Decay of quantum Loschmidt echo and fidelity in the broken phase of the Lipkin-Meshkov-Glick model

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Analytical expressions are derived for the fidelity and for the quantum Loschmidt echo in the broken phase of the Lipkin-Meshkov-Glick (LMG) model for large particle number N. A detailed comparison is given between this model and some other models for the decaying behaviors of the two quantities in the neighborhood of critical points. The two quantities show qualitatively different decaying behaviors on the two sides of the critical point. Specifically, in the broken phase they behave in a way similar to those in models such as the one-dimensional Ising chain in a transverse field. In contrast, in the symmetric phase, the similarity is to those in models such as the Dicke model.

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I. INTRODUCTION

In recent years, quantum phase transitions (QPTs) [1,2], at which the fundamental properties of the ground states of systems undergo dramatic changes, have attracted lots of attention. In addition to traditional concepts such as the order parameter, they can be also characterized by concepts borrowed from other fields, e.g., entanglement and concurrence [3–12]. In the neighborhood of a critical point λ_c, a small variation in the controlling parameter λ may induce large changes in the wave functions. This leads to the study of two other quantities at QPTs, one being the fidelity, defined as the overlap of the ground states corresponding to two neighboring values of λ (say, λ and λ'), and the other being the quantum Loschmidt echo (LE), which gives a measure of the deviation of quantum motions under λ and λ' for the same initial state. At QPTs, dramatic decay has been observed both in the fidelity (with respect to λ) [13–31] and in the LE (with respect to both λ and time) [32–39].

The classification of QPTs is an important topic and, up to now, a complete understanding of it has not yet been achieved. It would be of interest to study whether the decaying behaviors of the fidelity and of the LE may contribute to it. Indeed, at least two types of decaying behaviors have been observed at QPTs. On one hand, in systems such as the Dicke model [40] and the symmetric phase of the Lipkin-Meshkov-Glick (LMG) model [41], it has been found that the fidelity is a function of the ratio $(\lambda-\lambda_c)/(\lambda'-\lambda_c)$ only [36] and the LE has a power-law decay [37,38]. On the other hand, in systems such as the one-dimensional (1D) Ising chain in a transverse field, the fidelity depends separately on this ratio η and on the difference $(\lambda-\lambda')$ [29,30], and the LE has an exponential decay [32,33]. Interestingly, one may notice the following difference between these two types of systems, that is, in the Dicke model and in the LMG model the Hamiltonians are independent of the positions of the systems' particles, and, as a result, the spatial dimension d_s is in fact irrelevant, while in models such as the Ising chain the value of d_s is relevant. The above observations suggest that the decaying behaviors of the fidelity and of the LE could be useful in the classification of QPTs.

The situation with the broken phase of the LMG model is still unclear. Numerical simulations show that this model has the following property: The decay rate of the fidelity is linear in the particle number N in the broken phase, in contrast to the finite decay rate in the symmetric phase in the thermodynamic limit [26,31,42]. This looks peculiar, since, in most of the models that have been studied, the fidelity (also the LE) shows the same type of decaying behavior on the two sides of the critical point. In fact, as shown in Ref. [43], the purity also behaves differently on the two sides of the critical point of the LMG model. Also, the decaying behavior of the LE in the broken phase of the LMG model has yet to be studied.

Therefore, it would be of interest to study analytically both the fidelity and the LE in the broken phase of the LMG model. One may note that since the LE and the fidelity involve two values of the controlling parameter, analytical techniques developed in previous studies (e.g., in Refs. [12,44–48]) are not directly applicable to this study. In this paper, we derive analytical expressions of the LE and of the fidelity in this model for large N. Our results show that the decaying behaviors of the LE and of the fidelity in the broken phase are indeed qualitatively different from those in the symmetric phase. In fact, unexpectedly, they are somewhat similar to those observed in systems such as the 1D Ising chain in a transverse field.

