Probing quantum chaos with the entropy of decoherent histories

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Quantum chaos, a phenomenon that began to be studied in the last century, still does not have a rigorous understanding. By virtue of the correspondence principle, the properties of the system that lead to chaotic dynamics at the classical level must also be present in the underlying quantum system. In the classical case, the exponential divergence of nearby trajectories in time is described in terms of the Lyapunov exponent. However, in the quantum case, a similar description of chaos is strictly speaking impossible due to absence of trajectories. There are different approaches to remedy this situation, but the universal criterium of quantum chaos is absent. We propose the quantum chaos definition in the manner similar to classical one using decoherent histories as a quantum analog of trajectories. For this purpose we consider the model of open quantum kicked top interacting with environment, which is bosonic bath and illustrate this idea on it. Here environment plays the role of trajectory recording device. For kicked top model on classical level depending on the kick strength there is crossover between integrable and chaotic regimes. We show that for such a model the production of entropy of decoherent histories is radically different in the integrable and chaotic regimes. Thus, the entropy of an ensemble of quantum trajectories can be used as a signature of quantum chaos.

I. INTRODUCTION

Chaotic behavior plays a significant role in various fields of science (for example, it underlies classical thermodynamics [1–3] and hydrodynamics [4]). In classical systems, the chaos is characterized by the exponential sensitivity of the evolution of the system in time to initial conditions, but in quantum mechanics it is not possible to characterize chaos in the same way, since the concept of phase space trajectories loses its meaning due to the Heisenberg uncertainty principle. There are different approaches to the definition of quantum chaos: through the statistics of energy levels [5–8]; spectral form factors [8]; Loschmidt echo [14]; out-of-time ordered correlators (OTOC) [15–17]; in the context of quantum modeling through fidelity decay [13] and others. However, the true understanding of the nature of quantum chaos and the limits of using its various diagnostics, as well as the possible connection between them, is the subject of ongoing research, both theoretical and experimental. Until now, it has not been possible to present a universal criterion for determining quantum chaos and to rigorously understand this phenomenon. The methods of diagnosing quantum chaos have their drawbacks. For example, level statistics are poorly defined for small systems and there are specific examples, for which it doesn't work [10], OTOC does not work for billiard systems and in this case it is not possible to distinguish integrable behavior from chaotic [17]. Thus, interest in finding universal criteria for quantum chaos for classically chaotic systems, as well as understanding the nature of the appearance of this phenomenon, is motivated.

Interest in quantum chaos is caused by its wide application in explaining fundamental problems, such as: the thermalization mechanism in isolated systems, for which the eigenstates of quantum chaotic systems play a significant role [9, 18, 19]; quantum information scrambling [16]; in relation to open quantum systems, the influence of chaos on the processes of decoherence and dissipation [22–26], etc. At present, there are various experimental realizations of chaotic behavior, for example, in spin chains implemented with cold atoms [27] or spin chains on surfaces [20].

In this work we rely on the idea of Berry's work [21] which is that for the emergence of quantum chaos, the environment is important. Quantum decoherence that occurs in non-isolated systems inhibits the quantum suppression of chaos (due to the fact that quantum systems have discrete, quantized energy levels that control the evolution of dynamic quantities, therefore, this evolution cannot be truly chaotic). Thanks to the environment it is possible to introduce the concept of quantum trajectories of system as a record that is stored in certain degrees of freedom of the environment [41].

We consider a model where a quantum environment is connected to the open quantum system (OQS), which in some degrees of freedom records how the system behaved as it evolved over time. This is similar in spirit to the decoherent histories also known as consistent histories approach [29, 30, 36]. Therefore, we call the recorded information about OQS the decoherent history.

To correctly determine the decoherent histories, it is necessary to identify the degrees of freedom that carry information about how the OQS moved in the past. The formalism developed in this paper consists of several stages. First, the environment's degrees of freedom (later we can call them modes) that can carry information about the OQS are determined. There are infinitely

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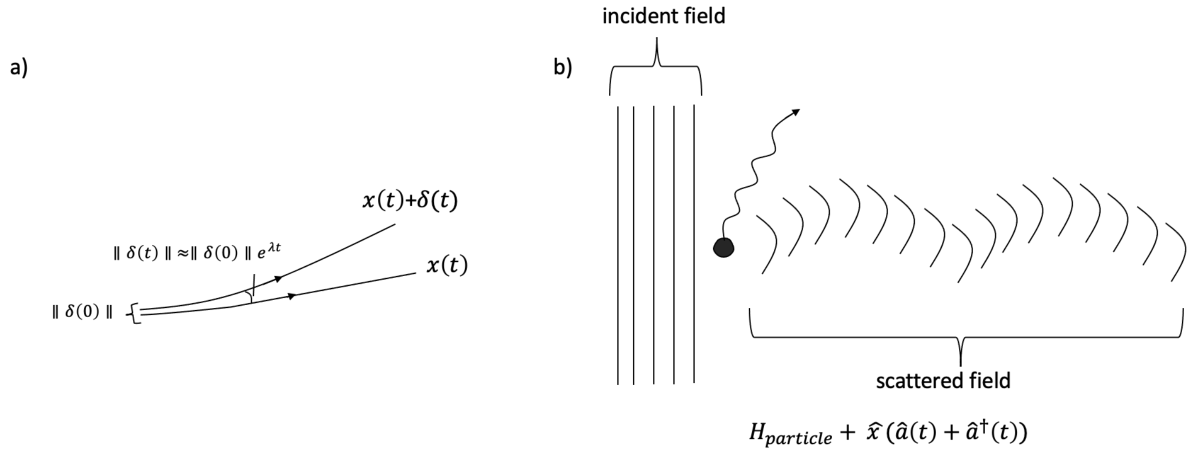


FIG. 1. a) Classical Lyapunov exponent through classical trajectories; b) Incident field is scattered by particle. Quantum analogue of trajectory is encoded in the scattered field, which can be modeled by coupling target particle to quantum environment through environment's operator $\widehat{a}(t)$.

many degrees of freedom in the environment, but only those degrees of freedom that have significantly interacted with the OQS can carry useful information. To do this, it is convenient to introduce the Lieb-Robinson light cone formalism [34], which describes the propagation of perturbation. Then the effectively interacting degrees of freedom will be inside the light cone. Secondly, from these degrees of freedom, the irreversibly decoupled ones are determined, since the trajectory record should not change at future times and should not depend on the future evolution of the OQS, another words, they must carry away information about the OQS and stop interacting. Knowing these degrees of freedom, we can measure them one after another and the sequence of measurement results is a quantum trajectory (decoherent history).

