Stability and fluctuations in black hole thermodynamics

George Ruppeiner*

Division of Natural Sciences, New College of Florida, 5800 Bay Shore Road, Sarasota, Florida 34243-2109, USA

(Received 12 December 2006; published 26 January 2007)

I examine thermodynamic fluctuations for a Kerr-Newman black hole in an extensive, infinite environment. This problem is not strictly solvable because full equilibrium with such an environment cannot be achieved by any black hole with mass M, angular momentum J, and charge Q. However, if we consider one (or two) of M, J, or Q to vary so slowly compared with the others that we can regard it as fixed, instances of stability occur, and thermodynamic fluctuation theory could plausibly apply. I examine seven cases with one, two, or three independent fluctuating variables. No knowledge about the thermodynamic behavior of the environment is needed. The thermodynamics of the black hole is sufficient. Let the fluctuation moment for a thermodynamic quantity X be $\sqrt{\langle(\Delta X)^2\rangle}$. Fluctuations at fixed M are stable for all thermodynamic states, including that of a nonrotating and uncharged environment, corresponding to average values $J = Q = 0$. Here, the fluctuation moments for J and Q take on maximum values. That for J is proportional to M. For the Planck mass it is $0.3990\hbar$. That for Q is $3.301e$, independent of M. In all cases, fluctuation moments for M, J, and Q go to zero at the limit of the physical regime, where the temperature goes to zero. With M fluctuating there are no stable cases for average $J = Q = 0$. But, there are transitions to stability marked by infinite fluctuations. For purely M fluctuations, this coincides with a curve which Davies identified as a phase transition.

DOI: 10.1103/PhysRevD.75.024037 PACS numbers: 04.70.Dy, 05.40.−a, 05.70.−a

I. INTRODUCTION

The fundamental simplicity of black holes has led to their thermodynamic description [1–3]. Viewed from outside, Kerr-Newman black holes can be characterized only by their mass M, angular momentum J, and charge Q. Detailed internal structure and history of formation are irrelevant in this ''no hair'' property. Such a drastic reduction of complexity is characteristic of thermodynamic systems.

In this description, black hole entropy is proportional to the area of the event horizon. With the appropriate multiplier, this entropy may be added to the entropy of any conventional thermodynamic system. The sum obeys a generalized second law of thermodynamics, with nondecreasing total entropy for a closed composite system.

But entropy is also connected to microscopic information (number of accessible microstates), raising the possibility of a thermodynamic fluctuation theory [4] based on Einstein's formula for the probability,

$$P \propto \exp(S_{\mathrm{tot}}/k_B), \tag{1}$$

where S_{tot} is the entropy of the closed system and k_B is Boltzmann's constant. While the nature of such information probably requires a future quantum theory of gravity, the thermodynamic formalism allows us to proceed knowing just the entropy.

Strictly speaking, a thermodynamic fluctuation approach is unviable since a Kerr-Newman black hole cannot come to equilibrium with any extensive, infinite environment. For no physical values of M, J, or Q is the full Hessian determinant[1] of the black hole entropy $S = S(M, J, Q)$ negative definite [5], as it must be to produce a local maximum in the total entropy. This puts the full problem, with (M, J, Q) all fluctuating, beyond the reach of the standard thermodynamic fluctuation formalism. Dynamically, physics ultimately favors extreme situations where the black hole either evaporates completely or grows without limit.

In reality, however, one of the variables M, J, or Q may be slow to change in time compared with the other two. Physically, it would be surprising if they all went at the same rate. This possibility offers the restricted class of problems with just two fluctuating variables, and the third effectively fixed or drifting very slowly out of equilibrium with the environment. There is also the possibility of a single fluctuating variable, with the other two fixed.

In this paper I consider all seven possible cases: fluctuating (M, J, Q), (J, Q), (M, Q), (M, J), M, J, and Q. Seven entropy Hessian determinants: p_1, p_2, p_3, p_1', p_2', p_2'', and p_1'', in various combinations, govern fluctuations. For stability, the relevant Hessian determinants must all be positive. As we will see, p_3 is always negative, p_1', p_2', and p_1'' are never negative, and p_1, p_2, and p_2'' may have either sign depending on the thermodynamic state. This means that the fluctuation case (M, J, Q) is not stable for any thermodynamic state, (J, Q), J, and Q are stable for all states, and (M, J), (M, Q), and M may be stable or unstable depending on the thermodynamic state.

*Electronic address: ruppeiner@ncf.edu

[1]By Hessian determinant I mean a determinant of a square matrix of second partial derivatives of S with respect to one or more of M, J, or Q. The full Hessian determinant includes all three of these independent variables.

This paper is arranged as follows. First, I summarize the basic thermodynamic fluctuation formalism as it applies to black holes. Second, I give a stability and fluctuation analysis of all seven cases above. Third, I discuss which cases might be physically relevant.

II. THERMODYNAMIC FLUCTUATION THEORY

A. Black hole thermodynamics

General relativity, including thermodynamic arguments, allows one to calculate the Kerr-Newman black hole entropy [3,6]

$$S(M, J, Q) = \frac{1}{8}(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2Q^2}). \quad (2)$$

Here M, J, and Q are expressed in length units, M and Q in cm, and J in cm^2. To convert to "real" units (Q in esu) write:

$$M\left(\frac{c^2}{G}\right), \qquad J\left(\frac{c^3}{G}\right), \quad \text{and} \quad Q\left(\frac{c^2}{G^{1/2}}\right), \quad (3)$$

where G is the constant of gravity and c is the speed of light [7]. In these geometrized units, $G = c = 1$.

To convert S to real units requires more discussion. Bekenstein [1] and Hawking [2] gave the conventional unit black hole entropy S_{bh},[2]

$$\frac{S_{bh}}{k_B} = \frac{1}{4}\left(\frac{A}{L_p^2}\right), \quad (4)$$

where A is the black hole area [6],

$$A = 4\pi(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2Q^2}), \quad (5)$$

L_p is the Planck length,

$$L_p \equiv \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-33} \text{ cm}, \quad (6)$$

and $\hbar$ is Planck's constant divided by 2π.[3] This leads to the conversion factor for S,

$$\frac{S_{bh}}{k_B} = S\left(\frac{8\pi}{L_p^2}\right). \quad (7)$$

As emphasized by Bekenstein [1], a theory containing both $\hbar$ and G has a quantum gravity flavor.[4]

The quantities M, J, and Q are conserved. This makes them natural coordinates for treating the interaction between two thermodynamic systems.

[2]Bekenstein first gave this in his Eq. (17), page 2338, but with a prefactor other than $\frac{1}{4}$. Hawking corrected this on his page 193.

[3]Thus, entropy in units of k_B is $1/4$ the Area in units of the Planck area; see [8] for a recent semipopular discussion.

[4]The entropy in this paper is the same as that of Davies [3], and differs from Tranah and Landsberg [5] by a factor of 8π.

Entropy, though additive between two systems, is not conserved. Neither is black hole entropy extensive. That is, if M, J, and Q are scaled up by a common factor, the entropy does not scale up by the same factor.