The paper is structured as follows. In Sec. II, we study the relation between eigenfunctions of the Hamiltonian in the LMG model, under different values of a controlling parameter. Then, in Sec. III, making use of results obtained in Sec. II, we derive an analytical expression of the fidelity and discuss its properties. Section IV is devoted to the derivation of an analytical expression of the LE, as well as detailed discussions of its properties. In Sec. V, we make a comparison between the decaying behaviors of the LE and of the fidelity in the LMG model and the behaviors found in other models. Finally, conclusions and discussions are given in Sec. VI.

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II. THE LMG MODEL AND RELATION BETWEEN ITS EIGENFUNCTIONS

We study the so-called two-orbital LMG model for N particles, which was first proposed to describe the shape-phase transition in nuclei physics [41]. In this section, we derive an expression for the relation between eigenfunctions of the Hamiltonian under different values of a controlling parameter, which will be used when computing the fidelity and the LE in the following sections.

A. The Hamiltonian and its diagonalization

In terms of the Pauli matrices σ_α ($\alpha = x,y,z$), the Hamiltonian in the LMG model can be written as [12,44–48]

$$H(\gamma,h) = -h\sum_{i=1}^{N}\sigma_z^i - \frac{\lambda}{N}\sum_{i<j}\left(\sigma_x^i\sigma_x^j + \gamma\sigma_y^i\sigma_y^j\right), \quad (1)$$

with parameters h and λ. Without loss of generality, we set $\lambda = 1$. We study the collective motion in this model and consider the parameter regime of $0 \leqslant \gamma < 1.0$ and $h > 0$. It proves convenient to write the Hamiltonian in terms of the total spin operator $S_\alpha = \sum_i \sigma_\alpha^i/2$, namely,

$$H(\gamma,h) = -\frac{2}{N}\left(S_x^2 + \gamma S_y^2\right) - 2hS_z + (1+\gamma)/2. \quad (2)$$

It is easy to verify that H is commutable with $\mathbf{S}^2$ and has a spin-flip symmetry, i.e.,

$$[H,\mathbf{S}^2] = 0 \quad \text{and} \quad \left[H, \prod_i \sigma_z^i\right] = 0. \quad (3)$$

In what follows, we consider only the maximum spin sector with $S = N/2$, to which the ground state belongs.

It is known that, in the thermodynamic limit $N \to \infty$, the system undergoes a QPT at a critical value $h_c = 1$, with a symmetric phase (SP) at $h > 1$ and a broken phase (BP) at $h < 1$ [41,44–50]. In the neighborhood of this critical point, the properties of the system change dramatically with a variation of h, but usually not with γ. Therefore, in what follows, we consider the variation of h only and take γ at a fixed value in the numerical simulations to be discussed below, namely, $\gamma = 0.5$. Consistently, we write the Hamiltonian as H_h.

To diagonalize the Hamiltonian, one may make use of the Holstein-Primakoff representation [51],

$$S_z = S - a^\dagger a, \quad (4)$$

$$S_+ = (2S - a^\dagger a)^{1/2} a, \quad (5)$$

$$S_- = a^\dagger (2S - a^\dagger a)^{1/2}, \quad (6)$$

where $S_\pm = S_x \pm iS_y$. The bosonic creation and annihilation operators obey $[a,a^\dagger] = 1$. Inserting these expressions into Eq. (2), one gets

$$\begin{aligned} H_h = &-\frac{1}{2}\left[\sqrt{1-\frac{a^\dagger a}{2S}}\,a + a^\dagger\sqrt{1-\frac{a^\dagger a}{2S}}\right]^2 \\ &+\frac{\gamma}{2}\left[\sqrt{1-\frac{a^\dagger a}{2S}}\,a - a^\dagger\sqrt{1-\frac{a^\dagger a}{2S}}\right]^2 \\ &-2h(S - a^\dagger a) + \frac{1+\gamma}{2}. \end{aligned} \quad (7)$$

