Time dilation of quantum clocks in a Newtonian gravitational field

Tommaso Favalli[1, 2, *] and Augusto Smerzi[1, †]

[1] *QSTAR, INO-CNR, and LENS, Largo Enrico Fermi 2, I-50125 Firenze, Italy*

[2] *Università degli Studi di Napoli "Federico II", Via Cinthia 21, I-80126 Napoli, Italy*

We consider two non-relativistic quantum clocks interacting with a Newtonian gravitational field produced by a spherical mass. In the framework of Page and Wootters approach, we derive a time dilation for the time states of the clocks. The delay is in agreement up to first order with the gravitational time dilation obtained from the Schwarzschild metric. This result can be extended by considering the relativistic gravitational potential: in this case we obtain the agreement with the exact Schwarzschild solution.

I. INTRODUCTION

We investigate, in the Page and Wootters (PaW) framework [1, 2], the time evolution of two quantum clocks A and B interacting with a Newtonian gravitational field produced by a spherical mass M. Our clock model has been introduced in [3–5] where the time states of the clock were chosen to belongs to the complement of an Hamiltonian derived by D. T. Pegg in 1998 [6].

We first show that free clocks, namely clocks not perturbed by the gravitational field, evolve synchronously. Then we consider two clocks located at different distances from the source of the gravitational field and we find that, as time (read by a far-away observer) goes on, the time states of the two clocks suffer a relative delay, with the clock in the stronger gravitational field ticking at a slower rate than the clock subject to a weaker field. The time dilation effect we find is in agreement, up to the first order of approximation, with the gravitational time dilation as obtained from the Schwarzschild metric.

In our analysis we initially do not consider relativistic corrections to the gravitational energy. We instead promote the masses of the clocks to operators using the mass-energy equivalence $m \to m + \hat{H}_{clock}/c^2$ [7]. Therefore our results originate from a non-relativistic quantum framework with the only exception of using the mass-energy equivalence, which is clearly a relativistic ingredient. As a last point we discuss the possibility of using the *relativistic gravitational potential* [8, 9] and we show that this choice leads to the agreement between our time dilation effect and the exact Schwarzschild solution.

In PaW theory time is a quantum degree of freedom which belongs to an ancillary Hilbert space. To describe the time evolution of a system S, PaW introduce a global space $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_S$, where $\mathcal{H}_C$ is the Hilbert space assigned to time. The flow of time emerges thanks to the entanglement between S and the temporal degree of freedom. In the following, the system S consists of the two additional clocks A and B, and we investigate their evolution with respect to the third clock C placed (for convenience) at infinite distance from the source of the gravitational field. The C clock, entangled with the system S, will thus play the role of a far-away reference clock not perturbed by the gravitational field. The PaW theory has recently attracted a lot of attention (see for example [3–5, 10–20]) and we provide a brief summary in Section II. In Section III we consider the clocks A and B described by Pegg's Hermitian operators complement of Hamiltonians with bounded, discrete spectra and equally-spaced energy levels. The clocks in this case have discrete time values. In Section IV we generalize to the case where the clocks A and B have continuous time values, namely described by Pegg's POVMs (with the Hamiltonians still maintaining a discrete energy spectrum).

II. GENERAL FRAMEWORK

We give here a brief rewiev of the PaW theory following the generalization proposed in [3–5]. We consider the global quantum system $|\Psi\rangle$ in a stationary state with zero eigenvalue:

$$\hat{H}\,|\Psi\rangle = 0 \tag{1}$$

where $\hat{H}$ is the Hamiltonian of the global system. We can then divide our global system into two non-interacting subsystems, the reference clock C and the system S. The total Hamiltonian can be written:

$$\hat{H} = \hat{H}_C + \hat{H}_S \tag{2}$$

where $\hat{H}_C$ and $\hat{H}_S$ are the Hamiltonians acting on C and S, respectively. We assume that $\hat{H}_C$ has a point-like spectrum, with non-degenerate eigenstates having rational energy ratios. More precisely, we consider d_C energy states $|E_i\rangle_C$ and E_i energy levels with $i = 0, 1, 2, ..., d_C - 1$ such that $\frac{E_i - E_0}{E_1 - E_0} = \frac{A_i}{B_i}$, where A_i and B_i are integers with no common factors. We obtain ($\hbar = 1$):

$$E_i = E_0 + r_i \frac{2\pi}{T_C} \tag{3}$$

where $T_C = \frac{2\pi r_1}{E_1}$, $r_i = r_1 \frac{A_i}{B_i}$ for $i > 1$ (with $r_0 = 0$) and r_1 equal to the lowest common multiple of the values of B_i. In this space we define the states

$$|t\rangle_C = \sum_{i=0}^{d_C - 1} e^{-iE_i t}\,|E_i\rangle_C \tag{4}$$

* favalli@lens.unifi.it

† augusto.smerzi@ino.it

where t can take any real value from t_0 to t_0+T_C. These states satisfy the key property $|t\rangle_C = e^{-i\hat{H}_C(t-t_0)}|t_0\rangle_C$ and furthermore can be used for writing the resolution of the identity in the C subspace:

$$\mathbb{1}_C = \frac{1}{T_C}\int_{t_0}^{t_0+T_C} dt\,|t\rangle\langle t|\,. \tag{5}$$

Thanks to property (5) the clock in C is represented by a POVM generated by the infinitesimal operators $\frac{1}{T_C}|t\rangle\langle t|\,dt$. This framework for the subspace C allow us to consider any generic Hamiltonian as Hamiltonian for the C subspace. Indeed, in the case of non-rational ratios of energy levels, the resolution of the identity (5) is no longer exact but, since any real number can be approximated with arbitrary precision by a ratio between two rational numbers, the residual terms and consequent small corrections can be arbitrarily reduced. This allow us to use the PaW framework with any Hamiltonian $\hat{H}_S$ for the system S. Indeed, with a generic spectrum for the Hamiltonian $\hat{H}_C$ we are sure that every energy state of S can be connected with an energy state of C satisfying (1) and no state of S will be excluded from the dynamics simply assuming $d_C \gg d_S$ and a sufficiently large T_C [3].

Let us now look at the heart of PaW theory. The condensed history of the system S is written through the entangled global stationary state $|\Psi\rangle \in \mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_S$, which satisfies the constraint (1), as follows:

$$|\Psi\rangle = \frac{1}{T_C}\int_0^{T_C} dt\,|t\rangle_C \otimes |\psi(t)\rangle_S \tag{6}$$

where we have choosen as initial time $t_0 = 0$. In this framework the relative state (in Everett sense [21]) of the subsystem S with respect to the time space C can be obtained via conditioning

$$|\psi(t)\rangle_S = \langle t|\Psi\rangle\,. \tag{7}$$

Note that, as mentioned before, equation (7) is the Everett *relative state* definition of the subsystem S with respect to the subsystem C. As pointed out in [11], this kind of projection has nothing to do with a measurement. Rather, $|\psi(t)\rangle_S$ is a state of S conditioned to having the state $|t\rangle_C$ in C. From equations (1), (2) and (7), it is possible to derive the Schrödinger evolution for the relative state of the subsystem S with respect to C [3]:

$$i\frac{\partial}{\partial t}|\psi(t)\rangle_S = \hat{H}_S|\psi(t)\rangle_S\,. \tag{8}$$

In the following we will consider the system S consisting of two clocks (A and B) and we will examine their evolution with respect to the clock in the space C which play the role of a far-away time reference.

