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Geometric Formulation of Classical Mechanics

Monography

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Dedication

To my family and friends

Acknowledgments

I wouldn't have gotten this far if it weren't mainly for my family, especially my mother and father, who have always been by my side in the most difficult times. I would also like to thank all my friends who were with me during this journey. I thank Babi for being my graduation partner and friend since literally day one. I thank Rafael do Lago, Talles, Lucca and Gustavinho for always motivating me with the most crazy discussions about physics. I thank Thiago, Aryadine, Igor and Henrique for being my biggest inspirations since I met them. I especially thank Felipe Perícole, for listening to me talk about symplectic geometry, day and night, and for so many other things. I thank Breno, Kora and so many others for being my partners at Kuka Bar and Woodstock. I also thank my friends in São Paulo for so many nightly games, which helped preserve my sanity. I am grateful to the incredible professor I met, who always motivated me to continue. I thank Rodrigo Paladino and Bebel, who were my teachers in high school and helped lay a solid foundation in my education for me to get here, and today I am proud to call them friends. I would also like to thank my advisor, Lucas Chibebe Céleri, for being so patient and attentive. Finally, I would like to thank all the people who have been with me from the beginning, until today.

ABSTRACT

This monograph aims to present symplectic geometry as the natural setting of classical mechanics, specially hamiltonian mechanics. Concepts of analytical mechanics were approached and reformulated using the mathematical tools of differential geometry. Purely geometric concepts such as flow integrals, phase flow and constants of motion as invariant functions were introduced.

Contents

List of Figures

Chapter 1

Introduction

Symplectic geometry is geometry on symplectic manifolds: even-dimensional manifolds equipped with a a nondegenerate differential 2-form, called a *symplectic structure*. It has its origins about two centuries ago, with Hamilton's formulation of mechanics, where pairs of generalized coordinates and velocities are replaced by pairs of generalized coordinates and momenta. In Hamiltonian mechanics, the underlying symplectic manifold is the phase space. Mechanics is geometry on phase space, and the symplectic structure is the object that encodes the laws of physics.

Although the origin of symplectic geometry being Hamiltonian mechanics, its development was extremely fast, and it has not only asserted its significance as a branch of differential geometry and topology (symplectic topology), but has also extended its influence into various domains. Beyond its standalone relevance in pure math, symplectic geometry is enriched through interactions with diverse fields, including dynamical systems, global analysis, mathematical physics, representation theory, partial differential equations, Riemannian geometry, and more.

The work will be divided into three important subjects: first, a brief review of analytical mechanics, where basic concepts of the Lagrangian and Hamiltonian formulation will be presented. Then, a chapter on differential geometry, to develop the mathematical apparatus necessary for the study of symplectic geometry, and finally, a chapter dedicated to show the connection between classical mechanics and symplectic geometry.

Chapter 2

Lagrangian and Hamiltonian Mechanics

The Lagrangian and Hamiltonian formulations are powerful descriptions of classical mechanics. The former was introduced by the mathematician Joseph-Louis Lagrange in his 1788 book, *Mécanique Analytique*, and it is based on a scalar formalism, different from Newton's vector formalism. The latter was developed in 1833 by Sir William Rowan Hamilton, as a reformulation of the Lagrangian formalism.

The main goal of this Chapter is to provide a brief introduction to these formulations of classical mechanics. We refer the reader to Refs. [1]

2.1 Lagrangian Mechanics

Physical systems often shows geometric constraints, also called holonomic constraints (such as a particle moving on the surface of a sphere, or a simple pendulum with a non-variable radius), which are algebraic equations imposed on the system, expressed as functions of position, and possibly time. Such constraints proves to be an inconvenience in the Newtonian formulation, where the use of redundant variables is necessary, and the constraint forces appear explicitly [2]. The Lagrangian formalism solves such problems, as it allows the description of physical systems based on a single scalar function: the *Lagrangian* function. This function is expressed in terms of coordinates called generalized coordinates, which are arbitrary coordinates independent of each other, chosen according to the system and its constraints.

In the Lagrangian formulation, the trajectory of a dynamical system can be fully determined by solving the *Euler-Lagrange equations* associated with the system. The Euler-Lagrange equations constitute a set of n second-order differential equations, where n is the number of degrees of freedom of the system in question. The *configuration* of a system is described by each value assigned to the generalized coordinates at a given instant of time. The *configuration space* is the cartesian space whose axes corresponds to the generalized coordinates. The trajectory of a system with n degrees of freedom is described in an n-dimensional configuration space.

2.1.1 Lagrangian Function and Hamilton's Principle

Given a dynamical system, we define a function called Lagrangian by

$$L = T - V, \tag{2.1}$$

where T and V are the kinetic and potential energy of the system. L can be a function of three variables, namely the generalized position x, the generalized velocity $\dot{x}$, and time t. The generalized position and velocity can both depends on time. We then define a quantity called *action* as the integral of the function L over time

$$S = \int_{t_0}^{t_1} dt L(x(t), \dot{x}(t), t). \tag{2.2}$$

The quantity S is a functional, that is, it associates a function of a determined class of functions to a real number.

With L and S defined, we can now state the stationary-action principle, also known as the *Hamilton's principle*

Theorem 1. *Given a physical system described by the function $L = L(x, \dot{x}, t)$, the true evolution of the system from time t_0 to t_1 is one that minimize (more generally, extremizes) the action S.*

2.1.2 Euler-Lagrange Equations

Since the action S is a functional of x, and knowing that S must be extremized, we will search for a differential equation for $x(t)$. Assuming that the curve x extremizes S, let's consider a neighboring curve $\bar{x}$, resulting from a variation of x, defined by

$$\bar{x}(t) = x(t) + \epsilon\eta(t), \tag{2.3}$$

where ϵ is any real parameter, and $\eta(t)$ is an arbitrary function, such that

$$\eta(t_0) = \eta(t_1) = 0, \tag{2.4}$$

because the varied curve must pass through the fixed points (x_0, t_0) and (x_1, t_1).

Constructing a Lagrangian with $\bar{x}$ and substituting in Eq.(2.2), we have a functional of $\bar{x}$, which is also a function of the parameter ϵ

$$S[\bar{x}] = \psi(\epsilon) = \int_{t_0}^{t_1} dt L(\bar{x}(t), \dot{\bar{x}}(t), t), \tag{2.5}$$

thus, we reduced the original problem to finding an extreme point of the real function ψ.

By hypothesis, when $\epsilon = 0$, $\bar{x} = x$, that is, the function ψ passes through an extremum because the curve x provides an extremum for the functional S. Then

$$\left.\frac{d\psi}{d\epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial x}\frac{\partial x}{\partial \epsilon} + \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \epsilon}\right) = 0. \tag{2.6}$$

Now, since

$$\frac{\partial x}{\partial \epsilon} = \eta, \qquad \frac{\partial \dot{x}}{\partial \epsilon} = \dot{\eta}, \tag{2.7}$$

we can write the (2.6) as

$$\left.\frac{d\psi}{d\epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\dot{\eta}\right) = 0. \tag{2.8}$$

By doing an integration by parts to eliminate $\dot{\eta}$, we finally get

$$\int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \eta = 0. \tag{2.9}$$

The above result leads us to the *Euler-Lagrange* equation

$$\frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}, \tag{2.10}$$

which is a second-order differential equation, equivalent to Newton's second law, but independent of the coordinate system. Due to Hamilton's principle enunciated in the previous section, a physical system must be described by the solution of this equation, since it provides an extremum for S.

