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Statistical mechanics of gravity and the thermodynamical origin of time

Carlo Rovelli
Department of Physics, University of Pittsburgh, Pittsburgh, 15260 USA
and
Dipartimento di Fisica, Universita' di Trento; INFN, Sezione di Padova, Italy

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Abstract. The lack of a statistical and thermodynamical theory of the gravitational field is pointed out. The possibility of developing such a theory is discussed, as well as its theoretical and cosmological relevance. The connection between this problem and the problem of physical time in (compact space) general relativity is emphasized. The idea that a preferred physical time variable is singled out by the statistical properties of the state is proposed. A scheme for a generally covariant statistical thermodynamics is put forward, by extending the Gibbs formalism to the presymplectic, or constrained, dynamical systems. This scheme is based on three equations. The first characterizes any statistical state; the second is the definition of a vector flow in terms of the statistical state; the third is an intrinsic definition of equilibrium. The vector flow of an equilibrium state is denoted thermodynamical time, and the suggestion is made that thermodynamical time is the internal time that carries all the 'common sense' characterizations of the notion of time. Finally, it is suggested that the issue of the second law of the thermodynamics could be reconsidered within this generally covariant framework.

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1. Thermodynamics and gravity

1.1. Definition of the problem

In this paper, we study the relation between general relativity, defined on a compact three-dimensional space, and thermodynamics. A dynamical system with a large number of degrees of freedom (a gas of particles or the electromagnetic field in a cavity) may admit statistical and thermodynamical descriptions. The question addressed in this paper is whether it is possible to develop a thermodynamical and statistical description of the gravitational field. There is a peculiar difficulty to face *in order to answer this question: the difficulty raised by the general covariance of* Einstein's theory. The present paper is focused on this difficulty and its consequences.

We do not consider here the statistical mechanics of matter on a fixed background geometry, that is, interacting with a given configuration of the gravitational field. Such a problem is defined by the assumption that matter is 'hot' while the gravitational field is 'cold', namely the microscopic degrees of freedom of matter share thermal energy but the thermal energy is not transmitted to gravitational degrees of freedom. This is a useful approximation, and has already been studied in detail [1]. Rather, we study here the generic situation in which thermal energy *is* transmitted to gravitational

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degrees of freedom: we wish to describe the regime in which also the gravitational field is 'hot' and its thermal interaction with matter is not neglected.

In such a situation, the gravitational field must be in a 'disordered' configuration, analogous to the configuration of the electromagnetic field in a room at finite temperature. We do not have control over the detailed microscopical configuration of the field. We must renounce giving a complete description of the gravitational microscopical degrees of freedom, and resort to describing the field in terms of macroscopical parameters (coarse graining). Following Gibbs [2], this can be done by imagining an ensemble of mental copies of the system, all with the same macroscopical features, but with different microscopical states. The ensemble is represented by a distribution function ρ on the phase space. Here, we study the possibility of defining distribution functions and their evolution in general relativity. As we will see, because of general covariance Gibbs formalism cannot be applied in its standard form.

A crude way to face the problem is to split the gravitational field into background metric and small fluctuations. But this procedure is just an approximation—it breaks general covariance and is consistent only as long as the thermal fluctuations of the gravitational field remain smaller than the background metric, namely in a low-temperature context. In order to treat the full thermodynamics of gravity a more comprehensive approach is required.

A word of caution is needed with regards to quantum mechanics. The statistical thermodynamics of a classical field is inconsistent (ultraviolet catastrophe). A statistical description of the classical gravitational field is likely to be plagued by the same problem. In this paper, this problem will be ignored: we remain in the regime of classical mechanics and disregard ultraviolet problems. In this way we may miss a crucial element of the puzzle, but we think that the relation between thermodynamics and classical general relativity is fundamental enough to deserve direct attention.

1.2. Motivations

Even disregarding quantum mechanics there may be reasons for which a statistical description of the gravitational field could turn out to be unfeasible (instability of gravitational systems, long range of the interaction, absence of a unique notion of energy, difficulty of defining the concept of equilibrium in a generally covariant framework). However, we are not aware of any conclusive demonstration that gravitational statistical mechanics is inconsistent, and we believe the problem is still open. In addition, we shall argue that a statistical description of the gravitational field may be impossible within the standard theory, but possible in some extended, generally covariant, formulation of statistical thermodynamics. Also, it is conceivable that thermodynamics for gravity makes sense only for low temperature and weak fields, but we think that there are at least three reasons for which thermodynamics beyond this low-temperature limit deserves to be explored.

First, there is a fruitful line of thought, championed for instance by Fermi and Landau, that claims that thermodynamics has a fundamental character and may lead to profound insights into nature. As is well known, the need for quantum theory and the Planck constant itself were historically deduced by investigating the thermodynamics of the classical fields. In this paper, we suggest that certain essential insights in the nature of time can indeed be reached by investigating the classical thermodynamics of general covariant theories.

The second reason is that the problem of a statistical description of gravity has many analogies with the problem of its quantum description. In one case we have a statistical distribution over geometries, in the other case a wavefunction over geometries. We may expect to learn something about the quantum problem from the thermal problem; hopefully, the latter is simpler.

The third reason for studying the problem is more concrete. According to the standard big-bang theory, the temperature was extremely high and the entropy extremely low in the first instants of the universe. Therefore, it is reasonable to suspect that far enough back in time the universe was outside the regime in which matter is hot and the gravitational field is cold. The gravitational field may have undergone wide thermal fluctuations (unless this is vetoed by something like Penrose's initial singularity hypothesis [10]). Now, in present-day physics there is no reasonable definition of temperature and entropy that holds in such a regime. Thus, when we consider temperature and entropy of the very early universe, it is very likely that we do not know what we are talking about. We return to this issue in appendix A and in a companion paper.

