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To cite this article: Stephen Boughn and Tony Rothman 2006 *Class. Quantum Grav.* **23** 5839

INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **23** (2006) 5839–5852 doi:10.1088/0264-9381/23/20/006

Aspects of graviton detection: graviton emission and absorption by atomic hydrogen

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Received 9 May 2006, in final form 27 July 2006
Published 20 September 2006
Online at stacks.iop.org/CQG/23/5839

Abstract
Graviton absorption cross sections and emission rates for hydrogen are calculated by both semi-classical and field theoretic methods. We point out several mistakes in the literature concerning spontaneous emission of gravitons and related phenomena, some of which are due to a subtle issue concerning gauge invariance of the linearized interaction Hamiltonian.

PACS numbers: 03.65.Sq, 04.30.Db, 04.30.Nk, 04.60.−m, 95.30.Cq, 95.30.Sf

1. Introduction

Two years ago, Dyson published a conjecture that no conceivable experiment performed in our universe could detect a single graviton [1]. Recently, in a companion paper [2] (henceforth RB), we have addressed Dyson's proposition and have been unable to find any clear-cut fundamental reason forbidding the detection of one graviton. However, when anything remotely resembling 'real' physics is taken into account, a detection becomes impossible, making Dyson's conjecture very likely true, at least without the introduction of exotic physics, such as extra dimensions.

In the process of checking our calculations against graviton spontaneous emission rates for hydrogen, we have discovered several mistakes in the literature involving both spontaneous emission rates and the response of elastic media to classical gravitational waves. Although some of the mistakes are evidently numerical, others involve a rather subtle issue of gauge invariance of the interaction Hamiltonian in linearized gravity. Our purpose here is to present these calculations in detail, thereby illuminating the source of the difficulties. To demonstrate the validity of our results, we compute transition rates from the 3d to 1s hydrogen states using both semi-classical and field theoretic methods in two different gauges. We also present the consistency check that led to the discovery of the mistakes: the requirement that detailed balance be satisfied among the spontaneous emission, stimulated emission and absorption rates. Finally, we compute the graviton ionization cross section of hydrogen, because in RB we chose this gravitational analogy to the ordinary photoelectric effect as the most relevant one in addressing Dyson's question.

2. Semi-classical analysis of spontaneous graviton emission

In the standard semi-classical treatment of spontaneous emission of photons from atomic dipole transitions (see, for example, [3]), the transition rate is found by beginning with the classical expression for power emitted by a dipole,

$$P = \frac{4k^2}{3c}|\mathbf{J}_0|^2, \tag{2.1}$$

in which $\mathbf{J}_0 \equiv \int \mathbf{J}(\mathbf{r})\, \mathrm{d}^3r$ is the total current and k is the wave number of the emitted radiation. One then attempts to interpret this expression in a quantum mechanical way by identifying the current density $\mathbf{J}$ (charge density times velocity) with the probability current $e|\Psi|^2\mathbf{p}/m$, for wavefunction Ψ and electron charge e. Because the quantum mechanical transition is between two states a and b, the standard treatment replaces $e|\Psi|^2$ with $e\Psi_a^*\Psi_b$, and so the total current becomes $\mathbf{J}_0 = \frac{e}{m}\int \Psi_a^*\mathbf{p}\Psi_b\, \mathrm{d}^3r$. Finally, the quantum mechanical transition rate is assumed to be

$$\Gamma = \frac{P}{\hbar\omega},$$

with P given by inserting the 'quantum' expression for $\mathbf{J}_0$ into equation (2.1). The justification for this argument is, to say the least, tenuous; therefore, it is important to verify the resulting transition rate in some other way, e.g., that it satisfies detailed balance or that a 'proper' field theoretic calculation yields the same result [3].

For the case of spontaneous graviton emission, we begin with the (quadrupole) gravitational analogue of equation (2.1),

$$P = \frac{2G\omega^6}{5c^5}\langle I_{jk}I_{jk}\rangle. \tag{2.2}$$

In this expression, $I_{jk} \equiv \int \rho(\mathbf{r})\left(x_jx_k - \frac{1}{3}\delta_{jk}r^2\right) \mathrm{d}^3r$ is the reduced mass quadrupole moment, $\mathbf{r} = \{x_j\}$ are Cartesian coordinates in the local inertial frame and ρ is the mass density of the radiating system [4]. The normalization for I_{jk}, as well as $\mathbf{J}$ above, is usually taken to be $I_{jk}(\mathbf{r}, t) = I_{jk}(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}$ Following the electromagnetic procedure, the semi-classical approach for gravity substitutes the electron mass m_e for e and replaces ρ by $m_\mathrm{e}\Psi_b^*\Psi_a$. (The equivalent mass density associated with the electric fields of the electron and proton is, for non-relativistic systems, small compared to the electron mass; we discuss this issue in more detail in section 5.) Therefore, the reduced quadrupole moment becomes

$$I_{jk} = m_\mathrm{e}\int \Psi_b^*\left(x_jx_k - \frac{1}{3}\delta_{jk}r^2\right)\Psi_a\, \mathrm{d}^3r. \tag{2.3}$$

To compare with previous work, we can now compute the transition rate from the 3d2 to the 1s state of hydrogen. In terms of the Bohr radius $a = \hbar^2/m_\mathrm{e}e^2$, the normalized wavefunctions are

$$\Psi_{1\mathrm{s}} = \frac{1}{\sqrt{\pi}a^{3/2}}\,\mathrm{e}^{-r/a}, \qquad \Psi_{3\mathrm{d}2} = \frac{1}{162\sqrt{\pi}}\frac{1}{a^{3/2}}\left(\frac{r^2}{a^2}\right)\mathrm{e}^{-r/3a}\sin^2\theta\,\mathrm{e}^{2\mathrm{i}\phi}. \tag{2.4}$$

Inserting these expressions into equation (2.3) and subsequently into equation (2.2) yields a transition rate of

$$\Gamma = P/\hbar\omega = \frac{3^8Gm_\mathrm{e}^2a^4\omega^5}{5\times 2^{13}\hbar c^5} = \frac{\alpha^6Gm_\mathrm{e}^3c}{360\hbar^2} = 5.7\times 10^{-40}\ \mathrm{s}^{-1}, \tag{2.5}$$

where $\alpha = e^2/\hbar c$ is the fine structure constant. It is straightforward to show that the transition rates for all 3d states (3d($\pm$2), 3d($\pm$1) and 3d0) are all the same.

