QSL from non-extensive relative entropies

In [2], the authors derived a bound for the quantum speed limit times from Rényi and Tsallis relative entropies. Our aim is obtaining an analytical expression for the QSL time. Starting with Rényi entropy, the QSL time is given by

$$\tau_\alpha^{\mathcal{R}}(\rho_t\|\rho_0) := \frac{|1-\alpha| R_\alpha(\rho_t\|\rho_0)}{\langle\!\langle \mathcal{G}_\alpha^R(t) \rangle\!\rangle_\tau}. \tag{1}$$

Let us proceed step by step, firstly calculating $\rho(t)$. It is

$$\rho(t) = U\rho_0 U^\dagger = e^{-iH_f t}\frac{e^{-\beta H_0}}{Z_0}e^{+iH_f t}. \tag{2}$$

We calculate now separately the exponentials, since this is the most cumbersome part. We have

$$\begin{aligned} \exp\left(-itH_f\right) &= \exp\left[-it(\hat{H}_0 \oplus \hat{H}_1)\right] \\ &= \exp\left[-it(\hat{\mathbb{H}}_0\hat{P}_0 \oplus \hat{\mathbb{H}}_1\hat{P}_1)\right] \\ &= \exp\left[-it\hat{\mathbb{H}}_0\hat{P}_0\right] \oplus \exp\left[-it\hat{\mathbb{H}}_1\hat{P}_1\right] \end{aligned} \tag{3}$$

with

$$\hat{P}_p = \frac{1}{2}\left(\hat{1} + (-1)^p e^{i\pi\hat{N}}\right) \tag{4}$$

being the projector in the $p = 0,1$ parity sector. Notice that each $\hat{H}_p$ has dimension 2^{L-1}. After Jordan-Wigner and Fourier transformations, we write $\hat{\mathbb{H}}_p$ in the form

$$\hat{\mathbb{H}}_p = \sum_{k\in\mathcal{K}_p}(h - J\cos(k))(\hat{c}_k^\dagger\hat{c}_k - \hat{c}_{-k}\hat{c}_{-k}^\dagger) - J(e^{-2i\theta}e^{ik}\hat{c}_k^\dagger\hat{c}_{-k}^\dagger + \text{H.c.}) \tag{5}$$

with

$$p = 0 \Longrightarrow \mathcal{K}_{p=0} = \left\{\pm\frac{\pi}{L}, \pm\frac{3\pi}{L}, \pm\frac{5\pi}{L}, \cdots, \pm\frac{(L-1)\pi}{L}\right\} = \mathbf{k}_0 \cup \{-\mathbf{k}_0\} \tag{6}$$

$$p = 1 \Longrightarrow \mathcal{K}_{p=1} = \left\{0, \pm\frac{2\pi}{L}, \pm\frac{4\pi}{L}, \cdots, \pm\frac{(L-2)\pi}{L}, \pi\right\} = \mathbf{k}_1 \cup \{-\mathbf{k}_1\} \cup \{0,\pi\} \tag{7}$$

For $p = 1$, we have $\hat{c}_{L+1} = \hat{c}_1$ and for $p = 0$, we have $\hat{c}_{L+1} = -\hat{c}_1$. The two terms with $k = 0$ and $k = \pi$

present in $p = 1$ (PBC) are

$$\hat{H}_{k=0} = -2J(\hat{n}_0 - \hat{n}_\pi) - 2h(\hat{n}_0 + \hat{n}_\pi - 2). \tag{8}$$