Define the black hole temperature T (proportional to the constant surface gravity), the angular velocity Ω, and the constant surface electric potential Φ by [3,5]

$$\frac{1}{T} \equiv \left(\frac{\partial S}{\partial M}\right)_{J,Q}, \quad (8)$$

$$-\frac{\Omega}{T} \equiv \left(\frac{\partial S}{\partial J}\right)_{M,Q}, \quad (9)$$

and

$$-\frac{\Phi}{T} \equiv \left(\frac{\partial S}{\partial Q}\right)_{M,J}. \quad (10)$$

Equations (2) and (8) yield

$$\frac{1}{T} = \frac{M}{4}\left(2 + \frac{2 - \beta}{\sqrt{1 - \alpha - \beta}}\right), \quad (11)$$

where the positive dimensionless quantities

$$\alpha \equiv \frac{J^2}{M^4}, \quad (12)$$

and

$$\beta \equiv \frac{Q^2}{M^2}. \quad (13)$$

Real T and S require

$$\alpha + \beta < 1. \quad (14)$$

Equality in Eq. (14) would clearly have $T \to 0$, which violates the third law of black hole thermodynamics forbidding naked singularities [9]. I refer to conditions satisfying Eq. (14) as the physical regime.

$S(M, J, Q)$ is a homogeneous function of the variables $(M, \sqrt{J}, Q)$ [6]. I shall use the term special homogeneous function (SHF) for a function of the form

$$M^a f(\alpha, \beta), \quad (15)$$

where the constant a and the function $f(\alpha, \beta)$ are dimensionless. S is an SHF since, by Eq. (2),

$$S = \tfrac{1}{8}M^2(2 - \beta + 2\sqrt{1 - \alpha - \beta}). \quad (16)$$

It is straightforward to prove: (1) if a function is an SHF, then its derivative with respect to any of M, J, or Q is an SHF, (2) multiplying and dividing two SHF's results in an SHF, and (3) adding or subtracting two SHF's results in an SHF. The last follows since addition or subtraction requires the SHF's to have the same units, and hence the same value of a. Since we may construct all the thermodynamic functions in this paper with the operations above starting with $S(M, J, Q)$, they must all be SHF's. Hence, we may represent the signs of our seven Hessian determinants with just two variables, rather than the three we might expect.

B. Black hole fluctuations

Consider now the environment surrounding the black hole. I regard it to be an extensive, infinite thermodynamic system of particles and photons with entropy S_e. As discussed by Hawking [2], for *finite* environments scenarios of full equilibrium with a single black hole exist. Pavón and Rubí [10] have worked out a thermodynamic fluctuation theory for such a case. Katz, Okamoto, and Kaburaki [11] have also analyzed stability for a number of ensembles involving Kerr-Newman black holes. But theories based on finite environments depend as well on the thermodynamics of the environment, which might not be known. By contrast, the more physically relevant extensive, infinite environments result in fluctuation expressions dependent only on the known thermodynamic properties of the black hole.

The total entropy

$$S_{\rm tot} = S + S_e \tag{17}$$

obeys a generalized second law of thermodynamics that for a closed system it not decrease [1–3]. I will build the thermodynamic fluctuation theory around the Gaussian expansion of $S_{\rm tot}$ [4].

Introduce the notation [12]

$$(X^1, X^2, X^3) \equiv (M, J, Q), \tag{18}$$

and

$$F_\alpha \equiv \frac{\partial S}{\partial X^\alpha}, \tag{19}$$

with corresponding properties of the environment denoted by the subscript e. Let us assume (incorrectly, as we will see) that the black hole and the environment are fully in equilibrium, with a local maximum for $S_{\rm tot}$. Consider a small fluctuation ΔX^α away from this equilibrium. To second order,

$$\Delta S_{\rm tot} = F_\mu \Delta X^\mu + F_{e\mu}\Delta X_e^\mu + \frac{1}{2}\frac{\partial F_\mu}{\partial X^\nu}\Delta X^\mu \Delta X^\nu + \frac{1}{2}\frac{\partial F_{e\mu}}{\partial X_e^\nu}\Delta X_e^\mu \Delta X_e^\nu, \tag{20}$$

where the coefficients are evaluated at the equilibrium state, which is set by the environment. The conservation laws demand

$$\Delta X^\alpha = -\Delta X_e^\alpha, \tag{21}$$

and a necessary condition for maximum entropy is

$$F_\alpha = F_{e\alpha}. \tag{22}$$

If the environment is very large, the second quadratic term in Eq. (20) is negligible compare with the first. To see this, fix the values of ΔX_e^α. As the extensive environment is scaled up to infinity at fixed $F_{e\alpha}$, X_e^α scales up without limit, and $\partial F_{e\alpha}/\partial X_e^\beta \to 0$. The ability to drop this quadratic term is a significant simplification offered by an infinite environment.

Equation (20) now becomes

$$\Delta S_{\rm tot} = -\tfrac{1}{2} g_{\mu\nu}\Delta X^\mu \Delta X^\nu, \tag{23}$$

where the symmetric matrix

$$g_{\alpha\beta} \equiv -\frac{\partial^2 S}{\partial X^\alpha \partial X^\beta}. \tag{24}$$

An essential point in this paper is that if we set one or two ΔX^α to zero, the sum Eq. (23) is only trivially modified. The corresponding F_α's need not be set equal as in Eq. (22). Entropy maximum requires that the matrix $g_{\alpha\beta}$ of the remaining variable(s) be positive definite. Conditions under which this might be satisfied are discussed in Sec. III.

The Gaussian approximation to the thermodynamic fluctuation theory results from Eqs. (1), (7), and (23) [4]. The probability (written for $n = 3$ variables) of finding the thermodynamic state in the range (X^1, X^2, X^3) to $(X^1 + dX^1, X^2 + dX^2, X^3 + dX^3)$ is

$$PdX^1dX^2dX^3 = \frac{\sqrt{\gamma}}{(2\pi)^{n/2}}\exp\left(-\frac{1}{2}\gamma_{\mu\nu}\Delta X^\mu \Delta X^\nu\right) \times dX^1 dX^2 dX^3, \tag{25}$$

where

$$\gamma_{\alpha\beta} \equiv \left(\frac{8\pi}{L_p^2}\right) g_{\alpha\beta}, \tag{26}$$

and γ is the determinant of $\gamma_{\alpha\beta}$. The modification if $n < 3$ is obvious.

All first fluctuation moments are zero [4]:

$$\langle \Delta X^\alpha \rangle = 0. \tag{27}$$

Second fluctuation moments are

$$\langle \Delta X^\alpha \Delta X^\beta \rangle = \gamma^{\alpha\beta}, \tag{28}$$

with $\gamma^{\alpha\beta}$ the components of the inverse $\gamma_{\alpha\beta}$ matrix.

C. Black hole phase transitions

Discussed have been possible phase transitions associated with black holes [3]; see Refs. [13,14] for recent references. In regular thermodynamic systems, phase transitions are usually presented in the context of microscopic properties coupling to macroscopic thermodynamics [15]. Such an approach is not yet possible with black holes. Although there is a thermodynamic picture, the microscopic picture is mostly missing. This considerably complicates the discussion.