In his standard text *Gravitation and Cosmology* [5], Weinberg presented a calculation of this transition rate using the same semi-classical approach. His result was $\Gamma = 2.5 \times 10^{-44}\ \mathrm{s}^{-1}$, which differs from ours by more than four orders of magnitude. Since the methods are identical, and the dimensional factors agree, the discrepancy is presumably due to a numerical error.

Indeed, in his book on quantum gravity [6], Kiefer obtained the result in equation (2.5), again using a semi-classical analysis. While this agreement may indicate that the above analysis has no computational errors, the semi-classical treatment itself 'can claim only a moderate amount of plausibility' [3]. Stronger support is afforded by demonstrating that the transition rate of equation (2.5) satisfies detailed balance. To do this, it is first necessary to evaluate the cross sections for absorption and stimulated emission.

3. Absorption, stimulated emission and detailed balance

Unlike spontaneous emission, the processes of absorption and stimulated emission can be treated adequately in the context of classical quantum mechanics, that is, without the use of quantum field theory. According to first-order perturbation theory, the transition probability between two hydrogenic states Ψ_a and Ψ_b is proportional to the square of the matrix element $\langle\Psi_b|H|\Psi_a\rangle$, where the interaction Hamiltonian H is derived from the interaction Lagrangian by its definition $H = pv - L$. For metric deviation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \ll 1$ and stress–energy tensor $T^{\mu\nu}$, the interaction Lagrangian density is given by [5, 7]

$$\mathcal{L} = \tfrac{1}{2} h_{\mu\nu} T^{\mu\nu}. \tag{3.1}$$

Although the action from which $\mathcal{L}$ is derived is a scalar, $\mathcal{L}$ itself is not gauge invariant. It is in fact this property that has led to the more serious conceptual errors mentioned in the introduction and which we will discuss in detail in section 5. The standard choice is to work in a *local inertial frame* (LIF) in which case the dominant term of the stress–energy tensor is just the mass–energy density, and so $\mathcal{L} \approx \frac{1}{2} h_{00} T^{00}$. When the generalized velocities are negligible, as in an LIF, the interaction Hamiltonian density is simply $\mathcal{H} = -\mathcal{L}$, and the corresponding interaction Hamiltonian for a localized mass m is therefore $H \approx -\frac{1}{2} m h_{00}$, if we assume that $h_{00}(\mathbf{r})$ is approximately constant in the region of nonvanishing mass density. Although LIFs are the most nearly Minkowskian and hence physical frame of reference, it is in the transverse–traceless, or TT, gauge that gravitational waves are most easily interpreted.

The TT gauge is a subset of what is variously called the Hilbert, Einstein, Fock, de Donder or Harmonic gauge, which is analogous to the Lorentz gauge of electromagnetism and defined by the requirement that ${h^\mu}_{\nu,\mu} = \frac{1}{2} {h^\mu}_{\mu,\nu}$. The TT gauge makes the further choice that $h_{\mu 0} = {h^\mu}_\mu = 0$. Then for an amplitude h and polarization tensor $e_{\mu\nu}$ the metric deviation can be written as $h_{\mu\nu} = h e_{\mu\nu}$, where $e_{00} = e_{\mu 0} = {e^\mu}_\mu = 0$. Consequently, a harmonic, plane gravitational wave (GW) can be expressed as

$$h_{jk}^{\mathrm{TT}} = h\, \mathrm{e}^{\mathrm{i}(k_l x^l - \omega t)} e_{jk} + \mathrm{c.c.}, \tag{3.2}$$

where $j, k, l = 1, 2, 3$ and $\omega = kc$. In this case, the Hilbert condition becomes $e_{kl} k^l = 0$, which for a plane wave propagating in the z-direction gives $e_{xx} = -e_{yy} = e_{xy} = e_{yx}$ and $e_{zj} = 0$. In what follows, we normalize the nonzero components of the polarization tensor to $|e_{ij}| = 1/\sqrt{2}$.

Independent of normalization, h_{00} in a LIF can be expressed in terms of the quantities h_{jk}^{TT} of the TT gauge. Since the duration of the interaction of the incident GW with the physical system (the hydrogen atom in this case) may be long, it is necessary to use a coordinate system that is locally inertial for an extended time. There are an infinite number of such coordinate systems; however, it can be shown that to second order in coordinate displacement h_{00} is the same in all such systems. Perhaps, the most useful of these are 'Fermi normal coordinates' in which [4]

$$h_{00} = -R_{0j0k}x^j x^k, \tag{3.3}$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor. To lowest order in $h_{\alpha\beta}$

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\beta\delta,\alpha\gamma} - h_{\alpha\gamma,\beta\delta}) \tag{3.4}$$

and is gauge invariant. Combining equations (3.2)–(3.4) yields

$$h_{00} = -\tfrac{1}{2}\omega^2 h\, \mathrm{e}^{\mathrm{i}(k_l x^l - \omega t)} x^j x^k e_{jk} + \text{c.c.}$$

The LIF interaction Hamiltonian therefore becomes

$$H = \tfrac{1}{4}m_{\mathrm{e}}\omega^2 h x^j x^k e_{jk}\, \mathrm{e}^{\mathrm{i}(k_l x^l - \omega t)} + \text{c.c.} \tag{3.5}$$

This is the Hamiltonian we will use in computing the matrix elements.

To calculate the transition rate, one also needs the flux of incident GWs, which is given by [4]

$$\mathcal{F} = \frac{c^3}{32\pi G}\langle h^{\mathrm{TT}}_{jk,0} h^{\mathrm{TT}}_{jk,0}\rangle, \tag{3.6}$$

where $\langle\,\rangle$ indicates an average over several cycles. Assuming equal amplitudes for the two polarizations and the normalization $|e_{ij}| = 1/\sqrt{2}$, we have

$$\mathcal{F} = \frac{c^3\omega^2}{8\pi G}h^2.$$

The transition probability per unit time between two discrete states is not constant in time if the incident radiation field and Hamiltonian are strictly monochromatic. So we assume that the radiation is spread over a range of frequencies with uncorrelated phases. In a small neighbourhood $\Delta\omega$ of each of these frequencies, the flux is given by $\mathrm{d}\mathcal{F} = I(\omega)\Delta\omega$ where $I(\omega)$ is the GW intensity. Therefore,

$$h^2 = \sum_{\omega} \frac{8\pi G}{c^3\omega^2} I(\omega)\Delta\omega.$$

The transition probability per unit time is then given by standard perturbation theory [3]:

$$\begin{aligned}\Gamma &= \frac{1}{t}\sum_{\omega}\frac{4|\langle\Psi_m|H|\Psi_n\rangle|^2 \sin^2\frac{1}{2}(\omega_{mn}-\omega)t}{\hbar^2(\omega_{mn}-\omega)^2}\\ &= \frac{2\pi G m_{\mathrm{e}}^2}{\hbar^2 c^3 t}\left|\int \Psi_m^*\, \mathrm{e}^{\mathrm{i}k_l x^l} x^j x^k e_{jk}\Psi_n\, \mathrm{d}^3 r\right|^2 \sum_{\omega}\frac{\omega^2 I(\omega)\Delta\omega \sin^2\frac{1}{2}(\omega_{mn}-\omega)t}{(\omega_{mn}-\omega)^2}.\end{aligned} \tag{3.7}$$

