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Quantum speed limit for a central system in Lipkin-Meshkov-Glick bath

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Abstract. We investigate the physical feature of quantum phase transition (QPT) in a Lipkin-Meshkov-Glick (LMG) model by calculating the quantum speed limit (QSL) time for a central qubit coupled to a LMG bath. We propose that with a suitable choice of the number of spins in bath and the coupling strength between central system and environment, the transition from no speed-up to speed-up of the open system evolution can be achieved at the QPT point, so the QSL is a potential candidate for witnessing QPT of a LMG model. By considering the physical reason for the acceleration, we study the non-Markovianity of the open system and verify that the non-Markovian property can induce the decrease of evolution time, which is also an indication that the phase transition of the system from Markovian to non-Markovian could occur at the QPT point.

1 Introduction

As an increasingly vital issue, quantum speed limit (QSL) time, defined as the minimum evolution time that a quantum system endures from an initial state to a final state, has aroused great interest in the areas of quantum computation [1], fast population transfer in quantum optics [2], quantum algorithms [3] and quantum optimal control protocols [4–6]. The notion of QSL derives from the Heisenberg uncertainty relation for energy and time, and characterizes the intrinsic upper bound of evolutional speed for a quantum system. The early investigations about this project were focused on the passage from a pure initial state to its orthogonal counterpart for closed systems [7–11], which could be unified as a lower bound of QSL time denoted by Mandelstam-Tamm (MT) [7] and Margolus-Levitin (ML) [11] types. Due to the fact that a realistic quantum system inevitably interacts with its external surroundings, how to calculate the QSL time for an open system under nonunitary dynamical evolution is essential. Recently, researchers have successfully come up with many approaches to extend the MT-ML bound to open system category [12–15]. Meanwhile, in view of the subsequent studies based on the speed limit of open quantum systems [16–19], it has been pointed out that the non-Markovian environment can induce accelerated evolution while the acceleration of quantum evolution would not occur under a Markovian environment [14,16–18]. So from another viewpoint, we can assume that the QSL time is a promising candidate for reflecting the property of open systems and scientists have successfully performed relevant experiment by using cavity QED [20].

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Quantum phase transition (QPT) [21], a critical change in the ground state of a many-body quantum system induced by pure quantum fluctuations and occurring at zero temperature, has been widely investigated in a number of contexts [22–31]. By tuning certain controlling parameters of the many-body environment, researchers can study the dynamical evolution of a system with QPT in order to detect the critical point through some special dynamical features. For example, Quan et al. [22] has examined the decay of Loschmidt echo (LE) of a two-level quantum system induced by its surrounding environment and shown that the behavior of LE can be obviously suppressed in the vicinity of the critical point. The correlation natures of quantum system, such as entanglement, quantum discord as well as Berry phase, can also characterize the critical point of a QPT [24–28,30]. In addition, the analysis in reference [31] has stressed that the fidelity susceptibility, which has direct relation with the mean variance of Hamiltonian of Ising system at the ground state, shows a divergence at the transition point and this can indicates a sudden change of energy fluctuation at the QPT point.

The Lipkin-Meshkov-Glick (LMG) model [32] was first introduced to describe the tunneling of bosons between two degenerate levels in nuclei. For its simple structure and unique phase transition feature, LMG model has been used to describe the magnetic properties of some acetate molecules [33] and many other physical systems, such as Bose-Einstein condensates [34] and small ferromagnetic particles [35]. A number of work has contributed to derive the exact solution of this model [36–38], moreover the physical realization of the system represented by LMG could offer a worthy test bed for investigations of quantum critical phenomena [39–42]. The behaviors of two interacting spins concurrence and spin squeezing of system at critical point were studied in reference [39]. It was also shown in reference [40] that the time dependent purity of the central system influenced by the LMG bath can clearly signify the QPT point. Another important finding is that near the critical region of the LMG model, the behaviors of the single-copy entanglement and the global geometric entanglement of the ground state are similar, which is contrary to usual one-dimensional spin systems [41]. Recently, people have researched the decay behaviors of the fidelity as well as quantum LE and show that they are useful to classify QPT of LMG model [42].