It is tempting to argue that the presence of diverging thermodynamic quantities signals phase transitions [3]. However, this connection is not guaranteed since it is easy to construct thermodynamic quantities which diverge for any state.[5] Perhaps, by analogy with fluid systems, the divergence of quantities such as heat capacities has some special status. But, in the absence of microscopic models, it is hard to be sure.

Interesting are discussions of phase transitions with some change of black hole topology [13,14]. But Kerr-Newman black holes have a spherical topology in the physical regime. Some authors draw conclusions from scaling relations among critical exponents [16]. The divergence of the curvature in a geometric approach to thermodynamics has also been connected to black hole phase transitions [13,14,17,18]. In this paper I add the divergence in second fluctuation moments of conserved quantities to the discussion. But, other than to suggest this, I draw no particular conclusions with respect to phase transitions.

III. SEVEN FLUCTUATION CASES

In this section, I analyze the seven possible fluctuation cases. The discussion is mostly formal, with considerations of physical significance deferred to the next section.

In reading difficult thermodynamic calculations, one can easily lose perspective in following simplification tricks. To avoid this, I work directly in M, J, and Q coordinates and use Mathematica to perform the computational tedium. Since stability and fluctuations are independent of the coordinates used to calculate them, we need never fear missing something essential by working in the "wrong" coordinates. The necessary quantities of analysis are the seven Hessian determinants p_1, p_2, p_3, p_1', p_2', p_2'', and p_1'' defined below. These are all SHF's, so I can represent their signs simply by finding their dependence on J and Q.

First, some general remarks about my results. All second fluctuation moments involving M, J, or Q in the stable regime go to zero at the limit of the physical regime ($\alpha + \beta = 1$). All cases involving fluctuations at fixed M are stable for all thermodynamic states. This includes physically perhaps the most important state with average $J = Q = 0$, corresponding to a nonrotating, uncharged environment. No case with fluctuating M is stable for this state. The switch from unstable to stable is accompanied by an infinity in $\sqrt{\langle(\Delta M)^2\rangle}$. For purely M fluctuations, this transition occurs at the Davies phase transition point.

A. (M, J, Q) fluctuating

Here, all three (M, J, Q) fluctuate and relevant is the full second order expansion for ΔS_{tot}, Eq. (23). Necessary and sufficient conditions [19] that the quadratic form for $-\Delta S_{\mathrm{tot}}$ be positive definite is that

$$p_1 \equiv g_{11} > 0, \tag{29}$$

[5]For example, if f is a regular thermodynamic function with value f_0 at thermodynamic state 0, $1/(f - f_0)$ diverges at 0.

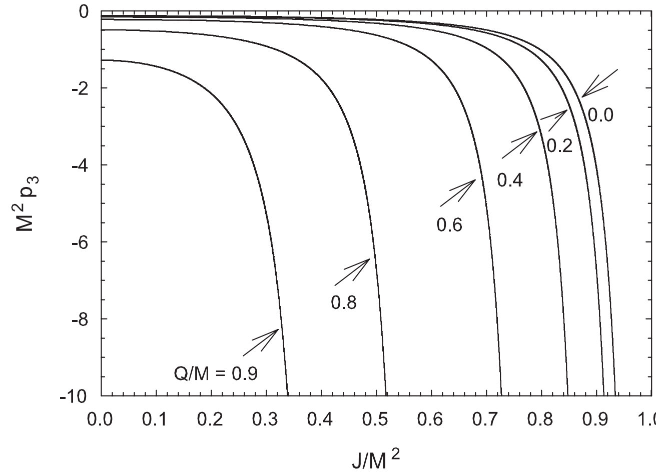

FIG. 1. $M^2 p_3$ as a function of J/M^2 for several values of Q/M. p_3 is negative for all values shown. The curves all diverge at the end of the physical regime $\alpha + \beta = 1$.

$$p_2 \equiv \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} > 0, \tag{30}$$

and

$$p_3 \equiv \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} > 0. \tag{31}$$

Consider first p_3. Direct evaluation of Eq. (31), using Eqs. (2) and (24), shows

$$p_3 = \frac{-K^4 - 3K^3 - 3K^2 - 3L^2K + 2K + L^4 - 5L^2 + 4}{64M^2K^5}, \tag{32}$$

where the combination of variables [5]:

$$K \equiv \sqrt{1 - \alpha - \beta}, \tag{33}$$

and

$$L \equiv \sqrt{1 + \alpha}. \tag{34}$$

Figure 1 shows $M^2 p_3$. The values displayed are all negative. Indeed, Tranah and Landsberg [5] proved that $p_3 < 0$ for all states in the physical regime, violating the inequality Eq. (31). There are thus no stable cases with (M, J, Q) all fluctuating. Figure 2 show this schematically.

Note also that p_3 diverges at the limit of the physical regime, $K \to 0$, as do all of the Hessian determinants in this section.

B. (J, Q) fluctuating, M fixed

Here, (J, Q) fluctuate at fixed M. By Eq. (23),

$$-2\Delta S_{\mathrm{tot}} = g_{22}(\Delta J)^2 + 2g_{23}\Delta J \Delta Q + g_{33}(\Delta Q)^2. \tag{35}$$

By Eq. (20), fixing the masses means that the temperatures of the black hole and the environment need not match.

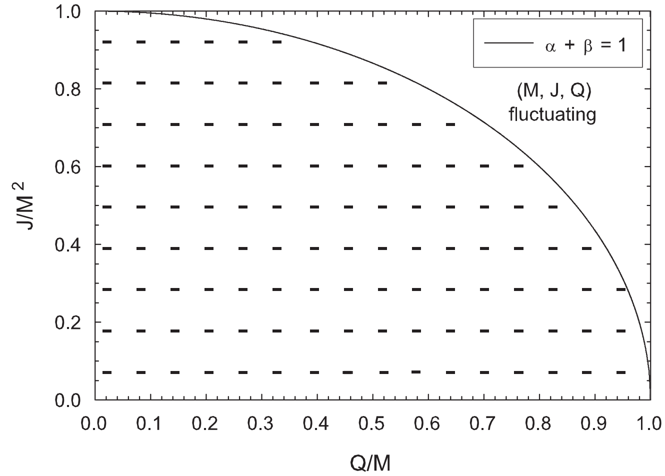

FIG. 2. Fluctuation regimes for (M, J, Q) all free to fluctuate. The curve marks the end of the physical regime $\alpha + \beta = 1$. The minus signs (signs of p_3) indicate that S_{tot} has no local maximum anywhere in the physical regime.

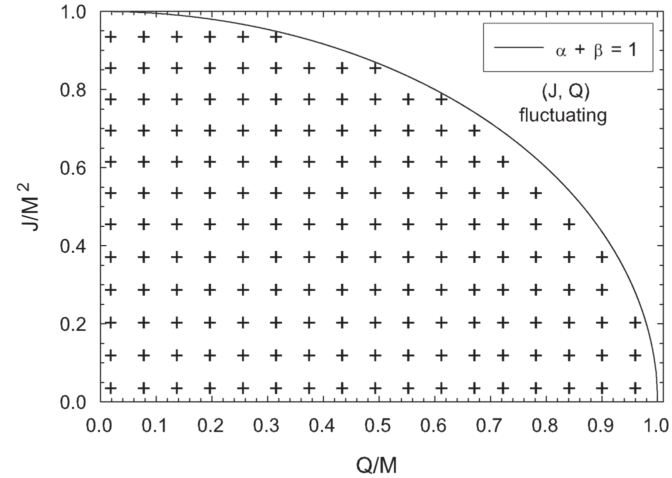

FIG. 3. Stable fluctuation regime for (J, Q) fluctuating at fixed M. The plus signs indicate the signs of both p_1' and p_2'. Clearly, S_{tot} may have a local maximum at any point in the physical regime.

