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Spectral Gap and Exponential Decay of Correlations

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Abstract: We study the relation between the spectral gap above the ground state and the decay of the correlations in the ground state in quantum spin and fermion systems with short-range interactions on a wide class of lattices. We prove that, if two observables anticommute with each other at large distance, then the nonvanishing spectral gap implies exponential decay of the corresponding correlation. When two observables commute with each other at large distance, the connected correlation function decays exponentially under the gap assumption. If the observables behave as a vector under the U(1) rotation of a global symmetry of the system, we use previous results on the large distance decay of the correlation function to show the stronger statement that the correlation function itself, rather than just the connected correlation function, decays exponentially under the gap assumption on a lattice with a certain self-similarity in (fractal) dimensions $D < 2$. In particular, if the system is translationally invariant in one of the spatial directions, then this self-similarity condition is automatically satisfied. We also treat systems with long-range, power-law decaying interactions.

1. Introduction

In non-relativistic quantum many-body systems, a folk theorem states that a nonvanishing spectral gap above the ground state implies exponentially decaying correlations in the ground state. Perhaps this has been the most popular folk theorem in this field since Haldane [1] predicted a "massive phase" in low dimensional, isotropic quantum systems. Quite recently, this statement was partially proved [2] for quantum lattice systems with a global U(1) symmetry in (fractal) dimensions $D < 2$. More precisely, a bound which decays to zero at large distance was obtained for correlation functions whose observables behave as a vector under the U(1)-rotation. Unfortunately, the bound is weaker than the expected exponential decay. On the other hand, exponential clustering of the correlations was also proved recently [3, 4] for quantum many-body lattice systems under the gap assumption. This is a non-relativistic version of Fredenhagen's theorem [5, 6] of relativistic quantum field theory. Clearly the following natural question arises: can this clustering property be combined with the above bound for the decay of the correlations to yield the tighter, exponentially decaying bound for the correlation functions themselves, rather than just for the connected correlation functions? We emphasize that these are different statements; given clustering, the decay of the correlation functions requires also that certain matrix elements vanish in the ground state sector.

In this paper, we address this problem and reexamine the above folk theorem by relying on the exponential clustering of the correlations. Our first step is to provide a rigorous proof of the exponential clustering. We extend the previous results in this case to treat long-range interactions including both power-law and exponentially decaying interactions. In the former case, all the upper bounds for the correlations become power-law bounds.

We then prove that ground state correlation functions of observables which transform as vectors under a U(1) symmetry decay exponentially or with a power law, depending on the form of the interaction, given an additional assumption on a certain self-similarity. In particular, if the system is translationally invariant in one of the spatial directions, this self-similarity condition is automatically satisfied. Therefore the corresponding correlation functions decay exponentially for translationally invariant systems on one-dimensional regular lattices. As a byproduct, we also prove that, if two observables anticommute with each other at large distance, then the corresponding correlation in the ground state decays exponentially under the gap assumption for a wide class of lattice fermion systems with exponentially decaying interactions in any dimensions. In this case, we do not need any other assumption except for those on the interactions and the spectral gap.

This paper is organized as follows: In the next section, we give the precise definitions of the models, and describe our main results. In Sect. 3, we prove the clustering of generic correlation functions under the gap assumption, and obtain the upper decaying bound for the fermionic correlations. The decay of the bosonic correlations are treated in Sect. 4. Appendix A is devoted to the proof of the Lieb-Robinson bound for the group velocity of the information propagation in the models with a long-range interaction decaying by power law.

2. Models and Main Results

We consider quantum systems on generic lattices [7]. Let Λ_s be a set of the sites, $x, y, z, w, \ldots$, and Λ_b a set of the bonds, i.e., pairs of sites, $\{x, y\}, \{z, w\}, \ldots$. We call the pair, $\Lambda := (\Lambda_s, \Lambda_b)$, the lattice. If a sequence of sites, $x_0, x_1, x_2, \ldots, x_n$, satisfies $\{x_{j-1}, x_j\} \in \Lambda_b$ for $j = 1, 2, \ldots, n$, then we say that the path, $\{x_0, x_1, x_2, \ldots, x_n\}$, has length n and connects x_0 to x_n. We denote by $\mathrm{dist}(x, y)$ the graph-theoretic distance which is defined to be the shortest path length that one needs to connect x to y. We denote by $|X|$ the cardinality of the finite set X. The Hamiltonian H_Λ is defined on the tensor product $\bigotimes_{x\in\Lambda_s} \mathcal{H}_x$ of a finite dimensional Hilbert space $\mathcal{H}_x$ at each site x. We assume $\sup_{\Lambda_s} \sup_x \dim \mathcal{H}_x \le N < \infty$. For a lattice fermion system, we consider the Fock space.

Consider the Hamiltonian of the form,

$$H_\Lambda = \sum_{X\subset\Lambda_s} h_X, \tag{2.1}$$

where h_X is the local Hamiltonian of the compact support X. We consider both power-law and exponentially decaying interactions h_X.

For the power-law decaying interactions h_X, we require the following conditions:

Assumption 2.1. *The interaction h_X satisfies*

$$\sum_{X \ni x,y} \|h_X\| \le \frac{\lambda_0}{[1+\mathrm{dist}(x,y)]^\eta} \tag{2.2}$$

with positive constants, λ_0 and η, and the lattice Λ equipped with the metric satisfies

$$\sum_{z \in \Lambda_s} \frac{1}{[1+\mathrm{dist}(x,z)]^\eta} \times \frac{1}{[1+\mathrm{dist}(z,y)]^\eta} \le \frac{p_0}{[1+\mathrm{dist}(x,y)]^\eta} \tag{2.3}$$

with a positive constant p_0.

Remark. If

$$\sup_{\Lambda_s} \sup_x \sum_{y \in \Lambda_s} \frac{1}{[1+\mathrm{dist}(x,y)]^\eta} < \infty, \tag{2.4}$$

then the inequality (2.3) holds as follows:

$$\begin{aligned}
&\sum_{z \in \Lambda_s} \frac{1}{[1+\mathrm{dist}(x,z)]^\eta} \times \frac{1}{[1+\mathrm{dist}(z,y)]^\eta} \\
&= \frac{1}{[1+\mathrm{dist}(x,y)]^\eta} \sum_{z \in \Lambda_s} \frac{[1+\mathrm{dist}(x,y)]^\eta}{[1+\mathrm{dist}(x,z)]^\eta [1+\mathrm{dist}(z,y)]^\eta} \\
&\le \frac{1}{[1+\mathrm{dist}(x,y)]^\eta} \sum_{z \in \Lambda_s} 2^\eta \frac{[1+\mathrm{dist}(x,z)]^\eta + [1+\mathrm{dist}(z,y)]^\eta}{[1+\mathrm{dist}(x,z)]^\eta [1+\mathrm{dist}(z,y)]^\eta} \\
&\le \frac{1}{[1+\mathrm{dist}(x,y)]^\eta} \sum_{z \in \Lambda_s} 2^\eta \left\{ \frac{1}{[1+\mathrm{dist}(x,z)]^\eta} + \frac{1}{[1+\mathrm{dist}(z,y)]^\eta} \right\},
\end{aligned} \tag{2.5}$$

where we have used the inequality, $[1 + \mathrm{dist}(x, y)]^\eta \le 2^\eta([1 + \mathrm{dist}(x, z)]^\eta + [1 + \mathrm{dist}(z, y)]^\eta)$. From the assumption (2.2) and the condition (2.4), one has

$$\sup_x \sum_{X \ni x} \|h_X\| |X| \le s_0 < \infty, \tag{2.6}$$

where s_0 is a positive constant which is independent of the volume of $|\Lambda_s|$.

Instead of these conditions, we can also require:

Assumption 2.2. *The interaction h_X satisfies*

$$\sup_x \sum_{X \ni x} \|h_X\| |X| [1+\mathrm{diam}(X)]^\eta \le s_1 < \infty, \tag{2.7}$$

where η is a positive constant, $\mathrm{diam}(X)$ *is the diameter of the set X, i.e.,* $\mathrm{diam}(X) = \max\{\mathrm{dist}(x,y) | \, x, y \in X\}$, *and s_1 is a positive constant which is independent of the volume of $|\Lambda_s|$.*

For exponentially decaying interactions h_X, we require one of the following two assumptions:

Assumption 2.3. *There exists a positive η satisfying the condition (2.4). The interaction h_X satisfies*

$$\sum_{X \ni x,y} \|h_X\| \leq \lambda_0 \exp[-(\mu + \varepsilon)\, \mathrm{dist}(x, y)] \tag{2.8}$$

with some positive constants, λ_0, μ and ε.

Remark. From the conditions, we have

$$\exp[-(\mu + \varepsilon)\, \mathrm{dist}(x, y)] \leq \frac{\lambda_0' \exp[-\mu\, \mathrm{dist}(x, y)]}{[1 + \mathrm{dist}(x, y)]^\eta} \tag{2.9}$$

with a positive constant λ_0', and

$$\sum_{z \in \Lambda_s} \frac{\exp[-\mu\, \mathrm{dist}(x, z)]}{[1 + \mathrm{dist}(x, z)]^\eta} \times \frac{\exp[-\mu\, \mathrm{dist}(z, y)]}{[1 + \mathrm{dist}(z, y)]^\eta} \leq \frac{p_0 \exp[-\mu\, \mathrm{dist}(x, y)]}{[1 + \mathrm{dist}(x, y)]^\eta} \tag{2.10}$$

with a positive constant p_0 in the same way as in the preceding remark.

Assumption 2.4. *The interaction h_X satisfies*

$$\sup_x \sum_{X \ni x} \|h_X\| |X| \exp[\mu\, \mathrm{diam}(X)] \leq s_1 < \infty, \tag{2.11}$$

where μ is a positive constant, and s_1 is a positive constant which is independent of the volume of $|\Lambda_s|$.

Remark. This assumption is milder than that in [6] by the absence of the factor $N^{2|X|}$ in the summand.

Further we assume the existence of a "uniform gap" above the ground state sector of the Hamiltonian H_Λ. The precise definition of the "uniform gap" is:

Definition 2.5 (Uniform gap). *We say that there is a uniform gap above the ground state sector if the spectrum $\sigma(H_\Lambda)$ of the Hamiltonian H_Λ satisfies the following conditions: The ground state of the Hamiltonian H_Λ is q-fold (quasi)degenerate in the sense that there are q eigenvalues, $E_{0,1}, \ldots, E_{0,q}$, in the ground state sector at the bottom of the spectrum of H_Λ such that*

$$\Delta\mathcal{E} := \max_{\mu,\mu'} \{|E_{0,\mu} - E_{0,\mu'}|\} \to 0 \quad \text{as} \quad |\Lambda_s| \to \infty. \tag{2.12}$$

Further the distance between the spectrum, $\{E_{0,1}, \ldots, E_{0,q}\}$, of the ground state and the rest of the spectrum is larger than a positive constant ΔE which is independent of the volume $|\Lambda_s|$. Namely there is a spectral gap ΔE above the ground state sector.