The previous studies promote us to consider the probability of characterizing QPT through the change of QSL time. Therefore, in this paper, we devote to present a detailed analysis of the QSL time of a central qubit system interacted with a LMG bath model [40]. We demonstrate how the LMG environment affects the speed of evolution of the central system. By researching the external influence from the bath, N and λ', on the QSL, it can be observed that the minimal evolution time of central qubit undergoes different behaviors in different phases, especially when the system is under proper qubit-bath coupling, the transition from no speed-up to speed-up will happen at the critical point of phase. Moreover, since the non-Markovianity of environment is the main reason for acceleration of evolution according to reference [14], we figure out the non-Markovianity of the model and note that the change from Markovian to non-Markovian is just at the QPT point, which is corresponding to the behavior of QSL time for central qubit.

The paper is organized as follows. In Section 2, the physical model is introduced and the reduced density matrix of the central qubit is obtained. In Section 3, we review the calculation method of QSL time and get the QSL time for the system coupled to LMG bath. Discussion about the result is also represented in this part. The fundamental physical mechanism of quantum speed-up evolution is argued in Section 4. Finally, conclusions are drawn in Section 5.

2 The LMG model

Here, we take an open quantum system, i.e., a central qubit interacting with the isotropic LMG bath environment into consideration. The total Hamiltonian of the model can be written as [32–43],

$$H = H_S + H_B + H_{SB}, \tag{1}$$

with

$$H_S = -\sigma^z, \tag{2a}$$

$$H_B = -\frac{\lambda}{N}\sum_{i<j}^{N}(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y) - \sum_{i=1}^{N}\sigma_i^z, \tag{2b}$$

$$H_{SB} = -\lambda'\sum_{i=1}^{N}(\sigma_i^x\sigma^x + \sigma_i^y\sigma^y), \tag{2c}$$

where σ^α and $\sigma_{i,j}^\alpha$, $\alpha = x, y, z$ $(i, j = 1, 2, 3, \ldots, N)$ are the Pauli matrices of the central system and the kth spin in the bath. H_{SB} represents the interacting Hamiltonian between the central qubit and its surrounding bath, with λ' denoting their coupling strength and the Hamiltonian of the central qubit is H_S. H_B is the self-Hamiltonian of the LMG model with λ, N denoting the coupling strength between the spins and the number of spins in the bath respectively. It is shown in the Hamiltonian that every spin in the LMG bath interacts with other spins in an x-y plane, which is considered as a long-range interaction.

By means of the ladder operators of the spin bath and the central qubit,

$$S_N^{\pm} = S_N^x \pm iS_N^y, \tag{3a}$$

$$s_{\pm} = s^x \pm is^y, \tag{3b}$$

we can rewrite the total Hamiltonian as,

$$H = -2s^z - \frac{\lambda}{N}\left(S_N^+S_N^- + S_N^-S_N^+ - N\right) - 2S_N^z - \lambda'\left(S_N^-s_+ + S_N^+s_-\right), \tag{4}$$

where $s^z = \sigma^z/2$ and

$$S_N^z = \frac{1}{2}\sum_{i=1}^{N}\sigma_i^z$$

are the spin of the central qubit and the total spin operator of the LMG bath, respectively. In the subspace spanned by Dicke states $|M\rangle = |N/2, M\rangle$ with $M = -N/2, \ldots + N/2$ [38,39], the ground state of the LMG model under ferromagnetic case ($\lambda > 0$) is [43],

$$|G\rangle = \begin{cases} \left|\frac{N}{2}, \frac{N}{2}\right\rangle & (0 < \lambda < 1), \\ \left|\frac{N}{2}, I\left(\frac{N}{2\lambda}\right)\right\rangle & (\lambda > 1), \end{cases} \tag{5}$$

where I is the integer part nearest to $N/2\lambda$. It is obvious that the system experiences a second-order QPT at the critical value $\lambda = 1$, i.e., the ground state of the bath has different symmetry property. The LMG bath is in the symmetry broken phase as $0 < \lambda < 1$ and in the symmetry phase as $\lambda > 1$.

