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Decay of Loschmidt Echo at a Critical Point in the Lipkin–Meshkov–Glick model *

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An analytical expression of the Loschmidt echo in the Lipkin–Meshkov–Glick model is derived in the thermodynamical limit. It is used in the study of the decaying behaviour of the echo at the critical point of a quantum phase transition of the model. It is shown that the echo has a power law decay for relatively long times.

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Quantum phase transitions (QPTs), i.e., fundamental changes of properties of the ground states of systems at zero-temperature, are crucial in the explanation of many properties of phase transitions observed experimentally at low temperature.[1] Conventional techniques dealing with thermal phase transitions, in particular, the concepts of order parameter and symmetry-breaking, are useful in the study of QPT, but can not reveal all their properties. Recently, quantum-information concepts such as entanglement[2−4] and fidelity[5−12] have also been found to be useful in characterizing QPT.

As shown by Quan *et al.*,[5] a special type of fidelity, the so-called quantum *Loschmidt echo* (LE), may undergo relatively fast and deep decay at critical points of QPT, hence may reveal positions of the critical points (also see Ref. [11]). The LE was first introduced by Peres[13] and has been studied extensively recently in both quantum chaotic and regular systems.[14,15] It is a measure of the stability of quantum motion under small perturbation, defined as the overlap of the time evolution of the same initial state under two slightly different Hamiltonians H and $H' = H + \epsilon V$. Specifically, the LE is $M(t) = |m(t)|^2$, where

$$m(t) = \langle \Psi_0 | \exp(iH't/\hbar) \exp(-iHt/\hbar) | \Psi_0 \rangle, \quad (1)$$

where H is an unperturbed Hamiltonian, V is a perturbation, and ϵ is a small quantity. In the study of QPT, $|\Psi_0\rangle$ is chosen as the ground state of H.[5,11,12] It is clear from the definition of LE that it is suitable in characterizing the sensitivity of quantum motion near a critical point.

Compared with the static fidelity as the overlap of ground states,[6−10] the LE is a dynamical quantity that is determined by properties of both ground states and some low-lying excited states, hence the LE may reveal more properties of QPT than static fidelity. A question of particular interest is about the decaying law of the LE in the vicinity of QPT, the knowledge of which may be useful in the classification of QPT.[12] There have been only a few analytical results concerning this problem: Direct analytical derivation shows that the LE has an initial parabolic decay in an Ising chain;[5] while a semiclassical analysis shows that the LE may have a power law or an exponential decay for relatively long times in certain different types of models.[12] Therefore, it would be of interest to derive analytical expression for the LE decay at QPT in some concrete model(s).

In this Letter, we study the LE decay in the Lipkin–Meshkov–Glick (LMG) model,[16] which was first proposed to describe shape phase transition in nuclei physics. Collective motion in the two-orbital LMG model can be studied analytically,[17] in particular, in the thermodynamic limit.[18,19] Recently, static properties like the fidelity susceptibility[20] and entanglement entropy[21] have been studied in the vicinity of the critical point of the QPT in this model; while its dynamical properties are still unclear.

In the following, we show that, in certain parameter regimes, an explicit analytical expression of the dynamical quantity LE can be derived in the thermodynamical limit in this model. The expression is found to be useful in the study of the decaying behavior of the LE in the vicinity of the critical point. In particular, it confirms a general semiclassical prediction given in Ref. [12], i.e., for relatively-long times the LE has a power-law decay near the QPT in systems like the LMG model. It is worth mentioning that this prediction has already been checked numerically in another model belonging to the same universal class,[12] namely, the Dicke model with the entanglement and fidelity having been studied recently.[22]

We discuss the LMG model in the thermodynamical limit. In the two-orbital LMG model for N interacting particles, in terms of Pauli matrices σ_α (α=x, y, z), the Hamiltonian can be written as[20]

$$H(\gamma, h) = -h \sum_{i=1}^{N} \sigma_z^i - \frac{1}{N} \sum_{i<j} (\sigma_x^i \sigma_x^j + \gamma \sigma_y^i \sigma_y^j). \quad (2)$$

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We consider the parameter regime, $0 \le |\gamma| < 1.0$ and $h > 0$. In studying the collective motion of the model, it would be convenient to write the Hamiltonian in terms of the total spin operator $S_\alpha = \sum_i \sigma_\alpha^i/2$,

$$H(\gamma, h) = -\frac{2}{N}(S_x^2 + \gamma S_y^2) - 2hS_z + (1+\gamma)/2, \quad (3)$$

where S indicates the largest eigenvalue of S_z, $S = N/2$.

