\begin{pmatrix}\langle(q(t)q(t'))_+\rangle_{\vartheta_0\vartheta}, & -\langle q(t')q(t)\rangle_{\vartheta_0\vartheta}\\ -\langle q(t)q(t')\rangle_{\vartheta_0\vartheta}, & \langle(q(t)q(t'))_-\rangle_{\vartheta_0\vartheta}\end{pmatrix}.$$

We note the identifications

$$G_r(t-t')=i\langle[q(t),q(t')]\rangle\eta_+(t-t')$$
$$G_a(t-t')=-i\langle[q(t),q(t')]\rangle\eta_-(t-t')$$
$$w(t-t_2,\,t'-t_2)=\langle\{q(t),q(t')\}\rangle.$$

It is also seen that the sum of the columns of G is proportional to $G_r(t-t')$, while the sum of the rows contains only $G_a(t-t')$.

We shall suppose that $G_r(t-t')$ can have no more than exponential growth, $\sim e^{\alpha(t-t')}$, as $t-t'\to\infty$. Then the complex Fourier transform

$$G(\zeta)=\int_{-\infty}^{\infty}d(t-t')e^{i\zeta(t-t')}G_r(t-t')$$

exists in the upper half-plane

$$\mathrm{Im}\zeta>\alpha$$

and is given explicitly by

$$G(\zeta)=[\omega_0{}^2-\zeta^2-A(\zeta)]^{-1}.$$

Here

$$A(\zeta)=\int_{-\infty}^{\infty}d(t-t')e^{i\zeta(t-t')}A_r(t-t')$$
$$=i\int_0^{\infty}d\tau e^{i\zeta\tau}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}(A_{-+}-A_{+-})(\omega)$$
$$=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{(A_{-+}-A_{+-})(\omega)}{\omega-\zeta}$$

or, since $(A_{-+}-A_{+-})(\omega)$ is an odd function of ω,

$$A(\zeta)=\int_0^{\infty}\frac{d\omega}{\pi}\frac{\omega(A_{-+}-A_{+-})(\omega)}{\omega^2-\zeta^2}.$$

We have already remarked on the generality of the representation

$$A_{-+}(\omega)/A_{+-}(\omega)=e^{\omega\beta(\omega)},\quad \beta(-\omega)=\beta(\omega),$$

and thus we shall write

$$(A_{-+}-A_{+-})(\omega)=a(\omega)\tanh[\tfrac{1}{2}\omega\beta(\omega)]$$
$$(A_{-+}+A_{+-})(\omega)=a(\omega)=a(-\omega)\geq 0,$$

which gives

$$G(\zeta)^{-1}=\omega_0^2-\zeta^2-\int_0^\infty \frac{d\omega}{\pi}\frac{\omega a(\omega)\tanh[\frac{1}{2}\omega\beta(\omega)]}{\omega^2-\zeta^2}.$$

Since this is an even function of ζ, it also represents the Fourier transform of G_a in the lower half-plane $\mathrm{Im}\zeta<-\alpha$.

If the effective temperature is positive and finite at all frequencies, $\beta(\omega)>0$, $G(\zeta)$ can have no complex poles as a function of the variable ζ^2. A complex pole at $\zeta^2=x+iy$, $y\neq 0$, is a zero of $G(\zeta)^{-1}$ and requires that

$$y\left[1+\int_0^\infty \frac{d\omega}{\pi}\frac{\omega a(\omega)\tanh[\frac{1}{2}\omega\beta(\omega)]}{(\omega^2-x)^2+y^2}\right]=0,$$

which is impossible since the quantity in brackets exceeds unity. On letting y approach zero, we see that a pole of $G(\zeta)$ can occur at a point $x=\omega'^2>0$ only if $a(\omega')=0$. If the external system responds through the oscillator coupling to any impressed frequency, $a(\omega)>0$ for all ω and no pole can appear on the positive real axis of ζ^2. As to the negative real axis, $G(\zeta)^{-1}$ is a monotonically decreasing function of $\zeta^2=x$ that begins at $+\infty$ for $x=-\infty$ and will therefore have no zero on the negative real axis if it is still positive at $x=0$. The corresponding condition is

$$\omega_0^2>\int_0^\infty \frac{d\omega}{\pi}a(\omega)\frac{\tanh[\frac{1}{2}\omega\beta(\omega)]}{\omega}.$$

Under these circumstances $\alpha=0$, for $G(\zeta)$, *qua* function of ζ^2, has no singularity other than the branch line on the positive real axis, and the ζ singularities are therefore confined entirely to the real axis. This is indicated by

$$G(\zeta)=\int_0^\infty d\omega^2\frac{B(\omega^2)}{\omega^2-\zeta^2}=\int_{-\infty}^\infty d\omega\epsilon(\omega)\frac{B(\omega^2)}{\omega-\zeta},$$

and $B(\omega^2)$ is the positive quantity

$$B(\omega^2)=\frac{(2\pi)^{-1}a(\omega)\tanh[\frac{1}{2}|\omega|\beta(\omega)]}{\left[\omega_0^2-\omega^2-P\int_0^\infty \frac{d\omega'^2}{2\pi}\frac{\tanh(\frac{1}{2}\omega'\beta(\omega'))}{\omega'^2-\omega^2}a(\omega')\right]^2+[\frac{1}{2}a(\omega)\tanh\frac{1}{2}\omega\beta(\omega)]^2}.$$

Some integral relations are easily obtained by comparison of asymptotic forms. Thus

$$\int_0^\infty d\omega^2 B(\omega^2)=1,$$

$$\int_0^\infty d\omega^2\omega^2 B(\omega^2)=\omega_0^2,$$

and

$$\int_0^\infty d\omega^2\omega^4 B(\omega^2)=\omega_0^4+\int_0^\infty \frac{d\omega^2}{2\pi}a(\omega)\tanh[\tfrac{1}{2}\omega\beta(\omega)]=\omega_0^4+\langle[i\dot{Q},Q]\rangle_\vartheta,$$

while setting $\zeta=0$ yields

$$\int_0^\infty d\omega^2\frac{B(\omega^2)}{\omega^2}=\left[\omega_0^2-\int_0^\infty \frac{d\omega}{\pi}a(\omega)\frac{\tanh\frac{1}{2}\omega\beta(\omega)}{\omega}\right]^{-1}$$

The Green's functions are recovered on using the inverse Fourier transformation

$$G(t-t')=\int_{-\infty}^\infty \frac{d\zeta}{2\pi}e^{-i\zeta(t-t')}G(\zeta),$$

where the path of integration is drawn in the half-plane of regularity. Accordingly,

$$G_r(t-t')=\int_0^\infty d\omega^2 B(\omega^2)\frac{\sin\omega(t-t')}{\omega}\eta_+(t-t')$$

and

$$G_a(t-t')=-\int_0^\infty d\omega^2 B(\omega^2)\frac{\sin\omega(t-t')}{\omega}\eta_-(t-t').$$

The integral relations mentioned previously can be expressed in terms of these Green's functions. Thus,

$$\int_0^\infty d\tau G_r(\tau)=\left[\omega_0^2-\int_0^\infty \frac{d\omega}{\pi}a\frac{\tanh(\frac{1}{2}\omega\beta)}{\omega}\right]^{-1},$$

while, in the limit of small positive τ,

$$G_r(\tau)-(1/\omega_0)\sin\omega_0\tau\sim(\tau^5/5!)\langle[i\dot{Q},Q]\rangle_\vartheta,$$

which indicates the initial effect of the coupling to the external system.