The analogue of the trajectory appears due to the fact that the system interacts with the environment Fig.1. The formation of quantum trajectories corresponds to the emergence of decoherent histories in the environment [40, 41].

The approach used in this work is based on the method of work [12], which allows one to model the dynamics of OQS beyond the limits of applicability of the Markov approximation [31, 32]. In this work, this approach is adapted and the modes of the environment, which contain information about the motion of the OQS, are microscopically derived. With the help of this, the concept of decoherent histories is constructed and the entropy of the ensemble of quantum trajectories [38, 39] is calculated. It is reasonable to assume that the entropy of the ensemble of these quantum trajectories will be radically different in the integrable and chaotic regimes, which was proved in this paper.

The paper is structured as follows. Sections II and III introduce the model in question. In Section IV we explain our treatment of decoherence history approach. Section V describes a method for derivation of the degrees of freedom of the environment, which contain information about the motion of the OQS. In Section VI we construct quantum trajectories (decoherent histories) and calculate the entropy of an ensemble of such trajectories. In Section VII we present our results. We conclude in Section VIII.

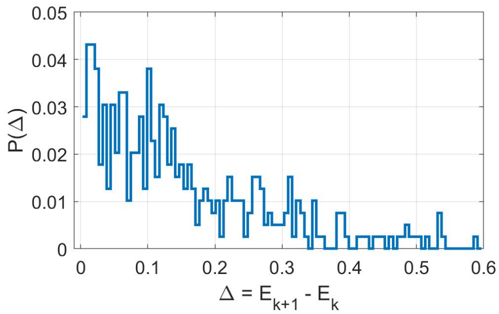
II. THE CONSIDERED CHAOTIC SYSTEM

We consider the model of a quantum kicked top [28] as OQS, which at the classical level has chaotic behavior for certain values of the kick strength K (hereinafter, the natural system of units is used everywhere: $\hbar = 1$). This model is well studied in the context of quantum chaos [8, 28]:

$$\widehat{H}_S = \frac{p}{\tau}\widehat{J}_y + \frac{K}{2j}\left(\widehat{J}_z - \beta\right)^2 \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \qquad (1)$$

The system is characterized by the angular momentum $\vec{J} = (J_x, J_y, J_z)$ with the corresponding commutators: $[J_i, J_j] = i\epsilon_{ijk}J_k$ (i, j, k run through x, y, z). The classical limit is reached by tending $j \to \infty$, $\hbar \to 0$ while preserving $\hbar j$. The first term is responsible for the precession around the y axis with the angular frequency $\frac{p}{\tau}$, the second one is related to the periodic sequence of kicks at the time distance τ.

By changing K, the motion of the system changes from integrable to chaotic. Fig.2 shows the level spacing distributions for different values of the kicked strength K.



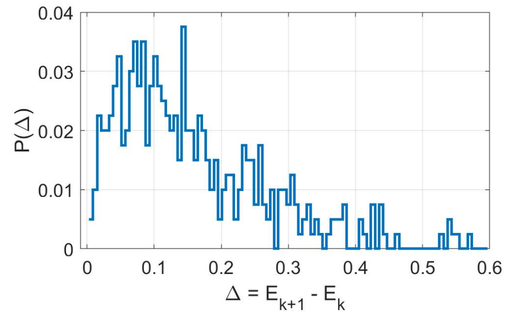


FIG. 2. Crossover between integrable (Poisson statistics) and chaotic (Wigner-Dyson statistics) motion; for the left statistics $K = 2$, for right $K = 3$.

The physical implementation of this model is provided by the system of interacting spins [33].

III. OPEN CHAOTIC QUANTUM SYSTEM

Our main idea is to introduce trajectories in the quantum case in order to obtain a way to diagnose the quantum chaos. To do this, it is necessary to connect the environment to the considered chaotic system (in this work the model of a quantum kicked top) (Section II). The role of the environment is played by a bosonic bath.

The complete Hamiltonian of the system is as follows:

$$\widehat{H} = \widehat{H}_S + \widehat{H}_E + \widehat{H}_{int} \tag{2}$$

where $\widehat{H}_S$, $\widehat{H}_E$, $\widehat{H}_{int}$ are the free Hamiltonians of the OQS (1) and the environment and the Hamiltonian of the interaction between them, respectively:

$$\widehat{H}_E = \int\limits_0^\infty \omega \widehat{a}^+(\omega)\widehat{a}(\omega)d\omega \tag{3}$$

$$\widehat{H}_{int} = \widehat{J}_y(\widehat{a}^+ + \widehat{a}), \quad \widehat{a} = \int\limits_0^\infty c(\omega)\widehat{a}(\omega)d\omega \tag{4}$$

where $\widehat{a}^+(\omega)$, $\widehat{a}(\omega)$ are bosonic environment's creation and annihilation operators with $[\widehat{a}(\omega), \widehat{a}^+(\tilde{\omega})] = \delta(\omega - \tilde{\omega})$, $c(\omega)$ is coupling. Such an interaction means that the environment records the trajectory of the projection of the y-component of the angular momentum of kicked top.

Fig.3 shows the behavior of a quantum kicked top in the case of integrable and chaotic motion. In the following sections we describe how does these results were obtained.