Since that k values appear in pairs $(k, -k)$, it's possible to rewrite equation (54) such that

$$\hat{\mathbb{H}}_0 = \sum_{k\in\mathbf{k}_0} \hat{H}_k; \quad \hat{\mathbb{H}}_1 = \sum_{k\in\mathbf{k}_1} \hat{H}_k + \hat{H}_{k=0} + \hat{H}_{k=\pi} \tag{9}$$

with

$$\hat{H}_k = 2(h - J\cos(k))\left(\hat{c}_k^\dagger \hat{c}_k + \hat{c}_{-k}\hat{c}_{-k}^\dagger\right) - 2J\sin(k)\left(ie^{-2i\theta}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger - ie^{2i\theta}\hat{c}_{-k}\hat{c}_k\right). \tag{10}$$

and

$$\hat{H}_{k=0} = (h-1)(2\hat{n}_0 - 1) \tag{11}$$

$$\hat{H}_{k=\pi} = (h+1)(2\hat{n}_\pi - 1). \tag{12}$$

Interestingly, the matrix $\hat{H}_k$ lives in a 4-dimensional space, spanned by the states

$$\{\hat{c}_k^\dagger \hat{c}_{-k}^\dagger |0\rangle, |0\rangle, \hat{c}_k^\dagger|0\rangle, \hat{c}_{-k}^\dagger|0\rangle\} = \{|2\rangle, |0\rangle, |1_k\rangle, |1_{-k}\rangle\} \tag{13}$$

and it has the matrix representation

$$H_k = \begin{pmatrix} 2(h - J\cos(k)) & -2iJe^{-2i\theta}\sin(k) & 0 & 0 \\ 2iJe^{2i\theta}\sin(k) & -2(h - J\cos(k)) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

In order to diagonalize H_k, which is our aim, we use the Bogoliubov transformation in the 2×2 non-diagonal block of H_k. The Jordan-Wigner creation and annihilation operators are related to the Bogoliubov operators by the following transformation

$$\begin{pmatrix} \hat{\gamma}_k \\ \hat{\gamma}_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\phi_k}{2}\right) & -i\sin\left(\frac{\phi_k}{2}\right) \\ -i\sin\left(\frac{\phi_k}{2}\right) & \cos\left(\frac{\phi_k}{2}\right) \end{pmatrix} \begin{pmatrix} \hat{c}_k \\ \hat{c}_{-k}^\dagger \end{pmatrix}. \tag{15}$$

where we define the Bogoliubov angle ϕ_k as

$$\tan(\theta_k) = \frac{\sin(k)}{h - \cos(k)}. \tag{16}$$

Fixing $\theta = \frac{\pi}{4}$, which means that the magnetic field lives in the $x - z$ plane and $J = 1$, the operador

$\hat{H}_k$ in equation (59) becomes

$$\hat{H}_k = \epsilon_k(h_f)\left(\hat{\gamma}_k^\dagger \hat{\gamma}_k + \hat{\gamma}_{-k}^\dagger \hat{\gamma}_{-k} - 1\right) = \epsilon_k(h_f)(\hat{n}_k + \hat{n}_{-k} - 1) \tag{17}$$

with the energies given by $\epsilon_k(h_f) = 2\sqrt{\sin^2(k) + [h_f - \cos(k)]^2}$ and the matrix representation

$$H_k = \begin{pmatrix} \epsilon_k(h_f) & 0 & 0 & 0 \\ 0 & -\epsilon_k(h_f) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

where we have used $\epsilon_{-k} = \epsilon_k$.

1 Discarding the negative parity subspace

The exponentials in equation (3) mixes the positive and negative parity subspaces, making the calculation extremely complicated. To avoid this, we discard the negative parity subspace $\mathbb{H}_1$.