The function $B(\omega^2)$ is bounded, and the Green's functions must therefore approach zero as $|t-t'|\to\infty$. Accordingly, all reference to the initial oscillator condition and to the time t_2 must eventually disappear. For sufficiently large $t-t_2$, $t'-t_2$, the function $w(t-t_2,$

$t'-t_2$) reduces to

$$w(t-t')=\int_{-\infty}^{\infty}d\tau d\tau' G_r(t-\tau)a(\tau-\tau')G_a(\tau'-t')$$

$$=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}G(\omega+i\epsilon)a(\omega)G(\omega-i\epsilon)\,|_{\epsilon\to 0}.$$

But

$$(1/2\pi)a(\omega)\,|G(\omega+i\epsilon)|^2=B(\omega^2)\coth[\tfrac{1}{2}|\omega|\beta(\omega)],$$

and, therefore,

$$w(t-t')=\int_0^{\infty}d\omega^2 B(\omega^2)\coth[\tfrac{1}{2}\omega\beta(\omega)]\frac{1}{\omega}\cos\omega(t-t').$$

The corresponding asymptotic form of the matrix $G(t-t_2, t'-t_2)$ is given by

$$G(t-t')=i\int_{-\infty}^{\infty}d\omega B(\omega^2)e^{-i\omega(t-t')}\begin{pmatrix}\eta_+(\omega)\eta_+(t-t')+\eta_-(\omega)\eta_-(t-t')+n, & -\eta_-(\omega)-n\\ -\eta_+(\omega)-n, & \eta_+(\omega)\eta_-(t-t')+\eta_-(\omega)\eta_+(t-t')+n\end{pmatrix},$$

with

$$n(\omega)=(e^{|\omega|\beta(\omega)}-1)^{-1},$$

which describes the oscillator in equilibrium at each frequency with the external system. When the temperature is frequency independent, this is thermal equilibrium. Note also that at zero temperature $n(\omega)=0$, and $G(t-t')_{++}$ is characterized by the temporal outgoing wave boundary condition—positive (negative) frequencies for positive (negative) time difference. The situation is similar for $G(t-t')_{--}$ as a function of $t'-t$.

It can no longer be maintained that placing $\beta_0=\beta$ removes all reference to the initial time. An interval must elapse before thermal equilibrium is established at the common temperature. This can be seen by evaluating the t_2 derivative of $w(t-t_2, t'-t_2)$:

$$\frac{\partial}{\partial t_2}w=-G_r(t-t_2)\int_{t_2}^{\infty}d\tau' a(t_2-\tau')G_a(\tau'-t')$$

$$-\int_{t_2}^{\infty}d\tau G_r(t-\tau)a(\tau-t_2)G_a(t_2-t')$$

$$+\frac{1}{\omega_0}\coth(\tfrac{1}{2}\omega_0\beta_0)$$

$$\times\left\{\frac{\partial}{\partial t_2}G_r(t-t_2)\int_{-\infty}^{\infty}d\tau' A_a(t_2-\tau')G_a(\tau'-t')\right.$$

$$\left.+\int_{-\infty}^{\infty}d\tau A_r(t-\tau)G_r(\tau-t_2)\frac{\partial}{\partial t_2}G_a(t_2-t')\right\},$$

for if this is to vanish, the integrals involving G_r, say, must be expressible as linear combinations of $G_r(t-t_2)$ and its time derivative, which returns us to the approximate treatment of the preceding section, including the approximate identification of ω_0 with the effective oscillator frequency. Hence $\vartheta_0=\vartheta$ does not represent the initial condition of thermal equilibrium between oscillator and external system. While it is perfectly clear that the latter situation is described by the matrix $G_\vartheta(t-t')$, a derivation that employs thermal equilibrium as an initial condition would be desirable.

The required derivation is produced by the device of computing the trace of the transformation function $\langle t_2'|t_2\rangle^{F\pm}$, in which the return path terminates at the different time $t_2'=t_2-T$, and the external force $F_-(t)$ is zero in the interval between t_2 and t_2'. The particular significance of the trace appears on varying the parameter λ that measures the coupling between oscillator and external system:

$$\frac{\partial}{\partial\lambda}\langle t_2'|t_2\rangle^{F\pm}=i\left\langle t_2'\left|\left[\int_{t_2}^{t_1}dtq_+Q_+(t)-\int_{t_2'}^{t_1}dtq_-Q_-(t)+G_\lambda(t_2')-G_\lambda(t_2)\right]\right|t_2\right\rangle^{F\pm}.$$

The operators G_λ are needed to generate infinitesimal transformations of the individual states at the corresponding times, if these states are defined by physical quantities that depend upon λ, such as the total energy. There is no analogous contribution to the trace, however, for the trace is independent of the representation, which is understood to be defined similarly at t_2 and t_2', and one could use a complete set that does not refer to λ. More generally, we observe that $G_\lambda(t_2')$ bears the same relation to the $\langle t_2'|$ states as does $G_\lambda(t_2)$ to the states at time t_2, and therefore

$$\mathrm{tr}\langle t_2'|G_\lambda(t_2')|t_2\rangle-\mathrm{tr}\langle t_2'|G_\lambda(t_2)|t_2\rangle=0.$$

Accordingly, the construction of an effective action operator can proceed as before, with appropriately modified ranges of time integration, and, for the external system, with

$$\langle Q(t)Q(t')\rangle=\frac{\mathrm{tr}\langle t_2'|Q(t)Q(t')|t_2\rangle}{\mathrm{tr}\langle t_2'|t_2\rangle}.$$

This trace structure implies that

$$\langle Q(t)Q(t_2)\rangle=\langle Q(t_2')Q(t)\rangle$$

or, since these correlation functions depend only on time differences, that

$$A_{-+}(t-t_2)=A_{+-}(t-t_2'),$$

which is also expressed by

$$A_{-+}(\omega)=e^{-i\omega T}A_{+-}(\omega).$$

The equations of motion for $t>t_2$ are given by

$$\left(\frac{d^2}{dt^2}+\omega_0^2\right)(q_--q_+)(t)-\int_{-\infty}^{\infty}dt'A_a(t-t')(q_--q_+)(t')=(F_--F_+)(t)$$

and

$$\left(\frac{d^2}{dt^2}+\omega_0^2\right)(q_++q_-)(t)-\int_{t_2}^{\infty}dt'A_r(t-t')(q_++q_-)(t')+i\int_{t_2}^{\infty}dt'a(t-t')(q_--q_+)(t')=(F_++F_-)(t)-2i\int_{t_2'}^{t_2}dt'A_{+-}(t-t')q_-(t').$$

These are supplemented by the equation for $q_-(t)$ in the interval from t_2' to t_2:

$$\left(\frac{d^2}{dt^2}+\omega_0^2\right)q_-(t)+i\int_{t_2'}^{t_2}dt'A_{--}(t-t')q_-(t')=-i\int_{t_2}^{\infty}dt'A_{-+}(t-t')(q_--q_+)(t'),$$

and the effective boundary condition

$$q_-(t_2')=q_+(t_2).$$

The equation for q_--q_+ is solved as before,

$$(q_--q_+)(t)=\int_{-\infty}^{\infty}dt'G_a(t-t')(F_--F_+)(t'),$$

whereas

$$\begin{aligned}(q_++q_-)(t)&=\int_{-\infty}^{\infty}dt'G_r(t-t')(F_++F_-)(t')\\&-i\int_{t_2}^{\infty}d\tau G_r(t-\tau)\int_{t_2}^{\infty}dt'a(\tau-t')(q_--q_+)(t')\\&-2i\int_{t_2}^{\infty}d\tau G_r(t-\tau)\int_{t_2-T}^{t_2}dt'A_{+-}(\tau-t')q_-(t')\\&+G_r(t-t_2)\frac{\partial}{\partial t_2}(q_++q_-)(t_2)\\&-\frac{\partial}{\partial t_2}G_r(t-t_2)(q_++q_-)(t_2),\end{aligned}$$

which has been written for external forces that are zero until the moment t_2 has passed.

