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The Cosmological Solution To Einstein's Field Equations

Monography

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The Cosmological Solution To Einstein's Field Equations

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Monograph presented to the Institute of Physics of the Federal University of Goiás as partial requirement to obtain the Bachelor's degree in Physics.

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Aos 05 dias do mês de setembro de 2022, a partir das 10 h 00, na sala 227 - IF2 do Instituto de Física da UFG, realizou-se a sessão pública de Defesa de Trabalho de Conclusão de Curso da estudante do curso de Física (Bacharelado), **Giulia Tessari de Moura,** matrícula 201504097, para apresentar sua monografia intitulada: "Solução Cosmológica das Equações de Einstein". A banca examinadora foi composta pelos professores Lucas Chibebe Celeri (IF/UFG), Rômulo Cesar Rougemont Pereira (IF/UFG) e Norton Gomes de Almeida (IF/UFG). A sessão pública de Defesa de TCC foi aberta pelo Presidente da Banca Examinadora, Professor Lucas Chibebe Celeri (Orientador), que na sequência passou a palavra para a estudante apresentar sua monografia. Após a exposição, a Banca Examinadora realizou a arguição da estudante. Ao finalizar a arguição, a Banca reuniu-se em sessão secreta a fim de concluir o julgamento da monografia. A Banca atribuiu à estudante a nota **9,75**, esta foi **APROVADA** na disciplina de TCC. Proclamados os resultados pelo Professor Lucas Chibebe Celeri (Presidente), foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos membros da Banca Examinadora.

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ABSTRACT

General Relativity is a theory about space, time and gravitation. It is one of the most well-verified theory in physics and it holds an important role in contemporary society by supporting the emergence of new technology, such as GPS, and studies to unfold. The GR contribution of interest to this work is in the field of Cosmology. This monograph aims to introduce some basic mathematical concepts in differential geometry and General Relativity in order to present the cosmological solution to Einstein's field equations, further relating the results to the observed cosmological data.

Keywords: General relativity; Differential geometry; Einstein field equations; Cosmology; Universe expansion; Cosmological constant.

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List of Abbreviations

GR	General Relativity.
IRF	Inertial Reference Frame.
MCRF	Momentarily Comoving Reference Frame.
RW	Robertson-Walker in ’Robertson-Walker metric’.
SR	Special Relativity.

Contents

Chapter 1

Introduction

As human beings, we are much inclined to describe the world around us through our own experiences and observations. Physics has its theoretical foundations reinforced by experimentation and it is not too far fetched to assume that Science as a whole, being an outstanding ensemble of human knowledge, can be understood as the polished understanding of our existence and all the subjectivity related to it. Our philosophical, individual life experiences and ideas heavily influence the rationalization of experimentation, being so that important concepts such as space and time vary throughout History as scientific knowledge evolves and gets refined.

Aristotle (384 B.C. - 322 B.C.) philosophically distinguished the celestial space from the earthly space. Everything in the Aristotelian Universe had its natural place and it could move in a natural or violent way. The natural cosmic movement was circular and uniform, while the natural terrestrial movement was linear [1]. Time was an universal order in which all changes were related, representing a 'number of change', relating the 'before' and 'after' of the 'now' [2]. All these concepts are very much acceptable if we think about their significance. Earth is clearly different from the night sky and all celestial objects 'move' in a circular, uniform way. If we gently throw a ball in a straight line, it tends to follow such a 'natural' path unless someone kicks it 'violently' out of its way. We can also sense that actions we did in our past influence our future, so actions throughout time are related.

Yet in the 17th century, due to the evolution of mathematical knowledge and improvement of the scientific method, it is possible to observe the reformulation of these Aristotelian ideas under the light of classical mechanics, developed by Isaac Newton (1643 - 1727). Since Earth was found not to be the center of the Universe, there are no longer any distinctions between earthly and cosmic space. Repeatability of experiments, the idea of reference frames and the rationalization of daily life experiences were the key to this new theory. In classical mechanics, as we commonly experience, space is seen as homogeneous and isotropic, that is the same as saying all its points are equivalent. It implies that the same experiment, under the same conditions, offers the same result no matter where it is performed. It also hints that physical properties of the space around us are constant, regardless of direction. Time is understood as being a one-way, homogeneous entity, meaning that laws of nature are the same for any instant of time and all systems evolve in the future direction [3]. It is possible to gather all these definitions and fuse them into a single structure: spacetime. It can be interpreted as a continuum of points which we call events, parts of space at an instant of time. And all these events have unique sets of four numbers, three spatial coordinates and one time coordinate, which make up what we call a coordinate system [4]. We call a person performing an experiment (measuring those four numbers of an event) an observer and each observer has their own points of reference to which they get their measurements, so they are on their individual reference frame.

This theoretical structure predicts the existence of an inertial reference frame, in which laws of nature are the same at any given time and the observer cannot affirm anything about their own movement through experiments, unlike an observer on an accelerated car or on a taking-off airplane (non-inertial reference frames). And as experimentation shows, any frame of reference with relative constant velocity v to another inertial reference frame is also inertial, since any constant value for v is equivalent to a null value, that is the same as being relatively at rest. With all this being said, a very important and fundamental concept is that of simultaneity. Two events $\mathcal{A}$ and $\mathcal{B}$ are said to be simultaneous when they 'happen at the same time', that is, if the difference between their measured time coordinates equals zero. Such events can occur at the act of measuring the length of an object, like a rod (where event $\mathcal{A}$ would be the top and event $\mathcal{B}$ would be the tip), or the simultaneous emission of two light rays from different sources. In the Newtonian spacetime, simultaneous events are coincident to all inertial reference frames. There is this natural, observer-independent simultaneity; different inertial reference frames cannot argue whether events were simultaneous or not, they simply agree on it [4]. And this elementary definition would be the main point of divergence between theories two centuries later.

Newton also had an interest in studying light, developing his *Opticks: or, A Treatise of the Reflexions, Refractions, Inflexions and Colours of Light* (1704). Even before Newton, others have already experimented with this very interesting subject. Ole Roemer (1644 - 1710), in 1676, became the first to measure the speed of light while observing the eclipses of Jupiter's moons, getting roughly to the correct value. It was revolutionary, since scholars considered this speed to be infinite or simply too high to be measured [5]. Since then, Hippolyte Fizeau (1819 - 1896), in 1851, got a much closer value (only a 5% difference) with the Fizeau–Foucault apparatus, using light beam reflections [6]. And more impressive than these incredible results was the fact that such speed was constant.

James Clerk Maxwell (1871 - 1939) genially developed an electromagnetic theory which resulted in a constant value for the speed of light and also agreed with measurements made by Weber and Kohlrausch (1856), Fizeau (1851) and Foucault [7]. It was an appalling result since, according to Classical Mechanics, two observers from different inertial reference frames in relative constant motion should obtain different values to the speed of light, such as it is for any velocity measurement. And well, it just so happens that such a difference is not observed. The speed of light in vacuum gives the same result $\mathbf{c} \approx 3,00 \times 10^8 m/s$ in any frame, regardless of its state of movement. Additionally, if it were possible to represent all inertial systems using Classical Mechanics, then it should be feasible to describe both electromagnetic phenomena and coordinate transformations between different observers in an invariant manner. Yet Maxwell's equations for electromagnetism [7] are not invariant under Galilean transformations (the transformation groups we use in classical mechanics) [8]. Take, for example, the interaction of a magnet and a conductor, both at constant, relative motion with respect to each other. In the conductor frame, it is at rest while the magnet is in relative motion, so there will be an electric field in the neighbourhood of the magnet, producing current on the conductor. But in the magnet frame, since the conductor is the one in motion, no electric field arises in the neighbourhood of the magnet [9]; so the observers cannot agree on which electromagnetic field is present. This deadlock could mean that either $i)$ Galilean transformations were valid, but not for electromagnetism, so there would be a preferred reference frame, also known as the ether frame, in which light would have $|\vec{v}| = c$ and such a frame could be found; or $ii)$ there was a relativity principle for both electromagnetism and classical mechanics and Maxwell equations would need to be altered to satisfy Galilean transformations; or lastly $iii)$ there was a relativity principle for both theories and the laws of mechanics needed to be changed [8]. Through the following decades, there were several attempts to confirm the ether hypothesis and to alter Maxwell's equations. Some worth mentioning are the Michelson-Morley experiment, the Lorentz-Fritzgerald contraction hypothesis, and emission theory [8]. All these assumptions fell to the ground due to the lack of results and there was nothing left to do other than accept that electromagnetism was valid and the well-known classical mechanics was unsuitable for describing the new phenomena, calling for the rise of a new theory which could provide a better description of the laws of nature.

Based on Maxwell's equations and the experimental evidence, Albert Einstein (1879 - 1955) formulated two postulates for his new special theory of relativity, in 1905. $i)$ Physical laws must be the same in all inertial reference frames; Preferred inertial reference frames do not exist. $ii)$ The speed of light in vacuum has the characteristic value c in all inertial reference frame, regardless of light source movement, and such value is the upper time limit for interactions to happen. Both Special Relativity and Classical Mechanics hold the same notion that inertial motion is that with no external forces acting on it, but the definition of an inertial reference frame is generalized to $iii)$ that in which the distance between two points is time-independent; space geometry at every instant t is Euclidean and all its points'clocks are synchronized and work at the same rate [10]. It basically means that space is flat at every instant and all measurements made at any point are correspondents. The time coordinate given by the clock at each point is also known as the proper time τ, a global inertial coordinate; that is, the coordinate given on the frame for which the subject point is at rest [4].

The evidence that measurements are observer-dependent leads us to consider 'time' as a coordinate instead of an universal entity alienated to all systems. Since there are no preferred inertial observers and coordinates vary, they have no intrinsic significance. To fully describe spacetime structure, it is needed to figure out what are its observer-independent, intrinsic properties. In special relativity, the only quantity with intrinsic significance is the spacetime interval Δs^2, which is a unique quantity for representing all measurements between events. Other physical quantities such as energy, momentum, velocity and acceleration are now mathematically represented by tensors, being frame-invariant through Lorentz transformations. Given that the speed of light act as a limiter for all interactions, the 'access' that an observer has to future and past events are limited by a light cone. Each event has a past and a future light cone which determines accessible events inside it. Simultaneous events are outside the light cone, neither in the past nor the future, and vary between reference frames. Since the time spent by a light signal to reach two different reference frames located on different distances from the source is not the same, observers will disagree whether some events were simultaneous or not, so simultaneity is now deemed observer-dependent. They only agree on the interval Δs^2, in a way that measurements on length and time between inertial observers balance themselves keeping the spacetime interval constant, leading to apparent time dilations and length contractions.

Special Relativity better explained some observed optical phenomena such as dragging effects, light aberration, the relativistic Doppler effect and many others. All its theoretical approach is based on the existence of inertial reference frames that fill all of spacetime, with coordinate always at rest relative to the origin, and all clocks run at the same rate relative to the origin's clock. But how truly practical are such frames? Can we construct an inertial reference frame on Earth? Since we are under the influence of a nonuniform gravitational field, clocks do not all run at the same rate and we cannot build a truly inertial reference frame. This fact was first suggested by Einstein through the idealization of the redshift experiment [10] and can be summarized as follows. A particle with an initial rest mass is dropped from the top of a tower at Earth's surface and falls freely until reaching an experimenter on the ground. They have a certain method that can change all the particle's energy into a single photon with the same energy, redirecting it upwards. Upon the photon's arrival at the top of the tower, the other experimenter again transforms the photon into a particle with a rest mass that should be the same as the first particle's rest mass to avoid perpetual motion due to energy gain. So we are led to predict that a photon climbing in Earth's gravitational field will lose energy and will consequently be redshifted ([10], p.112). Such redshift was measured directly in the Pound-Rebka experiment (1959), where gamma rays were emitted from the top of a tower and measured by a receiver at the bottom [11]. With improvements in technology between 1960 and 1990, the gravitational redshift became an effect that is central to society. The GPS navigation system incorporates vital corrections for the redshift, in the absence of which it would not remain accurate for more than a few minutes ([10], p.113). These experimental facts show us that gravitational fields are incompatible with global special relativity. We are only able to construct a local inertial reference frame in spacetime regions small enough so that gravitation nonuniformities effects are too little to measure. In this sense, local special relativity should be reformulated in terms of a more general theory.

Then, in 1914, Einstein formulated the mathematical basis and concepts of general relativity [12], extending special relativity to non-inertial reference frames. In addition to special relativity postulates, he added the Einstein equivalence principle, stating the equivalence between gravity and acceleration and establishing a freely falling frame as a local inertial reference frame. The elevator thought experiment gives good insight into this principle: Einstein proposed that if you were stuck inside an elevator that is isolated from the outside world, you would not be able to tell whether an object in free fall inside it was being pulled down by Earth's gravity or pulled up by the elevator accelerating rapidly, since the objects inside the elevator would accelerate at the same rate. Now suppose the cables pulling the elevator were cut off and you began to fall. As Galileo demonstrated in his famous experiment at the Leaning Tower of Pisa, all bodies given the same initial velocity follow the same trajectory in a gravitational field, regardless of their composition [10]. So you and the objects would fall at the same rate, maintaining a uniform velocity along with the elevator, being at relative rest to both the elevator and the objects, floating in the air. We can now relate to this scenario by remembering how we momentarily feel floating when we are inside an airplane and it shortly falls freely in order to lose altitude. In such conditions, it becomes clear that a freely falling frame is locally an inertial reference frame. In fact, it can be verified that photons are not redshifted in this frame ([10], p. 115). On the matter of gravity and acceleration, Einstein also wrote: "The centrifugal force which acts [...] upon a body is determined by precisely the same natural constant that also gives its action in a gravitational field. In fact, we have no means to distinguish a "centrifugal field" from a gravitational field."([12], p. 31). So by comparing transitions from an inertial reference frame to a rotating reference frame, Einstein exposed analogies to the geometry of surfaces and figured that a fitting description for generalizing special relativity spacetime would be that of a curved spacetime, with gravity as curvature itself. A suitable mathematical description was made through differential geometry, which is a way to describe mathematical objects with intrinsic curvature. Spacetime was then treated as a differentiable manifold with a metric that defines notions such as how to measure distance between events. Tensor calculus was developed in polar coordinates, showing that curvature implies the need of a connection to parallel transport vectors from one point to another in spacetime, defining the covariant derivative and also finding a curvature tensor. Other important tensors were found to represent physical quantities of interest in a frame-independent manner and, by doing so, deriving general tensor conservation laws valid in any system. Furthermore, general relativity is a geometrical theory of gravity, and the Einstein field equations are the generalized gravitational field equations. It was built analogously to Newton's gravity, but with tensor quantities. Mass density is still considered to be the source of gravity, but now it is also linked to energy and momentum. And such quantities are related to spacetime curvature through the Einstein tensor, clarifying how matter and gravity can influence spacetime geometry and vice-versa.

We have come a long way since Aristotelian times, but the desire to describe the laws of nature are still in our human essence. Since more tools have been developed, our description of the universe around us became increasingly more complex. It all comes to show that general relativity aims to offer a complete, generalized description of physical systems in a beautiful, concise manner. Along with general relativity, the dawn of the 20th century brought further insights into comprehending our vast universe, such as the emergence of cosmology, which is a wide branch of astronomy that involves the origin and evolution of the universe, studying its large scale properties as a whole. While other branches of astronomy deal with individual objects and phenomena or collections of objects, cosmology spans the entire universe from birth to death. It began in the early 1900s when scientists were debating whether the Milky Way contained the whole universe or whether it was simply one of many collections of stars. Edwin Hubble calculated the distance to a fuzzy nebulous object and determined that it laid outside of the Milky Way, proving our galaxy to be a small drop in the enormous universe [13]. Using general relativity to lay the framework, Hubble measured other galaxies and determined that they were moving away from us, leading him to conclude that the universe was not static but expanding. And that is the result that we aim to find.

In the present monograph, we intend to solve Einstein's field equations for the cosmological boundary conditions, that is, considering the whole Universe as our physical system. We are specially focused in first unhurriedly developing a justification for these equations. As we hope to have made clear with this introduction, human knowledge evolves and our tools to explain fundamental physical concepts become more complex and take the collective developed knowledge to a further stage. So we begin, in Chapter 2, by introducing the mathematical tools necessary to understand the least basic of differential geometry. It may not be a break-through subject since it dates back to the 19th century, but it is not common knowledge between undergrad students. The calculations and definitions presented will not be as formal as an authentic mathematical text like do Carmo (2016), but it will be sufficient to give an idea of how calculus is performed in intrinsically curved mathematical objects. The chapter culminates in the formulation of the Einstein tensor so that, in Chapter 3, we can discuss general relativity itself, trying to make a smooth transition from differential geometry to gravity and showing how the mathematical language connects to the physical interpretation. We will introduce important tensor quantities and conservation laws to justify the form of the Einstein field equations, also finding the Einstein tensor for a static spherically symmetric spacetime. Chapter 4, after briefly introducing some cosmological concepts and assumptions, is devoted to finally solve the field equations and discuss what the solutions tell us about our Universe, associating the results to the experimental data in the literature. Was there really a Big Bang? How is the Universe expanding? Is it possible for the Universe to be static? If not, how far is it from being static? Those are some punctual questions we will aim to answer. In Chapter 5, as its self-explanatory title elucidates, we will present an overview of the present work, along with a general take on what has been discussed and what the future can still hold for the presented discussion.

Some important notes: we will be using geometrized units along the text, so both the speed of light c and the gravitational constant G have the same value $c = G = 1$ ([10], pp. 186-187). We will also be alternating between covariant and contravariant notation, so be watchful of that while reading. Finally, note that we assume the reader has had previous contact with special relativity. Although some concepts have been discussed in this Introduction, other special relativity fundamentals, physical concepts and mathematical formulation will not be developed here. But we must highlight that it is plainly possible to find all the background needed in Refs. [4, 10, 14] or any other undergrad literature about this subject.

Chapter 2

Differential Geometry

Transitioning from special relativity to general relativity depends heavily on taking a different mathematical approach to describe a spacetime in which gravity is its curvature. Differential geometry is calculus applied to any geometry, being conveniently used to describe mathematical objects with intrinsic curvature. The main purpose of this chapter is to present the mathematical formulation of basic concepts in differential geometry for the following understanding of the next chapters. We will start off with vectors, dual vectors and tensors, which are frame-independent. Then we extend such definitions to a curved spacetime that looks locally flat, mathematically represented by a manifold. After that, we can define how to compare vectors at different points in a manifold through parallel transportation, using the covariant derivative, and also define curvature itself through the curvature tensor. Lastly, we will explore some important tensors and identities, culminating in the definition of the Einstein tensor.

2.1 A Little Background: Vectors, Dual Vectors, and Tensors in Flat Spacetime

A vector is an element of a vector space at a given point in spacetime, following the same linear algebra definition [15]. Note that here we make reference to four-vectors, which have one time component and three spatial components. In a more general way, if we consider a function $f(x^\alpha)$ and a curve[1] $x^\alpha(\sigma)$ (a precise notion of such concepts will be given latter), the directional derivative along the curve is

$$\frac{df}{d\sigma} = \lim_{\epsilon\to 0}\left[\frac{f\left(x^\alpha(\sigma+\epsilon)\right) - f\left(x^\alpha(\sigma)\right)}{\epsilon}\right] = \frac{dx^\alpha}{d\sigma}\frac{\partial f}{\partial x^\alpha}. \tag{2.1}$$

Let T_p be the tangent space at a point p and $\{\hat{e}_\alpha\}$ is a coordinate basis in T_p. So a tangent vector $\vec{t}$ to the curve is $\vec{t} = t^\alpha \hat{e}_\alpha$ with coordinate basis components $t^\alpha = \frac{dx^\alpha}{d\sigma}$. Since the directional derivative at the point labeled by σ and specified by $\vec{t}$ is $\frac{d}{d\sigma} = t^\alpha \frac{\partial}{\partial x^\alpha}$, so vectors and directional derivatives are in one-to-one correspondence [14]. In this way we can write a vector $\vec{V}$ as

$$\vec{V} \equiv V^\mu \frac{\partial}{\partial x^\mu} = V^\mu \hat{e}_\mu \tag{2.2}$$

[1]A curve is a mapping of an interval of a real line into a path in the manifold. A coordinate system then can be used to build the coordinate representation of the curve in $\mathbb{R}^n$ [10].