Returning to equation (3), we'll have

$$\begin{aligned} e^{-itH_f} &= \exp\left(-it \sum_{k\in\mathbf{k}_0} H_k\right) \\ &= \exp\left[-it \sum_{k\in\mathbf{k}_0} \epsilon_k(h_f)(\tilde{n}_k + \tilde{n}_{-k} - 1)\right] \\ &= \prod_{k\in\mathbf{k}_0} e^{-it\epsilon_k(h_f)(\tilde{n}_k + \tilde{n}_{-k} - 1)} \\ &= \prod_{k\in\mathbf{k}_0} \left[e^{-it\epsilon_k(h_f)}|\tilde{2}\rangle\langle\tilde{2}| + e^{it\epsilon_k(h_f)}|\tilde{0}\rangle\langle\tilde{0}| + 1|\tilde{1}_k\rangle\langle\tilde{1}_k| + 1|\tilde{1}_{-k}\rangle\langle\tilde{1}_{-k}|\right] \end{aligned} \tag{19}$$

$$e^{itH_f} = \prod_{k\in\mathbf{k}_0} \left[e^{it\epsilon_k(h_f)}|\tilde{2}\rangle\langle\tilde{2}| + e^{-it\epsilon_k(h_f)}|\tilde{0}\rangle\langle\tilde{0}| + 1|\tilde{1}_k\rangle\langle\tilde{1}_k| + 1|\tilde{1}_{-k}\rangle\langle\tilde{1}_{-k}|\right] \tag{20}$$

$$\rho_0 = \frac{e^{-\beta H_0}}{Z_0} = \frac{1}{Z_0} \prod_{k\in\mathbf{k}_0} \left[e^{-\beta\epsilon_k(h_f)}|\tilde{2}\rangle\langle\tilde{2}| + e^{\beta\epsilon_k(h_f)}|\tilde{0}\rangle\langle\tilde{0}| + 1|\tilde{1}_k\rangle\langle\tilde{1}_k| + 1|\tilde{1}_{-k}\rangle\langle\tilde{1}_{-k}|\right] \tag{21}$$

where $Z_0 = \mathrm{Tr}(e^{-\beta H_0}) = \prod_k 4\cosh^2\left(\frac{\beta\epsilon_k(h_0)}{2}\right)$. Multiplying all these terms, we'll have

$$\begin{aligned}\rho(t) =& \frac{1}{Z_0}\prod_{k\in\mathbf{k}_0} e^{-\beta\epsilon_k(h_0)}\cos^2\left(\frac{\Delta_k}{2}\right)|\tilde{2}\rangle\langle\tilde{2}| - ie^{-\beta\epsilon_k(h_0)-2it\epsilon_k(h_f)}\cos\left(\frac{\Delta_k}{2}\right)\sin\left(\frac{\Delta_k}{2}\right)|\tilde{2}\rangle\langle\tilde{0}| \\ &+ie^{+\beta\epsilon_k(h_0)+2it\epsilon_k(h_f)}\cos\left(\frac{\Delta_k}{2}\right)\sin\left(\frac{\Delta_k}{2}\right)|\tilde{0}\rangle\langle\tilde{2}| + e^{\beta\epsilon_k(h_0)}\cos^2\left(\frac{\Delta_k}{2}\right)|\tilde{0}\rangle\langle\tilde{0}| \\ &+ie^{-\beta\epsilon_k(h_0)+2it\epsilon_k(h_f)}\cos\left(\frac{\Delta_k}{2}\right)\sin\left(\frac{\Delta_k}{2}\right)|\tilde{0}\rangle\langle\tilde{2}| + e^{-\beta\epsilon_k(h_0)}\sin^2\left(\frac{\Delta_k}{2}\right)|\tilde{0}\rangle\langle\tilde{0}| \\ &+e^{\beta\epsilon_k(h_0)}\sin^2\left(\frac{\Delta_k}{2}\right)|\tilde{2}\rangle\langle\tilde{2}| - ie^{\beta\epsilon_k(h_0)-2it\epsilon_k(h_f)}\cos\left(\frac{\Delta_k}{2}\right)\sin\left(\frac{\Delta_k}{2}\right)|\tilde{2}\rangle\langle\tilde{0}| + 1|\tilde{1}_k\rangle\langle\tilde{1}_k| + 1|\tilde{1}_{-k}\rangle\langle\tilde{1}_{-k}|\end{aligned} \tag{22}$$