Perhaps the simplest procedure at this point is to ask for the dependence of the latter solution upon t_2, for fixed T. We find that

$$\begin{aligned}\frac{\partial}{\partial t_2}(q_++q_-)(t)=&-\int_{t_2}^{\infty}dt'G_r(t-t')A_r(t'-t_2)(q_++q_-)(t_2)\\&+i\int_{t_2}^{\infty}dt'G_r(t-t')a(t'-t_2)(q_--q_+)(t_2)\\&-2i\int_{t_2}^{\infty}dt'G_r(t-t')[A_{+-}(t'-t_2)q_-(t_2)\\&\qquad-A_{-+}(t'-t_2)q_+(t_2)],\end{aligned}$$

on using the relations

$$\int_{-\infty}^{\infty}d\tau A_r(t-\tau)G_r(\tau-t')=\int_{-\infty}^{\infty}d\tau G_r(t-\tau)A_r(\tau-t'),$$

$$A_{+-}(t-t_2')q_-(t_2')=A_{-+}(t-t_2)q_+(t_2).$$

Therefore,

$$(\partial/\partial t_2)(q_++q_-)(t)=0,$$

since, with positive time argument,

$$a-iA_r=2A_{-+}$$
$$a+iA_r=2A_{+-}.$$

The utility of this result depends upon the approach of the Green's functions to zero with increasing magnitude of the time argument, which is assured, after making the substitution $T\rightarrow i\beta$, under the circumstances we have indicated. Then we can let $t_2\rightarrow-\infty$ and obtain

$$(q_++q_-)(t)=\int_{-\infty}^{\infty}dt'G_r(t-t')(F_++F_-)(t')-i\int_{-\infty}^{\infty}dt'w(t-t')(F_--F_+)(t')$$

with

$$w(t-t')=\int_{-\infty}^{\infty}d\tau d\tau'G_r(t-\tau)a(\tau-\tau')G_a(\tau'-t'),$$

as anticipated.

Our results determine the trace ratio

$$\frac{\mathrm{tr}\langle t_2'|t_2\rangle^{F\pm}}{\mathrm{tr}\langle t_2'|t_2\rangle}=\frac{\mathrm{tr}\langle t_2|e^{iTH}|t_2\rangle^{F\pm}}{\mathrm{tr}e^{iTH}},$$

where H is the Hamiltonian operator of the complete system, and the substitution $T\rightarrow i\beta$ yields the transformation function

$$\langle t_2|t_2\rangle_\beta{}^{F\pm}=\exp\left[\tfrac{1}{2}i\int dtdt'F(t)G_\beta(t-t')F(t')\right]$$

with

$$G_\vartheta(t-t')=\tfrac{1}{2}G_r(t-t')\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}+\tfrac{1}{2}G_a(t-t')\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}+\tfrac{1}{2}iw(t-t')\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$$

and

$$w(t-t')=\int_{-\infty}^{\infty}d\omega B(\omega^2)\coth(\tfrac{1}{2}|\omega|\beta)e^{-i\omega(t-t')}.$$

We can also write

$$w(t-t')=\int_{-\infty}^{\infty}d\tau C(t-\tau)(G_a-G_r)(\tau-t'),$$

where

$$C(t-t')=\frac{i}{2\pi}P\int_{-\infty}^{\infty}d\omega\coth(\tfrac{1}{2}\omega\beta)e^{-i\omega(t-t')}$$
$$=\frac{1}{\beta}\coth\left[\frac{\pi}{\beta}(t-t')\right].$$

What is asserted here about expectation values in the presence of an external field $F(t)$ becomes explicit on writing

$$F_\pm(t)=f_\pm(t)+F(t)$$

and indicating the effect of $f_\pm(t)$ by equivalent time-ordered operators,

$$\left\langle\left(\exp\left[-i\int dtf_-(t)q(t)\right]\right)_-\times\left(\exp\left[i\int dtf_+(t)q(t)\right]\right)_+\right\rangle_\vartheta^F$$
$$=\exp\left\{\tfrac{1}{2}i\int dtdt'f(t)G_\vartheta(t-t')f(t')+i\int dtdt'(f_+-f_-)(t)G_r(t-t')F(t')\right\}.$$

Thus

$$\langle q(t)\rangle_\vartheta{}^F=\int_{-\infty}^{\infty}dt'G_r(t-t')F(t')$$

and the properties of $q-\langle q\rangle_\vartheta{}^F$, which are independent of F, are given by setting $F=0$ in the general result. In particular, we recover the matrix identity

$$G_\vartheta(t-t')=i\begin{pmatrix}\langle(q(t)q(t'))_+\rangle_\vartheta, & -\langle q(t')q(t)\rangle_\vartheta\\ \langle q(t)q(t')\rangle_\vartheta, & \langle(q(t)q(t'))_-\rangle_\vartheta\end{pmatrix}.$$

The relation between w and G_a-G_r can then be displayed as a connection between symmetrical product and commutator expectation values

$$\langle\{q(t),q(t')\}\rangle_\vartheta=\int_{-\infty}^{\infty}d\tau C(t-\tau)\left\langle\frac{1}{i}[q(\tau),q(t')]\right\rangle_\vartheta$$

In addition to the trace ratio, which determines the thermal average transformation function $\langle t_2|t_2\rangle_\vartheta{}^{F\pm}$ with its attendant physical information, it is possible to compute the trace

$$\mathrm{tr}\langle t_2'|t_2\rangle=\mathrm{tr}e^{iTH}\longrightarrow\mathrm{tr}e^{-\beta H}$$

which describes the complete energy spectrum and thereby the thermostatic properties of the oscillator in equilibrium with the external system. For this purpose we set $F_\pm=0$ for $t>t_2$ and apply an arbitrary external force $F_-(t)$ in the interval from t_2' to t_2. Moreover, the coupling term between oscillator and external system in the effective action operator is supplied with the variable factor λ (formerly λ^2). Then we have

$$\frac{\partial}{\partial\lambda}\mathrm{tr}\langle t_2'|t_2\rangle^{F-}$$
$$=-\tfrac{1}{2}\,\mathrm{tr}\left\langle t_2'\left|\int_{t_2'}^{t_2}dtdt'A_{--}(t-t')(q(t)q(t'))_-\right|t_2\right\rangle^{F-}$$
$$=-\tfrac{1}{2}i\int_{t_2'}^{t_2}dtdt'A_{--}(t-t')\frac{\delta}{\delta F_-(t')}\mathrm{tr}\langle t_2'|q_-(t)|t_2\rangle^{F-},$$

where $q_-(t)$ obeys the equation of motion

$$\left(\frac{d^2}{dt^2}+\omega_0{}^2\right)q_-(t)+i\lambda\int_{t_2'}^{t_2}dt'A_{--}(t-t')q_-(t')=F_-(t)$$

with the accompanying boundary condition

$$q_-(t_2')=q_+(t_2)=q_-(t_2),$$

which is a statement of periodicity for the interval $T=t_2'-t_2$. The solution of this equation is

$$q_-(t)=\int_{t_2'}^{t_2}dt'G(t-t')F_-(t'),$$

where the Green's function obeys

$$\left(\frac{d^2}{dt^2}+\omega_0{}^2\right)G(t-t')+i\lambda\int_{t_2'}^{t_2}d\tau A_{--}(t-\tau)G(\tau-t')=\delta(t-t')$$

and the requirement of periodicity. We can now place $F_-=0$ in the differential equation for the trace, and obtain

$$\frac{\partial}{\partial\lambda}\log\mathrm{tr}\langle t_2'|t_2\rangle=-\tfrac{1}{2}i\int_{t_2'}^{t_2}dtdt'A_{--}(t-t')G(t-t').$$