For all these reasons, we think it is reasonable to explore the thermodynamics of the full gravitational field. In the present paper we propose a scheme for such a theory. In section 2 we discuss the relevance of the 'issue of time' to our problem, and we introduce the main idea of the paper. In section 3, as a preliminary step, we translate standard statistical mechanics into the language of parametrized (presymplectic) systems. In section 4 we define generally covariant statistical mechanics. Section 5 contains the conclusions. Appendix A contains some speculations on the relevance of our results for the problem of the arrow of time. Appendix B extends the formalism of section 3 to time-dependent Hamiltonians. In a companion paper [11] we present some examples of application of the formalism developed here.

There are several important works that can be viewed as preliminary results towards the construction of a complete thermodynamics of gravity. Hawking radiation [3], the possibility of assigning entropy to black holes [4], the thermodynamics of 'gases' of black holes and of a black hole in a 'box' [6, 7] are surprising theoretical results. Related and suggestive results regard the connection between the thermal properties of a quantum-field state with the acceleration of the observer (Rindler), and the definition of temperature as a local quantity, by means of model 'thermometers' (Unruh) (see [8]). These results seem to indicate that there is a large and not yet explored territory, which relates general relativity, thermodynamics and quantum theory. A discussion of the problem is given by Penrose [9], where the lack of a general definition of gravitational entropy is stressed.

In the next section we argue that the problem of the thermodynamics of gravity is related to the so-called 'issue of time' in general relativity, namely the problem of determination of a preferred internal time variable. This problem has been discussed extensively and we cannot possibly survey here the various solutions proposed. We shall just mention some of these: the WKB state theory [16]; Misner's idea that an arbitrary fixed state defines a time direction in the configuration space of the system by the corresponding DeWitt current [17]; Ashtekar's result on the recovering of the Schrödinger equation from the Wheeler–DeWitt equation to second order around flat space [18]; the reference fluid theory [19]; Hartle's suggestion that time is a 'relict from the big bang' [20]. For a review, and for other references, see [22, 21]. In this paper, we put forward an entirely new idea for the solution of the problem. To our understanding, this idea may be related to, but is definitely different from all the other ideas about the origin of time of which we are aware.

2. Why the notion of time enters the problem

Let us begin to discuss why a standard statistical treatment of the gravitational field on a compact space is not viable. To run the standard machinery of equilibrium statistical thermodynamics the concept of energy is required. For instance, the canonical ensemble is defined [2] by the Gibbs equilibrium probability distribution on the phase space

$$\rho_e(p,q) = C\mathrm{e}^{-\beta H(p,q)} \tag{1}$$

where p and q are the phase space coordinates, and H is the energy. From this probability distribution, we compute the partition function

$$Z(\beta) = C\int \mathrm{d}p\,\mathrm{d}q\mathrm{e}^{-\beta H(p,q)} \tag{2}$$

from which all the thermodynamical properties of the system can be calculated. In (compact space) general relativity, there is no well defined energy H: indeed the canonical Hamiltonian energy vanishes. Thus, if we define the canonical ensemble of a general covariant theory by (1), where H is the generator of the coordinate-time translations, we obtain the meaningless result $Z(\beta) = 1$. The vanishing of the canonical Hamiltonian reflects the absence of a preferred time variable in general relativity. If we had a preferred internal physical-time variable we would have a well defined energy and we could run the standard machinery of statistical mechanics†. The issue of time is therefore at the roots of the relation between thermodynamics and gravity. The identification of physical time is a widely discussed problem in gravitational physics. Let us recall briefly the present state of this problem.

2.1. Notions of time in general relativistic mechanics: coordinate time and internal time

In a standard dynamical system, the equations of motion express the evolution of the physical observables as functions of time. This simple structure is not present in general relativity. In fact, the Einstein equations describe the evolution of the metric tensor in the coordinate-time variable t, but t is arbitrary because of general covariance. The physical predictions of the theory, that is the ones to be compared with experiments (tracks on photographic plates, correlations of light signals with local proper-time clocks ...) are coordinate-independent, and in particular are independent of t‡.

† In the asymptotically flat formalism, time and Hamiltonian are provided by the asymptotic Minkowski time and the ADM Hamiltonian. It is our feeling that the basic features of the physics of the gravitational field should not depend on its asymptotical properties, and that the asymptotically flat formulation tends to obscure the deepest physical insights provided by the discovery of general covariance. A formal definition of the (quantum-) thermal partition function for the gravitational field was proposed by Gibbons and Hawking in terms of a path integral over Euclidean sections of complex geometries, constrained to be defined between periodic boundaries on the Euclidean time [5]. This prescription is likely to become useless in the compact case. In spite of this difficulty, and of the well known difficulties to give meaning to these kinds of integrals (they cannot be computed in perturbation theory because of the non-renormalizability of the gravitational field [6]), the Gibbons–Hawking proposal is very interesting and a comparison with the ideas presented here would be useful.

‡ The consistency of the physical interpretation of a theory with gauge invariance demands that only gauge-invariant observables have physical meaning. Gauge-invariant observables are given by functions on phase space with vanishing Poisson brackets with all the constraints. Physical measuring procedures can be associated only to these observables. In particular these observables commute with the Hamiltonian constraint, which is the generator of t translations. Therefore their derivative with respect to the coordinate time t vanishes. Thus, no physical observable depends on the coordinate time (or any other coordinate).

On a given geometry, time directions are determined by the light cone structure of the geometry. This is not the same as having a formulation of the theory in the form of a set of variables that evolve in a single well defined 'time', but it is sufficient for a consistent interpretation of classical general relativity. After all, general relativity may be intrinsically incompatible with the very idea of a unique 'external' time in which everything evolves. However, we shall see that in a statistical (as well as quantum) context we are forced to sharpen this notion of time. While the coordinate time t, as we have seen, cannot be identified with the 'physical time' of non-relativistic mechanics, there is another notion of time that appears in general relativity: *internal* time.