Maximum entropy in the equilibrium state requires

$$p_1' \equiv g_{22} > 0, \tag{36}$$

and

$$p_2' \equiv \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} > 0. \tag{37}$$

Direct calculation shows

$$p_1' = \frac{K^2 + L^2 - 1}{4M^2K^3}, \tag{38}$$

and

$$p_2' = \frac{K^3 + (L^2 - 1)K + 1}{16M^2K^4}. \tag{39}$$

Since $K \geq 0$ and $L \geq 1$, we see that p_1' and p_2' are never negative in the physical regime, and fluctuations are stable. Figure 3 shows this schematically.

Physically, the environment most easily realized is nonrotating and uncharged. This corresponds to average values $J = Q = 0$. Here, direct calculation with Eq. (28) yields the fluctuation moments shown in Table I. Fluctuations in J scale up directly with M. At the Planck mass

$$M_p \equiv \sqrt{\frac{\hbar c}{G}} = 2.177 \times 10^{-5} \text{ g} = 1.616 \times 10^{-33} \text{ cm}, \tag{40}$$

J fluctuates at a little less than half $\hbar$.[6] Fluctuations in Q are independent of M, a little more than three fundamental charges.

Table I also shows the fluctuation moments along the axes $J = 0$ and $Q = 0$. For given K, the mass dependence is the same as for the origin. Along each coordinate axis, fluctuation moments for both J and Q are maximum at the origin, and decrease monotonically to zero at the end of the physical regime.

C. (M, Q) fluctuating, J fixed

Here, (M, Q) fluctuate at fixed J. By Eq. (23),

$$-2\Delta S_{\text{tot}} = g_{11}(\Delta M)^2 + 2g_{13}\Delta M \Delta Q + g_{33}(\Delta Q)^2. \tag{41}$$

TABLE I. Fluctuation moments for stable fluctuations at the origin and along the axes of J, Q space for (top to bottom) (J, Q), (M, Q), and (M, J) fluctuations. Unstable cases are indicated with a "...." In all the stable cases on the coordinate axes, off diagonal elements, e.g. $\langle \Delta M \Delta J \rangle$, are zero.

(J, Q)	$\sqrt{\langle(\Delta J)^2\rangle}/\hbar$	$\sqrt{\langle(\Delta Q)^2\rangle}/e$
$J = Q = 0$	$0.3990(\frac{M}{M_p})$	3.301
$J = 0$	$0.3990\sqrt{K}(\frac{M}{M_p})$	$4.669(\frac{K^3}{1+K^3})^{1/2}$
$Q = 0$	$0.3990\sqrt{K^3}(\frac{M}{M_p})$	$4.669(\frac{K}{1+K})^{1/2}$
(M, Q)	$\sqrt{\langle(\Delta M)^2\rangle}/m_e$	$\sqrt{\langle(\Delta Q)^2\rangle}/e$
$J = Q = 0$	...	...
$J = 0$	...	...
$Q = 0$	$6.742 \times 10^{21}\sqrt{\frac{-K^3}{(1+K)(-2+2K+K^2)}}$	$4.669(\frac{K}{1+K})^{1/2}$
(M, J)	$\sqrt{\langle(\Delta M)^2\rangle}/m_e$	$\sqrt{\langle(\Delta J)^2\rangle}/\hbar$
$J = Q = 0$	...	...
$J = 0$	$9.535 \times 10^{21}\sqrt{\frac{-K^3}{(1+K)^2(-1+2K)}}$	$0.3990\sqrt{K}(\frac{M}{M_p})$
$Q = 0$	...	...

[6] For a Planck mass black hole, the entropy in Eq. (7), and therefore the number of microstates, is of order unity. It is hard to imagine the character of a black hole with a smaller mass.

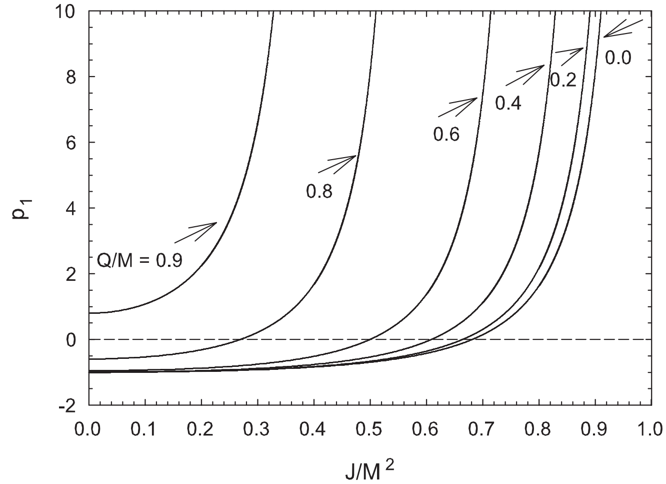

FIG. 4. p_1 as a function of J/M^2 for several values of Q/M. p_1 may be positive or negative. The curves all diverge at the end of the physical regime.

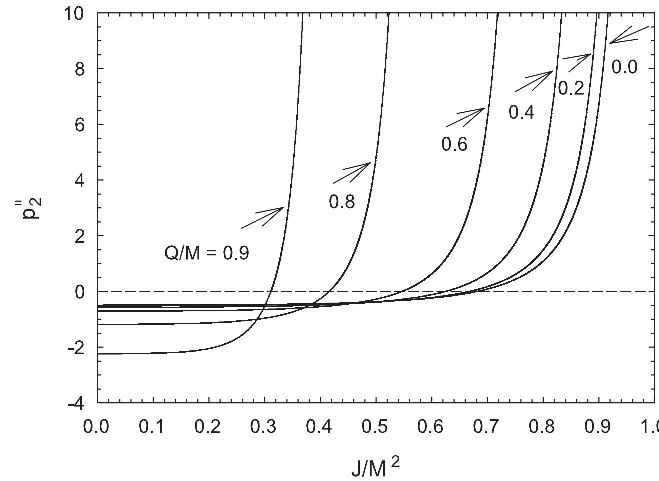

FIG. 5. p_2'' as a function of J/M^2 for several values of Q/M. p_2'' may be positive or negative. The curves all diverge at the end of the physical regime.

By Eq. (20), fixing J means that the angular velocities of the black hole and its environment need not match.