Consider that x^α and $x^{\alpha\prime}$ are two sets of coordinates that can be connected by a set of Lorentz transformations Λ. Then,

$$x^{\alpha\prime} = \Lambda^{\alpha\prime}{}_\beta x^\beta \qquad \Rightarrow \qquad t^{\alpha\prime} = \Lambda^{\alpha\prime}{}_\beta \frac{dx^\beta}{d\sigma}. \tag{2.3}$$

Since σ does not change, the vector $\vec{t}$ is invariant under Lorentz transformations:

$$\begin{aligned} \vec{t} = t^\alpha \hat{e}_\alpha = t^{\alpha\prime}\hat{e}_{\alpha\prime} = \Lambda^{\alpha\prime}{}_\beta t^\beta \hat{e}_{\alpha\prime} \\ \Rightarrow \hat{e}_{\alpha\prime} = \Lambda^\beta{}_{\alpha\prime}\hat{e}_\beta, \qquad \hat{e}_\alpha = \Lambda^{\beta\prime}{}_\alpha \hat{e}_{\beta\prime}. \end{aligned} \tag{2.4}$$

Comparing Eqs. (2.3) and (2.4), we can see that the basis vectors transform with the inverse transformations compared to coordinates. Also note that $\Lambda^\alpha{}_{\mu\prime}\,\Lambda^{\mu\prime}{}_\beta = \delta^\alpha_\beta$, where δ^α_β is the Kronecker delta. From the tangent space T_p we can construct a dual space T_p^* as a set of linear maps $T_p^* : T_p \to \mathbb{R}$. A dual vector $\tilde{\omega}$ is a linear map from vectors to real numbers [14]. For any two vectors $\vec{u}$, $\vec{v} \in T_p$ and any two numbers α, $\beta \in \mathbb{R}$, it holds

$$\tilde{\omega}(\alpha\vec{u} + \beta\vec{v}) = \alpha\tilde{\omega}(\vec{u}) + \beta\tilde{\omega}(\vec{v}), \tag{2.5}$$

where $\tilde{\omega}(\vec{u}),\ \ \tilde{\omega}(\vec{v}) \in \mathbb{R}$. Since T_p^* is also a vector space, for any $\tilde{\omega},\ \tilde{\theta} \in T_p^*$ and $\vec{v} \in T_p$,

$$(\alpha\tilde{\omega} + \beta\tilde{\theta})(\vec{v}) = \alpha\tilde{\omega}(\vec{v}) + \beta\tilde{\theta}(\vec{v}). \tag{2.6}$$

Let $\{\hat{\theta}^\mu\}$ be a basis for T_p^*. We demand that

$$\hat{\theta}^\alpha(\hat{e}_\beta) = \delta^\alpha{}_\beta, \qquad \tilde{\omega} = \omega_\mu \hat{\theta}^\mu. \tag{2.7}$$

The elements of T_p are called contravariant vectors, while elements of ${T_p}^*$ are covariant vectors[2]. Using components we obtain [10]

$$\tilde{\omega}(\vec{v}) = \omega_\mu \hat{\theta}^\mu\left(v^\gamma \hat{e}_\gamma\right) = \omega_\mu v^\gamma \hat{\theta}^\mu\left(\hat{e}_\gamma\right) = \omega_\mu v^\mu \tag{2.8}$$

$$\Rightarrow \omega_\mu v^\mu = \eta_{\mu\alpha}\omega^\alpha \eta^{\mu\beta} v_\beta = \eta_{\mu\alpha}\eta^{\mu\beta}\omega^\alpha v_\beta = \omega^\alpha v_\alpha. \tag{2.9}$$

Therefore, $\vec{v}(\tilde{\omega}) = \tilde{\omega}(\vec{v}) \in \mathbb{R}, {T_p}^{**} = T_p$. And, since the inverse transformations for basis vectors from both the vector space and the dual space are, respectively, $\omega_{\mu\prime} = \Lambda^\gamma{}_{\mu\prime}\omega_\gamma$ and $\hat{\theta}^{\mu\prime} = \Lambda^{\mu\prime}{}_\gamma \hat{\theta}^\gamma$, it implies that the relation $\vec{v}(\vec{\omega}) = \tilde{\omega}(\vec{v})$ is invariant under Lorentz transformations. Note that the gradient is a dual vector:

$$\tilde{d}\phi = \frac{\partial\phi}{\partial x^\mu}\hat{\theta}^\mu \qquad \Rightarrow \frac{\partial\phi}{\partial x^{\mu\prime}} = \frac{\partial x^\mu}{\partial x^{\mu\prime}}\frac{\partial\phi}{\partial x^\mu} = \Lambda^\mu{}_{\mu\prime}\frac{\partial\phi}{\partial x^\mu}, \tag{2.10}$$

since $dx^{\mu\prime} = \Lambda^{\mu\prime}{}_\mu dx^\mu$.

Consider V to be a finite dimensional vector space and let V^* be its dual vector space. Tensors can be defined as a generalization of vectors and dual vectors, being a multilinear map

$$T : \underbrace{V^* \times V^* \times \cdots \times V^*}_{r \text{ times}} \times \underbrace{V \times V \times \cdots \times V}_{s \text{ times}} \to \mathbb{R}. \tag{2.11}$$

[2] Vectors and dual vectors are not different representations of the same element, but distinct elements on their own.

A tensor of type $(\boldsymbol{r},\boldsymbol{s})$ maps r dual vectors and s vectors into real numbers [10]. Therefore, a $(\mathbf{0},\mathbf{1})$ tensor is simply a dual vector (covariant vector, one-form), since it maps a vector to a real number. Similarly, a $(\mathbf{1},\mathbf{0})$ tensor is a vector (contravariant vector), having one dual vector as an argument and mapping it to a real number. A tensor of type $(\mathbf{1},\mathbf{1})$ is a map from $V^* \times V$ to real numbers. If we fix $\vec{v} \in V$, $T(\cdot,\vec{v})$ is an element of V^{**}, and since $V^{**} = V$, $T(\cdot,\vec{v})$ in an element of V. So we have the linear map $(\mathbf{1},\mathbf{1}) : \vec{V} \to \vec{V}$ [4]. The collection of all tensors of type $(\boldsymbol{r},\boldsymbol{s})$ forms a vector space $\mathscr{T}(r,s)$, on which a basis can be defined as the tensor product of the individual basis $\{\hat{e}_\mu\}$ of V and $\{\hat{\theta}^\upsilon\}$ of V^*:

$$\hat{e}_{\mu_1} \otimes \cdots \otimes \hat{e}_{\mu_r} \otimes \hat{\theta}^{\upsilon_1} \otimes \cdots \otimes \hat{\theta}^{\upsilon_s}, \tag{2.12}$$
$$\text{with } n = \dim V = \dim V^* \Rightarrow \dim \mathscr{T}^r_s = n^{r+s}.$$

This leads us to the component notation

$$\boldsymbol{T} = T^{\mu_1 \ldots \mu_r}{}_{\upsilon_1 \ldots \upsilon_s} \hat{e}_{\mu_1} \cdots \hat{e}_{\mu_r} \otimes \hat{\theta}^{\upsilon_1} \ldots \hat{\theta}^{\upsilon_s}, \tag{2.13}$$

which is equivalent to

$$T^{\mu_1 \ldots \mu_r}{}_{\upsilon_1 \ldots \upsilon_s} = \boldsymbol{T}\left(\hat{\theta}^{\mu_1} \otimes \cdots \otimes \hat{\theta}^{\mu_r} \otimes \hat{e}_{\upsilon_1} \otimes \cdots \otimes \hat{e}_{\upsilon_s}\right), \tag{2.14}$$

since $\hat{\theta}^\mu\left(\hat{e}_\upsilon\right) = \delta^\mu_\upsilon$. Now we can define two important operations on tensors. The contraction on a tensor of type $(\boldsymbol{r},\boldsymbol{s})$ is a map $C : \mathscr{T}^r_s \to \mathscr{T}^{r-1}_{s-1}$ that acts on $\boldsymbol{T}$ as [4]

$$C(\boldsymbol{T}) = \sum_{\sigma=1}^{n} \boldsymbol{T}\left(\ldots,\hat{\theta}^\sigma,\ldots;\ldots,\hat{e}_\sigma,\ldots\right). \tag{2.15}$$

We observe that the operation $C(\boldsymbol{T})$ is independent of the basis

$$CT = T\left(e^j, e_j\right) = T\left(\Lambda^j{}_a e^a, e_b \Lambda^b{}_j\right) \tag{2.16}$$
$$= \Lambda^j{}_a \Lambda^b_j T\left(e^a, e_b\right) = \delta^b_a T\left(e^a, e_b\right) \tag{2.17}$$
$$= T\left(e^a, e_a\right) = CT, \tag{2.18}$$

so contraction is well defined [4]. Now, let $\boldsymbol{T}$ and $\boldsymbol{T}'$ be tensors such that $\boldsymbol{T} \in \mathscr{T}^r_s$ and $\boldsymbol{T}' \in \mathscr{T}^{r'}_{s'}$. We can build another tensor called the outer product of $\boldsymbol{T}$ and $\boldsymbol{T}'$, denoted by $\boldsymbol{T} \otimes \boldsymbol{T}' \in \mathscr{T}^{r+r'}_{s+s'}$. We have

$$\begin{aligned}&(\boldsymbol{T} \otimes \boldsymbol{T}')\left(\tilde{\omega}^1,\ldots,\tilde{\omega}^r,\ldots\tilde{\omega}^{r'};\vec{v}_1,\ldots,\vec{v}_s,\ldots,\vec{v}_{s'}\right)\\ &\quad = \boldsymbol{T}\left(\tilde{\omega}^1,\ldots,\tilde{\omega}^r;\vec{v}_1,\ldots,\vec{v}_s\right)\cdot\boldsymbol{T}'\left(\tilde{\omega}^{r+1},\ldots,\tilde{\omega}^{r'};\vec{v}_{s+1},\ldots,\vec{v}_{s'}\right).\end{aligned} \tag{2.19}$$

In terms of components, by writing $\boldsymbol{S} = \boldsymbol{T} \otimes \boldsymbol{T}'$, we obtain [4]

$$S^{\mu_1 \ldots \mu_{r+r'}}{}_{\alpha_1 \ldots \alpha_{s+s'}} = T^{\mu_1 \ldots \mu_r}{}_{\alpha_1 \ldots \alpha_s} T'^{\mu_{r+1} \ldots \mu_{r+r'}}{}_{\alpha_{s+1} \ldots \alpha_{s+s'}}. \tag{2.20}$$

From these definitions, it follows that tensor components transform under Lorentz transformations as

$$T^{\mu'_1 \ldots \mu'_r}{}_{\alpha'_1 \ldots \alpha'_s} = \Lambda^{\mu'_1}{}_{\mu_1} \cdots \Lambda^{\mu_{r'}}{}_{\mu_r} \cdot \Lambda^{\alpha_1}{}_{\alpha'_1} \cdots \Lambda^{\alpha_s}{}_{\alpha'_s} T^{\mu_1 \ldots \mu_r}{}_{\alpha_1 \ldots \alpha_s}, \tag{2.21}$$

and we can see that an upper index transforms like a vector, while a lower index transforms like a dual vector[3].

One important tensor is the metric tensor $\boldsymbol{g}$, which provides us the notion of infinitesimal squared distances, and is determined by the displacement tangent vector. Since these distances are quadratic in the displacement, the metric should be a linear map $\boldsymbol{g} : V_p \otimes V_p \to \mathbb{R}$ [4], i. e. a tensor of rank (order) $(0, 2)$. Furthermore, the metric should also be symmetric

$$g(\vec{V_1}, \vec{V_2}) = g(\vec{V_2}, \vec{V_1}) \quad \Rightarrow \quad g_{\mu\nu} = g_{\nu\mu} \tag{2.22}$$

and non-degenerate

$$g(\vec{V}, \vec{V_1}) = 0, \qquad \forall\, \vec{V} \in V_p \Rightarrow \vec{V_1} = 0. \tag{2.23}$$

This means that the metric is the inner product on the tangent space V_p at each point p. In a coordinate basis,[4]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \qquad dx^\mu dx^\nu = dx^\mu \otimes dx^\nu. \tag{2.24}$$

We have that $\boldsymbol{g}$ can be also viewed as a linear map from V_p into V_p^*, so $g(\cdot, \vec{V})$ is a dual vector. Such a map, because of the nondegeneracy, is both one-to-one and onto, thus having an inverse. As a consequence, we can use $\boldsymbol{g}$ to establish a one-to-one correspondence between vectors and dual vectors:

$$V^\mu = g^{\mu\nu} V_\nu, \qquad V_\mu = g_{\mu\nu} V^\nu, \tag{2.25}$$

and the dot product of two vectors $\vec{V}$ and $\vec{A}$ can be written as

$$\vec{V} \cdot \vec{A} = g_{\mu\nu} V^\mu A^\nu = V^\mu A_\mu = V_\nu A^\nu, \tag{2.26}$$

which leads us to

$$\vec{V} \cdot \vec{V} = g_{\nu\mu} V^\mu V^\nu \begin{cases} < 0, & \vec{V} \text{ is timelike,} \\ = 0, & \vec{V} \text{ is null,} \\ > 0, & \vec{V} \text{ is spacelike.} \end{cases} \tag{2.27}$$

A metric is not necessarily positive definite and the number of $+$ and $-$ signs of its eigenvalues is called the signature of the metric. For a 3D Euclidean space, the components of the metric, in the usual Cartesian reference system, are

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.28}$$

so vectors and dual vectors have exactly the same components (and we treat the gradient as simply a vector). In flat spacetime, the Minkowski metric has a Lorentzian signature and can be put in the form

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.29}$$

[3]The definitions of vectors, tensors and linear maps developed above depend on the concept of vector space, but since there is not a single vector space that covers all spacetime, we define these objects on the set of all vector spaces, resulting in vector and tensor fields defined on every point of spacetime.

[4]There is a sum implied on indices μ and ν.

For curved spacetimes, the metric components can vary greatly between reference frames, but the signature is always the same. We can verify that the trace of $\boldsymbol{g}$ is $Tr\,[\boldsymbol{g}] = g^{\mu\nu}g_{\mu\nu} = g^{\mu}{}_{\mu} = 4$, regardless of the components $g_{\mu\nu}$. We demand that it should be always possible to find a coordinate transformation $g_{\mu\nu} \to \eta_{\mu\nu}$ because experience has shown that space is locally flat and the mathematical theory is built surrounding such experimental fact. As coordinate transformations are symmetry group operations (Lorentz transformations), the trace is invariant. And since $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$, $Tr\,[\boldsymbol{\eta}] = \eta^{\mu\nu}\eta_{\mu\nu} = \eta^{\mu}{}_{\mu} = \delta^{\mu}{}_{\mu}(-1)^2 + 1^2 + 1^2 + 1^2 = 4$.

2.2 Manifolds, Tangent Vectors and Dual Vectors - A More Formal Approach

The best way to translate spacetime behaviour into a mathematical description is through an object called a manifold, which is an n-dimensional set that can be curved but that locally 'looks like' $\mathbb{R}^n$. Let us develop some important mathematical concepts and give spacetime a more precise description.

A topological space is the one in which each point lives in a neighborhood of other points in such a way continuity can be defined.

Definition 2.2.1 (Topological Space [4])**.** A topological space $\mathcal{M}$ is a set of points along with a collection $\mathcal{T}$ of open subsets $\{O_\alpha \subset \mathcal{M}\}$, called topology, such that

i) $\mathcal{M} \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$;
ii) $\bigcap\limits_{\alpha} O_\alpha$ is an open set (finitely many);
iii) $\bigcup\limits_{\alpha} O_\alpha$ is an open set (possibly infinite).

The relation between topological spaces is defined by a homeomorphism:

Definition 2.2.2 (Homeomorphism)**.** A homeomorphism between topological spaces $(\mathcal{M}, \mathcal{T})$ and $(\mathcal{M}', \mathcal{T}')$ is a map $\phi : \mathcal{M} \to \mathcal{M}'$ that is

i) Injective (one-to-one): for $p \neq q$, $\phi(p) \neq \phi(q)$;
ii) Surjective (onto): for each $p' \in \mathcal{M}'$, there exists $p \in \mathcal{M}$ such that $\phi(p) = p'$;
iii) ϕ and ϕ^{-1} are continuous.

Finally, we are able to define a manifold:

Definition 2.2.3 (Manifold)**.** An n-dimensional manifold is a topological space $\mathcal{M}$ in a manner that

a) $\mathcal{M}$ is locally homeomorphic to $\mathbb{R}^n$. For each $p \in \mathcal{M}$, there exists $O \subset \mathcal{M}$ $(p \in O)$ and a homeomorphism $\phi : O \to U$, where U is an open subset of $\mathbb{R}^n$;
b) $\phi_\alpha : O_\alpha \to U_\alpha$; $\phi_\beta : O_\beta \to U_\beta$. $O_\alpha \cap O_\beta \neq \emptyset$. It is required the map $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(O_\alpha \cap O_\beta) \to \phi_\alpha(O_\alpha \cap O_\beta)$, illustrated in Fig. (2.1), to be smooth.

We have that ϕ_α is a coordinate system in the sense that it provides the coordinates to label the points in the region O_α. If $x^\mu(p)$ are the coordinates of the point $p \in O_\alpha$,

$$\phi_\alpha(p) = (x^0, \ldots, x^{n-1}(p)) \equiv x^\mu(p). \tag{2.30}$$

Manifolds can be related to one another through maps:

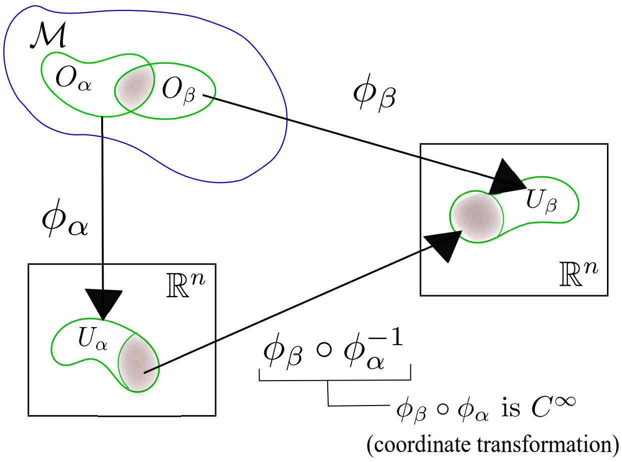

Figure 2.1: An illustration of the map $\phi_\beta \circ \phi_\alpha^{-1}$.

Definition 2.2.4 (Maps between Manifolds)**.** Let $\mathcal{M}$ and $\mathcal{M}'$ be manifolds and let $\{\phi_\alpha\}$ and $\{\phi'_\beta\}$ be the chart maps. A map $\Theta : \mathcal{M} \to \mathcal{M}'$ is smooth if, for each α and β, the map $\phi'_\beta \circ \Theta \circ \phi_\alpha{}^{-1}$ taking $U_\alpha \subset \mathbb{R}^n$ into $U_\beta{}' \subset \mathbb{R}^{n'}$ is smooth[5] for all ϕ. If a smooth map $\Theta : \mathcal{M} \to \mathcal{M}'$ is one-to-one, onto, and has a smooth inverse, then Θ is called a diffeomorphism and $\mathcal{M}$ and $\mathcal{M}'$ are called diffeomorphic manifolds, having identical manifold structure ([4], p.14). So a diffeomorphism is a smooth homeomorphism, an isomorphism between smooth manifolds.

Since tangent vectors represent changes in a given direction through differentiation, we may define the derivative of a function on a manifold in order to define vectors and dual vectors.

Definition 2.2.5 (Derivative of a Function)**.** Let $f : \mathcal{M} \to \mathbb{R}$ be a function and let $\phi \equiv x^\mu(p)$ be a coordinate system in the neighborhood of p. From this, we have $f \circ \phi^{-1} : U \to \mathbb{R}, U \subset \mathbb{R}^n$. Therefore, the derivative of a function f on $\mathcal{M}$ is[6]

$$\left.\frac{\partial f}{\partial x^\mu}\right|_p := \frac{\partial}{\partial x^\mu}\left(f \circ \phi^{-1}\right)\Big|_{\phi(p)} \quad . \tag{2.31}$$

Since we have already related tangent vectors to directional derivatives in Eq. (2.2), we can now write a more formal definition using Eq. (2.31):

Definition 2.2.6 (Tangent Vector)**.** A tangent vector X_p is an object that differentiates functions at a point $p \in \mathcal{M}$. If $\mathcal{F}$ is the set of all smooth functions on $\mathcal{M}$, then the tangent vector $X_p : \mathcal{F} \to \mathbb{R}$ satisfies

a) Linearity: $X_p(f + g) = X_p(f) + X_p(g)$, $f, g \in \mathcal{F}$;
b) $X_p(f) = 0$ if f is a constant function;
c) Leibniz Rule (analogous to the product rule): $X_p(fg) = f(p)X_p(g) + X_p(f)g(p)$, $f, g \in \mathcal{F}$.

We then define a tangent vector as

$$\partial_\mu|_p = \left.\frac{\partial}{\partial x^\mu}\right|_p \quad . \tag{2.32}$$

[5]These definitions are imported from the usual calculus on $\mathbb{R}^n$, where the notion of smoothness is well-defined. This is possible because the manifold is locally homeomorphic to $\mathbb{R}^n$, so calculus can be applied.

[6]Since we know how to differentiate functions on $\mathbb{R}^n$, the same follows for a manifold which is locally homeomorphic to $\mathbb{R}^n$.

