Gravitational decoherence by the apparatus in the quantum-gravity induced entanglement of masses

Fabian Gunnink, Anupam Mazumdar, Martine Schut, 1,2 and Marko Toroš³

¹ Van Swinderen Institute for Particle Physics and Gravity,

University of Groningen, 9747AG Groningen, the Netherlands

² Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence,

University of Groningen, 9747 AG Groningen, the Netherlands

³ School of Physics and Astronomy, University of Glasgow, Glasgow, G12 8QQ, UK

One of the outstanding questions in modern physics is how to test whether gravity is classical or quantum in a laboratory. Recently there has been a proposal to test the quantum nature of gravity by creating quantum superpositions of two nearby neutral masses, close enough that the quantum nature of gravity can entangle the two quantum systems, but still sufficiently far away that all other known Standard Model interactions remain negligible. However, the mere process of preparing superposition states of a neutral mass (the light system), requires the vicinity of laboratory apparatus (the heavy system). We will suppose that such a heavy system can be modelled as another quantum system; since gravity is universal, the lighter system can get entangled with the heavier system, providing an inherent source of gravitational decoherence. In this paper, we will consider two light and two heavy quantum oscillators, forming pairs of probe-detector systems, and study under what conditions the entanglement between two light systems evades the decoherence induced by the heavy systems. We conclude by estimating the magnitude of the decoherence in the proposed experiment for testing the quantum nature of gravity.

I. INTRODUCTION

The theory of General Relativity (GR) is one of the most well-tested theories of physics, successfully passing a number fundamental tests [1], with its latest success being the observation of gravitational waves [2]. However, at short-distance scales and early times, where quantum effects start playing an important role, GR breaks down [3], and a quantum theory of gravity is needed. There are several candidate quantum gravity (QG) theories, such as string theory [4] and loop quantum gravity [5], but despite theoretical progress, the connection with experiments has remained elusive [6].

Albeit the quantization of gravity is an often-used tool in theoretical physics, forming the backbone of candidate quantum-gravity theories, thus far, there is no definitive experimental evidence in support of the quantum nature of gravity. The reason is simple – the weakness of the gravitational force makes direct detection of gravitons a formidable challenge, a situation which will likely persist in the foreseeable future [7]. On the other hand, indirect tests of the quantum nature of gravity (with the first discussions dating back to Feynman [8]) have in recent years become a real prospect with the advances in precision sensing and metrology, opening the possibility of probing genuine quantum features of gravity with tabletop experiments.

In 2017 a simple experiment for a definitive test of the quantum nature of gravity was proposed in [9], along with its relevant background and feasibility studies (for a related work see [10]). The idea exploits the quantum-gravity-induced entanglement of masses (QGEM) to

discern between all classical models of gravity from the quantum one ¹. Two nearby masses, each delicately prepared in a spatial superposition, are placed close enough that that their mutual gravitational interaction can generate entanglement, but still far enough that all other interactions are strongly suppressed. The generated entanglement can be detected by measuring quantum correlations between the two masses, a genuinely quantum effect with no classical analogue, and, if detected, would provide the first definite evidence for the quantization of the gravitational field.

The argument for the entanglement-based test of the quantization of gravity can be summarized as follows. To generate matter-matter entanglement one requires a quantum interaction coupling the two systems; the quantum matter-matter gravitational interaction (which in the non-relativistic regime is the operatorvalued Newtonian potential) corresponds to the shift of the energy of the gravitational field, hence requiring the gravitational field itself to be a quantum operator, ruling out the possibility of a (real-valued) classical gravitational field [12]. Formally, entanglement between two quantum states cannot be increased with local operations and classical communications (LOCC) [13], as would be the case with a classical gravitational field, and hence, if gravitationally induced entanglement is detected, the gravitational interaction must

When talking about a theory of quantum gravity, we assume an effective quantum field theory where a massless spin-2 graviton acts as a force carrier for the gravitational force, and which behaves well at low energies [11]

be ostensibly quantum in nature. This argument has been discussed in detail within the context of perturbative quantum gravity [12, 14], the path-integral approach [15], and the Arnowitt-Desse-Meissner (ADM) formalism [16].

To probe directly the spin nature of the graviton, we will require a new setup where will seek the entanglement properties between the quantum state of a matter with regards to the photons in a cavity, as in the case of a light bending experiment due to the gravitational interaction, see [17]. This paper highlights the parameter space to study the entanglement between the quantum state of a photon and a matter.

However, in the current paper we will follow the experimental implementation of the protocol tow entangle the two spatially superposed masses. In fact, to realise such an experiment in reality there are many challenges, such as the creation of large superpositions with heavy masses [18–20], the isolation of the system from other quantum interactions [21–23] and reduction of noise [24]. The shielding of the system from other interactions will never be completely perfect, there will always be effects from the environment on the test masses. Specifically, the quantum masses may lose their coherence due to interaction with the environment. Methods for battling the decoherence have been proposed previously [25–27], and many sources of decoherence have been discussed, such as in [28, 29].

There is however one source of inherent decoherence which has thus far not been analyzed in detail. In order to witness the generated entanglement we require the presence of nearby experimental apparatus; while electromagnetic couplings between a neutral mass (the light system) and the lab equipment (the heavy system) can be suppressed with appropriate shielding, their mutual gravitational interaction is unavoidable, and scales unfavourably with the mass of the laboratory apparatus. The heavy laboratory equipment, which can be modelled quantum mechanically, can entangle with the two neutral masses, thus providing an unavoidable source of gravitational decoherence.

When we talk about the 'apparatus' or 'laboratory equipment' we refer to anything close to the experiment that can be quantum, such as the current carrying wires in the Stern-Gerlach setup [18–20, 30]. We call any such sources the 'heavy mass' $m_A \sim m_B \sim M$ (for the analysis we assume the heavy masses have a similar magnitude M), which depends on the specifications of the apparatus. The aim of this paper is to analyze this gravity-induced decoherence in presence of the heavy masses in a model independent fashion, an effect inherent to any experimental implementation but we do not exactly model the apparatus, and to quantify the attenuation of the entanglement between the two neutral spatially quantum superposed masses.

Decoherence is the information loss of a system to the

environment. Similarly, entanglement is generally seen as the sharing of information between two systems. In this paper we quantify decoherence with an entanglement measure, the concurrence, which quantifies how much the laboratory equipment and the test masses are entangled. Since we are computing an entanglement measure, this is a basis-independent approach. An often-used approach to analyze decoherence is to trace out the 'environment' system and find the entanglement between the test masses, which gives the entanglement between the test masses with the effect of the environment taken into account. This approach often has a preferred basis [31]. We briefly discuss this approach in Sec. VI, but when we talk about 'the decoherence' we refer to the entanglement between the apparatus and the test masses.

First, we will introduce the setup consisting of two heavy quantum harmonic oscillators (representing the laboratory apparatus) and two light quantum harmonic oscillators (representing the two probe systems), and introduce all the relevant interactions (Sec. II). We will then discuss how to calculate the entanglement using concurrence between two subsystems that are coupled by the quantized gravitational field within perturbative quantum gravity (Sec. III). Then we discuss the induced decoherence on the two light systems in the static limit (Sec. IV) and obtain higher order momentum corrections to this decoherence by considering the light systems beyond the static limit (Sec. V). We will end by analysing the allowed parameter space of the heavy systems under the condition that the entanglement between the light systems dominates the decoherence (Sec. VI) and finally we will conclude with a discussion of the found results (Sec. VII).