Let A_X, B_Y be observables with the support $X, Y \subset \Lambda_s$, respectively. We say that the pair of two observables, A_X and B_Y, is fermionic if they satisfy the anicommutation relation, $\{A_X, B_Y\} = 0$ for $X \cap Y = \emptyset$. If they satisfy the commutation relation, then we call the pair bosonic.

Define the ground-state expectation as

$$\langle \cdots \rangle_{0,\Lambda} := \frac{1}{q} \mathrm{Tr}\, (\cdots) P_{0,\Lambda}, \tag{2.13}$$

where $P_{0,\Lambda}$ is the projection onto the ground state sector. For the infinite volume,

$$\langle \cdots \rangle_0 := \text{weak}^*\text{-} \lim_{|\Lambda_s| \uparrow \infty} \langle \cdots \rangle_{0,\Lambda}\,, \tag{2.14}$$

where we take a suitable subsequence of finite lattices Λ going to the infinite volume so that the expectation converges to a linear functional for a set of quasilocal observables. Although the ground-state expectation thus constructed depends on the subsequence of the lattices and on the observables, our results below hold for any ground-state expectation thus constructed. Further, we denote by

$$\omega(\cdots) := \text{weak}^*\text{-} \lim_{|\Lambda_s| \uparrow \infty} \langle \Phi_\Lambda, (\cdots) \Phi_\Lambda \rangle \tag{2.15}$$

the ground-state expectation in the infinite volume for a normalized vector Φ_Λ in the sector of the ground state for finite lattice Λ.

Theorem 2.6 (Clustering of fermionic correlations). *Let A_X, B_Y be fermionic observables with a compact support. Assume that there exists a uniform spectral gap $\Delta E > 0$ above the ground state sector in the spectrum of the Hamiltonian H_Λ in the sense of Definition 2.5. Let ω be a ground-state expectation (2.15) in the infinite volume limit. Then the following bound is valid:*

$$\left| \omega(A_X B_Y) - \frac{1}{2} [\omega(A_X P_0 B_Y) - \omega(B_Y P_0 A_X)] \right|$$
$$\leq \text{Const.} \times \begin{cases} [1 + \mathrm{dist}(X, Y)]^{-\tilde{\eta}}, & for\ power - law\ decaying\ h_X; \\ \exp[-\tilde{\mu}\, \mathrm{dist}(X, Y)], & for\ exponentially\ decaying\ h_X, \end{cases} \tag{2.16}$$

where P_0 is the projection onto the sector of the infinite-volume ground state,[1] *and*

$$\tilde{\eta} = \frac{\eta}{1 + 2v_\eta/\Delta E} \quad and \quad \tilde{\mu} = \frac{\mu}{1 + 2v_\mu/\Delta E}. \tag{2.17}$$

Here v_η and v_μ are, respectively, an increasing function of η and μ, and give an upper bound of the group velocity of the information propagation.

Remark. Clearly there exists a maximum $\mu_{\max}$ such that the bound (2.8) holds for any $\mu \leq \mu_{\max}$. Combining this observation with (2.17), there exists a maximum $\tilde{\mu} = \max_{\mu \leq \mu_{\max}} \{\mu/(1 + 2v_\mu/\Delta E)\}$ which gives the optimal decay bound. When the interaction h_X is of finite range, one can take any large μ. But the upper bound v_μ of the group velocity exponentially increases as μ increases because v_μ depends on λ_0 of (2.8). In consequence, a finite $\tilde{\mu}$ gives the optimal bound.

[1] $\omega(\cdots P_0 \cdots)$ is also defined as a bilinear functional for a set of quasilocal observables in the weak* limit.

Formally applying the identity, $\langle A_X P_0 B_Y\rangle_0 = \langle B_Y P_0 A_X\rangle_0$, for the bound (2.16), we have the following decay bound for the correlation:[2]

Corollary 2.7. *Let A_X, B_Y be fermionic observables with a compact support. Assume that there exists a uniform spectral gap $\Delta E > 0$ above the ground state sector in the spectrum of the Hamiltonian H_Λ in the sense of Definition 2.5. Then the following bound is valid:*

$$|\langle A_X B_Y\rangle_0| \le \text{Const.} \times \begin{cases} [1+\text{dist}(X,Y)]^{-\tilde{\eta}}, & for\ power-law\ decaying\ h_X; \\ \exp[-\tilde{\mu}\,\text{dist}(X,Y)], & for\ exponentially\ decaying\ h_X, \end{cases} \tag{2.18}$$

in the infinite volume limit, where $\tilde{\eta}$, $\tilde{\mu}$ are as defined above.

Theorem 2.8 (Clustering of bosonic correlations). *Let A_X, B_Y be bosonic observables with a compact support. Assume that there exists a uniform spectral gap $\Delta E > 0$ above the ground state sector in the spectrum of the Hamiltonian H_Λ in the sense of Definition 2.5. Let ω be a ground-state expectation (2.15) in the infinite volume limit. Then the following bound is valid:*

$$\begin{aligned} &\left|\omega(A_X B_Y) - \frac{1}{2}\left[\omega(A_X P_0 B_Y) + \omega(B_Y P_0 A_X)\right]\right| \\ &\quad \le \text{Const.} \times \begin{cases} [1+\text{dist}(X,Y)]^{-\tilde{\eta}}, & for\ power-law\ decaying\ h_X; \\ \exp[-\tilde{\mu}\,\text{dist}(X,Y)], & for\ exponentially\ decaying\ h_X, \end{cases} \end{aligned} \tag{2.19}$$

where $\tilde{\eta}$, $\tilde{\mu}$ are as defined above.

Remark. Theorem 2.8 is a clustering bound for the connected correlation functions. We now make some additional definitions that will enable us, in certain cases, to prove the decay of $[\omega(A_X P_0 B_Y) + \omega(B_Y P_0 A_X)]/2$ so that Theorem 2.8 can be replaced with a stronger bound below, Theorem 2.10.

Definition 2.9 (Self-similarity). *Write $m = q^2$ with the degeneracy q of the ground state sector. We say that the system has self-similarity if the following conditions are satisfied: For any observable A of compact support and any given large $L > 0$, there exist transformations, $R_1, R_2, \ldots, R_m$, and observables, $B^{(1)}, B^{(2)}, \ldots, B^{(m)}$, such that the Hamiltonian H_Λ is invariant under the transformations, i.e., $R_j(H_\Lambda) = H_\Lambda$ for any lattice Λ with sufficiently large $|\Lambda_s|$, and that the observables satisfy the following conditions:*

$$B^{(j)} = R_j(A) \quad and \quad \left(B^{(j)}\right)^\dagger = R_j(A^\dagger) \quad for \quad j = 1, 2, \ldots, m, \tag{2.20}$$

$$\text{dist}(\text{supp}\, A, \text{supp}\, B^{(j)}) \ge L \quad for \quad j = 1, 2, \ldots, m, \tag{2.21}$$

and

$$\text{dist}(\text{supp}\, B^{(j)}, \text{supp}\, B^{(k)}) \ge L \quad for \quad j \ne k. \tag{2.22}$$

In Sect. 4, we will discuss other conditions similar to this self-similarity condition.

[2] See Sect. 3 for details.

Theorem 2.10. *Assume that the degeneracy q of the ground state sector of the Hamiltonian H_Λ is finite in the infinite volume limit, and that there exists a uniform spectral gap $\Delta E > 0$ above the ground state sector in the spectrum of the Hamiltonian H_Λ in the sense of Definition 2.5. Further assume that the system has self-similarity in the sense of Definition 2.9, and that there exists a subset $\mathcal{A}_b^s$ of bosonic observables with a compact support such that $R_j(\mathcal{A}_b^s) \subset \mathcal{A}_b^s = (\mathcal{A}_b^s)^\dagger$ for $j = 1, 2, \ldots, m$, and that $\langle A'_X B'_Y \rangle_0 \to 0$ as $\mathrm{dist}(X, Y) \to \infty$ for any pair of bosonic observables, $A'_X, B'_Y \in \mathcal{A}_b^s$. Let ω be a ground-state expectation (2.15) in the infinite volume limit, and let A_X, B_Y be a pair of bosonic observables satisfying $A_X \in \mathcal{A}_b^s$. Then the following bound is valid:*

$$|\omega(A_X B_Y)| \le \text{Const.} \times \begin{cases} [1 + \mathrm{dist}(X, Y)]^{-\tilde{\eta}}, & for\ power-law\ decaying\ h_X; \\ \exp[-\tilde{\mu}\ \mathrm{dist}(X, Y)], & for\ exponentially\ decaying\ h_X, \end{cases} \tag{2.23}$$

where $\tilde{\eta}$, $\tilde{\mu}$ are as defined above.

Remark. 1. If the finite system is translationally invariant in one of the spatial directions with a periodic boundary condition, then the self-similarity condition of Definition 2.9 is automatically satisfied by taking the translation as the transformation R_j. Thus we do not need an additional assumption for such systems.
2. Theorem 2.10 can be extended to a system having infinite degeneracy of the ground state sector in the infinite volume limit if the degeneracy for finite volume is sufficiently small compared to the volume of the system. See Theorem 4.1 in Sect. 4 for details.

In order to apply this theorem, we need to be able to show that $\langle A_X B_Y \rangle_0 \to 0$ as $\mathrm{dist}(X, Y) \to \infty$ in the infinite volume for any pair of bosonic observables, $A_X, B_Y \in \mathcal{A}_b^s$. However, this was proven [2] for quantum spin or fermion systems with a global U(1) symmetry on a class of lattices with (fractal) dimension $D < 2$ as defined in (2.26) below, so long as the observables behave as a vector under the U(1) rotation. A reader might think that the assumption of the existence of a nonvanishing spectral gap above the ground state sector automatically leads to the vanishing of the matrix elements in the ground state sector for such U(1)-vector observables. This is not necessarily true for a degenerate ground state.[3] For example, consider two spins, $\mathbf{S}_j = (S_j^{(1)}, S_j^{(2)}, S_j^{(3)})$, $j = 1, 2$, with spin 1/2, and its Hamiltonian,

$$H_{\text{toy}} = -\Delta E S_1^{(3)}, \quad \text{with } \Delta E > 0. \tag{2.24}$$

Clearly the ground states are given by the states with spin up for the first spin and with arbitrary configurations for the second spin, and there is a spectral gap ΔE above the degenerate ground states. The reason for considering the first spin is that it produces the two extra states with energy ΔE above the ground state and thus sets the magnitude of the gap; the first spin could be replaced by any other system with a Hamiltonian with a unique ground state and a gap ΔE. The system has U(1) symmetry and shows nonvanishing matrix elements in the ground state sector for U(1)-vector observables. Physically, the second spin represents what physicists would call a "local moment". We will discuss this type of toy model again in Remarks 4 and 5 at the end of Sect. 4.

[3] See [8] for a unique ground state in an infinite-volume sector.