The periodic Green's function is given by the Fourier series

$$G(t-t')=\frac{1}{T}\sum_{n=-\infty}^{\infty}\exp\left[-\frac{2\pi in}{T}(t-t')\right]G(n)$$

with

$$G(n)=\left[\omega_0{}^2-\left(\frac{2\pi n}{T}\right)^2-\lambda A(n)\right]^{-1}=G(-n)$$

and

$$A(n)=-\int_{t_2'}^{t_2}dt\exp\left[\frac{2\pi in}{T}(t-t')\right]iA_{--}(t-t')$$

$$=\int_0^\infty\frac{\omega d\omega}{\pi}\frac{(A_{-+}-A_{+-})(\omega)}{\omega^2-(2\pi n/T)^2},$$

where, it is to be recalled,

$$A_{-+}(\omega)=e^{-i\omega T}A_{+-}(\omega),$$

so that the integrand has no singularities at $\omega T=2\pi|n|$. Now we have

$$\frac{\partial}{\partial\lambda}\log\operatorname{tr}=\tfrac{1}{2}\sum_{-\infty}^{\infty}A(n)G(n)$$

$$=-\frac{1}{2}\frac{\partial}{\partial\lambda}\sum_{-\infty}^{\infty}\log\left[\omega_0{}^2-\left(\frac{2\pi n}{T}\right)^2-\lambda A(n)\right]$$

which, together with the initial condition

$$\lambda=0:\quad \operatorname{tr}e^{iTH}=(\operatorname{tr}_e e^{iTH_{\mathrm{ext}}})\sum_{n=0}^{\infty}e^{i(n+\frac{1}{2})\omega_0T}$$

$$=(\operatorname{tr}_e)(i/2\sin\tfrac{1}{2}\omega_0T),$$

yields

$$\operatorname{tr}e^{iTH}=(\operatorname{tr}_e)(i/2\sin\tfrac{1}{2}\omega_0T)$$

$$\times\exp\left\{-\tfrac{1}{2}\sum_{-\infty}^{\infty}\log\left[\frac{\omega_0{}^2-(2\pi n/T)^2-A(n)}{\omega_0{}^2-(2\pi n/T)^2}\right]\right\}.$$

We have already introduced the function

$$G^{-1}(\zeta)=\omega_0{}^2-\zeta^2-\int_0^\infty\frac{\omega d\omega}{\pi}\frac{(A_{-+}-A_{+-})(\omega)}{\omega^2-\zeta^2}$$

and examined some of its properties for real and positive $A_{-+}(\omega)$, $A_{+-}(\omega)$. This situation is recovered on making the substitution $T\rightarrow i\beta$, and thus

$$Z=\operatorname{tr}e^{-\beta H}=Z_e(1/2\sinh\tfrac{1}{2}\beta\omega_0)$$

$$\times\exp\left\{-\tfrac{1}{2}\sum_{-\infty}^{\infty}\log\left[\frac{G^{-1}(i2\pi n/\beta)}{\omega_0{}^2+(2\pi n/\beta)^2}\right]\right\},$$

the existence of which for all $\beta>0$ requires that $G^{-1}(\zeta)$ remain positive at every value comprised in $\zeta^2=-(2\pi n/\beta)^2$, which is to say the entire negative ζ^2 axis including the origin. The condition

$$G^{-1}(0)>0$$

is thereby identified as a stability criterion. To evaluate the summation over n most conveniently we shall give an alternative construction for the function $\log(G^{-1}(\zeta)/-\zeta^2)$, which, as a function of ζ^2, has all its singularities located on the branch line extending from 0 to ∞ and vanishes at infinity in this cut plane. Hence

$$\log(G^{-1}(\zeta)/-\zeta^2)=\frac{1}{\pi}\int_0^\infty d\omega^2\frac{\varphi(\omega)}{\omega^2-\zeta^2},$$

where the value

$$\varphi(0)=\pi$$

reproduces the pole of $G^{-1}(\zeta)/(-\zeta^2)$ at $\zeta^2=0$. We also recognize, on relating the two forms,

$$G^{-1}(\zeta)=(-\zeta^2)\exp\left[\frac{1}{\pi}\int_0^\infty d\omega^2\frac{\varphi(\omega)}{\omega^2-\zeta^2}\right]$$

$$=\omega_0{}^2-\zeta^2-\int_0^\infty\frac{d\omega^2}{2\pi}\frac{a(\omega)\tanh(\frac{1}{2}\omega\beta)}{\omega^2-\zeta^2},$$

that

$$-\tfrac{1}{2}a(\omega)\tanh(\tfrac{1}{2}\omega\beta)\cot\varphi(\omega)$$

$$=\omega_0{}^2-\omega^2-P\int_0^\infty\frac{d\omega'}{\pi}\frac{\omega'a(\omega')\tanh(\frac{1}{2}\omega'\beta)}{\omega'^2-\omega^2}.$$

The positive value of the right-hand side as $\omega\rightarrow 0$ shows that $\varphi(\omega)$ approaches the zero frequency limiting value of π from below, and the assumption that $a(\omega)>0$ for all ω implies

$$\pi\geq\varphi(\omega)>0,$$

where the lower limit is approached as $\omega\rightarrow\infty$.

A comparison of asymptotic forms for $G^{-1}(\zeta)$ shows that

$$\omega_0{}^2=\frac{1}{\pi}\int_0^\infty d\omega^2\varphi(\omega)=\int_0^\infty d\omega\left(-\frac{1}{\pi}\frac{d\varphi(\omega)}{d\omega}\right)\omega^2,$$

while

$$\int_0^\infty d\omega\left(-\frac{1}{\pi}\frac{d\varphi(\omega)}{d\omega}\right)\omega^4=\omega_0{}^4+2\langle[i\dot{Q},Q]\rangle_\beta.$$

The introduction of the phase derivative can also be performed directly in the structure of $G^{-1}(\zeta)$,

$$G^{-1}(\zeta)=\exp\left[\int_0^\infty d\omega\left(-\frac{1}{\pi}\frac{d\varphi(\omega)}{d\omega}\right)\log(\omega^2-\zeta^2)\right],$$

and equating the two values for $G^{-1}(0)$ gives

$$\int_0^\infty d\omega\left(-\frac{1}{\pi}\frac{d\varphi(\omega)}{d\omega}\right)\log\omega^2$$

$$=\log\left[\omega_0{}^2-\int_0^\infty\frac{d\omega}{\pi}a(\omega)\frac{\tanh\frac{1}{2}\omega\beta}{\omega}\right].$$

We now have the representation

$$\log\left[\frac{G^{-1}(i2\pi n/\beta)}{\omega_0{}^2+(2\pi n/\beta)^2}\right]$$

$$=\int_0^\infty d\omega\left(-\frac{1}{\pi}\frac{d\varphi(\omega)}{d\omega}\right)\log\frac{\omega^2+(2\pi n/\beta)^2}{\omega_0{}^2+(2\pi n/\beta)^2},$$

and the summation formula derived from the product form of the hyperbolic sine function,

$$\tfrac{1}{2}\sum_{-\infty}^{\infty}\log\frac{\omega^2+(2\pi n/\beta)^2}{\omega_0^2+(2\pi n/\beta)^2}=\log\left[\frac{\sinh\frac{1}{2}\omega\beta}{\sinh\frac{1}{2}\omega_0\beta}\right],$$

gives us the desired result

$$Z=Z_e\exp\left[-\int_0^\infty d\omega\left(-\frac{1}{\pi}\frac{d\varphi(\omega)}{d\omega}\right)\log 2\sinh\tfrac{1}{2}\omega\beta\right].$$

The second factor can be ascribed to the oscillator, with its properties modified by interaction with the external system. The average energy of the oscillator at temperature $\vartheta=\beta^{-1}$ is therefore given by

$$E=\frac{\partial}{\partial\beta}\int_0^\infty d\omega\left(-\frac{1}{\pi}\frac{d\varphi}{d\omega}\right)\log 2\sinh\tfrac{1}{2}\omega\beta$$

in which the temperature dependence of the phase $\varphi(\omega)$ is not to be overlooked. In an extreme high-temperature limit, such that $\omega\beta\ll 1$ for all significant frequencies, we have

$$E\simeq\frac{\partial}{\partial\beta}[\log\beta+\tfrac{1}{2}\log(\omega_0^2-\beta\langle Q^2\rangle_\vartheta)],$$

and the simple classical result $E=\vartheta$ appears when $\langle Q^2\rangle_\vartheta$ is proportional to ϑ. The oscillator energy at zero temperature is given by

$$E_0=\tfrac{1}{2}\int_0^\infty d\omega\left(-\frac{1}{\pi}\frac{d\varphi}{d\omega}\right)_{\vartheta=0}\omega,$$

and the oscillator contribution to the specific heat vanishes.