II. SETUP

Let us consider four massive systems, denoted by a, b, A and B with light masses m_a , m_b and heavy masses m_A , m_B , respectively. We wish to understand the entanglement of m_a , m_b via the quantum nature of gravity, while m_A , m_B would be responsible for gravitationally decohering the light masses. These massive systems are placed in harmonic traps located at $\pm \frac{d}{2}$ for systems a, b and located at $\pm \frac{D}{2}$ for systems A, B. We will assume D > d. Taking the harmonic oscillators to be well-localized, we obtain:

$$\hat{x}_a = -\frac{d}{2} + \delta \hat{x}_a , \qquad \hat{x}_b = \frac{d}{2} + \delta \hat{x}_b , \qquad (1)$$

$$\hat{x}_{a} = -\frac{d}{2} + \delta \hat{x}_{a} , \qquad \hat{x}_{b} = \frac{d}{2} + \delta \hat{x}_{b} , \qquad (1)$$

$$\hat{x}_{A} = -\frac{D}{2} + \delta \hat{x}_{A} , \qquad \hat{x}_{B} = \frac{D}{2} + \delta \hat{x}_{B} , \qquad (2)$$

with \hat{x}_i and $\delta \hat{x}_i$ the position operators and small equilibrium displacement for system i = a, b, A, B, respectively. We will further assume that all the masses are

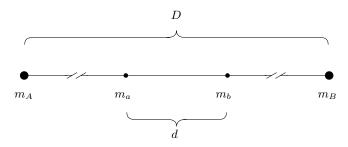


Figure 1: A graphical representation of the setup that visualizes the introduced parameters D, d. With a and b denoting the light systems and A and B denoting the heavy systems.

neutral to minimize the electromagnetic interactions. Although there will be dipolar interactions between all these systems. The Casimir induced dipole-dipole interactions between the two systems $a,\ A$ and $b,\ B$ can be minimised by placing a conducting plate, while the Casimir interaction between a light and a heavy system can be minimised by giving some hierarchy between D and d. The Hamiltonian for the matter system is given by:

$$\hat{H}_{\rm m} = \sum_{i=a,b,A,B} \frac{\hat{p}_i^2}{2m_i} + \frac{m_i \omega_i^2}{2} \delta \hat{x}_i \,, \tag{3}$$

with \hat{p}_i and ω_i the conjugate momenta and the trap's harmonic frequency for system i, respectively. The basis is chosen such that the matter systems are uncoupled which will simplify our computations ². The mode operators for the harmonic oscillator systems are given by:

$$\delta \hat{x}_j = \sqrt{\frac{\hbar}{2m_j\omega_j}} (j+j^{\dagger}), \ \hat{p}_j = i\sqrt{\frac{\hbar m_j\omega_j}{2}} (j-j^{\dagger}) \quad (6)$$

 2 As an initial system we can choose a Hamiltonian where there is a coupling between systems $a\ (b)$ and $A\ (B)$:

$$\hat{H}_{\rm m} = \sum_{i=1,2,3,4} \frac{\hat{p}_i'^2}{2m_i} + k_0(\delta \hat{x}_1^2 + \delta \hat{x}_2^2) + k_1(\delta \hat{x}_1 - \delta \hat{x}_2)^2 + k_2(\delta \hat{x}_3^2 + \delta \hat{x}_4^2) + k_3(\delta \hat{x}_3 - \delta \hat{x}_4)^2.$$
(4)

Then there exists a unitary transformation such that the Hamiltonian becomes decoupled. After the transformation the matter Hamiltonian can be written as

$$\hat{H}_{\rm m} = \hat{H}_a + \hat{H}_b + \hat{H}_A + \hat{H}_B \,. \tag{5}$$

with $\hat{H}_i=p_i^2+\omega_i^2\hat{x}_i^2$, and with $\omega_{a(b)}=k_{0(2)}^{1/2},$ $\omega_{A(B)}=[k_{0(2)}+2k_{1(3)}]^{1/2}$. The change of basis is given as:

$$\hat{x}_{a(b)} = [\hat{x}_{1(3)} + \hat{x}_{2(4)}]/\sqrt{2}, \qquad \hat{x}_{A(B)} = [\hat{x}_{1(3)} - \hat{x}_{2(4)}]/\sqrt{2} \,.$$

with j=a,b,A,B, and the operators satisfying the usual commutation relations ³. Thus the Hamiltonian can be written as:

$$\hat{H}_{\rm m} = \hbar \omega_a \hat{a}^{\dagger} \hat{a} + \hbar \omega_b \hat{b}^{\dagger} \hat{b} + \hbar \omega_A \hat{A}^{\dagger} \hat{A} + \hbar \omega_B \hat{B}^{\dagger} \hat{B} \,. \tag{7}$$

We now introduce a gravitational field and study the interaction Hamiltonian $\hat{H}_{\rm int}$ between the gravitational and matter fields.

We work in linearized gravity where the metric is given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $\eta_{\mu\nu}$ the flat Minkowski background with signature (-,+,+,+) and with $h_{\mu\nu}$ a perturbation which is small in magnitude around the Minkowski background. The metric fluctuations are then promoted to quantum operators:

$$\hat{h}_{\mu\nu} = \sqrt{\frac{16\pi G}{c^2}} \int d^3k \sqrt{\frac{\hbar}{2\omega_k (2\pi)^3}} \left(\hat{P}^{\dagger}_{\mu\nu} (\vec{k}) e^{-i\vec{k}\vec{r}} + \text{H.c.} \right),$$
(8)

where $\hat{P}_{\mu\nu}$ and $\hat{P}^{\dagger}_{\mu\nu}$ denote the graviton annihilation and creation operators, respectively, and satisfy the following commutation relations [32]:

$$[\hat{P}_{\mu\nu}(\vec{k}), \hat{P}_{\rho\sigma}^{\dagger}(\vec{k}')] = (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})\delta(\vec{k} - \vec{k}'). \quad (9)$$

In the weak field regime we can decompose the metric fluctuation operator into two modes: the spin-2 mode $\gamma_{\mu\nu}$ and the spin-0 mode $\gamma \equiv \eta_{\mu\nu}\gamma^{\mu\nu}$ [32] ⁴. This gives: $\hat{h}_{\mu\nu} = \hat{\gamma}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\hat{\gamma}$. Consequently the spin-2 and spin-0 decomposed parts of the graviton can be promoted to operators as well, and they are given in terms of the graviton creation- and annihilation operators [32]:

$$\hat{\gamma}_{\mu\nu} = \sqrt{\frac{16\pi G}{c^2}} \int d^3k \sqrt{\frac{\hbar}{2\omega_k (2\pi)^3}} \left(\hat{P}^{\dagger}_{\mu\nu}(\vec{k}) e^{-i\vec{k}\vec{r}} + \text{H.c.} \right),$$
(10)

$$\hat{\gamma} = 2\sqrt{\frac{16\pi G}{c^2}} \int d^3k \sqrt{\frac{\hbar}{2\omega_k (2\pi)^3}} \left(\hat{P}^{\dagger}(\vec{k}) e^{-i\vec{k}\vec{r}} + \text{H.c.} \right),$$
(11)

satisfying the commutation relations in eq. (9)⁵. The

$$\begin{split} [a,a] &= [b,b] = [A,A] = [B,B] = 0 \\ [a^{\dagger},a^{\dagger}] &= [b^{\dagger},b^{\dagger}] = [A^{\dagger},A^{\dagger}] = [B^{\dagger},B^{\dagger}] = 0 \\ [a,a^{\dagger}] &= [b,b^{\dagger}] = [A,A^{\dagger}] = [B,B^{\dagger}] = 1 \,. \end{split}$$

$$[\hat{P}(\vec{k}), \hat{P}^{\dagger}(\vec{k}')] = -\delta(\vec{k} - \vec{k}') \tag{12}$$

 $^{^3}$ These commutation relations are:

⁴ These two modes can be treated independently. $\gamma_{\mu\nu}$ is sometimes called the trace-reversed metric since $h=-\gamma$.

⁵ Following eq. (9) and the definition $\gamma \equiv \eta_{\mu\nu}\gamma^{\mu\nu}$, the additional commutation relation is:

gravity Hamiltonian is given by [32]:

$$\hat{H}_{g} = \int d^{3}k \, \hbar \omega_{k} \left(\frac{1}{2} \hat{P}_{\mu\nu}^{\dagger}(\vec{k}) \hat{P}_{\mu\nu}(\vec{k}) - \hat{P}^{\dagger}(\vec{k}) \hat{P}(\vec{k}) \right).$$
(13)

Now that both the matter and graviton systems have been introduced, we continue by studying their interaction and in the next section the consequential entanglement generation. The interaction term is given by the graviton coupling to the stress-energy tensor $T_{\mu\nu}$ (which specifies the matter system contents):

$$\hat{H}_{\rm int} = -\frac{1}{2} \int d^3 r \, \hat{h}^{\mu\nu}(\vec{r}) \hat{T}_{\mu\nu}(\vec{r}) \,. \tag{14}$$

We consider the two harmonically trapped particles a, bto be moving along the x-axis, while the two heavy traps A, B to be static. The systems A, B are taken to be static because we consider these systems to be the very massive apparatus which are approximated as harmonic oscillators. The four systems thus generate the following currents:

$$\hat{T}_{00}(\vec{r}) \equiv \sum_{n=a,b,A,B} m_n c^2 \delta(\vec{r} - \hat{r}_n) , \qquad (15)$$

$$\hat{T}_{ij}(\vec{r}) \equiv \sum_{n=a,b} \frac{\hat{p}_{n,\mu}\hat{p}_{n,\nu}}{E_i/c^2} \delta(\vec{r} - \hat{r}_n),$$
 (16)

with the position of the matter systems $\hat{r}_n = (\hat{x}_n, 0, 0)$, with $\hat{p}_{\mu} = (-E/c, \vec{p}), E = \sqrt{\vec{p}^2c^2 + m^2c^4}, \text{ and with } i, j = 1, 2, 3.$