In concrete gravitational experiments, time evolution is always described as the evolution of one physical variable (say the decreasing radius R of a binary system in which one star is a pulsar) as a function of another physical variable (say the counting T of the pulses of the pulsar, or the reading T' of a physical clock that follows proper time in our galaxy). Here, the variables T and T' are used as *internal time* variables. These are genuine functions on the phase space, which represent the objects used as clocks†.

The distinction between the coordinate time and the internal times has been emphasized in canonical quantum gravity because in the quantum formalism coordinate time disappears (as it does in the Hamilton–Jacobi formalism) and a Schrödinger equation can be recovered only as an evolution equation in an internal time [13, 15]. In the quantum context, the problem is whether or not it is possible to identify a *global* variable T, namely an internal time that can be used as a clock variable under *all* circumstances, everywhere on the phase space‡. To this purpose, T must be monotonically increasing along every orbit, a very non-trivial requirement. At present, no such variable T is known§. But if we disregard quantum theory, there is no need that internal times 'behaves as good clocks' under all circumstances; on the contrary, in general relativity we use different clocks, and therefore different internal times, in different physical contexts. Thus, at the level of classical mechanics this picture is satisfactory. Still, we believe there is a missing element in the puzzle, which emerges by considering thermodynamics, that we will discuss shortly.

First, however, let us recall how this complex pattern of time concepts is realized in the canonical theory. A standard Hamiltonian system is defined by a phase space and a Hamiltonian function that generates evolution in a unique external time parameter (in this paper we use 'Hamiltonian system' in this strict sense). General relativity *does not* admit a canonical formulation as a Hamiltonian system in this strict sense. It admits a canonical formulation as a constrained system with weakly vanishing Hamiltonian, or *presymplectic system*: presymplectic mechanics is an *extension* of Hamiltonian mechanics. Any Hamiltonian system can reformulated as a presymplectic system by promoting the time parameter to the role of phase space coordinate, namely enlarging the phase space to include time [23, 13]. In the resulting formulation time is on the same ground as the other variables. But there are systems like general

† There is a relation between the choice of an internal time T, which is a variable on the phase space that represents a clock, and the choice of a preferred coordinate system, with a preferred coordinate time variable t: given T, the corresponding choice of coordinates is implemented by imposing a gauge condition of the form $T(t) = t$.

‡ A global internal time is such that $T(t) = t$ is a good gauge fixing.

§ An attempt to define the canonical quantum formalism in the absence of a global internal time is given in [14].

relativity that are handed to us directly in presymplectic form. They are like systems that we have parametrized, but then we have forgotten which one of the variables was time. (They may even not admit a global 'deparametrization' at all.) This is how the canonical formalism of general relativity captures the physical fact that there is no preferred internal time in gravitational physics.

2.2. *Thermodynamics and time*

Let us now point out three difficulties:

Problem 1. A preferred internal time is needed for the definition of equilibrium thermodynamics. Let us go back to the problem raised at the beginning of this section, namely the impossibility of defining the Gibbs distribution due to the vanishing of the canonical Hamiltonian. The problem should now be clear: usual statistical mechanics is formulated for standard Hamiltonian systems, not for presymplectic systems. To apply standard statistical mechanics to general relativity we should first 'deparametrize' the theory. To do so we would need a *preferred* internal time. Therefore we could construct a standard statistical mechanics of the gravitational field only after having identified a preferred internal time variable. However, the discussion above indicates that there is no reason for selecting any variable as preferred time variable: every variable may be a good local internal time, and no variable is likely to be a good global internal time. This is the impasse at the roots of the difficulties of constructing a general covariant equilibrium thermodynamics.

Problem 2. The light cone structure undergoes thermal fluctuations. How do we reconstruct a timelike direction on the spacetime manifold, if the gravitational field is known only macroscopically, and therefore can only be described as a statistical superposition of metrics, each one with a different light cone structure? In a 'hot' physical regime, where we know the gravitational field only through some macroscopical parameters, the determination of the timelike directions via the light cone structure of the manifold is not viable, and we are forced to resort to the use of an internal time.

Problem 3. 'Common sense' time is a preferred internal time. According to the discussion in section 2.1, it seems as if *any* variable may be used as internal time. Namely, as if any variable could be used as a clock variable. But then, what is it that makes 'time' so special in the actual universe in which we live? Indeed, in spite of the fact that all variables are on the same footing in general relativity, there is the persistent idea in the literature that, after all, there should be a preferred internal time variable. There are many attempts in the literature to determine this preferred time and 'explain the origin of time'; we listed several of these attempts in section 1. Here, we propose a new idea about the existence of a 'preferred' time variable in a generally covariant theory.

A solution: thermodynamical time. The idea that we propose is the thermodynamical origin of time. First, let us see how this idea is related to problem 1. We have seen that the definition of a Gibbs statistical state depends on the choice of a preferred internal time. The idea that we put forth is that this logic can be reversed. We propose that it is precisely the statistical state of the system that *determines* which variable is the preferred internal time variable. In other words, there is no universal preferred internal time, but once a statistical state is given, then this statistical state defines a particular time variable. *The time variable determined in this way is denoted thermodynamical time.*

Second, let us discuss how this idea of the thermodynamical origin of time is related to problem 3. In non-relativistic and special relativistic physics, time is an external, absolute, 'flowing' quantity. But time is characterized by two domains of properties that we propose to keep distinct. First, time is the independent variable in terms of which motion is described. Let us call this 'mechanical time'. Second, time has the property of irreversibility, it defines the equilibrium of the system, it supports the psychological distinction between past and future, the psychological perception of the time flow, memory, and so on; we loosely denote these properties of time as the 'common sense properties' of time. Let us denote the variable that carries all these common sense properties of time as 'common sense time'—see [26] The relation between these common sense properties of time and thermodynamics has been extensively discussed [27]. Eddington has even argued that every realistic clock is essentially a thermodynamical clock and must make use of the second law of thermodynamics [28]. (A pendulum is not a clock unless it is equipped with a device that keeps track of the number of oscillations.)