Theorem 2.2.1 (Tangent Space)**.** The set of all tangent vectors at a point p forms an n-dimensional vector space, the tangent space T_p. The tangent vectors $\partial_\mu|_p$ provide a basis for T_p such as

$$X_p = X^\mu \, \partial_\mu|_p \,, \qquad X^\mu = X_p(x^\mu) \quad . \tag{2.33}$$

The proof to this theorem can be found in Wald, pp. 15-16 [4].

As a means to show to 'what' exactly tangent vectors are tangent to, let $\gamma : I \to \mathcal{M}$, with $I \subset \mathbb{R}$, be a curve on the manifold $\mathcal{M}$ that passes through p. The curve can be parameterized such that $\sigma(t = 0) = p \in \mathcal{M}$, as illustrated in Fig. (2.2).

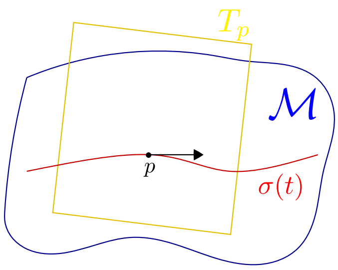

Figure 2.2: The tangent space T_p.

In a given coordinate system, we have $\phi \circ \sigma : \mathbb{R} \to \mathbb{R}^n$ parameterized by $x^\mu(t)$. From this, we can define

$$X^\mu = \left. \frac{dx^\mu(t)}{dt} \right|_{t=0} \,, \tag{2.34}$$

which implies

$$X_p = \left. \frac{dx^\mu(t)}{dt} \right|_{t=0} \left. \frac{\partial}{\partial x^\mu} \right|_p \,. \tag{2.35}$$

So a tangent vector is a directional derivative operator. When acting on a function, it provides how fast the function changes along the curve. The tangent space T_p is an intrinsic object to the manifold itself, being the space of all possible tangents to curves passing through p[7], forming a vector field:

Definition 2.2.7 (Vector Fields)**.** A vector field X is a smooth assignment of a tangent vector X_p to each point $p \in \mathcal{M}$. If $\mathcal{F}$ is the set of C^∞ functions of $\mathcal{M}$, then X is written as the map $X : \mathcal{F} \to \mathcal{F}$. The function $X(f)$ is defined as $[X(f)]\,(p) \equiv X_p(f)$. In coordinates, $X = X^\mu \frac{\partial}{\partial x^\mu}$.

And to every tangent space T_p there is an associated dual space:

Definition 2.2.8 (Dual Space)**.** At each point $p \in \mathcal{M}$, there can be defined a vector space T_p. Its dual, cotangent space T_p^* is the space of linear functionals ('cotangent vectors' or 'covectors'). If $\{\hat{e}_\mu\}$ is a basis for T_p, then $\{\hat{\theta}^\mu\}$ is a basis for T_p^* in such a way that $w = w_\mu \hat{\theta}^\mu$ for any $w \in T_p^*$.

Associated to such cotangent vectors, there can be defined a cotangent field Λ of a one-form:

[7] In general, for $p \neq q$, $T_p \neq T_q$.

Definition 2.2.9 (One-Form)**.** Let $f \in \mathcal{F}$ and define $df \in \Lambda$ by $df(X) = X(f)$. By introducing the set of coordinates x^μ on the manifold $\mathcal{M}$ and a basis ∂_μ of vector fields, we can choose $f = x^\mu$ in order to get

$$df(X) = X(f) \quad \Rightarrow \quad dx^\mu(\partial_\nu) = \partial_\nu(x^\mu) = \delta^\mu{}_\nu, \tag{2.36}$$

so dx^μ provides a basis for Λ dual to ∂_μ. In general, a one-form can be written as $\tilde{\omega} = \omega_\mu dx^\mu$. In such a basis, we have

$$df(X) = \frac{\partial f}{\partial x^\mu} dx^\mu \left(X^\nu \partial_\nu\right) = X^\mu \frac{\partial f}{\partial x^\mu} = X(f) \tag{2.37}$$

$$\Rightarrow df = \frac{\partial f}{\partial x^\mu} dx^\mu. \tag{2.38}$$

2.3 The Covariant Derivative and Parallel Transport

The partial derivative of a function f is the vector ∇f with components $(\nabla f)_\alpha = \partial f/\partial x^\alpha$. Some trouble arises when trying to define the derivative of a vector due to the fact that it involves the difference between vectors at different spacetime points. Since all operations (addition, subtraction, etc.) on vectors are defined only at one point, in order to compare two vectors at two different nearby points, we may parallel transport them to a single point [14], as illustrated in Fig. 2.3. Let p, with coordinates $\{x^\alpha\}$, be a point on flat space, and let $\vec{V}(x^\alpha)$ be a vector at p. Now consider a nearby point p', connected to p by the infinitesimal displacement $dx^\alpha = t^\alpha \varepsilon$ along the direction $\hat{t}$; p' has coordinates $\{x^\alpha + dx^\alpha\}$ and a tangent vector $\vec{V}(x^\alpha + dx^\alpha)$. To build the derivative, the vector on p' is parallel transported to itself back to p to give the vector $\vec{V}_{||}(x^\alpha)$ that is in the tangent space of p, so the subtraction from $\vec{V}(x^\alpha)$ can be performed [14].

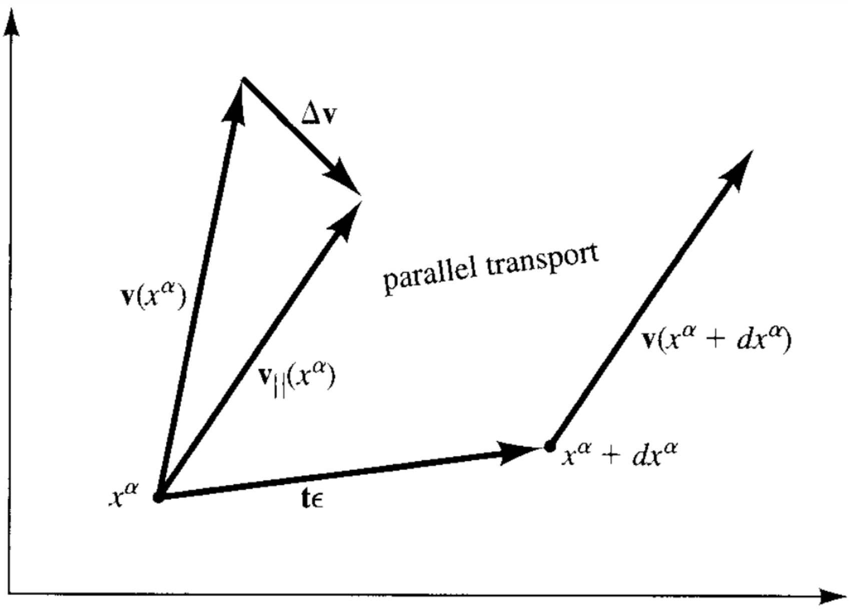

Figure 2.3: The derivative of a vector in flat space.
Illustration from [14], p. 431.

Therefore, parallel transport can also be defined on a local inertial reference frame in curved spacetimes for space is locally flat in our manifold. So the covariant derivative of a vector field $\vec{V}(x^\alpha)$ in the direction $\hat{t}$ in curved spacetime is defined as

$$\nabla_t \vec{V}\Big|_{x^\alpha} = \lim_{\varepsilon \to 0} \frac{\left[\vec{V}\left(x^\alpha + t^\alpha \varepsilon\right)\right]_{\parallel \text{ to } x^\alpha} - \vec{V}\left(x^\alpha\right)}{\varepsilon}. \tag{2.39}$$

Considering rectangular coordinates, the components V^α do not change as they are parallel transported [14]. Evaluating Eq. (2.39) is like evaluating the derivative of a function

$$(\nabla_t \vec{V})^\alpha = t^\beta \, \frac{\partial V^\alpha}{\partial x^\beta}, \tag{2.40}$$

where the tensor $\nabla \vec{V}$ has components

$$\nabla_\beta V^\alpha = \frac{\partial V^\alpha}{\partial x^\beta}. \tag{2.41}$$

But Eq. (2.41) is not valid in curvilinear coordinates[8] given that the components of a vector still change when parallel transported due to the variation on its angles with the basis vectors, pictured in Fig. 2.4.

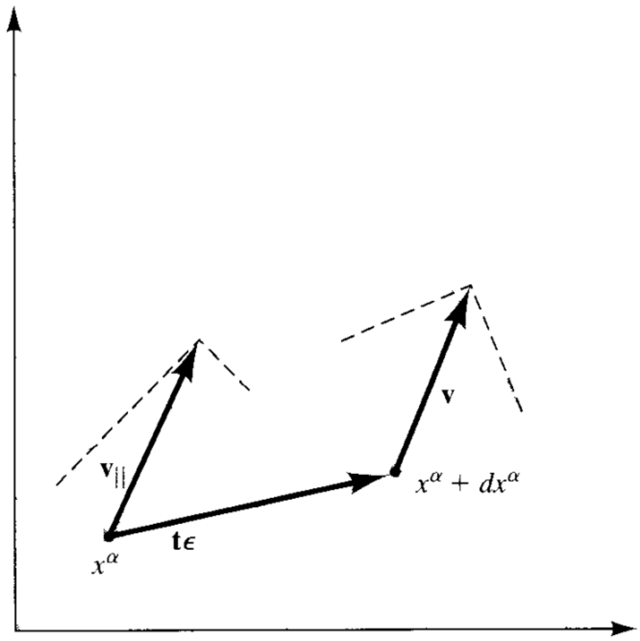

Figure 2.4: Components change under parallel transporting.
Illustration from [14], p. 432.

For example, consider that a vector $\vec{V}$ has components (V^r, V^θ) on the polar basis. Its derivatives will be given by

$$\frac{\partial \vec{V}}{\partial r} = \frac{\partial}{\partial r}\left(V^r \hat{e}_r + V^\theta \hat{e}_\theta\right) = \frac{\partial V^r}{\partial r}\hat{e}_r + V^r \frac{\partial \hat{e}_r}{\partial r} + \frac{\partial V^\theta}{\partial r}\hat{e}_\theta + V^\theta \frac{\partial \hat{e}_\theta}{\partial r}, \tag{2.42}$$

$$\frac{\partial \vec{V}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(V^r \hat{e}_r + V^\theta \hat{e}_\theta\right) = \frac{\partial V^r}{\partial \theta}\hat{e}_r + V^r \frac{\partial \hat{e}_r}{\partial \theta} + \frac{\partial V^\theta}{\partial \theta}\hat{e}_\theta + V^\theta \frac{\partial \hat{e}_\theta}{\partial \theta}; \tag{2.43}$$

[8]Since general relativity describes the intrinsic geometry of a curved spacetime, we must define derivatives on curvilinear coordinates.

and we can generalize the equations above to

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\hat{e}_\alpha + V^\alpha \frac{\partial \hat{e}_\alpha}{\partial x^\beta}. \tag{2.44}$$

By introducing the connection $\Gamma^\mu_{\alpha\beta}$, [9]

$$\frac{\partial \hat{e}_\alpha}{\partial x^\beta} = \Gamma^\mu_{\alpha\beta}\hat{e}_\mu, \tag{2.45}$$

which represent the μ-th component of $\partial \hat{e}_\alpha/\partial x^\beta$, we have that the components of the covariant derivative of $\vec{V}$, denoted by $\nabla\vec{V}$, can be written as

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\hat{e}_\alpha + V^\alpha\Gamma^\mu_{\alpha\beta}\hat{e}_\mu = \frac{\partial V^\alpha}{\partial x^\beta}\hat{e}_\alpha + V^\mu\Gamma^\alpha_{\mu\beta}\hat{e}_\alpha = \left(\frac{\partial V^\alpha}{\partial x^\beta} + V^\mu\Gamma^\alpha_{\mu\beta}\right)\hat{e}_\alpha$$

$$\Rightarrow (\nabla\vec{V})^\alpha_{\ \beta} = (\nabla_\beta\vec{V})^\alpha = \frac{\partial V^\alpha}{\partial x^\beta} + V^\mu\Gamma^\alpha_{\mu\beta} \tag{2.46}$$

In order to make it more compact, the following subscripted notation will be used

$$\frac{\partial V^\alpha}{\partial x^\beta} := V^\alpha_{\ ,\beta} \qquad \Rightarrow V^\alpha_{\ ;\beta} := V^\alpha_{\ ,\beta} + V^\mu\Gamma^\alpha_{\mu\beta} \tag{2.47}$$

and Eq. (2.46) will be written as

$$(\nabla\vec{V})^\alpha_{\ \beta} = (\nabla_\beta\vec{V})^\alpha = V^\alpha_{\ ;\beta}\ . \tag{2.48}$$

So the covariant derivative of a vector $\nabla\vec{V}$ is a $\mathbf{(1,1)}$ tensor field which maps the vector $\hat{e}_\beta$ into $\partial\vec{V}/\partial x^\beta$. Since scalars do not depend on basis vectors, the covariant derivative of a scalar ϕ is just its partial derivative (gradient) $\nabla_\alpha\phi = \partial\phi/\partial x^\alpha$, $\tilde{d}\phi = \nabla\phi$. For a one-form $\tilde{\omega}$, it can be shown that

$$(\nabla_\beta\tilde{\omega})_\alpha := (\nabla\tilde{\omega})_{\alpha\beta} := \omega_{\alpha;\beta} = \omega_{\alpha,\beta} - \omega_\mu\Gamma^\mu_{\alpha\beta}; \tag{2.49}$$

$$\nabla_\beta\left(\omega_\alpha V^\alpha\right) = \omega_{\alpha;\beta}V^\alpha + \omega_\alpha V^\alpha_{\ ;\beta}\ . \tag{2.50}$$

Additionally, for tensors of ranks $\mathbf{(2,0)}$, $\mathbf{(0,2)}$ and $\mathbf{(1,1)}$; the covariant derivative components can be found to be

$$\begin{aligned} \nabla_\beta S^{\mu\nu} &= S^{\mu\nu}_{\ \ ,\beta} + S^{\alpha\nu}\Gamma^\mu_{\alpha\beta} + S^{\mu\alpha}\Gamma^\nu_{\alpha\beta} \\ \nabla_\beta T_{\mu\nu} &= T_{\mu\nu,\beta} - T_{\alpha\nu}\Gamma^\alpha_{\mu\beta} - T_{\mu\alpha}\Gamma^\alpha_{\nu\beta} \\ \nabla_\beta P^\mu_{\ \nu} &= P^\mu_{\ \nu,\beta} + P^\alpha_{\ \nu}\Gamma^\mu_{\alpha\beta} - P^\mu_{\ \alpha}\Gamma^\alpha_{\nu\beta}. \end{aligned} \tag{2.51}$$

Such properties can all be summed up in the following definition:

Definition 2.3.1 (Covariant Derivative)**.** A derivative operator (covariant derivative) ∇ on a manifold $\mathcal{M}$ is a map $\nabla : \mathscr{T}^r_{\ s} \to \mathscr{T}^r_{\ s+1}$, where $\mathscr{T}^k_{\ l}$ are smooth tensor fields, and satisfies [4]:

[9] They represent what is called the connection since they are used to transport vectors from one tangent space to another, providing the 'connection' between both spaces.

1) Linearity: For all $\boldsymbol{T},\ \boldsymbol{S} \in \mathscr{T}^k_l$ and $a,\ b \in \mathbb{R}$,

$$\nabla_\mu \left(aT^{\alpha_1\cdots\alpha_{k_1}}{}_{\beta_1\cdots\beta_l} + bS^{\alpha_1\cdots\alpha_k}{}_{\beta_1\cdots\beta_l}\right) = a\nabla_\mu T^{\alpha_1\cdots\alpha_{k_1}}{}_{\beta_1\cdots\beta_l} + b\nabla_\mu S^{\alpha_1\cdots\alpha_k}{}_{\beta_1\cdots\beta_l}.$$

2) Leibnitz Rule: For all $\boldsymbol{T} \in \mathscr{T}^k_l, \boldsymbol{F} \in \mathscr{T}^{k'}_{l'}$,

$$\nabla_\upsilon \left[T^{\alpha_1\cdots\alpha_k}{}_{\beta_1\cdots\beta_l} F^{\mu_1\cdots\mu_{k'}}{}_{\nu_1\cdots\nu_{l'}}\right] = \nabla_\upsilon \left[T^{\alpha_1\cdots\alpha_k}{}_{\beta_1\cdots\beta_l}\right] F^{\mu_1\cdots\mu_{k'}}{}_{\nu_1\cdots\nu_{l'}} + T^{\alpha_1\cdots\alpha_k}{}_{\beta_1\cdots\beta_l} \left[\nabla_\upsilon F^{\mu_1\cdots\mu_{k'}}{}_{\nu_1\cdots\nu_{l'}}\right].$$

3) Commutativity with contraction: For all $\boldsymbol{T} \in \mathscr{T}^k_l$,

$$\nabla_\mu \left(T^{\alpha_1\cdots\gamma\cdots\alpha_k}{}_{\beta_1\cdots\gamma\cdots\beta_l}\right) = \nabla_\mu T^{\alpha_1\cdots\gamma\cdots\alpha_k}{}_{\beta_1\cdots\gamma\cdots\beta_l}.$$

4) Consistency with the definition of tangent vectors as directional derivatives:

For all functions $f \in \mathscr{F}$ and all tangent vectors $t^\alpha \in T_p$,

$$\vec{t}(f) = t^\alpha \nabla_\alpha f.$$

Now, we impose that the connection is compatible with the metric, which means that

$$g_{\alpha\mu;\beta} \equiv 0. \tag{2.52}$$

This also leads to a single connection in this space, which is symmetric

$$\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}. \tag{2.53}$$

The components of the connection are called the Christoffel symbols. After unraveling the terms on Eq. (2.52) using Eq. (2.47) and through some permutations of the indices, we can see that the Christoffel symbols are given by ([10], p. 134)

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right). \tag{2.54}$$

The covariant derivative of a tensor quantifies the instantaneous rate of change of a tensor field in comparison to what it would be if it were parallel transported. Parallel transport is a way of keeping a vector constant while transporting it along a certain path. Given a vector field $\vec{V}$ defined on a sphere, if vectors $\vec{V}$ at infinitesimally close points of the curve are parallel and of equal length, then $\vec{V}$ is said to be parallel transported along the curve. So in a locally inertial reference frame at a point p, along a curve $x^\alpha(\lambda)$ with the tangent $\vec{U} = d\vec{x}/d\lambda$,

$$\left.\frac{dV^\alpha}{d\lambda}\right|_p = \left.\frac{dV^\alpha}{dx^\beta}\frac{dx^\beta}{d\lambda}\right|_p = U^\beta V^\alpha{}_{,\beta} = 0\ . \tag{2.55}$$

And, since $\Gamma^\alpha_{\mu\nu} = 0$ at p in this frame, we can write Eq. (2.55) as

$$U^\beta V^\alpha{}_{,\beta} = U^\beta V^\alpha{}_{;\beta} = 0. \tag{2.56}$$

As $U^\beta V^\alpha{}_{;\beta} = 0$ is valid in any reference frame, a frame-invariant definition of parallel transport can be written.

Definition 2.3.2 (Parallel transport)**.** Given a curve $x^\alpha(\lambda)$ and its tangent $\vec{U} = d\vec{x}/d\lambda$, the parallel transport of $\vec{V}$ in the direction of $\vec{U}$ is defined by the condition

$$U^\beta V^\alpha{}_{;\beta} = 0 \Leftrightarrow \frac{d}{d\lambda}\vec{V} = \nabla_{\vec{U}}\vec{V} = 0. \text{ [10]} \tag{2.57}$$

[10]Notice that parallel transport is a path dependent concept, so there is no unique way of comparing two vectors at distinct points.

A very important type of curve is a straight line. In flat spacetime, two straight lines that are initially parallel remain parallel when extended, which is, the tangent to the curve at one point is parallel to the tangent at the previous point [10]. On curved spacetimes, local straight lines are those which parallel-transport their own tangent vector, called geodesics:

$$\nabla_{\vec{U}}\vec{U} = 0 \Rightarrow U^\beta U^\alpha{}_{;\beta} = U^\beta U^\alpha{}_{,\beta} + \Gamma^\alpha_{\mu\beta}U^\mu U^\beta = 0\,. \tag{2.58}$$

With a curve $x^\alpha(\lambda)$, we can write $U^\alpha = \frac{dx^\alpha}{d\lambda}$ and $U^\beta\frac{\partial}{\partial x^\beta} = \frac{d}{d\lambda}$, then Eq. (2.58) becomes

$$\nabla_{\vec{U}}\vec{U} = \frac{d}{d\lambda}\left(\frac{dx^\alpha}{d\lambda}\right) + \Gamma^\alpha_{\mu\beta}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda} = 0, \tag{2.59}$$

which is called the geodesic equation. It is a non-linear set of equations on $\{x^\alpha\}$, but that has a unique solution when initial conditions are given at $\lambda = \lambda_0$, so that the initial coordinates $x_0^\alpha = x^\alpha(\lambda_0)$ and the initial directional velocity $U_0^\alpha = (dx^\alpha/d\lambda)_{\lambda_0}$ define a unique geodesic. For a new parameter obtained through a linear transformation (an affine parameter), there is a new and unique geodesic. It is possible to show that the geodesic is a curve of extremal length ([16], pp.131-133) between any two points.