The following definition will be useful:

Definition 1. *Consider the Euler-Lagrange equation associated with the Lagrangian $L = L(x, \dot{x}, t)$ of a physical system. The quantity $p = \partial L / \partial \dot{x}$ is called the canonical momentum of the coordinate x.*

2.2 Conservation Laws

Here we show how some of the important conservation laws in classical mechanics emerges from this formalism.[3]

2.2.1 Conservation of Energy

The first conservation law we'll derive is due to the homogeneity of time: in a closed system, the Lagrangian does not explicitly depend on time. Let $L = L(q_i, \dot{q}_i)$. Differentiating L with respect to t, we have

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial x_i} \dot{x}_i + \frac{\partial L}{\partial \dot{x}_i} \ddot{x}_i \right). \tag{2.11}$$

Recognizing the first term within the sum as the time derivative of $\partial L / \partial \dot{x}$, we can rewrite the above expression as

$$\frac{dL}{dt} = \sum_i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial x_i} \dot{x}_i \right) \right], \tag{2.12}$$

where the Leibniz rule for derivatives was used. Therefore

$$\sum_i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial x_i} \dot{x}_i - L \right) \right] = 0, \tag{2.13}$$

that is, the quantity in brackets in the summation is conserved. This is the total energy of the system

$$\sum_i \left(\frac{\partial L}{\partial x_i} \dot{x}_i - L \right) = E. \tag{2.14}$$

2.2.2 Conservation of Momentum

The conserved quantity associated with the homogeneity of space is the linear momentum. This homogeneity implies that the mechanical properties of a closed system remain unchanged by a parallel displacement of the entire system. Let us consider an infinitesimal displacement $\vec{\delta r} = \vec{\epsilon}$ made in the system. As the mechanical properties of the system must remain the same, the infinitesimal variation in the Lagrangian must be equal to zero

$$\delta L = \sum_a \frac{\partial L}{\partial \vec{r_a}} \cdot \delta \vec{r}_a = \vec{\epsilon} \cdot \sum_a \frac{\partial L}{\partial \vec{r_a}} = 0. \tag{2.15}$$

As $\vec{\epsilon}$ is arbitrary, the sum must be equal to zero. From the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r_a}}}\right) = 0 \tag{2.16}$$

i.e., $\partial L/\partial \dot{\vec{r_a}} = \vec{p}$ is a constant of motion.

2.3 Hamiltonian Mechanics

In the Hamiltonian formulation, the equations of motion are the *Hamilton equations*, which constitute a set of $2n$ first order differential equations, obtained through the Hamiltonian function, whose dependencies are the generalized coordinates, the canonical momenta and time. While in the Lagrangian formulation there is a direct dependence between the generalized coordinates and the generalized velocities ($\dot{x} \equiv dx/dt$), in the Hamiltonian formulation the generalized coordinates and the canonical momenta are, a priori, independent of each other. The $2n$ dimensional space whose coordinate axes are the generalized coordinates and the canonical momenta is called phase space. A point in this space specifies the state of the system at an instant of time, and determines the entire phase trajectory of that system.

2.3.1 Legendre Transform

Let $y = y(x_1, x_2, ..., x_n)$ whose derivatives $\partial y/\partial x_i$ are p_i, with $i = 1, 2, ..., n$. A Legendre transform is a transformation that, when applied to y, provides a new function $\varphi = \varphi(p_1, p_2, ..., p_n)$ that contains all the information present in the original function.

The Legendre transform of y is given by

$$\varphi(p_1, p_2, ..., p_n) = y(x_1, x_2, ..., x_n) - \sum_{i=1}^{n} p_i x_i. \tag{2.17}$$

Transformations of this type are often used in thermodynamics to work with representations other than energy or entropy. In classical mechanics, the Legendre transformation is used to go from the Lagrangian formalism to the Hamiltonian formalism, which contains the same physical information as the previous formalism.

2.3.2 Hamiltonian Function and Hamilton's Canonical Equations

The Hamiltonian function $H = H(x, p, t)$ is obtained through a Legendre transformation of the Lagrangian function, where the generalized velocities are replaced with the canonical momenta

$$H(x, p, t) = \sum_{i=1}^{n} p_i \dot{x}_i - L(x, \dot{x}, t), \tag{2.18}$$

where the equations $p_i = \partial L / \partial \dot{x}_i$ for $\dot{x}_i$, can be solved in terms of x_i, p_i and t. Thus, x and p are treated on an equal footing, and we obtain an expression in which the canonical variables are mutually independent.

Now, for each degree of freedom of the system, there are two quantities that must be taken into account: the generalized coordinate and the canonical momentum. In the same way that in the Lagrangian formulation the set of n generalized coordinates generate the *configuration space*, the $2n$ canonical variables, in the Hamiltonian formulation, generate the *phase space*. The time evolution of a point in phase space is given by Hamilton's equations, and it's trajectory is called *phase trajectory*.

Differentiating (2.12), we have

$$dH = \sum_{i=1}^{n} (p_i d\dot{x}_i + \dot{x}_i dp_i) - \sum_{i=1}^{n} \left[\left(\frac{\partial L}{\partial x_i} dx_i + \frac{dL}{d\dot{x}_i} d\dot{x}_i \right) + \frac{\partial L}{\partial t} dt \right], \tag{2.19}$$

with

$$p_i = \frac{\partial L}{\partial \dot{x}_i}, \qquad \dot{p}_i = \frac{\partial L}{\partial x_i}. \tag{2.20}$$

Replacing the (2.14) in (2.13), we have the following

$$dH = \sum_{i=1}^{n} (\dot{x}_i dp_i - d\dot{p}_i dx_i) - \frac{\partial L}{\partial t} dt. \tag{2.21}$$

On the other hand, $H = H(x, p, t)$, so

$$dH = \sum_{i=1}^{n} \left(\frac{dH}{dx_i} dx_i + \frac{dH}{dp_i} dp_i \right) + \frac{\partial H}{\partial t} dt. \tag{2.22}$$

Comparing the (2.16) with (2.15), we obtain the Hamilton equations

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \tag{2.23}$$

and also

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{2.24}$$

In most cases of physical interest, the kinetic energy T in the Lagrangian is a purely quadratic function of the velocities, and V does not depend on the velocities. So we have

$$\sum_{i=1}^{n} p_i \dot{x}_i = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{x}_i} \dot{x}_i, \tag{2.25}$$

and by Euler's theorem for homogeneous functions,

$$\sum_{i=1}^{n} \frac{\partial T}{\partial \dot{x}_i} \dot{x}_i = 2T. \tag{2.26}$$

Therefore

$$H = \sum_{i=1}^{n} p_i \dot{x}_i - L = T + V, \tag{2.27}$$

that is, $H = E$ is the total energy of the system, expressed in terms of the generalized coordinates and canonical momenta. In a closed system, where H is time-independent and is identified as the total energy of the system, E is a conserved quantity.