Mechanical time and common sense time are identified in non-relativistic physics. In general relativistic physics, there is no preferred mechanical time. What about the second domain, namely the common sense properties of time? We propose the idea that in thermal relativistic physics there *is* a preferred time variable to which we may associate the common sense properties of time, namely a preferred common sense time. This common sense time has thermodynamical origin, and it is not unique in the theory but is determined by the state. If one knows how to define a suitable *thermodynamical time, the common sense time is the thermodynamical time defined* below. In the rest of this paper we construct a covariant statistical thermodynamics in which this idea is implemented.

We obtain this theory as follows. In section 3 we translate the standard Gibbs theory of the the statistical mechanics of Hamiltonian systems into the language of presymplectic mechanics. Then, in section 4, we *postulate* that these same equations also hold for the presymplectic systems that do not derive from the parametrization of Hamiltonian systems. One of the equations derived for the parametrized Hamiltonian systems relates the Gibbs equilibrium distribution to the Hamiltonian time flow. We *reinterpret* this equation as a *definition* of the thermodynamical time.

3. Preliminary: thermodynamics in the parametrized formalism

3.1. Parametrized Hamiltonian dynamics

Let us begin with a system with phase space Γ (with coordinates p, q, where $q = (q^i)$ and $p = (p_i)$ are vectors, may be infinite-dimensional), a standard symplectic form

$$\sigma = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i \tag{3}$$

and a Hamiltonian $H(p, q)$. The dynamical flow X_t^Γ (equal to the Hamilton equations) is given by the Hamiltonian vector field of the Hamiltonian. We recall that on a symplectic manifold with symplectic form σ, the Hamiltonian vector field

X of a function f is defined by $\sigma(X) = -\mathrm{d}f$. For an interesting review of the symplectic formalism see [24]. In coordinates, we have

$$X_t^\Gamma = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}. \tag{4}$$

Let us introduce the parametrized description of the system. We consider the extended phase space $\Gamma_{\mathrm{ex}} = \Gamma \times R^2$ with coordinates q, p, t, E, the symplectic form $\sigma_{\mathrm{ex}} = \sigma - \mathrm{d}E \wedge \mathrm{d}t$ and the constraint

$$C = E - H(p, q). \tag{5}$$

The symplectic space Γ_{ex}, equipped with the constraint C, contains all the physical information contained in the original system (Γ, σ, H), except for a single crucial detail: the time variable is confused with the other variables and there is no geometrical way to distinguish it.

It is convenient and more elegant to work at the intermediate level of the constraint surface $\mathcal{C}$, defined by $C = 0$. The 2-form induced on $\mathcal{C}$ by σ_{ex} is the presymplectic form

$$\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}H(p, q) \wedge \mathrm{d}t. \tag{6}$$

The presymplectic space $(\mathcal{C}, \omega)$ contains all the physical information contained in the triple $(\Gamma_{\mathrm{ex}}, \sigma_{\mathrm{ex}}, C)$. The solutions of the equations of motions of the system are obtained by integrating the null directions of ω; namely, they are given by the integral lines of the vector field Y defined by

$$\omega(Y) = 0. \tag{7}$$

Here the parentheses indicate the action of forms on vector fields. These integral lines are also denoted as the *orbits* of ω. In coordinates we have

$$Y = \frac{\partial}{\partial t} - \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}. \tag{8}$$

It is easy to infer from this expression the relation between the presymplectic formalism and the usual form of the Hamilton equations.

The presymplectic space $(\mathcal{C}, \omega)$ contains all the physical information on the original Hamiltonian system except the determination of the time variable. Let us now discuss how we can incorporate this information into the $(\mathcal{C}, \omega)$ space. This information is contained in the splitting of the coordinates (q, p, t) into two groups: the 'genuine' phase space coordinates (q, p) and the time coordinate t. That is, it is contained in a factorization of $\mathcal{C}$ in the form $\mathcal{C} = \Gamma' \times R$. Equivalently, it is contained in a vector field

$$X_t = \left.\frac{\partial}{\partial t}\right|_{q,p=\mathrm{constant}} \tag{9}$$

(the subscript t means time) plus the determination of one constant-t surface. Under this factorization the vector field Y splits into two components: $Y = Y_{\Gamma'} + Y_R$.

There are two ways of recovering the original phase space Γ. The first one consists in identifying Γ with Γ', namely with the space of the integral lines of X_t. Then $Y_{\Gamma'}$ is precisely the dynamical flow on the phase space: $Y_{\Gamma'} = X_t^{\Gamma}$, provided that Y is normalized so that $Y_R = X_t$. The second way is known in the framework of constrained systems: let Γ'' be the space of the orbits, and $\pi : C \to \Gamma''$ the natural projection. It is easy to prove that Γ'', equipped with the unique 2-form σ'' such that $\pi^* \sigma'' = \omega$, is isomorphic (as a symplectic manifold) to the original phase space (Γ, σ). If we reconstruct Γ in this way, the dynamical flow on Γ can be reconstructed (*à l'envers*) by using X_t (rather than Y). In fact, it is easy to see that X_t^{Γ} is obtained as

$$\pi_* X_t = X_t^{\Gamma} . \tag{10}$$

Thus, the vector field X_t on C contains the information on the identification of the time coordinate, and allows us to reconstruct the dynamical flow on the physical phase space Γ.