$$\begin{aligned}\rho(t) =& \frac{1}{Z_0}\prod_{k\in\mathbf{k}_0}\left[e^{-\beta\epsilon_k(h_0)}\cos^2\left(\frac{\Delta_k}{2}\right) + e^{\beta\epsilon_k(h_0)}\sin^2\left(\frac{\Delta_k}{2}\right)\right]|\tilde{2}\rangle\langle\tilde{2}| + \left[e^{\beta\epsilon_k(h_0)}\cos^2\left(\frac{\Delta_k}{2}\right) + e^{-\beta\epsilon_k(h_0)}\sin^2\left(\frac{\Delta_k}{2}\right)\right]|\tilde{0}\rangle\langle\tilde{0}| \\ &+2i\cosh(\beta\epsilon_k(h_0))e^{2it\epsilon_k(h_f)}\cos\left(\frac{\Delta_k}{2}\right)\sin\left(\frac{\Delta_k}{2}\right)|\tilde{0}\rangle\langle\tilde{2}| - 2i\cosh(\beta\epsilon_k(h_0))e^{-2it\epsilon_k(h_f)}\cos\left(\frac{\Delta_k}{2}\right)\sin\left(\frac{\Delta_k}{2}\right)|\tilde{2}\rangle\langle\tilde{0}| \\ &+ 1|\tilde{1}_k\rangle\langle\tilde{1}_k| + 1|\tilde{1}_{-k}\rangle\langle\tilde{1}_{-k}|\end{aligned}$$

$$\rho(t) = \frac{1}{Z_0}\prod_{k\in\mathbf{k}_0}\begin{pmatrix} e^{-\beta\epsilon_k(h_0)}\cos^2\left(\frac{\Delta_k}{2}\right) + e^{\beta\epsilon_k(h_0)}\sin^2\left(\frac{\Delta_k}{2}\right) & 2i\cosh(\beta\epsilon_k(h_0))e^{2it\epsilon_k(h_f)}\cos\left(\frac{\Delta_k}{2}\right)\sin\left(\frac{\Delta_k}{2}\right) \\ -2i\cosh(\beta\epsilon_k(h_0))e^{-2it\epsilon_k(h_f)}\cos\left(\frac{\Delta_k}{2}\right)\sin\left(\frac{\Delta_k}{2}\right) & e^{\beta\epsilon_k(h_0)}\cos^2\left(\frac{\Delta_k}{2}\right) + e^{-\beta\epsilon_k(h_0)}\sin^2\left(\frac{\Delta_k}{2}\right)\end{pmatrix} \oplus I_2. \tag{23}$$

Notice that $\mathrm{Tr}(\rho(t)) = 1$ as it must be and $\Delta_k = \tilde{\theta}_k - \theta_k$ is the difference between the final and the initial Bogoliubov angles. This result is quite amazing, $\rho(t)$ is given by the product of four dimensional matrices ρ_k, each of them acting in a different subspace of the Hilbert space. It implies that

$$[\rho_k, \rho_{k'}] = 0, \quad \text{if } k \neq k'. \tag{24}$$

We'll explore this property in what follows. We need to calculate the α power of $\rho(t)$. There is an expression for an α power of a generic 2×2 matrix

$$X = \begin{pmatrix} a & b \\ c & d\end{pmatrix}, \tag{25}$$

namely

$$X^\alpha = \left(\frac{\lambda_+^\alpha - \lambda_-^\alpha}{\lambda_+ - \lambda_-}\right)X - \left(\lambda_+\lambda_-\frac{\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}}{\lambda_+ - \lambda_-}\right)I_2 \tag{26}$$

where $\lambda_\pm$ are the eigenvalues of X given by

$$\lambda_\pm = \frac{\chi}{2} \pm \sqrt{\frac{\chi^2}{4} - \delta} \tag{27}$$

and $\chi = \mathrm{Tr}(X)$, $\delta = \det(X)$. In order to simplify the notation, let us give generic names to the terms in the expressions above. We rewrite equation (26) as