Here, the difference between the two electron energy levels is $\Delta E = \hbar\omega_{mn}$ and the sum is over the frequencies of the phase-independent GWs each within a band $\Delta\omega$. In the limit that $\Delta\omega$ is infinitesimally small, the summation can be replaced by an integral. Since the time factor has a sharp maximum at $\omega = \omega_{mn}$, $\omega^2 I(\omega)$ can be taken outside the integral and the limits extended to $\pm\infty$. Equation (3.7) then reduces to

$$\Gamma = \frac{\pi^2 G m_{\mathrm{e}}^2 \omega_{mn}^2 I(\omega_{mn})}{\hbar^2 c^3}\left|\int \Psi_m^*\, \mathrm{e}^{\mathrm{i}k_l x^l} x^j x^k e_j k \Psi_n\, \mathrm{d}^3 r\right|^2.$$

Finally, for the transition from the 3d to 1s states of hydrogen, the wavelength of the GW is much larger than the extent of the wavefunction so that $e^{ik_l x^l} \approx 1$ (the 'dipole' approximation). Then

$$\Gamma = \frac{\pi^2 G m_e^2 \omega_{mn}^2 I(\omega_{mn})}{\hbar^2 c^3} \left| \int \Psi_m^* x^j x^k e_{jk} \Psi_n \, d^3 r \right|^2$$
$$= \frac{\pi^2 G m_e^2 \omega_{mn}^2 I(\omega_{mn})}{\hbar^2 c^3} |D_{jk} e_{jk}|^2, \tag{3.8}$$

where

$$D_{ij} \equiv \int \Psi_b x_i x_j \Psi_a \, d^3 r. \tag{3.9}$$

For a GW propagating in the z-direction, the two linear polarizations are as given above and

$$|D_{jk} e_{jk}|^2 = 2 \left| \int \Psi_m^* xy \Psi_n \, d^3 r \right|^2 + \frac{1}{2} \left| \int \Psi_m^* (x^2 - y^2) \Psi_n \, d^3 x \right|^2. \tag{3.10}$$

To compute the mean transition rate for GWs incident from all directions, we simply average $|D_{jk} e_{jk}|^2$ over the sphere. The following is due to Dyson [8]. Since electron wavefunctions do not depend on the direction of the gravitational field, we can introduce two orthogonal unit vectors $\hat{\lambda}$, $\hat{\mu}$ and rewrite equation (3.10) as

$$|D_{jk} e_{jk}|^2 = 2 \left| \int \Psi_m^* \hat{\lambda} \cdot \mathbf{r} \mu \cdot \mathbf{r} \Psi_n \, d^3 r \right|^2 + \frac{1}{2} \left| \int \Psi_m^* (\hat{\lambda} \cdot \mathbf{r} \hat{\lambda} \cdot \mathbf{r} - \hat{\mu} \cdot \mathbf{r} \hat{\mu} \cdot \mathbf{r} \Psi_n \, d^3 x \right|^2.$$

The average of this expression over all directions is then

$$\langle |D_{jk} e_{jk}|^2 \rangle = \frac{1}{4\pi} \int d\Omega \bigg[2 \hat{\lambda}_i \hat{\mu}_j D_{ij} \hat{\lambda}_k \hat{\mu}_l D_{kl}^* + \hat{\lambda}_i \hat{\lambda}_j D_{ij} \hat{\lambda}_k \hat{\lambda}_l D_{kl}^*$$
$$- \frac{1}{2} \hat{\lambda}_i \hat{\lambda}_j D_{ij} \hat{\mu}_k \hat{\mu}_l D_{kl}^* - \frac{1}{2} \hat{\mu}_k \hat{\mu}_l D_{kl} \hat{\lambda}_i \hat{\lambda}_j D_{ij}^* \bigg], \tag{3.11}$$

where repeated indices are summed and

$$\langle \hat{\lambda}_i \hat{\lambda}_j D_{ij} \hat{\lambda}_k \hat{\lambda}_l D_{kl}^* \rangle = \langle \hat{\mu}_i \hat{\mu}_j D_{ij} \hat{\mu}_k \hat{\mu}_l D_{kl}^* \rangle.$$

Because $\hat{\mu}$ and $\hat{\lambda}$ are orthogonal, we can eliminate $\hat{\mu}$ by the following trick. Pick a direction for $\hat{\lambda}$, say $\hat{\lambda} = \hat{\mathbf{k}}$. Then in the usual spherical coordinates $\hat{\mu} = \hat{\mathbf{i}} \cos\phi + \hat{\mathbf{j}} \sin\phi$. The average $\langle \hat{\mu}_k \hat{\mu}_l \rangle$ over the unit circle in the plane perpendicular to $\hat{\lambda}$ can be seen to be $\langle \hat{\mu}_k \hat{\mu}_l \rangle = 1/2\delta_{kl} - 1/2\hat{\lambda}_k \hat{\lambda}_l$. Since this is a tensor equation, it is true in any coordinate system and, hence, for any direction $\hat{\lambda}$. Inserting this expression into equation (3.11) and making use of the identity

$$\langle \hat{\lambda}_i \hat{\lambda}_j \hat{\lambda}_k \hat{\lambda}_l \rangle \equiv \frac{1}{4\pi} \int d\Omega \, \hat{\lambda}_i \hat{\lambda}_j \hat{\lambda}_k \hat{\lambda}_l = \frac{1}{15} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

yields

$$\langle |D_{jk} e_{jk}|^2 \rangle = \tfrac{2}{5} \big(D_{jk} D_{jk}^* - \tfrac{1}{3} D_{jj} D_{kk}^* \big). \tag{3.12}$$

It is this average that must be used in equation (3.8) to compute Γ. For the transition from the 1s to the 3d2 state of hydrogen, the appropriate values of D_{jk} are, from equations (2.4) and (3.9),

$$D_{zz} = D_{xz} = D_{yz} = 0, \qquad D_{xx} = -D_{yy} = -i D_{xy} = \frac{3^4 a^2}{2^8}.$$

Substituting these values into equation (3.12) yields

$$\langle |D_{jk}e_{jk}|^2\rangle = \frac{8}{5}|D_{xx}|^2 = \frac{3^8a^4}{5\times 2^{13}}. \tag{3.13}$$

Finally, the absorption rate for the 1s to 3d2 transition follows from equation (3.8):

$$\Gamma = \frac{3^8\pi^2Gm_{\mathrm{e}}^2a^4\omega_{mn}^2I(\omega_{mn})}{5\times 2^{13}\hbar^2c^3}. \tag{3.14}$$

This expression also gives the stimulated emission rate between the 3d2 and 1s hydrogenic states.

