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# Unscrambling the physics of out-of-time-order correlators

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**Quantitative tools for measuring the propagation of information through quantum many-body systems, originally developed to study quantum chaos, have recently found many new applications from black holes to disordered spin systems.**

Suppose we had such exquisite control over a quantum many-body system that we could run time forwards and backwards while keeping the system well isolated. What features of quantum dynamics could we probe with such a quantum time machine? Recently, driven by theoretical developments in quantum matter and quantum gravity, many new responses to this question have emerged. These answers led to new insights into quantum chaos, the black hole information problem, thermalization and non-classicality in many-body physics, and bounds on quantum dynamics. Central to these developments is the concept of an out-of-time-order correlator (OTOC).

The OTOC can be defined through the following thought experiment: given a physical system described by a Hamiltonian $H$ and an initial state $|\psi\rangle$, we choose two Hermitian operators $W$ and $V$ and compare two states of the system that differ only in the time-ordering of the operations used to create them. Starting from $|\psi\rangle$, one state is obtained by applying $V$, evolving for time $t$, applying $W$, and finally evolving for time $-t$ to produce $\mathrm{e}^{iHt}W\mathrm{e}^{-iHt}V\,|\psi\rangle$. The other state is obtained by evolving for time $t$, applying $W$, evolving for time $-t$, and finally applying $V$ to produce $V\mathrm{e}^{iHt}W\mathrm{e}^{-iHt}\,|\psi\rangle$. The quantum overlap of these two states is the OTOC, $F(t) = \langle\psi|\,\mathrm{e}^{iHt}W\mathrm{e}^{-iHt}V\mathrm{e}^{iHt}W\mathrm{e}^{-iHt}V\,|\psi\rangle$: it probes the way $W$ and $V$ inhibit the cancellation between forward $\mathrm{e}^{-iHt}$ and backward $\mathrm{e}^{iHt}$ time evolution.

The first OTOCs appeared in the superconductivity literature almost fifty years ago[1], yet they have only been widely appreciated in the last five years. They have roots going back to Hahn's work on spin echoes[2]. There, the observed polarization revivals were attributed to the effective rewinding of time, and the eventual decay of those revivals were attributed to the inability of the experimenter to perfectly reverse the flow of time. This led to the concept of Loschmidt echoes[3–5], of which the OTOC is a variant, which were studied in many subsequent experiments—especially using nuclear magnetic resonance techniques[6–8]. In the near future, rapid progress in the quantum control of atoms, molecules, ions and photons will enable the coherent rewinding of time in qualitatively more complex quantum systems. This explains the considerable experimental interest in OTOCs, which led to a number of proposals[9–13] and the first experimental results[14–17].

In my view, the underlying theme behind much of this development is a new focus on the dynamics of quantum information. Perhaps surprisingly, the rewinding of time provides a very useful way of thinking about quantum information dynamics because it helps us probe how small perturbations spread over a system. Here I will focus on the spread and effective loss of quantum information in the context of thermalization: OTOCs turn out to illuminate many different aspects of quantum thermalization, including when and how it happens, how rapidly it can occur, and the role played by entanglement and non-classicality.

To make things concrete, let us imagine a spin chain with open boundary conditions consisting of interacting spin-1/2 particles. Consider two orthogonal initial pure states of the chain, which have approximately the same average energy but opposite expectation values for some local, non-conserved operator. If the system is chaotic, and if we consider the time evolution of these two states at a time longer than the thermalization time, the expectation value of the operator will be approximately the same in both time-evolved states. Furthermore, the expectation value in either state will also be approximately the same as that taken over the thermal state with the same average energy as the two initial states. As far as measurements of this operator are concerned, the two states at late times look the same and information seems to have been lost.

This process of thermalization can also be viewed through the lens of entanglement. For example, suppose one spin in our spin chain is initially entangled with another spin, a reference, which is then placed in an isolated box. In light of the interaction between the spins in the chain, the dynamics of the system will transform this initially simple two-spin entanglement into entanglement with increasingly complex many-body degrees of freedom. As time evolves, we must have access to, and fine control over, more and more degrees of freedom in the spin chain to recover the entanglement with the reference. At late time in a fully scrambled state, the entanglement with the reference can only be recovered from a set of slightly more than half the spins, although any such set will do. If sets of less than half sufficed, then we could recover the entanglement in two places, violating quantum no cloning.