2.4 Defining Curvature — The Riemann Curvature Tensor

In flat spacetime, the parallel transport of a vector along a curve leaves it unchanged and the covariant derivative of tensors commute. Furthermore, initially parallel geodesics remain parallel. Given that a vector changes basis coordinates while it is parallel transported on spherical coordinates, it might be possible to describe a curved space by defining curvature at each point. Let us consider an infinitesimal loop, so that the result of parallel transport of a vector along the loop depends on the enclosed curvature. This loop, represented by Fig. 2.5, has its sides defined by the coordinate lines $x^1 = a$, $x^1 = a + \delta a$, $x^2 = b$, and $x^2 = b + \delta b$.

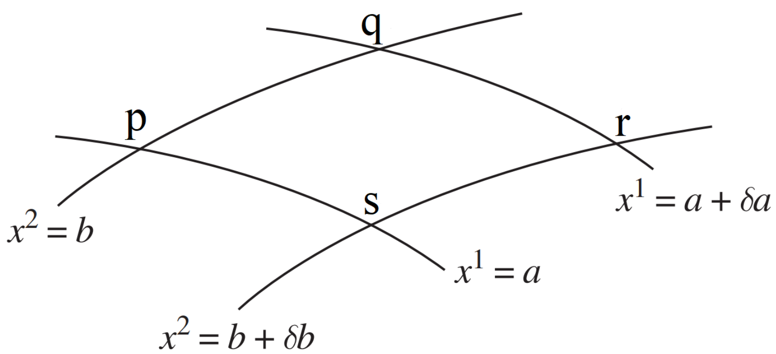

Figure 2.5: A small closed loop.

If a vector defined at the point p is parallel-transported to point q, $\nabla_{\hat{e}_1}\vec{V} = 0$ has components $\frac{\partial V^\alpha}{\partial x^1} = -\Gamma^\alpha_{\mu 1}V^\mu$. Integrating it from p to q,

$$\int_p^q \frac{\partial V^\alpha}{\partial x^1}dx^1 = V^\alpha(q) - V^\alpha(p) \quad \Rightarrow \quad V^\alpha(q) = V^\alpha(p) - \int_{x^2=b}\Gamma^\alpha_{\mu 1}V^\mu dx^1, \tag{2.60}$$

where the notation '$x^2 = b$' under the integral denotes the path pq. Similarly, integrating from q to r and from r to s, we have

$$V^\alpha(r) = V^\alpha(q) - \int_{x^1=a+\delta a} \Gamma^\alpha_{\mu 2} V^\mu dx^2 \,, \quad V^\alpha(s) = V^\alpha(r) + \int_{x^2=b+\delta b} \Gamma^\alpha_{\mu 1} V^\mu dx^1 \,, \tag{2.61}$$

$$\Rightarrow V^\alpha(p_{final}) = V^\alpha(s) + \int_{x^1=a} \Gamma^\alpha_{\mu 2} V^\mu dx^2. \tag{2.62}$$

Then the net change on the vector $V^\alpha(p)$ is

$$\begin{aligned}\delta V^\alpha = V^\alpha(p_{final}) - V^\alpha(p_{initial}) = &\int_{x^1=a} \Gamma^\alpha_{\mu 2} V^\mu dx^2 + \int_{x^2=b+\delta b} \Gamma^\alpha_{\mu 1} V^\mu dx^1 \\ &- \int_{x^1=a+\delta a} \Gamma^\alpha_{\mu 2} V^\mu dx^2 - \int_{x^2=b} \Gamma^\alpha_{\mu 1} V^\mu dx^1 \,. \end{aligned} \tag{2.63}$$

The terms in Eq. (2.63) would cancel in pairs if $\Gamma^\alpha_{\mu\beta}$ and V^μ were constants on the loop (as they would be in flat space). But in curved space they are not, so we can combine the integrals over similar integration variables and work to first order in the separation in the paths:

$$\begin{aligned}\delta V^\alpha &= \int_b^{b+\delta b} \left(\Gamma^\alpha_{\mu 2} V^\mu - \Gamma^\alpha_{\mu 2} V^\mu\right) dx^2 - \int_a^{a+\delta a} \left(\Gamma^\alpha_{\mu 1} V^\mu - \Gamma^\alpha_{\mu 1} V^\mu\right) dx^1 \\ &\cong \int_b^{b+\delta b} \left[\delta_a \frac{\partial}{\partial x^1} \left(\Gamma^\alpha_{\mu 2} V^\mu\right)\Big|_a - \delta_a \frac{\partial}{\partial x^1} \left(\Gamma^\alpha_{\mu 2} V^\mu\right)\Big|_a\right] dx^2 - \\ &\quad - \int_a^{a+\delta a} \left[\delta b \frac{\partial}{\partial x^1} \left(\Gamma^\alpha_{\mu 1} V^\mu\right)\Big|_b - \delta b \frac{\partial}{\partial x^2} \left(\Gamma^\alpha_{\mu 1} V^\mu\right)\Big|_b\right] dx^1 \end{aligned} \tag{2.64}$$

$$\Rightarrow \delta V^\alpha \cong - \int_b^{b+\delta b} \delta a \frac{\partial}{\partial x^1} \left(\Gamma^\alpha_{\mu 2} V^\mu\right) dx^2 + \int_a^{a+\delta a} \delta b \frac{\partial}{\partial x^2} \left(\Gamma^\alpha_{\mu 1} V^\mu\right) dx^1 \,. \tag{2.65}$$

Notice that we get to lowest order in Eq. (2.65) by only considering the negative terms inside the integrals from the previous step. And since we are working with such small quantities,

$$\int_a^{a+\delta a} f(x) dx \approx \delta a f(x) \quad \Rightarrow \quad \delta V^\alpha \approx \delta a \delta b \left[-\frac{\partial}{\partial x^1} \left(\Gamma^\alpha_{\mu 2} V^\mu\right) + \frac{\partial}{\partial x^2} \left(\Gamma^\alpha_{\mu 1} V^\mu\right)\right] \,. \tag{2.66}$$

Using $V^\alpha{}_{,b} = -\Gamma^\alpha_{\mu\beta} V^\mu$, we find $\left(\Gamma^\alpha_{\nu\lambda} V^\nu\right)_{,\beta} = \Gamma^\alpha_{\nu\lambda,\beta} V^\nu - \Gamma^\alpha_{\nu\lambda} \Gamma^\nu_{\mu\beta} V^\nu$. Substituting it into Eq. (2.66), and relabeling dummy indices, we get

$$\begin{aligned}\delta V^\alpha &\approx \delta a \delta b \left[-\Gamma^\alpha_{\mu 2,1} + \Gamma^\alpha_{\nu 2} \Gamma^\nu_{\mu 1} + \Gamma^\alpha_{\mu 1,2} - \Gamma^\alpha_{\nu 1} \Gamma^\nu_{\mu 2}\right] V^\mu \\ \Rightarrow \delta V^\alpha &= \delta a \delta b \left[\Gamma^\alpha_{\mu\sigma,\lambda} - \Gamma^\alpha_{\mu\lambda,\sigma} + \Gamma^\alpha_{\nu\lambda} \Gamma^\nu_{\mu\sigma} - \Gamma^\alpha_{\nu\sigma} \Gamma^\nu_{\mu\lambda}\right] V^\mu, \end{aligned} \tag{2.67}$$

for general coordinate lines x^σ and x^λ. By looking at the loop in Fig. 2.5, it is noticeable that the infinitesimal displacement along the loop will first be $\delta a \vec{e}_\sigma$, then $\delta b \vec{e}_\lambda$, then $-\delta a \vec{e}_\sigma$ followed by $-\delta b \vec{e}_\lambda$. The change on $\vec{V}$, δV^α, depends on the area of the loop $\delta a \delta b$, which means that δV^α is linear on $\delta a \vec{e}_\sigma$ and $\delta b \vec{e}_\lambda$, also being linear to V^α itself and the basis one-form $\tilde{\omega}^\alpha$ that gives δV^α from $\vec{V}$. Therefore, to define a tensor that gives the components of the net change of a vector after being parallel-transported around a loop, it is required to be a $(1, 3)$ tensor that, when supplied with arguments $\tilde{\omega}^\alpha$, $\vec{V}$, $\delta a \vec{e}_\mu$ and $\delta b \vec{e}_\nu$ gives δV^α. By looking at Eq. (2.67), such a tensor can be defined to be

$$R^\alpha{}_{\beta\mu\nu} := \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu} \Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\beta\mu} \,. \tag{2.68}$$

Since the curved aspect of spacetime is shown through it, it is called the Riemann curvature tensor $\boldsymbol{R}$. By using Eq. (2.54) we can write Eq. (2.68) in terms of the metric components in a locally inertial reference frame ([10], p. 159)

$$R^{\alpha}{}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu} \right), \tag{2.69}$$

and also obtain the lowered-index components $R_{\alpha\beta\mu\nu}$

$$\begin{aligned} R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}{}_{\beta\mu\nu} &= g_{\alpha\lambda} \frac{1}{2} g^{\lambda\sigma} \left(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu} \right) \\ \Rightarrow R_{\alpha\beta\mu\nu} &= \frac{1}{2} \left(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu} \right), \end{aligned} \tag{2.70}$$

to find, directly from Eq. (2.70), some useful tensor identities

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}, \tag{2.71}$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0. \tag{2.72}$$

Equations (2.71) and (2.72) show us that $R_{\alpha\beta\mu\nu}$ is antisymmetric in both first and second pair of indices, but it is symmetric in the pair exchange. Since $R_{\alpha\beta\mu\nu}$ characterizes the curvature of the manifold, a flat manifold demands

$$R^{\alpha}{}_{\beta\mu\nu} = 0. \tag{2.73}$$

Which means that, at a point p in a locally inertial reference frame (where spacetime is perceived to be flat), in cartesian coordinates, the covariant derivative is the same as the usual partial derivative. Such relation is very useful to our work since it simplifies the algebra and we can find, from an specific frame, general tensor equations valid in any frame. Let us see this in action by exploring the curvature tensor below. In a local inertial reference frame, we can differentiate Eq. (2.70) with respect to x^{λ}

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2} \left(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda} \right). \tag{2.74}$$

Using the symmetry $g_{\alpha\beta} = g_{\beta\alpha}$ and by permuting μ, ν and λ ciclically, we get that

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0. \tag{2.75}$$

On this specific frame, the usual partial derivative is the same as the covariant derivative, so Eq. (2.75) is equivalent to

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0\ , \tag{2.76}$$

which are called the Bianchi identities, a tensor equation valid in any reference frame. And from it, we can obtain the only independent contraction of the Riemann tensor([16], pp. 138-139) called the Ricci tensor:

$$R_{\alpha\beta} := R^{\mu}{}_{\alpha\mu\beta} = R_{\beta\alpha}, \tag{2.77}$$

an $(0, 2)$ symmetric tensor. By contracting the Ricci tensor with the metric we obtain the Ricci scalar

$$R := g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} g^{\alpha\beta} R_{\alpha\mu\beta\nu}. \tag{2.78}$$

If we contract the first and third indices of the Bianchi identities given in Eq. (2.76), we get

$$g^{\alpha\mu}\left[R_{\alpha\beta\mu\nu;\lambda}+R_{\alpha\beta\lambda\mu;\nu}+R_{\alpha\beta\nu\lambda;\mu}\right]=0$$
$$\Rightarrow R_{\alpha\beta\mu\nu}g^{\alpha\mu}{}_{;\lambda}+g^{\alpha\mu}R_{\alpha\beta\mu\nu;\lambda}+R_{\alpha\beta\lambda\mu}g^{\alpha\mu}{}_{;\nu}+g^{\alpha\mu}R_{\alpha\beta\lambda\mu;\nu}+R_{\alpha\beta\nu\lambda}g^{\alpha\mu}{}_{;\mu}+g^{\alpha\mu}R_{\alpha\beta\nu\lambda;\mu}=0\ . \tag{2.79}$$

Given the demand from Eq. (2.52) on the metric components and since $g^{\alpha\mu}$ is a function of $g_{\alpha\mu}$, we may also extend Eq. (2.52) to $g^{\alpha\beta}{}_{;\mu}=0$; which means that both components of the metric and its inverse can be taken in and out of covariant derivatives at will. Furthermore, if we take

$$g^{\alpha\mu}R_{\alpha\beta\lambda\mu;\nu}=-g^{\alpha\mu}R_{\alpha\beta\mu\lambda;\nu}=-R_{\beta\lambda;\nu} \tag{2.80}$$

along with the covariant derivative of Eq. (2.71)

$$R_{\alpha\beta\lambda\mu}=-R_{\alpha\beta\mu\lambda}\quad\Rightarrow\quad R_{\alpha\beta\lambda\mu;\nu}=-R_{\alpha\beta\mu\lambda;\nu}, \tag{2.81}$$

the contracted Bianchi identities on Eq. (2.79) are

$$R_{\beta\nu;\lambda}-R_{\beta\lambda;\nu}+R^{\mu}{}_{\beta\nu\lambda;\mu}=0. \tag{2.82}$$

Contracting again on both second and fourth indices results

$$g^{\beta\nu}\left[R_{\beta\nu;\lambda}-R_{\beta\lambda;\nu}+R^{\mu}{}_{\beta\nu\lambda;\mu}\right]=0\quad\Rightarrow\quad R_{\ ;\lambda}-R^{\mu}{}_{\lambda;\mu}-R^{\mu}{}_{\lambda;\mu}=0.$$

Since R is a scalar, $R_{;\lambda}\equiv R_{,\lambda}$ for all coordinates, so we have the twice-contracted Bianchi identities to be

$$\left(2R^{\mu}{}_{\lambda}-\delta^{\mu}{}_{\lambda}R\right)_{;\mu}=0\ . \tag{2.83}$$

If we define the symmetric tensor $\boldsymbol{G}$ in terms of the Riemann tensor and the metric [11] so that

$$G^{\alpha\beta}\equiv R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}R=G^{\beta\alpha}\ , \tag{2.84}$$

Eq. (2.83) is equivalent to

$$G^{\alpha\beta}{}_{;\beta}=0\ . \tag{2.85}$$

The tensor $\boldsymbol{G}$ is called the Einstein tensor since its fundamental importance to describe gravity, being related to the source of the field, was firstly noticed by Albert Einstein [12]. We can now properly discuss the main concepts of general relativity, leading to the formulation of Einstein's field equations.

[11] $G^{\alpha\beta}$ is divergence-free as an identity since it is derived from the metric and the curvature tensor

Chapter 3

General Relativity

In the Newtonian interpretation, space and time are universal and absolute entities. The theory's incompatibility with Maxwell's electromagnetism and the concept of action at a distance required some changes for a new theory to arise. Then special relativity, based on the fundamental assumptions that the laws of physics are the same in all inertial frames and that the speed of light in vacuum is constant, brought coherence to the issue. But still, gravity was out of the picture. Now we must explore general relativity: the generalization of special relativity, a geometrical theory of gravitation that refines Newton's law of gravity and provides a unified description of gravity as a geometric property of space and time (spacetime). We will first review some important definitions from the previous differential geometry chapter (such as defining space as a manifold) giving some physical meaning to them. Then we present Einstein's equivalence principle, which is the generalized assumption of how physical properties of spacetime are presented locally on inertial reference frames, also establishing gravity's similarity to local acceleration. Subsequently, we focus more on the conservation of some important physical quantities, such as matter density, energy and momentum fluxes; translated as tensors, which are invariant quantities and can be generalized to non-inertial reference frames. Such study will be done exploring fluid systems due to its convenience in representing energy-matter systems and its properties. After defining the fundamental stress-energy tensor, we will be able to establish a general relativity analogue to the Newtonian equation of the gravitational field, the Einstein's field equations, which are the ones we will solve in the next chapter for the boundary conditions of the whole Universe. In order to do so, we consider the Universe as a dynamic perfect fluid, changing through time. Therefore, in this chapter, we will lastly develop Einstein's equations for static perfect fluids in order to then find the analogues to dynamic fluids in our Cosmological Solution chapter.

3.1 Fundamental Concepts and Assumptions

The greater desire to understand and mathematically describe bodies in motion on non-inertial reference frames, under the influence of a gravitational field, made the use of differential geometry necessary, since moving from an inertial reference frame to a non-inertial one is similar to moving from cartesian coordinates to spherical coordinates. On this sense, spacetime (the set of all events) is a four-dimensional manifold with a metric. Such metric is locally measurable by rods and clocks and, as we know from special relativity, the distance along a rod between two nearby points is $|d\vec{x} \cdot d\vec{x}|^{\frac{1}{2}}$ and the time measured by a clock that experiences two events closely separated in time (proper time) is $|-d\vec{x}\cdot d\vec{x}|^{\frac{1}{2}}$ (more about special relativity in [10]). Any metric of special relativity can be put in the Minkowski form $\eta_{\alpha\beta}$ at any particular event by an appropriate choice of coordinates since, as being described by a manifold, is locally flat (homeomorphic to $\mathbb{R}^n$). To describe accelerated reference frames, inertial reference frames may be defined. Experimentation shows that freely falling particles all fall at the same rate in a gravitational field, regardless of their masses. They 'do not feel' the acceleration of gravity and move on a locally 'straight line', recognized as locally inertial reference frames. The weak equivalence principle expands this statement to a curved manifold, saying that freely falling particles move on timelike geodesics of spacetime. This can be generalized to any body of mass through Einstein's equivalence principle: any local physical experiment performed in a freely falling frame will have the same result as if it were performed in flat spacetime (special relativity). This statement is highly significant. Every free fall experiment has the same results as it would for an observer at rest or moving uniformly in deep space, far from all sources of gravity, also implying that effects which seem the same as those of gravity can be produced by an accelerated frame. An observer in a closed room cannot tell if objects are falling to the floor because the room is resting on the surface of the Earth or if the room is aboard a rocket in space which is accelerating at $9.81\,m/s^2$. Conversely, any effect observed in an accelerated frame should also be locally observed in a corresponding gravitational field.[1] It is also important to notice that what we are measuring and how we are measuring it will tell us what we should consider in our reference frame. Gravity's influence may be disregarded locally if our experiment demands less precision, but we may consider its effects otherwise. It all sums up to the fact that there is no such thing as a globally inertial reference frame, but the equivalence principle tells us that inertial reference frames, where space 'looks like' flat, can be locally found. And by recognizing it and giving our measurements tensorial equivalent quantities, some results found on such frames can be generalized to non-inertial reference frames. With that in mind, let us define matter, energy and momentum in terms of a tensor, generally describing the source of a gravitational field. And that can be suitably done through the study of fluids.

3.2 Conservation Laws in General Relativity

3.2.1 The Stress-Energy Tensor

A fluid can be defined as a flowing continuum. A continuum is a collection of numerous particles in which their individual dynamics cannot be specified, only measurements of the average properties of its elements are performed. A fluid element is a big enough collection of particles such that individual particles cannot be singled out, but also small enough so that its properties are all homogeneous. Therefore, in any point of the element, particles may present same average speed, same average kinetic energy and spacing [10]. For each fluid element, we go to the frame in which it is momentarily at rest (its total spatial momentum is zero), called the momentarily comoving reference frame (MCRF). It

[1]This principle allowed Einstein to predict several novel effects of gravity in 1907, such as the observed gravitational shift of light and tidal forces.

is specific to a single fluid element and Newtonian notions such as that one of ’force’ are valid. All scalar quantities associated with a fluid element in relativity (such as number density, energy density, and temperature) are defined to be their values in the MCRF [10]. Something that ’flows’ has weak forces parallel to the interface between its elements (’antislipping’ forces) when compared to pressure (direct ’push-pull’ force). The fluid which seems to be at rest is called dust, which is also the simplest fluid description. For dust, in a frame $\mathcal{O}$, the number density n (which is the number of particles per element of volume) is $n = N/(\Delta x \Delta y \Delta z)$. In a frame $\overline{\mathcal{O}}$ moving with constant velocity $\vec{v}$ in $\hat{x}$ relative to $\mathcal{O}$, the number density becomes $\overline{n} = N/(\sqrt{1-v^2}\Delta x \Delta y \Delta z) = N/(\Delta \overline{x} \Delta \overline{y} \Delta \overline{z})$, as shown in Fig. 3.1.