In order to diagonalize the Hamiltonian in the thermodynamic limit, following Ref. [18], one may perform a rotation,

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & 0 & \sin\theta_0 \\ 0 & 1 & 0 \\ -\sin\theta_0 & 0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} \bar{S}_x \\ \bar{S}_y \\ \bar{S}_z \end{pmatrix}, \quad (4)$$

where $\cos\theta_0 = h$ for $h < 1$ and $\cos\theta_0 = 1$ for $h > 1$. Making use of the Holstein–Primakoff transformation,[23] we have

$$\begin{aligned} \bar{S}_z &= S - a^\dagger a, \\ \bar{S}_+ &= (2S - a^\dagger a)^{1/2} a, \\ \bar{S}_- &= a^\dagger (2S - a^\dagger a)^{1/2}. \end{aligned} \quad (5)$$

One may write the Hamiltonian in terms of a and $a^\dagger$, where $a^\dagger$ and a are the bosonic creation and annihilation operators, respectively, satisfying $[a, a^\dagger] = 1$. Then, expanding the term $(2S - a^\dagger a)^{1/2}$ in the Hamiltonian in the power of $1/N$ and considering low-lying energy eigenstates in the thermodynamic limit, the Hamiltonian H can be written as

$$H(\gamma, h) = \Delta_B a^\dagger a + \Gamma_B ({a^\dagger}^2 + a^2), \quad (6)$$

where $\Delta_B = 2 + 2h\cos\theta_0 - 3\cos^2\theta_0 - \gamma$, $\Gamma_B = (\gamma - \cos^2\theta_0)/2$. In Eq. (6), we have omitted a term that is a constant for finite N and may go to infinity in the limit $N \to \infty$. We remark that Eq. (6), which is derived for low-lying energy eigenstates, is sufficient for our purpose of studying the LE of an initial ground state.

Finally, by a standard Bogoliubov transformation,

$$a_\Theta^\dagger = \cosh(\Theta/2) a^\dagger - \sinh(\Theta/2) a, \quad (7)$$

$$a_\Theta = -\sinh(\Theta/2) a^\dagger + \cosh(\Theta/2) a, \quad (8)$$

the Hamiltonian in Eq. (6) can be diagonalized,

$$H(\gamma, h) = \Delta a_\Theta^\dagger a_\Theta, \quad (9)$$

where

$$\Delta = 2[(h-1)(h-\gamma)]^{\frac{1}{2}}, \quad \tanh\Theta = \frac{1-\gamma}{2h-1-\gamma}, \quad (10)$$

for $h > 1$, and

$$\Delta = 2[(1-h^2)(1-\gamma)]^{\frac{1}{2}}, \quad \tanh\Theta = \frac{h^2-\gamma}{2-h^2-\gamma}, \quad (11)$$

for $h < 1$. An unimportant constant term in the expression (9) is also omitted. Thus, eigenstates of $H(\gamma, h)$ can be written as

$$|n\rangle_\Theta = \frac{1}{\sqrt{n!}} \left(a_\Theta^\dagger\right)^n |0\rangle_\Theta, \quad (12)$$

with eigenenergies $E_n = n\Delta$ for $n = 0, 1, 2, \cdots$.

Equations (10) and (11) show that when h approaches 1 from both sides, $\Delta \to 0$. This implies that the system is infinitely degenerate with zero energy gap at $h = 1$, having one energy level only. In other words, when h approaches 1 from both sides, all the levels join in the ground level at $h = 1$. Therefore, the system undergoes a quantum phase transition at the critical point $h_c = 1$. The phase with $h > 1$ is usually called the symmetric phase and the phase with $h < 1$ the broken phase.

Now, we study the LE of an initial ground state $|\Psi_0\rangle$ that is prepared in $|0\rangle_\Theta$, the ground state of H. From the definition (1) and the diagonal form of $H' \equiv H(\gamma', h')$ given by Eq. (9), we have

$$M(t) \simeq \left| \sum_{n=0}^{\infty} |{}_\Theta\langle 0|n\rangle_{\Theta'}|^2 \, e^{in\Delta(\gamma', h')t} \right|^2, \quad (13)$$

where we write the dependence of Δ on (γ', h') explicitly and set Planck constant to be unit, $\hbar = 1$. We have assumed that Eq. (9) holds for all the states $|n\rangle_{\Theta'}$ that have non-negligible overlap with $|0\rangle_\Theta$. This is valid when studying the LE induced by a small perturbation, i.e., when (γ, h) is sufficiently close to (γ', h').