The following physical situation has consequences that resemble the simple model of the previous section. For values of $\omega\lesssim\omega_0$, $a(\omega)\tanh(\frac{1}{2}\omega\beta)\ll\omega_0^2$, and $a(\omega)$ differs significantly from zero until one attains frequencies that are large in comparison with ω_0. The magnitudes that $a(\omega)$ can assume at frequencies greater than ω_0 is limited only by the assumed absence of rapid variations and by the requirement of stability. The latter is generally assured if

$$\frac{1}{\pi}\int_{\sim\omega_0}^\infty\frac{d\omega}{\omega}a(\omega)<\omega_0^2.$$

We shall suppose that the stability requirement is comfortably satisfied, so that the right-hand side of the equation for $\cot\varphi(\omega)$ is an appreciable fraction of ω_0^2 at sufficiently low frequencies. Then $\tan\varphi$ is very small at such frequencies, or $\varphi(\omega)\sim\pi$, and this persists until we reach the immediate neighborhood of the frequency $\omega_1<\omega_0$ such that

$$\omega_0^2-\omega_1^2-P\int_0^\infty\frac{d\omega}{\pi}\frac{\omega a(\omega)\tanh(\frac{1}{2}\omega\beta)}{\omega^2-\omega_1^2}=0.$$

That the function in question, $\mathrm{Re}G^{-1}(\omega+i0)$, has a zero, follows from its positive value at $\omega=0$ and its asymptotic approach to $-\infty$ with indefinitely increasing frequency. Under the conditions we have described, with the major contribution to the integral coming from high frequencies, the zero point is given approximately as

$$\omega_1^2\simeq\omega_0^2-\int_0^\infty\frac{d\omega}{\pi}a(\omega)\frac{\tanh(\frac{1}{2}\omega\beta)}{\omega},$$

and somewhat more accurately by

$$\omega_1^2=B\left[\omega_0^2-\int_0^\infty\frac{d\omega}{\pi}a(\omega)\frac{\tanh(\frac{1}{2}\omega\beta)}{\omega}\right],$$

where

$$B^{-1}=1+P\int_0^\infty\frac{d\omega}{2\pi}\frac{1}{\omega^2-\omega_1^2}\frac{d}{d\omega}[a(\omega)\tanh(\tfrac{1}{2}\omega\beta)].$$

As we shall see, B is less than unity, but only slightly so under the circumstances assumed.

In the neighborhood of the frequency ω_1, the equation that determines $\varphi(\omega)$ can be approximated by

$$-\tfrac{1}{2}a(\omega_1)\tanh(\tfrac{1}{2}\omega_1\beta)\cot\varphi(\omega)=B^{-1}(\omega_1^2-\omega^2)$$

or

$$\cot\varphi(\omega)=(\omega^2-\omega_1^2)/\gamma\omega_1\simeq(\omega-\omega_1)/\tfrac{1}{2}\gamma,$$

with the definition

$$\gamma=\tfrac{1}{2}Ba(\omega_1)[\tanh(\tfrac{1}{2}\omega_1\beta)/\omega_1]\ll\omega_1.$$

Hence, as ω rises through the frequency ω_1, φ decreases abruptly from a value close to π to one near zero. The subsequent variations of the phase are comparatively gradual, and φ eventually approaches zero as $\omega\to\infty$. A simple evaluation of the average oscillator energy can be given when the frequency range $\omega>\omega_1$ over which $a(\omega)$ is appreciable in magnitude is such that $\beta\omega\gg 1$. There will be no significant temperature variation in the latter domain and in particular ω_1 should be essentially temperature independent. Then, since $-(1/\pi)(d\varphi/d\omega)$ in the neighborhood of ω_1 closely resembles $\delta(\omega-\omega_1)$, we have approximately

$$E=\frac{\partial}{\partial\beta}\left[\log(2\sinh\tfrac{1}{2}\omega_1\beta)+\beta\int_{>\omega_1}^\infty d\omega\left(-\frac{1}{\pi}\frac{d\varphi}{d\omega}\right)\tfrac{1}{2}\omega\right]$$
$$=\omega_1\left(\frac{1}{e^{\beta\omega_1}-1}+\tfrac{1}{2}\right)+\int_{>\omega_1}^\infty d\omega\left(-\frac{1}{\pi}\frac{d\varphi}{d\omega}\right)\tfrac{1}{2}\omega,$$

which describes a simple oscillator of frequency ω_1, with a displaced origin of energy.

Note that with $\varphi(\omega)$ very small at a frequency slightly greater than ω_1 and zero at infinite frequency, we have

$$\int_{>\omega_1}^\infty d\omega\left(-\frac{1}{\pi}\frac{d\varphi}{d\omega}\right)\tfrac{1}{2}\omega\simeq\frac{1}{2\pi}\int_{>\omega_1}^\infty d\omega\,\varphi(\omega)>0.$$

Related integrals are

$$\omega_0^2-\omega_1^2 \simeq \frac{2}{\pi}\int_{>\omega_1}^{\infty} d\omega\,\omega\,\varphi(\omega)>0$$

and

$$\log B^{-1} \simeq \frac{2}{\pi}\int_{>\omega_1}^{\infty} \frac{d\omega}{\omega}\varphi(\omega)>0.$$

The latter result confirms that $B<1$. A somewhat more accurate formula for B is

$$B=\exp\left[-\int_{>\omega_1}^{\infty} d\omega\left(-\frac{1}{\pi}\frac{d\varphi(\omega)}{d\omega}\right)\log(\omega^2-\omega_1^2)\right].$$

If the major contributions to all these integrals come from the general vicinity of a frequency $\bar{\omega}\gg\omega_0$, we can make the crude estimates

$$\frac{1}{2\pi}\int_{>\omega_1}^{\infty} d\omega\,\varphi(\omega)\sim\frac{\omega_0^2}{\bar{\omega}}\ll\omega_1,\ \log B^{-1}\sim\left(\frac{\omega_0}{\bar{\omega}}\right)^2\ll 1.$$

Then neither the energy shift nor the deviation of the factor B from unity are particularly significant effects.