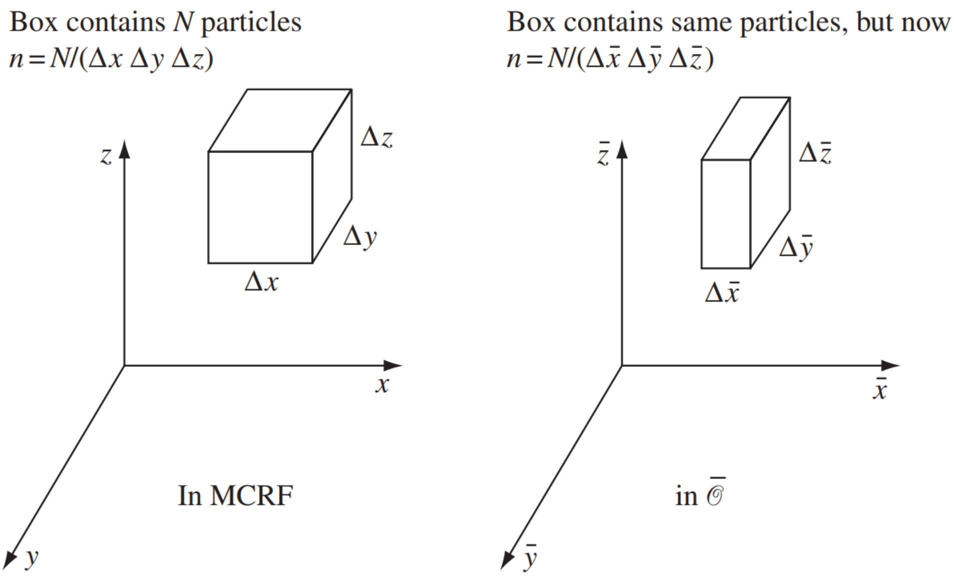

Figure 3.1: Lorentz contraction of fluid elements.
Illustration inspired by [10], p. 86.

In the rest frame, the particle flux is null, but in $\overline{\mathcal{O}}$, since we have that the volume is $\Delta y \Delta z \Delta \overline{t} = \Delta A \Delta \overline{t}$ and $N = nv\Delta y \Delta z \Delta \overline{t}/\sqrt{1-v^2}$, the particle flux in $\overline{x}$ is

$$(flux)^{\overline{x}} = \frac{nv\Delta \overline{t} \Delta A}{\sqrt{1-v^2}\Delta A \Delta \overline{t}} \quad \Rightarrow \quad (flux)^{\overline{x}} = \frac{nv}{\sqrt{1-v^2}} \,. \tag{3.1}$$

From this equation, we can define the number-flux four-vector (frame-independent)

$$\vec{N} = n\vec{U} \,, \quad \Rightarrow \quad \vec{N} \underset{\overline{\mathcal{O}}}{\rightarrow} \left(\frac{n}{\sqrt{1-v^2}}, \frac{nv^x}{\sqrt{1-v^2}}, \frac{nv^y}{\sqrt{1-v^2}}, \frac{nv^z}{\sqrt{1-v^2}} \right) , \tag{3.2}$$

where $\vec{U}$ is the four-velocity of the particles in $\overline{\mathcal{O}}$.[2] Note that $\vec{N}.\vec{N} = -n^2$ and n is a scalar, called the rest density, such that $n = (\vec{N}.\vec{N})^{\frac{1}{2}}$. If a surface is a solution to some equation such as $\phi(t, x, y, z) = const.$, then the one-form gradient $\tilde{d}\phi$ defines the surface $\phi = const.$ because it uniquely determines normal directions to the surface. Then we define the unit-normal one-form to be $\tilde{n} := \tilde{d}\phi/|\tilde{d}\phi|$, where the magnitude of $\tilde{d}\phi$ is $|\tilde{d}\phi| = |\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}|^{1/2}$. With that in mind, let ϕ

[2] In Newton, the number density was a scalar, invariant in all frames, but the flux was a frame-dependent three-vector. Now, there is only one four-vector for both quantities.

be a coordinate like $\overline{x}$. Then a surface of constant $\overline{x}$ has a unit-normal one-form $\tilde{d\overline{x}} \underset{\overline{\mathcal{O}}}{\rightarrow} (0,1,0,0)$. Therefore, the flux of particles across a surface $\overline{x} = \ const.$ is $\langle \tilde{d\overline{x}}, \vec{N} \rangle = N^{\alpha}(\tilde{d\overline{x}})_{\alpha} = N^{\overline{x}}$. If we choose $\phi = \overline{t}$, then we would simply have $N^{\overline{0}}$ which is the number density $\overline{n}$. So the number density, the number of particles contained in a unit volume at a given time, is the same as a timelike flux; the flux of particles across a surface of constant $\overline{t}$.

Now, let us represent energy and momentum in an invariant manner using tensors. An inertial frame can be defined by a one-form associated to its four-velocity $\boldsymbol{g} = (\vec{U},\ )$, having components $U_{\alpha} = \eta_{\alpha\beta}U^{\beta}$. In $\mathcal{O}$, it has components $U_0 = -1,\ U_i = 0$, which is equal to $-\tilde{dt}$, so we could similarly define a frame using $\tilde{dt}$. It may be pictured as a set of surfaces of $t = const.$ called the surfaces of simultaneity. So the energy of a particle whose four-momentum is $\vec{p}$ is $E := \langle \tilde{dt}, \vec{p} \rangle = p^0 = m$ [10]. The number density is simply n, so the energy density is $\rho = mn$. In $\overline{\mathcal{O}}$,

$$\overline{\rho} = \frac{m}{\sqrt{1-v^2}} . \frac{n}{\sqrt{1-v^2}} = \frac{mn}{1-v^2} = \frac{\rho}{1-v^2}. \tag{3.3}$$

This transformation involves two factors of $\Lambda^{\overline{0}}{}_{0} = (1-v^2)^{1/2}$, which indicates it is a component of a $\mathbf{(2,0)}$ tensor. Let us develop this statement. The definition of energy density requires two one-forms since the 'energy' part requires a one-form to select the zero component of the momentum four-vector and the 'density' part is defined through another one-form because it is a flux through a constant-time surface. Momentum density also involves two one-forms, one for density and another one for which momentum component that will be used. Additionally, momentum flux will also need two one-forms. We are talking about these quantities because, in general relativity, we cannot consider only mass to be the whole source of the gravitational field, we must take into account the 'bigger picture'. Mass in a system may present a flux, and since relativity relates energy to mass, it consequently may have an energy flux as well. Therefore, considering both matter and energy can offer us the opportunity to seek a more general description of the gravitational field and we are able to 'merge' those two things together by defining a tensor that offers all these quantities when supplied with the right arguments. And it must be a $\mathbf{(2,0)}$ tensor. Such essential, fundamental tensor is called the stress-energy tensor $\boldsymbol{T}$:

$$\mathbf{T}\left(\tilde{\mathrm{d}}x^{\alpha}, \tilde{\mathrm{d}}x^{\beta}\right) = T^{\alpha\beta} := \left\{ \begin{array}{c} \text{flux of the } \alpha \text{ component of four-momentum} \\ p^{\alpha} := \langle \tilde{d}x^{\alpha}, \vec{p} \rangle \text{ across a surface of constant } x^{\beta} \end{array} \right\}. \tag{3.4}$$

Let us make its significance clearer by detailing the different components of this tensor describing a system in the MCRF, in which Newton's laws are valid. The component T^{00} would be the flux of energy (zero momentum) across a surface of constant t, which is equivalent to the energy density of the system, same as its mass density. Therefore, T^{0i} is the energy flux across a surface $x^i = \ const.$ T^{i0} is the flux of i momentum (i component of the four-momentum p^{α}) on a surface $t = \ const.$, which is just the system's momentum density. And lastly, T^{ij} is the flux of i momentum across j surface. Momentum flux is called the stress in a system. Since force is the rate of change of momentum, momentum flux is basically force per unit area, which is the same as pressure. If we imagine a mass system at rest and divide it into several adjacent mass elements, T^{ij} represents forces between them. If such forces are perpendicular to the surfaces separating the elements, T^{ij} will be zero unless $i = j$.

By its definition, $\boldsymbol{T}$ is a symmetric tensor,

$$T^{\alpha\beta} = T^{\beta\alpha}, \tag{3.5}$$

which can be verified by proving such symmetry in only one frame (like the simplest MCRF) and extending it to all frames ([10], pp. 97-98).

3.2.2 Conserved Quantities

The stress-energy tensor does more than just generalizing mass density. The Einstein equivalence principle demands space to be locally flat, so that energy and momentum need to be locally conserved. Demanding that a quantity must be conserved is the same as requiring all its fluxes to be symmetrical. And since the stress-energy tensor contains all fluxes of matter, energy and momentum, the general conservation law in the MCRF can be written as ([10], pp. 98-99)

$$T^{\alpha\beta}{}_{,\beta} = 0 \,, \tag{3.6}$$

so that giving the components of $\boldsymbol{T}$ in a certain frame completely defines its state. We can generalize this equation to any frame by using the covariant derivative

$$T^{\alpha\beta}{}_{;\beta} = 0 \,. \tag{3.7}$$

Furthermore, the conservation of particles can be expressed as

$$\frac{\partial N^0}{\partial t} = -\frac{\partial N^x}{\partial x} - \frac{\partial N^y}{\partial y} - \frac{\partial N^z}{\partial z} \quad \Rightarrow \quad N^{\alpha}{}_{,\alpha} = (nU^{\alpha})_{,\alpha} = 0. \tag{3.8}$$

And since the covariant derivative is the same as the partial derivative in this frame, Eq. (3.8) can be written as

$$(nU^{\alpha})_{;\alpha} = 0, \tag{3.9}$$

valid in all frames, as implied by the Einstein equivalence principle, generalized to a curved spacetime.

Perfect Fluids

In Chapter 4, we will treat the Universe as a perfect fluid, so we might as well define it. A perfect fluid would be that of a 'perfect flow', where there is only pressure acting on its neighboring elements. In the MCRF, it has no viscosity and no heat conduction. No heat conduction means that energy can flow only if particles flow and the definition of $\boldsymbol{T}$ translates this statement as $T^{0i} = T^{i0} = 0$. Since viscosity is a force parallel to the interface between particles, no viscosity means there are only perpendicular forces to the interface[3], which leads to $T^{ij} = 0$ unless $i = j$. Therefore, the matrix representation of T^{ij} should be a diagonal matrix. And since T^{ij} is a force F^i per unit area dA^j, that is the same as the pressure p, the general form of T^{ij} is

$$T^{ij} = p\delta^i{}_j \qquad \Rightarrow \qquad \left(T^{\alpha\beta}\right) = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{3.10}$$

[3]Forces in the x direction will only cross an x surface, the same for y and z.

which can be simplified as

$$\begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} = \begin{pmatrix} \rho+p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{3.11}$$

$$\Rightarrow T^{\alpha\beta} = (\rho+p)(\vec{e_0}\vec{e_0})^{\alpha\beta} + p\eta^{\alpha\beta}. \tag{3.12}$$

The unit vector in the time direction $\vec{e_0}$, in the MCRF of the fluid element, is just its four-velocity $\vec{U}$. So $(\vec{e_0}\vec{e_0})^{\alpha\beta} = (\vec{U}\vec{U})^{\alpha\beta} = U^\alpha U^\beta$. Hence we have that the stress-energy tensor of a perfect fluid is

$$T^{\alpha\beta} = (\rho+p)U^\alpha U^\beta + p\eta^{\alpha\beta} \qquad \Rightarrow \boldsymbol{T} = (\rho+p)\vec{U}\otimes\vec{U} + p\boldsymbol{g}^{-1}. \tag{3.13}$$

From Eq. (3.6),

$$T^{\alpha\beta}{}_{,\beta} = \left[(\rho+p)U^\alpha U^\beta + p\eta^{\alpha\beta}\right]_{,\beta} = 0. \tag{3.14}$$

If the law of conservation of particles (3.8) is valid, we can write

$$\left[(\rho+p)U^\alpha U^\beta\right]_{,\beta} = \left[\frac{\rho+p}{n}U^\alpha n U^\beta\right]_{,\beta} = nU^\beta\left[\frac{\rho+p}{n}U^\alpha\right]_{,\beta}, \tag{3.15}$$

where n is the rest number density of particles. Since $\eta^{\alpha\beta}$ is a constant matrix, $\eta^{\alpha\beta}{}_{,\gamma} = 0$. We also have that $U^\alpha{}_{,\beta}U^\alpha = 0$ given that $U^\alpha U_\alpha = -1 \Rightarrow (U^\alpha U_\alpha)_{,\beta} = 0$. Consequently, Eq. (3.14) becomes

$$nU^\beta\left(\frac{\rho+p}{n}U^\alpha\right)_{,\beta} + p_{,\beta}\eta^{\alpha\beta} = 0. \tag{3.16}$$

If we multiply Eq. (3.16) by U_α and sum over α, we find the time component of the equation in the MCRF:

$$\begin{aligned} nU^\beta U_\alpha\left(\frac{\rho+p}{n}U^\alpha\right)_{,\beta} + p_{,\beta}\eta^{\alpha\beta}U_\alpha = 0 \qquad &\Rightarrow nU^\beta U_\alpha\left(\frac{\rho+p}{n}U^\alpha\right)_{,\beta} + p_{,\beta}U^\beta = 0 \\ \Rightarrow nU^\beta\left(\frac{\rho+p}{n}\right)_{,\beta}\cancelto{-1}{U_\alpha U^\alpha} + nU^\beta U_\alpha\left(\frac{\rho+p}{n}\right)&\cancelto{0}{U^\alpha{}_{,\beta}} + p_{,\beta}U^\beta = 0 \\ \Rightarrow U^\beta\left[-n\left(\frac{\rho+p}{n}\right)_{,\beta} + p_{,\beta}\right] = 0 \quad &\Rightarrow U^\beta\left[\rho_{,\beta} - \frac{\rho+p}{n}n_{,\beta}\right] = 0. \end{aligned} \tag{3.17}$$

Given that $p_{,\beta}U^\beta$ is the derivative of p along the world line of the fluid element, we can write Eq. (3.17) as

$$\frac{d\rho}{d\tau} - \frac{\rho+p}{n}\frac{dn}{d\tau} = 0. \tag{3.18}$$

From the first law of thermodynamics ([10], p. 95)

$$d\rho - \frac{\rho+p}{n}dn = nTdS, \tag{3.19}$$

with temperature T and entropy S, we get that the law of conservation of entropy in special relativity is

$$\frac{dS}{d\tau} = SU^\alpha{}_{,\alpha} = 0. \tag{3.20}$$

Given that the covariant derivative of a scalar like S is just its normal derivative, Eq. (3.20) is the same in general relativity. If the number of particles is conserved, then the specific entropy is related to heat flow, so, in a perfect fluid, if Eq. (3.9) is obeyed, then we should also have that entropy is constant in time during the flow of the fluid.

3.3 The Einstein Field Equations

Under the light of understanding gravity as curvature, general relativity elucidates how the gravitational field influences the behaviour of particles and, recursively, how both matter and energy determine the gravitational field. Newtonian gravity relates matter and gravity through the concept of a gravitational force $\vec{F_G}$ derived from a gravitational potential Φ as

$$\vec{F_G} = -m\nabla\Phi \; . \tag{3.21}$$

The source of the gravitational field obeys the equation

$$\nabla^2\Phi = 4\pi G\rho \; , \tag{3.22}$$

where G is the gravitational constant and ρ is the the mass density, being the source of the field. For a point particle with mass m, the solution is

$$\Phi = -\frac{Gm}{r} \; . \tag{3.23}$$

An analogue to the field equation in general relativity demands that Eq. (3.22) should take the form of a tensor equation. We opted to describe gravity and its action on matter through the convenient idea of a curved manifold with a metric, so to build an analogue field equation in general relativity, we must postulate a law which shows how the sources of the gravitational field determine the metric. Therefore, the generalization of Newton's gravitational potential should be the metric tensor $\boldsymbol{g}$. As previously discussed, the generalization of mass density is the stress energy tensor (3.4), which contains densities and fluxes of energy, momentum and will be the source of the field. Both $\boldsymbol{g}$ and $\boldsymbol{T}$ are symmetric tensors of same rank $\boldsymbol{(0,2)}$, so we could build a valid tensor equation from them. Therefore, the generalization of Eq. (3.22) would be

$$\boldsymbol{O}(\boldsymbol{g}) = k\boldsymbol{T} \; , \tag{3.24}$$

where $\boldsymbol{O}$ is a second-order differential operator acting on the metric tensor and the stress-energy tensor is multiplied by a constant k. We cannot simply write $\boldsymbol{O}$ to be the covariant derivative of the metric since we demanded metric compatibility in Eq. (2.52). But we know that, on the left-side of Eq. (3.24), $\boldsymbol{O}$ must produce a $\boldsymbol{(0,2)}$ tensor, it must be symmetric to match $\boldsymbol{T}$ and its components should be combinations of second derivatives of the metric; which is precisely the Ricci tensor in Eq. (2.77). Actually, we can generalize it even more. Any tensor of the form

$$O^{\alpha\beta} = R^{\alpha\beta} + \mu R g^{\alpha\beta} + \Lambda g^{\alpha\beta} \; , \tag{3.25}$$

with arbitrary constants μ and Λ, satisfies such conditions [10]. Note that $O^{\alpha\beta}$ contains all the curvature information we need: Eq. (3.24) is the mathematical embodiment of gravity as curvature, but it shows us that we cannot use the curvature tensor directly since it is a $(1,3)$ tensor. Instead, the most general way to describe curvature in order to match the source of the field is to use all tensors related to a curved manifold which are of desired ranking plus combinations of their components. Then Eq. (3.24) becomes

$$O^{\alpha\beta} = R^{\alpha\beta} + \mu R g^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta} \, . \tag{3.26}$$

From equations (2.52) and (3.7), we have that

$$O^{\alpha\beta}{}_{;\beta} = 0 \quad \Rightarrow \left(R^{\alpha\beta} + \mu R g^{\alpha\beta} + \Lambda g^{\alpha\beta}\right)_{;\beta} = 0 \quad \Rightarrow \left(R^{\alpha\beta} + \mu R g^{\alpha\beta}\right)_{;\beta} = 0. \tag{3.27}$$

For Eq. (3.27) to be an identity, we can compare it to the twice-contracted Bianchi identities from Eq. (2.83) and make $\mu = -\frac{1}{2}$. Finally, using the Einstein tensor defined on Eq. (2.84), the Einstein field equations can be written as

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R + \Lambda g^{\alpha\beta} = kT^{\alpha\beta} \quad \Rightarrow \quad G^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta} \tag{3.28}$$

$$\Rightarrow \boldsymbol{G} + \Lambda \boldsymbol{g} = k\boldsymbol{T}. \tag{3.29}$$

Equation (3.29) is the general relativity analogue to Newton's gravitational field equation (3.22) and it is the one we intend to solve in Chapter 4. By taking the weak field limit and demanding compatibility with Newtonian gravity, we get that $k = 8\pi G$ ([10], pp. 192 - 195). Since we are using geometrized units, $G = 1$ and Einstein's field equations become

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} = 8\pi T^{\alpha\beta}. \tag{3.30}$$

The constant Λ is called the cosmological constant. Einstein inserted it into the field equations because he was dissatisfied that otherwise his equations did not allow, apparently, for a static universe: gravity would cause a universe that was initially at dynamic equilibrium to contract. Observations of the expansion of the universe made by Hubble subsequently made him reject the term and regret he had ever invented it, calling it his "biggest blunder" [17]. However, recent astronomical observations suggest that it is small but not zero. We will return to the discussion around the meaning of the cosmological constant in Chapter 4.

3.4 The Einstein Tensor for a Static Spherically Symmetric Spacetime

The simple cosmological model that will be developed in the next chapter is isotropic and homogeneous, which means that it seems to be 'the same' from whatever direction we look at it, and even if matter was capable of heat conduction and viscosity, they would not occur since there are no temperature or force gradients. Therefore, a perfect fluid model with spherical symmetry is highly suitable to describe our system. Additionally, the Universe will be considered to be dynamic, so we must study the static case first. For this reason, let us find the Eintein tensor components in a static spherically symmetric spacetime.

3.4.1 Static Spherically Symmetric Spacetimes

On standard spherical coordinates (r, θ, ϕ), the line element of Minkowski space can be written as

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.31}$$

Notice that each surface of constant r or t is a two-dimensional spherical surface called a two-sphere. So if we set up $dr = dt = 0$ on Eq. (3.31), distances along curves confined on a two-sphere are given by

$$dl^2 = r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) := r^2 d\Omega^2, \tag{3.32}$$

where $d\Omega^2$ is the solid angle element. The circumference of a two-sphere is $2\pi r = 2\pi\sqrt{r^2}$, while the area is $4\pi r^2$. Since r^2 is the coefficient of $d\Omega^2$, any two-surface with the line element from Eq. (3.32), with r^2 independent of the angles θ and ϕ, has the intrinsic geometry of a two-sphere. Therefore, generalizing it to a curved manifold, a spherically symmetric spacetime is that in which every point is on a two-sphere with line element

$$dl^2 = f(r', t) \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.33}$$

The term $f(r', t)$ is an unknown function of coordinates r' and t of the manifold, being that the area of each sphere is $4\pi f(r', t)$. We define our radial coordinate r such as $f(r', t) := r^2$, representing a coordinate transformation from (r', t) to (r, t). Since it defines the radius of curvature and area of spheres, the coordinate r is called the "curvature coordinate" or "area coordinate". There is a no priori relation between the curvature coordinate and the proper distance from the center of the sphere to its surface, r is defined only by the spheres' properties.