Since we specified the movement of the oscillators a,b to be along the x-axis, the only non-zero T_{ij} components are $\hat{T}_{01},\,\hat{T}_{10}$ and $\hat{T}_{11}.$ Therefore, the only relevant $\hat{h}_{\mu\nu}$ components in the coupling are $\hat{h}_{00} =$ $\hat{\gamma}_{00} + \frac{1}{2}\hat{\gamma}$, $\hat{h}_{01} = \hat{h}_{10} = \hat{\gamma}_{01}$ and $\hat{h}_{11} = \hat{\gamma}_{11} - \frac{1}{2}\hat{\gamma}$. Writing the interaction Hamiltonian in terms of the decomposed metric perturbation, while exploiting the symmetries $\hat{T}_{01} = \hat{T}_{10}$ and $\hat{\gamma}_{01} = \hat{\gamma}_{10}$, gives:

$$\hat{H}_{\text{int}} = \int d^3r \left(\frac{1}{2} [\hat{\gamma}_{00}(\vec{r}) + \frac{1}{2} \hat{\gamma}(\vec{r})] \hat{T}_{00}(\vec{r}) + \frac{1}{2} [\hat{\gamma}_{11}(\vec{r}) - \frac{1}{2} \hat{\gamma}(\vec{r})] \hat{T}_{11}(\vec{r}) + \hat{\gamma}_{10}(\vec{r}) \hat{T}_{10}(\vec{r}) \right).$$
(17)

As explained in Ref. [12], the energy shift in the graviton vacuum due to the above interaction can only induce entanglement when the gravitational field is quantized, so if $h_{\mu\nu}$ or $\gamma_{\mu\nu}$, γ are operator-valued quantities. This can be explained in terms of the Local Operators and Classical Communications (LOCC) principle, which states that Local Operations or Classical Communication cannot increase the entanglement between the two systems. Only Quantum Communication can increase entanglement between the systems [12]. The graviton here acts as a quantum communicator between the two systems, and is therefore able to induce a coupling that entangles previously unentangled oscillators. This entanglement and decoherence are studied in the next sections.

III. ENTANGLEMENT VIA GRAVITON

We assume that initially the quantum matter systems are in the ground state (denoted by $|0\rangle_i$, with i specifying the system i = a, b, A, B):

$$|\psi_i\rangle = |0\rangle_a |0\rangle_b |0\rangle_A |0\rangle_B . \tag{18}$$

Since gravity will couple all the systems, it will induce interaction between the heavy and light oscillators, H_{hl} (which is scaled by a bookkeeping parameter λ), as a result the final state will evolve to:

$$|\psi_f\rangle = \frac{1}{\sqrt{\mathcal{N}}} \sum_{\substack{n_a, n_b \\ n_A, n_B}} C_{n_a n_b n_A n_B} |n_a\rangle |n_b\rangle |n_A\rangle |n_B\rangle .$$

The number states are denoted by $|n_i\rangle$, and the normalisation is given by $\mathcal{N} = \sum_{n_a, n_b n_A, n_B} |C_{n_a n_b n_A n_B}|^2$. In first order perturbation theory the coefficients for

the final wavefunction are given by:

$$C_{n_a n_b n_A n_B} = \lambda \frac{\langle n_a | \langle n_b | \langle n_A | \langle n_B | \hat{H}_{hl} | 0 \rangle | 0 \rangle | 0 \rangle | 0 \rangle}{\sum_{i=a,b,A,B} (E_{0_i} - E_{n_i})},$$
(20)

for the perturbed states, and $C_{0000} = 1$ for the unperturbed state. In the above equation E_{0i} is the groundstate energy and E_{n_i} denotes the nth excited state energy, for system i = a, b, A, B.

At this point it is important to take a note that \hat{H}_{hl} is a quantum operator. If it were classical, so not operator-valued, then for any perturbed coefficients $C_{n_a n_b n_A n_B} = 0$ due to the orthogonality of the states, thus the final wavefunction would be $|\psi_f\rangle$ = $|0\rangle |0\rangle |0\rangle |0\rangle$, the initial wavefunction. No entanglement can be generated in an initially unentangled system from a classical interaction. Since we are working in the framework of perturbative quantum field theory of gravity we expect an entanglement, which will be quantified by the concurrence of a biparte system, between the subsystems 1 and 2:

$$C \equiv \sqrt{2 - 2\operatorname{Tr}(\rho_1^2)}, \qquad (21)$$

⁶ Here we have left out the subscripts on the kets since it should be obvious from the state which system it concerns. In the remainder of the paper the order of the the states is always a, b, A, B.

where $\rho_1 = \text{Tr}_2(\rho)$ is the partial density matrix found by tracing out subsystem 2 in the full density matrix $\rho = |\psi_f\rangle \langle \psi_f|$. The larger the concurrence, the more strongly entangled the subsystems are, where a maximally entangled state gives the value $\sqrt{2}$ and an unentangled state gives the value 0.7.

The use of concurrence is limited to biparte systems though. As we are interested in the decoherence of the systems a, b due to their coupling to the more massive systems A, B, we choose the bi-partition such that subsystem 1 consists of the light particles a, b and system 2 consists of the heavy oscillators A, B. Since the entanglement and decoherence are two sides of the same coin, by studying the concurrence for this bi-partition we gain information about the effects of the apparatus (the heavy oscillators) on the coherence of the QGEM experiment (the two light particles). For the light-heavy bi-partition, the partial density matrix for the light system is

$$\rho_{1} = \frac{1}{N} \sum_{\substack{n_{a}, n_{b}, N_{a}, \\ N_{b}, n_{A}, n_{B}}} C_{n_{a}n_{b}n_{A}n_{B}} C_{N_{a}N_{b}n_{A}n_{B}}^{*} |n_{a} n_{b}\rangle \langle N_{a} N_{b}|,$$

using the notation $|n_a n_b\rangle = |n_a\rangle |n_b\rangle$. Inserting this expression into eq. (21), the heavy-light concurrence, denoted $C_{\rm hl}$, can be expressed in terms of the coefficients C:

$$C_{\rm hl} \equiv \left[2 - \frac{2}{N^2} \sum_{\substack{n_a, N_a, n_b, N_b \\ n_A, n_B, N_A, N_B}} C_{n_a n_b n_A n_B} C_{N_a N_b n_A n_B}^* \right]^{1/2} \times C_{n_a n_b N_A N_B} C_{N_a N_b N_A N_B}^* \right]^{1/2}.$$
(22)

Finding all the relevant expressions of the coefficients in eq. (20) would result in the quantification of decoherence/entanglement at first order in the perturbation theory. For this we need to find the interaction Hamiltonian between the heavy and light system, $\hat{H}_{\rm hl}$, this interaction is generated by the exchange of the virtual graviton.

The interaction between gravity and matter is given in eq. (17), from it we can compute the shift in energy to the graviton vacuum at second order in perturbation theory ⁸:

$$\Delta \hat{H}_g \equiv \int d^3k \, \frac{\langle 0|\,\hat{H}_{\rm int}|\vec{k}\rangle\langle\vec{k}|\hat{H}_{\rm int}|0\rangle}{E_0 - E_k} \,, \qquad (24)$$

with E_0 the energy of the vacuum state, $E_k = E_0 + \hbar \omega_k$ the energy of of the one-particle state $|\vec{k}\rangle$ representing the intermediate graviton, which is created from the vacuum with the graviton creation operators. The collection of projectors $|\vec{k}\rangle\langle\vec{k}|$ is given by (actually we give the normalized projectors):

$$|\vec{k}\rangle\langle\vec{k}| = \frac{1}{2}P_{00}^{\dagger}(\vec{k})|0\rangle\langle0|P_{00}(\vec{k}) + \frac{1}{2}P_{11}^{\dagger}(\vec{k})|0\rangle\langle0|P_{11}(\vec{k}) - P_{01}^{\dagger}(\vec{k})|0\rangle\langle0|P_{01}(\vec{k}) - P_{01}^{\dagger}(\vec{k})|0\rangle\langle0|P(\vec{k}).$$
(25)

For each projector summed in the above expression we can evaluate $\langle 0|\hat{H}_{\rm int}|\vec{k}\rangle$, with the interaction given in eq. (17).