In the interaction picture with respect to free bosonic environment:

$$\widehat{H}(t) = \widehat{H}_S(t) + \widehat{J}_y(\widehat{a}^+(t) + \widehat{a}(t)) \tag{5}$$

$$\widehat{a}(t) = \int\limits_0^\infty c(\omega)\widehat{a}(\omega)e^{-i\omega t}d\omega$$

In our work, it is convenient to represent the environment in the equivalent chain representation [11]. This is necessary in order to introduce the concept of the Lieb-Robinson light cone [34]. For a sufficiently wide class of spectral densities, there exists an unitary operator U that takes the system into a chain representation [11]. Using the unitary operator, the environment is represented as a chain where only neighboring modes interact:

$$a_n^+ = \int\limits_0^\infty U_n(\omega)\widehat{a}^+(\omega)d\omega \tag{6}$$

$$\begin{aligned}\widehat{H}(t) = \widehat{H}_S(t) + \widehat{J}_y h(\widehat{a_0}^+ + \widehat{a_0}) + \\ + \sum_{n=0}^\infty \left(\epsilon_n \widehat{a}_n^+\widehat{a}_n + h_n \widehat{a}_{n+1}^+\widehat{a}_n + h_n \widehat{a}_n^+\widehat{a}_{n+1}\right)\end{aligned} \tag{7}$$

with commutator $[a_i, a_j^+] = \delta_{ij}$. Knowing the spectral density, the coefficients ϵ_n and h_n and h can be calculated by recurrent formulas using orthogonal polynomials [11].

In the interaction picture with respect to free bosonic environment in chain representation we obtain:

$$\widehat{H}(t) = \widehat{H}_S(t) + \widehat{J}_y h(\widehat{a_0}^+(t) + \widehat{a_0}(t)) \tag{8}$$

IV. QUANTUM ENVIRONMENT AS THE RECORDER OF OQS TRAJECTORIES

In this work, the main idea is that we consider the environment as a recording device that records information about the movement of the OQS in some its degrees of freedom. Thus, in the environment there is a sequence of projection operators corresponding to these records (facts) about how the OQS moved. The definition of trajectories we introduce is related to the approach of decoherent histories [29, 30, 36].

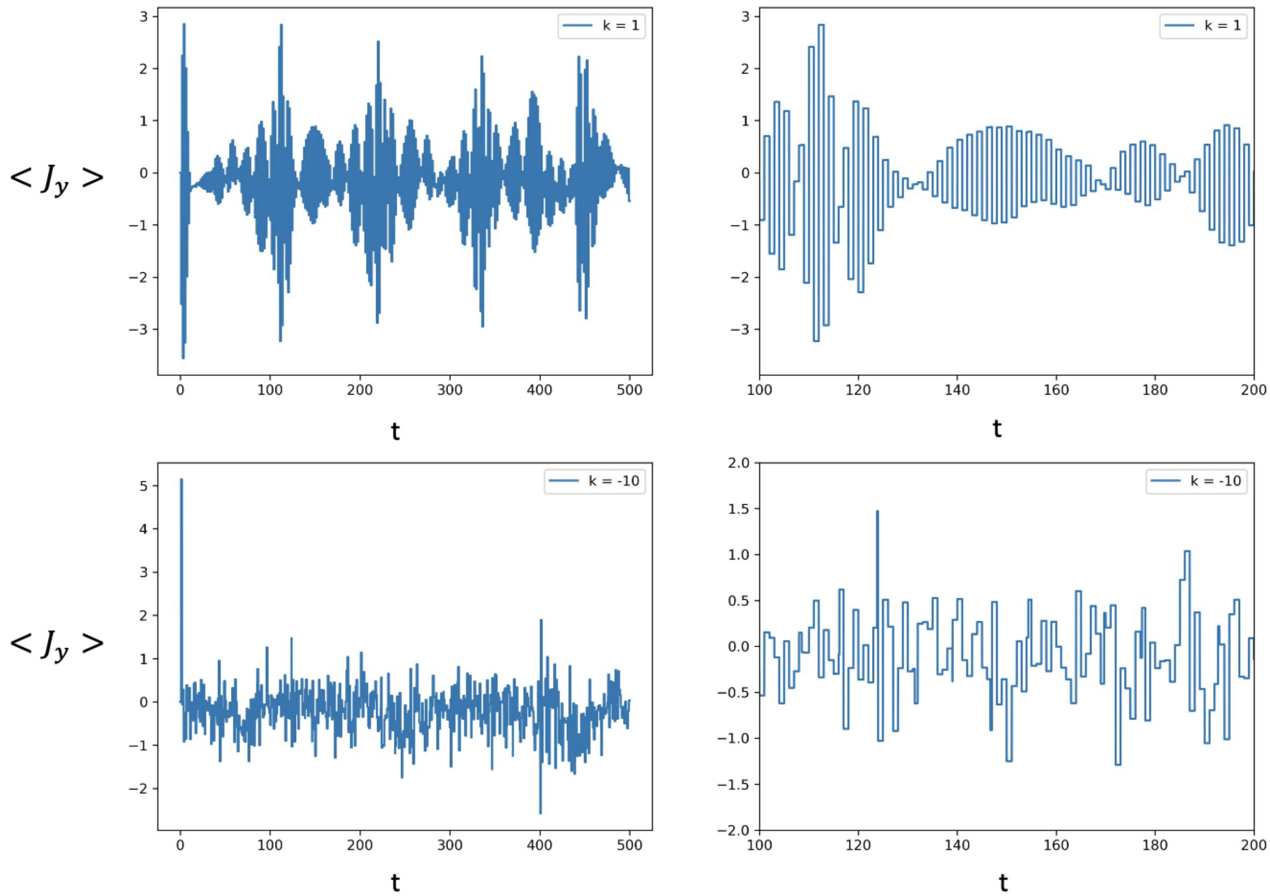


FIG. 3. Mean value of J_y versus time along the one trajectory. Top images are for regular motion $K = 1$; lower images, for $K = -10$; the right plots enlarge the left ones. Initial condition $|\Psi(0)\rangle = |J_y = 0\rangle \otimes |0\rangle_E$, $|0\rangle_E$ is vacuum state of the environment. The images are obtained with the following parameters for the environment: $\epsilon_n = 1$, $h_n = 0.2$, $h = 0.05$.