2.3.3 Poisson Bracket

Let $A = A(x_i(t), p_i(t), t)$ be an observable quantity. The total derivative of A with respect to time is

$$\frac{dA}{dt} = \sum_{i=1}^{n} \left(\frac{\partial A}{\partial x_i} \dot{x}_i + \frac{\partial A}{\partial p_i} \dot{p}_i \right) + \frac{\partial A}{\partial t}. \tag{2.28}$$

Recognizing $\dot{x}_i = (\partial H / \partial p_i)$ and $\dot{p}_i = -(\partial H / \partial x_i)$, we write

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}, \tag{2.29}$$

with

$$\{A, H\} = \sum_{i=1}^{n} \left(\frac{\partial A}{\partial x_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial x_i} \right). \tag{2.30}$$

The above expression defines the *Poisson brackets* of A and H. For the special case where $A = H = E$ and H is time-independent, we have $\partial H / \partial t = 0$, $\{H, H\} = 0$, so

$$\frac{dH}{dt} = 0, \tag{2.31}$$

thus recovering the conservation of energy.

Generally, given two dynamic variables, the Poisson brackets between them is defined as

$$\{B, C\} = \sum_{i=1}^{n} \left(\frac{\partial B}{\partial x_i} \frac{\partial C}{\partial p_i} - \frac{\partial B}{\partial p_i} \frac{\partial C}{\partial x_i} \right). \tag{2.32}$$

Hamilton's equations can be written as follows

$$\dot{x}_i = \{x_i, H\}, \qquad \dot{p}_i = \{p_i, H\} \tag{2.33}$$

Chapter 3

Differential Geometry

Differential geometry is the branch of mathematics concerned with studying the geometric properties of curves and mathematical objects called manifolds (the analogue of n-dimensional surfaces) using differential calculus. Its roots are in the 18th century, in cartography, currently having a strong connection with other areas of science, mainly with the study of topics related to physics.

3.1 Exterior Algebra

In this session, the concepts of k-vectors and k-forms and exterior product will be defined.[4]

3.1.1 k-vectors and k-forms

Let U be a vector space, U_0^k the space of tensors of the type $f : U \times ... \times U \rightarrow V$, with U taken k times, and U_k^0 the space of tensors of type $f : U^* \times ... \times U^* \rightarrow V$, with U^* taken k times.

Definition 1. *A k-vector of the vector space U is an antisymmetric element of U_0^k. The space of all said k-vectors is denoted by $\Lambda^k U$, with $\Lambda^0 U \equiv \mathbb{R}$.*

Definition 2. *A k-form of the vector space U is an antisymmetric element of U_k^0. The space of all k-forms is denoted by $\Lambda^K U^*$, with $\Lambda^0 U^* \equiv \mathbb{R}$.*

A k-form ω is also called an external form of degree k, and is a function of k-vectors, k-linear and antisymmetric

$$\omega(v_{i_1}, ..., v_{i_k}) = (-1)^n \omega(v_1, ..., v_k), \tag{3.1}$$

where $n = 0$ if the permutation of $i_1, ..., i_k$ is even, and $n = 1$ if the permutation is odd.

3.1.2 Antisymmetrization and Exterior Product

Given a tensor $f \in U_0^k$, it is possible to construct an antisymmetric tensor by introducing the antisymmetrizing operator A, as follows

$$Af = \frac{1}{k!} \sum_{\sigma} sign(\sigma) \sigma f \tag{3.2}$$

where the sum is over all permutations σ of $1, ..., k$, and $sign$ is the sign function for permutations: $sign(\sigma) = 1$ if the permutation is even , and $sign(\sigma) = -1$, if the permutation is odd.

The antisymmetrizing operator has the following properties:

1. A is a linear transformation: $U_0^k \to \Lambda^k U$
2. If f is already an antisymmetric tensor, $Af = f$, therefore $A^2 = A$.
3. If an element of U_0^k is symmetric in a pair of indices, $Af = 0$.

Using the antisymmetrizing operator, we can define a new operation for antisymmetric tensors. Let $f \in \Lambda^k beU$ and $g \in \Lambda^l U$. Then, the exterior product $f \wedge g \in \Lambda^{k+l}$ is defined by

$$f \wedge g = \frac{(k+l)!}{k!l!} A(f \otimes g). \tag{3.3}$$

For $u, v \in U$, we simply have: $u \wedge v = u \otimes v - v \otimes u$. Likewise, for $\alpha, \omega \in U^*$, we have: $\alpha \wedge \omega = \alpha \otimes \omega - \omega \otimes \alpha$.

The exterior product is *linear* in each variable, associative, and satisfies: $f \wedge g = (-1)^{kl} g \wedge f$.

If $x_i \in U$, $x_1 \wedge ... \wedge x_k$ is called a simple k-vector. Similarly, if $\omega_i \in U^*$, $\omega_1 \wedge ... \wedge \omega_k$ is a simple k-form.

In general, given n 1-external forms ω_i, and n vectors u_i, we have

$$\omega_1 \wedge ... \wedge \omega_n(u_1, ..., u_n) = det \begin{pmatrix} \omega_1(u_1) & . & . & \omega_n(u_1) \\ . & . & . & . \\ . & . & . & . \\ \omega_1(u_n) & . & . & \omega_n(u_n) \end{pmatrix}. \tag{3.4}$$

3.2 Differentiable Manifolds

A manifold is a n-dimensional generalization of a surface. It is defined as a topological space that locally resembles a euclidean space, which means, for a given point and it's neighbourhood on a n-dimensional manifold, one can construct a map that takes the point to an open subset of a n-dimensional euclidean space, $\mathbb{R}^n$. In this section, the concept of differential maps, curves and functions will be covered, and with that, it will be possible to build the notion of a tangent vector, tangent and cotangent spaces, and finally the tangent and cotangent bundles, manifolds on which the Lagrangian and Hamiltonian functions are defined, respectively.

3.2.1 Differentiable Maps

Let $f : \mathcal{M} \to \mathcal{N}$ be a differentiable map between a m-dimensional manifold $\mathcal{M}$ and a n-dimensional manifold $\mathcal{N}$. This kind of map is called a differential map, and it maps a point $z \in \mathcal{M}$ to $f(z) \in \mathcal{N}$.[5]

Definition 3. *f is class C^k when f^j is k times differentiable with respect to x^i around the neighbourhood of $z \in \mathcal{M}$.*

Definition 4. *A smooth function is a function of class C^∞.*

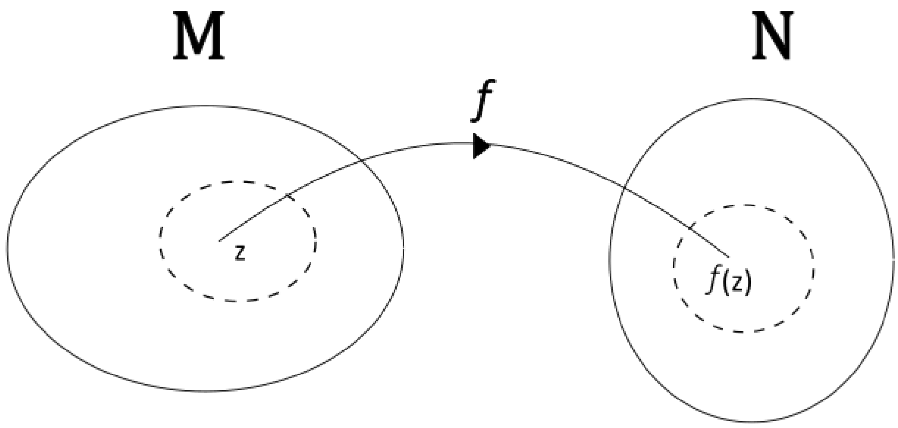

Figure 3.1: Map between manifolds.