We assume the initial state of the composed system of central qubit and LMG bath is separate and can be written as,

$$\rho_{tot}(0) = \rho_S(0) \otimes \rho_B(0). \tag{6}$$

So the time evolution of the total system is determined by

$$\rho_{tot}(t) = U(t)\rho_{tot}(0)U(t)^{\dagger}. \tag{7}$$

One can calculate the evolved density matrix of the central qubit by tracing over the environmental state, denoted by

$$\rho_s(t) = \mathrm{Tr}_E\left[\rho_{tot}(t)\right]. \tag{8}$$

The time evolution operator $U(t) = \exp(-iHt)$ in the invariant subspace H_M of H spanned by the ordered basis vector $\{|N/2, M\rangle \otimes |0\rangle, |N/2, M+1\rangle \otimes |1\rangle\}$ $(\{|0\rangle, |1\rangle\}$

are the eigenstates of the central qubit) is [40]

$$U(t) = \begin{bmatrix} a^2 e^{-ix_1 t} + b^2 e^{-ix_2 t}, \; ab(e^{-ix_1 t} - e^{-ix_2 t}) \\ ab(e^{-ix_1 t} - e^{-ix_2 t}), \; b^2 e^{-ix_1 t} + a^2 e^{-ix_2 t} \end{bmatrix} = \begin{bmatrix} U_{11} \; U_{12} \\ U_{21} \; U_{22} \end{bmatrix}, \tag{9}$$

where a and b are the expansion coefficients for the eigenstates of the total Hamiltonian under the basis vector, with

$$a = \varsigma / \sqrt{(\alpha - x_1)^2 + \varsigma^2},$$
$$b = (x_1 - \alpha) / \sqrt{(\alpha - x_1)^2 + \varsigma^2}. \tag{10}$$

The eigenvalues of H under subspace are

$$x_1 = \frac{1}{2}\left[(\alpha+\beta) + \sqrt{(\alpha-\beta)^2 + 4\varsigma^2}\right],$$
$$x_2 = \frac{1}{2}\left[(\alpha+\beta) - \sqrt{(\alpha-\beta)^2 + 4\varsigma^2}\right]. \tag{11}$$

α, β, and ς are the matrix elements of the total Hamiltonian which spans in the invariant subspace, i.e.,

$$H = \begin{pmatrix} \alpha & \varsigma \\ \varsigma & \beta \end{pmatrix}, \tag{12}$$

with

$$\alpha = -\frac{\lambda}{2N}\left(N^2 - 4M^2\right) - 2M - 1,$$
$$\beta = -\frac{\lambda}{2N}\left(N^2 - 4(M+1)^2\right) - 2(M+1) + 1,$$
$$\varsigma = -\lambda' \sqrt{N(N+2) - 4M(M+1)}. \tag{13}$$

Since the LMG model is at different ground states for the two phases, the reduced density matrix of the central qubit will be given in two forms. Therefore, when the initial state of the central system is chosen as

$$\rho_s(0) = c\,|0\rangle\langle 0| + d\,|0\rangle\langle 1| + d^*\,|1\rangle\langle 0| + (1-c)\,|1\rangle\langle 1|,$$

its evolutional state in the symmetric phase can be expressed as in the following form

$$\rho(t) = \begin{pmatrix} c\,|U_{11}|^2 + (1-c)\,|U'_{12}|^2 & dU_{11}U'^{*}_{22} \\ d^* U^*_{11} U'_{22} & c\,|U_{12}|^2 + (1-c)\,|U'_{22}|^2 \end{pmatrix}, \tag{14}$$

and in the symmetry broken phase, we have

$$\rho(t) = \begin{pmatrix} c + (1-c)\,|U_{12}|^2 & de^{i(N+1)t}U^*_{22} \\ d^* e^{-i(N+1)t} U_{22} & (1-c)\,|U_{22}|^2 \end{pmatrix}. \tag{15}$$

where U' is defined in another invariant subspace H_{M-1} of H spanned by $\{|N/2, M-1\rangle \otimes |0\rangle, |N/2, M\rangle \otimes |1\rangle\}$. In the next section, we will examine the limit evolution time of system based on the different forms of central qubit state in two phases and demonstrate that the two phases can be distinguished by the obviously different behaviors of QSL time, which is useful to mark the QPT point in the LMG bath under suitable conditions.