$$\langle 0|\,\hat{H}_{\rm int}\hat{P}_{00}(\vec{k})|\vec{0}\rangle = \mathcal{A}\sqrt{\frac{\hbar}{2\omega_k}}\hat{T}_{00}(\vec{k})\,,\tag{26}$$

$$\langle 0|\,\hat{H}_{\rm int}\hat{P}_{11}(\vec{k})|\vec{0}\rangle = \mathcal{A}\sqrt{\frac{\hbar}{2\omega_{k}}}\hat{T}_{11}(\vec{k})\,,\tag{27}$$

$$\langle 0| \hat{H}_{\rm int} \hat{P}_{01}(\vec{k}) | \vec{0} \rangle = \mathcal{A} \sqrt{\frac{\hbar}{2\omega_k}} \hat{T}_{01}(\vec{k}), \qquad (28)$$

$$\langle 0|\hat{H}_{\rm int}\hat{P}(\vec{k})|\vec{0}\rangle = \frac{\mathcal{A}}{2}\sqrt{\frac{\hbar}{2\omega_k}}\left[\hat{T}_{00}(\vec{k}) - \hat{T}_{11}(\vec{k})\right], \quad (29)$$

with $A \equiv \sqrt{16\pi G/c^2}$. $\hat{T}_{ij}(\vec{k})$ are the stress-energy tensor components in momentum space ⁹, which from eqs. (15),(16) are found to be:

$$\hat{T}_{00}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \left[m_A c^2 e^{-i\vec{k}\cdot\hat{r}_A} + m_B c^2 e^{-i\vec{k}\cdot\hat{r}_B} + E_a e^{-i\vec{k}\cdot\hat{r}_a} + E_b e^{-i\vec{k}\cdot\hat{r}_b} \right], \quad (31)$$

$$\hat{T}_{01}(\vec{k}) = -\frac{c}{(2\pi)^{3/2}} \left[\hat{p}_a e^{-i\vec{k}\cdot\hat{r}_a} + \hat{p}_b e^{-i\vec{k}\cdot\hat{r}_b} \right], \tag{32}$$

$$\hat{T}_{11}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \left[\frac{\hat{p}_a^2 c^2}{E_a} e^{-i\vec{k}\cdot\hat{r}_a} + \frac{\hat{p}_b^2 c^2}{E_b} e^{-i\vec{k}\cdot\hat{r}_b} \right]. \quad (33)$$

Filling in eqs. (31)-(33) and eqs. (26)-(29) into eq. (24) gives an expression for the graviton energy shift from the vacuum, $\Delta \hat{H}_g$. This expression can be simplified

$$\hat{T}_{ij}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3r \, e^{-i\vec{k}\cdot\vec{r}} \hat{T}_{ij}(\vec{r}) \,. \tag{30}$$

⁷ The concurrence can be related to the maybe better-known and more widly applicable von Neumann entropy via a simple relation [33].

⁸ The first order term corresponding to the emission/absorption of a graviton is given by $\langle 0|\hat{H}_{\rm int}|0\rangle$. This contribution vanishes since $\hat{H}_{\rm int}$ depends linearly on the graviton creation and annihi-

lation operators, and $\hat{P} |0\rangle = \hat{P}_{\mu\nu} |0\rangle = 0$, $\langle 0| \hat{P}^{\dagger} = \langle 0| \hat{P}^{\dagger}_{\mu\nu} = 0$. In the second order term (corresponding to the exchange of a virtual graviton) $\langle 0| \hat{H}_{\rm int} |\vec{k}\rangle$ is quadratically dependent on the creation an annihilation operators. Using the operator commutation rules shows that this contribution is non-vanishing.

⁹ The momentum-space stress-energy tensor components are given by the Fourier transform of the components in position space:

by performing the integral over \vec{k}^{10} . Furthermore we restrict the movement to the x-axis, meaning that $\hat{p}_{i,y} = \hat{p}_{i,z} = 0$, $\hat{p}_{i,x} \equiv \hat{p}_i$ and $\hat{r}_i = (\hat{x}_i, 0, 0)$ for i = a, b, A, B, to find the expression:

$$\begin{split} \Delta \hat{H}_g &= -\frac{\mathcal{A}^2}{16\pi c^2} \left[\frac{m_A E_a c^2 + m_A \frac{\hat{p}_a^2 c^4}{E_a}}{|\hat{x}_A - \hat{x}_a|} \right. \\ &+ \frac{m_A E_b c^2 + m_A \frac{\hat{p}_b^2 c^4}{E_b}}{|\hat{x}_A - \hat{x}_b|} + \frac{m_A m_B c^4}{|\hat{x}_A - \hat{x}_B|} \\ &+ \frac{E_a E_b - 4 \hat{p}_a \hat{p}_b c^2 + \hat{p}_a^2 c^2 \frac{E_b}{E_a} + \hat{p}_b^2 c^2 \frac{E_a}{E_b} + \frac{\hat{p}_a^2 \hat{p}_b^2 c^4}{E_a E_b}}{|\hat{x}_a - \hat{x}_b|} \\ &+ \frac{m_B E_a c^2 + m_B \frac{\hat{p}_a^2 c^4}{E_a}}{|\hat{x}_a - \hat{x}_B|} + \frac{m_B E_b c^2 + m_B \frac{\hat{p}_b^2 c^4}{E_b}}{|\hat{x}_b - \hat{x}_B|} \right]. \end{split}$$

Taking $m_a = m_b = m$ and $m_A = m_B = M$, and expanding eq. (35) in powers of $1/c^2$ gives the non-relativistic couplings among the 4 oscillators upto order $1/c^4$, and in first order in G, the full expression is presented in eq. (A1) ¹¹.

IV. QUANTIFYING THE DECOHERENCE

In this section we given an expression for the decoherence due to the gravitational interaction between the heavy and light systems. We find the decoherence using an entanglement measure, the concurrence, given in

¹⁰ This integration is simply

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{|\vec{k}|^2} e^{i\vec{k}\cdot\hat{r}} = \frac{1}{4\pi\hat{r}}, \tag{34}$$

and the expression was rewritten such that $\hat{r} = \hat{x}_i - \hat{x}_j$.

We can reach the *classical* point particle limit by substituting $\vec{r} \equiv \vec{x_i} - \vec{x_j}$ with the number-valued distances discussed in sec.

II, the potential becomes:

$$\Delta H_g = -G \left[\frac{m^2}{d} + \frac{M^2}{D} - \frac{8mM}{d^2 - D^2} \right]$$

$$- \frac{G}{c^2} \left[\frac{3p_a^2 - 8p_a p_b + 3p_b^2}{2d} - \frac{6DM(p_a^2 + p_b^2)}{(d^2 - D^2)m} \right]$$

$$- \frac{G}{c^4} \left[\frac{5p_a^4 - 18p_a^2 p_b^2 + 5p_b^4}{8dm^2} - \frac{20DM(p_a^4 + p_b^4)}{8(d^2 - D^2)m^3} \right]$$

$$+ \mathcal{O}\left(\frac{1}{c^6} \right). \tag{36}$$

If the heavy systems are not taken into account, i.e. M=0, eq. (36) reduces to the same expression found in Ref. [12] for the interaction between two harmonic oscillators. Furthermore, in the center-of-mass frame, i.e. $p \equiv p_a = -p_b$, eq. (36) gives a potential that matches known results for the non-relativistic potential between classical point particles [34–36].

eq. (23), which quantifies the information of the light system shared with the heavy system. We start by finding the first order interaction terms between the heavy and light systems. We can substitute the expressions (1),(2) for the position operators in terms of their displacements into the Hamiltonian in eq. (A1), and look at the lowest order coupling between the light and heavy matter systems ¹². These can be found by Taylor expanding the small displacements $\delta \hat{x}_i$, giving the lowest order interaction terms:

$$\hat{H}_{hl} = 16GmM \left[\frac{\delta \hat{x}_a \delta \hat{x}_A + \delta \hat{x}_b \delta \hat{x}_B}{(D-d)^3} + \frac{\delta \hat{x}_A \delta \hat{x}_b + \delta \hat{x}_a \delta \hat{x}_B}{(D+d)^3} \right]. \tag{38}$$

Note that in the above expression there is no coupling between the momentum and the position operators, even though the light system is taken to be non-static. This is because the lowest order coupling is between one heavy position/momentum operator and one light position/momentum operator. The coupling with momentum operators at this order appears as $-4G\hat{p}_a\hat{p}_b/dc^2$, it only gives a coupling between the two light particles instead of the light and heavy subsystems.

