Fluctuation-dissipation theorem and the Unruh effect of scalar and Dirac fields

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(Received 10 March 1999; published 30 August 1999)

We present a simple and systematic method to calculate Rindler noise, which is relevant to an analysis of the Unruh effect, by using the fluctuation-dissipation theorem. To do this, we calculate the dissipative coefficient *explicitly* from the equations of motion of the detector and the field. This method gives not only the correct answer but also a hint as to the origin of the apparent statistics inversion effect. Moreover, this method is generalized to the Dirac field by using the fermionic fluctuation-dissipation theorem. We can thus confirm that the fermionic fluctuation-dissipation theorem is working properly. [S0556-2821(99)00518-4]

PACS number(s): 04.62.+v

I. INTRODUCTION

For a uniformly accelerated observer in flat spacetime, the ordinary Minkowski vacuum, which is defined by using the ''frequency'' with respect to Minkowski time, can be seen as a thermal bath of particles [1]. (The temperature of the thermal bath is $\hbar a/2\pi$ where a is the acceleration of the observer.) These particles are considered as Rindler particles which are defined by using ''frequency'' with respect to Rindler time [2–4]. This effect is called the Unruh effect.

In order to see this effect, it is convenient to introduce a model of a ''particle detector'' called the DeWitt detector [5]. The DeWitt detector is a pointlike object with internal energy levels and moves along the world line $x(\tau)$ where τ is the proper time of the detector. The internal energy levels of the detector are $\{E_i\}$ and corresponding eigenstates are $\{|E_i\rangle\}$. That is,

$$H_D|E_i\rangle=E_i|E_i\rangle, \tag{1.1}$$

where H_D is the Hamiltonian of the internal structure of the detector. Also, the detector has an internal degree of freedom (''monopole moment'') $Q(\tau)$. The time evolution of $Q(\tau)$ in the Heisenberg picture is

$$Q(\tau)=e^{iH_D\tau/\hbar}Q(0)\quad e^{-iH_D\tau/\hbar}. \tag{1.2}$$

And then the detector is linearly coupled to a field $\phi(x)$ via this monopole. The interaction is described by the interaction Lagrangian

$$L_{int}=Q(\tau)\phi(\tau), \tag{1.3}$$

where $\phi(\tau)\equiv\phi(x(\tau))$ is the value of the field along the world line of the detector. (To be precise, we need adiabatic switching.)

The measurement process by this detector is as follows. Suppose that, at $\tau=-\infty$, the detector is in the ground state $|E_0\rangle$ and the field is in the Minkowski vacuum $|0_M\rangle$. After the detector-field interaction is switched on, the detector would not remain in $|E_0\rangle$, but would make a transition to an excited state. It means that the detector ''detects'' some particles.

The transition amplitude for the detector-field system to be found in $|E_1,\psi\rangle$ at $\tau=\infty$ is given by first order perturbation theory as

$$i\langle E_1,\psi|\int_{-\infty}^{\infty}d\tau Q(\tau)\phi(\tau)|E_0,0_M\rangle. \tag{1.4}$$

(Here, it has been assumed that the matrix element of Q is sufficiently small enough for the perturbation theory to be appropriate.) By Eq. (1.2), this can be written as

$$i\langle E_1|Q(0)|E_0\rangle\int_{-\infty}^{\infty}d\tau e^{i(E_1-E_0)\tau/\hbar}\langle\psi|\phi(\tau)|0_M\rangle. \tag{1.5}$$

Therefore, after summation over all final states of the field $|\psi\rangle$, the transition rate (the transition probability per unit proper time) from E_0 to E_1 is

$$|\langle E_1|Q(0)|E_0\rangle|^2\mathcal{F}\left(\frac{E_1-E_0}{\hbar}\right), \tag{1.6}$$

where

$$\mathcal{F}(\omega)=\int_{-\infty}^{\infty}d(\tau-\tau')e^{-i\omega(\tau-\tau')}g(\tau-\tau') \tag{1.7}$$

and

$$g(\tau-\tau')=\langle 0_M|\phi(\tau)\phi(\tau')|0_M\rangle. \tag{1.8}$$

Thus, the transition rate of the DeWitt detector is proportional to the ''response function'' $\mathcal{F}(\omega)$ which depends only on the field and the world line of the detector but not on the internal structure of the detector. Note that we may regard $g(\tau-\tau')$ as a kind of noise, ''the quantum noise in the Minkowski vacuum along the world line $x(\tau)$,'' and the response function as the power spectrum of this noise.

For the Unruh effect, the relevant noise is Rindler noise, where $x(\tau)$ is a uniformly accelerated world line. In two dimensions, the power spectrum of Rindler noise (the response of the detector) is exactly those of thermal noise in a thermal bath. However, in other dimensions, it was shown by Takagi [6] that there are some differences between Rindler noise and thermal noise. Specially, Rindler noise exhibits the phenomenon of the apparent inversion of statistics in odd dimensions.

In this paper, we present a simple and systematic method to reproduce these results by using the fluctuation-dissipation theorem [7–9] which is the basis of statistical mechanics for irreversible processes when the systems are slightly away from thermal equilibrium. This theorem states the relation between the spontaneous fluctuation of fields in thermal equilibrium and irreversible dissipation. Although the fluctuation-dissipation theorem has been formulated by various authors, we adopt the formulation by Callen and Welton [7], because their formulation is intuitively understandable and appealing. They showed that a general form of the fluctuation-dissipation theorem covers a wide range of phenomena such as the Einstein relation for Brownian motion, the Nyquist formula for voltage fluctuation in conductors, and the Planck distribution for photons.

Of course, the ''fluctuation-dissipation point of view'' has been already pointed out [10,6]. However, previous discussions concentrated on the vacuum expectation value of the commutator and anticommutator of the fields in connection with Huygens' principle. Although these commutators are related to dissipation, the notion of ''dissipation'' is not quite clear. In addition, the calculations involved in obtaining these commutators are quantum and essentially the same as those to obtain the ''fluctuation'' directly.

In contrast, in our application of the fluctuation-dissipation theorem to the Unruh effect, we calculate the dissipative coefficient *explicitly* from the equations of motion of the detector and the field. Then, by virtue of the theorem, we can immediately obtain Rindler noise. The entire calculation we need is *classical* and thus our calculation is completely *different* from the previous ones. By this calculation, we can get not only the correct answer but also a hint as to the origin of the apparent statistics inversion effect.

Moreover, our method is generalized to a Dirac field by using the fluctuation-dissipation theorem of a fermionic operator [11]. In the context of condensed matter physics, the fermionic fluctuation-dissipation theorem may not be directly applicable, because we usually measure bosonic quantities such as voltage or electric current. In this paper, we show that the fermionic fluctuation-dissipation theorem is indeed *applicable* to the Unruh effect and works *properly*.

To our knowledge, the present approach to the Unruh effect by using the bosonic and fermionic fluctuation-dissipation theorem as a cornerstone has not been discussed before.

II. RINDLER NOISE OF A REAL SCALAR FIELD

A. Thermal noise

In order to see how the fluctuation-dissipation theorem works not only for thermal noise but also for Rindler noise to be discussed later, we first consider thermal noise. The system consists of a real scalar field and a detector in n-dimensional flat spacetime. The detector is at rest and linearly coupled with the scalar field through an internal degree of freedom, $Q(t)$, there. The scalar field is initially in thermal equilibrium at temperature T. Thus, the action of the total system is

$$S=S_0(Q)+S_{int}(Q,\phi)+S_0(\phi), \tag{2.1}$$

where

$$S_0(Q)=\int dt L(Q,\dot{Q}), \tag{2.2}$$

$$\begin{aligned} S_{int}(Q,\phi)&=\int dt d\vec{x} Q(t)\phi(\vec{x},t)\delta(\vec{x}-\vec{x}_0)\\ &=\int dt Q(t)\phi(\vec{x}_0,t), \end{aligned} \tag{2.3}$$

$$S_0(\phi)=\int dt d\vec{x}\frac{1}{2}\left[(\partial_t\phi)^2-\sum_{i=1}^{n-1}(\partial_i\phi)^2-m^2\phi^2\right], \tag{2.4}$$

and $\vec{x}_0$ is the position of the detector. [$\vec{x}$ stands for an $(n-1)$-dimensional vector.] Here, we do not need an explicit form of $L(Q,\dot{Q})$ which is the Lagrangian for the (unperturbed) detector.