$$X^\alpha = c_1(\alpha) X - c_2(\alpha) I_2 \tag{28}$$

and $\rho(t)$ as

$$\rho(t) = \frac{1}{Z_0} \prod_{k \in \mathbf{k}_0} \left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ \overline{\rho_{12}} & \rho_{22} \end{pmatrix} \oplus I_2 \right]. \tag{29}$$

where the bar means complex conjugate. Now, we simply have

$$\begin{aligned} \rho^\alpha(t) &= \frac{1}{Z_0^\alpha} \prod_{k \in \mathbf{k}_0} \left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ \overline{\rho_{12}} & \rho_{22} \end{pmatrix}^\alpha \oplus I_2 \right] + \cdots \\ &= \frac{1}{Z_0^\alpha} \prod_{k \in \mathbf{k}_0} \left[\begin{pmatrix} c_1(\alpha)\rho_{11} - c_2(\alpha) & c_1(\alpha)\rho_{12} \\ c_1(\alpha)\overline{\rho_{12}} & c_1(\alpha)\rho_{22} - c_2(\alpha) \end{pmatrix} \oplus I_2 \right] \\ &= \frac{1}{Z_0^\alpha} \prod_{k \in \mathbf{k}_0} \rho_k(t, \alpha) \end{aligned} \tag{30}$$

where

$$c_1(\alpha) = \frac{\lambda_+^\alpha - \lambda_-^\alpha}{\lambda_+ - \lambda_-}, \quad c_2(\alpha) = \lambda_+ \lambda_- \frac{\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}}{\lambda_+ - \lambda_-} \tag{31}$$

and

$$\lambda_\pm = \cosh(\beta\epsilon_k(h_0)) \pm \frac{\sqrt{2}}{2} \sqrt{\cosh(2\beta\epsilon_k(h_0)) - \cos(2\Delta_k)}. \tag{32}$$

We also need $\rho_0^{1-\alpha}$ for the calculation of Rényi entropy. In the final hamiltonian basis, ρ_0 is written as

$$\begin{aligned} \rho_0 = \frac{1}{Z_0} \prod_{k \in \mathbf{k}_0} &\begin{pmatrix} e^{-\beta\epsilon_k(h_0)} \cos^2\left(\frac{\Delta_k}{2}\right) + e^{\beta\epsilon_k(h_0)} \sin^2\left(\frac{\Delta_k}{2}\right) & 2i\cosh(\beta\epsilon_k(h_0)) \cos\left(\frac{\Delta_k}{2}\right) \sin\left(\frac{\Delta_k}{2}\right) \\ -2i\cosh(\beta\epsilon_k(h_0)) \cos\left(\frac{\Delta_k}{2}\right) \sin\left(\frac{\Delta_k}{2}\right) & e^{\beta\epsilon_k(h_0)} \cos^2\left(\frac{\Delta_k}{2}\right) + e^{-\beta\epsilon_k(h_0)} \sin^2\left(\frac{\Delta_k}{2}\right) \end{pmatrix} \oplus I_2 \\ = \frac{1}{Z_0} \prod_{k \in \mathbf{k}_0} &\left[\begin{pmatrix} \rho_{11}^0 & \rho_{12}^0 \\ \overline{\rho_{12}^0} & \rho_{22}^0 \end{pmatrix} \oplus I_2 \right]. \end{aligned} \tag{33}$$