III. CLOCKS A AND B WITH DISCRETE TIME VALUES

We assume here the subsystem S consisting of two clocks, A and B, with discrete time values and we

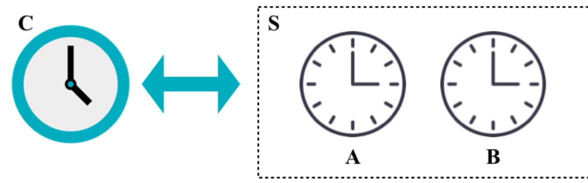

FIG. 1. The subsystem S consists of two clocks, A and B, and we study their evolution with respect to the clock C.

study their evolution with respect to the clock in the C subspace. In Section III.A we consider two free clock (namely not perturbed by the gravitational field), we define operators $\hat{\tau}$ and $\hat{\theta}$ complement of $\hat{H}_A$ and $\hat{H}_B$ and we show, as expected, that the clocks evolve synchronously. In Section III.B we place clocks A and B within the gravitational potential, at distance $x+h$ and x respectively from the center of a large mass M. In this new case operators $\hat{\tau}$ and $\hat{\theta}$ are complement of Hamiltonians $\hat{H'}_A$ and $\hat{H'}_B$ modified by the gravitational field and we will show how clock B ticks at a lower rate than A. Finally we bring A to an infinite distance from the mass.

A. Evolution of free clocks

As mentioned, we consider here the system S consisting of two clocks A and B. The global space reads now $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_S = \mathcal{H}_C \otimes \mathcal{H}_A \otimes \mathcal{H}_B$ where we assume $d_A = d_B = d$ and $d_C \gg d$. The global Hamiltonian reads:

$$\hat{H} = \hat{H}_C + \hat{H}_A + \hat{H}_B \tag{9}$$

where the clocks A and B are governed by Hamiltonians with bounded, discrete spectra and equally-spaced energy levels. Being the clocks A and B identical, we have:

$$\hat{H}_A = \hat{H}_B = \sum_{k=0}^{d-1}\frac{2\pi}{T}k\,|k\rangle\langle k| \tag{10}$$

where

$$\frac{2\pi}{T}k = E_k^{(A)} = E_k^{(B)} \tag{11}$$

are the equally-spaced energy eigenvalues. We simultaneously introduce the operators $\hat{\tau} = \sum_{m=0}^{d-1}\tau_m|\tau_m\rangle\langle\tau_m|$ and $\hat{\theta} = \sum_{l=0}^{d-1}\theta_l|\theta_l\rangle\langle\theta_l|$ complement of $\hat{H}_A$ and $\hat{H}_B$ respectively, with

$$|\tau_m\rangle_A = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}e^{-i\frac{2\pi}{T}k\tau_m}|k\rangle_A = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}e^{-i\frac{2\pi}{d}km}|k\rangle_A \tag{12}$$

and

$$|\theta_l\rangle_B = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1} e^{-i\frac{2\pi}{T}k\theta_l}|k\rangle_B = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1} e^{-i\frac{2\pi}{d}kl}|k\rangle_B \tag{13}$$

where we have defined: $\tau_m = m\frac{T}{d}$ and $\theta_l = l\frac{T}{d}$ ($m, l = 0, 1, 2, ..., d-1$). The Hermitian operator $\hat{\tau}$ is complement of the Hamiltonian $\hat{H}_A$ in the sense that $\hat{H}_A$ is generator of shifts in eigenvalues of $\hat{\tau}$ and, viceversa, $\hat{\tau}$ is the generator of energy shifts. The same holds for $\hat{\theta}$ with respect to $\hat{H}_B$. The quantity T is the time taken by the clocks to return to their initial state, indeed we have $|\tau_m + T\rangle_A = |\tau_m\rangle_A$ and $|\theta_l + T\rangle_B = |\theta_l\rangle_B$. We want teh clocks A and B to be independent, so we assume them in a product state in S. The global state $|\Psi\rangle$ satisfying the constraint (1) can be written:

$$\begin{aligned}|\Psi\rangle &= \frac{1}{T_C}\int_0^{T_C} dt\,|t\rangle_C \otimes |\psi(t)\rangle_S = \\ &= \frac{1}{T_C}\int_0^{T_C} dt\,|t\rangle_C \otimes |\varphi(t)\rangle_A \otimes |\phi(t)\rangle_B\end{aligned} \tag{14}$$

where $|\psi(t)\rangle_S = \langle t|\Psi\rangle = |\varphi(t)\rangle_A \otimes |\phi(t)\rangle_B$ is the relative state of $S = A + B$ at time t, namely the product state of A and B conditioned on having t in C.

Through the state $|\psi(t)\rangle_S$ we can investigate the time evolution of the clocks A and B. For the initial state of the clocks we choose the state

$$|\psi(0)\rangle_S = |\varphi(0)\rangle_A \otimes |\phi(0)\rangle_B = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|k\rangle_A \otimes \frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}|n\rangle_B \tag{15}$$

namely we consider the clocks A and B at time $t = 0$ to be in the time states $|\tau_0 = 0\rangle_A$ and $|\theta_0 = 0\rangle_B$. This implies that the state of S at generic time t reads:

$$\begin{aligned}|\psi(t)\rangle_S &= |\varphi(t)\rangle_A \otimes |\phi(t)\rangle_B = \\ &= \frac{1}{d}\sum_{k=0}^{d-1} e^{-i\frac{2\pi}{T}kt}|k\rangle_A \otimes \sum_{n=0}^{d-1} e^{-i\frac{2\pi}{T}nt}|n\rangle_B\end{aligned} \tag{16}$$

indicating the clocks evolving with the Schrödinger evolution. As times t goes on, the clocks "click"all the time states $|\tau_m\rangle_A$ and $|\theta_l\rangle_B$ until they reaches $t = T$, thus completing one cycle and returning to their initial state. As expected, the clocks evolve synchronously over time: considering indeed to be at time $t = m\frac{T}{d}$, we have

$$\begin{aligned}|\psi(t = m\tfrac{T}{d})\rangle_S &= |\varphi(t = m\tfrac{T}{d})\rangle_A \otimes |\phi(t = m\tfrac{T}{d})\rangle_B = \\ &= \frac{1}{d}\sum_{k=0}^{d-1} e^{-i\frac{2\pi}{d}km}|k\rangle_A \otimes \sum_{n=0}^{d-1} e^{-i\frac{2\pi}{d}nm}|n\rangle_B\end{aligned} \tag{17}$$

where A is clicking the $m-th$ state $|\tau_m\rangle_A$ and also B is clicking its $m-th$ state $|\theta_m\rangle_B$.