Now let us define the dimension of lattices which we consider. The "sphere", $S_r(x)$, centered at $x \in \Lambda_s$ with the radius r is defined as

$$S_r(x) := \{y \in \Lambda_s | \mathrm{dist}(y, x) = r\}. \tag{2.25}$$

Assume that there exists a "(fractal) dimension" $D \geq 1$ of the lattice Λ such that the number $|S_r(x)|$ of the sites in the sphere satisfies

$$\sup_{x \in \Lambda_s} |S_r(x)| \leq C_0 r^{D-1} \tag{2.26}$$

with some positive constant C_0. This class of the lattices is the same as in [9].

Consider spin or fermion systems with a global U(1) symmetry on the lattice Λ with (fractal) dimension $1 \leq D < 2$, and require the existence of a uniform gap above the ground state sector of the Hamiltonian H_Λ in the sense of Definition 2.5. Although the method of [2] can be applied to a wide class of such systems, we consider only two important examples, the Heisenberg and the Hubbard models. We take the set $\mathcal{A}_b^s$ to be the bosonic observables which behave as a vector under the U(1) rotation. In the rest of this section we use the results of [2] to show as in (2.30,2.32) that the correlation function for this class of observables in these models does decay to zero as $\mathrm{dist}(X, Y) \to \infty$. The bounds (2.30,2.32) provide only a slow bound on the decay. However, this slow bound on the decay suffices, in conjunction with the self-similarity condition of Definition 2.9 to apply Theorem 2.10. *Thus, under the self-similarity assumption as well as the gap assumption, all the upper bounds below (2.30,2.32) are replaced with exponentially decaying bounds by Theorem 2.10. In particular, a system with a translational invariance automatically satisfies the self-similarity condition as mentioned above. Therefore the corresponding correlations show exponential decay for translationally invariant systems on one-dimensional regular lattices.*

XXZ Heisenberg model. The Hamiltonian H_Λ is given by

$$H_\Lambda = H_\Lambda^{XY} + V_\Lambda(\{S_x^{(3)}\}) \tag{2.27}$$

with

$$H_\Lambda^{XY} = 2 \sum_{\{x,y\} \in \Lambda_b} J_{x,y}^{\mathrm{XY}} \left[S_x^{(1)} S_y^{(1)} + S_x^{(2)} S_y^{(2)} \right], \tag{2.28}$$

where $\mathbf{S}_x = (S_x^{(1)}, S_x^{(2)}, S_x^{(3)})$ is the spin operator at the site $x \in \Lambda_s$ with the spin $S = 1/2, 1, 3/2, \ldots$, and $J_{x,y}^{\mathrm{XY}}$ are real coupling constants; $V_\Lambda(\{S_x^{(3)}\})$ is a real function of the z-components, $\{S_x^{(3)}\}_{x \in \Lambda_s}$, of the spins. For simplicity, we take

$$V_\Lambda(\{S_x^{(3)}\}) = \sum_{\{x,y\} \in \Lambda_b} J_{x,y}^{\mathrm{Z}} S_x^{(3)} S_y^{(3)} \tag{2.29}$$

with real coupling constants $J_{x,y}^{\mathrm{Z}}$. Assume that there are positive constants, $J_{\max}^{\mathrm{XY}}$ and $J_{\max}^{\mathrm{Z}}$, which satisfy $|J_{x,y}^{\mathrm{XY}}| \leq J_{\max}^{\mathrm{XY}}$ and $|J_{x,y}^{\mathrm{Z}}| \leq J_{\max}^{\mathrm{Z}}$ for any bond $\{x, y\} \in \Lambda_b$.

Consider the transverse spin-spin correlation, $\left\langle S_x^+ S_y^- \right\rangle_0$, where $S_x^\pm := S_x^{(1)} \pm i S_x^{(2)}$. The following decay bound was proven [2]:

Theorem 2.11. *Assume that the fractal dimension D of (2.26) satisfies $1 \le D < 2$, and that there exists a uniform spectral gap $\Delta E > 0$ above the ground state sector in the spectrum of the Hamiltonian H_Λ of (2.27) in the sense of Definition 2.5. Then there exists a positive constant γ such that the transverse spin-spin correlation satisfies the bound,*

$$\left|\left\langle S_x^+ S_y^- \right\rangle_0\right| \le \text{Const.} \exp\left[-\gamma \{\mathrm{dist}(x,y)\}^{1-D/2}\right], \tag{2.30}$$

in the thermodynamic limit $|\Lambda_s| \to \infty$.

Remark. 1. The result can be extended to more complicated correlations such as the multispin correlation, $\left\langle S_{x_1}^+ \cdots S_{x_j}^+ S_{y_1}^- \cdots S_{y_j}^- \right\rangle_0$. If the system satisfies the self-similarity condition of Definition 2.9, then the upper bound, (2.30), can be replaced with a stronger exponentially decaying one by Theorem 2.10.

2. As is well known, the application of the Bethe-Ansatz method to the spin-1/2 antiferromagnetic XXZ chain shows a nonvanishing spectral gap above the two-fold degenerate ground state. (See, e.g., [10].) Since Haldane [1] predicted a "massive" phase in low dimensional, isotropic quantum systems, many examples have been found to have a spectral gap above the ground state sector.[4] For example, spin-1 open chain exhibits a spectral gap above the four-fold degenerate ground state [12]. Once these statements on the spectrum are justified, the exponential decay of the correlations follows from the present theorems.

Hubbard model [13, 14]: The Hamiltonian on the lattice Λ is given by

$$H_\Lambda = -\sum_{\{x,y\}\in\Lambda_b} \sum_{\alpha=\uparrow,\downarrow} \left(t_{x,y} c_{x,\alpha}^\dagger c_{y,\alpha} + t_{x,y}^* c_{y,\alpha}^\dagger c_{x,\alpha}\right) + V(\{n_{x,\alpha}\}) + \sum_{x\in\Lambda_s} \mathbf{B}_x \cdot \mathbf{S}_x, \tag{2.31}$$

where $c_{x,\alpha}^\dagger, c_{x,\alpha}$ are, respectively, the electron creation and annihilation operators with the z component of the spin $\mu = \uparrow, \downarrow$, $n_{x,\alpha} = c_{x,\alpha}^\dagger c_{x,\alpha}$ is the corresponding number operator, and $\mathbf{S}_x = (S_x^{(1)}, S_x^{(2)}, S_x^{(3)})$ are the spin operator given by $S_x^{(a)} = \sum_{\alpha,\beta=\uparrow,\downarrow} c_{x,\alpha}^\dagger \sigma_{\alpha,\beta}^{(a)} c_{x,\beta}$ with the Pauli spin matrix $(\sigma_{\alpha,\beta}^{(a)})$ for $a = 1,2,3$; $t_{i,j} \in \mathbf{C}$ are the hopping amplitude, $V(\{n_{x,\alpha}\})$ is a real function of the number operators, and $\mathbf{B}_x = (B_x^{(1)}, B_x^{(2)}, B_x^{(3)}) \in \mathbf{R}^3$ are local magnetic fields. Assume that the interaction $V(\{n_{x,\alpha}\})$ is of finite range in the sense of the graph theoretic distance.

Theorem 2.12. *Assume that the fractal dimension D of (2.26) satisfies $1 \le D < 2$, and that there exists a uniform spectral gap $\Delta E > 0$ above the ground state sector in the spectrum of the Hamiltonian H_Λ of (2.31) in the sense of Definition 2.5. Then the following bound is valid:*

$$\left|\left\langle c_{x,\uparrow}^\dagger c_{x,\downarrow}^\dagger c_{y,\uparrow} c_{y,\downarrow} \right\rangle_0\right| \le \text{Const.} \exp\left[-\gamma \{\mathrm{dist}(x,y)\}^{1-D/2}\right] \tag{2.32}$$

with some constant γ in the thermodynamic limit $|\Lambda_s| \to \infty$. If the local magnetic field has the form $\mathbf{B}_x = (0, 0, B_x)$, then we further have

$$\left|\left\langle S_x^+ S_y^- \right\rangle_0\right| \le \text{Const.} \exp\left[-\gamma' \{\mathrm{dist}(x,y)\}^{1-D/2}\right] \tag{2.33}$$

with some constant γ'.

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[4] For exactly solvable models, see, e.g., [11].

The proof is given in [2]. Clearly the Hamiltonian H_Λ of (2.31) commutes with the total number operator $\mathcal{N}_\Lambda = \sum_{x\in\Lambda_s}\sum_{\mu=\uparrow,\downarrow} n_{x,\mu}$ for a finite volume $|\Lambda_s| < \infty$. We denote by $H_{\Lambda,N}$ the restriction of H_Λ onto the eigenspace of $\mathcal{N}_\Lambda$ with the eigenvalue N. Let $P_{0,\Lambda,N}$ be the projection onto the ground state sector of $H_{\Lambda,N}$, and we denote the ground-state expectation by

$$\langle\cdots\rangle_{0,\nu} = \text{weak}^*\text{-}\lim_{|\Lambda_s|\uparrow\infty}\frac{1}{q_N}\mathrm{Tr}\,(\cdots)P_{0,\Lambda,N}, \tag{2.34}$$

where q_N is the degeneracy of the ground state, and ν is the limit of the filling factor $N/|\Lambda_s|$ of the electrons. Since the operators $S_x^\pm$ do not connect the sectors with the different eigenvalues N, the following is also valid [2]:

Theorem 2.13. *Assume that the fractal dimension D of (2.26) satisfies $1 \le D < 2$, and that there exists a uniform spectral gap $\Delta E > 0$ above the ground state sector in the spectrum of the Hamiltonian $H_{\Lambda,N}$ in the sense of Definition 2.5. Then the following bound is valid for the filling factor ν of the electrons:*

$$\left|\left\langle S_x^+ S_y^-\right\rangle_{0,\nu}\right| \le \text{Const.}\exp\left[-\gamma'\{\mathrm{dist}(x,y)\}^{1-D/2}\right] \tag{2.35}$$

with some constant γ' in the infinite volume limit.

Similarly to the above spin models, if the system satisfies the self-similarity condition of Definition 2.9, then these three upper bounds, (2.32), (2.33) and (2.35), can be replaced with a stronger exponentially decaying one by Theorem 2.10.

3. Clustering of Correlations

In order to prove the power-law and the exponential clustering, Theorems 2.6 and 2.8, we follow the method [3]. The key tools of the proof are Lemma 3.1 below and the Lieb-Robinson bound [6, 15] for the group velocity of the information propagation. The sketch of the proof is that the static correlation function can be derived from the time-dependent correlation function by the lemma, and the large-distance behavior of the time-dependent correlation function is estimated by the Lieb-Robinson bound. As a byproduct, we obtain the decay bound (2.18) for fermionic observables.