3.2.2 Coordinate Systems

Let $\mathcal{M}$ be a m-dimensional manifold on which every point $z \in \mathcal{M}$ has an open neighbourhood u homeomorphic to an open subset of $\mathbb{R}^m$. That is, there is a map φ that takes $z \in \mathcal{M}$ and its neighbourhood u to $\mathbb{R}^m$. This map is what is called a coordinate map.

Definition 5. *The pair* (u, φ) *is called a* coordinate system *or* chart.

Let (u, φ) and (u', ψ) be charts of $\mathcal{M}$ and $\mathcal{N}$, respectively. Then, write $\{x^i\}$, $i \in \{1, ..., m\}$, for coordinates of $\varphi(z)$, and $\{y^j\}$, $j \in \{1, ..., n\}$, for coordinates of $\psi(f(z))$.

The *coordinate presentation* of f is then written as

$$y^j = f^j(x^i). \tag{3.5}$$

This is a vector valued function, and it's called a change of coordinates

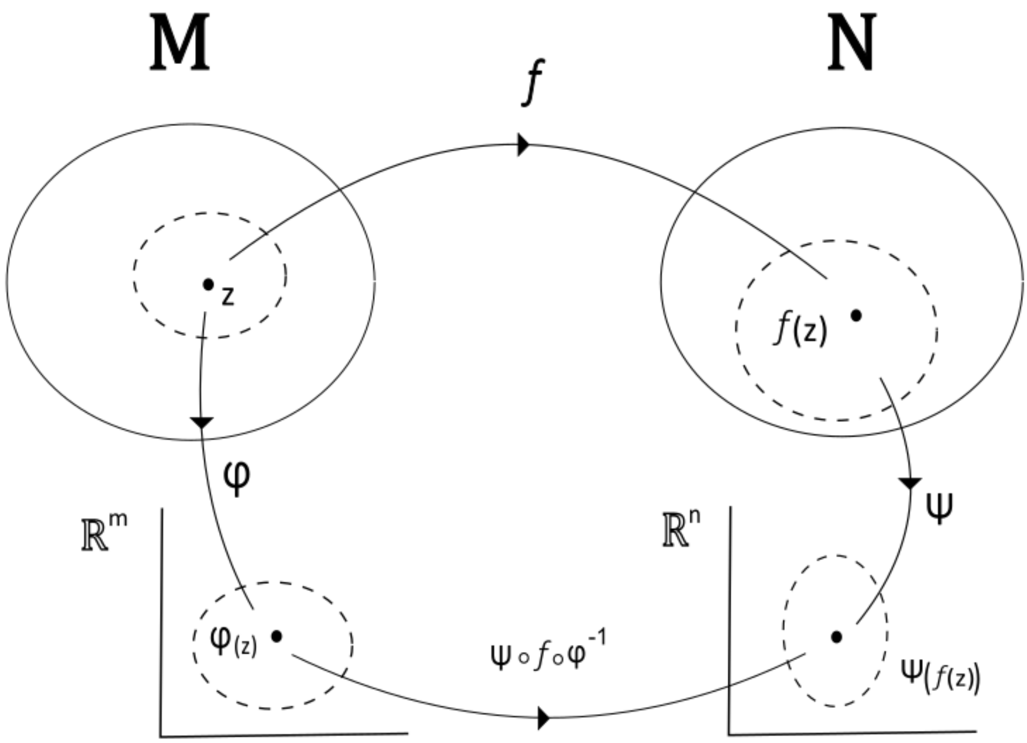

Figure 3.2: Coordinate presentation of a transformation between manifolds.

3.2.3 Curves and Functions

There are two kinds of maps that holds special relevance: curves and functions. These maps are fundamental to define the *tangent space*.

An open curve on $\mathcal{M}$ is a map that maps an open interval (a, b) in the real line onto the manifold. We denote this map as $C : (a, b) \to \mathcal{M}$, where $a < b \in \mathbb{R}$. The interval above can also be defined as $(a < t < b)$, where t is the parameter of the curve and $\{a, b, t\} \in \mathbb{R}$. A point through which the curve passes is denoted by $z = C(t)$.[5]

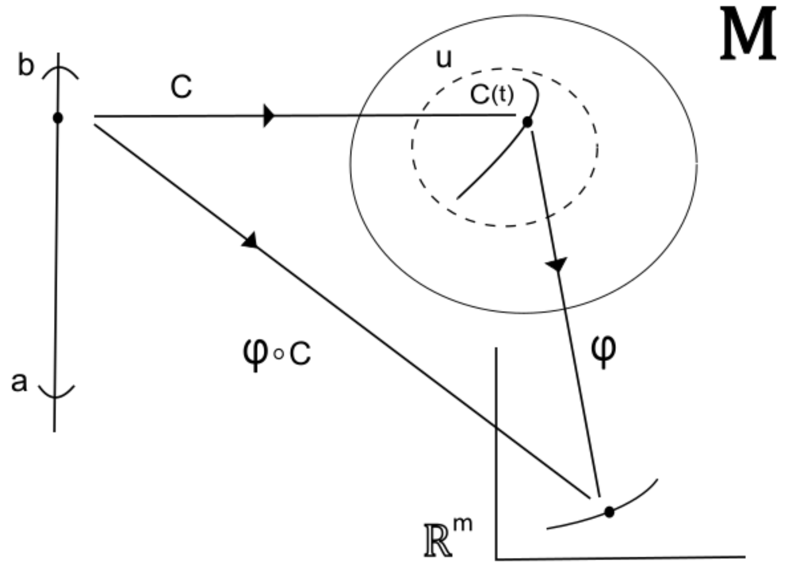

Figure 3.3: A curve on a manifold and its coordinate presentation.

A function f on $\mathcal{M}$ is a smooth map[1] that maps an open subset u of the manifold on the real line $\mathbb{R}$. We denote this map as $f : \mathcal{M} \to \mathbb{R}$. On a chart (u, φ) with local coordinates $\{x^1, ..., x^m\}$, the coordinate presentation of f is a map $f \circ \varphi : \mathbb{R}^m \to \mathbb{R}$.