From this action, we can derive the equations of motion

$$\left(\frac{\delta S_0}{\delta Q}\right)+\int d\vec{x}\phi(\vec{x},t)\delta(\vec{x}-\vec{x}_0)=0, \tag{2.5}$$

$$\partial_t^2\phi-\sum_i \partial_i^2\phi+m^2\phi-Q(t)\delta(\vec{x}-\vec{x}_0)=0, \tag{2.6}$$

where

$$-\left(\frac{\delta S_0}{\delta Q}\right)=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\right)-\left(\frac{\partial L}{\partial Q}\right). \tag{2.7}$$

By Fourier transformations,

$$\tilde{\phi}(\vec{k},\omega)\equiv\int dt d\vec{x}\phi(\vec{x},t)e^{i\vec{k}\cdot\vec{x}-i\omega t}, \tag{2.8}$$

$$\tilde{Q}(\omega)\equiv\int dt Q(t)e^{-i\omega t}, \tag{2.9}$$

these equations become

$$\left(\widetilde{\frac{\delta S_0}{\delta Q}}\right)+\int \frac{d\vec{k}}{(2\pi)^{n-1}}\tilde{\phi}(\vec{k},\omega)e^{-i\vec{k}\cdot\vec{x}_0}=0, \tag{2.10}$$

$$(-\omega^2+|\vec{k}|^2+m^2)\tilde{\phi}(\vec{k},\omega)-\tilde{Q}(\omega)e^{i\vec{k}\cdot\vec{x}_0}=0. \tag{2.11}$$

After elimination of $\tilde{\phi}(\vec{k},\omega)$ from these equations, one finds the effective equation of motion for the detector,

$$\left(\widetilde{\frac{\delta S_0}{\delta Q}}\right)+K_n(\omega)\tilde{Q}(\omega)=0, \tag{2.12}$$

where

$$K_n(\omega)=\int \frac{d\vec{k}}{(2\pi)^{n-1}}\frac{1}{-\omega^2+|\vec{k}|^2+m^2\pm i\epsilon}$$
$$=\frac{2^{2-n}\pi^{(1-n)/2}}{\Gamma\left(\frac{n-1}{2}\right)}\int_0^\infty \frac{\kappa^{n-2}d\kappa}{-\omega^2+\kappa^2+m^2\pm i\epsilon}. \tag{2.13}$$

Note that we have added $\pm i\epsilon$ terms to the denominator of the integrand to avoid the singularity. The sign is $+$ for $\omega>0$ and $-$ for $\omega<0$ due to causality. (The integrand, as a function of ω, must be analytic in the lower-half plane of the complex ω plane.) Thus, $K_n(\omega)$ has the imaginary part. This gives the friction term. We note that the conventional definition of the friction term contains the time derivative of $Q(t)$, which is converted to $i\omega$ by Fourier transformation. So the conventional definition of the dissipative coefficient $R_n(\omega)$ is given by

$$\mathrm{Im}\,K_n(\omega)=-\omega R_n(\omega). \tag{2.14}$$

The real part of $K_n(\omega)$ would diverge for higher n. But, fortunately, for our application of the fluctuation-dissipation theorem, we only need the imaginary part. [The divergence of the real part of $K_n(\omega)$ would generally be renormalized by the potential of the detector.]

By using

$$\frac{1}{x\pm i\epsilon}=\mathrm{P}\frac{1}{x}\mp i\pi\delta(x) \tag{2.15}$$

and

$$\delta(x^2-a^2)=\frac{1}{2|a|}[\delta(x+a)+\delta(x-a)], \tag{2.16}$$

one finds

$$\mathrm{Im}\;K_n(\omega)=\mp\frac{2^{1-n}\pi^{(3-n)/2}}{\Gamma\left(\frac{n-1}{2}\right)}(\sqrt{\omega^2-m^2})^{n-3}\theta(\omega^2-m^2). \tag{2.17}$$

Thus, the dissipative coefficient $R_n(\omega)$ is

$$R_n(\omega)=\frac{1}{|\omega|}\frac{2^{1-n}\pi^{(3-n)/2}}{\Gamma\left(\frac{n-1}{2}\right)}(\sqrt{\omega^2-m^2})^{n-3}\theta(\omega^2-m^2). \tag{2.18}$$

Especially, for the massless case, $m=0$, this becomes

$$R_n(\omega)=\frac{2^{1-n}\pi^{(3-n)/2}}{\Gamma\left(\frac{n-1}{2}\right)}|\omega|^{n-4}\quad(\text{for}\quad m=0). \tag{2.19}$$

By the fluctuation-dissipation theorem [7], this dissipation suggests the fluctuation of the scalar field if in thermal equilibrium. The fluctuation of the scalar field is defined by

$$\langle\phi(\vec{x}_0,t)\phi(\vec{x}_0,t)\rangle_\beta\equiv \mathrm{Tr}[e^{-\beta H}\phi(\vec{x}_0,t)\phi(\vec{x}_0,t)]/\mathrm{Tr}[e^{-\beta H}]$$
$$=\int dt'\langle\phi(\vec{x}_0,t)\phi(\vec{x}_0,t')\rangle_\beta\delta(t-t')$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega F_n(\omega), \tag{2.20}$$

where

$$F_n(\omega)=\int dt'\,e^{-i\omega(t-t')}\langle\phi(\vec{x}_0,t)\phi(\vec{x}_0,t')\rangle_\beta \tag{2.21}$$

is the power spectrum of thermal noise. Then, the fluctuation-dissipation theorem [7] says that

$$\int_{-\infty}^{\infty}d\omega F_n(\omega)=4\int_0^\infty d\omega\left[\frac{1}{2}\hbar\omega+\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\right]R_n(\omega)$$
$$=2\int_{-\infty}^{\infty}d\omega\frac{\hbar\omega R_n(|\omega|)}{e^{\beta\hbar\omega}-1} \tag{2.22}$$
$$=\frac{2^{2-n}\pi^{(3-n)/2}}{\Gamma\left(\frac{n-1}{2}\right)}\int_{-\infty}^{\infty}d\omega\frac{\hbar\omega|\omega|^{n-4}}{e^{\beta\hbar\omega}-1}$$
$$(\text{for}\quad m=0). \tag{2.23}$$

This is perfectly consistent with the previous result. (See, for example, Ref. [6].)