3 Quantum speed limit for central qubit evolution

In order to analyze the optimal evolution time of the central system coupled to LMG model, we resort to the definition of QSL time derived by Deffner and Lutz [14] which generalizes a unified lower bound, including both MT and ML types. By taking advantage of von Neumann trace inequality and Cauchy-Schwarz inequality, the QSL time representing the evolution from an initial state ρ_0 to its target state ρ_τ, governed by the time-dependent nonunitary master equation $\dot{\rho}_t = L_t \rho_t$, with positive generator L_t, can be given by:

$$\tau_{QSL} = \max\left\{\frac{1}{\Lambda^1_\tau}, \frac{1}{\Lambda^2_\tau}, \frac{1}{\Lambda^\infty_\tau}\right\} \sin^2\left[\mathcal{L}(\rho_0, \rho_\tau)\right], \tag{16}$$

where

$$\Lambda^p_\tau = \tau^{-1}\int_0^\tau \|L_t\rho_t\|_p dt \quad \text{and} \quad \|A\| = (\sigma_1^p + \ldots + \sigma_n^p)^{1/p}$$

denotes the Schatten p norm, with $\sigma_1, \ldots, \sigma_n$ being the singular values of A. The Bures angle between two states is $\mathcal{L}(\rho_0, \rho_\tau) = \arccos\sqrt{\mathrm{tr}(\rho_0\rho_\tau)}$. In light of the result in reference [42], it is clear that the sharpest bound of the quantum speed limit time is the ML-type bound based on the operator norm, i.e., $p = \infty$, of the nonunitary generator. As a consequence, we will measure the QSL time of the qubit by using the bound in terms of operator norm.

According to the theory of equation (6), we can first obtain the QSL time of central qubit induced by the LMG bath when $\lambda > 1$, i.e., in the symmetric phase. If we set the initial state of the qubit as the excited state $\rho_s(0) = |1\rangle\langle 1|$ for convenience, then the exact solution of the QSL time for central system is expressed as,

$$\tau_{QSL} = \frac{1 - |U'_{22}|^2}{\int_0^1 \left|\partial t\,|U'_{22}|^2\right| dt}, \tag{17}$$

with the actual evolution time being $\tau = 1$. Similarly, under the same initial condition, when the bath is in the symmetry broken phase we can get the form of the QSL time as,

$$\tau_{QSL} = \frac{1 - |U_{22}|^2}{\int_0^1 \left|\partial t\,|U_{22}|^2\right| dt}. \tag{18}$$

For the purpose of revealing how the QPT of LMG model can be signaled by the speed-up dynamic process of central qubit, we focus our analyses on the relation between

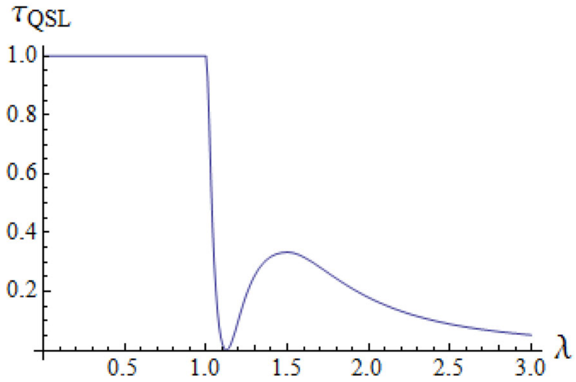


Fig. 1. The QSL time for the central qubit as a function of the coupling strength λ between qubits in the bath. The coupling strength between qubit and LMG bath is chosen as $\lambda' = 0.05$ and the number of spins in bath is $N = 100$. The actual evolution time is $\tau = 1$.

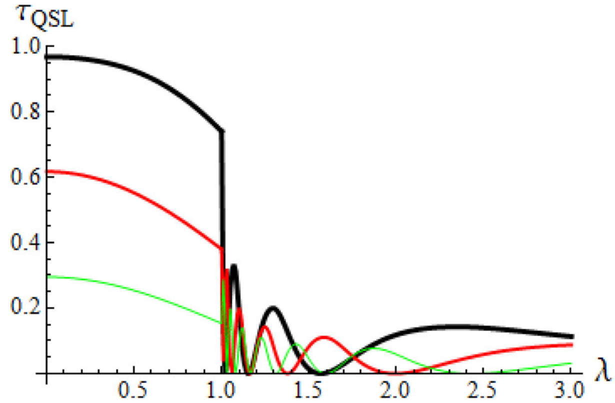


Fig. 2. The QSL time for the central qubit as a function of the coupling strength λ between qubits in the bath for different N. The coupling strength between qubit and LMG bath is chosen as $\lambda' = 0.05$ and the actual evolution time is $\tau = 1$. The curves with different thickness (from thick to thin) represent $N = 200$, $N = 300$, and $N = 400$.