This phenomenon of the spread of quantum information is sometimes called information scrambling. It can be usefully characterized in terms of the growth of Heisenberg operators, which is where the basic connection to rewinding time comes in. Given some operator $W$, the time-dependent Heisenberg operator $W(t) = \mathrm{e}^{iHt}W\mathrm{e}^{-iHt}$ has a structure reminiscent of the classical butterfly effect:

$$W(t) = \underbrace{\mathrm{e}^{iHt}}_{\text{backward}}\ \underbrace{W}_{\text{perturbation}}\ \underbrace{\mathrm{e}^{-iHt}}_{\text{forward}} \tag{1}$$

Even if $W$ is a simple local operator, say a spin operator on site 1, at later times $W(t)$ will have spread over many sites and will be qualitatively more complex. To understand the growth of $W(t)$, we expand it in terms of nested commutators,

$$W(t) = \sum_{\ell=0}^{\infty} \frac{(it)^{\ell}}{\ell!}[H, \ldots [H, W], \ldots] \tag{2}$$

Suppose the system's Hamiltonian $H$ consists of local $k$-body interactions, meaning interactions involving $k$ distinct neighbouring

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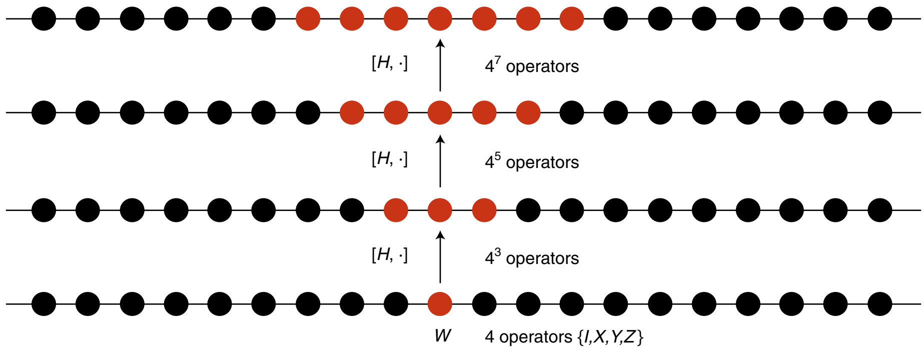


**Fig. 1 | Schematic of the growth of operators with time.** Heisenberg operators can be written as a sum of nested commutators (see equation (2)) with each iteration of the commutator being larger and more complicated than the previous iteration. As time increases, more and more terms in the expansion become important and the Heisenberg operator becomes larger and more complex.

spins. Consider now an operator $W$ that represents a physical effect that affects $r_W$ neighbouring spins in the chain. The operator that is defined by the commutator of the Hamiltonian with $W$, $[H,W]$ will instead generically affect $r_W + 2k - 1$ spins. Furthermore, if $W$ contains Pauli operators acting on $k_W$ spins at a time, then $[H,O]$ will generically contain Pauli operators acting on $k_W + k - 1$ spins at a time. Thus, as time increases and the higher order terms in equation (2) become important, the operator $W(t)$ expands in space and becomes increasingly complex, as sketched in Fig. 1.

OTOCs allow us to measure this growth of operators. The idea is to probe the spread of $W(t)$ with another operator $V$, which we take to be a simple spin operator located at some distance from $W$. This can be accomplished by looking at the expectation value of the squared commutator,

$$C(t) = \langle [W(t), \mathrm{V}]^{\dagger}[W(t), \mathrm{V}] \rangle \tag{3}$$

which will be initially zero for well-separated operators and which will become significantly different from zero once $W(t)$ has spread to the location of $V$. In the special case where $W$ and $V$ are Hermitian and unitary (this is not a crucial restriction), the squared commutator can be written as $C(t) = 2 - 2\mathrm{Re}[\langle W(t)VW(t)\mathrm{V}\rangle]$, where $(W(t)\mathrm{V}W(t)\mathrm{V}) = F(t)$ is the OTOC.

By contrast, time-ordered correlators like $W(t)V$ and response functions like $-i[W(t),V]$ typically decay rapidly, at least in thermalizing systems, so are not suitable for diagnosing operator growth. This decay is a consequence of thermalization wherein the system effectively loses memory of its initial state.

Now, if the spin chain is chaotic, then information is found to spread ballistically, as measured by the OTOC. The speed of the ballistic growth is called the butterfly velocity, $v_B$, so that $F(t) = 1$ for $x \gg v_B t$ and $F \approx 0$ for $x \ll v_B t$, where $x$ is the distance between the locations of $V$ and $W$. Because the size of commutator places upper bounds on the spread of influences in the system, the butterfly velocity can in many cases be regarded as an emergent speed limit for information propagation[18]. The time $t_*$ at which $F(t)$ is substantially different from unity is called the scrambling time; in local chaotic spin chains we have $t_* = x/v_B$.