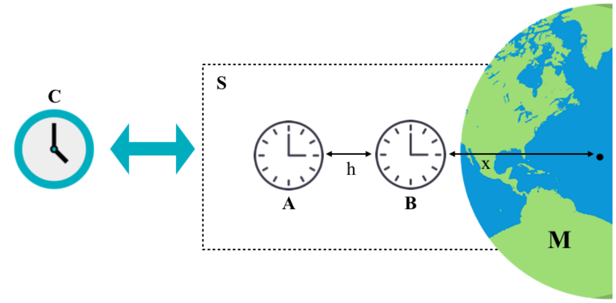

FIG. 2. The subsystem S consists of two clocks A and B both placed within the gravitational potential. B is at distance x from the center of the mass M, while A is at distance $x + h$.

B. A and B interacting with the gravitational field

We consider now the case in which clocks A and B are placed within a gravitational potential. In doing this we assume B at a distance x from the center of a spherical mass M and A placed at a distance $x + h$ (see Fig. 2). In the interaction between the clocks and the gravitational potential we consider the Newtonian potential $\phi(x) = -\frac{GM}{x}$, where G is the universal gravitational constant and the coordinate x is treated as a number. In calculating such interaction we promote the mass of the clocks to operator using the mass–energy equivalence: $m_A \to m_A + \hat{H}_A/c^2$ and $m_B \to m_B + \hat{H}_B/c^2$. We assume that the static mass of the clocks is negligibly small compared with the dynamical one and we focus only on the effects due to the internal degrees of freedom. Nevertheless, the fact of considering the contributions given by the static masses would only lead to an unobservable global phase factor in the evolution of the clocks.

For the Hamiltonians $\hat{H}_A$ and $\hat{H}_B$ we have again $\hat{H}_A = \hat{H}_B = \sum_{k=0}^{d-1}\frac{2\pi}{T}k|k\rangle\langle k|$, and so the global Hamiltonian now reads:

$$\begin{aligned}\hat{H} &= \hat{H}_C + \hat{H}_A + \hat{H}_B + \frac{\hat{H}_A}{c^2}\phi(x+h) + \frac{\hat{H}_B}{c^2}\phi(x) = \\ &= \hat{H}_C + \hat{H}'_A + \hat{H}'_B\end{aligned} \tag{18}$$

where

$$\hat{H}'_A = \hat{H}_A\left(1 + \frac{\phi(x+h)}{c^2}\right) = \hat{H}_A\left(1 - \frac{GM}{(x+h)c^2}\right) \tag{19}$$

and

$$\hat{H}'_B = \hat{H}_B\left(1 + \frac{\phi(x)}{c^2}\right) = \hat{H}_B\left(1 - \frac{GM}{xc^2}\right). \tag{20}$$

The Hamiltonians $\hat{H}'_A$ and $\hat{H}'_B$ can be written:

$$\hat{H}'_A = \sum_{k=0}^{d-1}\frac{2\pi}{T''}k|k\rangle\langle k| \tag{21}$$

and

$$\hat{H}'_B = \sum_{k=0}^{d-1} \frac{2\pi}{T'} k \left|k\right\rangle \left\langle k\right|, \tag{22}$$

where we have defined

$$T'' = \frac{T}{1 - \frac{GM}{(x+h)c^2}} \tag{23}$$

and

$$T' = \frac{T}{1 - \frac{GM}{xc^2}}. \tag{24}$$

In the subspace S we introduce again the operators $\hat{\tau}$ and $\hat{\theta}$ where now the first is complement of $\hat{H}'_A$ and the second is complement of $\hat{H}'_B$. The time states of the clocks read here:

$$\left|\tau_m\right\rangle_A = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{-i\frac{2\pi}{T''}k\tau_m} \left|k\right\rangle_A \tag{25}$$

with

$$\tau_m = m\frac{T''}{d} = m\frac{T}{d(1 - \frac{GM}{(x+h)c^2})} \tag{26}$$

and

$$\left|\theta_l\right\rangle_B = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{-i\frac{2\pi}{T'}k\theta_l} \left|k\right\rangle_B \tag{27}$$

with

$$\theta_l = l\frac{T'}{d} = l\frac{T}{d(1 - \frac{GM}{xc^2})}. \tag{28}$$

We notice that the presence of the gravitational field does not change the form of the time states. Indeed, through (26) and (28), we can rewrite (25) and (27) as

$$\left|\tau_m\right\rangle_A = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{-i\frac{2\pi}{d}km} \left|k\right\rangle_A \tag{29}$$

and

$$\left|\theta_l\right\rangle_B = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{-i\frac{2\pi}{d}kl} \left|k\right\rangle_B \tag{30}$$

which are the same of (12) and (13). The global state satisfying the constraint (1) can again be written as in (14) and we consider also here the clocks starting in the product state of the time states $\left|\tau_0 = 0\right\rangle_A$ and $\left|\theta_0 = 0\right\rangle_B$.

We can now look at the state $\left|\psi(t)\right\rangle_S$ investigating the time evolution of A and B. At generic time t we have:

$$\begin{aligned} &\left|\psi(t)\right\rangle_S = \left|\varphi(t)\right\rangle_A \otimes \left|\phi(t)\right\rangle_B = \\ &= \frac{1}{d} \sum_{k=0}^{d-1} e^{-i\frac{2\pi}{T}kt(1-\frac{GM}{(x+h)c^2})} \left|k\right\rangle_A \otimes \sum_{n=0}^{d-1} e^{-i\frac{2\pi}{T}nt(1-\frac{GM}{xc^2})} \left|n\right\rangle_B . \end{aligned} \tag{31}$$