Consider first the case of the bosonic observables. Let A_X, B_Y be bosonic observables with compact supports $X, Y \subset \Lambda_s$, respectively, and let $A_X(t) = e^{itH_\Lambda}A_X e^{-itH_\Lambda}$, where $t \in \mathbf{R}$ and H_Λ is the Hamiltonian for finite volume. Let Φ be a normalized vector in the ground state sector. The ground state expectation of the commutator is written as

$$\begin{aligned}\langle\Phi, [A_X(t), B_Y]\Phi\rangle &= \left\langle\Phi, A_X(t)(1-P_{0,\Lambda})B_Y\Phi\right\rangle - \left\langle\Phi, B_Y(1-P_{0,\Lambda})A_X(t)\Phi\right\rangle \\ &\quad + \left\langle\Phi, A_X(t)P_{0,\Lambda}B_Y\Phi\right\rangle - \left\langle\Phi, B_Y P_{0,\Lambda}A_X(t)\Phi\right\rangle.\end{aligned} \tag{3.1}$$

In terms of the ground state vectors $\Phi_{0,\nu}$, $\nu = 1, 2, \ldots, q$, with the energy eigenvalues, $E_{0,\nu}$, and the excited state vectors Φ_n with E_n, $n = 1, 2, \ldots$, one has

$$\begin{aligned}&\left\langle\Phi, A_X(t)(1-P_{0,\Lambda})B_Y\Phi\right\rangle \\ &\quad = \sum_{\nu,\nu'}\sum_{n\ne 0} a_\nu^* a_{\nu'}\left\langle\Phi_{0,\nu}, A_X\Phi_n\right\rangle\left\langle\Phi_n, B_Y\Phi_{0,\nu'}\right\rangle e^{-it(E_n-E_{0,\nu})},\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\left\langle \Phi, B_Y(1-P_{0,\Lambda})A_X(t)\Phi\right\rangle\\&\quad=\sum_{\nu,\nu'}\sum_{n\neq 0}a_\nu^* a_{\nu'}\left\langle \Phi_{0,\nu}, B_Y\Phi_n\right\rangle\left\langle \Phi_n, A_X\Phi_{0,\nu'}\right\rangle e^{it(E_n-E_{0,\nu'})},\end{aligned}\tag{3.3}$$

$$\begin{aligned}&\left\langle \Phi, A_X(t)P_{0,\Lambda}B_Y\Phi\right\rangle\\&\quad=\sum_{\nu,\nu'}\sum_{\mu}a_\nu^* a_{\nu'}\left\langle \Phi_{0,\nu}, A_X\Phi_{0,\mu}\right\rangle\left\langle \Phi_{0,\mu}, B_Y\Phi_{0,\nu'}\right\rangle e^{-it(E_{0,\mu}-E_{0,\nu})}\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}&\left\langle \Phi, B_Y P_{0,\Lambda}A_X(t)\Phi\right\rangle\\&\quad=\sum_{\nu,\nu'}\sum_{\mu}a_\nu^* a_{\nu'}\left\langle \Phi_{0,\nu}, B_Y\Phi_{0,\mu}\right\rangle\left\langle \Phi_{0,\mu}, A_X\Phi_{0,\nu'}\right\rangle e^{it(E_{0,\mu}-E_{0,\nu'})},\end{aligned}\tag{3.5}$$

where we have written

$$\Phi=\sum_{\nu=1}^{q}a_\nu\Phi_{0,\nu}.\tag{3.6}$$

In order to get the bound for $\langle\Phi, A_X(t=0)B_Y\Phi\rangle$, we want to extract only the "negative frequency part" (3.2) from the time-dependent correlation functions (3.1). For this purpose, we use the following lemma [3]:

Lemma 3.1. *Let $E\in\mathbf{R}$, and $\alpha>0$. Then*

$$\begin{aligned}\lim_{T\uparrow\infty}\lim_{\epsilon\downarrow 0}\frac{i}{2\pi}\int_{-T}^{T}\frac{e^{-iEt}e^{-\alpha t^2}}{t+i\epsilon}dt&=\frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{-\infty}^{0}d\omega\exp[-(\omega+E)^2/(4\alpha)]\\&=\begin{cases}1+\mathcal{O}(\exp[-\Delta E^2/(4\alpha)]) & for\ E\geq\Delta E;\\ \mathcal{O}(\exp[-\Delta E^2/(4\alpha)]) & for\ E\leq-\Delta E.\end{cases}\end{aligned}\tag{3.7}$$

Proof. Write

$$I(E)=\frac{i}{2\pi}\int_{-T}^{T}\frac{e^{-iEt}e^{-\alpha t^2}}{t+i\epsilon}dt.\tag{3.8}$$

Using the Fourier transformation,

$$e^{-iEt}e^{-\alpha t^2}=\frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{-\infty}^{\infty}\exp[-(\omega+E)^2/(4\alpha)]e^{i\omega t}d\omega,\tag{3.9}$$

we decompose the integral $I(E)$ into three parts as

$$I(E)=I_-(E)+I_0(E)+I_+(E),\tag{3.10}$$

where

$$I_-(E)=\frac{i}{2\pi}\frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{-T}^{T}dt\frac{1}{t+i\epsilon}\int_{-\infty}^{-\Delta\omega}d\omega\exp[-(\omega+E)^2/(4\alpha)]e^{i\omega t},\tag{3.11}$$

$$I_0(E) = \frac{i}{2\pi}\frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{-T}^{T} dt \frac{1}{t+i\epsilon}\int_{-\Delta\omega}^{\Delta\omega} d\omega \exp[-(\omega+E)^2/(4\alpha)]e^{i\omega t}, \tag{3.12}$$

and

$$I_+(E) = \frac{i}{2\pi}\frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{-T}^{T} dt \frac{1}{t+i\epsilon}\int_{\Delta\omega}^{\infty} d\omega \exp[-(\omega+E)^2/(4\alpha)]e^{i\omega t}, \tag{3.13}$$

where we choose $\Delta\omega = bT^{-1/2}$ with some positive constant b.

First let us estimate $I_0(E)$. Note that

$$\frac{1}{t+i\epsilon} = \frac{t}{t^2+\epsilon^2} - \frac{i\epsilon}{t^2+\epsilon^2}. \tag{3.14}$$

Using this identity, one has

$$I_0(E) = \frac{i}{2\pi}\frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{-\Delta\omega}^{\Delta\omega} d\omega \exp[-(\omega+E)^2/(4\alpha)]\int_{-T}^{T} dt \left[\frac{t\sin\omega t}{t^2+\epsilon^2} - \frac{i\epsilon\cos\omega t}{t^2+\epsilon^2}\right], \tag{3.15}$$

where we have interchanged the order of the double integral by relying on $|t| \le T < \infty$. Since the integral about t can be bounded by some constant, one obtains

$$|I_0(E)| \le \text{Const.} \times \alpha^{-1/2}\Delta\omega \le \text{Const.} \times \alpha^{-1/2}T^{-1/2}. \tag{3.16}$$

Therefore the corresponding contribution is vanishing in the limit $T \uparrow \infty$.

Note that

$$\frac{i}{2\pi}\int_{-T}^{T} dt \frac{e^{i\omega t}}{t+i\epsilon} = \begin{cases} \mathcal{O}(\omega^{-1}T^{-1}) & for\ \omega > 0; \\ e^{\epsilon\omega} + \mathcal{O}(\omega^{-1}T^{-1}) & for\ \omega < 0. \end{cases} \tag{3.17}$$

Using this, the function $I_+(E)$ of (3.13) can be evaluated as

$$|I_+(E)| \le \text{Const.} \times T^{-1/2}. \tag{3.18}$$

This is also vanishing in the limit.

Thus it is enough to consider only the integral $I_-(E)$. In the same way as the above, one has

$$I_-(E) = \frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{-\infty}^{-\Delta\omega} d\omega \exp[-(\omega+E)^2/(4\alpha)]e^{\epsilon\omega} + \mathcal{O}(T^{-1/2}). \tag{3.19}$$

Since $e^{\epsilon\omega} \le 1$ for $\omega < 0$, one has

$$\lim_{T\uparrow\infty}\lim_{\epsilon\downarrow 0} I_-(E) = \frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{-\infty}^{0} d\omega \exp[-(\omega+E)^2/(4\alpha)]. \tag{3.20}$$

Note that, for $E \le -\Delta E$,

$$\frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{-\infty}^{0} d\omega \exp[-(\omega+E)^2/(4\alpha)] \le \frac{1}{2}\exp[-\Delta E^2/(4\alpha)], \tag{3.21}$$

and, for $E \geq \Delta E$,

$$\begin{aligned}\frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{-\infty}^{0} d\omega \exp[-(\omega+E)^2/(4\alpha)] &= \frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{-\infty}^{\infty} d\omega \exp[-(\omega+E)^2/(4\alpha)] \\ &\quad -\frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{0}^{\infty} d\omega \exp[-(\omega+E)^2/(4\alpha)] \\ &= 1 + \mathcal{O}(\exp[-\Delta E^2/(4\alpha)]). \end{aligned} \tag{3.22}$$

Clearly these imply (3.7). □

From Lemma 3.1 and the expression (3.1) of the correlation function with (3.2) and (3.3), one has

$$\begin{aligned}&\lim_{T\uparrow\infty}\lim_{\epsilon\downarrow 0}\frac{i}{2\pi}\int_{-T}^{T} dt\,\frac{1}{t+i\epsilon}\,\langle\Phi, [A_X(t), B_Y]\Phi\rangle\, e^{-\alpha t^2} \\ &= \left\langle\Phi, A_X(1-P_{0,\Lambda})B_Y\Phi\right\rangle + \mathcal{O}(\exp[-\Delta E^2/(4\alpha)]) \\ &\quad + \lim_{T\uparrow\infty}\lim_{\epsilon\downarrow 0}\frac{i}{2\pi}\int_{-T}^{T} dt\,\frac{1}{t+i\epsilon}\left[\left\langle\Phi, A_X(t)P_{0,\Lambda}B_Y\Phi\right\rangle - \left\langle\Phi, B_Y P_{0,\Lambda}A_X(t)\Phi\right\rangle\right] e^{-\alpha t^2}\end{aligned} \tag{3.23}$$

for finite volume.

In the following, we treat only the power-law decaying interaction h_X because one can treat the exponentially decaying interactions in the same way. See also refs. [3, 4] in which the exponential clustering of the correlations is proved for finite-range interactions under the gap assumption along the same line as below.