The consistent histories, also known as decoherent histories (DH) formalism was introduced by Griffiths, Omnes, Gell-Mann, Hartle. This formalism is an interpretation of quantum mechanics that allows one to resolve/tame the main quantum paradoxes. DH approach is based on the assumption of the probabilistic nature of quantum time dependence [36].

A “history” is a set of events or prepositions, represented by projection operators $P^1_{\alpha_1}, ..., P^n_{\alpha_n}$ at a succession of times $t_1, ..., t_n$ time-ordered with unitary evolution between each projection [29, 38]:

$$C_{\alpha_1,...,\alpha_n} = P^n_{\alpha_n}(t_n)P^{n-1}_{\alpha_{n-1}}(t_{n-1})...P^1_{\alpha_1}(t_1) \tag{9}$$

where

$$P^i_{\alpha_i}(t_i) = e^{\frac{i}{\hbar}H_E(t_i-t_{i-1})}P^i_{\alpha_i}e^{-\frac{i}{\hbar}H_E(t_i-t_{i-1})} \tag{10}$$

Here we present this approach in relation to a free bath, in contrast to the original approach presented for the entire isolated system. We consider a bipartite system: an OQS and a bosonic bath with density matrix $\rho = |\Psi\rangle\langle\Psi|$ ($|\Psi\rangle$ — wave function OQS plus bath) and develop the decoherent histories approach only for a free bath.

The probability for history is given by [38]:

$$p(\alpha_1, ..., \alpha_n) = Tr(C_{\alpha_1,...,\alpha_n}\,\rho\, C^+_{\alpha_1,...,\alpha_n}) \tag{11}$$

For consistency the following conditions are necessary: (i) the sum of probabilities of the histories is unity $\sum_{\alpha_i} P^i_{\alpha_i} = I$; (ii) two distinct histories are mutually orthogonal $P^i_{\alpha_i}P^j_{\beta_j} = \delta_{\alpha_i\beta_j}P^i_{\alpha_i}$. Generalization of this condition is:

$$Tr(\widehat{C}_{\alpha_1,...,\alpha_n}\rho\,\widehat{C}^+_{\beta_1,...,\beta_n}) = 0\,, \quad \text{for } \alpha \neq \beta \tag{12}$$

In practice, it turns out that this condition is strictly impossible to achieve. However, it can be approximated arbitrarily well with respect to some given level of significance. Thus, we arrive at the condition of weak consistency:

$$Tr(\widehat{C}_{\alpha_1,...,\alpha_n}\rho\,\widehat{C}^+_{\beta_1,...,\beta_n}) \approx 0\,, \quad \text{for } \alpha \neq \beta \tag{13}$$

In the approach of decoherent histories, the question arises of how to build these projectors $P^i_{\alpha_i}$ and what observables and time moments to consider. There is some arbitrariness in the choice of this [39]. Moreover it is a difficult problem to construct them. Recently it was proposed to search for them on a quantum computer [35]. Our approach naturally resolves it. On the one hand, we have a physical model, the scattered field carries away information about the OQS motion, and on the other hand, we propose a formal consideration how these projectors may be found. Projectors must match the degrees of freedom of the environment and naturally arise from a properties of the environment. In the next Section we derive such degrees of freedom.

V. ENVIRONMENT DEGREES OF FREEDOM WHICH CARRY THE INFORMATION ABOUT THE TRAJECTORY

In this Section we describe our procedure for derivation the environment's degrees of freedom carrying useful information about the trajectory of the OQS.

A. Statistically significant interacting modes

The quantum environment is treated as a recording device. Its records can be measured and decoherent histories can be obtained. Decoherent histories can only be contained within a light cone, so it is necessary to be able to evaluate it. Below is an algorithm for computing the light cone a priori.

The light cone allows one to determine which degrees of freedom are significant and which are not. The region outside the light cone consists of degrees of freedom that will only be significantly excited at future times, or will never be excited at all. In particular, for each chain site in eq.(7), there is a point in time after which it becomes statistically significant for the evolution of the system.

In order to estimate which modes the OQS excited, it is necessary to introduce a measure that determines the influence of the OQS on the considered mode. For this, it is convenient to use the commutator $[\widehat{a}_0(t), \widehat{a}_j^+]$, which will show whether the operator $a_0(t)$ affects a_j. If the mode a_j is currently interacting with the OQS, then the norm of this commutator will be different from zero:

$$\|[\widehat{a}_0(t), \widehat{a}_j^+]\| = \sqrt{Tr\left([\widehat{a}_0(t), \widehat{a}_j^+][\widehat{a}_0(t), \widehat{a}_j^+]^+\right)} \quad (14)$$

where $\widehat{a_0}(t)$ is the degree of freedom with which the OQS interacts at time t. The operator $\widehat{a_0}(t)$ in the interaction picture can be expressed in terms of the original chain operators, according to:

$$\widehat{a}_0(t) = \sum_{k=0}^{\infty} \phi_k(t)\widehat{a}_k \quad (15)$$

where $\phi_k(t)$ is one-particle wave function, which satisfies the following first-quantized Schrödinger equation with the initial condition corresponding to the interaction quench at time $t = 0$:

$$\begin{cases} \partial_t \phi_k(t) = \frac{1}{i}\widehat{H}_1 \phi_k(t) \\ \phi_k(0) = \delta_{k0} \end{cases} \quad (16)$$

where

$$H_1 = -\begin{pmatrix} \epsilon_0 & h_0 & 0 & \dots & \dots \\ h_0 & \epsilon_1 & h_1 & 0 & \dots \\ 0 & h_1 & \epsilon_1 & h_2 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & h_{m(t)-1} & \epsilon_{m(t)} \end{pmatrix} \quad (17)$$

Here $m(t)$ is the number of environmental degrees of freedom (later we can call it modes) that have been excited due to co-evolution with the OQS over time t. The perturbation propagates along the Lieb-Robinson light cone [34] from the zero site a_0, with which the OQS is connected. Fig.4 shows the spread of the operator $\widehat{a}_0(t)$ over sites of chain.