To check the consistency of the absorption rate with the spontaneous emission rate calculated in section 2, we assume that gravitational radiation within a cavity is in thermal equilibrium with emitters and absorbers in the walls at temperature T. (While this *detailed balance* argument is 'valid' in a certain sense, it is straightforward to show that no such cavity can in principle be constructed for gravitational waves.) Equating the radiation absorbed per unit time with that emitted per unit time we have

$$N_{3\mathrm{d}2}(\Gamma_{\mathrm{sp}} + \Gamma_{\mathrm{st}}) = N_{1\mathrm{s}}\Gamma_{\mathrm{ab}},$$

where Γ_{sp}, Γ_{st} and Γ_{ab} are the transition rates for spontaneous emission, stimulated emission and absorption, $N_{3\,\mathrm{d}2}$ is the number of atoms in the 3d2 state and $N_{1\mathrm{s}}$ is the number atoms in the 1s state. Following Einstein, we expect $N_{3\,\mathrm{d}2}/N_{1\mathrm{s}} = \mathrm{e}^{-\hbar\omega_{mn}/kT}$. Now, substitute equation (3.14) and the *spontaneous* emission rate from equation (2.5) into the above expression. Solving for $I(\omega_{mn})$ yields

$$I(\omega_{mn}) = \frac{\hbar\omega_{mn}^3}{\pi^2c^2}(\mathrm{e}^{\hbar\omega_{mn}/kT} - 1)^{-1}$$

which is consistent with the intensity of black body radiation. This indicates that the ratio $\Gamma_{\mathrm{sp}}/\Gamma_{\mathrm{ab}}$ is correct and that the result yielded by the semi-classical approach in section 2 is as valid as equation (3.14).

The absorption rate equation (3.14) can be expressed in terms of an integrated cross section,

$$\Gamma = \frac{1}{\hbar\omega}\int \sigma(\omega)I(\omega)\,\mathrm{d}\omega. \tag{3.15}$$

Clearly, the absorption cross section is sharply peaked near ω_{mn}, and so $\int\sigma(\omega)\,\mathrm{d}\omega = \hbar\omega\Gamma/I(\omega_{mn})$. If we define an average cross section as $\langle\sigma\rangle = \int\sigma(\omega_{mn})\,\mathrm{d}\omega/\omega_{mn}$, then from equation (3.14):

$$\langle\sigma\rangle = \frac{3^8\pi^2Gm_{\mathrm{e}}^2a^4\omega_{mn}^2}{5\times 2^{13}\hbar c^3}.$$

For the transition between the 1s and 3d2 states $\omega_{mn} = \frac{4e^2}{9\hbar a}$, yielding

$$\sigma_{\mathrm{abs}} = \frac{3^4\pi^2}{5\times 2^9}\frac{G\hbar}{c^3} = 0.31\ell_{\mathrm{Pl}}^2. \tag{3.16}$$

Here, $\ell_{\mathrm{Pl}} \sim 10^{-33}$ cm is the Planck length. Surprisingly, all the physical constants associated with the hydrogen atom have disappeared from the cross section, leaving only the square of the Planck length and a numerical constant of order unity. We return to this important point in section 6.

The absorption rate (3.14) was for unpolarized GWs averaged over all incoming directions. For completeness, the absorption rate for one polarization of a gravitational wave incident in the θ, ϕ direction is readily shown to be

$$\Gamma = \frac{3^8\pi^2Gm_{\mathrm{e}}^2a^4\omega_{mn}^2I(\omega_{mn})}{2^{16}\hbar^2c^3}[(1+\cos^2\theta)^2\cos^2 2\phi + 4\cos^2\theta\sin^2 2\phi].$$

The rate for the other polarization is obtained by interchanging $\sin^2 2\phi$ and $\cos^2 2\phi$.

4. Field theoretic calculation of spontaneous graviton emission

As an independent check of equation (2.5) we next compute the transition rate via a field theoretic approach, that is, in terms of gravitons. This also allows a comparison with the result of Lightman *et al*, who in problem 18.18 of their well-known *Problem Book in General Relativity and Gravitation* [9] also used field theoretic methods to compute the 3d0 to 1s transition rate in hydrogen.

To quantize gravitational waves in the linearized theory, we follow the standard procedure of decomposing the metric perturbations into plane waves:

$$h_{jk} = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},\alpha} h_{\mathbf{k},\alpha}\, \mathrm{e}^{\mathbf{k},\alpha}_{jk}\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \text{c.c.} \tag{4.1}$$

Here, $h_{\mathbf{k},\alpha}$ are the Fourier amplitudes, α indicates the polarization, e_{jk} is the polarization tensor and box normalization with volume V is assumed. The energy density of GWs corresponding to the flux $\mathcal{F}$ given by equation (3.6) is

$$\mathrm{d}E/\mathrm{d}V = \frac{c^2}{32\pi G}\langle h^{\mathrm{TT}}_{jk,0} h^{\mathrm{TT}}_{jk,0}\rangle.$$

Substituting equation (4.1) into this expression and integrating over all space gives the total energy in the GWs. Noting that

$$\left\langle \int \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}\, \mathrm{e}^{\mathrm{i}(\mathbf{k}'\cdot\mathbf{r}-\omega' t)}\, \mathrm{d}^3 r \right\rangle = 0$$

and

$$\left\langle \int \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}\, \mathrm{e}^{-\mathrm{i}(\mathbf{k}'\cdot\mathbf{r}-\omega' t)}\, \mathrm{d}^3 r \right\rangle = V \delta_{\mathbf{k}\mathbf{k}'}\delta_{\omega\omega'}$$

then

$$E = \frac{c^2\omega^2}{16\pi G} \sum_{\mathbf{k},\alpha,\alpha'} \mathrm{e}^{\mathbf{k},\alpha}_{jk}\, \mathrm{e}^{\mathbf{k},\alpha'}_{jk} h_{\mathbf{k},\alpha} h^*_{\mathbf{k},\alpha'}.$$