3.4.2 The Metric

Let us now define our coordinates and find a metric form. Consider spheres r amd $r + dr$. We demand that a line of $\theta = \phi = \ const.$ is orthogonal to both two-spheres, which means that the line has a tangent basis vector $\hat{e}_r$ and $\hat{e}_r.\hat{e}_\theta = \hat{e}_r.\hat{e}_\phi = 0$, with $\hat{e}_\theta$ and $\hat{e}_\phi$ lying on the spheres. That implies that our metric g has components $g_{r\theta} = g_{r\phi} = 0$. But since the whole spacetime must be spherically symmetric, a line $r = \theta = \phi = \ const.$ must be orthogonal to the two-spheres so there shouldn't be a preferred direction in space. This means that $\hat{e}_t.\hat{e}_\theta = \hat{e}_t.\hat{e}_\phi = 0$ and $g_{t\theta} = g_{t\phi} = 0$. Therefore, the general metric of a spherically symmetric spacetime can be put in the form

$$ds^2 = g_{ij}dx^i dx^j = g_{00}dt^2 + 2g_{0r}drdt + g_{rr}dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.34}$$

A static spacetime is that in which it is possible to find a time coordinate t in a way that (i) all metric components are independent of t and (ii) the geometry is unchanged by reverse time transformation $t \to -t$. A spacetime with the property (i) but not necessarily (ii) is called stationary. Like the case of a rotating star, in which time reversal changes the sense of rotation, but the metric components remain constant in time. In a static spacetime, property (ii) implies that the coordinate transformation $(t,r,\theta,\phi) \to (-t,r,\theta,\phi)$ has $\Lambda^{\bar{0}}{}_0 = -1$ and $\Lambda^i{}_j = \delta^i{}_j$, so

$$\begin{cases} g_{\bar{0}\bar{0}} = \left(\Lambda^0{}_{\bar{0}}\right)^2 g_{00} = g_{00}, \\ g_{\bar{0}\bar{r}} = \Lambda^0{}_{\bar{0}}\Lambda^r{}_{\bar{r}} g_{0r} = -g_{0r}, \\ g_{\bar{r}\bar{r}} = \left(\Lambda^r{}_{\bar{r}}\right)^2 g_{rr} = g_{rr}. \end{cases} \tag{3.35}$$

An unchanged geometry implies that $g_{\bar{\alpha}\bar{\beta}} = g_{\alpha\beta}$, so $g_{0r} \equiv 0$. Given that $g_{00} < 0$ and $g_{rr} > 0$ at any point, we can write, from Eq. (3.34), the metric of a static spherically symmetric spacetime as

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Psi}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.36}$$

where $\Phi(r)$ and $\Psi(r)$ are functions of the radial coordinate r.

3.4.3 The Einstein Tensor

The Einstein tensor components are given in Eq. (2.84). In order to compute them, we must first find the Ricci tensor components and then the Ricci scalar. To do so, we first need to compute the Riemann curvature tensor and contract it, which can only be done by finding the Christoffel symbols. Therefore, let us begin all these calculations. The metric components for the static spherically symmetric spacetime can be found directly from Eq. (3.36). And since $\{g_{\alpha\beta}\}$ is a diagonal matrix, its inverse can be found directly:

$$(g_{\alpha\beta}) = \begin{pmatrix} -e^{2\Phi} & 0 & 0 & 0 \\ 0 & e^{2\Psi} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \qquad \left(g^{\alpha\beta}\right) = \begin{pmatrix} -e^{-2\Phi} & 0 & 0 & 0 \\ 0 & e^{-2\Psi} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}. \tag{3.37}$$

The Christoffel symbols can be computed directly from Eq. (2.54). Since there are no crossed terms from the metric components, the only ones which need to be computed are

$$\Gamma^t_{\mu\nu} = \frac{1}{2}g^{tt}\left(g_{t\mu,\nu} + g_{t\nu,\mu} - g_{\mu\nu,\beta}\right), \qquad \Gamma^r_{\mu\nu} = \frac{1}{2}g^{rr}\left(g_{r\mu,\nu} + g_{r\nu,\mu} - g_{\mu\nu,\beta}\right), \tag{3.38}$$

$$\Gamma^\theta_{\mu\nu} = \frac{1}{2}g^{\theta\theta}\left(g_{\theta\mu,\nu} + g_{\theta\nu,\mu} - g_{\mu\nu,\beta}\right) \qquad \Gamma^\phi_{\mu\nu} = \frac{1}{2}g^{\phi\phi}\left(g_{\phi\mu,\nu} + g_{\phi\nu,\mu} - g_{\mu\nu,\beta}\right). \tag{3.39}$$

From (3.37), the second derivatives of the metric components are

$$g_{tt,r} = \frac{d}{dr}(-e^{2\Phi}) = -2e^{2\Phi}\Phi', \quad g_{rr,r} = 2e^{2\Psi}\Psi', \quad g_{\theta\theta,r} = 2r, \tag{3.40}$$

$$g_{\phi\phi,r} = 2r\sin^2\theta, \qquad g_{\phi\phi,\theta} = r^2 2\sin\theta\cos\theta = r^2\sin 2\theta, \tag{3.41}$$

where $\Phi' := d\Phi/dr$ and $\Psi' := d\Psi/dr$. By replacing indices values on the Christoffel symbols from Eqs. (3.38) and (3.39) and using the second derivatives on Eqs. (3.40) and (3.41), we find that the resulting non-zero Christoffel symbols are

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \Phi', \qquad \Gamma^r_{tt} = e^{2(\Phi-\Psi)}\Phi', \qquad \Gamma^r_{rr} = \Psi', \qquad \Gamma^r_{\theta\theta} = -re^{-2\Psi}, \tag{3.42}$$

$$\Gamma^r_{\phi\phi} = -r\sin^2\theta e^{-2\Psi}, \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = r^{-1}, \tag{3.43}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = r^{-1}, \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta. \tag{3.44}$$

Let us now decide on which components of the Riemann tensor $R_{\alpha\beta\mu\nu}$ we will compute, since it has a total of $4^4 = 256$ components. We have that $R_{\alpha\alpha\mu\nu} = 0 = R_{\alpha\beta\nu\nu}$ given the symmetries in Eq. (2.71) and we can reduce the number of independent components: we choose pairs of $\alpha \neq \beta$ and, accounting for the fact that order doesn't matter, there are 6 possible combinations. Similarly, there are also 6 pairs of indices $\mu \neq \nu$. This gives us $6 \times 6 = 36$ possible terms, which can be laid out in a 6×6 matrix, but because of the symmetry $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$ there are still some dependent terms in those 36, so we divide the number of off-diagonal elements by two to get $5 \times 6/2 + 6 = 21$ independent terms. Furthermore, the cyclic identity in Eq. (2.72) tells us that one of these 21 terms can be determined from the other two terms, giving us a total of 20 independent components of the Riemann tensor. The illustration of this train of thought can be seen below.

$$
\begin{array}{cccccc}
R_{trtr} & R_{trt\theta} & R_{trt\phi} & R_{trr\theta} & R_{trr\phi} & \underline{R_{tr\theta\phi}} \\
R_{t\theta tr} & R_{t\theta t\theta} & R_{t\theta t\phi} & R_{t\theta r\theta} & \underline{R_{t\theta r\phi}} & R_{t\theta\theta\phi} \\
R_{t\phi tr} & R_{t\phi t\theta} & R_{t\phi t\phi} & \underline{R_{t\phi r\theta}} & R_{t\phi r\phi} & R_{t\phi\theta\phi} \\
R_{r\theta tr} & R_{r\theta t\theta} & R_{r\theta t\phi} & R_{r\theta r\theta} & R_{r\theta r\phi} & R_{r\theta\theta\phi} \\
R_{r\phi tr} & R_{r\phi t\theta} & R_{r\phi t\phi} & R_{r\phi r\theta} & R_{r\phi r\phi} & R_{r\phi\theta\phi} \\
R_{\theta\phi tr} & R_{\theta\phi t\theta} & R_{\theta\phi t\phi} & R_{\theta\phi r\theta} & R_{\theta\phi r\phi} & R_{\theta\phi\theta\phi}
\end{array}
\Rightarrow
\begin{array}{cccccc}
R_{trtr} & R_{trt\theta} & R_{trt\phi} & R_{trr\theta} & R_{trr\phi} & \underline{R_{tr\theta\phi}} \\
\cancel{R_{t\theta tr}} & R_{t\theta t\theta} & R_{t\theta t\phi} & R_{t\theta r\theta} & \underline{R_{t\theta r\phi}} & R_{t\theta\theta\phi} \\
\cancel{R_{t\phi tr}} & \cancel{R_{t\phi t\theta}} & R_{t\phi t\phi} & \underline{R_{t\phi r\theta}} & R_{t\phi r\phi} & R_{t\phi\theta\phi} \\
\cancel{R_{r\theta tr}} & \cancel{R_{r\theta t\theta}} & \cancel{R_{r\theta t\phi}} & R_{r\theta r\theta} & R_{r\theta r\phi} & R_{r\theta\theta\phi} \\
\cancel{R_{r\phi tr}} & \cancel{R_{r\phi t\theta}} & \cancel{R_{r\phi t\phi}} & \cancel{R_{r\phi r\theta}} & R_{r\phi r\phi} & R_{r\phi\theta\phi} \\
\cancel{R_{\theta\phi tr}} & \cancel{R_{\theta\phi t\theta}} & \cancel{R_{\theta\phi t\phi}} & \cancel{R_{\theta\phi r\theta}} & \cancel{R_{\theta\phi r\phi}} & R_{\theta\phi\theta\phi}.
\end{array}
$$

The underlined terms are those to which the cyclic identity applies. From these remaining 20 components, it is easy to verify that the only non-zero components of the curvature tensor are the diagonal terms

$$
(R_{\alpha\beta\mu\nu}) = \begin{pmatrix}
R_{trtr} & 0 & 0 & 0 & 0 & 0 \\
0 & R_{t\theta t\theta} & 0 & 0 & 0 & 0 \\
0 & 0 & R_{t\phi t\phi} & 0 & 0 & 0 \\
0 & 0 & 0 & R_{r\theta r\theta} & 0 & 0 \\
0 & 0 & 0 & 0 & R_{r\phi r\phi} & 0 \\
0 & 0 & 0 & 0 & 0 & R_{\theta\phi\theta\phi}
\end{pmatrix}. \tag{3.45}
$$

The other $256-36 = 220$ components are determined by Eq. (2.71). Therefore, from Eqs. (2.68), (3.42) and (3.44), the six non-zero components of the Riemann curvature tensor are

$$
R_{trtr} = g_{tt}R^t{}_{rtr} = g_{tt}\left(\Gamma^t_{rr,t} - \Gamma^t_{rt,r} + \Gamma^t_{\sigma t}\Gamma^\sigma_{rr} - \Gamma^t_{\sigma r}\Gamma^\sigma_{rt}\right) = e^{2\Phi}\left[\Phi'' + (\Phi')^2 - \Phi'\Psi'\right], \tag{3.46}
$$

$$
R_{t\theta t\theta} = r\Phi' e^{2(\Phi-\Psi)}, \quad R_{t\phi t\phi} = r\Phi'\sin^2\theta e^{2(\Phi-\Psi)}, \quad R_{r\theta r\theta} = g_{rr}R^r{}_{\theta r\theta} = r\Psi', \tag{3.47}
$$

$$
R_{r\phi r\phi} = r\Psi'\sin^2\theta, \quad R_{\theta\phi\theta\phi} = g_{\theta\theta}R^\theta{}_{\phi\theta\phi} = r^2\sin^2\theta\left(1 - e^{-2\Psi}\right). \tag{3.48}
$$

Now, let us obtain the components the Ricci tensor. Since there are no crossed terms in the metric components, if $\alpha \neq \beta$ then Eq. (2.84) (with components in the covariant form) becomes $G_{\alpha\beta} = R_{\alpha\beta}$. Through some direct (but time-consuming) calculation, it can be verified, using Eqs. (2.77) and (2.69), that $R_{\alpha\beta}$ = 0 unless $\alpha = \beta$. Therefore, the non-zero Einstein tensor components that will be computed are

$$
G_{tt} = R_{tt} - \frac{1}{2}g_{tt}R, \qquad G_{rr} = R_{rr} - \frac{1}{2}g_{rr}R \tag{3.49}
$$

$$
G_{\theta\theta} = R_{\theta\theta} - \frac{1}{2}g_{\theta\theta}R, \qquad G_{\phi\phi} = R_{\phi\phi} - \frac{1}{2}g_{\phi\phi}R, \tag{3.50}
$$

and the Ricci tensor resulting components are

$$R_{tt} = R^{\sigma}{}_{t\sigma t} = R^{r}{}_{trt} + R^{\theta}{}_{t\theta t} + R^{\phi}{}_{t\phi t} = e^{2(\Phi-\Psi)}\left[\Phi'' + (\Phi')^2 - \Psi'\Phi' + \frac{2\Phi'}{r}\right], \tag{3.51}$$

$$R_{rr} = R^{t}{}_{rtr} + R^{\theta}{}_{r\theta r} + R^{\phi}{}_{r\phi r} = -\Phi'' - (\Phi')^2 + \Phi'\Psi' + \frac{2\Psi'}{r}, \tag{3.52}$$

$$R_{\theta\theta} = R^{t}{}_{\theta t\theta} + R^{r}{}_{\theta r\theta} + R^{\phi}{}_{\theta\phi\theta} = e^{-2\Psi}\left(-r\Phi' + r\Psi' - 1\right) + 1, \tag{3.53}$$

$$R_{\phi\phi} = R^{t}{}_{\phi t\phi} + R^{r}{}_{\phi r\phi} + R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta R_{\theta\theta}. \tag{3.54}$$

Then, using the inverse metric components from (3.37) and Eqs. (3.51)-(3.54), the Ricci scalar is

$$\begin{aligned} R &= g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \\ \Rightarrow R &= 2e^{-2\Psi}\left[-\Phi'' - (\Phi')^2 + \Psi'\Phi' + \frac{2\Psi' - 2\Phi'}{r} - \frac{1}{r^2}\right] + \frac{2}{r^2}. \end{aligned} \tag{3.55}$$

Using the results from (3.37) and Eqs. (3.55), (3.51)-(3.54) in Eqs. (3.49) and (3.50), in a static spherically symmetric spacetime, the Einstein tensor has components

$$\begin{cases} G_{tt} = \frac{1}{r^2}e^{2\Phi} + e^{2(\Phi-\Psi)}\left[\frac{2\Psi'}{r} - \frac{1}{r^2}\right] = \frac{1}{r^2}e^{2\Phi}\frac{d}{dr}\left[r\left(1 - e^{-2\Psi}\right)\right] \\ G_{rr} = -\frac{1}{r^2}e^{2\Psi}\left(1 - e^{-2\Psi}\right) + \frac{2}{r}\Phi' \\ G_{\theta\theta} = r^2 e^{-2\Psi}\left[\Phi'' + (\Phi')^2 + \frac{\Phi'}{r} - \Phi'\Psi' - \frac{\Psi'}{r}\right] \\ G_{\phi\phi} = \sin^2\theta G_{\theta\theta}. \end{cases} \tag{3.56}$$

Now let us proceed to solve the Einstein field equations to the cosmological boundary conditions.

Chapter 4

The Cosmological Solution to Einstein's Field Equations

Cosmology is a branch of astronomy that involves the study of the Universe as a whole, diving into its history, evolution, composition and dynamics. The primary aim of research in cosmology is to understand the large-scale structure of the Universe, but we can also derive some conclusions about small-scale organizations, connecting galaxies to stars and stars to planets and human life, all in a big, complex chain of events. The interaction between cosmology and other scientific branches is a rich area of research and our ability to understand the Universe on large scales depends in an essential way on general relativity. The Newtonian theory of gravity is sufficient as long as the mass of a system is small compared to its size (the ratio between them is way smaller than 1). When this ratio becomes closer to 1, it can be the case of the radius becoming small faster than the mass (neutron stars and black holes, for example) or if the system mass increases faster than its radius, which is the case in the cosmological scale. The Universe has roughly the same density at any point, so when we consider larger radius, the mass increases with the volume and the mass-radius ratio gets so large that GR becomes needed [10].

And it is with this mindset that finding the solution to the Einstein field equations for the Universe has fundamental significance, since it makes sense of all the experimental data about the accelerated expansion measured today. In this chapter, we will study its solution for the whole Universe by first establishing proper boundary conditions. Then we will be able to define a general cosmological spacetime metric and discuss the three different possible types of universes that arise in such context. Along with the results from the previous chapter, we will find the Einstein tensor for a dynamic spherically symmetric spacetime and solve Einstein's field equations using the perfect fluid stress-energy tensor, obtaining equations that show us how the Universe scale factor changes through time, exploring an early Universe and what it says about our present time. Given the experimental data, we will then relate it to the calculations and discuss which is the most probable type of our Universe, along with what the cosmological constant means and how it accelerates the Universe expansion.

4.1 Boundary Conditions and the Cosmological Metric

4.1.1 The Obervable Universe and its Boundary Conditions

The observable Universe, illustrated on Fig. (4.1), was found to be homogeneous and isotropic, meaning that it looks the same, on average, in every direction we look. This accessible part of the Universe is fundamentally bound by the limits of our light cone, which is called the particle horizon. Additionally, some regions of space are so distant that are beyond the past light-cone of our reference frame, thus no information can reach us. That is called the optical horizon. It is not a fundamental limit since the more we travel to the future, further the past light-cone reaches and we can access some of these early observations of the Universe. Some of those previously unobserved processes that were recently measured are gravitational waves, which is an outstanding experimental confirmation of a general relativity prediction [18]. Furthermore, we admit that these regions which become accessible in time will be similar to those that have been previously observed. Such concept in cosmology is called the assumption of mediocrity, meaning that every unknown region is very similar to the insides of our reference frame light cone [10]. This Universe revealed to be expanding, so every galaxy,

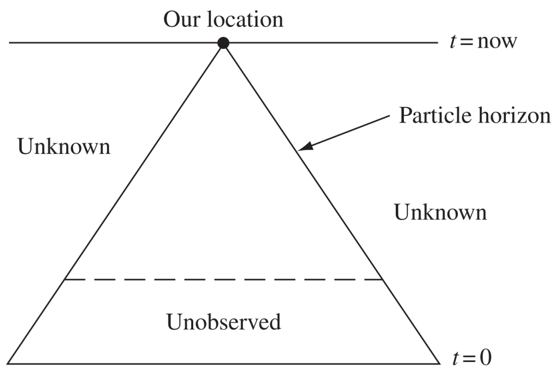

Figure 4.1: A spacetime diagram of the Universe's past light-cone.
Illustration from [10].

on average, is receding at a speed proportional to its distance from us. This recessional velocity is called the Hubble flow (from Edwin Hubble) and it can be easily visualized as the inflating balloon on Fig. (4.2). If we draw equally spaced dots on it and watch the balloon as it expands, the distance on the surface between two dots will grow at a rate proportional to the extent of the distance. So any point will have the same vision of other points receding the same way, and the Earth reference frame is no more special than any other, meaning that the homogeneity of the distribution of points with time is preserved due to this proportionality. Since there is no measured velocity anisotropy whatsoever, we can write this recessional velocity v in terms of the distance d as

$$v = Hd, \tag{4.1}$$

where H is the Hubble constant whose present value is measured to be [10]

$$H = (71 \pm 4)\ km/s.Mpc = (2.3 \pm 0.1) \times 10^{-18} s^{-1} \tag{4.2}$$

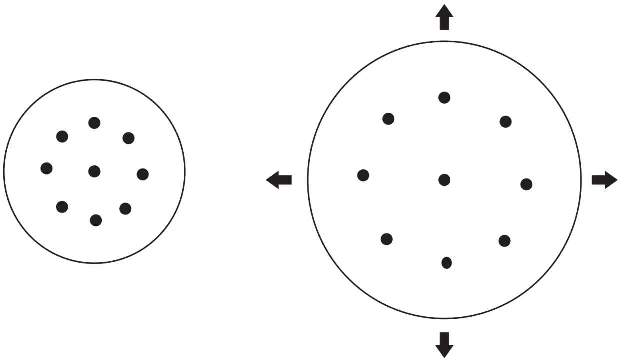

Figure 4.2: The expanding balloon representation.
Illustration from [10].

and associated to it, there is the Hubble time

$$t_H = H^{-1} = [(2.3 \pm 0.1) \times 10^{-18} s^{-1}]^{-1} = (4.3 \pm 0.2) \times 10^{17} s \tag{4.3}$$

which is equal to approximately 14 billion years! This is the time-scale for the cosmological expansion. Note that to describe the expanding Universe in this manner, our cosmological reference frame is the one that is expanding with the Universe (galaxies recessional velocities are lesser than 1). It is the same to assume that, in our neighborhood, there exists a preferred choice of time generating homogeneous and isotropic hypersurfaces. Then Eq. (4.1) would be valid in the local inertial frame of any observer who is at rest with respect to these hypersurfaces at any point [10]. With such considerations in mind, let us build the cosmological metric.