The approximation of $\mathrm{Re}G(\omega+i0)$ as $B^{-1}(\omega_1^2-\omega^2)$ evidently holds from zero frequency up to a frequency considerably in excess of ω_1. Throughout this frequency range we have

$$-\tfrac{1}{2}a(\omega)\tanh(\tfrac{1}{2}\omega\beta)\cot\varphi(\omega)=B^{-1}(\omega_1^2-\omega^2)$$

or

$$\cot\varphi(\omega)=(\omega_1^2-\omega^2)/\gamma\omega$$

with

$$\gamma(\omega)=\tfrac{1}{2}Ba(\omega)\tanh(\tfrac{1}{2}\omega\beta)/\omega.$$

If in particular $\beta\omega_1\ll 1$, the frequencies under consideration are in the classical domain and γ is the frequency independent constant

$$\gamma=\tfrac{1}{4}Ba(0)\beta.$$

To regard γ as constant for a quantum oscillator requires a suitable frequency restriction to the vicinity of ω_1. The function $B(\omega^2)$ can be computed from

$$B(\omega^2)=\frac{2}{\pi}\frac{\sin^2\varphi(\omega)}{a(\omega)\tanh(\frac{1}{2}\omega\beta)}$$
$$=\frac{1}{\pi}\frac{B}{\gamma\omega}\frac{1}{\cot^2\varphi+1},$$

and accordingly is given by

$$B(\omega^2)=B\frac{1}{\pi}\frac{\gamma\omega}{(\omega^2-\omega_1^2)^2+(\gamma\omega)^2}$$
$$=B\frac{\gamma\omega}{\pi}\left|\frac{1}{\omega^2+i\gamma\omega-\omega_1^2}\right|^2.$$

The further concentration on the immediate vicinity of ω_1, $|\omega-\omega_1|\sim\gamma$, gives

$$B(\omega^2)=\frac{B}{2\omega_1}\frac{1}{\pi}\frac{\frac{1}{2}\gamma}{(\omega-\omega_1)^2+(\frac{1}{2}\gamma)^2}$$

which clearly identifies $B<1$ with the contribution to the integral $\int d\omega^2 B(\omega^2)$ that comes from the vicinity of this resonance of width γ at frequency ω_1, although the same result is obtained without the last approximation. The remainder of the integral, $1-B$, arises from frequencies considerably higher than ω_1 according to our assumptions.

There is a similar decomposition of the expressions for the Green's functions. Thus, with $t>t'$,

$$G_r(t-t')\simeq B\int_{\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\frac{1}{-\omega^2-i\gamma\omega+\omega_1^2}$$
$$+\int_{\gg\omega_1^2}^{\infty} d\omega^2 B(\omega^2)\frac{\sin\omega(t-t')}{\omega}.$$

The second high-frequency term will decrease very quickly on the time scale set by $1/\omega_1$. Accordingly, in using this Green's function, say in the evaluation of

$$\langle q(t)\rangle_\delta{}^F=\int_{-\infty}^{\infty} dt' G_r(t-t')F(t')$$

for an external force that does not vary rapidly in relation to ω_1, the contribution of the high-frequency term is essentially given by

$$F(t)\int_0^{\infty} d(t-t')\int_{\gg\omega_1^2}^{\infty} d\omega^2 B(\omega^2)\frac{\sin\omega(t-t')}{\omega}$$
$$=F(t)\int_{\gg\omega_1^2}^{\infty} d\omega^2\frac{B(\omega^2)}{\omega^2}.$$

But

$$\int_{\gg\omega_1^2}^{\infty} d\omega^2\frac{B(\omega^2)}{\omega^2}\simeq\left[\omega_0^2-\int_0^{\infty}\frac{d\omega}{\pi}a(\omega)\frac{\tanh\frac{1}{2}\omega\beta}{\omega}\right]^{-1}-\frac{B}{\omega_1^2},$$
$$\simeq 0,$$

and the response to such an external force is adequately described by the low-frequency part of the Green's function. We can represent this situation by an equivalent differential equation

$$\left(\frac{d^2}{dt^2}+\gamma\frac{d}{dt}+\omega_1^2\right)\langle q(t)\rangle_\delta{}^F=BF(t)$$

which needs no further qualification when the oscillations are classical but implies a restriction to a frequency interval within which γ is constant, for quantum oscillations. We note the reduction in the effectiveness of the external force by the factor B. Under the circumstances outlined this effect is not important and we shall place B equal to unity.

One can make a general replacement of the Green's functions by their low-frequency parts:

$$G_r(t-t') \to e^{-\frac{1}{2}\gamma(t-t')}\frac{1}{\omega_1}\sin(\omega_1(t-t'))\eta_+(t-t')$$

$$G_a(t-t') \to -e^{-\frac{1}{2}\gamma|t-t'|}\frac{1}{\omega_1}\sin(\omega_1(t-t'))\eta_-(t-t'),$$

if one limits the time localizability of measurements so that only time averages of $q(t)$ are of physical interest. This is represented in the expectation value formula by considering only functions $f_\pm(t)$ that do not vary too quickly. The corresponding replacement for $w(t-t')$ is

$$w(t-t') \to \coth(\tfrac{1}{2}\omega_1\beta)e^{-\frac{1}{2}\gamma|t-t'|}\frac{1}{\omega_1}\cos\omega_1(t-t'),$$

and the entire matrix $G_\vartheta(t-t')$ obtained in this way obeys the differential equation

$$\left(\frac{d^2}{dt^2}+\gamma\frac{d}{dt}+\omega_1^2\right)\left(\frac{d^2}{dt'^2}+\gamma\frac{d}{dt'}+\omega_1^2\right)G_\vartheta(t-t')$$

$$=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(\frac{d^2}{dt^2}+\omega_1^2\right)\delta(t-t')$$

$$+\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\gamma\frac{d}{dt}\delta(t-t')+\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\tfrac{1}{2}ia\delta(t-t'),$$

where $a=a(\omega_1)$.

The simplest presentation of results is again to be found in the Langevin viewpoint, which directs the emphasis from the coordinate operator $q(t)$ to the fluctuating force defined by

$$\left(\frac{d^2}{dt^2}+\gamma\frac{d}{dt}+\omega_1^2\right)q(t)=F(t)+F_f(t),$$

which is to say

$$F_f(t)=Q(t)+\gamma\frac{d}{dt}q(t)+(\omega_1^2-\omega_0^2)q(t).$$

This change is introduced by the substitution

$$f_\pm(t)=\left(\frac{d^2}{dt^2}-\gamma\frac{d}{dt}+\omega_1^2\right)k_\pm(t),$$

and the necessary partial integrations involve the previously established lemma on time-ordered operators, which here asserts that

$$\left(\exp\left[i\int dtfq\right]\right)_+=\left(\exp\left[i\int dtk(F+F_f)\right]\right)_+$$

$$\times\exp\left\{\tfrac{1}{2}i\int dt[(\omega_1^2-\omega_0^2)k^2+\omega_1^2k^2-(dk/dt)^2]\right\}.$$

We now find

$$\left\langle\left(\exp\left[-i\int dtk_-F_f\right]\right)_-\left(\exp\left[i\int dtk_+F_f\right]\right)_+\right\rangle_\vartheta$$

$$=\exp\left[-\tfrac{1}{2}\int dtdt'k(t)\zeta(t-t')k(t')\right]$$

with

$$\zeta(t-t')=\tfrac{1}{2}a\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\delta(t-t')$$

$$-i(\omega_0^2-\omega_1^2)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\delta(t-t')$$

$$+i\gamma\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\frac{d}{dt}\delta(t-t').$$

The latter matrix can also be identified as

$$\zeta(t-t')=\begin{pmatrix}\langle(F_f(t)F_f(t'))_+\rangle_\vartheta, & -\langle F_f(t')F_f(t)\rangle_\vartheta\\ -\langle F_f(t)F_f(t')\rangle_\vartheta, & \langle(F_f(t)F_f(t'))_-\rangle_\vartheta\end{pmatrix}.$$

In the classical limit

$$\left\langle\exp\left[i\int dtkF_f\right]\right\rangle_\vartheta=\exp\left[-\tfrac{1}{4}a\int dtk^2\right]$$

and

$$\tfrac{1}{4}a=\gamma\vartheta.$$

If a comparison is made with the similar results of the previous section it can be appreciated that the frequency range has been extended and the restriction $\omega_1\simeq\omega_0$ removed.