B. Rindler noise

Next, we consider Rindler noise. The system again consists of a real scalar field and a detector in n-dimensional flat spacetime. However, the detector is now uniformly accelerated; i.e., the world line of the detector is

$$x^0(\tau)=a^{-1}\sinh a\tau, \tag{2.24}$$

$$x^1(\tau)=a^{-1}\cosh a\tau, \tag{2.25}$$

$$x^i(\tau)=\text{const.}\quad(i=2,\ldots,n-1), \tag{2.26}$$

where τ is the proper time of the detector and a is the acceleration. Therefore, it is convenient to take the Rindler coordinates [12]

$$x^0=\xi\sinh\eta, \tag{2.27}$$

$$x^1=\xi\cosh\eta, \tag{2.28}$$

$$x^i=x^i\quad(i=2,\ldots,n-1). \tag{2.29}$$

In these coordinates, the world line of the detector is

$$\eta(\tau)=a\tau, \tag{2.30}$$

$$\xi(\tau)=a^{-1}, \tag{2.31}$$

$$x^i(\tau)=\text{const} \quad (i=2,\ldots,n-1). \tag{2.32}$$

The detector is again linearly coupled with the scalar field through an internal degree of freedom, $Q(\tau)$, there. However, the scalar field is initially in the ground state in this case. The action of the total system is

$$S=S_0(Q)+S_{int}(Q,\phi)+S_0(\phi), \tag{2.33}$$

where

$$S_0(Q)=\int d\tau L(Q,\dot{Q}), \tag{2.34}$$

$$\begin{aligned} S_{int}(Q,\phi)&=\int d\tau d\eta d\xi d\boldsymbol{x} Q(\tau)\phi(\xi,\boldsymbol{x},\eta)\delta(\xi-a^{-1})\\ &\quad\times\delta(\eta-a\tau)\delta(\boldsymbol{x}-\boldsymbol{x}_0)\\ &=\int d\tau Q(\tau)\phi(a^{-1},\boldsymbol{x}_0,a\tau), \end{aligned} \tag{2.35}$$

$$\begin{aligned} S_0(\phi)=\int d\eta d\xi d\boldsymbol{x}\frac{1}{2}\xi\Bigg[&\frac{1}{\xi^2}(\partial_\eta\phi)^2-(\partial_\xi\phi)^2\\ &-\sum_{i=2}^{n-1}(\partial_i\phi)^2-m^2\phi^2\Bigg], \end{aligned} \tag{2.36}$$

and $\boldsymbol{x}_0$ is the position of the detector. [$\boldsymbol{x}$ stands for an $(n-2)$-dimensional vector.] $S_0(\phi)$ is the same as Eq. (2.4) but written in Rindler coordinates. Again, we do not need an explicit form of $L(Q,\dot{Q})$ which is the Lagrangian for the (unperturbed) detector.

The equations of motion derived from this action are

$$\begin{aligned} \left(\frac{\delta S_0}{\delta Q}\right)+\int d\eta d\xi d\boldsymbol{x}\phi(\xi,\boldsymbol{x},\eta)\delta(\xi-a^{-1})&\\ \times\delta(\eta-a\tau)\delta(\boldsymbol{x}-\boldsymbol{x}_0)=0& \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} &\frac{1}{\xi^2}\partial_\eta^2\phi-\frac{1}{\xi}\partial_\xi(\xi\partial_\xi\phi)-\sum_i\partial_i^2\phi+m^2\phi\\ &-\frac{1}{\xi}\int d\tau Q(\tau)\\ &\quad\times\delta(\xi-a^{-1})\delta(\eta-a\tau)\delta(\boldsymbol{x}-\boldsymbol{x}_0)=0, \end{aligned} \tag{2.38}$$

where

$$-\left(\frac{\delta S_0}{\delta Q}\right)=\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{Q}}\right)-\left(\frac{\partial L}{\partial Q}\right). \tag{2.39}$$

In this case, we consider the transformations

$$\begin{aligned} \Phi(\nu,\boldsymbol{k},\Omega)\equiv\frac{1}{\pi}\int_0^\infty\frac{d\xi}{\xi}\int d\eta d\boldsymbol{x}\sqrt{2\nu\sinh\pi\nu}&\\ \times K_{i\nu}(M_k\xi)e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\Omega\eta}\phi(\xi,\boldsymbol{x},\eta),& \end{aligned} \tag{2.40}$$

$$\tilde{Q}(\omega)\equiv\int d\tau Q(\tau)e^{-i\omega\tau}, \tag{2.41}$$

where

$$M_k\equiv\sqrt{m^2+|\boldsymbol{k}|^2} \tag{2.42}$$

and $K_{i\nu}(z)$ is a modified Bessel function of imaginary order which satisfies

$$\left\{z^2\frac{d^2}{dz^2}+z\frac{d}{dz}-(z^2-\nu^2)\right\}K_{i\nu}(z)=0. \tag{2.43}$$

(Note that Ω is a frequency with respect to Rindler time η and ω is a frequency with respect to the proper time of the detector τ.) By the orthogonality relation [2]

$$\frac{1}{\pi^2}\int_0^\infty\frac{dx}{x}K_{i\mu}(x)K_{i\nu}(x)=\frac{\delta(\mu-\nu)}{2\nu\sinh\pi\nu}, \tag{2.44}$$

we can write the inverse transformation of Eq. (2.40) as

$$\begin{aligned} \phi(\xi,\boldsymbol{x},\eta)=\frac{1}{(2\pi)^{n-1}\pi}\int_0^\infty d\nu\int d\Omega d\boldsymbol{k}\sqrt{2\nu\sinh\pi\nu}&\\ \times K_{i\nu}(M_k\xi)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}+i\Omega\eta}\Phi(\nu,\boldsymbol{k},\Omega).& \end{aligned} \tag{2.45}$$

Then, the equations of motion become

$$\begin{aligned} \left(\widetilde{\frac{\delta S_0}{\delta Q}}\right)+\frac{1}{(2\pi)^{n-2}\pi a}\int_0^\infty d\nu\int d\boldsymbol{k}\sqrt{2\nu\sinh\pi\nu}&\\ \times K_{i\nu}(M_k/a)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_0}\Phi(\nu,\boldsymbol{k},\omega/a)=0,& \end{aligned} \tag{2.46}$$

$$\begin{aligned} &(-\Omega^2+\nu^2)\Phi(\nu,\boldsymbol{k},\Omega)\\ &-\frac{1}{\pi}\sqrt{2\nu\sinh\pi\nu}K_{i\nu}(M_k/a)e^{i\boldsymbol{k}\cdot\boldsymbol{x}_0}\tilde{Q}(\Omega a)=0. \end{aligned} \tag{2.47}$$

After elimination of $\Phi(\nu,\boldsymbol{k},\Omega)$ from these equations, one finds the effective equation of motion for the detector,

$$\left(\widetilde{\frac{\delta S_0}{\delta Q}}\right)+\mathcal{K}_n(\omega)\tilde{Q}(\omega)=0, \tag{2.48}$$

where

$$\mathcal{K}_n(\omega)=\frac{1}{2^{n-2}\pi^n a}\int_0^\infty d\nu\int d\boldsymbol{k}\frac{2\nu\sinh\pi\nu}{-(\omega/a)^2+\nu^2\pm i\epsilon}\times[K_{i\nu}(M_k/a)]^2$$
$$=\frac{2^{3-n}}{\pi^{(n+2)/2}\Gamma\left(\frac{n-2}{2}\right)a}\int_0^\infty d\nu\frac{2\nu\sinh\pi\nu}{-(\omega/a)^2+\nu^2\pm i\epsilon}\times\int_0^\infty d\kappa\kappa^{n-3}[K_{i\nu}(M_\kappa/a)]^2. \tag{2.49}$$

Again, we have added the $\pm i\epsilon$ term to the denominator of the integrand. Especially, for the massless case, $m=0$, we can use a closed form of integration given by the formula [13]

$$\int_0^\infty dx x^{n-3}[K_\mu(x)]^2=\frac{2^{n-5}}{\Gamma(n-2)}\left[\Gamma\left(\frac{n-2}{2}\right)\right]^2\times\Gamma\left(\frac{n}{2}-1+\mu\right)\Gamma\left(\frac{n}{2}-1-\mu\right). \tag{2.50}$$