In order to estimate the left-hand side, we recall the Lieb-Robinson estimate (A.1) in Appendix A,

$$\left\|\frac{1}{t}[A_X(t), B_Y]\right\| \leq \text{Const.} \times \frac{1}{(1+r)^\eta}\frac{e^{v|t|}-1}{|t|}, \tag{3.24}$$

for $r > 0$, where we have written $r = \mathrm{dist}(X, Y)$. Using this estimate, the integral can be evaluated as

$$\begin{aligned}&\left|\int_{-T}^{T} dt\,\frac{\langle\Phi, [A_X(t), B_Y]\Phi\rangle}{t+i\epsilon}e^{-\alpha t^2}\right| \\ &\leq \left|\int_{|t|\leq c\ell} dt\,\frac{\langle\Phi, [A_X(t), B_Y]\Phi\rangle}{t+i\epsilon}e^{-\alpha t^2}\right| + \left|\int_{|t|>c\ell} dt\,\frac{\langle\Phi, [A_X(t), B_Y]\Phi\rangle}{t+i\epsilon}e^{-\alpha t^2}\right| \\ &\leq \text{Const.} \times \frac{1}{(1+r)^{\eta-cv}} + \frac{\text{Const.}}{\sqrt{\alpha}\ell}\exp[-\alpha c^2\ell^2],\end{aligned} \tag{3.25}$$

where c is a positive, small parameter, and $\ell = \log(1+r)$, and we have used

$$\int_{|t|\leq c\ell}\frac{e^{v|t|}-1}{|t|}dt \leq 2e^{cv\ell}. \tag{3.26}$$

In order to estimate the integral in the right-hand side of (3.23), we consider the matrix element $\langle\Phi_{0,\nu}, A_X(t)P_{0,\Lambda}B_Y\Phi_{0,\nu'}\rangle$ because the other matrix elements in the ground state can be treated in the same way. Using Lemma 3.1, one has

$$\begin{aligned}&\lim_{T\uparrow\infty}\lim_{\epsilon\downarrow 0}\frac{i}{2\pi}\int_{-T}^{T}dt\frac{1}{t+i\epsilon}\langle\Phi_{0,\nu}, A_X(t)P_{0,\Lambda}B_Y\Phi_{0,\nu'}\rangle e^{-\alpha t^2}\\&=\sum_{\mu=1}^{q}\langle\Phi_{0,\nu}, A_X\Phi_{0,\mu}\rangle\langle\Phi_{0,\mu}, B_Y\Phi_{0,\nu'}\rangle\frac{1}{2\pi}\sqrt{\frac{\pi}{\alpha}}\int_{-\infty}^{0}d\omega\exp[-(\omega+\Delta\mathcal{E}_{\mu,\nu})^2/(4\alpha)],\end{aligned}\tag{3.27}$$

where $\Delta\mathcal{E}_{\mu,\nu}=E_{0,\mu}-E_{0,\nu}$. Using the assumption (2.12) and the dominated convergence theorem, we have that, for any given $\varepsilon>0$, there exists a sufficiently large volume of the lattice Λ_s such that

$$\left|\lim_{T\uparrow\infty}\lim_{\epsilon\downarrow 0}\frac{i}{2\pi}\int_{-T}^{T}dt\frac{e^{-\alpha t^2}}{t+i\epsilon}\langle\Phi_{0,\nu}, A_X(t)P_{0,\Lambda}B_Y\Phi_{0,\nu'}\rangle-\frac{1}{2}\langle\Phi_{0,\nu}, A_XP_{0,\Lambda}B_Y\Phi_{0,\nu'}\rangle\right|<\varepsilon.\tag{3.28}$$

Combining this observation, (3.23) and (3.25), and choosing $\alpha=\Delta E/(2c\ell)$, one obtains

$$\begin{aligned}&\left|\omega(A_XB_Y)-\frac{1}{2}[\omega(A_XP_0B_Y)+\omega(B_YP_0A_X)]\right|\\&\le\text{Const.}\times\frac{1}{(1+r)^{\eta-cv}}+\text{Const.}\times\exp\left[-\frac{c\Delta E}{2}\ell\right]\end{aligned}\tag{3.29}$$

in the infinite volume limit, where the ground-state expectation ω is given by (2.15). Choosing $c=\eta/(v+\Delta E/2)$, we have

$$\left|\omega(A_XB_Y)-\frac{1}{2}[\omega(A_XP_0B_Y)+\omega(B_YP_0A_X)]\right|\le\frac{\text{Const.}}{[1+\text{dist}(X,Y)]^{\tilde\eta}},\tag{3.30}$$

with $\tilde\eta=\eta/(1+2v/\Delta E)$. In the same way, we have

$$\begin{aligned}&\left|\omega(A_XB_Y)-\frac{1}{2}[\omega(A_XP_0B_Y)+\omega(B_YP_0A_X)]\right|\\&\le\text{Const.}\times\exp[-\tilde\mu\,\text{dist}(X,Y)]\end{aligned}\tag{3.31}$$

for the exponentially decaying interaction h_X, where $\tilde\mu=\mu/(1+2v/\Delta E)$. This proves Theorem 2.8. The corresponding bound for finite-range interactions was already obtained in [4]. Using the definition (2.13) of the expectation $\langle\cdots\rangle_{0,\Lambda}$ and the identity,

$$\langle A_XP_0B_Y\rangle_{0,\Lambda}=\langle B_YP_0A_X\rangle_{0,\Lambda},\tag{3.32}$$

for the integral in the right-hand side of (3.23), we obtain

$$\begin{aligned}&\left|\langle A_XB_Y\rangle_{0,\Lambda}-\langle A_XP_{0,\Lambda}B_Y\rangle_{0,\Lambda}\right|\\&\le\text{Const.}\times\begin{cases}[1+\text{dist}(X,Y)]^{-\tilde\eta}, & for\ power-law\ decaying\ h_X;\\ \exp[-\tilde\mu\,\text{dist}(X,Y)], & for\ exponentially\ decaying\ h_X\end{cases}\end{aligned}\tag{3.33}$$

for any finite lattice $\Lambda_s\supset X, Y$ in the same way as in the above.

Next consider the case that the pair, A_X, B_Y, is fermionic. Note that

$$\begin{aligned}
&\left\langle \Phi_{0,\nu}, \{A_X(t), B_Y\}\Phi_{0,\nu'}\right\rangle \\
&\quad = \left\langle \Phi_{0,\nu}, A_X(t)(1-P_{0,\Lambda})B_Y\Phi_{0,\nu'}\right\rangle + \left\langle \Phi_{0,\nu}, B_Y(1-P_{0,\Lambda})A_X(t)\Phi_{0,\nu'}\right\rangle \\
&\qquad + \left\langle \Phi_{0,\nu}, A_X(t)P_{0,\Lambda}B_Y\Phi_{0,\nu'}\right\rangle + \left\langle \Phi_{0,\nu}, B_Y P_{0,\Lambda}A_X(t)\Phi_{0,\nu'}\right\rangle. \qquad (3.34)
\end{aligned}$$

Since the difference between bosonic and fermionic observables is in the signs of some terms, one has

$$\begin{aligned}
&\left|\omega(A_X B_Y) - \frac{1}{2}\left[\omega(A_X P_0 B_Y) - \omega(B_Y P_0 A_X)\right]\right| \\
&\quad \le \text{Const.} \times \begin{cases} [1+\text{dist}(X,Y)]^{-\tilde{\eta}}, & for\ power-law\ decaying\ h_X; \\ \exp[-\tilde{\mu}\,\text{dist}(X,Y)], & for\ exponentially\ decaying\ h_X. \end{cases} \qquad (3.35)
\end{aligned}$$

In particular, thanks to the identity (3.32), we obtain

$$|\langle A_X B_Y\rangle_0| \le \text{Const.} \times \begin{cases} [1+\text{dist}(X,Y)]^{-\tilde{\eta}}, & for\ power-law\ decaying\ h_X; \\ \exp[-\tilde{\mu}\,\text{dist}(X,Y)], & for\ exponentially\ decaying\ h_X. \end{cases} \qquad (3.36)$$

This is nothing but the desired bound. We stress that, for infinite degeneracy of the infinite-volume ground state, this upper bound is also justified in the same argument with the dominated convergence theorem.

4. Vanishing of the Matrix Elements in the Ground State

The aim of this section is to prove the bound (2.23) for the correlation and discuss an extension of Theorem 2.10 to a system having infinite degeneracy of infinite-volume ground state. The latter result is summarized as Theorem 4.1 below. We will give only the proof of Theorem 4.1 because Theorem 2.10 is proved in the same way. By the clustering bounds (3.30) and (3.31), it is sufficient to show that all the matrix elements, $\left\langle \Phi_{0,\nu'} A_X \Phi_{0,\nu}\right\rangle$, in the sector of the ground state are vanishing. The key idea of the proof is to estimate the absolute values of the matrix elements by using the self-similarity condition and the decay bound (4.2) below of the correlations at a sufficiently large distance.

We denote by q_Λ the degeneracy of the sector of the ground state for the finite lattice Λ, and we allow $q_\Lambda \to \infty$ as $|\Lambda_s| \uparrow \infty$. We write $m = q_\Lambda^2$ for short. To begin with, we write the bound (3.33) as

$$\left|\langle A_X B_Y\rangle_{0,\Lambda} - \left\langle A_X P_{0,\Lambda} B_Y\right\rangle_{0,\Lambda}\right| \le G_0(\text{dist}(X,Y)), \qquad (4.1)$$

where we have written the upper bound of the right-hand side by the function G_0 of the distance. We assume that the following bound holds:

$$\left|\langle A_X B_Y\rangle_{0,\Lambda}\right| \le G_1(\text{dist}(X,Y)) \qquad (4.2)$$

with an upper bound G_1 which is vanishing at the infinite distance. Further we define $\tilde{G}_\Lambda$ as

$$\tilde{G}_\Lambda(A_X, B_Y) := \max\{G_0(\text{dist}(X,Y)), G_1(\text{dist}(X,Y))\}. \qquad (4.3)$$

Theorem 4.1. *Let ω be a ground-state expectation (2.15) in the infinite volume limit, and let A_X, B_Y be a pair of bosonic observables with compact supports X, Y. Assume that there exists a uniform spectral gap $\Delta E > 0$ above the ground state sector in the spectrum of the Hamiltonian H_Λ in the sense of Definition 2.5. Suppose that, for any given $\epsilon > 0$, there exists $M_0 > 0$ such that, for any large lattice Λ satisfying $|\Lambda_s| \ge M_0$, there exists a set of observables, $B^{(j)}$, $j = 1, 2, \ldots, m$, and a set of transformations, R_j, $j = 1, 2, \ldots, m$, satisfying the following conditions: Any pair of the observables, $A_X, B^{(1)}, \ldots, B^{(m)}$, is bosonic,*

$$B^{(j)} = R_j(A), \quad (B^{(j)})^\dagger = R_j(A_X^\dagger) \quad \textit{and} \quad R_j(H_\Lambda) = H_\Lambda, \tag{4.4}$$

and

$$q_\Lambda^3 \max_{\substack{i,j\in\{0,1,\ldots,m\}:\\ i\ne j}} \left\{ \tilde{G}_\Lambda \left(\left(B^{(i)} \right)^\dagger, B^{(j)} \right) \right\} < \epsilon, \tag{4.5}$$

where we have written $B^{(0)} = A_X^\dagger$. Then we have the bound,

$$|\omega(A_X B_Y)| \le \text{Const.} \times \begin{cases} [1 + \text{dist}(X, Y)]^{-\tilde{\eta}}, & for\ power-law\ decaying\ h_X; \\ \exp[-\tilde{\mu}\,\text{dist}(X, Y)], & for\ exponentially\ decaying\ h_X, \end{cases} \tag{4.6}$$

in the infinite volume limit.