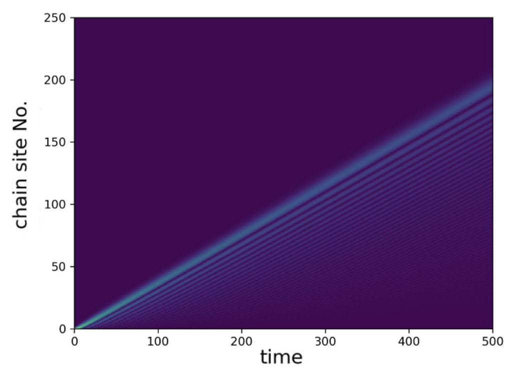


FIG. 4. Wave function $\phi_k(t)$ propagating the interaction operator $a_0(t)$ over time. The color matches $|\phi_k(t)|$. It can be seen that the perturbation propagates along the light cone.

For the simplest case of a linear environment, commutator (14) will be a C-number, since $[\widehat{a}_i, \widehat{a}_j^+] = \delta_{ij}$. Thus, instead of the trace from the product of the commutators, one can calculate their vacuum average:

$$C_j(t) = \begin{cases} Tr\left([\widehat{a}_0(t), \widehat{a}_j^+][\widehat{a}_0(t), \widehat{a}_j^+]^+\right) \\ \langle 0|[\widehat{a}_0(t), \widehat{a}_j^+][\widehat{a}_0(t), \widehat{a}_j^+]^+|0\rangle \end{cases} = $$

$$= M_j(t) \begin{cases} Tr1 \\ \langle 0|0\rangle \end{cases} \quad (18)$$

This function is OTOC [16]. The condition $C_j(t) \geq 0$ will mean that the $\phi_j(t)$ mode coupled with the OQS at a given time. If $C_j(t)$ is negligible, then the excitation of this mode due to the OQS is also negligible.

The light cone is determined not by the instantaneous intensity of the mode interaction, but rather by the average intensity of the mode interaction over the time interval from 0 to t. During the time t, only those modes enter the light cone that interact significantly on average over the entire interval. Therefore, it is necessary to consider only statistically significant interactions during the chosen time interval and eliminate sudden short-term excitations of environmental modes, which will make a negligible contribution. Thus, the condition is given by OTOC averaged over time:

$$\langle C_j^+(t)\rangle = \int_0^t C_j(\tau)d\tau \tag{19}$$

And we consider the modes that are effectively coupled and interact with OQS, influencing their joint evolution.

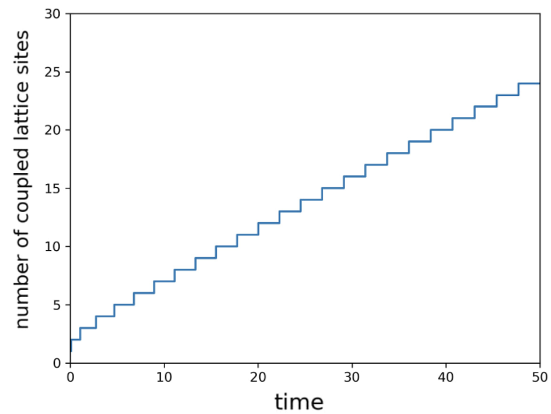


FIG. 5. The chain sites coupled to the OQS, depending on time, form a forward light cone. Coupled modes are defined according to eq.(22).

B. Records may be nonlocal

Information about the OQS is not necessarily carried by local chain degrees of freedom a_j, these can be their arbitrary linear combinations, so it is necessary to be able to take into account the statistical significance of such linear combinations (nonlocal with respect for chain); so it is necessary to be able to take into account the statistical significance and determine whether they fall inside the light cone. The average statistical significance of $\chi = \sum_{k=0}^{\infty}\chi_k|k\rangle$ state ($|k\rangle$ is quantum localized in k chain site) with corresponding creation and annihilation operators $\widehat{\chi}^+ = \sum_{k=0}^{\infty}\chi_k\widehat{a}_k^+$ is therefore:

$$\int_0^t \langle 0|[\widehat{a}_0(\tau), \sum_k \chi_j\widehat{a}_j^+][\widehat{a}_0(\tau), \sum_l \chi_l\widehat{a}_l^+]^+|0\rangle d\tau =$$

$$= \langle\chi|\int_0^t |\phi(\tau)\rangle\langle\phi(\tau)|d\tau|\chi\rangle = \langle\chi|\rho_+(t)|\chi\rangle \tag{20}$$

with

$$\rho_+(t) = \int_0^t |\phi(\tau)\rangle\langle\phi(\tau)|d\tau \tag{21}$$

We introduce a metric that determines whether the contribution of the χ state is significant or not:

$$g_+(\chi,t) = \langle\chi|\rho_+(t)|\chi\rangle - a_{cut} \tag{22}$$

if $g_+(\chi,t) < 0$, then the contribution of this mode can be neglected with threshold a_{cut}. Modes lying inside the light cone, that is satisfying condition $g_+(\chi,t) > 0$ contain information about the OQS (the kicked top). Fig.5 shows the modes (chain sites) coupled to the OQS over time determined according to eq.(22).