Stability requires

$$p_1 > 0, \tag{42}$$

and

$$p_2'' \equiv \begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{vmatrix} > 0. \tag{43}$$

Direct calculation shows

$$p_1 = -\frac{(2K - L^2)(K^2 + 2K + L^2)}{4K^3}, \tag{44}$$

and

$$p_2'' = -\frac{K^4 - (L^2 - 4)K^3 + (L^2 + 2)K^2 - (L^4 + 2L^2 - 4)K + 2(L^4 - 5L^2 + 4)}{16K^4}. \tag{45}$$

Figure 4 shows p_1. It has regimes of positive and negative values separated by a curve of zeros which follows from setting the numerator in Eq. (44) to zero. The only solution in the physical regime is $2K = L^2$, which may be written

$$\alpha^2 + 6\alpha + 4\beta = 3. \tag{46}$$

As emphasized by Davies [3], Eq. (46) marks a diverging heat capacity

$$C_{J,Q} \equiv T\left(\frac{\partial S}{\partial T}\right)_{J,Q}. \tag{47}$$

This connects to p_1 via[7]

$$p_1 = \frac{1}{T^2 C_{J,Q}}. \tag{48}$$

Figure 5 shows p_2'', which has regimes of positive and negative values separated by a curve of zeros.[8] Fig. 6 shows this curve as well as the curve $p_1 = 0$. Also shown is the regime of stable fluctuations where both p_1 and p_2'' are positive. Clearly, the only coordinate axis in J, Q space with stable fluctuations is the J axis. Table I shows the stable (M, Q) fluctuation moments for $Q = 0$, with m_e the electron mass. Fluctuations in M start unstable at the origin, but become stable when K exceeds 0.7321 ($J/M^2 = 0.6813$).

[7]There is a minor typographical error in Ref. [5]. On the right hand side of Eq. (3.17), C should be L.

[8]This curve of zeroes corresponds to diverging $C_{J,\Phi}$ in Ref. [5].

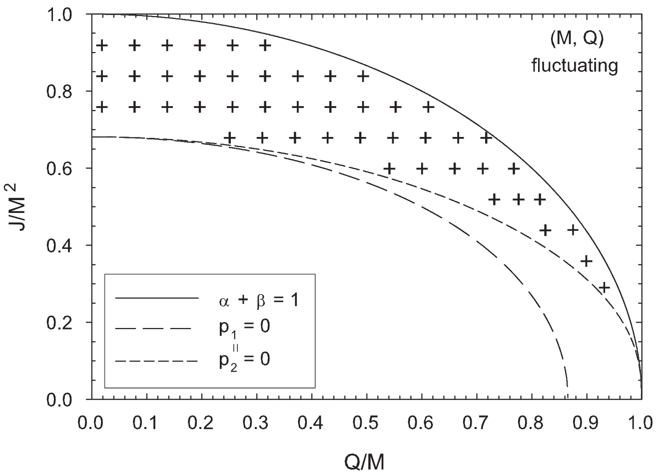

FIG. 6. Stable fluctuation regime for (M, Q) at fixed J. The physical regime where p_1 and p_2'' are both positive is indicated by + signs.

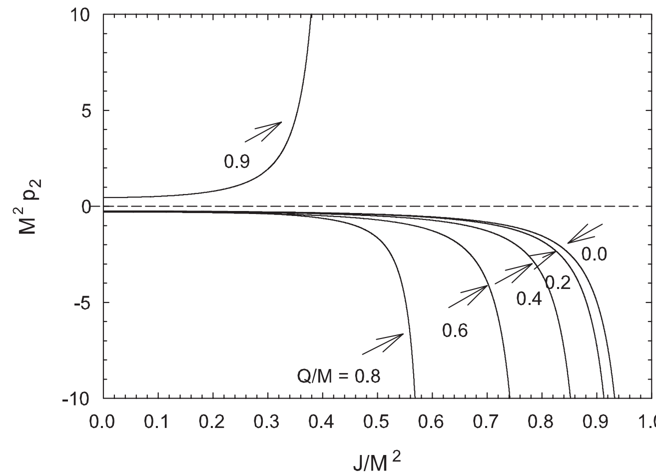

FIG. 7. $M^2 p_2$ as a function of J/M^2 for several values of Q/M. p_2 may be positive or negative. The curves all diverge at the end of the physical regime.

For given K, neither fluctuation moment depends on M. Fluctuations in M expressed in electron units are huge, requiring many electron masses to produce a sizable change in $\Delta S_{\rm tot}$. In contrast, very few electron charges are required to vary Q by its fluctuation moment. This pattern is more general, and leads me in the next section to consider mass as a drifting parameter not governed by the statistics of thermodynamic fluctuations.[9]

All the fluctuation moments diverge along the curve $p_2'' = 0$ because on inverting the coefficient matrix in Eq. (41) p_2'' goes on the bottom. There are no anomalies in the fluctuation moments along the Davies infinity curve $p_1 = 0$, which is not in the stable regime.

D. (M, J) fluctuating, Q fixed

Here, (M, J) fluctuate at fixed Q. By Eq. (23),

$$-2\Delta S_{\rm tot} = g_{11}(\Delta M)^2 + 2g_{12}\Delta M \Delta J + g_{22}(\Delta J)^2. \quad (49)$$

By Eq. (20), fixing Q means that the potentials of the black hole and its environment need not match.

Stability requires

$$p_1 > 0, \quad (50)$$

and

$$p_2 > 0. \quad (51)$$

Direct calculation shows

$$p_2 = -\frac{2K^3 + 3K^2 + 2(L^2 - 1)K + 3L^2 - 4}{16M^2K^4}. \quad (52)$$

[9]However, if the charged entities producing fluctuations were predominantly, say, quantum black holes with the Planck mass, and we substituted M_p for m_e in Table I, the multiplier for $\sqrt{\langle(\Delta M)^2\rangle}/M_p$ would be the far more reasonable 0.3990.

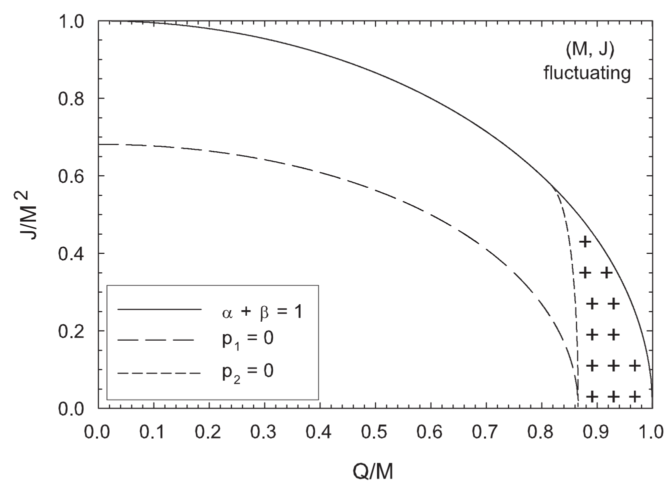

FIG. 8. Stable fluctuation regime for (M, J) fluctuating at fixed Q. The physical regime where p_1 and p_2 are both positive is indicated by + signs.

Figure 7 shows $M^2 p_2$. It has both positive and negative values. p_2 is zero[10] in the physical regime if and only if

$$\alpha = \frac{(3 - 4\beta)\beta^2}{4(\beta - 1)^2}. \quad (53)$$

This curve of zeros is shown in Fig. 8, along with that for p_1. There is a regime in the physical region where both p_1 and p_2 are positive and fluctuations are stable.