Proof. From the bound (3.30) or (3.31) and the Schwarz inequality,

$$|\omega(A_X P_0 B_Y)|^2 \le \omega(A_X P_0 A_X^\dagger)\omega(B_Y^\dagger P_0 B_Y), \tag{4.7}$$

it is sufficient to show $\omega(A_X P_0 A_X^\dagger) = \omega(A_X^\dagger P_0 A_X) = 0$. Further, we have

$$\left\langle \Phi, A P_{0,\Lambda} A^\dagger \Phi \right\rangle \le q_\Lambda \left\langle A P_{0,\Lambda} A^\dagger \right\rangle_{0,\Lambda} = q_\Lambda \left\langle A^\dagger P_{0,\Lambda} A \right\rangle_{0,\Lambda} \tag{4.8}$$

for any ground state vector Φ with norm one and any observable A on the finite lattice Λ. Therefore we estimate $q_\Lambda \left\langle A_X P_{0,\Lambda} A_X^\dagger \right\rangle_{0,\Lambda}$.

Note that, from the clustering bound (4.1), (4.2) and (4.3), we have

$$\begin{aligned} \left| \langle A_X P_{0,\Lambda} B_Y \rangle_{0,\Lambda} \right| &\le \left| \langle A_X B_Y \rangle_{0,\Lambda} \right| + \left| \langle A_X B_Y \rangle_{0,\Lambda} - \langle A_X P_{0,\Lambda} B_Y \rangle_{0,\Lambda} \right| \\ &\le 2\tilde{G}_\Lambda(A_X, B_Y). \end{aligned} \tag{4.9}$$

We define

$$B_i^{(j)} := \left\langle \Phi_{0,\nu'}, B^{(j)} \Phi_{0,\nu} \right\rangle \tag{4.10}$$

for $j = 0, 1, \ldots, m$ and for the finite lattice Λ, where we have written $i = (\nu', \nu)$ with $i = 1, 2, \ldots, m$ for short. Since $(B_1^{(j)}, B_2^{(j)}, \ldots, B_m^{(j)})$ is an m-dimensional vector, there exist complex numbers, C_j, $j = 0, 1, \ldots, m$, such that, at least, one of C_j is nonvanishing and that

$$\sum_{j=0}^{m} C_j B_i^{(j)} = 0. \tag{4.11}$$

Let ℓ be the index which satisfies $|C_\ell| = \max\{|C_0|, |C_1|, \ldots, |C_m|\}$. Clearly, we have

$$B_i^{(\ell)} = -\sum_{j\neq\ell} \frac{C_j}{C_\ell} B_i^{(j)}. \tag{4.12}$$

Therefore

$$\begin{aligned}
\left\langle (B^{(\ell)})^\dagger P_{0,\Lambda} B^{(\ell)} \right\rangle_{0,\Lambda} &= \frac{1}{q_\Lambda} \sum_{i=1}^m \left| B_i^{(\ell)} \right|^2 = -\sum_{j\neq\ell} \frac{C_j}{C_\ell} \frac{1}{q_\Lambda} \sum_{i=1}^m (B_i^{(\ell)})^* B_i^{(j)} \\
&\le m \max_{j\neq\ell} \left\{ \left| \left\langle (B^{(\ell)})^\dagger P_{0,\Lambda} B^{(j)} \right\rangle_{0,\Lambda} \right| \right\} \\
&\le 2q_\Lambda^2 \max_{j\neq\ell} \left\{ \tilde{G}_\Lambda \left((B^{(\ell)})^\dagger, B^{(j)} \right) \right\},
\end{aligned} \tag{4.13}$$

where we have used the inequality (4.9) for getting the last bound. When $\ell = 0$, we obtain

$$q_\Lambda \left\langle A_X P_{0,\Lambda} A_X^\dagger \right\rangle_{0,\Lambda} \le 2\epsilon \tag{4.14}$$

from $B^{(0)} = A_X^\dagger$ and the assumption (4.5). When $\ell \neq 0$, we reach the same conclusion by using the relation,

$$\left\langle A_X^\dagger P_{0,\Lambda} A_X \right\rangle_{0,\Lambda} = \left\langle R_\ell(A_X^\dagger) P_{0,\Lambda} R_\ell(A) \right\rangle_{0,\Lambda} = \left\langle \left(B^{(\ell)} \right)^\dagger P_{0,\Lambda} B^{(\ell)} \right\rangle_{0,\Lambda}, \tag{4.15}$$

which is derived from the assumption (4.4). $\square$

Remark. 1. The advantage of Theorem 4.1 is that it is easier to find $B^{(j)}$ and R_j because of the finiteness of the lattice. Actually one can construct $B^{(j)}$, R_j and Λ satisfying the requirement by connecting m copies of a small, finite lattice to each other at their boundaries. But, if the degeneracy q_Λ exceeds $\sqrt{|\Lambda_s|}$, we cannot find the observables, $B^{(j)}$, and the transformations, R_j. Therefore our argument does not work in such cases.

2. Under the weaker assumption,

$$q_\Lambda^2 \max_{\substack{i,j\in\{0,1,\ldots,m\}:\\ i\neq j}} \left\{ \tilde{G}_\Lambda \left(\left(B^{(i)} \right)^\dagger, B^{(j)} \right) \right\} < \epsilon, \tag{4.16}$$

than (4.5), we can obtain the bound,

$$|\langle A_X B_Y \rangle_0| \le \text{Const.} \times \begin{cases} [1 + \operatorname{dist}(X, Y)]^{-\tilde{\eta}}, & for\ power-law\ decaying\ h_X; \\ \exp[-\tilde{\mu} \operatorname{dist}(X, Y)], & for\ exponentially\ decaying\ h_X, \end{cases} \tag{4.17}$$

in the infinite volume limit.

3. Consider the situation of the above Remark 2 or the case with a finite degeneracy of the infinite-volume ground state. Then, instead of introducing the transformations R_j, we can directly require

$$\left\langle A_X^\dagger P_0 A_X \right\rangle_0 = \left\langle \left(B^{(j)}\right)^\dagger P_0 B^{(j)} \right\rangle_0 \quad \text{for } j = 1, 2, \ldots, m, \tag{4.18}$$

in the infinite volume limit, and at infinite distance between the observables A_X and $B^{(j)}$.

4. Let us show that the self-similarity condition is indispensable to show the vanishing of matrix elements in the ground state sector. For this purpose, we introduce additional spin degrees of freedom for the present Hamiltonian H_Λ. We assume that the additional two spins are located at the origin and obey the Hamiltonian H_{toy} of (2.24), and assume that there is no interaction between the two systems of H_Λ and H_{toy}. Clearly the total system still exhibits a spectral gap above the ground state sector with a slightly different degeneracy, and the correlations exhibit the same decay as in the system without the additional two spins. However, this system does not satisfy the self-similarity condition because of the existence of the additional spins, and in this case one of the additional spins has nonvanishing matrix elements in the ground state sector as we discussed in Sect. 2.
5. For the degeneracy of the ground state sector, it is very instructive to consider the magnetic model which is given by n copies of the toy model (2.24). Namely the Hamiltonian is given by

$$H_{\text{toy},n} = -\Delta E \sum_{\ell=1}^{n} S_{1,\ell}^{(3)}, \tag{4.19}$$

where $\mathbf{S}_{j,\ell} = (S_{j,\ell}^{(1)}, S_{j,\ell}^{(2)}, S_{j,\ell}^{(3)})$ are the spin-1/2 operators on the ladder, $j = 1, 2$; $\ell = 1, 2, \ldots, n$, and ΔE is a positive constant. Clearly the model exhibits the spectral gap ΔE above the ground state with 2^n-fold degeneracy. However, even though the system is self-similar, we cannot apply our theorems to this model because the ground state degeneracy is too large. In fact, the vectors $\mathbf{S}_{2,\ell}$ show nonvanishing matrix elements in the ground state sector. However,

$$\left\langle S_{2,m}^{(\alpha)} S_{2,n}^{(\beta)} \right\rangle_{0,\Lambda} = 0 \quad \text{for } m \neq n. \tag{4.20}$$

A. Lieb-Robinson Bound for Group Velocity

Quite recently, Nachtergaele and Sims [6] have extended the Lieb-Robinson bound [15] to a wide class of models with long-range, exponentially decaying interactions. In this appendix, we further extend the bound to the power-law decaying interactions. We also tighten the bound on the exponentially decaying case. (See Assumption 2.4 compared to that in [6].) However, in our proof, the time t must be real.

In the following, we treat only the case with bosonic observables and with the power-law decaying interaction h_X because the other cases including the previous results can be treated in the same way.

Theorem A.1. *Let A_X, B_Y be a pair of bosonic observables with the compact support, X, Y, respectively. Assume that the system satisfies the conditions in Assumption 2.1 or 2.2. Then*

$$\|[A_X(t), B_Y]\| \le C\|A_X\|\|B_Y\||X||Y|\frac{e^{v|t|}-1}{[1+\mathrm{dist}(X,Y)]^{\eta}} \quad \textit{for } \mathrm{dist}(X,Y) > 0, \quad \text{(A.1)}$$

where the positive constants, C and v, depend only on the interaction of the Hamiltonian and the metric of the lattice.

Remark. The same bound for fermionic observables is obtained by replacing the commutator with the anticommutator in the left-hand side.

For exponentially decaying interaction h_X, the following bound is valid:

Theorem A.2. *Let A_X, B_Y be a pair of bosonic observables with the compact support, X, Y, respectively. Assume that the system satisfies the conditions in Assumption 2.3 or 2.4. Then*

$$\begin{aligned}&\|[A_X(t), B_Y]\|\\ &\quad \le C\|A_X\|\|B_Y\||X||Y|\exp[-\mu\,\mathrm{dist}(X,Y)]\left[e^{v|t|}-1\right] \quad \text{for } \mathrm{dist}(X,Y) > 0, \quad \text{(A.2)}\end{aligned}$$

where the positive constants, C and v, depend only on the interaction of the Hamiltonian and the metric of the lattice.

Remark. For the proof under Assumption 2.3, we rely on the inequalities, (2.9) and (2.10). Assumption 2.4 is milder than that in ref. [6] as remarked in Sect. 2.