There is a drawback to the light cone defined in a chain basis, namely that the environment's degrees of freedom are not statistically independent. By analogy with the Kotelnikov theorem, truly independent degrees of freedom will appear in a basis where the speed of propagation of the light cone is minimal, and the intervals between the times of appearance of modes are proportional to the width of the spectral density of the environment as the bandwidth of the environment as a recording device [12]. Since a Lieb-Robinson metric (22) is defined for arbitrary mode we can turn to the basis where the light cone propagates with a minimum speed which we call the minimal light cone. Further, unless otherwise stated, we will work with him. The information about OQS is recorded in nonlocal environment's degrees of freedom.

A detailed algorithm for obtaining the minimal light cone is derived in the work [12]. We will denote these modes coupled to OQS for time interval $[0,T]$ as $\kappa_1^{in},...,\kappa_{m_{in}(T)}^{in}$ and the discrete times of their appearance as $t_1^{in},...,t_{m_{in}(T)}^{in}$.

The total joint state of the quantum kicked top and bosonic bath $|\Psi(t)\rangle$ effectively evolves with Hamiltonian:

$$\widehat{H}_{eff}(t) = \widehat{H}_S(t) + \\ + \sum_{l=1}^{m_{in}(t)}\left\{\widehat{J}_y\langle\phi(t)|\kappa_l^{in}\rangle\widehat{\kappa}_l^{in} + \widehat{J}_y\langle\phi(t)|\kappa_l^{in}\rangle^*\widehat{\kappa}_l^{in+}\right\} \tag{23}$$

Entanglement between degrees of freedom is neglected when their statistical significance is below a threshold.

C. Irreversibly decoupled modes – stable records

Records carrying information about the OQS must be stable facts, so it is necessary to consider modes that are irreversibly decoupled from the OQS.

Two different cases of outgoing (decoupled) modes are possible: (a) the modes, which have never interacted with the OQS; (b) modes were interacting with the OQS and then irreversibly decoupled from it. The first situation does not contain any information about the OQS, and we discard these modes from consideration. However it is necessary to track the evolution of the (b). The mode decoupled from the OQS at time t_l^{out} must be a linear combination of κ_1^{in}, κ_2^{in}, ..., $\kappa_{m_{in}(t_l^{out})}^{in}$ they must be in the subspace of modes coupled to the OQS for the time interval $[0, t_l^{out}]$. It is these modes that will store information about the trajectory of the OQS.

Analogically eq.(20), for outgoing modes, the measure of statistical significance at the time t, which determines the decoupled of the mode from the OQS is

$$\langle C^-(t,\chi)\rangle = \langle\chi| \int_t^T |\phi(\tau)\rangle\langle\phi(\tau)|d\tau|\chi\rangle = \langle\chi|\rho_-(t)|\chi\rangle \quad (24)$$

with

$$\rho_-(t) = \int_t^T |\phi(\tau)\rangle\langle\phi(\tau)|d\tau \quad (25)$$

A mode can be considered irreversibly decoupled if the OTOC averaged over future times is negligible.

We are interested in modes that carry information about the trajectory of the OQS, so the condition of lack of statistical significance for them is:

$$g_-(\kappa^{in}, t) = \langle\kappa^{in}|\rho_-(t)|\kappa^{in}\rangle - a_{cut} < 0 \quad (26)$$

These modes can be found by analogy with the search for coupled modes in minimal light cone by some unitary rotation of the basis of coupled modes $\kappa_1^{in}, ..., \kappa_{m_{in}(T)}^{in}$. We will denote irreversibly decoupled modes for time interval $[0, T]$ as $\kappa_1^{out}, ..., \kappa_{m_{out}(T)}^{out}$ and the discrete times of their decoupled as $t_1^{out}, ..., t_{m_{out}(T)}^{out}$. The information about the OQS contains irreversibly decoupled modes that previously interacted with it. For more details see work [12].

D. Relevant modes

By the time t_k^{out}, when the k-st κ_k^{out} mode was decoupled, $m_{in}(t_k^{out}) - 1$ modes remained coupled. We call these modes relevant modes, because they are statistically significant for future evolution. At time t_k^{out} there is $k-1$ irreversibly decoupled modes, $m_{in}(t_k^{out})$ coupled modes and their difference are relevant modes $r(t_k^{out})$:

$$r(t_k^{out}) = m_{in}(t_k^{out}) - k + 1 \quad (27)$$

The total system state $|\Psi(t)\rangle$ evolves over the time interval $[t_k, t_k^{out}]$, where t_k is the time of the previous mode coupled or decoupled with relevant modes.

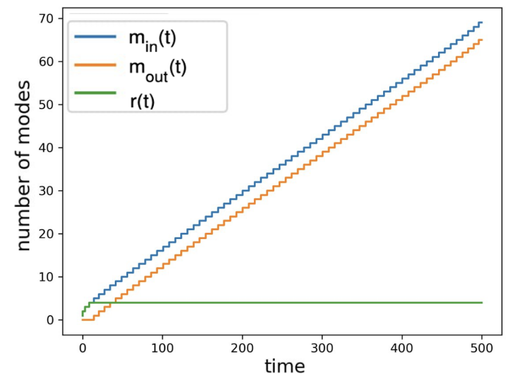


FIG. 6. The number of modes in the system over time, $m_{in}(t)$, $m_{out}(t)$, $r(t)$ — coupled, irreversibly decoupled and relevant modes, respectively.

Fig.6 shows the number of connected modes, disconnected modes and relevant modes over time. It can be seen that the number of relevant modes saturates and practically does not change during the evolution of the system.

E. Relation to decoherent histories approach

The problem with the decoherent histories approach is related to the inability to achieve the consistency condition (12). Our approach suggests an effective solution to this problem. Since the records are contained in irreversibly decoupled degrees of freedom, and it is over them that the projectors are taken, then the corresponding terms from the Hamiltonians will be thrown out. Thus, firstly, the sum of the probabilities will always be one (i), and secondly, they will always be orthogonal (ii).