The fact that X_t determines the dynamical flow on Γ'' may generate some confusion: the physical motion is 'along the orbits of Y ', and X_t takes orbits into different orbits, and therefore solutions of the equations of motion, into different solutions: how can it be interpreted as a time flow? Since X_t plays a major role in what follows, we conclude this subsection by clarifying this point. The phase space Γ admits two alternative interpretations. Let Γ' be the space of the states of the system at a fixed time, and Γ'' the space of the solutions of the equations of motion. Γ' and Γ'' are isomorphic, and the phase space Γ can be equivalently interpreted either as representing Γ' or Γ''. The first interpretation is more common, but the second is also discussed in the literature; in a precise sense they correspond to the Schrödinger and Heisenberg interpretations of the Hilbert space of quantum mechanics. In the context of the parametrized systems, Γ' and Γ'' naturally arise as the factorization of C under the flows of X_t and Y respectively. If we interpret the phase space *à la* Schrödinger as the space of the states at a given time, then the dynamical flow maps the state of a system at time t into the state of the system at time $t + \mathrm{d}t$; in this case the dynamical flow is just the projection of Y. On the other hand, if we interpret the phase space *à la* Heisenberg as the space of the solutions of the equations of motion, then the meaning of the dynamical flow is more subtle: it maps a given solution of the equations of motion into another solution which differs from the previous one in the fact that it is shifted ahead in time by an amount $\mathrm{d}t$†. The importance of this interpretation *à la* Heisenberg of the time evolution is that it can be extended to systems that do not have a well defined *global* internal time.

3.2. *Statistical mechanics in the parametrized formalism*

A generic statistical state of the system is defined by a probability distribution ρ on Γ. In general, the distribution function evolves in time. Let $\rho(q, p, t)$ be the distribution

† As an example consider the harmonic oscillator with angular frequency ω and mass m. Γ' has coordinates (p, q) and the dynamical flow maps (p, q) into $(p - m\omega^2 q\, \mathrm{d}t, q + m^{-1} p\, \mathrm{d}t)$. The solutions of the equations of motion are $q(t) = A \sin(\omega t + \phi)$ and are therefore coordinatized by $A = p^2 + q^2$ and $\phi = \tanh^{-1}(m\omega q/p) - \omega t$, which are *both* constants of the motion in the (q, p, t) space. Thus Γ'' is the space of the couples (A, ϕ). On Γ'', the dynamical flow maps the solution (A, ϕ) into the solution $(A, \phi - \omega\, \mathrm{d}t)$, which is a different solution of the equations of motion shifted in time with respect to the previous one. Therefore if we reconstruct the phase space as the space of the orbits of Y, then X_t can be directly interpreted as the dynamical flow on this space.

function at time t. Note that $\rho(q,p,t)$ is a function on $\mathcal{C}$. The t dependence of $\rho(q,p,t)$ is determined by the well known equation

$$\frac{\partial}{\partial t}\rho(t) = \{\rho(t), H\}\,. \tag{11}$$

It is very remarkable that this equation admits an intrinsic geometrical form and can be expressed in terms of $(\mathcal{C},\omega)$ alone, with no reference to X_t. It is in fact equivalent to the equation

$$Y(\rho) = 0 \tag{12}$$

or

$$\{\rho, C\} = 0\,. \tag{13}$$

Therefore $\rho(q,p,t)$ can be obtained as the pullback under π of ρ from the phase space Γ to the constraint surface $\mathcal{C}$: it is constant along the orbits of ω. This is the first result that we will later extend to the general case.

Next, consider a peculiar statistical state, namely the equilibrium Gibbs distribution ρ_e, defined in (1). Now, it is easy to verify that

$$\omega(X_t) = -\frac{1}{\beta}\,\mathrm{d}\ln\rho_e\,. \tag{14}$$

Namely, there is a relation between the time vector field X_t and the equilibrium probability distribution. Thus, we obtain the following crucial result, which is extremely relevant for what follows: *the Hamiltonian vector field of the logarithm of an equilibrium distribution is proportional to the Hamiltonian time flow.*

Now, suppose we are given the parametrized formulation of the system in the form $(\mathcal{C},\omega)$, *without giving* X_t, namely without specifying which variable is the time variable. Suppose we are also given the additional information that the system is in the statistical state ρ_e, and we are explicitly told that ρ_e is an equilibrium state. Then we can reconstruct the time flow X_t (up to an overall (constant) $1/\beta$ rescaling): indeed, the time flow is proportional to the Hamiltonian vector field of $\ln\rho$. Thus: *an equilibrium statistical state contains information about the time flow.* In section 4 this result will be used for defining the preferred time flow for the systems without an *a priori* definition of time. To that purpose, however, we need an intrinsic definition of equilibrium, which is studied in the next subsection.

3.3. Equilibrium

In this subsection we study an intrinsic characterization of an equilibrium probability distribution. An equilibrium state is time-invariant. Since what we are searching is a way to read out the time variable from the statistical state (and since any distribution is invariant under a flow, as is shown below), we need an alternative characterization of equilibrium, not depending on the *a priori* knowledge of the time flow.