Hence, $\rho_0^{1-\alpha}$ will be

$$\begin{aligned}\rho_0^{1-\alpha} &= \frac{1}{Z_0^{1-\alpha}} \prod_{k\in\mathbf{k}_0} \left[\begin{pmatrix} c_1^0(1-\alpha)\rho_{11}^0 - c_2^0(1-\alpha) & c_1^0(1-\alpha)\rho_{12}^0 \\ c_1^0(1-\alpha)\overline{\rho_{12}^0} & c_1^0(1-\alpha)\rho_{22}^0 - c_2^0(1-\alpha) \end{pmatrix} \oplus I_2 \right] \\ &= \frac{1}{Z_0^{1-\alpha}} \prod_{k\in\mathbf{k}_0} \rho_k^0(1-\alpha).\end{aligned} \tag{34}$$

Interestingly, the eigenvalues of ρ_0, $\{\lambda_+^0, \lambda_-^0\}$, are equal to the eigenvalues of $\rho(t), \{\lambda_+, \lambda_-\}$.

In the calculation of Rényi entropy,

$$R_\alpha(\rho(t)\|\rho_0) = \frac{1}{\alpha-1}\log[\mathrm{Tr}[\rho^\alpha(t)\rho_0^{1-\alpha}]], \tag{35}$$

we have to be careful with the commutativity of the matrices involved. Although the products appearing in $\rho^\alpha(t)$ and $\rho_0^{1-\alpha}$ run over the same k values, we can't write the product between $\rho^\alpha(t)$ and $\rho_0^{1-\alpha}$ putting together their arguments inside only one product over k. That is,

$$\rho^\alpha(t)\rho_0^{1-\alpha} = \frac{1}{Z_0}\prod_{k\in\mathbf{k}_0}\rho_k(t,\alpha)\prod_{k\in\mathbf{k}_0}\rho_k^0(1-\alpha) \neq \frac{1}{Z_0}\prod_{k\in\mathbf{k}_0}\left[\rho_k(t,\alpha)\rho_k^0(1-\alpha)\right] \tag{36}$$

This is because the argument of the product in $\rho^\alpha(t)$ doesn't commute with the argument of the product in $\rho_0^{1-\alpha}$.

In order to calculate the trace, let us call the elements in the set $\mathbf{k}_0 = \{k(1), k(2), ..., k(m)\}$, where $m = |\mathbf{k}_0|$. Since $\rho^\alpha(t)$ and $\rho_0^{1-\alpha}$ are products of four-dimensional matrices, then so $\rho^\alpha(t)\rho_0^{1-\alpha}$ also is. Therefore, we have

$$\begin{aligned}\mathrm{Tr}[\rho^\alpha(t)\rho_0^{1-\alpha}] &= [\rho^\alpha(t)\rho_0^{1-\alpha}]_{11} + [\rho^\alpha(t)\rho_0^{1-\alpha}]_{22} + [\rho^\alpha(t)\rho_0^{1-\alpha}]_{33} + [\rho^\alpha(t)\rho_0^{1-\alpha}]_{44} \\ &= [\rho^\alpha(t)\rho_0^{1-\alpha}]_{11} + [\rho^\alpha(t)\rho_0^{1-\alpha}]_{22} + 2.\end{aligned} \tag{37}$$

where the last equality is possible since $\rho^\alpha(t)$ and $\rho_0^{1-\alpha}$ are block diagonal matrices and the second block of them is a 2×2 identity. All we have to do is proceed calculating each of these terms. However, it turns out impossible to obtain an analytical formula which holds for any L value. All we can do is obtaining expression for particular values of chain lengths.

1.1 The case $L = 2$

In the case $L = 2$, the set $\mathbf{k}_0$ only has one value, namely, $\mathbf{k}_0 = \{\pi/2\}$ and we have