For the normalization $|e_{kj}| = 1/\sqrt{2}$, we have $\sum_{jk} \mathrm{e}^{\mathbf{k},\alpha}_{jk}\, \mathrm{e}^{\mathbf{k},\alpha'}_{jk} = \delta_{\alpha\alpha'}$, and

$$E = \frac{c^2\omega^2}{16\pi G} \sum_{\mathbf{k},\alpha} |h_{\mathbf{k},\alpha}|^2.$$

Next identify $h^*_{\mathbf{k},\alpha}$ and $h_{\mathbf{k},\alpha}$ with the raising and lowering operators, $a^\dagger_{\mathbf{k},\alpha}$ and $a_{\mathbf{k},\alpha}$, such that the classical energy and quantum Hamiltonian agree with one another, i.e.,

$$E = H = \frac{1}{2}\sum_{\mathbf{k},\alpha} \hbar\omega\big(a^\dagger_{\mathbf{k},\alpha} a_{\mathbf{k},\alpha} + a_{\mathbf{k},\alpha} a^\dagger_{\mathbf{k},\alpha}\big) = \sum_{\mathbf{k},\alpha}\left(N_{\mathbf{k},\alpha} + \frac{1}{2}\right)\hbar\omega,$$

where $N_{\mathbf{k},\alpha} = a^\dagger_{\mathbf{k},\alpha} a_{\mathbf{k},\alpha}$ is the number operator. One can always reset the zero of the energy scale so that $H = \sum_{\mathbf{k},\alpha} \hbar\omega N_{\mathbf{k},\alpha}$. The only way that $E = H$ is if we make the identifications

$$h_{\mathbf{k},\alpha} \to \frac{1}{c}\sqrt{\frac{16\pi G\hbar}{\omega}}\, a_{\mathbf{k},\alpha}$$

and

$$h^*_{\mathbf{k},\alpha} \to \frac{1}{c}\sqrt{\frac{16\pi G\hbar}{\omega}}\, a^\dagger_{\mathbf{k},\alpha}.$$

Then

$$h_{jk} \to \frac{1}{c\sqrt{V}} \sum_{\mathbf{k},\alpha} \sqrt{\frac{16\pi G\hbar}{\omega}} \left[a_{\mathbf{k},\alpha}\, \mathrm{e}^{\mathbf{k},\alpha}_{jk} \exp \mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t) + a^{\dagger}_{\mathbf{k},\alpha}\, \mathrm{e}^{\mathbf{k},\alpha}_{jk} \exp -\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega t)\right]. \quad (4.2)$$

We now consider the transition rate between two hydrogen states that involves the emission of a single graviton. According to the *golden rule*, the transition rate per solid angle is given by

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{2\pi}{\hbar} |\langle 1|\langle f|H|i\rangle|0\rangle|^2 \rho. \quad (4.3)$$

Here, $|i\rangle|0\rangle$ is the initial electron and graviton state (with no gravitons), $|f\rangle|1\rangle$ is the final electron and graviton state (with one graviton) and

$$\rho = \frac{V\omega^2}{(2\pi)^3 \hbar c^3}$$

is the graviton energy density of states per solid angle, again box normalized. From equations (3.2) and (3.5), the interaction Hamiltonian (in the local inertial frame) can be written as

$$H = \frac{m_{\mathrm{e}}\omega^2}{4} h_{jk} x^j x^k, \quad (4.4)$$

where h_{jk} is expressed in the TT gauge. In order to compare directly with the results of Lightman *et al* [9], we compute the spontaneous transition between the 3d0 and 1s states of hydrogen, although as pointed out earlier, the transition rates from all the 3d*m* states are identical. Substituting equation (4.2) into equation (4.4) the transition matrix element becomes

$$\langle 1|\langle 1\mathrm{s}|H|3\mathrm{d}0\rangle|0\rangle = \frac{m_{\mathrm{e}}\omega^2}{4c} \sqrt{\frac{16\pi G\hbar}{V\omega}} \int \mathrm{d}^3 r\, \Psi^*_{3\,\mathrm{d}0} e_{jk} x^j x^k \Psi_{1\mathrm{s}}, \quad (4.5)$$

where $\omega = (E_{3\mathrm{d}0} - E_{1\mathrm{s}})/\hbar$. Substituting equation (4.5) into equation (4.3) yields

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{m_{\mathrm{e}}^2 \omega^5 G}{4\pi\hbar c^5} |D_{jk} e_{jk}|^2,$$

where D_{ij} is defined in equation (3.9). As before, $|D_{jk}e_{jk}|^2$ should be averaged over all directions. With equation (3.12) we then have

$$\Gamma = \frac{2m_{\mathrm{e}}^2\omega^5 G}{5\hbar c^5} \left(D_{jk} D^*_{jk} - \frac{1}{3} D_{jj} D^*_{kk} \right).$$

For the 3d0 and 1s hydrogen wavefunctions, we find that $D_{xx} = D_{yy} = -\frac{1}{2} D_{zz} = -a^2\sqrt{3^7/2^{15}}$ and, therefore,

$$\Gamma = \frac{3^8 G m_{\mathrm{e}}^2 a^4 \omega^5}{5 \times 2^{13} \hbar c^5}, \quad (4.6)$$

which is precisely the same as the semi-classical result in equation (2.5). It is this agreement that provides the strongest confirmation of the semi-classical treatment.

The *Problem Book* authors obtained a result that is larger than this one by about an order of magnitude. In part, the disagreement is due to numerical and normalization errors. Nevertheless, when these mistakes are corrected their result still differs from equation (4.6). The underlying reason can be traced to the fact that whereas our calculation was carried out in the local inertial frame, Lightman *et al* [9] worked entirely within the TT gauge. One can recover the proper result in the TT gauge; however, much care must be taken to correctly interpret the interaction Hamiltonian in that gauge, as will become apparent in the following section.