Physically, the canonical equilibrium distribution is defined by imagining that the system interacts with a heat reservoir. In view of the applications that we have in mind, it is more appropriate to refer to a more 'internal' definition of equilibrium. Thus, we shall refer to equilibrium as a situation in which every small but still macroscopic component of the system is in equilibrium, in the usual sense, with the rest of the system. The idea of a local definition of temperature is implicit in Unruh's work [8]. Let us separate the system in a small (but macroscopic) subsystem S' with coordinates p', q' and the rest of the system S'', with coordinates p'', q''. We assume that the interaction between the subsystem and the rest is small; this can be seen as a definition of macroscopical subsystem, as well as a property necessary for the consistency of a statistical description. Thus we have $H(p,q) = H''(p'',q'') + H''(p'',q'')$ up to (surface) terms macroscopically small. The equilibrium probability distribution factorizes:

$$\rho(p,q) = \rho'(p',q') \quad \rho''(p'',q'') \,. \tag{15}$$

The fact that equilibrium can be defined in terms of factorization of the density distribution is emphasized for instance by Landau and Lifshitz [29]. Now, let X_t be the time flow. Its projection on Γ can be decomposed into the sum of two components X_t' and X_t'', with components in the (q'',p'') and in the (q'',p'') directions respectively. Then, from the factorization property (15) it follows that the Lie brackets of X_t' and X_t'' vanish:

$$[X_t', X_t''] = 0 \,. \tag{16}$$

Equivalently,

$$X_t'(\rho'') = 0 \qquad \text{and} \qquad X_t''(\rho') = 0 \,. \tag{17}$$

Thus, given an arbitrary separation of the system into two macroscopic subsystem, an equilibrium state has the property that the time vector fields of the two subsystems commute.

4. Generally covariant thermodynamics

In the previous section we have reached three results concerning the statistical mechanics of standard Hamiltonian systems expressed in presymplectic form. First, any statistical state satisfies (12). Second, an equilibrium state is related to the time flow on the constraint surface, and this relation is given by (14). Third, a system in equilibrium can be decomposed into macroscopic subsystems, and the time flow decomposes into commuting vector fields (16). In appendix B, we show that these results also remain true in a system in which the Hamiltonian is time-dependent.

We now consider those presymplectic systems which are *not* standard Hamiltonian systems and are not obtained by parametrizing a standard Hamiltonian system. They are usually given to us as constrained systems with a vanishing canonical Hamiltonian, as general relativity. On the constraint surface we have only the $(\mathcal{C},\omega)$ structure with its orbits, and there is no preferred flow X_t. They do not have a preferred time variable. As discussed in section 2, the physical interpretation that we give to these systems is consistent with the absence of a preferred time flow: there is no preferred mechanical time variable, the dynamics is expressed as the correlation of dynamical variables with any arbitrary chosen internal time. An internal time may not be globally defined; it may serve as internal time only on a small portion of the phase space or along a small portion of the orbits, without challenging the physical interpretation.

A presymplectic system can be uniquely associated to a standard Hamiltonian system only if a *global* internal time exists, and if it is unique. Existence and uniqueness of this global internal time may be relevant for the quantum theory, but we are not concerned here with this problem. From our point of view any (local) internal time is on the same footing, as far as the mechanical aspects of the problem are concerned.

Now, our purpose is to define the 'equilibrium thermodynamics' of these presymplectic systems. The strategy that we follow here is to use the three results derived for the presymplectic systems that are obtained parametrizing a standard Hamiltonian system, and to *postulate* that they hold, in general, for every presymplectic system, whether or not there is an associated Hamiltonian system.

The key idea concerns the second of these results, namely the relation between an equilibrium statistical state ρ_e and the time vector field X_t: in the general case, we propose to read this equation as the *definition* of the preferred time vector field, in terms of a given statistical state. We put forth the physical hypothesis that in the general case it is X_t determined in this way that captures our physical notion of 'time flow', including in particular its common sense features. Namely, *we postulate that the common sense time, in a given equilibrium statistical state, is determined by the statistical state itself via equation (14).*

Let us have a phase space Γ_{ex}, with a large number of degrees of freedom; the dynamics is fixed by a Hamiltonian constraint

$$C \sim 0 \tag{18}$$

(here $\sim$ means weakly vanishing, in the sense of Dirac). Let $(\mathcal{C}, \omega)$ be the constraint surface. We define a statistical state of the system as a probability distribution ρ on $\mathcal{C}$. The first postulate for a covariant thermodynamics will be the following:

$$\{\rho, C\} \sim 0\,. \tag{19}$$

Note that we are not requiring this equation to hold as a definition of equilibrium. Rather, this equation must be satisfied by *any* statistical state of the system. Equation (19) can be equivalently rewritten in an intrinsic way on $\mathcal{C}$ by requiring that for every Y in the kernel of ω

$$Y(\rho) = 0\,. \tag{20}$$

Now, let ρ be a statistical state of the system. Let X_ρ be the vector field on the constraint surface defined by the equation

$$\omega(X_\rho) = -\mathrm{d} \ln \rho \tag{21}$$

namely the Hamiltonian vector field of the logarithm of the distribution function. (X_ρ is defined up to an arbitrary addition of a vector proportional to Y.) We denote X_ρ as the flow associated to ρ. Note that the equation

$$X_\rho(\rho) = 0 \tag{22}$$

follows from the definition. Therefore there is always an internal flow, with respect to which *any* given probability distribution is stationary.

Now let us get to the concept of equilibrium. Assume that the system can be separated into two macroscopic components. Assume there is a small component with phase space coordinates q', p' and a large component with coordinates q'', p'', and the vector field X_ρ has components X'_ρ and X''_ρ in the two subsystems. Then we say that the two subsystems are in equilibrium if

$$[X'_\rho, X''_\rho] \sim 0. \tag{23}$$

We say that a system is in equilibrium if any one of its macroscopical subsystems is in equilibrium with the rest of the system. An equilibrium state is denoted ρ_e†. Equation (17), which follow from the equilibrium property, can be interpreted as the statement that the distribution function of each subsystem is stationary with respect to the vector field defined by the rest of the system.

We call the flow associated to an equilibrium state *thermodynamical time*, and we denote it as X_t. Thus X_t is determined by

$$\omega(X_t) = -\mathrm{d}\ln\rho_e \tag{24}$$

In summary, in a dynamical system formulated in the parametrized formalism:

(i) a statistical state ρ is given by a probability distribution on the constraint surface that satisfies (19);

(ii) every statistical state ρ defines a flow X_ρ via (24);

(iii) an equilibrium statistical state ρ_e is a state in which the flow satisfies (23) for every splitting into macroscopical subsystems;

(iv) the time flow of an equilibrium state is called the thermodynamical time.