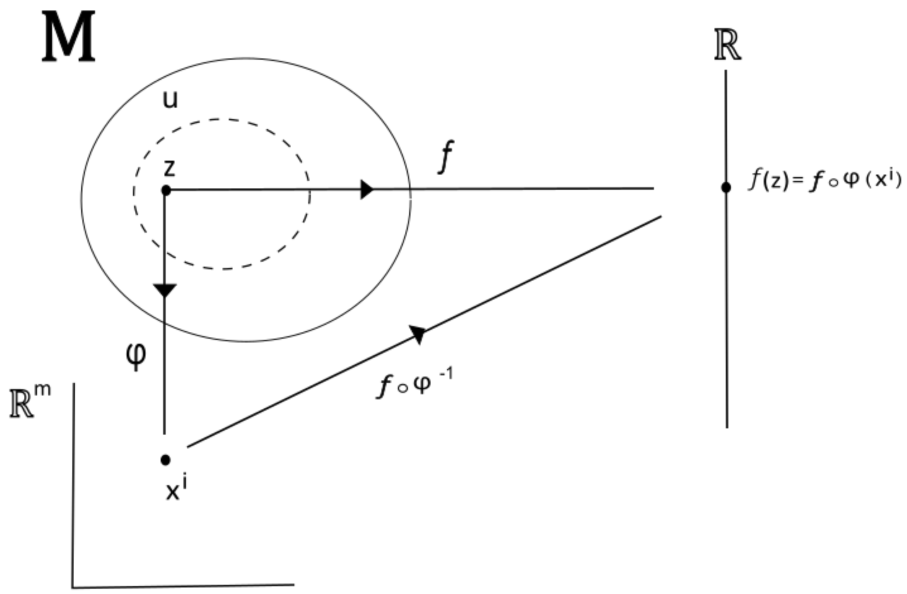

Figure 3.4: Function on a manifold and its coordinate presentation.

Note that a function is not the same as a coordinate map; a coordinate map is an open subset of the manifold mapped into an open subset of a m-dimensional euclidean space. A function simply associates a m-dimensional point on the manifold into a real number.

3.2.4 Tangent Space

Let $C : (a, b) \to \mathcal{M}$ be a curve on a m-dimensional manifold $\mathcal{M}$, and let $f : \mathcal{M} \to \mathbb{R}$ be a function on the same manifold. A tangent vector v at a point $z = C(t = 0)$ is a *directional derivative* of a function *along* the curve C

$$v = \left.\frac{df(C(t))}{dt}\right|_{t=0}. \tag{3.6}$$

In terms of local coordinates, we write

$$\left.\frac{df(C(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\partial f}{\partial x^i}\frac{dx^i}{\mathrm{d}t}\right|_{t=0}. \tag{3.7}$$

Define $\mathrm{d}x^i \mathrm{d}t = v^i$, then

$$\frac{df}{dt} = v^i \frac{\partial f}{\partial x^i} \equiv v[f], \tag{3.8}$$

where v is given by

$$v = v^i \frac{\partial}{\partial x^i}. \tag{3.9}$$

[1]The letter f will be used both for transformations between manifolds, and for certain functions, depending on the context.

Note that, in this way, a vector is defined by the way it acts on functions on the manifold.

The space of all possible tangent vectors at the point $z = C(t = 0)$ is a vector space called *tangent space at z*, denoted TM_z. This vector space has the same dimension as the manifold on which it is constructed, and its basis vectors are $\partial/\partial x^i = e_i$. So, rewriting equation (3.9)

$$v = v^i e_i. \tag{3.10}$$

Definition 6. *Let $f : \mathcal{M} \to \mathcal{N}$ be a transformation between manifolds. The derivative of f, denoted f_* is the transformation between tangent spaces*

$$f_{*z} : TM_z \to TN_{f(z)} \tag{3.11}$$

The transformation f_* is often called a *pushforward*.

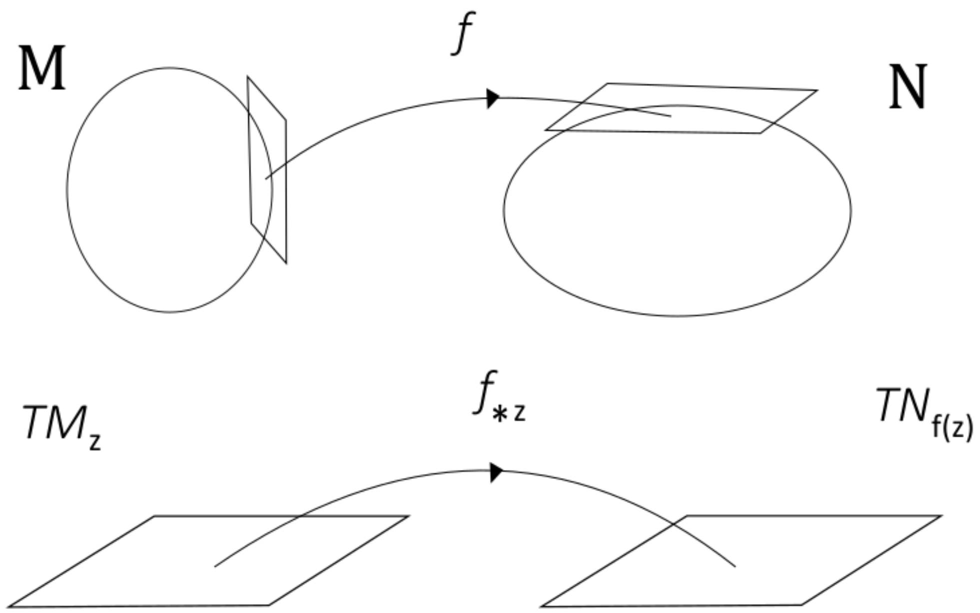

Figure 3.5: A pushforward $f_{*z} : TM_z \to TN_{f(z)}$

The union $\bigcup TM_z$ of all the tangent spaces in $\mathcal{M}$ also has structure of a differentiable manifold, and it is called the *tangent bundle* TM of $\mathcal{M}$. Note that a point in TM is a pair (z, v), where z is a point on the manifold and v is a vector in TM_z. So the tangent bundle TM is a $2m$-dimensional manifold built from the manifold $\mathcal{M}$ and its tangent spaces. The tangent vectors v can be thought as the "instantaneous velocity vectors" at the point z.

Definition 7. *The function $L : TM \to \mathbb{R}$, i.e, a function that maps points and velocities into a real number is called a Lagrangian.*

Definition 8. *A map $C : \mathbb{R} \to \mathcal{M}$ is called a motion in the Lagrangian system.*

The map $\Pi : TM \to \mathcal{M}$ that corresponds each vector $v \in TM_z$ to a point $z \in u$, is called *natural projection*. The map $\Pi^{-1}(z)$ is the inverse of the natural projection at point z, and it is the tangent space TM_z. This space is the "*fiber*" of the tangent bundle at point z. Generally:

Definition 9. *A* fiber bundle *is a triplet {$\mathcal{E}$, $\mathcal{B}$, Π }, where $\mathcal{E}$ is the total space, $\mathcal{B}$ is a base space, and* Π *is a map that corresponds each point on $\mathcal{B}$ to a point on $\mathcal{E}$.*

By the definition above, the tangent bundle of a manifold $\mathcal{M}$ is the triplet $\{TM, \mathcal{M}, \Pi\}$, where the projection $\Pi(p) = z$, and $p \in TM$.

Definition 10. *A cross section of a fiber bundle is a transformation S of the kind: $S : \mathcal{B} \to \mathcal{E}$, where the projection $\Pi(S(z)) = I$, where I is the identity.*

Definition 11. *A* vector field *V on a manifold $\mathcal{M}$ is a cross section of the tangent bundle TM. A vector field associates a point z on a manifold to a tangent vector v_z at TM_z.*

3.3 Differential Forms

Differential forms gives the notion of "oriented area", or "signed area". This is necessary so that we can measure areas and volumes in the manifold, since manifolds are not equipped with a notion of size.