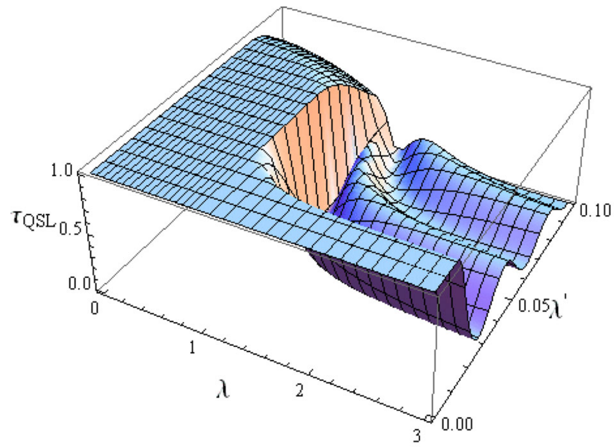


Fig. 3. The QSL time for the central qubit as a function of the coupling of spins in LMG bath λ and the coupling between qubit and bath λ'. The number of spins LMG bath is chosen as $N = 100$ and the actual evolution time is $\tau = 1$.

QSL time and the feature of LMG bath. In Figure 1, the QSL time is plotted as a function of the coupling strength among spins in the bath. Our numerical results for the QSL on the basis of the analytical expressions above show that the system endures a sudden transition from no speedup to speedup of quantum evolution at the critical point in the weak qubit-bath coupling regime. Note that the QSL time is tight, since it reaches the actual evolution time when LMG environment is in symmetry broken phase. However, system experiences a dramatic acceleration in the process of evolution, leading to the decrease of QSL time in symmetric phase surrounding. Thus it reveals the fact that near the critical point the QSL time is very sensitive to the coupling strength between spins in bath, which is to say the critical behavior of the QSL time can be an ideal witness for QPT in LMG model.

Although the similar form can still exist as the number of spins in the bath rising, however, it could not lead to a tight bound as $\lambda \leqslant 1$. We draw a size dependence of QSL time as a function of coupling strength under the same qubit-bath coupling condition in Figure 1. From Figure 2, it is clear that as the total influence from external environment becoming strong through enlarging the size N, the evolution speed of the central system will increase, causing the QSL time reduction, which is in agreement with the findings in references [19,20]. In both of the two phases, the QSL time cannot reach the actual evolution time and its value declines steadily in the symmetry broken phase, whereas the trend turns to along an oscillating trace in the symmetric phase. Here, the QPT point can only be characterized by distinguishing these two different behaviors.

In light of the above analyses for the critical phenomena, we can speculate that the environmental factors from LMG bath have a close relation with the behavior of QSL time on either side of the QPT point. So it is necessary for us to further consider the influence from both the system-environment coupling and spin-spin coupling in bath on the QSL time. Let us take the $N = 100$ as an example. Figure 3 exhibits QSL plotted as a function of the strength of qubit-bath coupling and spin-spin coupling in LMG environment. The picture reveals a comprehensive view about the behaviors of QSL in both phases. We can see that in the symmetry broken phase, the evolution of central system keeps at a stable speed, i.e., the QSL is tight, as the interaction between qubit and spin small ($\lambda' \leqslant 0.8$), while the QSL time could not be reachable at the actual driving time in a region where the strength of qubit-spin coupling becomes large. The QSL's response for coupling strength λ' is more sensitive in the symmetric phase. In an extremely weak coupling regime ($\lambda' \leqslant 0.1$), the time of evolution of central system can stay immune to the external disturbance and remain at the actual evolution time, which indicates that the moderate environmental influence for quantum control is essential. When we continue increasing the spin-environment coupling strength ($\lambda' > 0.1$), the QSL time reduces abruptly and demonstrates oscillating feature, presenting a poorer and more conservative bound of QSL. From the picture it is obvious that the critical phenomenon from no acceleration to acceleration in Figure 1 can only happen as the value of λ' being suitable. Considering the results in Figure 2, we can easily predict that for larger size of N, the qubit-spin coupling has more obvious effect on QSL, i.e., with the change of λ' the evolution of central qubit will tend to be faster corresponding to smaller values of QSL time. Until when the environment approaches to an infinite size of degrees of freedom $N \to \infty$, the QSL time will be very small in both phases and the critical phenomenon at QPT point cannot be distinguished.