Thus, one obtains, for Eq. (2.49),

$$\mathcal{K}_n(\omega)=\frac{a^{n-3}\Gamma\left(\frac{n-2}{2}\right)}{2\pi^{n/2}\Gamma(n-2)}\int_0^\infty d\nu\times\frac{1}{-(\omega/a)^2+\nu^2\pm i\epsilon}\left|\frac{\Gamma\left(\frac{n}{2}-1+i\nu\right)}{\Gamma(i\nu)}\right|^2, \tag{2.51}$$

where we have used

$$|\Gamma(iy)|^2=\frac{\pi}{y\sinh\pi y}\quad(y\quad\text{real}). \tag{2.52}$$

(Although the κ integral is absent for $n=2$, this expression is also valid for $n=2$.) By using

$$\Gamma(2z)=\frac{2^{2z}}{2\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{2.53}$$

one finds that the dissipative coefficient $\mathcal{R}_n(\omega)$ is

$$\mathcal{R}_n(\omega)\equiv\frac{\operatorname{Im}\mathcal{K}_n(\omega)}{-\omega}=\frac{2^{1-n}\pi^{(3-n)/2}}{\Gamma\left(\frac{n-1}{2}\right)}\frac{a^{n-2}}{\omega^2}\left|\frac{\Gamma\left(\frac{n}{2}-1+i\frac{\omega}{a}\right)}{\Gamma\left(i\frac{\omega}{a}\right)}\right|^2\quad(\text{for}\quad m=0) \tag{2.54}$$

and that the ratio of this Rindler case to the thermal case, Eq. (2.19), is given by

$$r_n(\omega)=\mathcal{R}_n(\omega)/R_n(\omega) \tag{2.55}$$
$$=\left|\frac{\omega}{a}\right|^{2-n}\left|\frac{\Gamma\left(\frac{n}{2}-1+i\frac{\omega}{a}\right)}{\Gamma\left(i\frac{\omega}{a}\right)}\right|^2. \tag{2.56}$$

By using the elementary formulas

$$\Gamma(z+1)=z\Gamma(z), \tag{2.57}$$

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right)=\frac{\pi}{\cos\pi z}, \tag{2.58}$$

and Eq. (2.52), one finds, for even n,

$$\left|\Gamma\left(\frac{n}{2}-1+i\frac{\omega}{a}\right)\right|^2=\left[\left(\frac{n}{2}-2\right)^2+\left(\frac{\omega}{a}\right)^2\right]\times\left[\left(\frac{n}{2}-3\right)^2+\left(\frac{\omega}{a}\right)^2\right]\cdots\left[1^2+\left(\frac{\omega}{a}\right)^2\right]\times\left(\frac{\omega}{a}\right)^2\left|\Gamma\left(i\frac{\omega}{a}\right)\right|^2$$
$$\equiv d_n(\omega)\left|\frac{\omega}{a}\right|^{n-2}\left|\Gamma\left(i\frac{\omega}{a}\right)\right|^2, \tag{2.59}$$

and, for odd n,

$$\left|\Gamma\left(\frac{n}{2}-1+i\frac{\omega}{a}\right)\right|^2=\left[\left(\frac{n}{2}-2\right)^2+\left(\frac{\omega}{a}\right)^2\right]\times\left[\left(\frac{n}{2}-3\right)^2+\left(\frac{\omega}{a}\right)^2\right]\cdots\left[\left(\frac{1}{2}\right)^2+\left(\frac{\omega}{a}\right)^2\right]\times\left|\Gamma\left(\frac{1}{2}+i\frac{\omega}{a}\right)\right|^2$$
$$\equiv d_n(\omega)\left|\frac{\omega}{a}\right|^{n-2}\left(\frac{\omega}{a}\right)^{-1}\left|\Gamma\left(\frac{1}{2}+i\frac{\omega}{a}\right)\right|^2. \tag{2.60}$$

Then, the ratio in Eq. (2.56) can be written as

$$r_n(\omega)=\begin{cases}d_n(\omega)\quad(n\quad\text{even}),\\ \tanh(\pi\omega/a)d_n(\omega)\quad(n\quad\text{odd}).\end{cases} \tag{2.61}$$

Note that

$$\lim_{a\to0}r_n(\omega)=1, \tag{2.62}$$

that is,

$$\lim_{a\to0}\mathcal{R}_n(\omega)=R_n(\omega). \tag{2.63}$$

By the fluctuation-dissipation theorem, this dissipation means the fluctuation of the scalar field if in ''thermal'' equilibrium. Although the scalar field is in the vacuum state in this case, we can view it as in thermal equilibrium at temperature $T=\hbar a/2\pi$ by the ''thermalization theorem.'' There are various versions of the theorem. (For a review, see Ref. [6].)

For example, since the positive Rindler wedge R_+ (a quarter of Minkowski spacetime, $x^1>|x^0|$) is causally disconnected from the negative Rindler wedge R_- (another quarter of Minkowski spacetime, $x^1<-|x^0|$), a uniformly accelerated observer who will be permanently confined within R_+ is not concerned with the degrees of freedom associated with R_-. Then, by tracing out over these degrees from the Minkowski vacuum, the observer gets the thermal density matrix at temperature $T=\hbar a/2\pi$ [4]. More specifically [14,15,6],

$$\langle 0_M|O^{(+)}|0_M\rangle = \mathrm{Tr}[e^{-(2\pi/a)aH_R^{(+)}}O^{(+)}]/\mathrm{Tr}[e^{-(2\pi/a)aH_R^{(+)}}], \tag{2.64}$$

where $O^{(+)}$ is an operator only for R_+ and $H_R^{(+)}$ is the Rindler Hamiltonian (the generator of η translation, i.e., the boost) restricted to there. Note that, for a uniformly accelerated observer, the generator of the τ translation is $aH_R^{(+)}$. That is, as far as the uniformly accelerated observer is concerned, the expectation value in the Minkowski vacuum of the operator can be seen as an ensemble average over the density matrix which is the same form as the canonical ensemble at temperature $T=\hbar a/2\pi$. This effective canonical ensemble is true even for interacting fields [16,17].

Equivalently, one can see the thermal character from the periodicity of the propagator in imaginary time [18,15]. Moreover, it was shown by Sewell [19] in the context of axiomatic quantum field theory that Rindler noise satisfies the Kubo-Martin-Schwinger (KMS) condition [8,20] (which is the definition of the thermal equilibrium with systems with infinite numbers of degrees of freedom) at temperature $T=\hbar a/2\pi$ for a general interacting field of any spin in any dimension. The intuitive explanation of Sewell's theorem is found in Ref. [10].

From these various versions of the thermalization theorem, one might conclude without calculations that Rindler noise is the same as thermal noise at temperature $T=\hbar a/2\pi$. However, these results *only* mean that a uniformly accelerated observer would see the Minkowski vacuum as the thermal equilibrium state but do *not* mean that the uniformly accelerated detector would respond in the same way as it would do at rest in the thermal bath, as was emphasized by Unruh and Wald [4] and Takagi [6]. The thermalization theorem, itself, says nothing about the noise. In fact, Rindler noise is different from thermal noise in several points.