We return from these extended considerations on thermal equilibrium and consider one extreme example of negative temperature for the external system. This is described by

$$a(\omega)=a\delta(\omega-\omega_1),\quad \omega>0$$

and

$$-\beta(\omega_1)=|\beta|>0.$$

With the definition

$$(1/\pi)\omega_1a\tanh(\tfrac{1}{2}\omega_1|\beta|)=(\omega_1\mu)^2,$$

we have

$$G(\zeta)=\left[\omega_0^2-\zeta^2+\frac{(\omega_1\mu)^2}{\omega_1^2-\zeta^2}\right]^{-1}$$

$$=\frac{\omega_1^2-\zeta^2}{[\zeta^2-\frac{1}{2}(\omega_0^2+\omega_1^2)]^2+(\omega_1\mu)^2-[\frac{1}{2}(\omega_0^2-\omega_1^2)]^2}.$$

As a function of ζ^2, $G(\zeta)$ now has complex poles if

$$\tfrac{1}{2}|\omega_0^2-\omega_1^2|<\omega_1\mu.$$

We shall suppose, for simplicity, that $\omega_1=\omega_0$ and $\mu\ll\omega_0$. Then the poles of

$$G(\zeta)=\tfrac{1}{2}[(\omega_0^2+i\omega_0\mu-\zeta^2)^{-1}+(\omega_0^2-i\omega_0\mu-\zeta^2)^{-1}]$$

are located at $\zeta=\pm(\omega_0+\frac{1}{2}i\mu)$ and $\zeta=\pm(\omega-\frac{1}{2}i\mu)$. Accordingly, $G(\zeta)$ is regular outside a strip of width $2\alpha=\mu$. The associated Green's functions are given by

$$G_r(t-t')=\cosh(\tfrac{1}{2}\mu(t-t'))(\omega_0)^{-1}\sin(\omega_0(t-t'))\eta_+(t-t'),$$

$$G_a(t-t')=-\cosh(\tfrac{1}{2}\mu(t-t'))(\omega_0)^{-1}\sin(\omega_0(t-t'))\eta_-(t-t'),$$

and the function $w(t-t_2, t'-t_2)$, computed for $\omega_0(t-t_2)$, $\omega_0(t'-t_2)\gg1$, is

$$\begin{aligned}w(t-t_2,t'-t_2)&\\ \simeq(\omega_0)^{-1}\cos(\omega_0(t-t'))&[\coth(\tfrac{1}{2}\omega_0|\beta|)\sinh(\tfrac{1}{2}\mu(t-t_2))\\ &\times\sinh(\tfrac{1}{2}\mu(t'-t_2))+\coth(\tfrac{1}{2}\omega_0\beta_0)\\ &\times\cosh(\tfrac{1}{2}\mu(t-t_2))\cosh(\tfrac{1}{2}\mu(t'-t_2))].\end{aligned}$$

After the larger time intervals $\mu(t-t_2)$, $\mu(t'-t_2)\gg1$, we have

$$w(t-t_2,t'-t_2)\sim(2\omega_0)^{-1}e^{+\frac{1}{2}\mu(t-t_2)}e^{+\frac{1}{2}\mu(t'-t_2)}\cos\omega_0(t-t')\times[n_0+(1-e^{-\omega_0|\beta|})^{-1}],$$

with

$$n_0=(e^{\omega_0\beta_0}-1)^{-1}.$$

When t is in the vicinity of a time t_1, such that the amplification factor

$$k\simeq\tfrac{1}{2}e^{\frac{1}{2}\mu(t_1-t_2)}\gg1,$$

the oscillator is described by the classical coordinate

$$q(t)=k[q_s(t)+q_n(t)].$$

Here

$$q_s(t)=\int_{t_2}^{\infty}dt'\frac{1}{\omega_0}\sin(\omega_0(t-t'))F(t')e^{-\frac{1}{2}\mu(t'-t_2)}$$

and

$$q_n(t)=q_1\cos\omega t+q_2\sin\omega t,$$

where q_1 and q_2 are characterized by the expectation value formula

$$\langle e^{i(q_1f_1+q_2f_2)}\rangle=\exp[-(\nu/\omega_0)\tfrac{1}{2}(f_1^2+f_2^2)],$$

in which

$$\nu=n_0+(1-e^{-\omega_0|\beta|})^{-1}.$$

Accordingly, the probability of observing q_1 and q_2 within the range dq_1, dq_2 is

$$p(q_1q_2)dq_1dq_2=\frac{1}{2\pi}\frac{\omega_0}{\nu}\exp\left[-\frac{\omega_0}{\nu}\frac{1}{2}(q_1^2+q_2^2)\right]dq_1dq_2$$

$$=\frac{\omega_0}{\nu}\exp\left(-\frac{1}{2}\frac{\omega_0}{\nu}q_n^2\right)q_ndq_n\frac{1}{2\pi}d\varphi,$$

where q_n and φ are the amplitude and phase of $q_n(t)$. Despite rather different assumptions about the external system, these are the same conclusions as before, apart from a factor of $\frac{1}{2}$ in the formula for the gain.

GENERAL THEORY

The whole of the preceding discussion assumes an external system that is only slightly influenced by the presence of the oscillator. Now we must attempt to place this simplification within the framework of a general formulation. A more thorough treatment is also a practical necessity in situations such as those producing amplification of the oscillator motion, for a sizeable reaction in the external system must eventually appear, unless a counter mechanism is provided.

It is useful to supplement the previous Lagrangian operator with the term $q'(t)Q$, in which $q'(t)$ is an arbitrary numerical function of time, and also, to imagine the coupling term qQ supplied with a variable factor λ. Then

$$\begin{aligned}\frac{\partial}{\partial\lambda}\langle t_2|t_2\rangle^{F\pm q\pm'}&=i\left\langle\left|\int dt(q_+Q_+-q_-Q_-)\right|\right\rangle,\\ &=-i\int_{t_2}^{t_1}dt\left(\frac{\delta}{\delta F_+(t)}\frac{\delta}{\delta q_+'(t)}-\frac{\delta}{\delta F_-(t)}\frac{\delta}{\delta q_-'(t)}\right)\\ &\qquad\times\langle t_2|t_2\rangle^{F\pm q\pm'},\end{aligned}$$

provided that the states to which the transformation function refers do not depend upon the coupling between the systems, or that the trace of the transformation function is being evaluated. A similar statement would apply to a transformation function with different terminal times. This differential equation implies an integrated form, in which the transformation function for the fully coupled system ($\lambda=1$) is expressed in terms of the transformation function for the uncoupled system ($\lambda=0$). The latter is the product of transformation functions for the independent oscillator and external system. The relation is

$$\langle t_2|t_2\rangle^{F\pm}=\exp\left[-i\int_{t_2}^{t_1}dt\left(\frac{\delta}{\delta F_+}\frac{\delta}{\delta q_+'}-\frac{\delta}{\delta F_-}\frac{\delta}{\delta q_-'}\right)\right]\times\langle t_2|t_2\rangle_{\mathrm{osc}}^{F\pm}\langle t_2|t_2\rangle_{\mathrm{ext}}^{q\pm'}|_{q_\pm'=0},$$

and we have indicated that $q_\pm'$ is finally set equal to zero if we are concerned only with measurements on the oscillator.