4 Memory effect of LMG bath

Just as the analyses in references [14–18], the fundamental acceleration mechanism is the non-Markovian feature of environment. The backflow of information from reservoir to system can raise the capacity of potential speedup for quantum evolution and cause the decrease of QSL time. The measure of non-Markovianity is based on the trace distance between two distinguishable states ρ_1 and ρ_2 [44],

$$D(\rho_1, \rho_2) = \frac{1}{2}\mathrm{tr}\, |\rho_1 - \rho_2| \,. \tag{19}$$

So the information flow can be expressed as the gradient of trace distance,

$$\sigma(t, \rho_{1,2}(0)) = \partial t D(\Phi_t \rho_1(0), \Phi_t \rho_2(0)) \,, \tag{20}$$

where Φ_t $(t \in [0, \tau])$ denotes the dynamical map for system evolution and its positive value represents information flowing back to the system. Then the non-Markovianity is defined as the total amount of information flow [45]

$$\mathcal{N}(\Phi) = \max_{\rho_1, \rho_2} \int_{\sigma > 0} \sigma(t, \rho_{1,2}(0))\, dt. \tag{21}$$

In accordance with [46,47], the optimal initial state pair $(\rho_1(0), \rho_2(0))$ can be attained as the antipodal points on the equator of the Bloch sphere, i.e., $\rho_s^1(0) = |1\rangle\langle 1|$ and $\rho_s^2(0) = |0\rangle\langle 0|$. Hence, we can truncate the time-integral in $\mathcal{N}$ from 0 to τ and calculate the non-Markovianity of the LMG system according to the guidance of reference [16] by transforming the form of equation (14) as:

$$\mathcal{N}(\Phi) = \frac{1}{2}\left[\int_0^1 |\partial_t |D(t)||\, dt + |D(\tau)| - 1\right], \tag{22}$$

where $D(t) = |U_{22}|^2$ when the LMG environment is in the symmetry broken phase and $D(t) = ||U_{11}|^2 - |U'_{12}|^2|$ is in the condition of symmetric phase for the LMG bath.

To evaluate the speed-up phenomenon induced by non-Markovian property, we depict the non-Markovianity of open system according to equation (22). It is obvious from Figure 4 that the information reflux demonstrates different forms with respect to two phases. In the symmetry broken phase, there is no backflow of information to central qubit so that the non-Markovianity remains zero, which corresponds to the previous research that system would not accelerate in the Markovian environment. The degree of non-Markovianity increases a lot in the symmetric phase and this behavior is in accordance with the result of decrease of QSL time in the same phase. Also we can see that the transition point $\lambda = 1$ denoting from Markovian to non-Markovian environment is the QPT point for LMG bath, which is identical with the above conclusion for QSL time. So we can attribute the quantum speedup of the central qubit to the non-Markovian nature of the LMG bath.

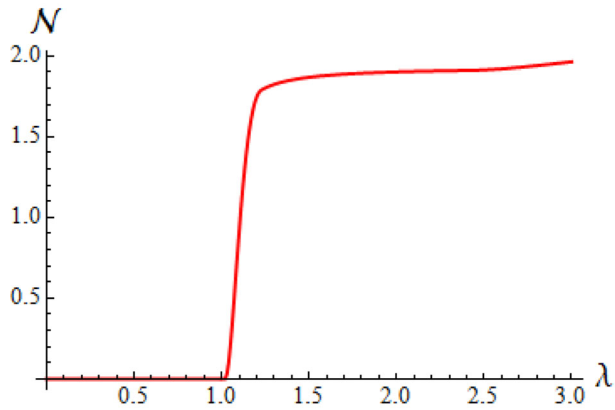
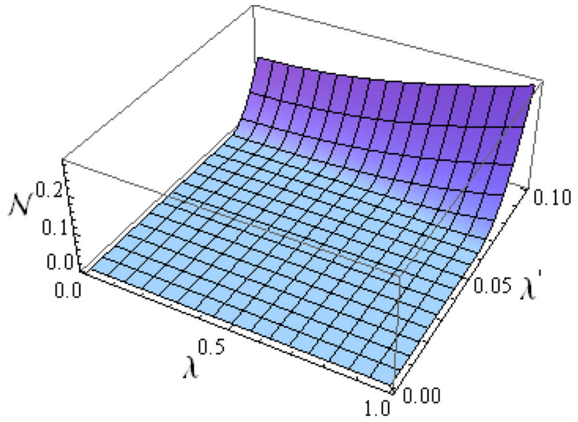