Now, we shall calculate the Rindler noise. By using Eq. (2.54), the thermalization theorem, and the fluctuation-dissipation theorem, we can take a shortcut to calculate it. The fluctuation of the scalar field is defined by

$$\langle 0_M|\phi(\tau)\phi(\tau)|0_M\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \mathcal{F}_n(\omega), \tag{2.65}$$

where $\phi(\tau)=\phi(\xi(\tau),\boldsymbol{x}(\tau),\eta(\tau))$ is the value of the field along the world line of the detector and

$$\mathcal{F}_n(\omega) = \int d\tau' e^{-i\omega(\tau-\tau')}\langle 0_M|\phi(\tau)\phi(\tau')|0_M\rangle \tag{2.66}$$

is the power spectrum of Rindler noise. Then, by using Eq. (2.64), the fluctuation-dissipation theorem says that, for the massless case,

$$\begin{aligned}\int_{-\infty}^{\infty} d\omega \mathcal{F}_n(\omega) &= 4\int_0^{\infty} d\omega \left[\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\right]\mathcal{R}_n(\omega) \\ &= 2\int_{-\infty}^{\infty} d\omega \frac{\hbar\omega\mathcal{R}_n(|\omega|)}{e^{\beta\hbar\omega}-1} \\ &= 2\int_{-\infty}^{\infty} d\omega \frac{\hbar\omega R_n(|\omega|)}{e^{\beta\hbar\omega}-1} r_n(|\omega|) \\ &= 2\int_{-\infty}^{\infty} d\omega \frac{\hbar\omega R_n(|\omega|)}{e^{\beta\hbar\omega}-(-1)^n} d_n(\omega) \quad (\text{for } m=0),\end{aligned} \tag{2.67}$$

where

$$d_2(\omega)=1, \tag{2.68}$$

$$d_3(\omega)=1\times\frac{\omega}{|\omega|}, \tag{2.69}$$

$$d_4(\omega)=1, \tag{2.70}$$

$$d_5(\omega)=\left[1+\left(\frac{a}{2\omega}\right)^2\right]\times\frac{\omega}{|\omega|}, \tag{2.71}$$

$$d_6(\omega)=\left[1+\left(\frac{a}{\omega}\right)^2\right]. \tag{2.72}$$

This result including the statistical inversion in the denominator of Eq. (2.67) is consistent with that of Takagi [6], except for a minor difference in $d_n(\omega)$ for odd n. (We believe that our result is the correct one.) Note that, in Ref. [6], the case of the complex scalar field was calculated, but the result is the same for real scalar fields as will be explained in the next section.

We have derived Eq. (2.67) on the basis of Eq. (2.64) and the fluctuation-dissipation theorem of a bosonic operator. This clearly shows that the statistics inversion is an ''apparent'' one not based on the basic principle of statistical mechanics but based on the ''temperature'' dependence of the dissipative coefficient.

III. GENERALIZATION TO COMPLEX SCALAR FIELD

Now, we generalize the analysis to that of a complex scalar field. The situation is the same as the real scalar case, except that Q and ϕ become non-Hermitian.

A. Thermal noise

For the thermal noise, Eqs. (2.3) and (2.4) are replaced by

$$S_{int}(Q,\phi)=\int dtd\vec{x}[Q(t)\phi(\vec{x},t)+Q^\dagger(t)\phi^\dagger(\vec{x},t)]\delta(\vec{x}-\vec{x}_0)$$
$$=\int dt[Q(t)\phi(\vec{x}_0,t)+Q^\dagger(t)\phi^\dagger(\vec{x}_0,t)], \tag{3.1}$$

$$S_0(\phi)=\int dtd\vec{x}\left[|\partial_t\phi|^2-\sum_{i=1}^{n-1}|\partial_i\phi|^2-m^2|\phi|^2\right]. \tag{3.2}$$

Thus, the effective equation of motion for the detector is

$$\left(\widetilde{\frac{\delta S_0}{\delta Q^\dagger}}\right)+K_n(\omega)\tilde{Q}(\omega)=0, \tag{3.3}$$

where $K_n(\omega)$ is the same function as the real scalar case, Eq. (2.13). Therefore, the dissipative coefficient is the *same* as the real scalar case, Eq. (2.19).

Then, we can use the fluctuation-dissipation theorem of a non-Hermitian operator [11]. The fluctuation of the field is now defined by

$$\langle\phi(\vec{x}_0,t)\phi^\dagger(\vec{x}_0,t)\rangle_\beta=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega F_n(\omega), \tag{3.4}$$

where

$$F_n(\omega)=\int dt' e^{-i\omega(t-t')}\langle\phi(\vec{x}_0,t)\phi^\dagger(\vec{x}_0,t')\rangle_\beta, \tag{3.5}$$

and the theorem says that

$$\int_{-\infty}^{\infty}d\omega F_n(\omega)=4\int_0^{\infty}d\omega\frac{\hbar\omega}{2}\left\{\left[1+\frac{1}{e^{\beta\hbar\omega}-1}\right]R_n(\omega)+\left[\frac{1}{e^{\beta\hbar\omega}-1}\right]R_n(-\omega)\right\}$$
$$=2\int_{-\infty}^{\infty}d\omega\frac{\hbar\omega R_n(-\omega)}{e^{\beta\hbar\omega}-1} \tag{3.6}$$
$$=\frac{2^{2-n}\pi^{(3-n)/2}}{\Gamma\left(\frac{n-1}{2}\right)}\int_{-\infty}^{\infty}d\omega\frac{\hbar\omega|\omega|^{n-4}}{e^{\beta\hbar\omega}-1}\quad(\text{for}\quad m=0). \tag{3.7}$$

B. Rindler noise

Similarly, for Rindler noise, Eqs. (2.35) and (2.36) are replaced by

$$S_{int}(Q,\phi)=\int d\tau d\eta d\xi d\boldsymbol{x}[Q(\tau)\phi(\xi,\boldsymbol{x},\eta)+Q^\dagger(\tau)\phi^\dagger(\xi,\boldsymbol{x},\eta)]\delta(\xi-a^{-1})\delta(\eta-a\tau)\times\delta(\boldsymbol{x}-\boldsymbol{x}_0)$$
$$=\int d\tau[Q(\tau)\phi(a^{-1},\boldsymbol{x}_0,a\tau)+Q^\dagger(\tau)\phi^\dagger(a^{-1},\boldsymbol{x}_0,a\tau)], \tag{3.8}$$

$$S_0(\phi)=\int d\eta d\xi d\boldsymbol{x}\xi\left[\frac{1}{\xi^2}|\partial_\eta\phi|^2-|\partial_\xi\phi|^2-\sum_{i=2}^{n-1}|\partial_i\phi|^2-m^2|\phi|^2\right], \tag{3.9}$$

and the effective equation of motion becomes

$$\left(\widetilde{\frac{\delta S_0}{\delta Q^\dagger}}\right)+\mathcal{K}_n(\omega)\tilde{Q}(\omega)=0, \tag{3.10}$$

where $\mathcal{K}_n(\omega)$ is the same function as the real scalar case, Eq. (2.49). So the dissipative coefficient is the *same* as the real scalar case, Eq. (2.54), again.

The fluctuation of the field is now defined by

$$\langle 0_M|\phi(\tau)\phi^\dagger(\tau)|0_M\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\mathcal{F}_n(\omega), \tag{3.11}$$

where

$$\mathcal{F}_n(\omega)=\int d\tau' e^{-i\omega(\tau-\tau')}\langle 0_M|\phi(\tau)\phi^\dagger(\tau')|0_M\rangle, \tag{3.12}$$

and by using the fluctuation-dissipation theorem of a non-Hermitian operator [11] and Eq. (2.64), one finds that

$$\int_{-\infty}^{\infty}d\omega\mathcal{F}_n(\omega)=4\int_0^{\infty}d\omega\frac{\hbar\omega}{2}\left\{\left[1+\frac{1}{e^{\beta\hbar\omega}-1}\right]\mathcal{R}_n(\omega)+\left[\frac{1}{e^{\beta\hbar\omega}-1}\right]\mathcal{R}_n(-\omega)\right\}$$
$$=2\int_{-\infty}^{\infty}d\omega\frac{\hbar\omega\mathcal{R}_n(-\omega)}{e^{\beta\hbar\omega}-1} \tag{3.13}$$
$$=2\int_{-\infty}^{\infty}d\omega\frac{\hbar\omega R_n(-\omega)}{e^{\beta\hbar\omega}-(-1)^n}d_n(\omega)\quad(\text{for}\quad m=0). \tag{3.14}$$

in agreement with the previous analysis [6].