4.1.2 The Cosmological Metric

Adopting comoving coordinates in the MCRF, in order to simplify calculations, we consider that each galaxy in our neighborhood has no random velocity and each set of coordinates on the galaxies' frames are the spatial ones $\{x^i, i = 1, 2, 3\}$ and the proper time t. The expansion of the Universe for a reference frame that moves along with it can be measured as a change of proper distance between galaxies, indicating that the metric has time-dependent coefficients (the system is not static). So at a moment t_0, the hypersurfaces of constant time have the line element

$$dl^2(t_0) = h_{ij}(t_0) dx^i dx^j, \tag{4.4}$$

with h_{ij} being the time-dependent coefficients. All h_{ij}s must increase at the same rate to correspond the observed isotropic expansion, so the expansion measurement on the hypersurface at a moment t_1 will be

$$dl^2(t_1) = f(t_1, t_0) h_{ij}(t_0) dx^i dx^j \tag{4.5}$$
$$= h_{ij}(t_1) dx^i dx^j. \tag{4.6}$$

In general, we can write the line element for any t as

$$dl^2(t) = R^2(t) h_{ij} dx^i dx^j, \tag{4.7}$$

where $R^2(t)$ is a overall scale factor with $R^2(t_0) = 1$ and h_{ij} represents a constant metric equal to that of the hypersurface at t_0. Now extending the constant-time hypersurface line element to the full spacetime

$$ds^2 = -dt^2 + g_{0i}dtdx^i + R^2(t)h_{ij}dx^i dx^j, \tag{4.8}$$

where $g_{00} = -1$ because t is the proper time along a line $dx^i = 0$. Since the definition of simultaneity at constant t agrees with that of a local Lorentz frame (inertial reference frame), than the base vectors $\vec{e_0}$ and $\vec{e_i}$ of the comoving coordinates must be orthogonal

$$g_{0i} = \vec{e_0}.\vec{e_i} = 0 \tag{4.9}$$

and we are left with a metric without those crossed terms

$$ds^2 = -dt^2 + R^2(t)h_{ij}dx^i dx^j. \tag{4.10}$$

Isotropy implies a spherical symmetry with respect to the origin of the coordinates, meaning that all points are equivalent and the origin can be located at any point. From the spherically symmetric metric in Eq. (3.36), using the relation expressed in Eq. (4.7), the line element in Eq. (4.10) can be written as

$$dl^2 = e^{2\Psi(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.11}$$

Now we must find the full cosmological space metric by determining $e^{2\Psi(r)}$ according to our boundary conditions. Homogeneity implies that the metric should be spherically symmetric not only with respect to the origin, but also to every point. This means that every scalar must be independent of the position at a fixed time, so the Ricci scalar R must have the same value on every point. Instead of computing the Ricci scalar for our metric, we can just obtain the trace G of the Einstein tensor $\boldsymbol{G}$ and demand it to be a constant, since the relation $G = -\frac{1}{2}R$ is valid. Using the expressions in Eq. (3.56) for the Einstein tensor components, it is easy to see that G_{ij} for the line element in Eq. (4.11) are the same as the ones for the metric in Eq. (3.36) if we set $\Phi = 0$. Therefore, we obtain

$$\begin{cases} G_{rr} = \dfrac{1}{r^2} - \dfrac{1}{r^2}e^{2\Psi} = -\dfrac{1}{r^2}e^{2\Psi}(1 - e^{-2\Psi}), \\ G_{\theta\theta} = -re^{-2\Psi}\Psi', \\ G_{\phi\phi} = -re^{-2\Psi}\Psi'\sin^2\theta = sin^2\theta G_{\theta\theta}. \end{cases} . \tag{4.12}$$

Then the trace is

$$\begin{aligned} G &= g^{ij}G_{ij} = g^{rr}G_{rr} + g^{\theta\theta}G_{\theta\theta} + g^{\phi\phi}G_{\phi\phi} \\ &= e^{-2\Psi}\left[-\frac{1}{r^2}e^{2\Psi}\left(1 - e^{-2\Psi}\right)\right] + \frac{1}{r^2}\left(-re^{-2\Psi}\Psi'\right) + \frac{1}{r^2\sin^2\theta}\left(-\sin^2\theta re^{-2\Psi}\Psi'\right) \\ &= -\frac{1}{r^2}\left(1 - e^{-2\Psi} + 2re^{-2\Psi}\Psi'\right) = -\frac{1}{r^2}\left[1 + e^{-2\Psi}\left(2r\Psi' - 1\right)\right] \\ &= -\frac{1}{r^2}\left[1 - \left(re^{-2\Psi}\right)'\right] = -\frac{1}{r^2}\left[r\left(1 - e^{-2\Psi}\right)\right]' \end{aligned} \tag{4.13}$$

and it is set to be a constant k

$$k = -\frac{1}{r^2}\left[r\left(1 - e^{-2\Psi}\right)\right]'. \tag{4.14}$$

By integrating Eq. (4.14), it is possible to obtain the component g_{rr} of the cosmological metric since, from (3.37), $g_{rr} = e^{2\Psi}$:

$$
\begin{aligned}
& -kr^2 = \left[r(1-e^{-2\Psi})\right]' \\
& \Rightarrow -\int kr^2 dr = r(1-e^{-2\Psi}) \\
& \Rightarrow -\frac{k}{3}r^3 + A = r(1-e^{-2\Psi}) \Rightarrow e^{-2\Psi} = 1 + \frac{1}{3}kr^2 - \frac{A}{r}
\end{aligned}
$$

$$
\Rightarrow g_{rr} = e^{2\Psi} = \frac{1}{1+\frac{1}{3}kr^2 - \frac{A}{r}}. \tag{4.15}
$$

Because we are in an inertial reference frame, it implies local flatness at $r = 0$, so $g_{rr}(r = 0) = 1 \Rightarrow A = 0$. If we define a curvature constant as $\kappa = -\frac{k}{3}$, then our line element on Eq. (4.11) becomes

$$
g_{rr} = \frac{1}{1-\kappa r^2} \qquad \Rightarrow dl^2 = \frac{dr^2}{1-\kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)
$$

$$
\Rightarrow dl^2 = \frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2. \tag{4.16}
$$

So plugging Eq. (4.16) into Eq. (4.10), the full cosmological spacetime metric is

$$
ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right], \tag{4.17}
$$

known as the Robertson-Walker metric. We must guarantee that this metric is isotropic and homogeneous for all κ, so all possibilities resume on evaluating it for $\kappa = (-1, 0, +1)$. Since the modulus of the curvature constant can be reabsorbed through a reparametrization of the radial coordinate, we only take the signal into account in this evaluation [4].

i) For $\kappa = 0$, at any moment $t = t_0$, we have

$$
ds^2 = -dt^2 + R^2(t_0)dr^2 + R^2(t_0)r^2 d\Omega^2 \tag{4.18}
$$

$$
\Rightarrow dl^2 = R^2(t_0)dr^2 + R^2(t_0)r^2 d\Omega^2 = d(r')^2 + (r')^2 d\Omega^2 \tag{4.19}
$$

with $r' = R(t_0)r$. That on Eq. (4.19) is the same as the flat Euclidian space metric in spherical coordinates, describing a flat Robertson-Walker universe, clearly homogeneous and isotropic [10].

ii) For $\kappa = +1$, let

$$
d\chi^2 = \frac{dr^2}{1-r^2}, \qquad \chi = \arcsin r \Rightarrow r = \sin\chi. \tag{4.20}
$$

So the line element at $t = t_0$ is

$$
dl^2 = R^2(t_0)d\chi^2 + R^2(t_0)\sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.21}
$$

which is the metric of a three-sphere with radius $R(t_0)$, being a closed, spherical Robertson-Walker metric. This is the most appropriate mathematical description of the cosmological expansion balloon analogy.

iii) For $\kappa = -1$, Eq. (4.20) becomes

$$d\chi^2 = \frac{dr^2}{1+r^2}, \qquad \chi = \int \frac{1}{\sqrt{1+r^2}} dr = arcsinh(\tfrac{r}{1}) + C \Rightarrow r = \sinh\chi \tag{4.22}$$

at $t = t_0$, by setting $C = 0$. And then we have

$$dl^2 = R^2(t_0)(d\chi^2 + \sinh^2\chi d\Omega^2), \tag{4.23}$$

which is the open, hyperbolic Robertson-Walker metric. As χ increases, the circumference grows more rapidly with the proper radius than in flat space, meaning there is no natural end to it [10]. It is possible to show that, in appropriate coordinates, the intrinsic geometry of a hyperbola in Minkowski spacetime $t^2 - x^2 - y^2 - z^2 = const. > 0$ is identical with that of Eq. (4.23), so the hyperbolic Robertson-Walker metric represents the geometry of a hypersurface embedded in Minkowski spacetime; it is a hypersurface of events that all have the same timelike interval from the origin. A Lorentz transformation leaves the metric, and hence the hyperbola unchanged, but changes the origin of spatial coordinates. And since any point on the hyperbola can be made into this origin by choosing the correct transformation, intervals on the hyperbola are Lorentz invariant and the hypersurface defined by Eq. (4.23) is indeed homogeneous and isotropic ([10], p.345).

Since Eq. (4.17) is isotropic and homogeneous for all κ, we can develop some cosmological dynamics and solve the Einstein field equations in order to find some physical interpretations surrounding these universes evolution and obtain a better understanding of the observed expansion of our observable Universe.

4.2 Cosmological Dynamics and the Solution to Einstein's Field Equations

4.2.1 The divergence of the Stress-Energy tensor

Given that our Universe is not static, its evolution, for each κ, depends on a function of time: the scale factor $R(t)$. By solving Einstein's equations we can determine how this factor behaves and, therefore, the physical implications for the evolution of the Universe. Provided its isotropy and homogeneity, let us consider the Universe as a perfect fluid at rest with all its properties depending on time. Therefore, its mass density and pressure are also time functions $\rho = \rho(t), \;\; p = p(t)$. From Eq. (3.13), we have that the time-component of the divergence of the stress-energy tensor for the cosmological fluid is[1]

$$\begin{aligned} \nabla_\beta(T^{\alpha\beta}) &= T^{\alpha\beta}{}_{;\,\beta} = T^{\alpha\beta}{}_{,\,\beta} + T^{\gamma\beta}\,\Gamma^{\alpha}{}_{\gamma\beta} + T^{\alpha\gamma}\,\Gamma^{\beta}{}_{\gamma\beta} \\ \Rightarrow \nabla_\alpha(T^{0\alpha}) &= T^{0\alpha}{}_{;\,\alpha} = T^{0\alpha}{}_{,\,\alpha} + T^{0\mu}\,\Gamma^{\alpha}{}_{\mu\alpha} + T^{\mu\alpha}\,\Gamma^{0}{}_{\mu\alpha}. \end{aligned} \tag{4.24}$$

[1] It is a common practice to write the t-component of a tensor using the index 0, so we will alternate between both notations in this chapter.

For the Robertson-Walker metric in Eq. (4.17), the metric components can be written in a matrix form as

$$(g_{\alpha\beta}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2(t)}{1-\kappa r^2} & 0 & 0 \\ 0 & 0 & R^2(t) r^2 & 0 \\ 0 & 0 & 0 & R^2(t) r^2 \sin^2\theta \end{pmatrix}, \qquad (g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}. \tag{4.25}$$

Developing the same calculations we did in Sec. 3.4.3, the non-vanishing Christoffel symbols can be computed to be

$$\Gamma^0{}_{jk} = \frac{\dot{R}}{R} g_{jk}, \qquad \Gamma^j{}_{0k} = \frac{\dot{R}}{R}\delta^j{}_k, \qquad \Gamma^r{}_{rr} = \frac{\kappa r}{1-\kappa r^2}, \qquad \Gamma^r{}_{\theta\theta} = -r(1-\kappa r^2), \tag{4.26}$$

$$\Gamma^r{}_{\phi\phi} = -r(1-\kappa r^2)\sin^2\theta, \qquad \Gamma^\theta{}_{r\theta} = \Gamma^\phi{}_{r\phi} = \frac{1}{r}, \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \tag{4.27}$$

$$\Gamma^\phi{}_{\theta\phi} = \cot\theta, \tag{4.28}$$

for $\{j,k\} = \{r,\theta,\phi\}$. Therefore, using Eqs. (4.24) and (4.26)-(4.28), the cosmological fluid stress-energy tensor divergence time-component is

$$T^{00}{}_{;\,0} = T^{00}{}_{,\,0} + T^{0\mu}\,\Gamma^0{}_{\mu 0} + T^{\mu 0}\,\Gamma^0{}_{\mu 0} = \dot{\rho}, \qquad T^{0r}{}_{;\,r} = T^{\mu r}\,\Gamma^0{}_{\mu r} + T^{0\mu}\,\Gamma^r{}_{\mu r} = (p+\rho)\frac{\dot{R}}{R},$$

$$T^{0\theta}{}_{;\theta} = T^{\mu\theta}\,\Gamma^0{}_{\mu\theta} + T^{0\mu}\,\Gamma^\theta{}_{\mu\theta} = (p+\rho)\frac{\dot{R}}{R}, \qquad T^{0\phi}{}_{;\phi} = T^{\mu\phi}\,\Gamma^0{}_{\mu\phi} + T^{0\mu}\,\Gamma^\phi{}_{\mu\phi} = (p+\rho)\frac{\dot{R}}{R},$$

$$\Rightarrow T^{0\alpha}{}_{;\alpha} = T^{00}{}_{;\,0} + T^{0r}{}_{;\,r} + T^{0\theta}{}_{;\theta} + T^{0\phi}{}_{;\phi}$$

$$\Rightarrow T^{0\alpha}{}_{;\alpha} = \dot{\rho} + 3(p+\rho)\frac{\dot{R}}{R} = 0, \tag{4.29}$$

in which we can see a relation between the system's energy and the scale factor. If we multiply Eq. (4.29) by R^3 and apply some algebraic manipulation, we obtain the explicit relation

$$R^3 T^{0\alpha}{}_{;\alpha} = R^3\dot{\rho} + 3(p+\rho)\frac{\dot{R}}{R}R^3 = 0 \Rightarrow R^3\dot{\rho} = -3(p+\rho)\dot{R}R^2 \Rightarrow R^3\dot{\rho} - 3\rho\dot{R}R^2 = -3\rho\dot{R}R^2$$

$$\Rightarrow \frac{d}{dt}(\rho R^3) = -p\frac{d}{dt}(R^3). \tag{4.30}$$

4.2.2 Matter-dominated and Radiation-dominated cosmologies

Under the light of an expanding Universe, we can interpret the left hand side of Eq. (4.30) as the rate of change of the system's total energy, while the right hand side is the work it does as it expands. We may also differentiate between a matter-dominated cosmology, the present epoch, and a radiation-dominated cosmology, representing the early Universe [10]. In the present epoch, the Universe is observed to contain cold, non-relativistic matter particles with very small random velocities, so its general behaviour is the same of dust we described in Sec. 3.2.1, having no pressure. So for the matter-dominated cosmology we have

$$p = 0 \qquad \Rightarrow \frac{d}{dt}(\rho R^3) = 0. \tag{4.31}$$

In the early Universe, the cosmological fluid energy density is in the form of radiation, with highly relativistic particles having significant velocities, similar to a photon gas [10]. Considering that all these particles n have the same speed v and mass m, we can express p and ρ as functions of v, m and n and show that $p = \frac{1}{3}\rho$ ([16], pp. 85-87). Then Eq. (4.30) for the radiation-dominated cosmology becomes

$$\frac{d}{dt}(\rho R^3) = -\frac{1}{3}\rho\frac{d}{dt}(R^3) \quad \Rightarrow \dot{\rho}R^3 + 3\rho R^2\dot{R} = -\rho R^2\dot{R} \quad \Rightarrow \dot{\rho}R^3 + 4\rho R^2\dot{R} = 0 \quad \times (R)$$
$$\Rightarrow \dot{\rho}R^4 + 4\rho R^3\dot{R} = 0 \qquad \Rightarrow \frac{d}{dt}(\rho R^4) = 0. \tag{4.32}$$

Now, let us solve the Einstein field equations from Eq. (3.30) for the cosmological fluid.

4.2.3 Solving Einstein Field Equations - the Acceleration of the Scale Factor

Isotropy guarantees that $G_{tj} = 0$, for all j. We can also see that $G_{jk}\ \alpha\ g_{jk}$, with $\{j,k\} = \{r,\theta,\phi\}$. That will lead us to two independent components of the Einstein tensor. Given that the Bianchi identities from Eq. (2.76) provide a relation between them, we only need to compute one component G_{tt}. Using the Christoffel symbols from Eqs. (4.26)-(4.28), the non-vanishing components of the curvature tensor are

$$R^j{}_{tjt} = -\left[\frac{\dot{R}}{R}\right]_{,t} - \left(\frac{\dot{R}}{R}\right)^2, \quad R^t{}_{jtj} = \left[\frac{\dot{R}}{R}g_{jj}\right]_{,t} - \left(\frac{\dot{R}}{R}\right)^2 g_{jj}, \quad R^j{}_{kjk} = \left(\frac{\dot{R}}{R}\right)^2 g_{kk} + \frac{\kappa}{R^2}g_{kk}, \tag{4.33}$$

so the components of the Ricci tensor are

$$R_{tt} = R^r{}_{trt} + R^\theta{}_{t\theta t} + R^\phi{}_{t\phi t} = -3\left[\frac{\dot{R}}{R}\right]_{,t} - 3\left(\frac{\dot{R}}{R}\right)^2, \tag{4.34}$$

$$R_{jj} = R^t{}_{jtj} + R^k{}_{jkj} = \left[\frac{\dot{R}}{R}g_{jj}\right]_{,t} + \left(\frac{\dot{R}}{R}\right)^2 g_{jj} + \frac{2\kappa}{R^2}g_{jj}. \tag{4.35}$$

Through Eqs. (4.25), (4.34) and (4.35), we have that the time-component of the Einstein tensor is

$$G_{tt} = R_{tt} - \frac{1}{2}g_{tt}\left(g^{tt}R_{tt} + g^{jj}R_{jj}\right) = \frac{1}{2}R_{tt} + \frac{1}{2}\left(g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi}\right)$$

$$\begin{aligned} G_{tt} = -\frac{3}{2}\left[\frac{\dot{R}}{R}\right]_{,t} - \frac{3}{2}\left(\frac{\dot{R}}{R}\right)^2 + \frac{1}{2}\Bigg\{ & g^{rr}\left[\left(\frac{\dot{R}}{R}g_{rr}\right)_{,t} + \left(\frac{\dot{R}}{R}\right)^2 g_{rr} + \frac{2\kappa}{1-kr^2}\right] + \\ & +g^{\theta\theta}\left[\left(\frac{\dot{R}}{R}g_{\theta\theta}\right)_{,t} + \left(\frac{\dot{R}}{R}\right)^2 g_{\theta\theta} + 2\kappa r^2\right] + \\ & +g^{\phi\phi}\left[\left(\frac{\dot{R}}{R}g_{\phi\phi}\right)_{,t} + \left(\frac{\dot{R}}{R}\right)^2 g_{\phi\phi} + 2kr^2\sin^2\theta\right]\Bigg\} \end{aligned}$$

$$\Rightarrow G_{tt} = 3\left(\frac{\dot{R}}{R}\right)^2 + 3\,\frac{\kappa}{R^2}. \tag{4.36}$$

In Eq. (3.30), we can consider that the cosmological constant is just another contribution to the whole stress-energy tensor, given by [10]

$$T_\Lambda^{\alpha\beta} = -(\Lambda/8\pi)g^{\alpha\beta}. \tag{4.37}$$

From this point of view, we can write an energy density and a pressure associated to the cosmological constant [10], being respectively

$$\rho_\Lambda = \frac{\Lambda}{8\pi}, \qquad p_\Lambda = -\rho_\Lambda. \tag{4.38}$$