Equation (31) provides again the Schrödinger evolution for A and B where we can notice a different delay in the phases of the two clocks. This different delay indicates the two clocks ticking different time states at a given time t as read by the clock C. To better see this point we can consider to be at time $t = m\frac{T}{d}$ as read by C. Equation (31) becomes:

$$\begin{aligned} &\left|\psi(t = m\frac{T}{d})\right\rangle_S = \left|\varphi(t = m\frac{T}{d})\right\rangle_A \otimes \left|\phi(t = m\frac{T}{d})\right\rangle_B = \\ &= \frac{1}{d} \sum_{k=0}^{d-1} e^{-i\frac{2\pi}{d}km(1-\frac{GM}{(x+h)c^2})} \left|k\right\rangle_A \otimes \sum_{n=0}^{d-1} e^{-i\frac{2\pi}{d}nm(1-\frac{GM}{xc^2})} \left|n\right\rangle_B \end{aligned} \tag{32}$$

showing that no one, between A and B, has clicked the $m-th$ state. Clock B has indeed clicked a number of states $m' = m\left(1 - \frac{GM}{xc^2}\right)$ and A has clicked a number of states $m'' = m\left(1 - \frac{GM}{(x+h)c^2}\right)$ $(m', m'' \in \mathbb{R})$. From this we easily derive:

$$m' = m''\left(1 - \frac{GM}{xc^2}\right)\left(1 - \frac{GM}{(x+h)c^2}\right)^{-1} \tag{33}$$

which is in agreement with the time dilation between two clocks at a (radial) distance h from each other as obtained in the first order expansion of the Schwarzschild metric. We notice that the "agreement" only regards the time dilation effect: the x and $x + h$ coordinates represent here simply the distance between the clocks and the center of the spherical mass M. We do not define the Schwarzschild radial coordinate, as we do not define coordinate time and proper time. We are indeed working in a purely quantum non-relativistic framework, with the only exception of having promoted the masses of the clocks to operators using the mass-energy equivalence. Expanding the second term in the right-hand side of (33) and neglecting terms $\sim \left(\frac{GM}{xc^2}\right)^2$, the equation reads

$$\begin{aligned} m' &\simeq m''\left(1 - \frac{GM}{xc^2}\right)\left(1 + \frac{GM}{(x+h)c^2}\right) = \\ &\simeq m''\left(1 - \frac{GMh}{x(x+h)c^2}\right) \end{aligned} \tag{34}$$

which, for $h \ll x$, becomes:

$$m' \simeq m''\left(1 - \frac{ah}{c^2}\right) \tag{35}$$

where we have defined the gravitational acceleration as $a = \frac{GM}{x^2}$. This time dilation effect is in agreement with the relativistic result in the case, for example, of two clocks placed on the earth and separated by a vertical distance h, sufficiently small relative to the radius of the planet. We notice here that our results follows from having a finite T. If we consider indeed clocks with unbounded time states, we would not find the same effect.

We can also calculate the gravitational redshift as produced by our framework. Considering indeed the frequecy ν as proportional to the spacing between two neighboring energy levels of the clocks, we have $\nu = 1/T$ for a non-perturbed clock. For the clocks A and B within the gravitational field we can write:

$$\delta\nu = \nu_A - \nu_B = \frac{1}{T''} - \frac{1}{T'} \tag{36}$$

that, for $h \ll x$, becomes

$$\begin{aligned}\delta\nu &= \frac{1}{T}\left[\left(1-\frac{GM}{(x+h)c^2}\right)-\left(1-\frac{GM}{xc^2}\right)\right]\\ &= \frac{1}{T}\frac{GM}{c^2}\left(\frac{1}{x}-\frac{1}{x+h}\right) \simeq \frac{1}{T}\frac{GMh}{x^2c^2}.\end{aligned} \tag{37}$$

Considering again $a = \frac{GM}{x^2}$ and neglecting terms of the order $\sim \left(\frac{GM}{xc^2}\right)^2$, we obtain

$$\frac{\delta\nu}{\nu_B} \simeq \frac{ah}{c^2} \tag{38}$$

that is (at the first order of approximation) in agreement with the relativistic result [22]. Equation (38) clearly holds even when considering the spacing between any two energy levels and not two neighbors.

As a last point, we consider the clock B again placed within the gravitational potential at a distance x from origin of the gravitational field, but we bring clock A at an infinite distance from M (see Fig. (3)). We implement this scenario simply by taking $h \longrightarrow \infty$. In this limiting case the Hamiltonian of clock A becomes $\hat{H'}_A \longrightarrow \hat{H}_A = \sum_{k=0}^{d-1} \frac{2\pi}{T} k \left|k\right\rangle \left\langle k\right|$ and equation (33) reduces to:

$$m' = m''\left(1 - \frac{GM}{xc^2}\right). \tag{39}$$

This expression is again in agreement with the first order expansion ofthe gravitational time dilation as derived from the Schwarzschild metric.

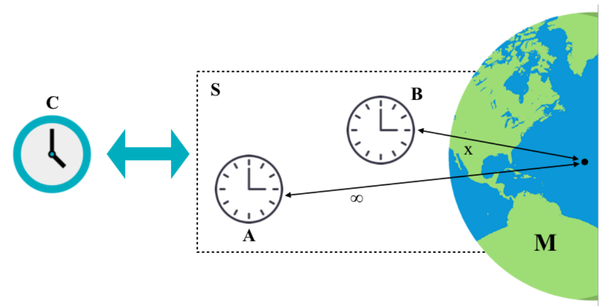

FIG. 3. The subsystem S consists of two clocks A and B. B is placed within the gravitational potential at distance x from the center of the spherical mass M, while A is non-perturbed at an infinite distance from the mass.

IV. CLOCKS A AND B WITH CONTINUOUS TIME VALUES

We replicate here the discussion made in Section III, but considering the clocks A and B represented by Pegg's POVM with continuous time values. As in the previous Section, we first study the time evolution of free clocks, then we will introduce the gravitational field and we study two cases: we will first consider A and B within the gravitational potential at a distance h from each other, then we bring A at infinite distance from the mass.

A. Evolution of free clocks

The system S consists here of two free clocks A and B with continuos time values. The global Hilbert space reads $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_S = \mathcal{H}_C \otimes \mathcal{H}_A \otimes \mathcal{H}_B$ where we assume $d_A = d_B = d$ and $d_C \gg d$. The global Hamiltonian reads:

$$\hat{H} = \hat{H}_C + \hat{H}_A + \hat{H}_B \tag{40}$$

and the clocks are governed again by Hamiltonians with bounded, discrete spectra and equally-spaced energy levels. Namely we have:

$$\hat{H}_A = \hat{H}_B = \sum_{k=0}^{d-1} \frac{2\pi}{T} k \left|k\right\rangle \left\langle k\right| \tag{41}$$

where $\frac{2\pi}{T}k = E_k^{(A)} = E_k^{(B)}$ are the equally-spaced energy eigenvalues. In this paragraph we don not consider the Hermitian operators $\hat{\tau}$ and $\hat{\theta}$ complement of $\hat{H}_A$ and $\hat{H}_B$. Rather we introduce the time states:

$$\left|\tau_f\right\rangle_A = \sum_{k=0}^{d-1} e^{-i\frac{2\pi}{T}k\tau_f} \left|k\right\rangle_A = \sum_{k=0}^{d-1} e^{-i2\pi kf} \left|k\right\rangle_A \tag{42}$$

and

$$\left|\theta_g\right\rangle_B = \sum_{k=0}^{d-1} e^{-i\frac{2\pi}{T}k\theta_g} \left|k\right\rangle_B = \sum_{k=0}^{d-1} e^{-i2\pi kg} \left|k\right\rangle_B \tag{43}$$

where we have defined $\tau_f = fT$ and $\theta_g = gT$, with f and g taking any real values in the interval $[0,1]$. Assuming the clocks A and B in a product state in S, the global state $|\Psi\rangle$ satisfying the constraint (1) can again be written:

$$\begin{aligned}|\Psi\rangle &= \frac{1}{T_C}\int_0^{T_C} dt \left|t\right\rangle_C \otimes \left|\psi(t)\right\rangle_S =\\ &= \frac{1}{T_C}\int_0^{T_C} dt \left|t\right\rangle_C \otimes \left|\varphi(t)\right\rangle_A \otimes \left|\phi(t)\right\rangle_B\end{aligned} \tag{44}$$

where $\left|\psi(t)\right\rangle_S = \langle t|\Psi\rangle = \left|\varphi(t)\right\rangle_A \otimes \left|\phi(t)\right\rangle_B$ is the relative state of $S = A + B$ at time t, namely the product state of A and B conditioned on having t in C.