We will now use the mode operators in eq. (6) to write \hat{H}_{hl} in terms of the mode operators j, j^{\dagger} with j = a, b, A, B. The resulting Hamiltonian is:

$$\hat{H}_{hl}^{\text{op}} = \frac{8G\hbar\sqrt{Mm}}{\sqrt{\omega_l\omega_h}} \left[\frac{a^{\dagger}A^{\dagger} + b^{\dagger}B^{\dagger}}{(D-d)^3} + \frac{a^{\dagger}B^{\dagger} + A^{\dagger}b^{\dagger}}{(D+d)^3} \right], \tag{39}$$

where all irrelevant terms (the terms that annihilate the vacuum) have been left out for simplicity. Filling this Hamiltonian into eq. (20), we find the only non-zero coefficients are ¹³ (where we assumed $\omega_a = \omega_b = \omega_l$ and

$$C_{ll} = \frac{Gm}{d^3\omega_l^2} + \frac{2Gm}{c^2d} \,, (37)$$

where the superscript (1) indicates that we have taken the first order coupling, which consists of a static contribution (from the position operator coupling) and a non-static (from momentum operator coupling) contribution, with the momentum contribution being suppressed by $1/c^2$.

Any terms $|00 n_A n_B\rangle$ and $|n_a n_b 00\rangle$ are left because they arise from the self-interaction within the light and heavy subsystems, respectively, and are therefore not relevant to our analysis.

Since we are considering a bipartite heavy-light system, only the interaction between heavy and light is taken into account to find the decoherence. Any heavy-heavy or light-light interaction can be viewed as 'self-interaction' since it only causes entanglement within the subsystem. However, the strength of the light-light entanglement is important to analyse the decoherence effects of the heavy system. Taking into account only the light-light couplings in eq. (A1) and following the same procedure as described in this section, we find the concurrence between the two light oscillators at lowest order to be:

 $\omega_A = \omega_B = \omega_h$ for simplicity, and set $\lambda = 1$):

$$C_{1010} = C_{0101} = -\frac{\mathfrak{g}_{-}}{\omega_h + \omega_l},$$
 (40)

$$C_{0110} = C_{1001} = -\frac{\mathfrak{g}_+}{\omega_h + \omega_l} \tag{41}$$

with

$$\mathfrak{g}_{\pm} = \frac{8G}{(D \pm d)^3} \frac{\sqrt{mM}}{\sqrt{\omega_h \omega_l}}.$$
 (42)

The final state given in eq. (19) (up to the first order in the perturbation theory) is thus given by:

$$|\psi_f\rangle = \frac{1}{\sqrt{\mathcal{N}}} \left(|0000\rangle - \frac{\mathfrak{g}_-}{\omega_h + \omega_l} |1010\rangle - \frac{\mathfrak{g}_+}{\omega_h + \omega_l} |0110\rangle - \frac{\mathfrak{g}_+}{\omega_h + \omega_l} |1001\rangle - \frac{\mathfrak{g}_-}{\omega_h + \omega_l} |0101\rangle \right), \tag{43}$$

with the normalization $\mathcal{N} = 1 + 2(\mathfrak{g}_{-}^2 + \mathfrak{g}_{+}^2)/(\omega_h + \omega_l)^2$, and using notation $|n_a\rangle |n_b\rangle |n_A\rangle |n_B\rangle = |n_a n_b n_A n_B\rangle$. The final state is an entangled state between the ground states and first excited states of the light and heavy subsystems. Due to the pair-wise interactions taken here, in each of the perturbed states one heavy and one light system are in the first excited states. Using eq. (21) the concurrence is found to be:

$$C_{\rm hl}^{(1)} = \sqrt{2 - 2 \frac{1 + 2 \frac{(\mathfrak{g}_{-}^2 + \mathfrak{g}_{+}^2)^2}{\omega^4} + 8 \frac{\mathfrak{g}_{-}^2 \mathfrak{g}_{+}^2}{\omega^4}}{(1 + 2 \frac{\mathfrak{g}_{-}^2 + \mathfrak{g}_{+}^2}{\omega^2})^2}},$$
 (44)

where $\omega \equiv \omega_h + \omega_l$ for simplicity, and the superscript (1) denotes that we have taken the lowest order contributions to the entanglement. In the limit where $\mathfrak{g}_{\pm}/\omega \ll 1$ the concurrence becomes

$$C_{\rm hl}^{(1)} \approx 2\sqrt{2}\sqrt{\frac{\mathfrak{g}_+^2 + \mathfrak{g}_-^2}{\omega^2}}.$$
 (45)

We now consider two special cases representing different experimental setups: $D\gg d$ and D=2d. Taking the limit $D\gg d$, we can Taylor expand the couplings $\mathfrak{g}_{\pm}\approx \frac{8G}{D^3}\frac{\sqrt{mM}}{\sqrt{\omega_h\omega_l}}\left(1\mp 3\frac{d}{D}+\mathcal{O}\!\left(\frac{d^2}{D^2}\right)\right)$. The expression for the concurrence simplifies to: 14

$$C_{\rm hl}^{(1)}(D\gg d) \approx \frac{32G}{(\omega_h+\omega_l)D^3}\sqrt{\frac{Mm}{\omega_h\omega_l}}$$
. (46)

$$\mathcal{C}_{\rm hl}^{(1)}(D\gg d)\approx 2\sqrt{2}\frac{\mathfrak{g}}{\omega}\sqrt{2+42\frac{d^2}{D^2}}\approx 4\frac{\mathfrak{g}}{\omega}\left(1+\frac{21}{2}\frac{d^2}{D^2}\right)\,.$$

The terms of order $\mathcal{O}((d/D)^2)$ can thus be neglected if $D \gg d$.

The degree of entanglement grows with the masses of the apparatus and the test masses (M, m, respectively), it grows inversely with the harmonic trap frequencies and it grows inversely (inverse cubic) with the distance between the test mass and the apparatus.

We now explore another possible configuration of the four oscillators where the spacing between any adjacent oscillators will be constant at a distance d, by setting D=2d. In this case the concurrence in eq. (45) simplifies to:

$$C_{\rm hl}^{(1)}(D=2d) \approx \frac{32\sqrt{365}G}{27(\omega_h + \omega_l)d^3} \sqrt{\frac{Mm}{\omega_h \omega_l}}.$$
 (47)

We note that the dependence on the masses, frequencies and distance between the oscillators is identical to the behaviour of the concurrence in eq. (46).

Since the interaction which goes with 1/r is stronger at shorter distances, eq. (46) is expected to be smaller which is seen in the $1/D^3$ which (due to the chosen setup $D\gg d$) gives a small concurrence. Note that we are ignoring all the self-interactions within the two subsystems, we are mainly interested in the low energy limit for the entanglements.

Instead of limiting D and choosing a specific setup to simplify the results, we could also note that for the two couplings \mathfrak{g}_{\pm} , the coupling between two neighbouring oscillators, \mathfrak{g}_{-} , will dominate the coupling between two maximally separated oscillators, \mathfrak{g}_{+} , which is due to the couplings being inversely cubic dependent on the distances. We can thus simplify the expression of the concurrence by taking the coupling $\mathfrak{g} = \sqrt{\mathfrak{g}_{+}^2 + \mathfrak{g}_{-}^2} \approx \mathfrak{g}_{-}$, giving:

$$C_{\rm hl}^{(1)}(\mathfrak{g}) \approx \frac{16\sqrt{2}G}{(D-d)^3(\omega_h + \omega_l)} \sqrt{\frac{2mM}{\omega_h \omega_l}}.$$
 (48)

This limit does not determine our setup, but neglects the couplings between $m_a - M_B$ and $m_b - M_A$.

These three limits work in different domains of D. In figure 2 we compare the different approximations as a function of D. The range of D shown is from d (which is taken to be of the order 10^{-4} m, following [12]) to 10^{-3} m, the lines continue to be a constant for a larger D. As one would expect, the concurrence $\mathcal{C}_{D=2d}$ is the worst approximation (except when D=2d). The concurrence $\mathcal{C}_{\mathfrak{g}}$ performs the best across the whole range. Although the concurrence $\mathcal{C}_{D\gg d}$ starts performing well around $D\sim 10^{-3}$ m as well. We've explored and analysed these limits in order to be able to perform an analytical analysis in Sec. VI.