Fig. 4. The non-Markovianity of the open system as a function of the coupling strength λ between qubits in LMG bath with spin numbers $N = 100$. The coupling strength between bath and central qubit is chosen as $\lambda' = 0.05$ and the actual evolution time is $\tau = 1$.

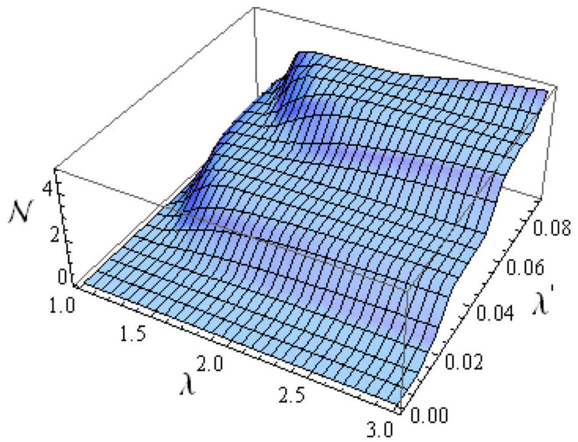
For further illustrating the relation between non-Markovianity and QSL time, with the consequence of environmental coupling dependence of QSL time in Figure 3, we investigate the effect from external couplings on non-Markovianity as shown in Figure 5. In the symmetry broken phase, there is information flow coming back to the central system at large qubit-bath coupling and it verifies that the Markovian process can turn to a non-Markovian one by enlarging the strength of interaction between qubit and spins in bath. It is also clear to see that the transition point from Markovian to non-Markovian is equivalent to the point from no speedup to speedup of the system as Figure 3 has shown. In the symmetric phase, we can discover the same correspondence between QSL time and non-Markovianity as that in symmetry broken phase. Therefore it proves that non-Markovian feature is the intrinsic mechanism for the quantum acceleration phenomenon of the central system in LMG bath.

5 Conclusion

We have investigated the QSL time of a central qubit system coupled to a LMG bath. Our results show that the minimal evolution time of the central system can characterize the QPT of LMG environment depending on its obviously different behaviors in two phases. Specifically, by exploring the relationship among QSL time, spin-spin coupling strength in bath and system-environment interaction



(a)



(b)

Fig. 5. The non-Markovianity of the open system as a function of the coupling strength λ of spins in LMG bath and the coupling between qubit and bath λ'. (a) and (b) are the situations in the symmetry broken phase and symmetric phase, respectively. The number of spins in LMG bath is chosen as $N = 100$ and the actual evolution time is $\tau = 1$.

and adding the factor of spin numbers in bath, we find that for suitable values of qubit-bath coupling strength λ' and N, system has no potential capacity for further acceleration; that is to say, the QSL time is the actual evolution time in the symmetry broken phase, while in the symmetric phase system can possess a significant speedup. In addition, we also study the main physical factor for the phenomenon through calculating the non-Markovianity of the LMG bath. It is clarified that non-Markovian effect can lead to faster time evolution for central qubit and the change from Markovian to non-Markovian corresponds to the change of QSL time at the same critical point. As a consequence, we can mention that both the system and environment experience a phase transition at the phase transition point. The generalization of our results to a wide variety of quantum open system is a valuable issue for the research about quantum thermodynamics by utilizing QSL time. Additionally, a latest extension of QSL time to the generation of nonclassicalty characterized by quantumness has been researched [48]. The new bound deduced in the work is proven to be tight and more reachable than the standard QSL under pure dephasing model both in Markovian and non-Markovian environment, which deserves further study.

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