From Fig. 8, the only coordinate axis with stable fluctuations is the Q axis. Table I shows the stable (M, J) fluctuation moments. Fluctuations start unstable at the origin, but for $J = 0$ become stable at $K = 1/2$ ($Q/M = 0.8660$). This stable regime corresponds to the black hole

[10]This curve of zeroes corresponds to diverging $C_{\Omega,Q}$ in Ref. [5].

in a nonrotating environment. For given K, the fluctuation moment for M is mass independent while that for J scales up in proportion to mass.

Off the coordinate axes, the fluctuation moments all diverge along the curve $p_2 = 0$, since p_2 is the determinant of the coefficients in Eq. (49). There are no anomalies in the fluctuation moments along the Davies curve $p_1 = 0$, which is not in the stable regime.

E. *M* fluctuating, *J*, *Q* fixed

Here, M fluctuates at fixed J and Q. By Eq. (23),

$$-2\Delta S_{\mathrm{tot}} = g_{11}(\Delta M)^2. \tag{54}$$

Stability requires

$$p_1 > 0. \tag{55}$$

Figure 9 shows the regime of stable fluctuations. The origin is unstable, but each coordinate axis has a stable regime, shown in Table II. Fluctuations in M become stable along the Q axis at $K = 1/2$ ($Q/M = 0.8660$), and along the J axis at $K = 0.7321$ ($J/M^2 = 0.6813$). For given K, the fluctuation moment for M is mass independent.

Off the coordinate axes, the M fluctuation moment diverges on the Davies anomaly curve $p_1 = 0$.

F. *J* fluctuating, *M*, *Q* fixed

Here, J fluctuates at fixed M and Q. By Eq. (23),

$$-2\Delta S_{\mathrm{tot}} = g_{22}(\Delta J)^2. \tag{56}$$

Stability requires

$$p_1' > 0. \tag{57}$$

But p_1' is never negative in the physical regime, so all fluctuations are stable. Figure 10 shows the regime of stable fluctuations.

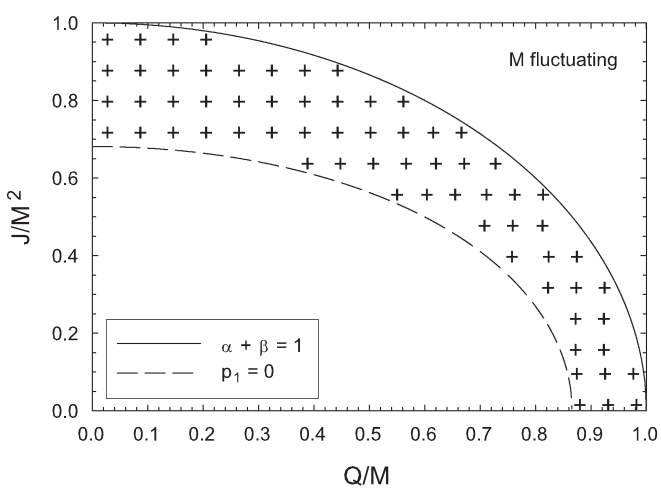

FIG. 9. Stable fluctuation regime for M fluctuating at fixed J and Q. The physical regime where p_1 is positive is indicated by $+$ signs.

TABLE II. Fluctuation moments for stable fluctuations at the origin and along the axes of J, Q space for (top to bottom) M, J, and Q fluctuations. Unstable cases are indicated with a "…"

M	$\sqrt{\langle(\Delta M)^2\rangle}/m_e$
$J = Q = 0$	…
$J = 0$	$9.535 \times 10^{21}\sqrt{\frac{-K^3}{(1+K)^2(-1+2K)}}$
$Q = 0$	$6.742 \times 10^{21}\sqrt{\frac{-K^3}{(1+K)^2(-2+2K+K^2)}}$
J	$\sqrt{\langle(\Delta J)^2\rangle}/\hbar$
$J = Q = 0$	$0.3990(\frac{M}{M_p})$
$J = 0$	$0.3990\sqrt{K}(\frac{M}{M_p})$
$Q = 0$	$0.3990\sqrt{K^3}(\frac{M}{M_p})$
Q	$\sqrt{\langle(\Delta Q)^2\rangle}/e$
$J = Q = 0$	3.301
$J = 0$	$4.669\sqrt{\frac{K^3}{1+K^3}}$
$Q = 0$	$4.669\sqrt{\frac{K}{1+K}}$

Table II shows fluctuations in J at the origin and along the axes. For given K they scale up in proportion to M and are maximum at the origin.

G. *Q* fluctuating, *M*, *J* fixed

Here, Q fluctuates at fixed M and J. By Eq. (23),

$$-2\Delta S_{\mathrm{tot}} = g_{33}(\Delta Q)^2. \tag{58}$$

Stability requires

$$p_1'' \equiv g_{33} > 0. \tag{59}$$

We find

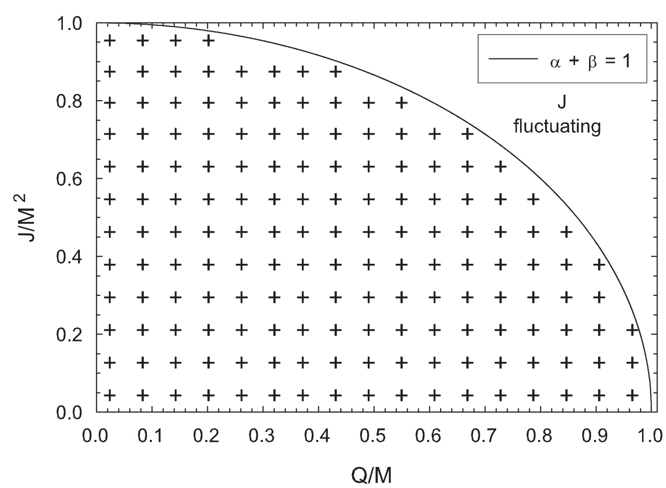

FIG. 10. Stable fluctuation regime for J fluctuating at fixed M and Q. The physical regime where p_1' is positive is indicated by $+$ signs.

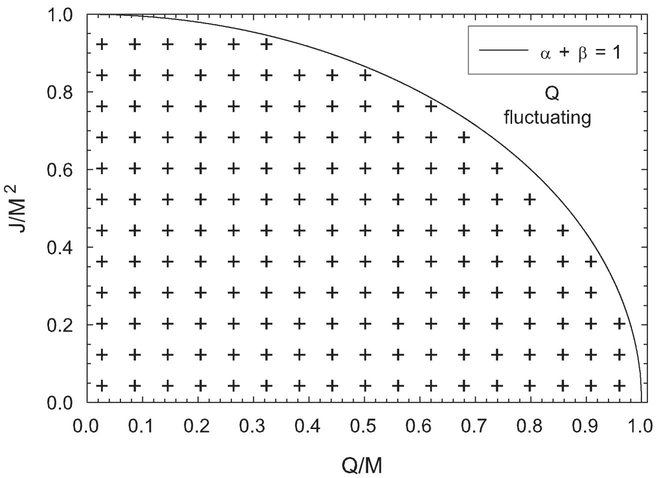

FIG. 11. Stable fluctuation regime for Q fluctuating at fixed M and J. The physical regime where p_1'' is positive is indicated by $+$ signs.

$$p_1'' = \frac{K^3 - L^2 + 2}{4K^3}. \tag{60}$$

Since $K \geq 0$ and $L \leq \sqrt{2}$, p_1'' is never negative, and all fluctuations are stable. Figure 11 shows the regime of stable fluctuations.