The concurrence quantifies the entanglement due to the coupling between the light and heavy system. Since entanglement and decoherence are two sides of the same coin, the concurrence between the subsystems gives a hint about the decoherence behaviour of the test masses

¹⁴ We have neglected terms of order $\mathcal{O}((d/D)^2)$, which might seem incorrect due to the expression being under the square root, but it turns out that when keeping these second order terms, the approximation $\mathfrak{g}_{\pm}/\omega \ll 1$ simplifies the concurrence to:

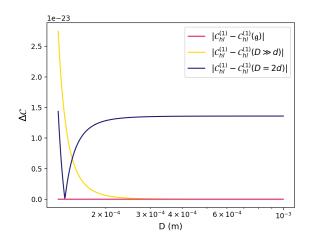


Figure 2: Difference between the concurrence in eq. (45) and the approximate concurrences in eqs. (46), (47) and (48) as a function of the distance D. For $d = 10^{-4}$ m, $m = 10^{-14}$ kg, $\omega_l = 10^8$ Hz, $\omega_h = 10^8$ Hz, $M = 10^{-8}$ kg.

due to the presence of the apparatus. The concurrence quantifies the shared information between two subsystems. Decoherence is the loss of information within the subsystem due to the shared information with the second system, which is the environment. It is also clear that if there is no interaction between the heavy and light subsystems ($\mathfrak{g}_{\pm}=0$), then there is no decoherence. Minimizing the mass M and maximizing the trap frequency ω and distance D minimize the decoherence from the apparati.

V. CONTRIBUTION FROM HIGHER ORDER COUPLINGS

At first order the coupling between the heavy and light systems is only between the position operators. Therefore we also have a look at the next order coupling, which additionally contains coupling with momentum operators. Filling in the eqs. (1) and (2) for the position operators in terms of their displacements, into the Hamiltonian given in eq. (A1), we can then Taylor expand the resulting Hamiltonian. Taking only the next order coupling between the light and heavy matter systems ¹⁵, gives the interaction terms as in the Hamiltonian in eq. (A2). This expression contains the

couplings between three operators: two light momentum/position operators and one heavy position operator, or two heavy position operators and one light position operator. The relevant non-zero coefficients for the final wavefunction defined in eq. (20) can be found the same way as before, by filling in the mode operators of eq.(6) into the interaction Hamiltonian. This gives:

$$C_{0102} = C_{1020} = \frac{\mathfrak{g}_1^-}{2\omega_h + \omega_l},$$
 (49)

$$C_{0120} = C_{1002} = -\frac{\mathfrak{g}_1^+}{2\omega_h + \omega_l},$$
 (50)

$$C_{0201} = C_{2010} = \frac{\mathfrak{g}_3^- - \mathfrak{g}_2^-}{\omega_h + 2\omega_l},$$
 (51)

$$C_{0210} = C_{2001} = \frac{\mathfrak{g}_2^+ - \mathfrak{g}_3^+}{\omega_h + 2\omega_l},$$
 (52)

with the six different couplings defined by:

$$\mathfrak{g}_1^{\pm} = \frac{12\sqrt{2}G}{\omega_h(D \pm d)^4} \sqrt{\frac{m\hbar}{\omega_l}}, \qquad (53)$$

$$\mathfrak{g}_2^{\pm} = \frac{12\sqrt{2}G}{\omega_l(D \pm d)^4} \sqrt{\frac{M\hbar}{\omega_h}},\tag{54}$$

$$\mathfrak{g}_3^{\pm} = \frac{3G\omega_l}{\sqrt{2}c^2(D\pm d)^2}\sqrt{\frac{M\hbar}{\omega_h}}.$$
 (55)

The "-"-labelled couplings arise due to interactions between neighbouring heavy and light oscillators, while the "+"-labelled couplings arise due to maximally separated heavy and light oscillators. Moreover, we underline the fact that the \mathfrak{g}_3 couplings represent the interaction of two momentum operators with a position operator, while the \mathfrak{g}_1 and \mathfrak{g}_2 couplings are attributable to the product of three position operators.

Recalling that $C_{0000} = 1$, the perturbed wavefunction up to first-order from eq. (19) is given by:

$$|\psi_{f}\rangle = \frac{1}{\sqrt{N}} \left[|0000\rangle + \frac{\mathfrak{g}_{1}^{-}}{2\omega_{h} + \omega_{l}} (|0102\rangle + |1020\rangle) - \frac{\mathfrak{g}_{1}^{+}}{2\omega_{h} + \omega_{l}} (|0120\rangle + |1002\rangle) + \frac{\mathfrak{g}_{3}^{-} - \mathfrak{g}_{2}^{-}}{\omega_{h} + 2\omega_{l}} (|0201\rangle + |2010\rangle) + \frac{\mathfrak{g}_{2}^{+} - \mathfrak{g}_{3}^{+}}{\omega_{h} + 2\omega_{l}} (|0210\rangle + |2001\rangle) \right], \quad (56)$$

where the normalization constant is now given by $\mathcal{N}=1+2\left[\frac{(\mathfrak{g}_1^-)^2+(\mathfrak{g}_1^+)^2}{(2\omega_h+\omega_l)^2}+\frac{(\mathfrak{g}_3^--\mathfrak{g}_2^-)^2+(\mathfrak{g}_2^+-\mathfrak{g}_3^+)^2}{(\omega_h+2\omega_l)^2}\right]$. The concurrence is calculated using its definition in eq. (23) and presented in eq. (A3) in the Appendix. The expression simplifies based on the assumption that the characteristic couplings over the associated frequency is signifi-

 $^{^{15}}$ The heavy-heavy and light-light couplings can be seen as self-interactions for the light-heavy bipartition used to calculate $\mathcal{C}_{\rm hl}.$

cantly smaller than one, so when:

$$\frac{\mathfrak{g}_1^{\pm}}{2\omega_h + \omega_l} \ll 1, \qquad \frac{\mathfrak{g}_{2,3}^{\pm}}{\omega_h + 2\omega_l} \ll 1. \tag{57}$$

In this regime the concurrence simplifies to:

$$C_{\rm hl}^{(2)} \approx 2\sqrt{\frac{(\mathfrak{g}_1^-)^2 + (\mathfrak{g}_1^+)^2}{(2\omega_h + \omega_l)^2} + \frac{(\mathfrak{g}_2^- - \mathfrak{g}_3^-)^2 + (\mathfrak{g}_2^+ - \mathfrak{g}_3^+)^2}{(\omega_h + 2\omega_l)^2}}.$$
(58)

Again, the concurrences quantify the decoherence of the light oscillators due to the heavy oscillators. From eq. (A3) we see that the concurrence decreases as the couplings \mathfrak{g}_i are set to zero, with the concurrence being zero when there is no more coupling between the system and the environment, meaning that there is no loss of coherence in the system.

In order to get a better idea of the parameter dependence we explore the approximation where we note that some couplings dominate other couplings, which is the best approximation over the range of D considered here. In this case the couplings $\mathfrak{g}_{1,2,3}^-$ dominate the respective $\mathfrak{g}_{1,2,3}^+$ couplings, however this doesn't simplify the concurrence as much as the other approximations. Therefore we additionally use that the coupling \mathfrak{g}_3 is suppressed by a factor $1/c^2$ (for typical values of the distances and trap frequencies). The concurrence then simplifies to:

$$C_{\rm hl}^{(2)}(\mathfrak{g}) \approx \frac{24G\sqrt{2\hbar}}{(D-d)^4\omega_h\omega_l}\sqrt{\frac{m\omega_l}{(\omega_l+2\omega_h)^2} + \frac{M\omega_h}{(2\omega_l+\omega_h)^2}}.$$
(59)

We see that the second order coupling contribution is suppressed by $\sqrt{\hbar}$, and is inversely quartic dependent on the separation.

In figure 3 we plot the different order contributions to the concurrence given in eqs. (48) and (59) for different ω_h and as a function of D. The light oscillator system is taken to be as in Refs. [9, 12]. The heavy frequencies are taken over a range $10^7-10^9\,\mathrm{Hz}$, which are experimentally viable [37]. The heavy mass is taken to be $10^{-8}\,\mathrm{kg}$, such that M>m. We see that the first order concurrence dominates the next order concurrence with about ten orders of magnitude. As D increases the concurrence goes to zero and both order concurrences both becomes zero eventually. This plot shows clearly that the next order coupling contributions to the decoherence are negligible.