5. Gauge properties of the interaction Hamiltonian

In his pioneering work on gravitational wave detection, Weber [10] used the equation of geodesic deviation to deduce the gravitational force density of weak gravitational fields acting on non-relativistic matter. In a local inertial frame, the gravitational force density f_g^j is

$$f_g^j = -\rho R^j_{0k0} x^k, \tag{5.1}$$

where ρ is the mass density of the detector and R^j_{0k0} are components of the Riemann curvature tensor. Then the equation of motion of the detector is $\rho \partial^2 x^j / \partial t^2 = f^j + f_g^j$, where f^j is the total classical (i.e., non-relativistic) force density on the detector mass elements. This relation implicitly assumes that the gravitational wave has negligible effect on the physics of the detector, which is a reasonable assumption because the LIF is the most nearly Minkowskian frame. Since gravity couples to the mass–energy density of matter, one expects that the gravitational modification of the classical forces in the detector will be proportional to the binding energy density of the system. In fact, it can be shown that this is the case [11]. While equation (5.1) strictly holds true only in a LIF, R^j_{0k0} is already first order in $h_{\mu\nu}$, and so the right-hand side of the equation is, to first order, invariant under infinitesimal coordinate, or gauge, transformations. It is in this sense that the expression for the gravitational force in equation (5.1) is gauge invariant.

As an alternative formulation to Weber's, Dyson [7] introduced the interaction Lagrangian density, $\mathcal{L} = \frac{1}{2} h_{\mu\nu} T^{\mu\nu}$, which we used in sections 3 and 4. It follows directly from the general definition of the stress–energy tensor as the functional derivative of the matter action, I_m, with respect to $g_{\mu\nu}$ [5]:

$$\delta I_m = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{g(x^\mu)} T^{\mu\nu}(x^\mu) \delta g_{\mu\nu}(x^\mu).$$

In this expression, $g(x^\mu)$ is the determinant of $-g_{\mu\nu}(x^\mu)$ and $g_{\mu\nu}$ is to be considered an external field rather than a dynamical variable. For any metric that is close to Minkowskian, i.e., $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $\sqrt{g(x^\mu)} \approx 1$, one can easily see that the action differs from that of a free particle by $\frac{1}{2} \int \mathrm{d}^4 x \, T^{\mu\nu}(x^\mu) h_{\mu\nu}(x^\mu)$. Therefore, the interaction Lagrangian density is given by

$$\mathcal{L}_\mathrm{I} = \tfrac{1}{2} h_{\mu\nu} T^{\mu\nu}. \tag{5.2}$$

The Euler–Lagrange equations following from a Lagrangian with this interaction term are the same as implied by the equation of geodesic deviation only if $h_{\mu\nu}$ is expressed in the LIF and $T^{\mu\nu}$ is the classical stress–energy tensor of the system. The reason is that the Lagrangian (5.2) is not invariant under infinitesimal coordinate (gauge) transformations $x'_\mu = x_\mu + \xi_\mu$, even though the action from which it is derived is a scalar quantity. Of course, this does not mean that the computed motions of particles depend on a particular gauge. $T^{\mu\nu}$ in equation (5.2) includes the effects of the classical, non-gravitational forces acting on the particle. In non-LIF gauges, e.g., the TT gauge, these forces can be significantly modified by the gravitational wave and the modifications must be taken into account in calculating particle motions.

As a specific example, consider a system of particles held together by electromagnetic forces. In principle, the action must therefore include the electromagnetic stress–energy, which is coupled to the gravitational field. In the LIF gauge, gravitationally induced electromagnetic forces are smaller than the tidal forces and can be neglected. On the other hand, in a non-LIF, the solution to the Einstein–Maxwell equations includes gravitationally induced electromagnetic forces that are comparable to the tidal effects and, therefore, they must be taken into account. (A simple example of such a system was treated by Boughn [11].)

Dyson's treatment of elastic systems in the presence of gravitational waves [7] is based on the interaction Lagrangian (5.2) as expressed in the TT gauge, which does not constitute a LIF. Gravitationally induced modifications to the internal stresses must therefore be taken into account. Dyson does not do this, however, and concludes that 'the response [of an elastic solid] depends on irregularities in the shear-wave modulus, and is strongest at free surfaces.' As a consequence, Dyson's analysis implies that a self-gravitating, compressible perfect fluid (a system without shear) should not interact with a gravitational wave. But in fact the Sun is a reasonable approximation to such a system at periods comparable to sound travel times across the Sun and yet does couple to incident gravitational waves [12, 13].

Similarly, in computing their transition rate in hydrogen, the *Problem Book* authors [9] worked in the TT gauge but did not include electromagnetic stresses in the interaction Lagrangian. Recall that in special relativity, the Minkowski metric can be written as $d\tau^2 = -\eta_{\mu\nu}\,dx^\mu\,dx^\nu = dt^2(1-v^2)$. This leads to the free-particle Lagrangian $L = -m\sqrt{1-v^2}$. For a charged particle interacting with both an electromagnetic and a gravitational field, we replace $\eta_{\mu\nu}$ by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and add the usual electromagnetic interaction term, giving

$$L = -m\sqrt{1 - \eta_{jk}v^j v^k - h_{jk}v^j v^k} - q\Phi + qv^j A_j, \tag{5.3}$$

where Φ and $\mathbf{A}$ are the electromagnetic scalar and vector potentials and we assume the $h_{\mu 0} = 0$ condition of the TT gauge. For non-relativistic systems, equation (5.3) can be expanded to lowest order in v^j,

$$L = -m + \tfrac{1}{2}m(\eta_{jk}v^j v^k + h_{jk}v^j v^k) - q\Phi + qv^j A_j. \tag{5.4}$$

Note that the h_{jk} term in this expression is just the interaction Lagrangian of equation (5.2). The Hamiltonian is formed through the standard prescription $H \equiv \pi_\alpha \dot{q}^\alpha - L$, where $\pi_\alpha \equiv \partial L/\partial \dot{q}^\alpha$ and q^α are the generalized momenta and velocities. Working to first order in h_{jk}, the canonical momenta are

$$\pi_j = m(\eta_{jk}v^k + h_{jk}v^k) + qA_j.$$

To first order, the inverse of $(\eta_{jk} + h_{jk})$ is $(\eta^{jk} - h^{jk})$, where $h^{jk} \equiv h_{jk}$. Then solving for v^j gives

$$v^j = \frac{1}{m}(\eta^{jk} - h^{jk})(\pi_k - qA_k)$$

and the Hamiltonian becomes

$$H = \frac{(\eta^{jk} - h^{jk})(\pi_j - qA_j)(\pi_k - qA_k)}{2m} + m + q\Phi. \tag{5.5}$$

The constant m is not physically relevant and, as one sees, the Hamiltonian (5.5) follows the usual minimal substitution rule, $\pi \to \pi - q\mathbf{A}$. To implement equation (5.5), π_j is identified with the quantum mechanical operator $-i\hbar\partial/\partial x^j$ and h_{jk} is identified with the graviton raising and lower operators according to equation (4.2). For non-relativistic systems, magnetic fields are much smaller than electric fields and so, in the Coulomb gauge, the vector potential $\mathbf{A}$ can be ignored. In this case, the interaction part of the Hamiltonian operator becomes

$$H_{\mathrm{I}} = \frac{\hbar^2}{2m}h^{jk}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}. \tag{5.6}$$

When this expression is substituted into the matrix element of equation (4.3), the result is a transition rate $16/25$ times that of equation (4.6) and is the rate presented in the *Problem Book*, modulo numerical and normalization errors.