Let us consider for the moment just the external system with the perturbation $q'Q$, the effect of which is indicated by[6]

$$\langle t_2|t_2\rangle^{q_\pm'}=\left\langle t_2\left|\left(\exp\left[-i\int dt q_-'Q\right]\right)_-\right.\right.\times\left.\left.\left(\exp\left[i\int dt q_+'Q\right]\right)_+\right|t_2\right\rangle.$$

We shall define

$$Q_+(t,q_\pm')=\frac{\langle t_2|Q_+(t)|t_2\rangle^{q_\pm'}}{\langle t_2|t_2\rangle^{q_\pm'}}=\frac{1}{i}\frac{\delta}{\delta q_+'(t)}\log\langle t_2|t_2\rangle^{q_\pm'}$$

and similarly

$$Q_-(t,q_\pm')=\frac{\langle t_2|Q_-(t)|t_2\rangle^{q_\pm'}}{\langle t_2|t_2\rangle^{q_\pm'}}=-\frac{1}{i}\frac{\delta}{\delta q_-'(t)}\log\langle t_2|t_2\rangle^{q_\pm'}.$$

When $q_\pm'(t)=q'(t)$, we have

$$Q_+(t,q')=Q_-(t,q')=\langle t_2|Q(t)|t_2\rangle^{q'},$$

which is the expectation value of $Q(t)$ in the presence of the perturbation described by $q'(t)$. This is assumed to be zero for $q'(t)=0$ and depends generally upon the history of $q'(t)$ between t_2 and the given time.

The operators $q_\pm(t)$ are produced within the transformation function by the functional differential operators $(\pm 1/i)\ \delta/\delta F_\pm(t)$, and since the equation of motion for the uncoupled oscillator is

$$\left(\frac{d^2}{dt^2}+\omega_0^2\right)q(t)=F(t),$$

we have

$$\left(\frac{\partial^2}{\partial t^2}+\omega_0^2\right)\left(\pm\frac{1}{i}\right)\frac{\delta}{\delta F_\pm(t)}\langle t_2|t_2\rangle^{F\pm}=\exp\left[-i\int dt\left(\frac{\delta}{\delta F_+}\frac{\delta}{\delta q_+'}-\frac{\delta}{\delta F_-}\frac{\delta}{\delta q_-'}\right)\right]\times F_\pm(t)\langle t_2|t_2\rangle_{\mathrm{osc}}{}^{F\pm}\langle t_2|t_2\rangle_{\mathrm{ext}}{}^{q_\pm'}|_{q_\pm'=0}.$$

On moving $F_\pm(t)$ to the left of the exponential, this becomes

$$F_\pm(t)\langle t_2|t_2\rangle^{F\pm}+\exp[\quad]\langle t_2|t_2\rangle_{\mathrm{osc}}{}^{F\pm}\times\left(\pm\frac{1}{i}\right)\frac{\delta}{\delta q_\pm'(t)}\langle t_2|t_2\rangle_{\mathrm{ext}}{}^{q_\pm'}|_{q_\pm'=0}.$$

But

$$\left(\pm\frac{1}{i}\right)\frac{\delta}{\delta q_\pm'(t)}\langle t_2|t_2\rangle_{\mathrm{ext}}{}^{q_\pm'}=Q_\pm(t,q_\pm')\langle t_2|t_2\rangle_{\mathrm{ext}}{}^{q_\pm'},$$

and furthermore,

$$\exp[\quad]Q(t,q_\pm')\langle t_2|t_2\rangle_{\mathrm{osc}}{}^{F\pm}\langle t_2|t_2\rangle_{\mathrm{ext}}{}^{q_\pm'}|_{q_\pm'=0}=Q\left(t,\pm\frac{1}{i}\frac{\delta}{\delta F_\pm}\right)\langle t_2|t_2\rangle^{F\pm},$$

which leads us to the following functional differential equation for the transformation function $\langle t_2|t_2\rangle^{F\pm}$, in which a knowledge is assumed of the external system's reaction to the perturbation $q_\pm'(t)$:

$$\left[\left(\frac{\partial^2}{\partial t^2}+\omega_0^2\right)\left(\pm\frac{1}{i}\right)\frac{\delta}{\delta F_\pm(t)}-Q_\pm\left(t,\pm\frac{1}{i}\frac{\delta}{\delta F_\pm}\right)-F_\pm(t)\right]\times\langle t_2|t_2\rangle^{F\pm}=0.$$

Throughout this discussion one must distinguish between the $\pm$ signs attached to particular components and those involved in the listing of complete sets of variables.

The differential equations for time development are supplemented by boundary conditions which assert, at a time t_1 beyond which $F_+(t)=F_-(t)$, that

$$\left(\frac{\delta}{\delta F_+(t_1)}+\frac{\delta}{\delta F_-(t_1)}\right)\langle t_2|t_2\rangle^{F\pm}=0$$

while, for the example of the transformation function $\langle t_2|t_2\rangle_{\beta_0}{}^{F\pm}$, we have the initial conditions

$$\left[\left(\frac{\delta}{\delta F_+}-\frac{\delta}{\delta F_-}\right)(t_2)+\frac{i}{\omega_0}\coth(\tfrac{1}{2}\omega_0\beta_0)\frac{\partial}{\partial t_2}\left(\frac{\delta}{\delta F_+}+\frac{\delta}{\delta F_-}\right)(t_2)\right]\langle t_2|t_2\rangle^{F\pm}=0$$

and

$$\left[\frac{\partial}{\partial t_2}\left(\frac{\delta}{\delta F_+}-\frac{\delta}{\delta F_-}\right)(t_2)-i\omega_0\coth(\tfrac{1}{2}\omega_0\beta_0)\left(\frac{\delta}{\delta F_+}+\frac{\delta}{\delta F_-}\right)(t_2)\right]\langle t_2|t_2\rangle^{F\pm}=0.$$

The previous treatment can now be identified as the

[6] Such positive and negative time-ordered products occur in a recent paper by K. Symanzik [J. Math. Phys. 1, 249 (1960)], which appeared after this paper had been written and its contents used as a basis for lectures delivered at the Brandeis Summer School, July, 1960.

approximation of the $Q_\pm(t,q_\pm')$ by linear functions of $q_\pm'$,

$$Q_+(t,q_\pm')=i\int dt'[A_{++}(t-t')q_+'(t')-A_{+-}(t-t')q_-'(t')]$$

$$Q_-(t,q_\pm')=i\int dt'[A_{-+}(t-t')q_+'(t')-A_{--}(t-t')q_-'(t')],$$

wherein the linear equations for the operators $q_\pm(t)$ and their meaning in terms of variations of the $F_\pm$ have been united in one pair of functional differential equations. This relation becomes clearer if one writes

$$\langle t_2|t_2\rangle^{F\pm}=\exp[iW(F_\pm)]$$

and, with the definition

$$q_\pm(t,F_\pm)=(\pm)\frac{\delta}{\delta F_\pm(t)}W(F_\pm)$$

$$=\left(\pm\frac{1}{i}\right)\frac{\delta}{\delta F_\pm(t)}\log\langle t_2|t_2\rangle^{F\pm},$$

converts the functional differential equations into

$$\left(\frac{\partial^2}{\partial t^2}+\omega_0^2\right)q_\pm(t,F_\pm)-Q_\pm\left(t,\,q_\pm\pm\frac{1}{i}\frac{\delta}{\delta F_\pm}\right)-F_\pm(t)=0.$$

The boundary conditions now appear as

$$(q_+-q_-)(t_1,F_\pm)=0$$

and

$$(q_++q_-)(t_2,F_\pm)+\frac{i}{\omega_0}\coth(\tfrac{1}{2}\omega_0\beta_0)\frac{\partial}{\partial t_2}(q_+-q_-)(t_2,F_\pm)=0$$

$$\frac{\partial}{\partial t_2}(q_++q_-)(t_2,F_\pm)-i\omega_0\coth(\tfrac{1}{2}\omega_0\beta_0)(q_+-q_-)(t_2,F_\pm)=0.$$

When the $Q_\pm$ are linear functions of $q_\pm$, the functional differential operators disappear[7] and we regain the linear equations for $q_\pm(t)$, which in turn imply the quadratic form of $W(F_\pm)$ that characterizes the preceding discussion.

[7] The degeneration of the functional equations into ordinary differential equations also occurs when the motion of the oscillator is classical and free of fluctuation.