In this section we have calculated the decoherence due to next order momentum and position couplings of the system and environment. We saw that the dominant contribution comes from the coupling of the position operators, not the position-momentum operator

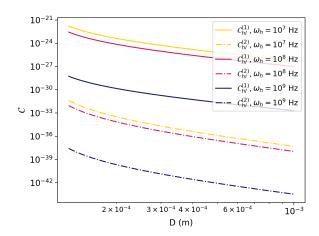


Figure 3: Concurrence as a function of the separation D. For $m \sim 10^{-14}$ kg, $d \sim 10^{-4}$ m and $\omega_l \sim 10^8$ Hz. For $M = 10^{-8}$ kg, and for different values of $\omega_h = 10^7, 10^8, 10^9$ Hz. The solid lines represent the concurrence due to the first order couplings in eq. (48). The dashdotted represent the concurrence due to the next order couplings in eq. (59).

coupling, meaning that the momentum of test particles does not increase the decoherence comparably. In eq. (37) we saw that the momentum-contributions (at first order) also doesn't increase the light-light concurrence, C_{ll} much, they are suppressed by a factor $1/c^2$. The contribution of the momentum terms in the decoherence scales as $\sqrt{\hbar}/c^2$, which is approximately an order of $1/c^2$ smaller. One might therefore think that by adding momentum terms the system can be better protected against decoherence because the light-light entanglement increases an order c^2 more compared to the light-heavy entanglement. However, as seen in figure 3 contributions of this order are negligible for these experimental parameters. So looking purely at this scaling it is not necessarily useful to give the particles momentum such that the light-light entanglement is increased more relative to the light-heavy entanglement.

Additionally we saw that these next order couplings entangle states where one of the light oscillators is in the first excited state and one of the heavy oscillators are in the second excited state. This contribution is however dominated by the first order position couplings, which give rise to entanglement with first excited states.

VI. RESTRICTIONS ON THE EXPERIMENTAL PARAMETERS

In the above sections we found the decoherence from the heavy oscillators on the light oscillators. We will

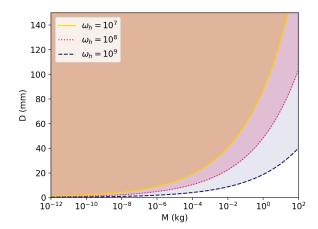


Figure 4: Distance D as a function of the heavy oscillator mass M, given in eq. (60), for different values of ω_h . The shaded area above each curve indicates the parameter space such that the light-light concurrence dominates the decoherence (for a static systems and upto first order in the couplings). For $m \sim 10^{-14} \, \mathrm{kg}$, $d \sim 10^{-4} \, \mathrm{m}$ and $\omega_l \sim 10^8 \, \mathrm{Hz}$.

now compare this decoherence to the concurrence between the two light test masses. By requiring that the concurrence $C_{ll} > C_{hl}$ and solving the inequality, we aim to restrict the parameter space of the heavy system. As we have seen that the momentum terms in C_{ll} and the second order couplings giving $C_{hk}^{(2)}$ are heavily suppressed, we simply compare the C_{ll} and $C_{hl}^{(1)}$ in the static case. So we require the first term in eq. (37) to be larger than eq. (48) (which uses the approximation that one of the coupling terms can be neglected, which was shown to be the best approximation across the range of D considered). The resulting inequality is:

$$D > \left(\frac{16\sqrt{2M}\omega_l^2}{\sqrt{m\omega_l\omega_h}(\omega_l + \omega_h)}\right)^{1/3} d + d. \tag{60}$$

This inequality is plotted in figure 4, where the light oscillator system parameter are chosen as found in previous works, $m \sim 10^{-14}\,\mathrm{kg},\ d \sim 10^{-4}\,\mathrm{m}$ and $\omega_l \sim 10^8\,\mathrm{Hz}$ [12]. In this figure the area above the curve is the parameter space such that the light-light entanglement dominates the decoherence. The range of M is chosen such that $M\gg m$. We see that as ω_h increases, the allowed parameter space increases. Furthermore, a heavier apparatus mass requires a higher separation D for the internal entanglement to dominate, as one would expect.

The results derived from figure 4 can be considered the results for the 'static case', where the light oscillator system is considered to have no momentum. We can also consider the case in which it does have momentum contributions, still at first order in the couplings. This results in the inequality:

$$D > \left(\frac{1}{\frac{1}{d^3\omega_I^2} + \frac{2}{dc^2}} \frac{16\sqrt{2M}}{\sqrt{m\omega_l\omega_h}(\omega_l + \omega_h)}\right)^{1/3} + d, \quad (61)$$

which is very similar to the one in eq. (60). The second term in the denominator of the first fraction is the contribution from the momentum coupling in the light system. If this term is taken to be zero (so reduce to the static case), then we regain the eq. (60). For the parameter space of the light system considered here $(d \sim 10^{-4} \,\mathrm{m}, \,\omega_l \sim 10^8 \,\mathrm{Hz})$, the momentum contribution is of the order 10^{-12} and is heavily suppressed by the first term (which is of the order 10^{-4}). At this range of experimental parameters, the contribution to the entanglement from the momentum coupling within the light system is so heavily suppressed that it does not change the parameter space much.

A plot of the allowed parameter space was given in figure 4. We require a separation of the order of centimeters for masses upto 10^2 kg if $\omega_h \geq \omega_l$. If $\omega_h < \omega_l$ then we require a separation of the order of centimeters for masses upto the order of kilograms, and a separation of the order of decimeters for higher masses.

The analysis we have done so far has compared the entanglement between the heavy and light system with the entanglement between the two light systems in the absence of the heavy system. Comparing these two entanglements has provided a way to put restrictions on the parameter space. However, we should also have a look at the at the entanglement of the two light systems in the presence of the heavy systems. By tracing out the heavy systems we can take the effects of the heavy system into account while finding the concurrence within the light system.

We consider the density matrix of the full system within which we want to find the concurrence to be the two light systems, given by the density matrix ρ_1 in eq. (22). Since this represents a mixed state, we cannot use the definition of the concurrence given in eq. (23). Instead, we use the definition for the concurrence for mixed states: [40, 41]

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{62}$$

where the λ_i 's are the ordered eigenvalues (highest to lowest) of the matrix $\sqrt{\sqrt{\rho_1}\tilde{\rho}_1\sqrt{\rho_1}}$ with $\tilde{\rho}_1=(Y\otimes Y)\rho_1^*(Y\otimes Y)$, where ρ_1^* is the complex conjugate of ρ_1 , and Y is the Pauli matrix 16 . In figure 5 we plot the concurrence between the two light systems with the effect

¹⁶ Previously we made the bi-partition light-heavy, where the to-

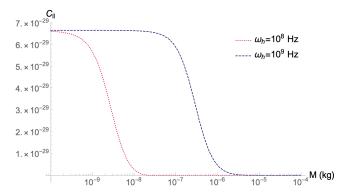


Figure 5: The concurrence between the two light oscillators in the presence of two heavy system (decoherence), as a function of the heavy system's mass, M. For a light system with parameters $m \sim 10^{-14}\,\mathrm{kg},\,d \sim 10^{-4}\,\mathrm{m}$ and $\omega_l \sim 10^8\,\mathrm{Hz}$. The heavy system has $D=10^{-3}\,\mathrm{m}$ and $\omega_h=10^8-10^9\,\mathrm{Hz}$.

of the heavy systems taken into account, as a function of the heavy mass M. As expected, we see that as the heavy mass increases, the coupling between the heavy and light system increases and thus the entanglement between the two light systems decreases due to decoherence. As the heavy mass goes to zero, we recover the value for a static system given by C_{ll} in eq. (37). For the distance D to be of the order of millimeters, the system fully decoheres if $M > 10^{-8}$ kg for $\omega_h = 10^8$ Hz and for $M > 10^{-6}$ kg for $\omega_h = 10^9$ Hz. This is inline with the parameter space plotted in figure 4 (at the order of millimeters, the lines have the mass values mentioned above). Additionally from this plot we could require that the decoherence reduces the entanglement to maximally 80% of the original value, which would require the heavy mass to be approximately of order 10^{-9} kg or smaller for $\omega_h = 10^8 \, \text{Hz}$. Knowing the experimental parameters of the heavy system can provide us with information about the expected coherence of the light system.

VII. DISCUSSION

In this paper we explored the effect of the apparatus on the QGEM setup in terms of decoherence, by approximating both the apparatuses, and the test masses as harmonic oscillators. Considering the apparatuses to be heavy, static oscillators and the test masses to be light non-static oscillators, we found the decoherence due the apparatuses on the test masses.

Decoherence and entanglement are two sides of the same coin, therefore when we say 'decoherence' we mean the entanglement between the heavy and the light system. This entanglement quantifies how much information is shared between the systems and thus how much information is leaked from the light system to the environment (the heavy system), which is called decoherence. The decoherence results are given as concurrences in eqs. (45) and (58). Since we quantify the decoherence with an entanglement measure, this is a basis-independent result.