If we apply this theory to a parametrized system obtained from a Hamiltonian system, then we recover the usual notions. A statistical state is a standard probability distribution (including its time dependence), a Gibbs state is an equilibrium state, the thermodynamical time is (the $1/\beta$ rescaling of) the Hamiltonian time. If we apply the theory to a presymplectic dynamical system, we have a self-consistent notion of equilibrium; and if the system is in equilibrium, *the state of the system picks up a preferred internal time variable: the thermodynamical time.*

The central idea proposed in this paper is that the time of non-relativistic physics has a set of properties (which we have denoted common sense properties) which have thermodynamical origin; in relativistic physics, there is no preferred time variable as far as mechanics is concerned, but there *is* a preferred internal variable that carries these common sense properties: this is the thermodynamical time‡. Note that we are not conjecturing that the thermodynamical time is a Hamiltonian time (namely that any presymplectic system admits a deparametrization where X_t is the Hamiltonian time flow). The hypothesis here is that the time variable characterized by the common sense properties of time is, in general, a thermodynamical time, and not necessarily the Hamiltonian time of a Hamiltonian dynamical system. According to this hypothesis, what we perceive as time is, essentially, the thermodynamical time

† The definition of equilibrium proposed in this paper can probably be refined or improved. This may be done without affecting the general scheme. A very interesting suggestion was given to me by one of the referees: we should require that the subsystems be preserved by the dynamics—that $[X'_\rho, Y] \sim [X''_\rho, Y] \sim 0$.

‡ The fact that it is *possible* to extract an internal variable t from a general covariant theory like general relativity, such that the dynamics expressed in t has a Newtonian form, is known. See for instance [25].

(or a rescaling of it). Also, note that we are not claiming that our postulates for equilibrium statistical states single out preferred directions of phase space as time directions: on the contrary, *any* direction of phase space can (locally) play the role of thermodynamical time, if the system happens to be in the right corresponding state.

Using a pictorial analogy, we may say that we suggest that we relativize the time direction in the same way in which the 'up' direction was relativized in going from Aristotelian to Newtonian physics: a point of space had a universal 'up' direction in Aristotelian physics, while in Newtonian physics all directions are *a priori* equal: the 'up' direction of a point is determined by the gravitational force at that point, namely by the physical configuration of the massive bodies around the point, i.e. by the state of the system. Similarly, we suggest that there is no preferred time direction in phase space; the time direction is determined by the (coarse-grained) description of the physical fields, namely by the (statistical) state of the system.

To be clearer, we are not claiming in this paper that the axioms presented single out a preferred statistical state that determines a preferred time, but the very opposite: we propose that *there is no preferred physical time, in general.* What we are saying, is that in certain concrete physical situations, the statistical state that we expect could better describe the system (on physical grounds, in a specific physical situation), and the variable that we would like to single out as the most reasonable physical time of the system (again on physical grounds, in that specific physical situation), are related in the way we have described.

It may seem strange that time is determined by an equilibrium state, since in an equilibrium state the system loses track of the direction (the versus, the arrow) of time. However we are not concerned with the versus of the time flow: we are concerned with the identification of the variable that represent time, which is a very different problem. A large number of authors in different contexts have noted that the equations of mechanics are surprisingly unaware of which variable is the time variable. In non-relativistic mechanics this symmetry between the independent time variable and the dependent dynamical variables can be seen as a curiosity. In a gravitational context, not only is the time variable on the same ground as the others, but we do not even know how to recognize it. The theory seems to indicate that there is no explanation for the peculiar properties of the time variable at the mechanical level. We suggest that such an explanation should be searched at the thermodynamical level. We propose the idea that the very concept of time is meaningful only in a thermodynamical context.

This suggestion is supported in this paper by an explicit relation between statistical properties of the system and a time flow, namely by (24). In a companion paper [11] we shall see that this equation is surprisingly effective: it *does* allow us to derive the quantity that we we would like to call 'time', in covariant physical theories *from* physically, or experimentally, reasonable ansätze about the statistical state that describes the system. Thus we urge the reader to consider the examples in the companion paper as a concretization of the theory proposed here. The success in the examples is probably the best support, at this point, for the proposed theory. We leave for future work the actual derivation of a consistent set of properties of this thermodynamical time that would support our suggestion.

5. Conclusions

We have analysed the problem of the construction of statistical thermodynamics in a generally covariant context. We have noticed the difficulty due to the absence of a *preferred internal time variable: standard equilibrium thermodynamics can be defined* only after the selection of a preferred internal time. We have proposed reversing such a relation between time and thermodynamics, and we have suggested that: *it is the thermal state itself that defines the variable which plays the role of preferred internal time.* Thus, we suggest that time has a thermodynamical origin.

We propose three equations for the definition of a covariant thermodynamics. On the constraint surface $(\mathcal{C}, \omega)$ of the dynamical system representing the general covariant theory, every statistical state of the system is given by a distribution function ρ that must satisfy (19). This equation requires that the distribution function has vanishing Poisson brackets with all the constraints of the theory. Every such state defines a vector field X_ρ by (24). We define an equilibrium state as a state in which X_ρ satisfies (23); namely the distribution of a subsystem is stationary in the flow defined by the rest of the system. This scheme extends standard statistical mechanics to systems that do not have a Hamiltonian formulation. As such, it could be useful in other contexts besides gravity.

The flow of an equilibrium state is denoted as *the thermodynamical time* determined by the equilibrium state. We have made the hypothesis that the common sense or non-general relativistic time that has the standard attributes of time, including the thermodynamical ones, should be identified with (a rescaling of) this thermodynamical time†.