Differential forms are k-linear, where k is the order or degree of the form, skew-symmetric and form an alternating algebra.

3.3.1 Exterior Derivative

Let u be an open set on a manifold $\mathcal{M}$ of dimension n. The antisymmetric space of all external forms of degree k in u will be denoted by $\Lambda^k(u)$. By definition, $\Lambda^0(u)$ is the set of all differentiable functions in u. The goal of this subsection is to define an operator d, which acts on a k-form ω, transforming it into a $k+1$-form, dω

$$\mathrm{d} : \Lambda^k(u) \to \Lambda^{k+1}(u). \tag{3.12}$$

This operator is called "exterior derivative", and has the following properties:

1. It is linear: $\mathrm{d}(\alpha + \omega) = \mathrm{d}\alpha + \mathrm{d}\omega$;

2. $\mathrm{d}(\alpha \wedge \omega) = \mathrm{d}\alpha \wedge \omega + (-1)^k \alpha \wedge \mathrm{d}\omega$, where k is the degree of α.

3. It is closed, i.e: $\mathrm{d}(\mathrm{d}\omega) = 0$.

4. $\mathrm{d}f = (\partial f / \partial x^i)\mathrm{d}x^i$, where f is a 0-form (a function) in u.

We will use this operator to define differential forms on a manifold.

3.3.2 1-forms

Let $\mathbb{R}^n$ be an n-dimensional vector space. A differential form of order 1, or simply a differential "1-form", is a linear function $\omega : \mathbb{R}^n \to \mathbb{R}$. The space of 1-forms is a vector space $(\mathbb{R}^n)^*$ called the *dual space* to $\mathbb{R}^n$.

The dual vector v^* to v is also called the *cotangent vector* to v. The dual space to a tangent space at z, TM_z, is the cotangent space TM_z^*. Similarly to what was done in section 3.2.4, the union of all cotangent spaces $\bigcup TM_z^*$ has structure of a differentiable manifold, and it is called the *cotangent bundle* TM^* of $\mathcal{M}$. Later, the *phase space* of a physical system will be identified as the cotangent bundle of its configurational space tangent bundle. Thus, the phase space is dual to the tangent bundle.

Let $\mathcal{M}$ be a manifold, and $f : \mathcal{M} \to \mathbb{R}$ be a function on the manifold. The simplest 1-form is the differential function f, df; if $f = f(x^i)$, then, using the d operator:

$$df = \frac{\partial f}{\partial x^i} dx^i \tag{3.13}$$

The acting of v on a function f is: $v[f] = v^i(\partial f/\partial x^i)$. Then, the action of df on V is defined via

$$df(v) \equiv v[f] = v^i \frac{\partial f}{\partial x^i}. \tag{3.14}$$

In other words, v acts on a function f and returns a real number, and the differential of the same function f acts on v in an identical way. In a sense, the element dx^i acting on v "extracts" the i-th component of v.

Since $df/dt = dx^i/dt(\partial f/\partial x^i)$, we can express df as a linear combination

$$df = \frac{\partial f}{\partial x^i} dx^i \equiv \frac{\partial f}{\partial x^i} e^i, \tag{3.15}$$

where e^i is a basis and $e^i(e_j) = \delta^i_j$.

Note that, due to third property, $d(df) = 0$. In this case, it is said that df is a closed form, and an exact differential of the function f.

3.3.3 2-forms

Let $\alpha = g_1(x_1, ..., x_n)dx^1 + ... + g_n(x_1, ..., x_n)dx^n$ be a differential 1-form. The g_i functions are the components of the form. Acting d on α, we have

$$d\alpha = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} dx^j \right) \wedge dx^i. \tag{3.16}$$

This is the general expression for the exterior differentiation of 1-forms, which gives 2-forms. Note that $dx^i \wedge dx^i = 0$. Differential 2-forms gives a way of measuring oriented areas.

If the 1-form α comes from the external differential of a function, that is, if $\alpha = df$, due to the exactness of df, α would not produce a 2-form.

3.3.4 k-forms

A differential k-form ω^k at point z on the manifold $\mathcal{M}$ is an external k-form on the tangent space TM_z; if ω^k is defined at each point of the manifold, it is said that ω^k is a k-form on the manifold $\mathcal{M}$.

Given a point $z \in u$, where u is a neighbourhood, and a map $\psi : u \to \mathbb{R}^m$. In local coordinates $\{x_i\} = \{x_1, ..., x_m\}$, the set $\{dx_1, ..., dx_m\}$ forms a basis of the space of 1-forms at TM_z.

The exterior product $dx_{i_1} \wedge, ..., \wedge dx_{i_k}$, $i_1 < ... < i_k$ forms a basis of the space of the exterior k-forms at TM_z.

Generally, a k-form has the following aspect

$$\omega^k = \sum_{i_1 < ... < i_k} \phi(x) dx_{i_1} \wedge ... \wedge dx_{i_k}, \tag{3.17}$$

where $\phi(x)$ is a smooth function.

Definition 12. *Let v be a vector on TM_z, and ω be a k-form on TM_z. The interior product between v and ω is*

$$i_v \omega(v_1, ..., v_{k-1}) = \omega(v, v_1, ..., v_{k-1}). \tag{3.18}$$

Definition 13. *Let $\mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds. Let f be a differential map that maps points $z \in \mathcal{M}$ to points $z' \in \mathcal{N}$. Let ω be a differential k-form on $\mathcal{N}$. The transformation f induces a k-form on $\mathcal{M}$, from ω, denoted $f^*\omega$. Its action on k tangent vectors $\{v_1, ..., v_k\} \in TM_z$ is defined by*

$$(f^*\omega)(v_1, ..., v_k) = \omega(f_* v_1, ... f_* v_k), \tag{3.19}$$

where f_* is the differential of the function f. This induced map is called a pullback.

3.3.5 Integration of forms in manifolds

Let $\mathcal{M}$ be a manifold, and $(U, \varphi = (x^1, ..., x^m))$ a coordinate chart. By definition, a m-form ω is integrable on $\mathcal{M}$ if its components $\omega_{1,...,m}$ are integrable on $\mathbb{R}^m$. Therefore

$$\int_M \omega = \int_U \omega = \int_{\mathbb{R}} ... \int_{\mathbb{R}} \omega_{1,...,m} dx^1 ... dx^m. \tag{3.20}$$

Let ω be a k-form, and $\mathrm{d}\omega$ its exterior derivative. Stokes theorem states that:

$$\int_D \omega = \int_{\mathcal{M}} \mathrm{d}\omega, \tag{3.21}$$

where D is the border of the manifold, with dimension $m - 1$. In $\mathbb{R}$, stokes theorem acquire the familiar look, where the operations of divergence and curl appears.

3.4 Symplectic Manifolds

A symplectic manifold $\mathcal{M}^{2n}$ is an even-dimensional manifold, equipped with a symplectic structure.

Definition 14. *A symplectic structure in $\mathcal{M}^{2n}$ is a closed, non-degenerate differential 2-form*

- *$d\omega^2$=0*

- *$\omega^2(v_1, v_2) = 0$ for all v_2 implies $v_1 = 0$.*

Therefore, the pair (M^{2n}, ω^2) is a symplectic manifold.