We assume that the volume $|\Lambda_s|$ of the lattice Λ is finite. If it is necessary to consider the infinite volume limit, we take the limit after deriving the desired Lieb-Robinson bounds which hold uniformly in the size of the lattice. Let A, B be observables supported by compact sets, $X, Y \subset \Lambda_s$, respectively. The time evolution of A is given by $A(t) = e^{itH_\Lambda}Ae^{-itH_\Lambda}$. First, let us derive the inequality (A.12) below for the commutator $[A(t), B]$. We assume $t > 0$ because the negative t can be treated in the same way. Let $\epsilon = t/N$ with a large positive integer N, and let

$$t_n = \frac{t}{N}n \quad \text{for } n = 0, 1, \ldots, N. \quad \text{(A.3)}$$

Then we have

$$\|[A(t), B]\| - \|[A(0), B]\| = \sum_{i=0}^{N-1} \epsilon \times \frac{\|[A(t_{n+1}), B]\| - \|[A(t_n), B]\|}{\epsilon}. \quad \text{(A.4)}$$

In order to obtain the bound (A.12) below, we want to estimate the summand in the right-hand side. To begin with, we note that the identity, $\|U^*OU\| = \|O\|$, holds for any observable O and for any unitary operator U. Using this fact, we have

$$\begin{aligned}\|[A(t_{n+1}), B]\| - \|[A(t_n), B]\| &= \|[A(\epsilon), B(-t_n)]\| - \|[A, B(-t_n)]\|\\ &\le \|[A+i\epsilon[H_\Lambda, A], B(-t_n)]\| - \|[A, B(-t_n)]\| + \mathcal{O}(\epsilon^2)\\ &= \|[A+i\epsilon[I_X, A], B(-t_n)]\| - \|[A, B(-t_n)]\| + \mathcal{O}(\epsilon^2)\end{aligned}$$
$$\text{(A.5)}$$

with

$$I_X = \sum_{Z:Z\cap X\neq\emptyset} h_Z, \tag{A.6}$$

where we have used

$$A(\epsilon) = A + i\epsilon[H_\Lambda, A] + \mathcal{O}(\epsilon^2) \tag{A.7}$$

and the triangle inequality. Further, by using

$$A + i\epsilon[I_X, A] = e^{i\epsilon I_X} A e^{-i\epsilon I_X} + \mathcal{O}(\epsilon^2), \tag{A.8}$$

we have

$$\begin{aligned} \|[A + i\epsilon[I_X, A], B(-t_n)]\| &\le \left\|[e^{i\epsilon I_X} A e^{-i\epsilon I_X}, B(-t_n)]\right\| + \mathcal{O}(\epsilon^2) \\ &= \left\|[A, e^{-i\epsilon I_X} B(-t_n) e^{i\epsilon I_X}]\right\| + \mathcal{O}(\epsilon^2) \\ &\le \|[A, B(-t_n) - i\epsilon[I_X, B(-t_n)]]\| + \mathcal{O}(\epsilon^2) \\ &\le \|[A, B(-t_n)]\| + \epsilon\, \|[A, [I_X, B(-t_n)]]\| + \mathcal{O}(\epsilon^2). \end{aligned} \tag{A.9}$$

Substituting this into the right-hand side in the last line of (A.5), we obtain

$$\begin{aligned} \|[A(t_{n+1}), B]\| - \|[A(t_n), B]\| &\le \epsilon\, \|[A, [I_X, B(-t_n)]]\| + \mathcal{O}(\epsilon^2) \\ &\le 2\epsilon \|A\|\, \|[I_X(t_n), B]\| + \mathcal{O}(\epsilon^2). \end{aligned} \tag{A.10}$$

Further, substituting this into the right-hand side of (A.4) and using (A.6), we have

$$\begin{aligned} \|[A(t), B]\| - \|[A(0), B]\| &\le 2\|A\| \sum_{n=0}^{N-1} \epsilon \times \|[I_X(t_n), B]\| + \mathcal{O}(\epsilon) \\ &\le 2\|A\| \sum_{Z:Z\cap X\neq\emptyset} \sum_{n=0}^{N-1} \epsilon \times \|[h_Z(t_n), B]\| + \mathcal{O}(\epsilon). \end{aligned} \tag{A.11}$$

Since $h_Z(t)$ is the continuous function of the time t for a finite volume, the sum in the right-hand side converges to the integral in the limit $\epsilon \downarrow 0$ ($N \uparrow \infty$) for any fixed finite lattice Λ. In consequence, we obtain

$$\|[A(t), B]\| - \|[A(0), B]\| \le 2\|A\| \sum_{Z:Z\cap X\neq\emptyset} \int_0^{|t|} ds\, \|[h_Z(s), B]\|\,. \tag{A.12}$$

We define

$$C_B(X, t) := \sup_{A\in\mathcal{A}_X} \frac{\|[A(t), B]\|}{\|A\|}, \tag{A.13}$$

where $\mathcal{A}_X$ is the set of observables supported by the compact set X. Then we have[5]

$$C_B(X,t) \le C_B(X,0) + 2 \sum_{Z:Z\cap X\ne\emptyset} \|h_Z\| \int_0^{|t|} ds\, C_B(Z,s) \tag{A.14}$$

from the above bound (A.12).

We recall that the observables, A and B, are, respectively, supported by the compact sets, $X, Y \subset \Lambda_s$. Assume $\mathrm{dist}(X,Y) > 0$. Then we have $C_B(X,0) = 0$ from the definition of $C_B(X,t)$, and note that

$$C_B(Z,0) \le \begin{cases} 2\|B\|, & for\ Z\cap Y \ne \emptyset; \\ 0, & \text{otherwise.} \end{cases} \tag{A.15}$$

Using these facts and the above bound (A.14) iteratively, we obtain

$$\begin{aligned} C_B(X,t) &\le 2 \sum_{Z_1:Z_1\cap X\ne\emptyset} \|h_{Z_1}\| \int_0^{|t|} ds_1\, C_B(Z_1,s_1) \\ &\le 2 \sum_{Z_1:Z_1\cap X\ne\emptyset} \|h_{Z_1}\| \int_0^{|t|} ds_1\, C_B(Z_1,0) \\ &\quad +2^2 \sum_{Z_1:Z_1\cap X\ne\emptyset} \|h_{Z_1}\| \sum_{Z_2:Z_2\cap Z_1\ne\emptyset} \|h_{Z_2}\| \int_0^{|t|} ds_1 \int_0^{|s_1|} ds_2\, C_B(Z_2,s_2) \\ &\le 2\|B\|(2|t|) \sum_{Z_1:Z_1\cap X\ne\emptyset, Z_1\cap Y\ne\emptyset} \|h_{Z_1}\| \\ &\quad +2\|B\|\frac{(2|t|)^2}{2!} \sum_{Z_1:Z_1\cap X\ne\emptyset} \|h_{Z_1}\| \sum_{Z_2:Z_2\cap Z_1\ne\emptyset, Z_2\cap Y\ne\emptyset} \|h_{Z_2}\| \\ &\quad +2\|B\|\frac{(2|t|)^3}{3!} \sum_{Z_1:Z_1\cap X\ne\emptyset} \|h_{Z_1}\| \sum_{Z_2:Z_2\cap Z_1\ne\emptyset} \|h_{Z_2}\| \\ &\quad \times \sum_{Z_3:Z_3\cap Z_2\ne\emptyset, Z_3\cap Y\ne\emptyset} \|h_{Z_3}\| + \cdots . \end{aligned} \tag{A.16}$$

Proof of Theorem A.1 under Assumption 2.1. The first sum in the power series (A.16) is estimated as

$$\begin{aligned} \sum_{Z_1:Z_1\cap X\ne\emptyset, Z_1\cap Y\ne\emptyset} \|h_{Z_1}\| &\le \sum_{x\in X}\sum_{y\in Y}\sum_{Z_1\ni x,y} \|h_{Z_1}\| \\ &\le \frac{\lambda_0 |X||Y|}{[1+\mathrm{dist}(X,Y)]^\eta} \end{aligned} \tag{A.17}$$

[5] Since the local interaction h_Z with $Z \subset X$ does not change the support X of A in the time evolution, we can expect that the sum in the right-hand side of (A.14) can be restricted to the set Z satisfying $Z \cap X \ne \emptyset$ and $Z \setminus X \ne \emptyset$. However, this restriction does not affect the resulting Lieb-Robinson bound. Therefore we omit the discussion.

from the assumption (2.2). The second, double sum is estimated as

$$\begin{aligned}
&\sum_{Z_1:Z_1\cap X\neq\emptyset}\|h_{Z_1}\|\sum_{Z_2:Z_2\cap Z_1\neq\emptyset, Z_2\cap Y\neq\emptyset}\|h_{Z_2}\|\\
&\quad\le\sum_{x\in X}\sum_{y\in Y}\sum_{z_{12}\in\Lambda_s}\sum_{Z_1\ni x,z_{12}}\|h_{Z_1}\|\sum_{Z_2\ni z_{12},y}\|h_{Z_2}\|\\
&\quad\le\sum_{x\in X}\sum_{y\in Y}\sum_{z_{12}\in\Lambda_s}\frac{\lambda_0}{[1+\mathrm{dist}(x,z_{12})]^\eta}\frac{\lambda_0}{[1+\mathrm{dist}(z_{12},y)]^\eta}\\
&\quad\le\frac{\lambda_0^2 p_0|X||Y|}{[1+\mathrm{dist}(X,Y)]^\eta},
\end{aligned}\tag{A.18}$$

where we have used the assumptions (2.2) and (2.3). Similarly, the third, triple sum can be estimated as

$$\begin{aligned}
&\sum_{Z_1:Z_1\cap X\neq\emptyset}\|h_{Z_1}\|\sum_{Z_2:Z_2\cap Z_1\neq\emptyset}\|h_{Z_2}\|\sum_{Z_3:Z_3\cap Z_2\neq\emptyset, Z_3\cap Y\neq\emptyset}\|h_{Z_3}\|\\
&\quad\le\sum_{x\in X}\sum_{y\in Y}\sum_{z_{12}\in\Lambda_s}\sum_{z_{23}\in\Lambda_s}\sum_{Z_1\ni x,z_{12}}\|h_{Z_1}\|\sum_{Z_2\ni z_{12},z_{23}}\|h_{Z_2}\|\sum_{Z_3\ni z_{23},y}\|h_{Z_3}\|\\
&\quad\le\sum_{x\in X}\sum_{y\in Y}\sum_{z_{12}\in\Lambda_s}\sum_{z_{23}\in\Lambda_s}\frac{\lambda_0}{[1+\mathrm{dist}(x,z_{12})]^\eta}\frac{\lambda_0}{[1+\mathrm{dist}(z_{12},z_{23})]^\eta}\frac{\lambda_0}{[1+\mathrm{dist}(z_{23},y)]^\eta}\\
&\quad\le\sum_{x\in X}\sum_{y\in Y}\sum_{z_{12}\in\Lambda_s}\frac{\lambda_0}{[1+\mathrm{dist}(x,z_{12})]^\eta}\frac{\lambda_0^2 p_0}{[1+\mathrm{dist}(z_{12},y)]^\eta}\\
&\quad\le\frac{\lambda_0^3 p_0^2|X||Y|}{[1+\mathrm{dist}(X,Y)]^\eta}.
\end{aligned}\tag{A.19}$$