In order to derive an explicit expression for ${}_\Theta\langle 0|n\rangle_{\Theta'}$ in Eq. (13), we consider the two cases: (1) the symmetric phase with arbitrary values of $h, h' > 1$ and (2) the special broken phase with $h = h' < 1$. In both cases, the two systems $H(\gamma, h)$ and $H(\gamma', h')$ have the same creation and annihilation operators a and $a^\dagger$, which can be seen from Eqs. (4) and (5). Then, from Eqs. (7) and (8), we obtain

$$a_{\Theta'}^\dagger = \cosh\left(\frac{\Theta' - \Theta}{2}\right) a_\Theta^\dagger - \sinh\left(\frac{\Theta' - \Theta}{2}\right) a_\Theta, \quad (14)$$

$$a_{\Theta'} = -\sinh\left(\frac{\Theta' - \Theta}{2}\right) a_\Theta^\dagger + \cosh\left(\frac{\Theta' - \Theta}{2}\right) a_\Theta. \quad (15)$$

Substituting Eq. (15) into the relation $a_{\Theta'}|0\rangle_{\Theta'} = 0$ and expanding $|0\rangle_{\Theta'}$ in $|n\rangle_\Theta$, we find the following expansion of $|0\rangle_{\Theta'}$,

$$|0\rangle_{\Theta'} = \frac{1}{\sqrt{C}} \sum_{n=0}^{S} \sqrt{\frac{(2n-1)!!}{(2n)!!}} \tanh^n \left(\frac{\Theta' - \Theta}{2}\right) |2n\rangle_\Theta, \quad (16)$$

where C is a normalization constant,

$$C = \sum_{n=0}^{S} \frac{(2n-1)!!}{(2n)!!} \tanh^{2n}(\Theta'/2 - \Theta/2). \quad (17)$$

Then, substituting Eq. (16) into Eq. (13), we get

$$M(t)=\left|\frac{1}{C}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\tanh^{2n}\left(\frac{\Theta'-\Theta}{2}\right)\right.$$
$$\left.\times\, e^{i2n\Delta(\gamma',h')t}\right|^2.$$

After some algebra, this expression of $M(t)$ can be simplified and we find the following simple expression of the LE,

$$M(t)=\left[1+q^2\sin^2\left(\Delta(\gamma',h')t\right)\right]^{-1/2}, \tag{18}$$

where

$$q=\left|\frac{2\tanh(\Theta'/2-\Theta/2)}{1-\tanh^2(\Theta'/2-\Theta/2)}\right|. \tag{19}$$

We remark that, as mentioned above, Eq. (18) holds in both the case of symmetric phase with $h,h'>1$ and the case of $h=h'$ in the broken phase with $h,h'<1$. In the case of $h\neq h'$ in the broken phase, a and $a^\dagger$ are different in the two systems $H(\gamma,h)$ and $H(\gamma',h')$, and it turns out that deriving a concise expression for the LE in this case is much more difficult than that for Eq. (18); we are not to discuss this case in this study.

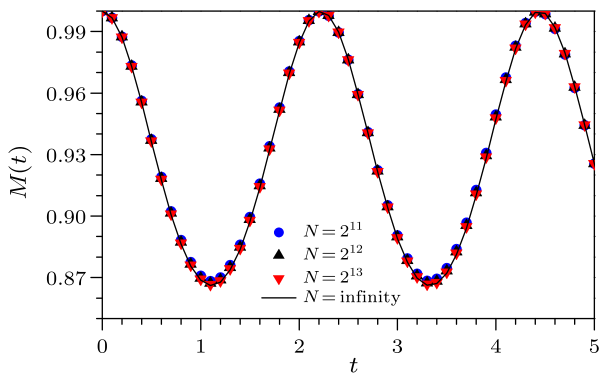

Fig. 1. (Color online) Variation of the LE $M(t)$ with time for $h=1.1$, and $h'=h+0.4$ in the symmetric phase with $\gamma=\gamma'=0.5$. The solid curve represents the analytical prediction of the LE given in Eq. (18) for the thermodynamic limit. Circles and triangles up and down give numerically computed LE for systems with $N=2^{11}$, 2^{12}, and 2^{13}, respectively.