This factor can be accounted for by including the purely electromagnetic part of the Lagrangian density

$$\mathcal{L}_{\mathrm{EM}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu}, \tag{5.7}$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the usual electromagnetic field tensor [14]. To lowest order in h^{jk}, this becomes

$$\mathcal{L}_{\mathrm{EM}} = -\frac{1}{16\pi} \eta^{\alpha\mu} \eta^{\beta\nu} F_{\alpha\beta} F_{\mu\nu} + \frac{1}{8\pi} h^{jk} \eta^{\mu\alpha} F_{\mu j} F_{\alpha k}. \tag{5.8}$$

The purely electromagnetic part of the Hamiltonian density then has the form

$$\mathcal{H}_{\mathrm{EM}} = (\partial \mathcal{L}_{\mathrm{EM}} / \partial A_{j,0}) A_{j,0} - \mathcal{L}_{\mathrm{EM}}. \tag{5.9}$$

It is straightforward to show that to first order in h^{jk} the total Hamiltonian becomes (see, for example, chapter 57 of [3])

$$H = \frac{(\eta^{jk} - h^{jk})(\pi_j - qA_j)(\pi_k - qA_k)}{2m} + \frac{1}{8\pi} \int (|\mathbf{E}|^2 \, \mathrm{d}^3x + h^{jk} E_j E_k) \, \mathrm{d}^3x, \tag{5.10}$$

where we have again invoked the Coulomb gauge and ignored terms involving the magnetic field, so the electric field is given by $E_j = A_{0,j}$. Finally, the interaction part of the Hamiltonian can be written as

$$H_{\mathrm{I}} = \frac{\hbar^2}{2m} h_{jk} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} + \frac{1}{8\pi} \int h_{jk} E^j E^k \, \mathrm{d}^3x, \tag{5.11}$$

since $h^{jk} = h_{jk}$ and to lowest order in h_{jk} there is no distinction between upper and lower indices in the other fields.

In this expression, E^j is the sum of electric fields of the proton and the electron, $E^j_{\mathrm{p}} + E^j_{\mathrm{e}}$, so the integrand of equation (5.11) is $\left(E^j_{\mathrm{p}} + E^j_{\mathrm{e}}\right)\left(E^k_{\mathrm{p}} + E^k_{\mathrm{e}}\right)$. The integrals of $E^j_{\mathrm{p}} E^k_{\mathrm{p}}$ and $E^j_{\mathrm{e}} E^k_{\mathrm{e}}$ over all space are constants, independent of the location of the particles. Therefore, the contribution of these terms to the matrix element (see equation (4.3)) between the 1s and 3d0 states vanishes. The only electromagnetic terms in the interaction Hamiltonian that contribute to the transition rate are

$$H_{\mathrm{EM,I}} = \frac{1}{8\pi} h_{jk} \int \left(E^j_{\mathrm{p}} E^k_{\mathrm{e}} + E^k_{\mathrm{p}} E^j_{\mathrm{e}}\right) \mathrm{d}^3x, \tag{5.12}$$

where we again assume that $h_{jk}(\mathbf{r})$ is approximately constant in the region of significant electric fields. Consider the first term

$$E^j_{\mathrm{p}} E^k_{\mathrm{e}} = -e^2 \frac{x^j \left(x^k - x^k_{\mathrm{e}}\right)}{r^3 |\mathbf{r} - \mathbf{r}_{\mathrm{e}}|^3} = e^2 \frac{x^j}{r^3} \nabla^k \frac{1}{|\mathbf{r} - \mathbf{r}_{\mathrm{e}}|}. \tag{5.13}$$

Here e is the electron charge, $\mathbf{r}$ is the field point, $\mathbf{r}_{\mathrm{e}}$ is the electron coordinate and the proton is assumed to be located at $\mathbf{r} = 0$. Via the addition theorem for spherical harmonics, $|\mathbf{r} - \mathbf{r}_{\mathrm{e}}|^{-1}$ can be expressed as [14]

$$\frac{1}{|\mathbf{r} - \mathbf{r}_{\mathrm{e}}|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell + 1} \frac{r_<}{r_>^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi).$$

Noting that the matrix element between the 1s and 3d0 states will pick out only the $\ell = 2, m = 0$ term, we are thus left with

$$\frac{1}{|\mathbf{r} - \mathbf{r}_{\mathrm{e}}|} = \frac{4\pi r_<^2}{5 r_>^3} Y^*_{20}(\theta, \phi) Y_{20}(\theta_{\mathrm{e}}, \phi_{\mathrm{e}}), \tag{5.14}$$

where $r_<$ ($r_>$) is the smaller (larger) of r and r_e. Substituting equation (5.14) into equation (5.13) and integrating over all space yields, after a tedious calculation,

$$\int \mathrm{d}^3x\, E_p^x E_e^x = \int \mathrm{d}^3x\, E_p^y E_e^y = -\frac{1}{2}\int \mathrm{d}^3x\, E_p^z E_e^z = -\frac{\pi}{3}\frac{e^2(3\cos^2\theta_e - 1)}{r_e}. \tag{5.15}$$

The integrals vanish for $j \neq k$ and, therefore, the two terms in equation (5.12) are equal. Substituting equation (5.15) into equation (5.12) and adding the result to the electron Hamiltonian (see equation (5.6)) yields the interaction part of the Hamiltonian. Another tedious computation via equation (4.3) gives, finally, a transition rate of

$$\Gamma = P/\hbar\omega = \frac{3^8 G m_e^2 a^4 \omega^5}{5 \times 2^{13}\hbar c^5} = \frac{\alpha^6 G m_e^3 c}{360\hbar^2}, \tag{5.16}$$

which is precisely the same as the LIF field theoretic and semi-classical results (4.6) and (2.5).