Table II shows fluctuations in Q at the origin and along the axes. For given K, they are independent of M and are maximum at the origin.

IV. PHYSICAL CONSIDERATIONS

In this section, I give a limited discussion of which fluctuation cases might have physical relevance.

Start by making an imperfect analogy to a classical thermodynamics problem. Consider a binary fluid mixture of two noninteracting components. The mixture is separated into two parts by a rigid semipermeable membrane which lets through one of the fluid components, but not the other. One of the parts is finite in size, and the other is an infinite environment.

Each of the two parts can be characterized by three independent conserved thermodynamic variables, the internal energy and the two particle numbers. These variables would be X^α and X_e^α in the formalism of Sec. II B. There are three conjugate variables F_α and $F_{e\alpha}$ for each part. Pick X^3, with conjugate variable F_3, as the fixed particle number.

The two parts in contact will exchange energy X^1 and particles X^2 until they reach equilibrium where $S_{\rm tot}$ is maximized, subject to the constraint $X^3 = \text{const}$. A maximum requires that the first order terms in Eq. (20) sum to zero. Since $\Delta X^3 = -\Delta X_e^3 = 0$, there is no need to require $F_3 = F_{e3}$. But fluctuating X^1 and X^2 do require $F_\alpha = F_{e\alpha}$ for $\alpha = 1, 2$. These two conditions, and $X_3 = \text{const}$, set the equilibrium state. The environment's second order term in Eq. (20) is negligible for the same reason as before (extensive, infinite environment), and we get the Gaussian fluctuation theory Eq. (25) with just two fluctuating independent variables.

In reality, the membrane will not be perfect, and X^3 will drift slowly. But we can continue to use thermodynamic fluctuation theory Eq. (25) as a good approximation simply by adjusting the equilibrium values of X^1 and X^2 and the expansion coefficients $\gamma_{\alpha\beta}$ as X^3 drifts. This discussion extends easily to the case where the membrane is impermeable to both components, transmitting only heat.

The black hole problem here is an imperfect realization of this binary fluid problem. First, there is no membrane impermeable to M, J, or Q. They all fluctuate. Neither could we reasonably select a specially prepared environment to control one of the variables (e.g., a massless, spinless, or a chargeless gas). The black hole will presumably create whatever particles it likes near its event horizon, and equilibrium is unlikely until the environment is populated by particles in the same proportion to those created.

But this does not prevent us from moving ahead. Davies[11] pointed out that for charged nonquantum black holes the spin down rate is very slow compared with electric discharge processes. This suggests (M, Q), M, or Q fluctuations. Davies also pointed out that if the black hole is in a thermal radiation bath, J and Q are fixed if superadiance can be neglected, suggesting M fluctuations. A big contribution by Davies was to show that M fluctuations have a regime of stability (shown here in Fig. 9) despite the fact that the heat capacity of self gravitating systems is usually negative.

There is no obvious measure for comparing the relative significance of changes of different quantities. My approach is to estimate the contribution to the second order term in $\Delta S_{\rm tot}$ of the various properties of a slow electron added to the black hole. Properties which contribute little are judged to be disconnected from thermodynamic fluctuations, and taken to be slowly drifting.

If a slow electron falls into a black hole, M will change by the electron mass m_e, J roughly by the electron spin angular momentum, about $\hbar$, and Q by the electron charge e. So put

$$(dM, dJ, dQ) = (m_e, \hbar, e). \tag{61}$$

Now pick a state (M, J, Q), and compare the variation in $-2\Delta S_{\rm tot}$ in Eq. (23) on trying in turn each of the three (dM, dJ, dQ). Of course, there is much more to fluctuations than slow electrons, but the numbers shown below are so extreme that my simple argument might be at least representative of the full picture.

The diagonal elements of $-2\Delta S_{\rm tot}$ are $(g_{11}dM^2, g_{22}dJ^2, g_{33}dQ^2)$. Define the normalized quantities

[11] See page 511 of Ref. [3].

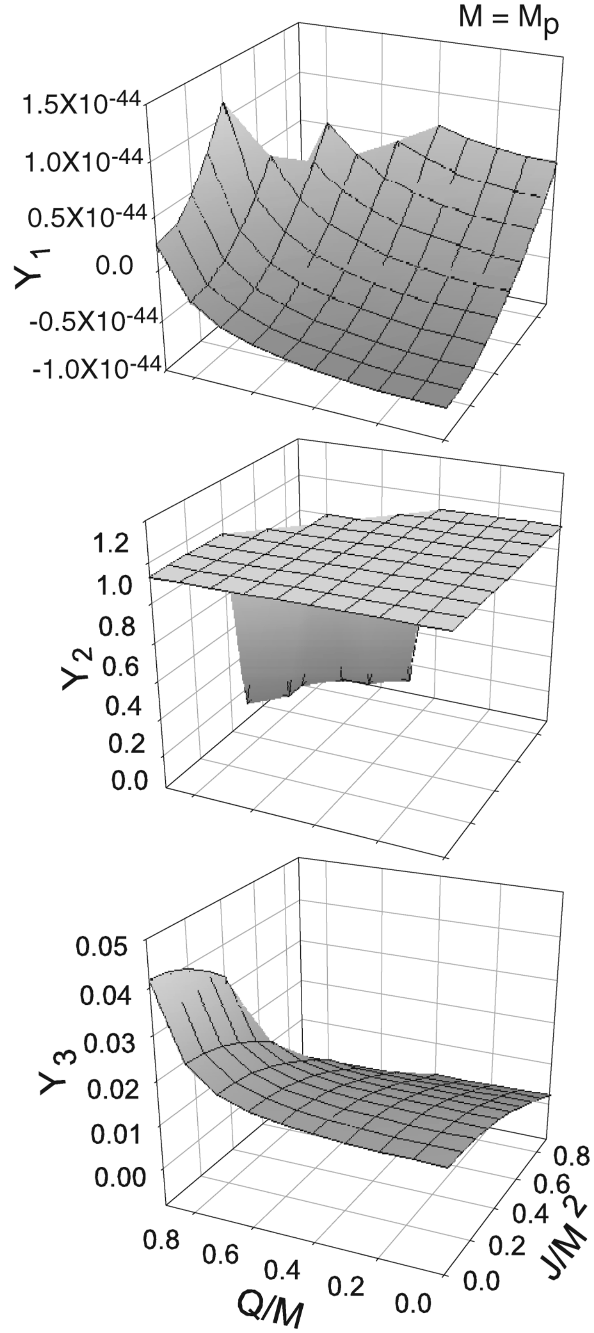

FIG. 12. Y_1 (for dM), Y_2 (for dJ), and Y_3 (for dQ) as functions of J/M^2 and Q/M for a Planck mass black hole $M = M_p$. Values outside the physical regime are set to zero. It is seen that dM has a negligible effect on $-2\Delta S_{\rm tot}$, indicating that (J, Q) fluctuations are the most significant here.