One can also study the detailed time-dependence of the OTOC, which in some systems at early times[19,20] goes like

$$F(t) = 1 - \varepsilon e^{\lambda_L t} + \dots \tag{4}$$

with $\varepsilon$ some small model and operator dependent prefactor. The exponent $\lambda_L$ has been conjectured to be a quantum analogue of a Lyapunov exponent[20]. Furthermore, in a chaotic system at thermal equilibrium, a close cousin of the OTOC obeys a dynamical bound arising from unitarity and analyticity that translates into a bound on the exponent, which, when it exists, is of the form $\lambda_L \leq 2\pi k_B T/\hbar$ (ref. [21]). Remarkably, in a version of quantum gravity known as AdS/CFT (anti-de Sitter/conformal field theory) duality in which bulk quantum gravity is 'holographically dual' to boundary quantum field theory, it was found that black holes saturate this bound[19].

A related outcome is found in the so-called Sachdev–Ye–Kitaev (SYK) model[20,22]. As with black holes, OTOCs in the SYK model saturate the chaos bound at low temperature. Given this similarity, a natural question is, does maximal chaos imply the existence of a holographic dual description with black holes and local gravitational physics? However, while a sector of the dynamics in SYK matches that of local gravity, SYK has many additional degrees of freedom that probably render its holographic dual effectively non-local[23,24].

As an aside, systems exhibiting the kind of exponential growth of OTOCs described by equation (4) are typically in some kind of semi-classical limit or have a large number of local degrees of freedom. An example is a gauge theory with many colours of gauge bosons, which in the classical limit is dual to gravity. In systems like a spin chain with a finite number of local degrees of freedom, much remains to be understood about the way OTOCs grow in time. One proposed direction is to focus on the case where $W$ and $V$ are unbounded, leading to a notion of weak quantum chaos[25].

OTOCs are also continuing to shed light on the black hole information problem. They are closely related to the so-called Hayden–Preskill protocol[26,27], which describes when information can be recovered from old black holes under the fanciful condition that we have collected all the early Hawking radiation and stored it safely in a powerful quantum computer. Until recently it was not known how to decode the Hayden-Preskill protocol, but now decoders have been found in which the OTOC signals when the decoding is possible[28,29]. Remarkably, the gravitational interpretation of the decoding is the passage through a traversable wormhole[28,30]—a kind of quantum teleportation.

In contrast to these fast scrambling systems, some quantum systems generically fail to thermalize, a phenomenon known as many-body localization[31,32]. OTOCs again provide a way to characterize the corresponding slow growth of operators. In contrast to the ballistic spreading characteristic of a chaotic spin chain, operators spread only logarithmically with time in a many-body localized state[33–36]. The physics is a slow dephasing process that is induced by the weak dependence of the energy of local spins on other distant ones. The OTOC exhibits power law rather than exponential time dependence, leading to a scrambling time which is exponential in separation, $t_* \sim e^{x/\xi}$ for some localization length $\xi$. On the border between localization and delocalization, operators exhibit even more unusual growth. For example, in a marginal many-body localized state, characterized by a stronger dependence of local spin energy on distant spins, one calculation finds a scrambling time having the form of a stretched exponential, $t_* \sim e^{\sqrt{x/\xi}}$ (ref. [37]).

Besides probing thermalization or the lack thereof, OTOCs, being directly related to the variance of the commutator $[W(t),V]$, can be used to probe out-of-equilibrium fluctuations[12]. Inspired by a tool from non-equilibrium statistical mechanics known as Jarzynski's equality, which relates work fluctuations and free energy differences, OTOCs have been shown to obey a Jarzynski-like equality[13]. Non-commuting observables cannot generically be assigned a joint probability distribution. However, their correlations can sometimes be viewed as arising as from a quasi-probability, a complex valued distribution that only obeys some of the rules of probability. Non-positive values of a quasi-probability can signal non-classicality, and such a relation has been found for OTOCs[38]. Another benefit of these relations to out-of-equilibrium statistical mechanics is that they provide more ways to access the OTOC experimentally[12,13,38].

OTOCs are now rapidly morphing from their early incarnation as probes of quantum chaos. They have touched a large number of topics of current theoretical interest and have generated considerable experimental efforts. Recent nuclear magnetic resonance experiments used OTOCs to see chaotic dynamics in very small systems[15] and evidence of localization in larger chains[16]. OTOCs in collective spin systems have also been studied using trapped ions[14] and momentum states of a Bose–Einstein condensate[17], with a view towards probing chaos in the quantum kicked top. So far all the experiments are amenable to direct numerical simulation, but the next generation of experiments are likely to push beyond this threshold. New horizons seem to abound—for example, in a novel metrological scheme[39] where sensitivity to initial conditions is used to amplify a signal to nearly the Heisenberg limit, the measurement can be interpreted in terms of an OTOC.

I think we should not be surprised to see many more developments coming soon, especially as more complex coherent quantum many-body systems become experimentally accessible.

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