The cotangent bundle of a manifold $\mathcal{N}$ of dimension n has the natural structure of a manifold of dimension $2n$. A point in the cotangent bundle is a differential 1-form in the tangent space to $\mathcal{N}$, at any point z. Let $\{x^1, ..., x^n\}$ be a set of local coordinates of a point z in $\mathcal{N}$, then such a 1-form is given by n components y. Then $\{x^1, ..., x^n, y_1, ..., y_n\}$ gives the set of local coordinates of a point in TN^*.

The bundle TN^* has a symplectic structure, written in terms of the local coordinates described above. Let the canonical 1-form be $\omega^1 = \sum_i^n y_i dx^1$. Then, taking its exterior derivative

$$d\omega^1 = \sum_i^n \left(\frac{\partial y_i}{\partial x_i} dx_i \wedge dx_i + \frac{\partial y_i}{\partial y_i} dy_i \wedge dx_i \right); \tag{3.22}$$

By using the fact that $dx_i \wedge dx_i = 0$ we obtain

$$d\omega^1 = -\omega^2 = \sum_i^n dx^i \wedge dy_i, \tag{3.23}$$

which is the natural symplectic structure of the cotangent bundle.

Chapter 4

Geometric Formulation of Classical Mechanics

The phase space has a natural symplectic structure, thus being a symplectic manifold. Hamiltonian mechanics is nothing more than geometry in phase space. The vector field in phase space that corresponds to a differential of a function H is a Hamiltonian vector field X_H. A vector field on a manifold determines the phase flow. For the Hamiltonian field, the phase flow integration preserves the symplectic structure of the phase space.

In the setting of symplectic geometry, the differential 2-form denoted ω determines the "laws of classical mechanics". It is the mathematical object that turns energy H, into dynamics X_H.[6]

4.1 Canonical Form

There is, in a symplectic manifold, a standard canonical 1-form, also called *tautological* 1*-form*. Let $\mathcal{Q}$ be a manifold of dimension n and TQ^* its cotangent bundle of dimension $2n$, equipped with a symplectic structure. Consider the points $q \in \mathcal{Q}$, and $x \in \; TQ^*$, where $x = (q, p)$ and let Π be a natural projection

$$\Pi : TQ^* \to \mathcal{Q}, \tag{4.1}$$

given $\Pi(q, p) = q$. The derivative of Π is

$$\Pi_* : T(TQ^*)_{(p,q)} \to T\mathcal{Q}_q, \tag{4.2}$$

and it makes the transition from the tangent bundle of TQ^* to the tangent bundle of $\mathcal{Q}$ (that is, it associates vectors from one manifold to another through a pushforward).

Consider a vector $u \in T(TQ^*)_{(q,p)}$. Define a 1-form λ as

$$\lambda_{(q,p)}(u) = p(\Pi_* u). \tag{4.3}$$

Let $(U, \varphi(q) = (q^i))$ be a map on the manifold $\mathcal{Q}$, and $(U', \psi(q, p) = (q^i, p_i))$ a map on the manifold TQ^*; the point (q, p) has coordinates $(q^1, ..., q^i, p_1, ..., p_i)$. Acting the natural projection

$$\Pi(q^i, p_i) = (q^i). \tag{4.4}$$

Then, the action of the derivative is

$$\Pi_* \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_j} \right) = \left(\frac{\partial}{\partial q^i} \right), \tag{4.5}$$

with matrix representation

$$\Pi_* = \begin{pmatrix} 1 & 0 & . & 0 & 0 & . & . & 0 \\ 0 & 1 & . & 0 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & . & . & 1 & 0 & . & . & 0 \end{pmatrix}, \tag{4.6}$$

that is, a matrix with n-lines and $2n$-columns.

Writing $u \in T(TQ^*)$ in the standard basis

$$u = \sum_i a^i \frac{\partial}{\partial q^i} + b^j \frac{\partial}{\partial p_j}, \tag{4.7}$$

acting Π_*, we obtain

$$\Pi_* u = \sum_i a^i \frac{\partial}{\partial q^i}. \tag{4.8}$$

The dual basis in $T(TQ^*)^*$ is $(\mathrm{d}q^i, \mathrm{d}p_j)$, so, by Eq. (4.3)

$$\lambda = \sum_i p_i dq^i. \tag{4.9}$$

The form λ is the tautological form. The canonical 2-form is, then

$$\mathrm{d}\lambda = -\omega = -\sum_i \left(\frac{\partial p_i}{\partial q^j} dq^j + \frac{\partial p_i}{\partial p^j} dp^j \right) \wedge dq^i, \tag{4.10}$$

so

$$\omega = \sum_i dq^i \wedge dp_i. \tag{4.11}$$

Since the coefficients of this 2-form remain constant, it can be regarded as a matrix operating on R^{2n}

$$\mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{4.12}$$

that is called symplectic matrix.

Definition 1. *A symplectomorphism of symplectic manifolds* $(\mathcal{M}, \omega)$*,* $(\mathcal{N}, \theta)$ *is defined as a diffeomorphism that satisfies* $f^*(\theta) = \omega$*.*

A symplectomorphism is also called a "canonical transformation", where a region of the (q, p) phase space is mapped to a region of a new (Q, P) phase space, preserving the symplectic structure. A canonical transformation *always* obeys the following [2]

$$\mathbf{M}^T \mathbf{J} \mathbf{M} = \mathbf{J}, \tag{4.13}$$

Where the matrix $\mathbf{M}$ is the Jacobian matrix of the transformation. Equation (4.13) is a necessary condition to verify if a given transformation is a symplectomorphism. It preserves the symplectic matrix, thus, preserves the symplectic form.

With the tautological form and the symplectic structure, and with the definition of symplectomorphism, the *Darboux* theorem follows:

Theorem 1. *Let $(\mathcal{M}, \omega)$ be a symplectic manifold ; for every $z \in \mathcal{M}$ there is a chart (U, φ) with $z \in U$, such that φ is a symplectomorphism between (U, ω) and $\mathbb{R}^{2n}$.*

Darboux theorem states that *every* symplectic manifold is locally symplectomorphic to the standard symplectic structure on Euclidean space. So, there is no local symplectic geometry, all symplectic manifolds looks locally the same.

4.2 Hamiltonian Vector Field

Let $\mathcal{Q}$ be the configurational manifold of a physical system, that is, each point in the manifold represents a configuration of the system at time t. The dimension of the manifold depends on the number of constraints j, that is, if $n = 3N - j$, with N being the number of particles of the system, dim$\mathcal{Q} = n$, which is the number of degrees of freedom of the system. The natural coordinates of the tangent bundle TQ will be identified as (q^i), and of the cotangent bundle TQ^* as (p_i). The cotangent bundle of the configurational manifold is the phase space of the system. The phase space $TQ^* = \mathcal{M}$ is a symplectic manifold with symplectic structure ω.