From these observations, we have

$$\begin{aligned}
C_B(X,t)&\le\frac{2\|B\||X||Y|}{[1+\mathrm{dist}(X,Y)]^\eta}\left\{2|t|\lambda_0+\frac{(2|t|)^2}{2!}\lambda_0^2p_0+\frac{(2|t|)^3}{3!}\lambda_0^3p_0^2+\cdots\right\}\\
&=\frac{2p_0^{-1}\|B\||X||Y|}{[1+\mathrm{dist}(X,Y)]^\eta}\{\exp[2\lambda_0p_0|t|]-1\}.
\end{aligned}\tag{A.20}$$

Consequently, we obtain

$$\|[A(t),B]\|\le\frac{2p_0^{-1}\|A\|\|B\||X||Y|}{[1+\mathrm{dist}(X,Y)]^\eta}\{\exp[2\lambda_0p_0|t|]-1\}\tag{A.21}$$

from (A.13). □

Proof of Theorem A.1 under Assumption 2.2 The first sum in the power series (A.16) is estimated as

$$\begin{aligned}
\sum_{Z_1:Z_1\cap X\neq\emptyset, Z_1\cap Y\neq\emptyset}\|h_{Z_1}\|&\le\sum_{x\in X}\sum_{y\in Y}\sum_{Z_1\ni x,y}\|h_{Z_1}\|\\
&\le\sum_{x\in X}\sum_{y\in Y}\sum_{Z_1\ni x,y}\|h_{Z_1}\|[1+\mathrm{dist}(x,y)]^{-\eta}[1+\mathrm{diam}(Z_1)]^\eta\\
&\le[1+\mathrm{dist}(X,Y)]^{-\eta}|X||Y|s_0,
\end{aligned}\tag{A.22}$$

where

$$s_0 = \sup_x \sum_{Z \ni x} \|h_Z\| [1 + \mathrm{diam}(Z)]^\eta. \tag{A.23}$$

Clearly, this constant s_0 is finite from the assumption (2.7). The second, double sum is estimated as

$$\begin{aligned}
&\sum_{Z_1 : Z_1 \cap X \neq \emptyset} \|h_{Z_1}\| \sum_{Z_2 : Z_2 \cap Z_1 \neq \emptyset, Z_2 \cap Y \neq \emptyset} \|h_{Z_2}\| \\
&\leq \sum_{x \in X} \sum_{y \in Y} \sum_{z_{12} \in \Lambda_s} \sum_{Z_1 \ni x, z_{12}} \|h_{Z_1}\| \sum_{Z_2 \ni z_{12}, y} \|h_{Z_2}\| \\
&\leq \sum_{x \in X} \sum_{y \in Y} \sum_{z_{12} \in \Lambda_s} [1 + \mathrm{dist}(x, z_{12})]^{-\eta} [1 + \mathrm{dist}(z_{12}, y)]^{-\eta} \\
&\quad \times \sum_{Z_1 \ni x, z_{12}} \|h_{Z_1}\| [1 + \mathrm{diam}(Z_1)]^\eta \sum_{Z_2 \ni z_{12}, y} \|h_{Z_2}\| [1 + \mathrm{diam}(Z_2)]^\eta \\
&\leq [1 + \mathrm{dist}(X, Y)]^{-\eta} \sum_{x \in X} \sum_{y \in Y} \sum_{z_{12} \in \Lambda_s} \sum_{Z_1 \ni x, z_{12}} \|h_{Z_1}\| [1 + \mathrm{diam}(Z_1)]^\eta \\
&\quad \times \sum_{Z_2 \ni z_{12}, y} \|h_{Z_2}\| [1 + \mathrm{diam}(Z_2)]^\eta \\
&\leq [1 + \mathrm{dist}(X, Y)]^{-\eta} \sum_{x \in X} \sum_{y \in Y} \sum_{Z_1 \ni x} \|h_{Z_1}\| [1 + \mathrm{diam}(Z_1)]^\eta \\
&\quad \times \sum_{Z_2 \ni y} \|h_{Z_2}\| |Z_2| [1 + \mathrm{diam}(Z_2)]^\eta \\
&\leq [1 + \mathrm{dist}(X, Y)]^{-\eta} |X| |Y| s_0 s_1,
\end{aligned} \tag{A.24}$$

where we have used the assumption (2.7) and the inequality,

$$\begin{aligned}
&[1 + \mathrm{dist}(x, z)]^{-\eta} [1 + \mathrm{dist}(z, y)]^{-\eta} \\
&\quad = [1 + \mathrm{dist}(x, z) + \mathrm{dist}(z, y) + \mathrm{dist}(x, z)\mathrm{dist}(z, y)]^{-\eta} \\
&\quad \leq [1 + \mathrm{dist}(x, z) + \mathrm{dist}(z, y)]^{-\eta} \\
&\quad \leq [1 + \mathrm{dist}(x, y)]^{-\eta},
\end{aligned} \tag{A.25}$$

for any $z \in \Lambda_s$. Similarly, the third, triple sum can be estimated as

$$\begin{aligned}
&\sum_{Z_1 : Z_1 \cap X \neq \emptyset} \|h_{Z_1}\| \sum_{Z_2 : Z_2 \cap Z_1 \neq \emptyset} \|h_{Z_2}\| \sum_{Z_3 : Z_3 \cap Z_2 \neq \emptyset, Z_3 \cap Y \neq \emptyset} \|h_{Z_3}\| \\
&\leq \sum_{x \in X} \sum_{y \in Y} \sum_{z_{12} \in \Lambda_s} \sum_{z_{23} \in \Lambda_s} \sum_{Z_1 \ni x, z_{12}} \|h_{Z_1}\| \sum_{Z_2 \ni z_{12}, z_{23}} \|h_{Z_2}\| \sum_{Z_3 \ni z_{23}, y} \|h_{Z_3}\| \\
&\leq [1 + \mathrm{dist}(X, Y)]^{-\eta} \sum_{x \in X} \sum_{y \in Y} \sum_{z_{12} \in \Lambda_s} \sum_{z_{23} \in \Lambda_s} \sum_{Z_1 \ni x, z_{12}} \|h_{Z_1}\| [1 + \mathrm{diam}(Z_1)]^\eta \\
&\quad \times \sum_{Z_2 \ni z_{12}, z_{23}} \|h_{Z_2}\| [1 + \mathrm{diam}(Z_2)]^\eta \sum_{Z_3 \ni z_{23}, y} \|h_{Z_3}\| [1 + \mathrm{diam}(Z_3)]^\eta \\
&\leq [1 + \mathrm{dist}(X, Y)]^{-\eta} |X| |Y| s_0 s_1^2.
\end{aligned} \tag{A.26}$$

From these observations, we have

$$C_B(X,t) \leq \frac{2s_0 s_1^{-1}\|B\||X||Y|}{[1+\mathrm{dist}(X,Y)]^\eta}\sum_{n=1}^{\infty}\frac{(2s_1|t|)^n}{n!}$$
$$= \frac{2s_0 s_1^{-1}\|B\||X||Y|}{[1+\mathrm{dist}(X,Y)]^\eta}\{\exp[2s_1|t|]-1\}. \tag{A.27}$$

As a result, we obtain

$$\|[A(t),B]\| \leq \frac{2s_0 s_1^{-1}\|A\|\|B\||X||Y|}{[1+\mathrm{dist}(X,Y)]^\eta}\{\exp[2s_1|t|]-1\} \tag{A.28}$$

from (A.13). □

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References

1. Haldane, F.D.M.: Continuum dynamics of the 1-D Heisenberg antiferromagnet: identification with the O(3) nonlinear sigma model. Phys. Lett. **93 A**, 464–468 (1983); Nonlinear field theory of large-spin Heisenberg antiferromagnets: semiclassically quantized solitons of the one-dimensional easy-axis Néel state. Phys. Rev. Lett. **50**, 1153–1156 (1983)
2. Koma, T.: Spectral gap and decay of correlations in U(1)-symmetric lattice systems in dimensions $D < 2$. http://arxiv.org/list/math-ph/0505022, 2005
3. Hastings, M.B.: Lieb-Schultz-Mattis in higher dimensions. Phys. Rev. B**69**, 104431(13 pages) (2004)
4. Hastings, M.B.: Locality in quantum and Markov dynamics on lattices and networks. Phys. Rev. Lett. **93**, 140402(4 pages) (2004)
5. Fredenhagen, K.: A remark on the cluster theorem. Commun. Math. Phys. **97**, 461–463 (1985)
6. Nachtergaele, B., Sims, R.: Lieb-Robinson bounds and the exponential clustering theorem. http://arxiv.org/list/math-ph/0506030, 2005
7. Dhar, D.: Lattices of effectively nonintegral dimensionality. J. Math. Phys. **18**, 577–585 (1977); Tasaki, H.: Critical phenomena in fractal spin systems. J. Phys. A: Math. Gen. **20**, 4521–4529 (1987)
8. Wreszinski, W.F.: Goldstone's theorem for quantum spin systems of finite range. J. Math. Phys. **17**, 109–111 (1976); Landau, L., Fernando Perez J., Wreszinski, W.F.: Energy gap, clustering, and the Goldstone theorem in statistical mechanics, J. Stat. Phys. **26**, 755–766 (1981); Wreszinski, W.F.: Charges and symmetries in quantum theories without locality. Forts. Phys. **35**, 379–413 (1987)
9. Koma, T., Tasaki, H.: Classical XY model in 1.99 dimensions. Phys. Rev. Lett. **74**, 3916–3919 (1995)
10. Johnson, J.D.: A survey of analytic results for the 1-D Heisenberg magnets. J. Appl. Phys. **52**, 1991–1992 (1981)
11. Majumdar, C.K., Ghosh, D.K.: On next nearest-neighbor interaction in linear chain. I, II. J. Math. Phys. **10**, 1388–1402 (1969); Majumdar, C.K.: Antiferromagnetic model with known ground state. J. Phys. C **3**, 911–915 (1970); Affleck, I., Kennedy, T., Lieb, E.H., Tasaki, H.: Valence bond ground state in isotropic quantum antiferromagnets. Commun. Math. Phys. **115**, 477–528 (1988); Fannes, M., Nachtergaele, B., Werner, R.F.: Finitely correlated states on quantum spin chains. Commun. Math. Phys. **144**, 443–490 (1992)
12. Kennedy, T.: Exact diagonalisations of open spin-1 chains. J. Phys. Cond. Matt. **2**, 5737–5745 (1990)
13. Koma, T., Tasaki, H.: Decay of superconducting and magnetic correlations in one- and two-dimensional Hubbard models. Phys. Rev. Lett. **68**, 3248–3251 (1992)
14. Macris, N., Ruiz, J.: A remark on the decay of superconducting correlations in one- and two-dimensional Hubbard models. J. Stat. Phys. **75**, 1179–1184 (1994)
15. Lieb, E.H., Robinson, D.W.: The finite group velocity of quantum spin systems. Commun. Math. Phys. **28**, 251–257 (1972)

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