We mirrored the calculation in Ref. [12], and highlighted again the necessity for the quantum nature of gravity for there to be entanglement in the final state if the initial state is pure (not entangled). If the shift in the gravitational vacuum were not quantized (so not operator-valued but a c-number), then the final state would simply be the un-entangled final state. As in Ref. [12] we obtained the results from the perturbation theory up-to first order in the wave function, and we evaluated the shift in the graviton vacuum energy upto the second order.

We computed the concurrence for the first and the second order couplings, showing the dominant terms and their dependencies on the experimental parameters. The first order coupling concurrence was found to be very dominant over the higher order contributions. In order of relevance, a large separation between the test masses and the apparatuses high trap frequencies, and low masses will reduce the decoherence. We explored the limits $D \gg d$ and D = 2d corresponding to different setups, resulting in the same dependence on the experimental parameters, but resulting in a bigger decoherence for the D = 2d setup due to the smaller D. We also approximated the concurrence by assuming that the the nearest neighbour coupling dominates, which turned out to be the best approximation, and we used this to restrict the parameter space for the apparatus.

We also explored the first order momentum contributions to the decoherence, which appeared in the next order couplings and are therefore suppressed by a factor $\sqrt{\hbar}$ compared to the momentum contributions to the light-light entanglement, which entered at the lowest order couplings. Because the momentum couplings appear at first order in the light-light entanglement,

tal 16×16 density matrix is pure. Therefore we were allowed to use the pure state definition of concurrence along this bipartition. Now we have traced out the heavy system, so for the bi-partition light-light we have a total density matrix that is mixed, and we need to use the mixed state definition. Note that this is different from taking a bi-partition where system 1 contains one light system and system 2 contains the heavy systems and the other light system, which has a total pure state, but which regards the heavy systems as a part of the quantum system as opposed to as the environment.

making the system non-static could potentially have increased the light-light entanglement more relative to the light-heavy entanglement, making the system in that sense more resistant to decoherence. However, we found that relative to the static contributions to the entanglement, the momentum contribution is negligible and too heavily suppressed, for the experimental parameters considered here. So adding momentum cannot be used in this regime to make the system more robust to decoherence, in the sense that although the increase in light-light entanglement is larger compared to the increase in the light-heavy entanglement, these increases are negligible compared to the static entanglement in both cases.

By requiring the decoherence to be smaller than the light-light concurrence, we found that the separation D will be of the order of centimetres for the masses upto $M \sim 100\,\mathrm{kg}$ if the trap frequency is larger than $10^8\,\mathrm{Hz}$. A smaller trap frequency for the same range of masses requires a larger separation. Of course, a larger separation D, a smaller mass M, and a higher frequency ω_h , decrease the decoherence. This was confirmed by figure 5 in which we plotted the light-light entanglement under the influence of interactions with the environment.

By modelling the the apparatuses as harmonic oscillators, we are able to make an approximate prediction about the allowed separation between the detectors and the test masses that does not completely destroy the coherence of the test particles. For example, the typical spacing of ion traps are of the order of millimetres,

which is smaller then the scale found here, and the decoherence is smaller than the light-light entanglement only for masses M upto 10^{-6} kg, for the considered frequencies (as seen from figure 4).

Setting one of the heavy masses to be zero, $M_B = 0$, we can also use our method to find the decoherence due to a single massive oscillator. At no point in the calculations have we assumed that M > m, therefore the resulting decoherence rates hold for any mass M. However, in the range where M < m, we expect the light-light entanglement to be dominant since the gravitational coupling happens via the masses, assuming that the distances are such that D > d. These light sources of decoherence might become relevant at very short distances. Similarly we haven't explored masses of $M \sim m$, where the coupling between the heavy and light system is of the same scale. These sources are expected to become relevant at $D \sim d$. Our results for the decoherence rate are general and can also be used for other mass ranges. We have modelled the mass Mas a coherent state, future research could also explore different type of states, such as thermal states.

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Appendix A: Equations

Interaction Hamiltonian between the heavy and light system, after a non-relativistic expansion upto order $1/c^4$ and at first order in G:

$$\Delta \hat{H}_{g} = -G \left[\frac{mM}{|\hat{x}_{a} - \hat{x}_{A}|} + \frac{mM}{|\hat{x}_{a} - \hat{x}_{B}|} + \frac{mM}{|\hat{x}_{A} - \hat{x}_{b}|} + \frac{mM}{|\hat{x}_{b} - \hat{x}_{B}|} + \frac{m^{2}}{|\hat{x}_{a} - \hat{x}_{b}|} + \frac{M^{2}}{|\hat{x}_{a} - \hat{x}_{B}|} \right]$$

$$- \frac{G}{c^{2}} \left[\frac{3M}{2m} \left(\frac{\hat{p}_{a}^{2}}{|\hat{x}_{a} - \hat{x}_{A}|} + \frac{\hat{p}_{a}^{2}}{|\hat{x}_{a} - \hat{x}_{B}|} + \frac{\hat{p}_{b}^{2}}{|\hat{x}_{A} - \hat{x}_{b}|} + \frac{\hat{p}_{b}^{2}}{|\hat{x}_{b} - \hat{x}_{B}|} \right) + \frac{3\hat{p}_{a}^{2} - 8\hat{p}_{a}\hat{p}_{b} + 3\hat{p}_{b}^{2}}{2|\hat{x}_{a} - \hat{x}_{b}|} \right]$$

$$- \frac{G}{c^{4}} \left[\frac{5M}{8m^{3}} \left(\frac{\hat{p}_{a}^{4}}{|\hat{x}_{a} - \hat{x}_{A}|} + \frac{\hat{p}_{a}^{4}}{|\hat{x}_{a} - \hat{x}_{B}|} + \frac{\hat{p}_{b}^{4}}{|\hat{x}_{A} - \hat{x}_{b}|} + \frac{\hat{p}_{b}^{4}}{|\hat{x}_{b} - \hat{x}_{B}|} \right) + \frac{5\hat{p}_{a}^{4} - 18\hat{p}_{a}^{2}\hat{p}_{b}^{2} + 5\hat{p}_{b}^{4}}{8m^{2}|\hat{x}_{a} - \hat{x}_{b}|} \right] + \mathcal{O}\left(\frac{1}{c^{5}}\right). \tag{A1}$$

Interaction Hamiltonian between the heavy and light system in terms of the displacement operators, only showing terms containing three operators (which are the second order interactions, as indicated by the superscript (2)):

$$\begin{split} \hat{H}_{hl}^{(2)} &= 48GmM \left[\frac{\delta \hat{x}_a (\delta \hat{x}_A)^2 - (\delta \hat{x}_a)^2 \delta \hat{x}_A}{(D-d)^4} + \frac{\delta \hat{x}_a (\delta \hat{x}_B)^2 - (\delta \hat{x}_a)^2 \delta \hat{x}_B}{(d+D)^4} \right. \\ &\qquad \qquad + \frac{\delta \hat{x}_A (\delta \hat{x}_b)^2 - (\delta \hat{x}_A)^2 \delta \hat{x}_b}{(d+D)^4} + \frac{\delta \hat{x}_b (\delta \hat{x}_B)^2 - (\delta \hat{x}_b)^2 \delta \hat{x}_B}{(D-d)^4} \right] \\ &\qquad \qquad + \frac{6GM}{c^2 m} \left[\frac{(\hat{p}_b)^2 \delta \hat{x}_A + (\hat{p}_a)^2 \delta \hat{x}_B}{(d+D)^2} - \frac{(\hat{p}_a)^2 \delta \hat{x}_A + (\hat{p}_b)^2 \delta \hat{x}_B}{(D-d)^2} \right]. \end{split} \tag{A2}$$

Next order concurrence between the heavy and light system, due to the interaction displayed in the above two Hamiltonians:

$$C_{\text{hl}}^{(2)} = \sqrt{2 - \frac{2}{\mathcal{N}^2} \left[1 + \frac{2 \left[(\mathfrak{g}_3^- - \mathfrak{g}_2^-)^2 + (\mathfrak{g}_2^+ - \mathfrak{g}_3^+)^2 \right]^2 + 8 \left(\mathfrak{g}_3^- - \mathfrak{g}_2^- \right)^2 \left(\mathfrak{g}_2^+ - \mathfrak{g}_3^+ \right)^2}{\left(\omega_h + 2\omega_l \right)^4} + \frac{2 \left[(\mathfrak{g}_1^+)^2 + (\mathfrak{g}_1^-)^2 \right]^2 + 8 (\mathfrak{g}_1^+ \mathfrak{g}_1^-)^2}{\left(2\omega_h + \omega_l \right)^4} \right]}{(A3)}$$