Through the state $|\psi(t)\rangle_S$ we investigate again the time evolution of the clocks A and B. For the initial state of the clocks we choose the state

$$|\psi(0)\rangle_S = |\varphi(0)\rangle_A \otimes |\phi(0)\rangle_B = \frac{1}{d}\sum_{k=0}^{d-1}|k\rangle_A \otimes \sum_{n=0}^{d-1}|n\rangle_B \tag{45}$$

namely we consider the clocks A and B at time $t=0$ to be in the time states $|\tau_{f=0}\rangle_A$ and $|\theta_{g=0}\rangle_B$ (apart from a normalization constant). This implies that the state of S at time $t=fT$ reads:

$$\begin{aligned}|\psi(t=fT)\rangle_S &= |\varphi(t=fT)\rangle_A \otimes |\phi(t=fT)\rangle_B = \\ &= \frac{1}{d}\sum_{k=0}^{d-1} e^{-i2\pi kf}|k\rangle_A \otimes \sum_{n=0}^{d-1} e^{-i2\pi nf}|n\rangle_B \end{aligned}\tag{46}$$

indicating the clocks evolving together. Also in this case we can indeed see that A and B are clicking simultaneously the time states $|\tau_f\rangle_A$ and $|\theta_f\rangle_B$.

B. A and B interacting with the gravitational field

We consider now the case of clocks A and B with continuous time values both interacting with the Newtonian gravitaional field. Clock B is placed at a distance x from the center of the mass M, while A is at a distance $x+h$. We have again $\hat{H}_A = \hat{H}_B = \sum_{k=0}^{d-1}\frac{2\pi}{T}k\,|k\rangle\langle k|$, and so the global Hamiltonian reads:

$$\begin{aligned}\hat{H} &= \hat{H}_C + \hat{H}_A + \frac{\hat{H}_A}{c^2}\phi(x+h) + \hat{H}_B + \frac{\hat{H}_B}{c^2}\phi(x) = \\ &= \hat{H}_C + \hat{H}'_A + \hat{H}'_B\end{aligned}\tag{47}$$

where

$$\hat{H}'_A = \hat{H}_A\left(1-\frac{GM}{(x+h)c^2}\right) = \sum_{k=0}^{d-1}\frac{2\pi}{T''}k\,|k\rangle\langle k| \tag{48}$$

and

$$\hat{H}'_B = \hat{H}_B\left(1-\frac{GM}{xc^2}\right) = \sum_{k=0}^{d-1}\frac{2\pi}{T'}k\,|k\rangle\langle k| \tag{49}$$

with $T''=T/(1-\frac{GM}{(x+h)c^2})$ and $T'=T/(1-\frac{GM}{xc^2})$. As in the case of previous Section, we notice that the presence of the gravitational field does not change the form of the time states for the clocks. Indeed we have:

$$|\tau_f\rangle_A = \sum_{k=0}^{d-1} e^{-i\frac{2\pi}{T''}k\tau_f}|k\rangle_A = \sum_{k=0}^{d-1} e^{-i2\pi kf}|k\rangle_A \tag{50}$$

and

$$|\theta_g\rangle_B = \sum_{k=0}^{d-1} e^{-i\frac{2\pi}{T'}k\theta_g}|k\rangle_B = \sum_{k=0}^{d-1} e^{-i2\pi kg}|k\rangle_B \tag{51}$$

where we now considered $\tau_f = fT''$ and $\theta_g = gT'$ with $f,g\in[0,1]$. Definitions (50) and (51) for the time states are therefore unchanged from (42) and (43).

Choosing the product state (45) as initial condition for the clocks, we can look at the evolution of A and B. Assuming clock C to be in $t=fT$, equation (46) becomes:

$$\begin{aligned}&|\psi(t=fT)\rangle_S = |\varphi(t=fT)\rangle_A \otimes |\phi(t=fT)\rangle_B = \\ &= \frac{1}{d}\sum_{k=0}^{d-1} e^{-i2\pi kf(1-\frac{GM}{(x+h)c^2})}|k\rangle_A \otimes \sum_{n=0}^{d-1} e^{-i2\pi nf(1-\frac{GM}{xc^2})}|n\rangle_B\,.\end{aligned}\tag{52}$$

Equation (52) shows that, when the clock C reads time $t=fT$, clock A has not reached the state $|\tau_f\rangle_A$ and B has not reached the state $|\theta_f\rangle_B$. Rather A is clicking the state $|\tau_{f''}\rangle_A$ with $f''=f\left(1-\frac{GM}{(x+h)c^2}\right)$ and B is clicking the state $|\theta_{f'}\rangle_B$ with $f'=f\left(1-\frac{GM}{xc^2}\right)$. This implies

$$f' = f''\left(1-\frac{GM}{xc^2}\right)\left(1-\frac{GM}{(x+h)c^2}\right)^{-1} \tag{53}$$

which is the same of equation (33) in this new case of clocks with continuous time values. As we did in Section III.B, neglecting terms of the order $\sim\left(\frac{GM}{xc^2}\right)^2$, writing the gravitational acceleration $a=\frac{GM}{x^2}$ and for $h\ll x$, we can derive:

$$f' \simeq f''\left(1-\frac{ah}{c^2}\right) \tag{54}$$

where we can see the clock B ticking slower then clock A in agreement with the relativistic result.

Also in this case, the result obtained in Section III.B regarding gravitational redshift can be derived. The spectrum of the clock Hamiltonians has indeed not changed, and we can still define the frequency $\nu=1/T$ for a non-perturbed clock. For the clocks A and B within the gravitational field we have again:

$$\frac{\delta\nu}{\nu_B} = \frac{\nu_A-\nu_B}{\nu_B} \simeq \frac{ah}{c^2} \tag{55}$$

where $\nu_A = 1/T''$, $\nu_B = 1/T'$ and a the gravitational acceleration at the distance x from the center of the planet.