$$
\begin{aligned}
[\rho^\alpha(t)\rho_0^{1-\alpha}]_{11} &= \sum_{n=1}^{2}[\rho^\alpha(t)]_{1n}[\rho_0^{1-\alpha}]_{n1} \\
&= \sum_{n=1}^{2}[\rho_{\frac{\pi}{2}}(t,\alpha)]_{1n}[\rho^0_{\frac{\pi}{2}}(1-\alpha)]_{n1} \\
&= [\rho_{\frac{\pi}{2}}(t,\alpha)]_{11}[\rho^0_{\frac{\pi}{2}}(1-\alpha)]_{11} + [\rho_{\frac{\pi}{2}}(t,\alpha)]_{12}[\rho^0_{\frac{\pi}{2}}(1-\alpha)]_{21} \\
&= \frac{1}{Z_0}\Big(c_1(\alpha)\rho_{11} - c_2(\alpha)\Big)\Big(c_1^0(1-\alpha)\rho^0_{11} - c_2^0(1-\alpha)\Big)_{\frac{\pi}{2}} + \Big(c_1(\alpha)c_1^0(1-\alpha)\rho_{12}\overline{\rho^0_{12}}\Big)_{\frac{\pi}{2}} \\
&= \frac{1}{Z_0}\Big(c_1(\alpha)\rho_{11} - c_2(\alpha)\Big)\Big(c_1^0(1-\alpha)\rho_{11} - c_2^0(1-\alpha)\Big)_{\frac{\pi}{2}} + \Big(c_1(\alpha)c_1^0(1-\alpha)e^{2it\epsilon_k(h_f)}|\rho^0_{12}|^2\Big)_{\frac{\pi}{2}}
\end{aligned} \tag{38}
$$

2 Bounding the entropy

Due to the difficulty involved in the calculation of an exact expression, we resort to a bound for the entropy, namely [Celeri]

$$
R_\alpha(\rho_\tau\|\rho_0) \leq \frac{\tau\langle\!\langle \mathcal{G}_\alpha^R(t)\rangle\!\rangle_\tau}{|1-\alpha|} \tag{39}
$$

where $\langle\!\langle \bullet \rangle\!\rangle_\tau = \tau^{-1}\int_0^\tau \bullet\, dt$ and

$$
\mathcal{G}_\alpha^R(t) = \Phi_\alpha^{\mathcal{R}}\|\rho_0^{1-\alpha}\|_2\ \|[H_t,\rho_0^\alpha]\|_2 \tag{40}
$$

with $\|A\|_2 = \sqrt{\mathrm{Tr}(A^\dagger A)}$ being the Schatten 2-norm and $\Phi_\alpha^{\mathcal{R}} = |1 + (1-\alpha)\ln(\lambda_{\min}(\rho_0))|^{-1}$. The term $\lambda_{\min}(\rho_0)$ means the smallest eigenvalue of ρ_0. In our case, $H_t = H(h_f) = H_f$ is the final hamiltonian and doesn't depend on the time, then $\langle\!\langle \mathcal{G}_\alpha^R(t)\rangle\!\rangle_\tau = \mathcal{G}_\alpha^R$.

We start calculating $\|\rho_0^{1-\alpha}\|_2$. First, since ρ_0 is hermitian, then $\rho_0^{1-\alpha}$ will also be hermitian. Therefore, we have

$$
\begin{aligned}
\|\rho_0^{1-\alpha}\|_2 &= \sqrt{\mathrm{Tr}\left[(\rho_0^{1-\alpha})^\dagger\rho_0^{1-\alpha}\right]} \\
&= \frac{1}{Z_0^{1-\alpha}}\sqrt{\mathrm{Tr}\left[\prod_k \rho_k^0(1-\alpha)\prod_k \rho_k^0(1-\alpha)\right]}
\end{aligned} \tag{41}
$$