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Appendix A. A speculation on the arrow of time

A long-standing mystery in the physics of time is the so-called problem of the arrow of time, or the problem of the asymmetry of time [27, 30]. The time asymmetry of our universe can be traced back to the fact that the the universe was extremely out of equilibrium long ago [29, 10]. Within standard big-bang theory, a rough estimate of the 'initial' entropy of the universe divided by the 'maximal allowed entropy' provides a number of the order of 10^{123}, according to [9]. Is it possible to explain why the initial state is so highly improbable?

As we discussed, to extend the notion of a entropy all the way to the big bang is very likely to be inconsistent. Indeed, the approximation in which thermodynamics is defined on a fixed 'cold' background geometry breaks down close to the big bang: when the gravitational field undergoes wild thermal fluctuations, the Robertson–Walker time coordinate t_{RW} is not a good time coordinate and entropy defined with

† Time and temperature, which in this paper we suggest may be deeply related, *are* related from the point of view of etymology. The English adjective 'temporal', as well as the French 'temps' and the Italian 'tempo', are related to the Latin *tempus*, with a proper meaning of 'cut', 'section', 'part or division of', and a general meaning of 'time'. Temperature is connected (via a late mediaeval use of the word, that indicated the moderation, or due proportion, of the climate of a region) to the Latin *temperatura*, 'due, appropriate proportion', 'harmonious, proportionate relation', and the corresponding verb *temperare*, 'to mix in due proportions' (!). The relation between *temperare* and *tempus*, via the common idea of 'partition in due proportion', is generally (but not universally) accepted by philologists.

respect to t_{RW} is not physically meaningful. Thus, the issue of the initial entropy cannot be analysed in terms of the (non-covariant) choice of an arbitrary background geometry; it should be investigated in the context of a covariant thermodynamics like the one constructed in this paper.

If, as we are proposing in this paper, the very notion of time has a thermodynamical origin, then the problem of the low initial entropy and the problem of the arrow of time turn out to be in some sense reversed. The task is not to justify why entropy increases along a given time, but to understand how a thermal state determines a time flow with a strong asymmetry. In a situation of true equilibrium there is no preferred direction of the time flow. The time asymmetry in the universe indicates that the universe is not in an equilibrium state. On the other hand, the universe is well thermalized within large components; in particular, the cosmic background radiation is close to equilibrium. Let us separate the degrees of freedom of the universe into two macroscopical components. One of these components, say the cosmic background radiation, is in equilibrium within itself, the other, say the gravitational degrees of freedom, is not. In a non-equilibrium context, the different macroscopical subsystems define different time flows on the phase space. Let us assume that there is a time flow X_t defined by the equilibrium component. Then we may develop a reasonable thermodynamical description in this time X_t. However, since the other component is not in equilibrium, (22) does not hold. In such a situation, the entropy

$$S = \int_\Gamma \rho \, \ln \rho \tag{A1}$$

is not conserved:

$$X_t(S) \neq 0 \,. \tag{A2}$$

Thus this thermodynamical state defines a time flow, but entropy is not constant in this time flow; thermal physics turns out to be time-asymmetric. In other words, in the scheme proposed, the fact that entropy grows in one of the directions of time is a consequence of the fact that the degrees of freedom that define the thermodynamical time are not in equilibrium with other, weakly interacting, degrees of freedom. The possibility of making these speculations precise by developing a quantitative model of a non-equilibrium state will be considered elsewhere.

Appendix B. Time-dependent Hamiltonians

It is useful to consider systems with a time-dependent Hamiltonian $H(p,q,t)$. Suppose the system is in a Boltzmann distribution at $t = 0$:

$$\rho(p,q,t=0) = \mathrm{e}^{-\beta H(p,q,t=0)} \tag{B1}$$

and then it evolves freely. The probability distribution changes according to

$$\frac{\partial}{\partial t}\rho(p,q,t) = \{\rho, H\} \,. \tag{B2}$$

By integrating this equation we get ρ at later times. Note that ρ will not in general be of the form $\exp[-\beta H(t)]$ (with a fixed temperature β), because this is not a solution of the evolution equation. However we can always write ρ in the form

$$\rho(p,q,t) = \mathrm{e}^{-\beta(p,q,t)H(p,q,t)} \tag{B3}$$

where $\beta(p,q,t)$ is a suitable function on the phase space. The equation that determines β is

$$\frac{\partial\beta}{\partial t} = \{\beta, H\} - \frac{\beta}{H}\frac{\partial H}{\partial t} \tag{B4}$$

or, equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}(\beta H) = 0\,. \tag{B5}$$

Can we meaningfully talk of equilibrium in this case ? Assume, again, that the system is composed by two weakly interacting subsystems, and that we may write

$$\rho(q,p,t) = \mathrm{e}^{-\beta'(t)H'(p',q',t)}\mathrm{e}^{-\beta''(t)H''(p'',q'',t)}\,. \tag{B6}$$

Then, we may say that the two subsystems are in equilibrium if they have the same temperature, namely if

$$\beta' = \beta''\,. \tag{B7}$$

If this condition is realized, then there would be no transfer of energy from the two subsystems if we add some small additional coupling.

Let us now consider the parametrized version of the time-dependent case. Recall that one of the main advantages of the parametrized formalism is precisely the fact that it handles naturally and in a geometrical way the time-dependent Hamiltonians. Indeed the only difference is that now on the constraint surface the presymplectic form has the structure

$$\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}H(p,q,t) \wedge \mathrm{d}t\,. \tag{B8}$$

Note that the evolution equation for the probability distribution is nothing but the condition (12) above. If we separate the system in two subsystems then the condition for the equilibrium still holds. Therefore the presymplectic formalism extends immediately to time-dependent Hamiltonians without any additional modification.

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