Bringing now clock A at infinite distance from M ($h\longrightarrow\infty$), we have again $\hat{H}'_A \longrightarrow \hat{H}_A = \sum_{k=0}^{d-1}\frac{2\pi}{T}k\,|k\rangle\langle k|$ and equation (53) becomes:

$$f' = f''\left(1-\frac{GM}{xc^2}\right). \tag{56}$$

Again we found clock B ticking slower in agreement with the first order expansion of the gravitational time dilation as derived from the Schwarzschild metric.

V. DISCUSSION

The agreement between our results and the relativistic solution clearly holds up to the first order of approximation in the Taylor expansion of the Schwarzschild metric. This does not include, for example, the case of clocks placed close to the event horizon of a black hole. Nevertheless, we can extend our analysis considering the expression of the relativistic gravitational potential within the global Hamiltonian. Indeed, in this case, the potential energy V of a clock placed at a distance x from the center of the spherical mass M reads [8, 9]:

$$V = m_{clock}c^2 \left[\left(1 - \frac{2GM}{xc^2}\right)^{\frac{1}{2}} - 1 \right]. \tag{57}$$

Therefore, considering the case of A and B in the gravitational field (at distances $x+h$ and x respectively from the center of the spherical mass M) and promoting the mass of the clocks to operator using the mass-energy equivalence, we have the global Hamiltonian:

$$\begin{aligned} \hat{H} &= \hat{H}_C + \hat{H}_A + \hat{H}_A \left[\left(1 - \frac{2GM}{(x+h)c^2}\right)^{\frac{1}{2}} - 1 \right] + \hat{H}_B + \hat{H}_B \left[\left(1 - \frac{2GM}{xc^2}\right)^{\frac{1}{2}} - 1 \right] = \\ &= \hat{H}_C + \hat{H}_A \left(1 - \frac{2GM}{(x+h)c^2}\right)^{\frac{1}{2}} + \hat{H}_B \left(1 - \frac{2GM}{xc^2}\right)^{\frac{1}{2}} = \hat{H}_C + \hat{H}'_A + \hat{H}'_B \end{aligned} \tag{58}$$

where now

$$\hat{H}'_A = \hat{H}_A \left(1 - \frac{2GM}{(x+h)c^2}\right)^{\frac{1}{2}} = \sum_{k=0}^{d-1} \frac{2\pi}{T''} k \left|k\right\rangle \left\langle k\right| \tag{59}$$

and

$$\hat{H}'_B = \hat{H}_B \left(1 - \frac{2GM}{xc^2}\right)^{\frac{1}{2}} = \sum_{k=0}^{d-1} \frac{2\pi}{T'} k \left|k\right\rangle \left\langle k\right| \tag{60}$$

with $T'' = T\left(1 - \frac{2GM}{(x+h)c^2}\right)^{-\frac{1}{2}}$ and $T' = T\left(1 - \frac{2GM}{xc^2}\right)^{-\frac{1}{2}}$.

We can now investigate the time evolution of A and B. At generic time t (as read by the clock C) we have:

$$\begin{aligned} &\left|\psi(t)\right\rangle_S = \left|\varphi(t)\right\rangle_A \otimes \left|\phi(t)\right\rangle_B = \\ &\propto \sum_{k=0}^{d-1} e^{-i\frac{2\pi}{T}kt\left(1 - \frac{2GM}{(x+h)c^2}\right)^{\frac{1}{2}}} \left|k\right\rangle_A \otimes \sum_{n=0}^{d-1} e^{-i\frac{2\pi}{T}nt\left(1 - \frac{2GM}{xc^2}\right)^{\frac{1}{2}}} \left|n\right\rangle_B . \end{aligned} \tag{61}$$

In the case of clocks with discrete time values equation (61) implies that, when A clicks its $m'' - th$ time state, clock B has clicked a number of states:

$$m' = m'' \left(1 - \frac{2GM}{xc^2}\right)^{\frac{1}{2}} \left(1 - \frac{2GM}{(x+h)c^2}\right)^{-\frac{1}{2}} \tag{62}$$

which, in the limit $h \longrightarrow \infty$, becomes

$$m' = m'' \left(1 - \frac{2GM}{xc^2}\right)^{\frac{1}{2}} . \tag{63}$$

Similarly, for the case of clocks with continuous time values equation (61) implies that, when A clicks the time state $\left|\tau_{f''}\right\rangle_A$, the clock B is clicking the time state $\left|\theta_{f'}\right\rangle_B$ with:

$$f' = f'' \left(1 - \frac{2GM}{xc^2}\right)^{\frac{1}{2}} \left(1 - \frac{2GM}{(x+h)c^2}\right)^{-\frac{1}{2}} \tag{64}$$

which, in the limit $h \longrightarrow \infty$, again becomes

$$f' = f'' \left(1 - \frac{2GM}{xc^2}\right)^{\frac{1}{2}} . \tag{65}$$

In conclusion, the use of the relativistic gravitational potential leads to a time dilation which is in agreement with the exact Schwarzschild solution. We notice that, taking the first order expansion of the relativistic gravitational potential, we recover the results of Sections III and IV.

VI. CONCLUSIONS

In this work we have explored the effects of gravity on quantum clocks. We have considered two clocks, A and B, interacting with a Newtonian gravitational field, and we have studied their evolution with respect to the clock C, taken as a far-away reference via the PaW mechanism. We have performed our investigation in the case of A and B having discrete and continuous time values.

In both cases we have first considered two free clocks and shown that, as expected, they evolve synchronously over time. Then we have studied the scenario in which the clocks are placed in the gravitational field, at different distances from the origin of the field. We have shown that, as time t goes on (as read by the far-away clock C), the states of the two clocks suffer a different delay in the phases depending on the intensity of the gravitational potential. This delay in the phases translates into the two clocks ticking at different rates. The time dilation is in accordance with the first order expansion of the gravitational time dilation as derived from the Schwarzschild metric. Finally, we have discussed the possibility of using the relativistic gravitational potential within the global Hamiltonian and we have shown that this choice leads to the agreement between our time dilation effect and the exact result obtained with the Schwarzschild metric.

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Appendix A: Conditional Probabilieties and Main Values of the Observables for Discrete Clocks

We investigate here the behavior of specific quantities, as conditional probabilities and mean values of the observables. We first study the case of A and B as free clocks, then we compare the results with those obtained by considering B within the gravitational field and A placed far away, not perturbed by the field. In this Appendix we work with cloks with discrete time values, thus starting from the formalism described in Section III.