From this analysis we conclude that one can work with an interaction Hamiltonian in any gauge as long as all the relevant interactions are included in the stress–energy tensor. However, it is also clear that in this case working in the LIF leads to a much simpler analysis. For classical systems, it is undoubtedly also possible to work in non-LIF gauges; however, in the case of elastic media considered by Dyson [7], for example, the best way to do this is not immediately apparent.

6. Graviton ionization cross section of hydrogen

In RB, an important step was to calculate the ionization cross section for hydrogen. As we saw in section 3, the graviton absorption cross section was $\sigma \sim \ell_{\mathrm{Pl}}^2$. That all dimensions other than the Planck length drop out of the cross section is at first surprising, but can be understood simply as follows. The classical cross section for a system of mass m, size ℓ and frequency ω is roughly [2, 4] $\sigma \sim Gm\ell^2\omega/c^3$. Assuming the Nicholson–Bohr quantization condition, the angular momentum for such a system near its ground state is $L \sim m\ell^2\omega \sim \hbar$. Thus,

$$\sigma \sim \frac{G\hbar}{c^3} = \ell_{\mathrm{Pl}}^2 \approx 10^{-66}\ \mathrm{cm}^2,$$

and we see that the Planck-length-squared cross section is solely a result of angular momentum quantization.

Because in RB we decided on the gravitational analogy of the photoelectric effect as a method for detecting gravitons, it was necessary to compute the gravito-ionization cross section for hydrogen in the ground state. One expects it to be similar in magnitude to the above; however, the ordinary photoionization cross section does have large numerical factors 'of order unity' and is also strongly dependent on photon energy. Therefore, in this section we compute the ionization cross section for gravitons with energies in the range 13.6 eV $\ll E \ll 2.5 \times 10^4$ eV, energies for which the non-relativistic Born approximation is appropriate. In fact, as detailed in RB, there are many strong astrophysical sources of gravitons in this energy range.

As in section 3, we take the interaction Hamiltonian in a LIF to be

$$H = \tfrac{1}{4} m_e \omega^2 h x^j x^k e_{jk} \exp \mathrm{i}(k_l x^l - \omega t) + \mathrm{c.c.} \tag{6.1}$$

and compute the matrix element between an initial hydrogen ground state, Ψ_i, and a plane wave final state (the Born approximation), Ψ_f, i.e.,

$$\Psi_i = \frac{1}{\sqrt{\pi}a^{3/2}}\,\mathrm{e}^{-r/a}, \qquad \Psi_f = \frac{1}{L^{3/2}}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \tag{6.2}$$

where the plane wave is box normalized with dimension L. The transition probability per unit time between these two states is given by the *golden rule*,

$$\Gamma = \frac{2\pi}{\hbar}\rho(k)|\langle f|H|i\rangle|^2, \tag{6.3}$$

where k is the wave number of the emerging electron and $\rho(k)$ is the energy density of final states:

$$\rho(k) = \frac{m_e k L^3}{2\pi^2\hbar^2}. \tag{6.4}$$

As in section 3, we average $|\langle f|H|i\rangle|^2$ over all directions using equation (3.12). For an incident GW of amplitude h, this result is

$$|\langle f|H|i\rangle|^2 = \frac{3\times 2^{11}\pi}{5}\frac{h^2\omega^4 m_e^2 a^7(a^4k^4)}{L^3(1+a^2k^2)^8}.$$

Equations (6.3) and (6.4) then give

$$\Gamma = \frac{3\times 2^{10}}{5}\frac{h^2\omega^4 m_e^3 a^{11}k^5}{\hbar^3(1+a^2k^2)^8}.$$

By definition, the ionization cross section is $\sigma = \Gamma\hbar\omega/\mathcal{F}$, where $\mathcal{F} = \frac{c^3\omega^2\hbar^2}{8\pi G}h^2$ is the GW flux from section 3. Thus,

$$\sigma = \frac{3\times 2^{13}\pi}{5}\frac{G\omega^3 m^3 a^{11}k^5}{c^3\hbar^2(1+a^2k^2)^8}. \tag{6.5}$$

ω can be eliminated from this expression by using the Einstein photoelectric relation (conservation of energy), which requires that the incident graviton energy equal the sum of the electron binding energy and the kinetic energy of the emerging electron, or $\hbar\omega = e^2/2a + \hbar^2k^2/2m_e$. Hence,

$$\sigma = \frac{3\times 2^{10}\pi}{5}\frac{(ka)^5}{(1+k^2a^2)^5}\frac{G\hbar}{c^3}. \tag{6.6}$$

For $ak \gg 1$, the dependence of this result on final electron momentum is in fact identical to that of the ordinary photoionization cross section. We also see that the ionization cross section is, modulo a dimensionless factor, equal to the Planck length squared; for energetic gravitons with $\hbar\omega \gg 13.6$ eV, however, the dimensionless factor can be quite small. Smolin [15] used a field theoretic argument to estimate the ionization cross section of a bound electron and obtained a result that is also proportional to ℓ_{Pl}^2. However, the dependence on ka is quite different, with the cross section increasing rapidly for large ka rather than decreasing rapidly as indicated in equation (6.6).

The differential ionization cross section for linearly polarized gravitons is also not difficult to compute and we state it here for completeness. For one polarization we obtain

$$\mathrm{d}\sigma/\mathrm{d}\Omega = 3^2\times 2^5\frac{(ka)^5}{(1+k^2a^2)^5}\sin^2 2\phi\sin^4\theta\frac{G\hbar}{c^3};$$

letting $\sin^2 2\phi \to \cos^2 2\phi$ gives the other polarization.

7. Conclusion

As discussed in the introduction, the original motivation for this paper was Dyson's conjecture that a single graviton could never be detected in the real universe. In RB, we employed the above gravito-electric cross section to show that if one is limited only by the mass, energy content and age of the universe, one can design a highly idealized *gedankenexperiment* that could detect *some* gravitons in the lifetime of the universe. As soon as one begins to consider detector physics and background noise, though, detecting even a single graviton becomes impossible. In that sense, Dyson's conjecture appears correct.

Although one might argue that the detailed calculations presented here are not entirely necessary to address Dyson's conjecture, it has become clear that the physics of gravitational wave–matter interaction, in particular the gauge properties of the interaction Hamiltonian, presents enough subtleties to catch even experienced practitioners off guard. As this will undoubtedly happen from time to time in the future, we feel it is of some importance to elucidate this matter as clearly as possible.

Acknowledgments

We would like to thank Freeman Dyson for inspiring this work and for generously making available his unpublished notes. We also thank Jim Peebles for useful discussions and the Physics Department at Princeton University, where much of this work was carried out.

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