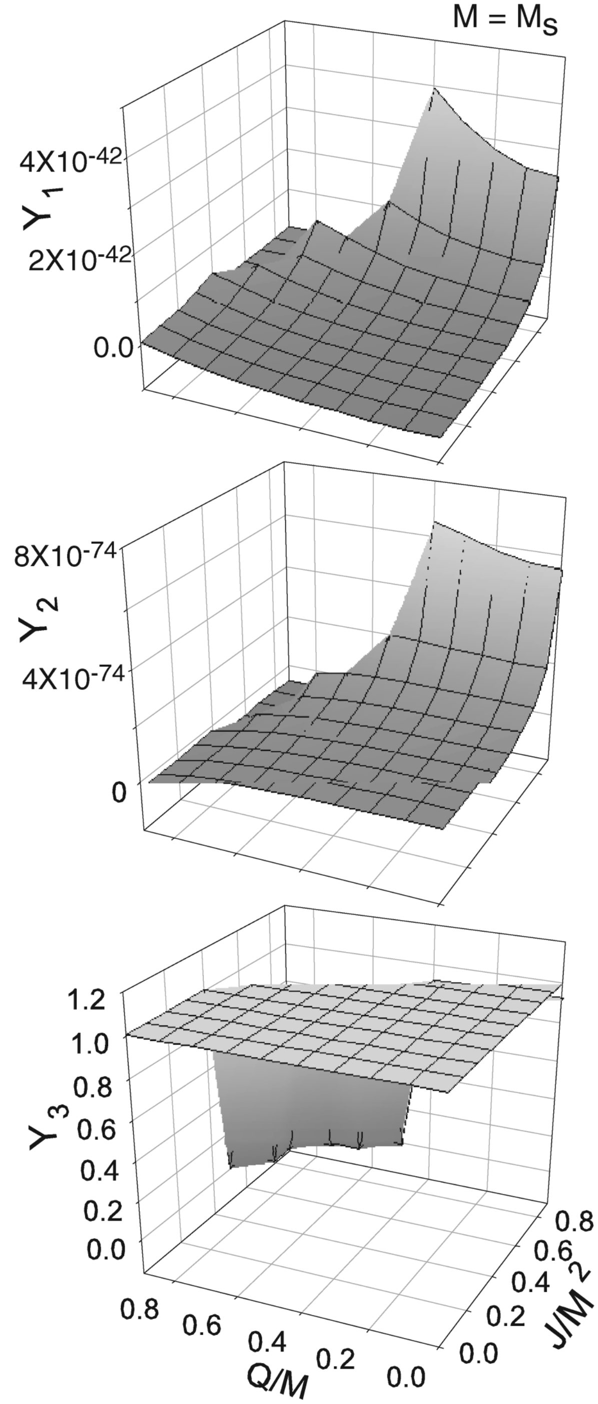

FIG. 13. Y_1 (for dM), Y_2 (for dJ), and Y_3 (for dQ) as functions of J/M^2 and Q/M for a solar mass black hole $M = M_s$. It is seen that dJ and dM have negligible effect on $-2\Delta S_{\rm tot}$, indicating that Q fluctuations are the most significant here.

$$Y_\alpha \equiv \frac{g_{\alpha\alpha}(dX^\alpha)^2}{\sqrt{\sum_{\mu=1}^{3}[g_{\mu\mu}(dX^\mu)^2]^2}}. \tag{62}$$

The second derivatives $g_{11} = p_1$, $g_{22} = p_1'$, and $g_{33} = p_1''$ are given by Eqs. (38), (44), and (60).

The relative sizes of Y_α depend on M. Consider first the Planck mass black hole. For $M = M_p$, Fig. 12 shows each Y_α for all physical J and Q. dM makes a negligible contribution to fluctuations, indicating (J, Q) fluctuations. This corresponds to Fig. 3 where all physical states have stable fluctuations.

Figure 13 shows the results for a solar mass black hole

$$M = M_s = 147\,700 \text{ cm}. \tag{63}$$

Here, dJ is by far the least important, with dM making a larger, though still insignificant, contribution. dQ dominates and Q fluctuations, shown in Fig. 11, seem appropriate. These are stable for all states.

V. CONCLUSIONS

I have worked out thermodynamic fluctuation theory for the Kerr-Newman black hole in an extensive, infinite environment. Such an environment has the advantage that its character is irrelevant beyond the values of the three parameters specifying its thermodynamic state. This is significant because the precise constitution of the universe is unknown (dark matter, etc). The known thermodynamics of the black hole is all that enters the structure of the theory.

Although the full problem where all (M, J, Q) fluctuate is not stable, it seems reasonable to consider limited cases where one or two of these conserved variables vary so slowly as to be considered fixed. These were all examined.

All fluctuations with M fixed are stable everywhere in the physical regime. Scenarios with M fixed may be physically the most relevant, as I argued with fluctuating particles having roughly the properties of an electron. For a Planck mass black hole, (J, Q) fluctuations seem most significant. As M increases, Q fluctuations become dominant. Physically, perhaps most important are fluctuations in a nonrotating and uncharged environment, with average $J = Q = 0$. Here fluctuations are maximum and the second fluctuation moment for Q is $3.301e$, independent of M. The second fluctuation moment for J scales up linearly with M. For the Planck mass it is $0.3990\hbar$. In cases where M fluctuates, the transition from unstable (near $J = Q = 0$) to stable is accompanied by infinite fluctuation moments.

[1] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973); **9**, 3292 (1974).
[2] S. W. Hawking, Phys. Rev. D **13**, 191 (1976).
[3] P. C. W. Davies, Proc. R. Soc. A **353**, 499 (1977).
[4] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, New York, 1977).
[5] D. Tranah and P. T. Landsberg, Collective Phenomena **3**, 81 (1980).
[6] L. Smarr, Phys. Rev. Lett. **30**, 71 (1973).
[7] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[8] J. D. Bekenstein, Sci. Am., p. 58 (2003).
[9] B. Carter, in *Black Holes*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973).
[10] D. Pavón and J. M. Rubí, Phys. Lett. A **99**, 214 (1983); Gen. Relativ. Gravit. **17**, 387 (1985).
[11] J. Katz, I. Okamoto, and O. Kaburaki, Classical Quantum Gravity **10**, 1323 (1993).
[12] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics* (Wiley, New York, 1985).
[13] G. Arcioni and E. Lozano-Tellechea, Phys. Rev. D **72**, 104021 (2005).
[14] J. E. Åman and N. Pidokrajt, Phys. Rev. D **73**, 024017 (2006).
[15] M. E. Fisher, in *Proceedings of the Gibbs Symposium*, edited by D. G. Caldi and G. D. Mostow (American Mathematical Society and American Physical Society, Providence, Rhode Island, 1990).
[16] O. Kaburaki, Phys. Lett. A **217**, 315 (1996).
[17] S. Ferrara, G. W. Gibbons, and R. Kallosh, Nucl. Phys. **B500**, 75 (1997).
[18] R. G. Cai and J. H. Cho, Phys. Rev. D **60**, 067502 (1999).
[19] H. Eves, *Elementary Matrix Theory* (Dover, New York, 1966).