IV. GENERALIZATION TO A DIRAC FIELD

Next, we generalize to a Dirac field, using the fluctuation-dissipation theorem of a fermionic operator [11]. The situation is the same as the scalar case. However, the internal degree of freedom of the detector is now a spinor Θ and the field is a Dirac field ψ.

A. Thermal noise

For thermal noise, the action of the total system is

$$S=S_0(\Theta)+S_{int}(\Theta,\psi)+S_0(\psi), \tag{4.1}$$

where

$$S_0(\Theta)=\int dt L(\Theta,\dot{\Theta}), \tag{4.2}$$

$$\begin{aligned} S_{int}(\Theta,\psi)&=\int dt d\vec{x}[\bar{\Theta}(t)\psi(\vec{x},t)+\bar{\psi}(\vec{x},t)\Theta(t)]\delta(\vec{x}-\vec{x}_0)\\ &=\int dt[\bar{\Theta}(t)\psi(\vec{x}_0,t)+\bar{\psi}(\vec{x}_0,t)\Theta(t)], \end{aligned} \tag{4.3}$$

$$S_0(\psi)=\int dt d\vec{x}\bar{\psi}[i\gamma^\mu\partial_\mu-m]\psi, \tag{4.4}$$

and

$$\bar{\psi}\equiv\psi^\dagger\gamma^0, \quad \bar{\Theta}\equiv\Theta^\dagger\gamma^0. \tag{4.5}$$

Here, γ^μ are the γ matrices which satisfy

$$\{\gamma^\mu,\gamma^\nu\}=-2\eta^{\mu\nu}. \tag{4.6}$$

The equations of motion become

$$\left(\frac{\delta_L S_0}{\delta\bar{\Theta}}\right)+\int d\vec{x}\psi(\vec{x},t)\delta(\vec{x}-\vec{x}_0)=0, \tag{4.7}$$

$$\partial_t^2\psi-\sum_i\partial_i^2\psi+m^2\psi-[i\gamma^\mu\partial_\mu+m]\Theta(t)\delta(\vec{x}-\vec{x}_0)=0, \tag{4.8}$$

where L denotes the left derivative. After Fourier transformation and elimination of the field, one finds that the effective equation of motion for the detector is

$$\left(\widetilde{\frac{\delta_L S_0}{\delta\bar{\Theta}}}\right)+K_{1/2,n}(\omega)\tilde{\Theta}(\omega)=0, \tag{4.9}$$

where

$$K_{1/2,n}(\omega)=\int\frac{d\vec{k}}{(2\pi)^{n-1}}\frac{-\omega\gamma^0+\vec{k}\cdot\vec{\gamma}+m}{-\omega^2+|\vec{k}|^2+m^2\pm i\epsilon}. \tag{4.10}$$

(The subscript ''1/2'' denotes the spin of the field.) Since the $\vec{k}\cdot\vec{\gamma}$ term is odd in $\vec{k}$, we can relate $K_{1/2,n}(\omega)$ to that of the scalar case, Eq. (2.13):

$$K_{1/2,n}(\omega)=[-\omega\gamma^0+m]K_n(\omega). \tag{4.11}$$

Therefore, we define the dissipative coefficient as

$$\begin{aligned} R_{1/2,n}(\omega)&\equiv-\frac{\operatorname{Im}K_{1/2,n}(\omega)}{-\omega}\\ &=-[-\omega\gamma^0+m]R_n(\omega), \end{aligned} \tag{4.12}$$

where $R_n(\omega)$ is that of the scalar case, Eq. (2.18). This extra $-$ sign is a convention; it was chosen so that the dissipative coefficient is ''positive'' in the sense that

$$\operatorname{Tr}\gamma^0 R_{1/2,n}(\omega)\propto\omega R_n(\omega)\geqslant 0 \quad \text{for} \quad \omega\geqslant 0, \tag{4.13}$$

and this convention is the same as in Ref. [11]. We also define

$$\bar{R}_{1/2,n}(-\omega)\equiv-R_{1/2,n}(-\omega)=[\omega\gamma^0+m]R_n(-\omega) \tag{4.14}$$

and

$$\operatorname{Tr}\gamma^0\bar{R}_{1/2,n}(-\omega)\propto\omega R_n(-\omega)\geqslant 0 \quad \text{for} \quad \omega\geqslant 0. \tag{4.15}$$

Then, we can use the fluctuation-dissipation theorem of a fermionic operator [11]. The thermal fluctuation of the Dirac field is defined by

$$\begin{aligned} \langle\psi(\vec{x}_0,t)\bar{\psi}(\vec{x}_0,t)\rangle_\beta&\equiv\operatorname{Tr}[e^{-\beta H}\psi(\vec{x}_0,t)\bar{\psi}(\vec{x}_0,t)]/\operatorname{Tr}[e^{-\beta H}]\\ &=\int dt'\langle\psi(\vec{x}_0,t)\bar{\psi}(\vec{x}_0,t')\rangle_\beta\delta(t-t')\\ &=\frac{1}{2\pi}\int dt'd\omega e^{-i\omega(t-t')}\\ &\quad\times\langle\psi(\vec{x}_0,t)\bar{\psi}(\vec{x}_0,t')\rangle_\beta. \end{aligned} \tag{4.16}$$

The power spectrum of the Dirac field is defined by

$$F_{1/2,n}(\omega)\equiv\Delta_n^{-1}\operatorname{Tr}\left[\gamma^0\int dt'e^{-i\omega(t-t')}\langle\psi(\vec{x}_0,t)\bar{\psi}(\vec{x}_0,t')\rangle_\beta\right], \tag{4.17}$$

where Δ_n is the dimension of the γ matrices. Therefore,

$$\operatorname{Tr}[\gamma^0\langle\psi(\vec{x}_0,t)\bar{\psi}(\vec{x}_0,t)\rangle_\beta]=\frac{\Delta_n}{2\pi}\int_{-\infty}^{\infty}d\omega F_{1/2,n}(\omega). \tag{4.18}$$

Then, the theorem [11] says that

$$\int_{-\infty}^{\infty} d\omega F_{1/2,n}(\omega) = \frac{4}{\Delta_n}\mathrm{Tr}\Bigg[\gamma^0\int_0^{\infty} d\omega\frac{\hbar\omega}{2}\times\Bigg\{\Bigg[1-\frac{1}{e^{\beta\hbar\omega}+1}\Bigg]R_{1/2,n}(\omega)+\Bigg[\frac{1}{e^{\beta\hbar\omega}+1}\Bigg]\overline{R}_{1/2,n}(-\omega)\Bigg\}\Bigg]$$
$$=\frac{2}{\Delta_n}\int_{-\infty}^{\infty} d\omega\frac{\hbar\omega}{e^{\beta\hbar\omega}+1}\mathrm{Tr}[\gamma^0\overline{R}_{1/2,n}(-\omega)]$$
$$=2\int_{-\infty}^{\infty} d\omega\frac{\hbar\omega^2 R_n(-\omega)}{e^{\beta\hbar\omega}+1}. \tag{4.19}$$

This result is consistent with the previous result. (See, for example, Ref. [6].) The differences between this case and the scalar case are the power of ω in the numerator, which comes from the fact that the Dirac equation is first order with respect to the time derivative, and the Fermi distribution factor, which is provided by the fluctuation-dissipation theorem of a fermionic operator.

B. Rindler noise

For Rindler noise, one might define the Rindler noise of a Dirac field as

$$\langle 0_M|\psi(\tau)\overline{\psi}(\tau')|0_M\rangle. \tag{4.20}$$

However, this noise is not stationary, i.e., is not a function of $\tau-\tau'$. Instead, we define the Rindler noise of a Dirac field as [6,15]

$$\langle 0_M|\hat{\psi}(\tau)\overline{\hat{\psi}}(\tau')|0_M\rangle, \tag{4.21}$$

where

$$\hat{\psi}(\tau)=\exp\Bigg[-\frac{1}{2}a\tau\gamma^0\gamma^1\Bigg]\psi(\tau)=[\cosh(a\tau/2)-\gamma^0\gamma^1\sinh(a\tau/2)]\psi(\tau)\equiv S_\tau\psi(\tau). \tag{4.22}$$

This transformation is the Lorentz transformation, the boost, from the laboratory frame to the instantaneously comoving frame of a uniformly accelerated observer at τ.