In fact, the problem of decoherent histories lies in the fact that it is not taken into account that after interaction quench the OQS is renormalized (by analogy with the electron in high energy physics). In this paper, we solve this problem by assuming that the renormalizable OQS consists of a bare OQS and relevant modes with which it interacts significantly at future times. And the irreversibly decoupled modes (the stable records) are really those degrees of freedom that contain information (facts) and can be measured. In this case, a weak consistency is obtained, which converges exponentially quickly in the number of relevant modes.

VI. SIMULATING DECOHERENT HISTORIES

Knowing the degrees of freedom in which the environment records information about the trajectory of the kicked top, they can be measured. The measurement statistics will give an ensemble of quantum trajectories — decoherent histories.

Before $t = t_k^{out}$ the mode κ_k^{out} was coupled to the OQS. It was in an entangled state with the OQS due to the Schmidt decomposition:

$$|\Psi(t_k^{out})\rangle = \\ = \sum_q c_q(k) |\Psi_{coll}^{(q)}(t_k^{out})\rangle_{rel} \otimes |\Psi_J^{(q)}(t_k^{out})\rangle_{\kappa_k^{out}} \quad (28)$$

where index rel means OQS with relevant modes, and κ_k^{out} refers to the newly formed irreversibly decoupled mode, q enumerates the basis elements for the such mode.

Since this κ_k^{out} mode is irreversibly decoupled, the amplitudes $c_q(k)$ do not depend on time, these are invariants. A flow of motion invariants arises; they cease to depend on time effectively by the threshold of significance. Thus, form (28) is invariant at all future times and an invariant entanglement structure arises for future evolution. It has also been confirmed numerically. The emerging invariant structure of entanglement carries an ensemble of decoherent histories.

According to the von Neumann measurement model [37], one can collapse the wave function (28) and interpret the equation as the k-th quantum jump at time $t = t_k^{out}$: $|\Psi(t_k^{out})\rangle \to |\Psi_{coll}^{(q)}(t_k^{out})\rangle$ with probability $|c_q(k)|^2$. Such quantum jump are irreversible in time.

By the time t, $m_{out}(t)$ modes has been irreversibly decoupled. Each mode decoupled is accompanied by a quantum jump, which is obtained by recurrently applying the measurement procedure:

$$\begin{aligned} &|\Psi(t_1^{out})\rangle \to |\Psi_{coll}^{(q_1)}(t_1^{out})\rangle_{rel} \\ &|\Psi_{coll}^{(q_1)}(t_2^{out})\rangle_{rel} \to |\Psi_{coll}^{(q_1 q_2)}(t_2^{out})\rangle_{rel} \\ &|\Psi_{coll}^{(q_1 q_2)}(t_3^{out})\rangle_{rel} \to |\Psi_{coll}^{(q_1 q_2 q_3)}(t_3^{out})\rangle_{rel} \end{aligned} \quad (29)$$

Therefore, $m_{out}(t)$ quantum jumps occur before time t. They are characterized by the history of choices $h = (q_1, q_2, \ldots, q_k) = \{q_k\}_{k:\, t_k^{out} \leq t}$, appearing with probabilities:

$$P(q_1, q_2, \ldots, q_k) \ = \prod_{k:\, t_k^{out} \leq t} |c_{q_k}(k)|^2 \quad (30)$$

This is the proposed definition of decoherent histories. In this case, the average of observables over all decoherent histories h up to the time t corresponds to the full many-particle quantum dynamics of the OQS in terms of the significance threshold.

Thus, in the environment projectors operator (eq.(9)) in decoherent histories approach (Sec.IV) are naturally appear as:

$$P_{\alpha_k}^k = Tr_{\kappa_k^{out}}(|\Psi(t_k^{out})\rangle\langle\Psi(t_k^{out})|) \quad (31)$$

A. The entropy of an ensemble of decoherent histories

The statistical ensemble of quantum jump histories is encoded in an emerging invariant entanglement structure (28) that does not change at future times. By analogy with a tape recorder that records and does not change the recorded data.

Summarizing, to observe a trajectory a measuring device is needed. By adding the environment, considered as a recording device, information about the trajectory is recorded in the stream of irreversibly decoupled degrees of freedom.

We can introduce the definition of the entropy of an ensemble of decoherent histories (30):

$$S = - \sum_{\tilde{q}=(q_1,\ldots q_N)} P(\tilde{q}) \ln(P(\tilde{q})) \quad (32)$$

VII. THE ENTROPY OF DECOHERENT HISTORIES AS A MARKER OF QUANTUM CHAOS

We proposed that entropy of an ensemble of decoherent histories (32) may be a criteria of quantum chaos. In this Section we present our main results.

The entropy was calculated taking into account the simplified assumption of the presence of ergodicity for quantum trajectories. In the sense that averaging over all trajectories is equivalent to averaging within one sufficiently long trajectory over all choices. It is the averaging over one trajectory that was used in our work.

As soon as a degree of freedom that irreversibly decoupled appeared, a quantum jump was performed for it. In Fig.7 the probability distribution of quantum jumps $|c_q|^2$ (all possible choices) is presented.

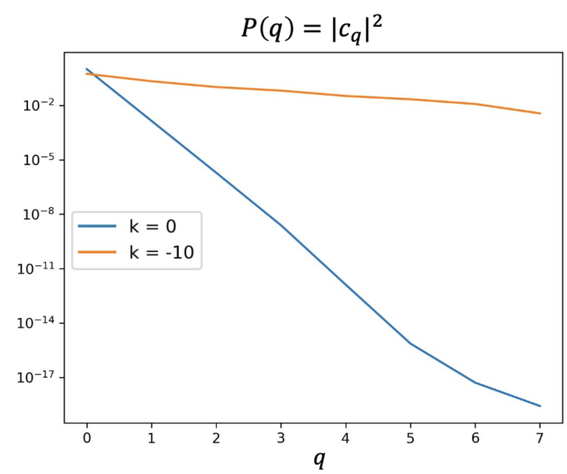


FIG. 7. Quantum jump probability distribution $|c_q|^2$ in two cases for $K = 0$ (blue curve) and for $K = -10$ (orange curve). 7 quanta participated in the dynamics.