To check the validity of Eq. (18), we compare its prediction with numerically computed $M(t)$ for large but finite N. As shown in Figs. 1 and 2, the agreements are good. In calculating $M(t)$ for finite N, we use its definition in Eq. (1) and employ the method of direct numerical diagonalization of the Hamiltonian.

Equation (18) shows that the LE is a periodic function, with period T determined by the system $H(h',\gamma')$,

$$T=\pi/\Delta(\gamma',h'). \tag{20}$$

For h' near the critical point $h_c=1$, according to Eqs. (10) and (11), the period has the following scaling behavior,

$$T=A\left|h'-1\right|^{-1/2}, \tag{21}$$

where $A=\pi/(2\sqrt{h'-\gamma'})$ for $h'>1$ in the symmetry phase and $A\simeq\pi/(2\sqrt{2-2\gamma'})$ for $h'<1$ in the broken phase. The period becomes infinite in the limit $h'\to h_c=1$.

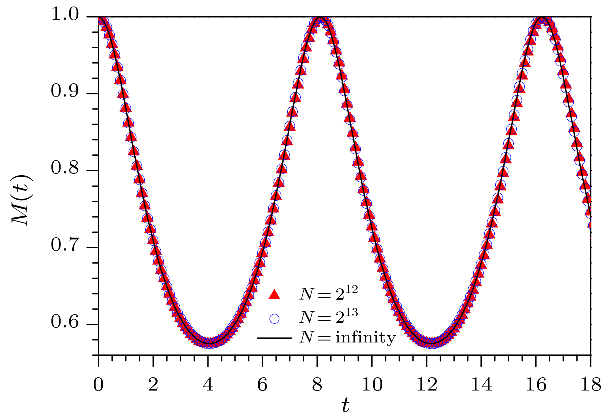

Fig. 2. (Color online) The same as Fig. 1 but for $h=h'=0.5$ in the broken phase with $\gamma=0.5,\gamma'=\gamma+0.45$. Triangles and circles are for $N=2^{12}$ and 2^{13}, respectively.

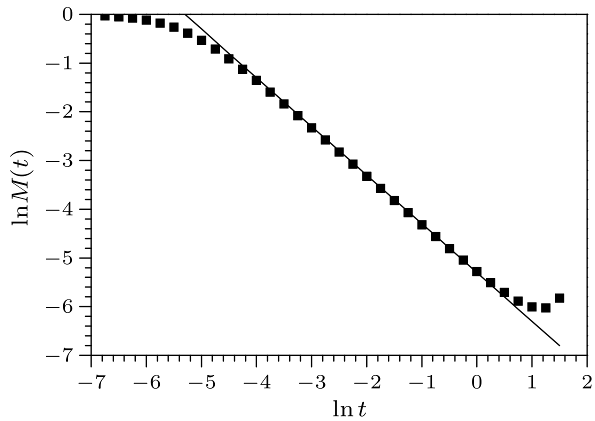

Fig. 3. (Color online) Decay of the LE near the critical point, $h-1=10^{-5}$, $h'=h+0.1$, $\gamma=\gamma'=0.5$. Solid squares represent the predictions of Eq. (18) and the solid line is a fit given by a $1/t$ decay.

It is also of interest to study the decay of the LE $M(t)$ for $t<T/2$. For sufficiently small $|h-1|$, Eq. (18) shows that for short times satisfying $qt\Delta(\gamma,h')\ll 1$, the LE has an initial parabolic decay, a result that can also be obtained by a perturbation theory,

$$M(t)\simeq 1-\frac{1}{2}q^2\Delta^2(\gamma,h')t^2. \tag{22}$$

For intermediate times satisfying $1\ll qt\Delta(\gamma,h')\ll q$, the LE has a power law decay,

$$M(t)\simeq\frac{1}{q\Delta(\gamma,h')t}\sim t^{-1}, \tag{23}$$

as illustrated in Fig. 3.

The prediction of power law decay in Eq. (23) confirms a general semiclassical analysis given in Ref. [12], which shows that the LE may have a power law decay at the QPT of a system when the following requirements are satisfied: That is, the system has an infinitely-degenerate ground level at the critical point and has a classical counterpart with one-degree of freedom. It is easy to see that the LMG model satisfies these requirements.

In summary, we have studied the LE of initial ground states in the LMG model near the critical point. An analytical expression of the LE is derived for certain parameter regimes in the thermodynamical limit. In the neighborhood of the critical point, the LE has an initial parabolic decay followed by a $1/t$ decay, and oscillates for a long time.

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