Thus, we write the action in terms of

$$\hat{\psi}(\xi,\boldsymbol{x},\eta)\equiv S_{\eta/a}\psi(\xi,\boldsymbol{x},\eta), \tag{4.23}$$

$$\hat{\Theta}(\tau)\equiv S_\tau\Theta(\tau). \tag{4.24}$$

Then, the action of the total system is

$$S=S_0(\hat{\Theta})+S_{int}(\hat{\Theta},\hat{\psi})+S_0(\hat{\psi}), \tag{4.25}$$

where

$$S_0(\hat{\Theta})=\int d\tau L(\hat{\Theta},\dot{\hat{\Theta}}), \tag{4.26}$$

$$S_{int}(\hat{\Theta},\hat{\psi})=\int d\tau d\eta d\xi d\boldsymbol{x}[\overline{\hat{\Theta}}(\tau)\hat{\psi}(\xi,\boldsymbol{x},\eta)+\overline{\hat{\psi}}(\xi,\boldsymbol{x},\eta)\hat{\Theta}(\tau)]\delta(\xi-a^{-1})\delta(\eta-a\tau)\times\delta(\boldsymbol{x}-\boldsymbol{x}_0), \tag{4.27}$$

$$S_0(\hat{\psi})=\int d\eta d\xi d\boldsymbol{x}\xi\overline{\hat{\psi}}\Bigg[i\gamma^0\frac{1}{\xi}\partial_\eta+i\gamma^1\Bigg(\frac{1}{2\xi}+\partial_\xi\Bigg)+i\sum_{i=2}^{n-1}\gamma^i\partial_i-m\Bigg]\hat{\psi}. \tag{4.28}$$

This $S_0(\hat{\psi})$ is the same as Eq. (4.4) but written in Rindler coordinates and $\hat{\psi}$. (If we introduce the vielbein, calculate the spin connection in Rindler space, and define the spinor with respect to the local Lorentz transformation, we would get this action.)

The equations of motion become

$$\Bigg(\frac{\delta_L S_0}{\delta\overline{\hat{\Theta}}}\Bigg)+\int d\eta d\xi d\boldsymbol{x}\hat{\psi}(\xi,\boldsymbol{x},\eta)\delta(\xi-a^{-1})\delta(\eta-a\tau)\times\delta(\boldsymbol{x}-\boldsymbol{x}_0)=0 \tag{4.29}$$

and

$$\frac{1}{\xi^2}\partial_\eta^2\hat{\psi}+\frac{1}{\xi^2}\gamma^0\gamma^1\partial_\eta\hat{\psi}+\frac{1}{4\xi^2}\hat{\psi}-\frac{1}{\xi}\partial_\xi(\xi\partial_\xi\hat{\psi})-\sum_i\partial_i^2\hat{\psi}+m^2\hat{\psi}-\Bigg[i\gamma^0\frac{1}{\xi}\partial_\eta+i\gamma^1\Bigg(\frac{1}{2\xi}+\partial_\xi\Bigg)+i\sum_i\gamma^i\partial_i+m\Bigg]\frac{1}{\xi}\times\int d\tau\hat{\Theta}(\tau)\delta(\xi-a^{-1})\delta(\eta-a\tau)\delta(\boldsymbol{x}-\boldsymbol{x}_0)=0. \tag{4.30}$$

In order to deal with the $\gamma^0\gamma^1$ term of the second equation, we define the projection operators

$$\gamma_\pm\equiv\Bigg(\frac{1\pm\gamma^0\gamma^1}{2}\Bigg), \tag{4.31}$$

which satisfy

$$\gamma_\pm^2=\gamma_\pm,\quad \gamma_+\gamma_-=0,\quad \gamma_++\gamma_-=1,\quad \gamma_\pm(\gamma^0\gamma^1)=\pm\gamma_\pm. \tag{4.32}$$

By using these projection operators, we can divide $\hat{\psi}$ into two parts:

$$\hat{\psi}=\hat{\psi}_++\hat{\psi}_-, \tag{4.33}$$

where $\hat{\psi}_\pm\equiv\gamma_\pm\hat{\psi}$. Then, the second equation becomes

$$\frac{1}{\xi^2}\left(\partial_\eta \pm \frac{1}{2}\right)^2 \hat{\psi}_\pm - \frac{1}{\xi}\partial_\xi(\xi\partial_\xi\hat{\psi}_\pm) - \sum_i \partial_i^2\hat{\psi}_\pm + m^2\hat{\psi}_\pm - \gamma_\pm\left[i\gamma^0\frac{1}{\xi}\partial_\eta + i\gamma^1\left(\frac{1}{2\xi}+\partial_\xi\right) + i\sum_i \gamma^i\partial_i + m\right]\frac{1}{\xi} \times \int d\tau \hat{\Theta}(\tau)\delta(\xi - a^{-1})\delta(\eta - a\tau)\delta(\boldsymbol{x}-\boldsymbol{x}_0) = 0. \tag{4.34}$$

In this case, instead of $K_{i\nu}(M_k\xi)$, we consider $K_{i\nu\pm 1/2}(M_k\xi)$ as

$$\hat{\Psi}_\pm(\nu,\boldsymbol{k},\Omega) \equiv \frac{1}{\pi}\int_0^\infty \frac{d\xi}{\xi}\int d\eta d\boldsymbol{x} K_{i\nu\pm 1/2}(M_k\xi) \times e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\Omega\eta}\hat{\psi}_\pm(\xi,\boldsymbol{x},\eta). \tag{4.35}$$

(See, for example, Ref. [21].) By the orthogonality relation

$$\frac{1}{\pi^2}\int_0^\infty \frac{dx}{x} K_{i\mu\pm 1/2}(x) K_{i\nu\pm 1/2}(x) = \frac{\delta(\mu-\nu)}{(\mp 2i\nu - 1)\cosh\pi\nu}, \tag{4.36}$$

which can be obtained from Eq. (2.44) by the shift $i\nu \to i\nu \pm \frac{1}{2}$, we can invert Eq. (4.35) as

$$\hat{\psi}_\pm(\xi,\boldsymbol{x},\eta) = \frac{1}{(2\pi)^{n-1}\pi}\int d\nu d\Omega d\boldsymbol{k}(\mp 2i\nu - 1)\cosh\pi\nu \times K_{i\nu\pm 1/2}(M_k\xi)e^{-i\boldsymbol{k}\cdot\boldsymbol{x} + i\Omega\eta}\hat{\Psi}_\pm(\nu,\boldsymbol{k},\Omega). \tag{4.37}$$

If we specialize to the massless case, then after a straightforward calculation, one finds that the effective equation of motion for the detector is

$$\left(\frac{\delta_L S_0}{\delta\bar{\hat{\Theta}}}\right) + \mathcal{K}_{1/2,n}(\omega)\tilde{\hat{\Theta}}(\omega) = 0, \tag{4.38}$$

where

$$\mathcal{K}_{1/2,n}(\omega) = \frac{a^{n-2}\Gamma\left(\frac{n-2}{2}\right)}{8\pi^{n/2+1}\Gamma(n-2)}\int d\nu\left[\frac{1}{-\omega/a+\nu+i\epsilon} - \frac{1}{\omega/a+\nu-i}\right]\frac{(2\nu-i)\cosh\pi\nu}{2\omega/a - i}\left(\frac{n-3}{2} - i\frac{\omega}{a}\right)(-\gamma^0+\gamma^1) \times\Gamma\left(\frac{n+1}{2}-1+i\nu\right)\Gamma\left(\frac{n-1}{2}-1-i\nu\right) - \left[\frac{1}{-\omega/a+\nu+i\epsilon} - \frac{1}{\omega/a+\nu+i}\right]\frac{(2\nu+i)\cosh\pi\nu}{2\omega/a+i}\left(\frac{n-3}{2}+i\frac{\omega}{a}\right) \times(\gamma^0+\gamma^1)\Gamma\left(\frac{n-1}{2}-1+i\nu\right)\Gamma\left(\frac{n+1}{2}-1-i\nu\right). \tag{4.39}$$