Now, since each $\rho_k^0(1-\alpha)$ acts on a different subspace, they commute with each other and we have

$$\begin{aligned}
\left\|\rho_0^{1-\alpha}\right\|_2 &= \frac{1}{Z_0^{1-\alpha}}\sqrt{\mathrm{Tr}\left[\prod_{k\in\mathbf{k}_0}\rho_k^0(1-\alpha)^2\right]} \\
&= \frac{1}{Z_0^{1-\alpha}}\sqrt{\prod_{k\in\mathbf{k}_0}\mathrm{Tr}\left[\rho_k^0(1-\alpha)^2\right]} \\
&= \frac{1}{Z_0^{1-\alpha}}\prod_{k\in\mathbf{k}_0}\sqrt{\left[c_1^0(1-\alpha)\rho_{11}^0(k)-c_2^0(1-\alpha)\right]^2+\left[c_1^0(1-\alpha)\rho_{22}^0(k)-c_2^0(1-\alpha)\right]^2+2(c_1^0)(1-\alpha)^2|\rho_{12}^0(k)|^2+2} \\
&= \frac{1}{Z_0^{1-\alpha}}\prod_{k\in\mathbf{k}_0}\sqrt{2(c_1^0)^2\sin^2(\Delta_k)+2(c_2^0)^2-4c_1^0c_2^0\cosh(\beta\epsilon_k(h_0))+2c_1^0\cosh(2\beta\epsilon_k(h_0))+2},
\end{aligned} \tag{42}$$

in the expression above we dropped out the α dependence in c_1^0 and c_2^0 for the sake of brevity. The term $\|[H_f,\rho_0^\alpha]\|_2$ reads

$$\begin{aligned}
\left\|[H_f,\rho_0^\alpha]\right\|_2 &= \sqrt{\mathrm{Tr}\left([H_f,\rho_0^\alpha]^\dagger[H_f,\rho_0^\alpha]\right)} \\
&= \sqrt{\mathrm{Tr}\left(-[H_f,\rho_0^\alpha]^2\right)} \\
&= \sqrt{\mathrm{Tr}\left(H_f(\rho_0^\alpha)^2H_f+\rho_0^\alpha(H_f)^2\rho_0^\alpha-H_f\rho_0^\alpha H_f\rho_0^\alpha-\rho_0^\alpha H_f\rho_0^\alpha H_f\right)}.
\end{aligned} \tag{43}$$

The commutator can be written as

$$\begin{aligned}
[H_f,\rho_0^\alpha] &= \left[\sum_{k'\in\mathbf{k}_0}H_{k'},\prod_{k\in\mathbf{k}_0}\rho_k^0(\alpha)\right] \\
&= \sum_{k'\in\mathbf{k}_0}\left[H_{k'},\prod_{k\in\mathbf{k}_0}\rho_k^0(\alpha)\right] \\
&= \sum_{k'\in\mathbf{k}_0}\left\{\left([H_{k'},\rho_{k(1)}^0]\prod_{k=k(2)}^{k(m)}\rho_k^0\right)+\left(\rho_{k(1)}^0[H_{k'},\rho_{k(2)}^0]\prod_{k=k(3)}^{k(m)}\rho_k^0\right)+\cdots+\left(\prod_{k=k(1)}^{k(m-1)}\rho_k^0[H_{k'},\rho_{k(m)}^0]\right)\right\} \\
&= \sum_{k'\in\mathbf{k}_0}\left\{[H_{k'},\rho_{k(1)}^0(\alpha)]\prod_{k=k(2)}^{k(m)}\rho_k^0(\alpha)+\sum_{i=2}^{m}\prod_{k=k(1)}^{k(i-1)}\rho_k^0(\alpha)[H_{k'},\rho_{k(i)}^0(\alpha)]\prod_{k=k(i+1)}^{k(m)}\rho_k^0(\alpha)\right\}
\end{aligned} \tag{44}$$

in which the commutator

$$[H_{k'},\rho_k^0(\alpha)] = \begin{pmatrix} 0 & 2i\epsilon_{k'}(h_f)\cosh(\beta\epsilon_k(h_0))\sin(\Delta_k) & 0 & 0 \\ 2i\epsilon_{k'}(h_f)\cosh(\beta\epsilon_k(h_0))\sin(\Delta_k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{45}$$