1. A and B free clocks

We recall here that we are working in the global space $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_S = \mathcal{H}_C \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, so the global state satisfying constraint (1) is:

$$\begin{aligned} |\Psi\rangle &= \frac{1}{T_C} \int_0^{T_C} dt\, |t\rangle_C \otimes |\psi(t)\rangle_S = \\ &= \frac{1}{T_C} \int_0^{T_C} dt\, |t\rangle_C \otimes |\varphi(t)\rangle_A \otimes |\phi(t)\rangle_B . \end{aligned} \tag{A1}$$

We assume that the clock in the C subspace reads time $t = \tau_m$ and we work with the state of the system S: $|\psi(t=\tau_m)\rangle_S = \langle t=\tau_m|\Psi\rangle = |\varphi(t=\tau_m)\rangle_A \otimes |\phi(t=\tau_m)\rangle_B$. We consider two quantities, namely the conditional probability $P(\theta_l|\tau_m)$ of obtaining θ_l on B conditioned to having found τ_m on A, and the mean value of $\langle\theta\rangle(\tau_m)$. We have for the probability $P(\theta_l|\tau_m)$:

$$P(\theta_l|\tau_m) = \frac{\langle\psi(t=\tau_m)|\tau_m\rangle \langle\tau_m| \otimes |\theta_l\rangle \langle\theta_l|\psi(t=\tau_m)\rangle}{\langle\psi(t=\tau_m)|\tau_m\rangle \langle\tau_m|\psi(t=\tau_m)\rangle} \tag{A2}$$

which leads to the result

$$P(\theta_l|\tau_m) = \frac{1}{d^2} \sum_{n=0}^{d-1} \sum_{k=0}^{d-1} e^{-i(\theta_l - \tau_m)\frac{2\pi}{T}(n-k)}. \tag{A3}$$

This probability is well defined, indeed it easy to verify that $\sum_{l=0}^{d-1} P(\theta_l|\tau_m) = 1$ for each given τ_m. Furthermore, through probability (A3), we can calculate the mean value $\langle\theta\rangle(\tau_m)$ as follows:

$$\langle\theta\rangle(\tau_m) = \sum_{l=0}^{d-1} \theta_l P(\theta_l|\tau_m) = \frac{T}{d^3} \sum_{n=0}^{d-1} \sum_{k=0}^{d-1} f(n-k) e^{i\tau_m \frac{2\pi}{T}(n-k)} \tag{A4}$$

where $f(n-k) = \sum_{l=0}^{d-1} l e^{-i\frac{2\pi l}{d}(n-k)}$. In this particular case in which the two clock are identical and both non-perturbed we easily find

$$P(\theta_m|\tau_m) = 1 \tag{A5}$$

and

$$\langle\theta\rangle(\tau_m) = \tau_m = \theta_m. \tag{A6}$$

Equations (A5) and (A6) indicate what we had already seen in (16), namely that the two clocks, as expected, run together over time.

2. B interacting with the gravitational field

We consider now the case of clock B placed at a distance x from the center of a large, spherical mass M while A is placed far away, at infinite distance from the mass. The time states of the clocks are therefore given by (12) and (27) for A and B respectively.

We assume now again that the clock in the C subspace reads time $t = \tau_m$ and we work with the state of the system S: $|\psi(t=\tau_m)\rangle_S = \langle t=\tau_m|\Psi\rangle = |\varphi(t=\tau_m)\rangle_A \otimes |\phi(t=\tau_m)\rangle_B$. We look at the quantities (A3) and (A4) in this new case where B is placed within the gravitatinal field. We have:

$$P(\theta_l|\tau_m) = \frac{1}{d^2} \sum_{n=0}^{d-1} \sum_{k=0}^{d-1} e^{-i(\theta_l - \tau_m)\frac{2\pi}{T'}(n-k)} \tag{A7}$$

and

$$\langle\theta\rangle(\tau_m) = \sum_{l=0}^{d-1} \theta_l P(\theta_l|\tau_m) = \frac{T'}{d^3} \sum_{n=0}^{d-1} \sum_{k=0}^{d-1} f(n-k) e^{i\tau_m \frac{2\pi}{T'}(n-k)} \tag{A8}$$

where $f(n-k)$ is again $f(n-k) = \sum_{l=0}^{d-1} l e^{-i\frac{2\pi l}{d}(n-k)}$ and, we recall, $T' = T/(1 - \frac{GM}{xc^2})$. Through (A7) and (A8) we can again see the effect of the gravitational field in the clock B with respect to clock A. Let us consider two simple examples: we will look at clocks A and B in the case $d = 2$ and $d = 3$.

In the case $d = 2$ we have:

$$P(\theta_l|\tau_m) = \frac{1}{2}\left[1 + \cos(\pi(l - m(1 - \frac{GM}{xc^2})))\right] \tag{A9}$$

that leads to $P(\theta_m|\tau_m) = \frac{1}{2}[1 + \cos(\pi m \frac{GM}{xc^2})]$. So, assuming $\cos(z) \simeq 1 - \frac{z^2}{2}$, we find: $P(\theta_m|\tau_m) \simeq 1 - \frac{m^2\pi^2}{4}\left(\frac{GM}{xc^2}\right)^2$ which depends on m (a larger m implies a reduced probability) and on the intensity of the gravitational potential.

Going instead to calculate $\langle\theta\rangle(\tau_m)$ we have:

$$\langle\theta\rangle(\tau_m) = \frac{T'}{4}\left[1-\cos(m\pi(1-\frac{GM}{xc^2}))\right]$$

$$\Rightarrow \langle\theta\rangle(\tau_0) = 0 = \theta_0$$
$$\Rightarrow \langle\theta\rangle(\tau_1) \simeq \frac{T'}{2}\left[1-\frac{\pi^2}{4}\left(\frac{GM}{xc^2}\right)^2\right] = \theta_1\left[1-\frac{\pi^2}{4}\left(\frac{GM}{xc^2}\right)^2\right]. \tag{A10}$$

We can notice that, during the period of time in which the main value of A moved from $\tau_0 = 0$ to τ_1, the mean value $\langle\theta\rangle$ of the clock B moved to $\theta_1\left[1-\frac{\pi^2}{4}\left(\frac{GM}{xc^2}\right)^2\right]$, that is it doesn't reach clicking the first eigenvalue θ_1. So, also considering the main values, we find the clock B delayed in ticking with respect to the clock A by a factor that depends again on the intensity of the gravitational potential.

The same conclusion can be reached considering slightly more complex clocks, i.e. with d=3. In this case we obtain:

$$P(\theta_l|\tau_m) = \frac{1}{9}\left[3+4\cos(\frac{2\pi}{3}(l-m(1-\frac{GM}{xc^2})))+2\cos(\frac{4\pi}{3}(l-m(1-\frac{GM}{xc^2})))\right] \tag{A11}$$

that for $l = m$ becomes $P(\theta_m|\tau_m) = \frac{1}{9}[3+4\cos(\frac{2\pi m}{3}\frac{GM}{xc^2})+2\cos(\frac{4\pi m}{3}\frac{GM}{xc^2})]$. So, considering again the second order of approximation $\cos(z) \simeq 1-\frac{z^2}{2}$, we have: $P(\theta_m|\tau_m) \simeq 1-\frac{8\pi^2 m^2}{27}\left(\frac{GM}{xc^2}\right)^2$.