Note that the singularity on the real axis of ν is only at $\nu = \omega/a$. To avoid this singularity, we add the $+i\epsilon$ term. (The sign of the $i\epsilon$ term can be determined by causality, as above. Then, the sign is $+$ for both $\omega > 0$ and $\omega < 0$.) From this $+i\epsilon$ term, we can obtain the dissipative coefficient

$$\mathcal{R}_{1/2,n}(\omega) \equiv -\frac{\mathrm{Im}\,\mathcal{K}_{1/2,n}(\omega)}{-\omega} = \frac{2^{1-n}\pi^{(3-n)/2}}{\Gamma\left(\frac{n-1}{2}\right)}\frac{a^{n-1}}{\omega^2}\coth(\pi\omega/a) \times\left|\frac{\Gamma\left(\frac{n+1}{2}-1+i\frac{\omega}{a}\right)}{\Gamma\left(i\frac{\omega}{a}\right)}\right|^2\gamma^0 = \alpha_n\mathcal{R}_{n+1}(\omega)\coth(\pi\omega/a)\gamma^0, \tag{4.40}$$

where $\mathcal{R}_{n+1}(\omega)$ is that of the scalar case, Eq. (2.54), in $n+1$ dimensions, and

$$\alpha_n = \frac{2\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}. \tag{4.41}$$

Here, we have again included the extra $-$ sign as a convention in the definition of Eq. (4.40). We also define

$$\overline{\mathcal{R}}_{1/2,n}(-\omega) \equiv -\mathcal{R}_{1/2,n}(-\omega) = \alpha_n\mathcal{R}_{n+1}(-\omega)\coth(\pi\omega/a)\gamma^0. \tag{4.42}$$

Then, we can use the fluctuation-dissipation theorem of a fermionic operator [11]. The power spectrum of the Dirac field is defined by

$$\mathcal{F}_{1/2,n}(\omega) \equiv \Delta_n^{-1}\,\mathrm{Tr}\left[\gamma^0 \int dt'\, e^{-i\omega(t-t')}\langle 0_M|\hat{\psi}(\tau)\bar{\hat{\psi}}(\tau')|0_M\rangle\right] \tag{4.43}$$

and, then,

$$\mathrm{Tr}[\gamma^0\langle 0_M|\hat{\psi}(\tau)\bar{\hat{\psi}}(\tau)|0_M\rangle] = \frac{\Delta_n}{2\pi}\int_{-\infty}^{\infty} d\omega\,\mathcal{F}_{1/2,n}(\omega). \tag{4.44}$$

Thus, by using Eq. (2.64), the theorem [11] says that

$$\begin{aligned}\int_{-\infty}^{\infty} d\omega\,\mathcal{F}_{1/2,n}(\omega) &= \frac{4}{\Delta_n}\mathrm{Tr}\left[\gamma^0\int_0^{\infty} d\omega\,\frac{\hbar\omega}{2}\left\{\left[1-\frac{1}{e^{\beta\hbar\omega}+1}\right]\mathcal{R}_{1/2,n}(\omega)\right.\right.\\ &\quad \left.\left.+\left[\frac{1}{e^{\beta\hbar\omega}+1}\right]\overline{\mathcal{R}}_{1/2,n}(-\omega)\right\}\right]\\ &= \frac{2}{\Delta_n}\int_{-\infty}^{\infty} d\omega\,\frac{\hbar\omega}{e^{\beta\hbar\omega}+1}\,\mathrm{Tr}[\gamma^0\overline{\mathcal{R}}_{1/2,n}(-\omega)]\\ &= 2\alpha_n\int_{-\infty}^{\infty} d\omega\,\frac{\hbar\omega\mathcal{R}_{n+1}(-\omega)}{e^{\beta\hbar\omega}-1}.\end{aligned} \tag{4.45}$$

That is, the power spectrum of the Rindler noise of a massless Dirac field in n dimensions is proportional to that of a massless scalar field, Eq. (2.67), in $n+1$ dimensions. From the analysis of the scalar case, we can thus show the statistical inversion. This result nicely agrees with that of Takagi [6].

V. CONCLUSION

We have presented a simple and systematic method to evaluate Rindler noise. We have first calculated the dissipative coefficient *explicitly* from the equations of motion of the detector and the field. Then, by using the fluctuation-dissipation theorem, we have obtained the Rindler noise, which is relevant to the analysis of the Unruh effect. This method is generalized to the Dirac field, by using the fermionic fluctuation-dissipation theorem [11]. These results are perfectly consistent with previously known results including apparent statistical inversion in odd dimensions.

Although Rindler noise can be calculated *directly* from the action of the field as Takagi [6] did, we emphasize that there are several advantages in calculating the noise *indirectly*. That is, to introduce the detector, calculate the dissipative coefficient and then use the fluctuation-dissipation theorem, as we have done in this paper.

First of all, it contains only *classical* calculations except for the fluctuation-dissipation theorem. To calculate the dissipative coefficient, we only have to eliminate the degree of freedom of the field from the classical equations of motion of the detector and field. Thus, the calculations become much simpler.

Next, this method gives a hint as to the origin of the apparent statistics inversion effect. While the dissipative coefficient for thermal noise does not depend on the temperature in an ideal setting, those for Rindler noise inevitably depend on the ''temperature'' (the acceleration). This temperature dependence of the dissipative coefficient destroys the simple Bose (or Fermi) distribution provided by the fluctuation-dissipation theorem and causes the phenomenon of the ''apparent'' inversion of statistics in odd dimensions [6]. Why does this difference happen? From our calculation, the answer is obvious. Note that the entire effect of the field on the detector is represented by the influence functional [9] which is constructed from the action of the field, the detector-field interaction, and the initial condition of the field. (The influence functional for a scalar field in the Minkowski vacuum, coupled to a uniformly accelerating DeWitt detector, was derived by Anglin [22].) In the thermal case, the temperature is the initial condition. On the other hand, in the Rindler case, the temperature is in the detector-field interaction as the acceleration. Because it is only the action of the field and the detector-field interaction that are needed to obtain the effective equation of motion and thus the dissipative coefficient, the dissipative coefficient for thermal noise does not depend on the temperature in our idealized treatment but those for Rindler noise inevitably depend on the temperature. (Of course, this point of view would not explain everything about the apparent statistics inversion effect.)

Finally, we can see that the fermionic fluctuation-dissipation theorem [11] works properly. In the context of condensed matter physics, it is generally difficult to use the fermionic fluctuation-dissipation theorem because we usually measure bosonic quantities such as voltage or electric current. However, we have applied this fermionic version of the theorem to the Unruh effect and have obtained the right result including apparent statistical inversion in odd dimensions. Thus, we have confirmed that the fermionic fluctuation-dissipation theorem is indeed working properly.

Since the relation between Rindler and Minkowski coordinates is very similar to the relation between Schwarzschild and Kruskal coordinates, there is hope that the above method could be applicable to Hawking radiation [23,24] also. This would be a future work.

ACKNOWLEDGMENTS

The author thanks K. Fujikawa for introducing him to this subject and helpful discussions. The author also thanks Y. Nomura for discussions and encouragement.

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