The figure shows that for the integrable case at $K = 0$ the probability distribution is very narrow, while for the chaotic regime $K = -10$ the jump probability distribution is very wide.

This procedure was repeated for the entire time interval. One random implementation of the choice of quantum transitions was considered. In Fig.8 the instantaneous production of entropy is shown depending on the number of the quantum jump. In integrable and chaotic regimes, entropy along one trajectory behaves in radically different ways.

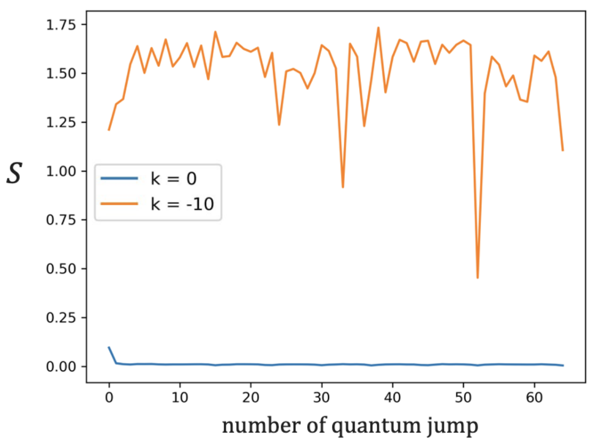


FIG. 8. Instantaneous entropy production along one trajectory for $K = 1$ (blue curve) and for $K = -10$ (orange curve).

When q jumps have already happened and the moment of the next jump has come, we can expand the wave function of the system in terms of the Schmidt expansion (28) and from the previous set of significant modes select a new set of significant modes and a mode that is irreversibly decoupled (on which the projection is carried out):

$$|\Psi(t)\rangle = \sum_{P_{q+1}} c_{P_{q+1}} |\Psi_{coll}(t, P_1, ..., P_q, P_{q+1})\rangle_{rel} \otimes |\Psi_J(t)\rangle_{P_{q+1}}$$

At the $q+1$ step, a new distribution of quantum jumps arises (a set of alternatives). Entropy for one jump will increase:

$$\Delta S = -\sum_{P_{q+1}} c_{P_{q+1}} \ln(c_{P_{q+1}}) \quad (33)$$

Average entropy production for one trajectory per quantum jump:

$$\langle \Delta S \rangle = \frac{1}{n} \sum_n \Delta S \quad (34)$$

where n is total number of the quantum jumps. In Fig.9 represents the average entropy production per quantum jump. It can be seen that its behavior changes strongly when passing from the integrable case there is practically no increase in entropy to the chaotic case there is a strong increase in entropy.

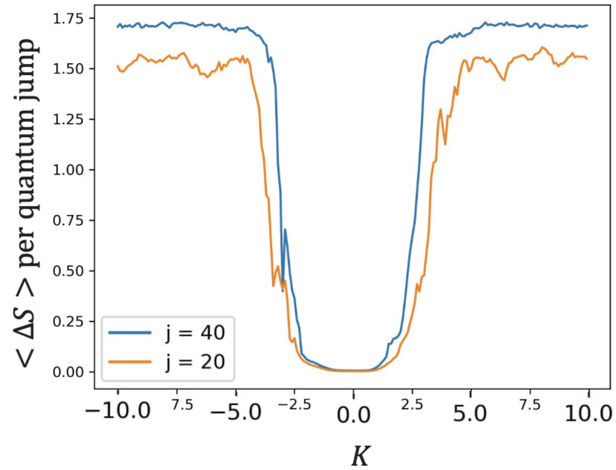


FIG. 9. Average entropy production (34) depending on the value of the kicked strength K. One can see a sharp increase in entropy production in the region of crossover between integrable and chaotic dynamics. The calculation was carried out for two different quantum numbers $j = 20$ (orange curve), for $j = 40$ (blue curve).

It was confirmed that in the integrable case the trajectories behave more regularly and the entropy practically does not increase, while in the transition to the chaotic case the trajectories mix strongly and the entropy grows rather sharply. Moreover, with an increase in j, the entropy growth angle increases. Thus, it is assumed that the entropy production along one trajectory can be a criterion of a quantum chaos.

VIII. CONCLUSIONS

The main idea was to introduce the definition of quantum chaos by analogy with the classical definition through the divergence of nearby trajectories.

Quantum trajectories can be introduced by connecting the system to the environment. The quantum environment in this case is analogous to a recording device. The role of the information carrier in the quantum environment is played by the degrees of freedom that are irreversibly decoupled from the OQS the stable records, which periodically arise during the evolution of the system in time.

In our work, we firstly offer the novel way of finding the degrees of freedom of the environment, that carries information about the trajectory by dint of averaging OTOC (19) and (24). Secondly, based on it we introduce the definition of trajectories and as a criterion for quantum chaos, we propose to use the entropy of the ensemble of given trajectories (32).

Thus, one can consider environment as a measuring device that autonomously selects the time of measurement and the preferred basis without the intervention of a human experimenter. And in turn during the evolution in real time, measuring one after another irreversibly decoupled modes with a certain probability, a sequence of measurements is obtained, which results is a quantum trajectory.

It was confirmed that for a regular motion, decoherent histories behave relatively regularly, while for a chaotic motion, the recorded particle trajectory is more fluctuating and irregular. The entropy of an ensemble of such trajectories grows faster for the chaotic case than for the integrable one. It is also possible to observe a noticeable sharp increase in entropy during the transition of the system dynamics from integrable to chaotic at values of the kicked strength K from 2 to 3 (Fig.9). Thus, this approach made it possible to fix the phenomenon of quantum chaos for the model of a quantum kicked top. We propose to connect any considered chaotic system to the environment and use the entropy of the ensemble of decoherent histories as a criterion of chaos.

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