With $\alpha = \beta = t$, Eq. (3.30) becomes

$$G_{tt} + \Lambda g_{tt} = 8\pi T_{tt}. \tag{4.39}$$

Substituting Eq. (4.36) in Eq. (4.39) we obtain

$$\begin{aligned}
\frac{3\dot{R}^2}{R^2} + \frac{3\kappa}{R^2} - \Lambda = 8\pi\rho \qquad \times\left(\tfrac{R^2}{3}.\tfrac{1}{2}\right) \\
\Rightarrow \frac{1}{2}\dot{R}^2 + \frac{\kappa}{2} - \frac{\Lambda R^2}{6} = \frac{4}{3}\pi R^2\rho \quad \Rightarrow \frac{1}{2}\dot{R}^2 = -\frac{\kappa}{2} + \frac{4}{3}\pi R^2\left(\rho + \frac{\Lambda}{8\pi}\right) \\
\Rightarrow \frac{1}{2}\dot{R}^2 = -\frac{\kappa}{2} + \frac{4}{3}\pi R^2(\rho_m + \rho_\Lambda),
\end{aligned} \tag{4.40}$$

known as the Friedmann equation, where ρ_m is the energy density of matter and radiation. Since we are in a matter-dominated epoch, the solution to Eq. (4.31) is $\rho m(t) = C/R^3(t)$, where C is a constant of integration. Then $[R^2\rho m](t) = C/R(t)$ decreases with time as $R(t)$ increases due to the observed Hubble expansion. In Eq. (4.40), the term $R^2\rho_\Lambda$ increases rapidly while $R^2\rho_m$ decreases, so $\dot{R}(t)$ grows directly proportional to $R(t)$ and we can see that the Universe is not only expanding, but it is expanding in an accelerated manner, propelled by a positive cosmological constant [10]. Let us now determine the acceleration of the scale factor explicitly. By differentiating Eq. (4.40) with respect to time, we have

$$\begin{aligned}
\frac{d}{dt}(\dot{R}\dot{R}) = \frac{8\pi}{3}\frac{d}{dt}(R^2\rho_m) + \frac{8\pi}{3}\frac{d}{dt}(R^2\rho_\Lambda) \\
\Rightarrow 2\ddot{R}\dot{R} = \frac{8\pi}{3}\left[2R\dot{R}(\rho_m + \rho_\Lambda) + R^2(\dot{\rho_m}^2 + \dot{\rho_\Lambda}^2)\right] \ \times\left(\tfrac{1}{\dot{R}R}\right) \\
\Rightarrow \frac{\ddot{R}}{R} = \frac{4\pi}{3}\left[2(\rho_m + \rho_\Lambda) + \frac{R}{\dot{R}}(\dot{\rho_m} + \dot{\rho_\Lambda})\right].
\end{aligned} \tag{4.41}$$

From Eq. (4.30),

$$\dot{\rho}R^3 + \rho 3R^2\dot{R} = -p3R^2\dot{R} \quad \Rightarrow \dot{\rho} = -\frac{3\dot{R}}{R}(p+\rho) \quad \Rightarrow \dot{\rho_m} + \dot{\rho_\Lambda} = -\frac{3\dot{R}}{R}(p_m + \rho_m + p_\Lambda + \rho_\Lambda).$$

And since $p_\Lambda = -\rho_\Lambda$,

$$\dot{\rho_m} + \dot{\rho_\Lambda} = -\frac{3\dot{R}}{R}(p_m + \rho_m). \tag{4.42}$$

Therefore, using equations (4.42) and (4.41), the acceleration of the scale factor is

$$\begin{aligned}
\frac{\ddot{R}}{R} = \frac{4\pi}{3}\left\{2(\rho_m + \rho_\Lambda) + \frac{R}{\dot{R}}\left[-\frac{3\dot{R}}{R}(p_m + \rho_m)\right]\right\} = \frac{4\pi}{3}(2\rho_m + 2\rho_\Lambda - 3\rho_m - 3\rho_\Lambda) \\
\Rightarrow \frac{\ddot{R}}{R} = -\frac{4\pi}{3}(\rho_m + 2p_\Lambda + 3p_m) \qquad \Rightarrow \frac{\ddot{R}}{R} = -\frac{4\pi}{3}(\rho + 3p),
\end{aligned} \tag{4.43}$$

with total energy density $\rho = \rho_m$ and total pressure $p = \frac{2}{3}p_\Lambda + p_m$. By analyzing Eq. (4.43), it is clear that a positive cosmological constant contribution to the expansion with a big enough negative pressure, as defined in Eq. (4.38), indicates an accelerated Universe expansion. A negative pressure on a fluid has the physical meaning of tension, just as the expanding balloon introduces tension on its stretched rubber as it inflates (the component of the stress tensor along the band is negative) [10].

4.2.4 The Universe - Future and Early Scenarios

Now, let us investigate different future scenarios based on the value of a positive cosmological constant for each curvature. The Friedmann equation (4.40) can be interpreted as an energy equation

$$\underbrace{\frac{1}{2}\dot{R}^2}_{kinetic\ energy} \quad \underbrace{-\frac{4}{3}\pi R^2(\rho_m + \rho_\Lambda)}_{potential\ energy} \quad = \quad \underbrace{-\frac{\kappa}{2}}_{total\ energy}, \tag{4.44}$$

so the dynamics of R will be restrained to Eq. (4.44). We also have that $\mathbf{\dot{R} > 0}$ and $\mathbf{\Lambda = 0, +1 \Rightarrow \rho_\Lambda \geq 0}$. Let us also assume that the matter content of the Universe has positive energy density. **For $\kappa = -1$,** the total energy will be positive. The potential energy has the term $R^2\rho_m$ that falls as R^{-1} and the term $R^2\rho_\Lambda$ that increases as R^2, which gives us a negative potential energy. The kinetic energy will always increase, since $\dot{R}$ grows with R, and be greater than the negative potential energy to give a positive total energy. If $\rho_\Lambda = 0$, the expansion will occur at a faster pace, but either way, there will be a hyperbolic Universe as described in Eq. (4.23) that expands endlessly. **For $\kappa = 0$,** it will be a flat expanding Universe, as in Eq. (4.19). If $\rho_\Lambda > 0$, the kinetic energy will increase to balance the rising negative potential energy, so the Universe would keep expanding. If $\rho_\Lambda = 0$, since matter density ρ_m will decrease as fast as R^{-3}, as R approaches infinity, the growing negative potential energy could asymptotically slow down to a zero expansion rate [10]. Lastly, **for $\kappa = +1$,** the total energy becomes negative. There will be an expanding closed Universe, with the spherical metric in Eq. (4.21), that, for $\rho_\Lambda = 0$, reaches a maximum expansion radius and than re-collapses. If $\rho_\Lambda > 0$, its future behaviour depends on the balance of ρ_Λ and ρ_m [10].

The same energy analogy can be applied to the case of a negative cosmological constant: from Eq. (4.43), a big enough positive pressure would lead to negative acceleration, which shows us that the Universe would be rapidly contracting at a certain point and, regardless of curvature, the kinetic energy would always attain to zero, making the Big Crunch[2] an inevitable future. The observable data suggests that the cosmological constant should be small and positive [19] and that the observable Universe is flat and accelerated [20].

Besides examining possible futures, we may also look back at the early Universe, exploring the possibilities of the Big Bang scenario. For it to happen, it would have been a moment where the scale factor R had a null value at a finite time in the past. On Eq. (4.40), as R gets smaller, the matter term becomes more significant than the curvature term $-\kappa/2$ since it grows proportional to R^{-1} on

[2]The Big Crunch is a hypothetical scenario in which the expansion of the Universe would eventually be reversed and the Universe would collapse upon itself, ultimately causing the cosmic scale factor to reach zero. It could be followed by a reformation of the Universe starting with another Big Bang.

the matter-dominated epoch and to R^{-2} during the early Universe epoch. Therefore, all three kinds of universes have very similar beginning scenarios and the possibility of reaching a point where $R = 0$ in the past depends only on the behaviour of matter [10]. Let us calculate a function of R for the radiation-dominated Universe (early Universe) and simplify it by setting $\Lambda = 0$. We neglect the κ term on Eq. (4.40) and we can write $\rho = BR^{-4}$ for an arbitrary constant B, so it becomes

$$\dot{R}^2 = \frac{8\pi}{3} B R^{-2} \quad \Rightarrow \quad R\frac{dR}{dt} = \left(\frac{8}{3}\pi B\right)^{1/2} \tag{4.45}$$

and has the solution

$$R^2 = \left(\frac{32}{3}\pi B\right)^{1/2} (t - T) \tag{4.46}$$

with T as the constant of integration. Therefore, at a finite time in the past, $R = 0$ was reached and our zero of time can be adjusted so that $R = 0$ when $t = T = 0$. So in a radiation-dominated cosmology without a cosmological constant, the Universe had an expansion rate of $R(t) \; \alpha \; t^{1/2}$. If we assume that the Universe was matter-dominated at times early enough for one to be able to neglect κ in the Friedmann equation and to have zero cosmological constant, we can integrate Eq. (4.31) to become

$$\int_t^{t_0} \frac{d}{dt}(\rho R^3) = 0 \quad \Rightarrow \rho_0 R^3(t_0) - \rho(t) R^3(t) = 0$$
$$\rho(t) = \rho_0 \frac{1}{R^3(t)}, \tag{4.47}$$

with t_0 being the time of measurement (that is very close to the beginning of the Universe) and $R(t_0) = R_0 = 1$. Inserting this relation in Eq. (4.40) and integrating from R to R_0 and from t to t_0, we find

$$\frac{1}{2}\dot{R}^2 = \frac{4}{3}\pi R^2 \rho_m = \frac{4}{3}\pi R^2 \frac{\rho_0}{R^3(t)} \quad \Rightarrow \dot{R} = \frac{8\pi}{3}\frac{\rho_0}{R^{1/2}}$$
$$\int_R^{R_0} dR' \sqrt{R'} = \frac{8\pi}{3}\rho_0 \int_t^{t_0} dt \Rightarrow R = \left[1 + 4\pi\rho_0(t - t_0)\right]^{2/3}.$$

If we choose our early time to be $t_0 = \frac{1}{4\pi\rho_0}$ and use the freedom to shift the origin of the time axis, we find an expansion rate of

$$R = (4\pi\rho_0)^{2/3} t^{2/3} \qquad \Rightarrow R(t) \; \alpha \; t^{2/3}. \tag{4.48}$$

And if there is a positive cosmological constant, the solutions in Eq. (4.46) and Eq. (4.48) will not suffer big changes since ρ_Λ will only increase the value of $\dot{R}$ ant any given value of R, so the moment $R = 0$ just shifts a little closer to the present epoch [10].

Thus, with a non-negative cosmological constant, along with the assumption that matter density is positive, the Einstein field equations indicate that the Universe began at a finite time in the past with $R = 0$, the so-called Big Bang. Since time simply began at the Big Bang, it is not possible to know what happened at earlier times (within the context of GR). Furthermore, for a negative cosmological constant, the radiation term dominates the potential energy term, which overwhelms the total energy for sufficiently small R, regardless of curvature $\kappa = \{-1, 0, +1\}$. So the presently expanding universe must have also come from the Big Bang.

4.2.5 Einstein's Static Solution

Einstein added the cosmological constant so that his equations would admit a static solution $\dot{R} = 0$, so let us verify if such solution is possible from Eq. (4.43). We may also guarantee that it is an equilibrium solution, meaning that the system is at a maximum or a minimal of the potential so that the dynamics of $\dot{R}$ will not change. Because the Universe is matter-dominated at the present time, we can use Eq. (4.47), $\rho(t) = \rho_0 \frac{R_0^3}{R^3(t)}$, where the subscript '0' refers to the static solution. Substituting this relation on the Eq. (4.40) we have

$$\frac{1}{2}\dot{R}^2 = -\frac{\kappa}{2} + \frac{4}{3}\pi\frac{\rho_0 R_0^3}{R} + \frac{4}{3}\pi\rho_\Lambda R^2 \qquad \Rightarrow \dot{R}^2 = -\kappa + \frac{8\pi}{3}\frac{\rho_0 R_0^3}{R} + \frac{8}{3}\pi\rho_\Lambda R^2. \tag{4.49}$$

And differentiating Eq. (4.49) with respect to time gives us

$$\begin{aligned} 2\dot{R}\ddot{R} &= 0 - \frac{8\pi}{3}\frac{\rho_0 R_0^3 \dot{R}}{R^2} + \frac{8}{3}\pi\rho_\Lambda 2R\dot{R} \qquad \times \left(\tfrac{1}{2\dot{R}}\right) \\ \ddot{R} &= \frac{8\pi}{3}\rho_\Lambda R - \frac{4\pi}{3}\rho_0 R_0^3 R^{-2}. \end{aligned} \tag{4.50}$$

Since we are looking for a static solution,

$$\ddot{R} = \frac{8\pi}{3}\rho_\Lambda R(t_0) - \frac{4\pi}{3}\rho_0 R_0^3 R(t_0)^{-2} = 0 \quad \Rightarrow \frac{8\pi}{3}\rho_\Lambda R_0 = \frac{4\pi}{3}\rho_0 R_0^3 R_0^{-2} \quad \Rightarrow \rho_\Lambda = \frac{1}{2}\rho_0. \tag{4.51}$$

That means that for Einstein's static solution, the cosmological constant energy density has to be half of the matter energy density. If we substitute the resultant Eq. (4.51) on the first derivative in Eq. (4.49) for $R(t) = R(t_0)$

$$\begin{aligned} \dot{R}^2 = -\kappa + \frac{8\pi}{3}\frac{\rho_0 R_0^3}{R} + \frac{8}{3}\pi\rho_\Lambda R^2 \qquad &\Rightarrow \dot{R(t_0)}^2 = -\kappa + \frac{8\pi}{3}\frac{\rho_0 R_0^3}{R_0} + \frac{8}{3}\pi\frac{\rho_0}{2}R_0^2 = 0 \\ &\Rightarrow \kappa = \frac{8\pi}{3}\rho_0 R_0^2 + \frac{4\pi}{3}\rho_0 R_0^2 = 4\pi\rho_0 R_0^2, \end{aligned} \tag{4.52}$$

we see that the first time-derivative vanishes for this value of κ at the static solution. Furthermore, if we obtain the second derivative of the right-hand-side (**RH**) of Eq. (4.40) with respect to R

$$\frac{d^2\mathbf{RH}}{dR^2} = \frac{d}{dt}\left[\frac{8\pi}{3}R(\rho_m + \rho_\Lambda)\right] = \frac{8\pi}{3}(\rho_m + \rho_\Lambda), \tag{4.53}$$

and evaluate it at the static solution, where $\rho_m = \rho_0$ and $\rho_\Lambda = \rho_0/2$, we have

$$\frac{d^2\mathbf{RH}}{dR^2} = \frac{8\pi}{3}\frac{3\rho_0}{2} = 4\pi\rho_0 > 0, \tag{4.54}$$

assuming that the Universe has positive matter density. The positive result above indicates that the potential is a minimum and the static solution is stable.

4.2.6 The Critical Density

The Universe would be static if the relation in Eq. (4.51) could be measured. Although, experimentation has already confirmed the Hubble flow and the Einstein equations agree with the expansion, so how far is the present epoch Universe from being static? What is the present ratio between matter density and the cosmological constant density? If we divide Eq. (4.40) by $\frac{4\pi R^2}{3}$, we have that

$$\frac{3\dot{R}^2}{8\pi R^2} = -\frac{3\kappa}{8\pi R^2} + \rho_m + \rho_\Lambda. \tag{4.55}$$

Let us recall the Hubble flow. Measurements of the expansion of the universe were made using the redshift of galaxies. If we consider a galaxy is at a fixed coordinate position on some hypersurface at constant cosmological time t, we receive the light emitted at t_0. If we consider our galaxy to be fixed at a position coordinate χ at time t, it can be given as a function of the proper distance $d_0 = d(t_0)$ from our reference frame at the present time t_0, being

$$\chi = R(t)d_0 \tag{4.56}$$

If we differentiate Eq. (4.56) in relation to time,

$$v = \dot{\chi} = \dot{R(t)}d_0 = \frac{\chi \dot{R(t)}}{R(t)}. \tag{4.57}$$

Since the Hubble parameter is given by Eq. (4.1), the current rate of change of proper distance between our reference frame at the origin and the galaxy at fixed χ is

$$v = \frac{\chi \dot{R(t)}}{R(t)} = H\chi \qquad \Rightarrow \quad H(t) = \frac{\dot{R}(t)}{R(t)}. \tag{4.58}$$

This means that the Hubble parameter is the instantaneous relative rate of the Universe. If we substitute Eq. (4.58) in Eq. (4.55), we get

$$\frac{3H^2}{8\pi} = -\frac{3\kappa}{8\pi R^2} + \rho_m + \rho_\Lambda. \tag{4.59}$$

Since the last two terms in Eq. (4.59) are densities, we can interpret the other terms in this manner as well, so that the Hubble expansion has an associated energy density $\rho_H = 3H^2/8\pi$ and the spatial curvature parameter contributes with the energy density $\rho_\kappa = -3\kappa/8\pi R^2$. Then Eq. (4.59) becomes

$$\rho_H = \rho_\kappa + \rho_m + \rho_\Lambda, \tag{4.60}$$

called the critical density equation, since different values for the Hubble energy density ρ_H work as a limiting factor for values of the curvature energy density ρ_κ and, therefore, determine the cosmological metric. From Eq. (4.60), if the physical energy density $\rho = \rho_m + \rho_\Lambda$ is greater than ρ_H, then ρ_κ must be negative, making the curvature parameter κ positive, resulting in a RW metric for a closed universe. On the contrary, if $\rho < \rho_H$, than ρ_κ must be positive, the curvature parameter κ would be negative and we would have an open, hyperbolic universe. If we divide Eq. (4.60), evaluated at present time t_0, by the critical density ρ_c

$$\rho_c = \frac{3}{8\pi}H_0^2, \tag{4.61}$$

with H_0 from Eq. (4.2). We then get

$$1 = \Omega_\kappa + \Omega_m + \Omega_\Lambda, \qquad \Omega_n := \frac{\rho_n}{\rho_c}, \tag{4.62}$$

and the observational data ([10], pp. 359-360) points to

$$\Omega_\Lambda = 0.7, \quad \Omega_m = 0.3, \quad \Omega_\kappa = 0. \tag{4.63}$$

So the measured value of the cosmological constant energy density is about twice that of the matter energy density, so we are near to but not exactly at Einstein's static solution. The results indicate that our Universe is flat and is dominated by the energy density of a positive cosmological constant instead of the matter energy density. To put into numbers, using the scaled Hubble constant $h = H_0/100km.s^{-1}Mpc^{-1} = 0.71$ and leaving the geometrized units system for a moment, the critical energy density is

$$\rho_c = \frac{3}{8\pi G}H_0^2 = \frac{3h^2 \times 10^{10}m^2s^{-2}Mpc^{-2}}{8\pi 6,674 \times 10^{-11}m^3kg^{-1}s^{-2}} = \frac{3 \times 10^{21}h^2m^2kg}{8\pi 6,674m^3(3,1 \times 10^{22}m)^2}$$
$$\Rightarrow \rho_c = 1,86 \times 10^{-26}h^2kgm^{-3} = 9,4 \times 10^{-27}kgm^{-3}. \tag{4.64}$$

Therefore, the matter energy density in our Universe is $\rho_m = 0.3\rho_c = 2.8 \times 10^{-27}kgm^{-3}$, being a larger result than the measured matter density obtained through observations of stars and galaxies. Studies investigating the formation of elements in the early universe indicate that the density of baryonic matter (matter made of protons and neutrons) has only $\Omega_b = 0.04$, so most of the matter in the universe is non-baryonic, does not emit light, and can only be studied indirectly through its gravitational effects, also referred to as 'dark matter' ([10], pp. 359).

Chapter 5

Conclusions and perspectives

The present work aimed to solve Einstein's field equations for the cosmological boundary conditions. After developing an initial discussion surrounding physical knowledge evolution through the centuries and then building up the necessary mathematical background, we hope to have made the justification for Einstein's equations clear. We also expect that the solution to such equations found in this work resonates with the reader, both in a personal and academic level, complementing their insights on the subject.

Through the results analysis and association with the referenced present observational data, we have found that we live in a likely flat, expanding universe, dominated by a positive small cosmological constant. By solving the Einstein field equations and considering a non-negative cosmological constant, it all indicates that the Universe began at a finite time in the past. We mathematically see that the cosmological constant is driving the universal expansion, but "what" it exactly means and "how" it is done is still a matter of debate. Some hypothesis discussing dark matter and dark energy [10] arose, but it is still not a consensus and needs further experimental backup. By exploring the static solution, we see that matter density should be higher to cease the expansion, so it will likely not happen. Also the Universe's critical density indicates that most of matter composition is non-baryonic and does not emit light, so there is still a great room for experimentation and future discoveries to unfold such mysteries.

Other fields of interest in General Relativity that were not discussed here also propel other researches and interesting discussions. Some worth mentioning are the study of quasars and compact X-ray sources, diving into gravitational collapse and attempting to describe the behaviour of strong gravitational fields [4]. There is also the ultimate goal of developing a quantum theory for gravitation and studies about the creation of particles by black holes seem to be promising [4]. Furthermore, we could not forget to mention the outstanding results of gravitational waves detection [18] and the first ever photograph taken from a black hole [21]. Developments like these only add even more thrilling fuel to the scientific community to keep moving forward. There are still plenty of questions which demand answers and it is impossible to hide the excitement of living in an era that we might watch them unfold.

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