In the same way we obtain for the mean value $\langle\theta\rangle(\tau_m)$:

$$\langle\theta\rangle(\tau_m) = \frac{T'}{27}[9-6\cos(\frac{2\pi m}{3}(1-\frac{GM}{xc^2}))-2\sqrt{3}\sin(\frac{2\pi m}{3}(1-\frac{GM}{xc^2}))+$$
$$-3\cos(\frac{4\pi m}{3}(1-\frac{GM}{xc^2}))+\sqrt{3}\sin(\frac{4\pi m}{3}(1-\frac{GM}{xc^2}))]. \tag{A12}$$

Equation (A12) implies that:

$$\Rightarrow \langle\theta\rangle(\tau_0) = 0 = \theta_0$$
$$\Rightarrow \langle\theta\rangle(\tau_1) \simeq \frac{T'}{3}\left[1-\frac{10\sqrt{3}}{27}\left(\frac{2\pi}{3}\frac{GM}{xc^2}\right)^3\right] = \theta_1\left[1-\frac{10\sqrt{3}}{27}\left(\frac{2\pi}{3}\frac{GM}{xc^2}\right)^3\right] \tag{A13}$$
$$\Rightarrow \langle\theta\rangle(\tau_2) \simeq \frac{2T'}{3}\left[1-\frac{5}{36}\left(\frac{4\pi}{3}\frac{GM}{xc^2}\right)^2\right] = \theta_2\left[1-\frac{5}{36}\left(\frac{4\pi}{3}\frac{GM}{xc^2}\right)^2\right].$$

Again we can easily see how, with the passage of time and so with respect to the τ_m values, the mean value $\langle\theta\rangle(\tau_m)$ increases its delay in ticking by a factor which depends on the intensity of the gravitational potential.

Appendix B: Conditional Probabilieties and Main Values of the Observables for Continuous Clocks

We replicate here (more briefly) the discussion given in Appendix A but considering the case of clocks A and B with continuous time values. The formalism we adopt is therefore the one described in Section IV.

We start considering the case of A and B as free clocks and the global state satisfying (1) is given by (44). We assume that the clock in the C subspace reads time $t = \tau_f$ and we work with the state of S: $|\psi(t=\tau_f)\rangle_S = \langle t=\tau_f|\Psi\rangle = |\varphi(t=\tau_f)\rangle_A \otimes |\phi(t=\tau_f)\rangle_B$. We search the conditional probability density $P(\theta_g|\tau_f)$ of obtaining θ_g on B conditioned to having found τ_f on A, and the mean value of $\langle\theta\rangle(\tau_f)$. The probability $P(\theta_g|\tau_f)$ can be calculated as:

$$P(\theta_g|\tau_f) = \frac{1}{T}\frac{\langle\psi(t=\tau_f)|\tau_f\rangle\langle\tau_f|\otimes|\theta_g\rangle\langle\theta_g|\psi(t=\tau_f)\rangle}{\langle\psi(t=\tau_f)|\tau_f\rangle\langle\tau_f|\psi(t=\tau_f)\rangle} \tag{B1}$$

which leads to the result

$$P(\theta_g|\tau_f) = \frac{1}{Td}\sum_{n=0}^{d-1}\sum_{k=0}^{d-1}e^{-i2\pi(g-f)(n-k)}. \tag{B2}$$

This probability density is well defined (indeed we have $\int_0^1 dg\ P(\theta_g|\tau_f) = 1$ for each given τ_f) and it has a maximum for $g = f$ where it take the value $P(\theta_{g=f}|\tau_f) = \frac{d}{T}$. To beter understand the behavior of $P(\theta_g|\tau_f)$ we can consider the simple case of $d = 2$, thus obtaining: $P(\theta_g|\tau_f) = \frac{1}{T}\left[1 + \cos(2\pi(g - f))\right]$, indicating that the probability density $P(\theta_g|\tau_f)$ oscillates around the value $1/T$ reaching a maximum when $g = f$.

Through the probability density (B2), we can calculate the mean value $\langle\theta\rangle(\tau_f)$ obtaining:

$$\langle\theta\rangle(\tau_f) = \int_0^1 dg\ \theta_g P(\theta_g|\tau_f) = \frac{T}{2} + \frac{iT}{2\pi d}\sum_{n\neq k}\frac{e^{i2\pi f(n-k)}}{n-k}. \tag{B3}$$

In the simple case of $d = 2$, equation (B3) becomes:

$$\langle\theta\rangle(\tau_f) = \frac{T}{2}\left[1 - \frac{1}{\pi}\sin(2\pi f)\right] \tag{B4}$$

which shows the mean value $\langle\theta\rangle(\tau_f)$ oscillating around the value $T/2$ as a function of f.

We can now consider the case in which B is placed within the gravitational field, where the time state of the clocks are given by (42) and (51) for A and B respectively. The probability density (B2) becomes:

$$P(\theta_g|\tau_f) = \frac{1}{T'd}\sum_{n=0}^{d-1}\sum_{k=0}^{d-1} e^{-i2\pi(g-f(1-\frac{GM}{xc^2}))(n-k)} \tag{B5}$$

from which we immediately notice that the maximum is obtained for $g = f(1 - \frac{GM}{xc^2})$, where the probability takes the value d/T'. Looking for the probability that clocks A and B click the same time value, we easily find:

$$P(\theta_{g=f}|\tau_f) = \frac{1}{T'd}\sum_{n=0}^{d-1}\sum_{k=0}^{d-1} e^{-i2\pi f\frac{GM}{xc^2}(n-k)}. \tag{B6}$$

From (B6), in the case of $d = 2$, follows $P(\theta_{g=f}|\tau_f) = \frac{1}{T'}\left[1 + \cos(2\pi f\frac{GM}{xc^2})\right]$ which, approximating the cosine function as $\cos(z) \simeq 1 - \frac{z^2}{2}$, becomes: $P(\theta_{g=f}|\tau_f) \simeq \frac{2}{T'}\left[1 - \pi^2 f^2\left(\frac{GM}{xc^2}\right)^2\right]$. This latter equation shows $P(\theta_{g=f}|\tau_f)$ depending on f and on the intensity of the gravitational field. Still working directly with $d = 2$, we have for the main value $\langle\theta\rangle(\tau_f)$:

$$\langle\theta\rangle(\tau_f) = \frac{T'}{2}\left[1 - \frac{1}{\pi}\sin(2\pi f(1 - \frac{GM}{xc^2}))\right] \tag{B7}$$

which differs from (B4) because of the presence of the term $(1 - \frac{GM}{xc^2})$ multiplying f within the sine function. The function $\langle\theta\rangle(\tau_f)$ is therefore "stretched" by the action of the gravitational field on clock B.

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