Definition 2. *Let X_H be a vector field in $(\mathcal{M}, \omega^)$. X_H it is called a local Hamiltonian vector field if $i_{X_H}\omega^2 = 0$ is a closed differential form, that is:*

$$di_{X_H}\omega = 0. \tag{4.14}$$

Definition 3. *Let $H : \mathcal{M}^{2n} \to \mathbb{R}$ be the Hamiltonian function. If the interior product of X_H and ω, in addition to being a closed differential form, is equal to the differential of the Hamiltonian function, X_H is called a* global *Hamiltonian vector field*

$$i_{X_H}\omega = dH. \tag{4.15}$$

The differential canonical 2-form is $\omega = \sum_i dq_i \wedge dp^i$. Let X_H be a vector field. Then

$$i_{X_H}\omega(V) = \omega(X_H, V), \tag{4.16}$$

which is the acting of $dq \wedge dp$ on the pair (X_H, V)

$$\sum_i dq^i \wedge dp_i(X_H, V) = \sum_i (dq^i \otimes dp_i - dp_i \otimes dq^i)(X_H, V) = \sum_i dq^i(X_H)dp_i(V) - dp_i(X_H)dq^i(V), \tag{4.17}$$

i.e,

$$i_{X_H}\omega = \sum_i \left(dq^i(X_H)dp_i - dp_i(X_H)dq^i \right). \tag{4.18}$$

The Hamiltonian function is $H = H(q, p)$. Then

$$dH = \sum_i \left(\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i \right) \tag{4.19}$$

We want $i_{X_H}\omega^2 = dH$. Therefore, the Hamiltonian field in local coordinates writes

$$X_H = \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right). \tag{4.20}$$

From the relation above, it is possible to see that the components of the Hamiltonian vector field are the Hamilton's canonical equations.

It is usual to write $JdH = X_H$, where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{4.21}$$

.

Therefore, the differentiable function $H : M^{2n} \to \mathbb{R}$ uniquely defines a vector field X_H, such that

$$dH(V) = \omega^2(X_H, V). \tag{4.22}$$

4.3 Poisson Brackets

Generally, any vector field associated to a function on a manifold have components

$$V_f = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \right), \tag{4.23}$$

in canonical coordinates.

The Poisson brackets of two 1-forms on a symplectic manifold is defined by:

$$\{\alpha, \beta\} = -i_{[V_\alpha, V_\beta]}\omega^2. \tag{4.24}$$

For two differentiable functions f and g

$$\{f, g\} = \omega(V_f, V_g). \tag{4.25}$$

By the definition of interior product: $i_{V_f}\omega(V_g) = \omega(V_f, V_g)$, the above equation can be written as

$$\{f, g\} = i_{V_f}\omega(V_g). \tag{4.26}$$

In other words, the Poisson brackets of two functions is the contraction of the vector field V_f with the symplectic form.

Acting ω on the pair (X_H, V_f)

$$\omega(X_H, V_f) = \sum_i dq(X_H)dp(V_f) - dp(X_H)dq(V_f) = -\sum_i \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial f}{\partial p}; \tag{4.27}$$

From the equation above, we have the expression for $\{H, f\}$

$$\{H, f\} = \sum_i \left(\frac{\partial H}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} \right). \tag{4.28}$$

The Poisson bracket are used to describe how physical observables evolve over time. It quantifies the influence of one observable on another in terms of their time evolution in phase flow. Its representation as the symplectic form gives another insight about its nature: the Poisson bracket between two observables is preserved under canonical transformations.

4.4 Flow Integral

In this section, we will define *phase flow*, enunciate Liouville's theorem, and finally constants of motion [7]

4.4.1 Phase Flow

Consider an arbitrary phase space $(\mathcal{M}, \omega)$, with dim$\mathcal{M} = 2n$. The canonical coordinates are $\{q^i, p_i\}$. The one-parameter group of transformations is called phase flow:

$$g^t : (q_i(0), p^i(0)) \to (q_i(t), p^i(t)), \tag{4.29}$$

onde $q^i(t)$ e $p_i(t)$ são as soluções pras equações de Hamilton.

The *volume form* Ω is defined as

$$\Omega = dq^1 \wedge ... \wedge dq^n \wedge dp_1 \wedge ... \wedge dp_n; \tag{4.30}$$

The volume is, then:

$$V = \int_D \Omega = \int_{\mathbb{R}} dq_1...dq_n dp_1...dp_n. \tag{4.31}$$

Now follow Liouville's theorem:

Theorem 2. *Phase flow preserves the volume V of a D domain of phase space.*

Therefore, by Liouville's theorem, the elements of the group g^t are canonical transformations: if f is a canonical transformation, the pullback $f^*\omega$ implies $f^*\omega^n$. Then:

$$(f^*\omega)^n = \omega^n = \Omega, \tag{4.32}$$

and in fact, the volume is conserved under canonical transformations.

4.4.2 Flow Integral

Let $\mathcal{M}$ be the phase space, f a canonical transformation, $z \in \mathcal{M}$ and $v \in TM_z$. Let c be a curve defined in phase space. We know that

$$df(z) = \frac{d}{dt}(f(c(t))). \tag{4.33}$$

Consider the phase flow g^t, defined by the Hamiltonian vector field. Of course, $g^t : \mathcal{M} \to \mathcal{M}$, and $C(t) = g^t(x)$. Then

$$df(X_H) = \{H, f\}. \tag{4.34}$$

Comparing with equation 4.33, we have that

$$\{H, f\}(z) = df(z) = \frac{d}{dt}(f(c(t))). \tag{4.35}$$

that is, the transformation f is a *constant of motion* in the phase flow if and only if $\{H, f\} = 0$.

With the result above, we see that conserved quantities are obtained when the poisson bracket of its function is zero in the phase flow. In particular, the law of conservation of energy is written:

$$\{H, H\} = \omega(X_H, X_H) = 0; \tag{4.36}$$

H is the first integral of the Hamiltonian phase flow.

Chapter 5

Conclusions

This work has aimed to develop the mathematical apparatus and motivate the use of symplectic geometry as a tool to attack classical mechanics. It is expected that the reader has been able to understand key concepts of differential geometry and symplectic geometry, and why this setting is the natural framework to Hamiltonian mechanics.

First, the key concepts of analytical mechanics was discussed, such as the calculus of variations and conservation theorems. These concepts form the fundamental basis for the geometrization of mechanics. Then the powerful tooling of differential geometry was introduced in a compact manner; The focus of this work was not on rigorous proofs and demonstration of all mathematical theorems, but rather that the reader can equip himself with the mathematics necessary to understand phase space mechanics as geometry.

The main objective of this work was to present the geometric formulation of mechanics, where the concepts of analytical mechanics presented in chapter 2 were reformulated using the tools of differential geometry.

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Appendices

Appendix A

One Parameter Group

A one parameter group is a continuous group

$$\varphi : \mathbb{R} \to \mathcal{G} \tag{A.1}$$

from the real line $\mathbb{R}$ (considered as an additive group) to some other topological group G. If φ is injective, then $\varphi(\mathbb{R})$, the image, will be a subgroup of G that is isomorphic to $\mathbb{R}$ as an additive group.

The operation of a one-parameter group on a set is referred to as a flow. A smooth vector field on a manifold induces a local flow at a given point, forming a one-parameter group of local diffeomorphisms that guides points along integral curves of the vector field. Utilizing the local flow of a vector field allows the definition of the flow integral and poisson brackets along the vector field.