In the ground state of the broken phase, $a^\dagger a$ gives a contribution of the order N, hence, in Ref. [45], a further rotation is performed for the total spin operator. Here, we adopt an alternative technique which will make it easier to derive the relation between eigenstates of different H_h. That is, we perform the following transformation of the creation and annihilation operators,

$$b^\dagger = a^\dagger - \beta, \quad (8)$$

with $\beta \geqslant 0$. We assume that $\beta \sim N^\delta$ with $0 \leqslant \delta \leqslant 1/2$. Substituting Eq. (8) into the Hamiltonian in Eq. (7), we obtain

$$\begin{aligned} H_h = &-\frac{k}{4S}\{[g(b+\beta) + (b^\dagger+\beta)g]^2 \\ &-\gamma[g(b+\beta) - (b^\dagger+\beta)g]^2\} \\ &+2hb^\dagger b + 2h\beta(b^\dagger + b) + 2h\beta - 2hS + \frac{1+\gamma}{2}, \end{aligned} \quad (9)$$

where

$$g = \sqrt{1 - \frac{b^\dagger b + \beta(b^\dagger + b)}{k}}, \quad (10)$$

with $k = 2S - \beta^2$.

Then, under the condition that $b^\dagger b \ll N$, we expand the function g in the power of $1/k$ and substitute the result into the Hamiltonian H_h in Eq. (9). This gives an expansion of H_h, composed of terms proportional to β^l/k^m with $l \geqslant 0$ and $m \geqslant 0$. It is easy to see that $\beta^l/k^m \sim N^{l\delta - m}$. Under the assumption of $\delta \in [0,1/2]$, for low-lying states and for large N, one may neglect all the terms for which $m > l/2$ and get

$$\begin{aligned} H_h = &\left(2h + \frac{2\beta^2}{S}\right)b^\dagger b + \frac{2\beta^2 - k}{4S}(b^\dagger + b)^2 \\ &+\frac{k\gamma}{4S}(b^\dagger - b)^2 + \frac{\beta^2}{2S}[b^{\dagger 2} + b^2] \\ &+\left[2h\beta - \frac{k\beta}{S}\left(1 - \frac{\beta^2}{k}\right)\right](b^\dagger + b) \\ &+2h\beta^2 - 2hS - \frac{\beta^2 k}{S} + \frac{1+\gamma}{2}. \end{aligned} \quad (11)$$

The linear terms in Eq. (11) can be eliminated at the following values of β,

$$\beta = 0 \quad \text{or} \quad \beta = \sqrt{S(1-h)}. \quad (12)$$

Note that these two values of β are consistent with the above assumption of $\beta \sim N^\delta$. Then, the Hamiltonian is written as

$$H_h = \Delta b^\dagger b + \Gamma(b^{\dagger 2} + b^2), \quad (13)$$

where a constant term has been omitted and

$$\Delta = 2h + \frac{6\beta^2 - k(1+\gamma)}{2S}, \quad (14)$$

$$\Gamma = \frac{4\beta^2 - k(1-\gamma)}{4S}. \quad (15)$$

The Hamiltonian in Eq. (13) is bilinear and can be diagonalized by a standard Bogoliubov transformation,

$$c^\dagger = \cosh\left(\frac{\Theta}{2}\right)b^\dagger - \sinh\left(\frac{\Theta}{2}\right)b, \tag{16}$$

$$c = -\sinh\left(\frac{\Theta}{2}\right)b^\dagger + \cosh\left(\frac{\Theta}{2}\right)b. \tag{17}$$

The result is

$$H_h = \varepsilon_h c^\dagger c, \tag{18}$$

where again some constant term has also been omitted, and

$$\varepsilon_h = \sqrt{\Delta^2 - 4\Gamma^2}, \tag{19}$$

$$\tanh\Theta = -\frac{2\Gamma}{\Delta}. \tag{20}$$

To write ε_h and $\tanh\Theta$ explicitly, in the case of $\beta = 0$,

$$\varepsilon_h = 2\left[(h-1)(h-\gamma)\right]^{1/2}, \tag{21}$$

$$\tanh\Theta = \frac{1-\gamma}{2h-1-\gamma}, \tag{22}$$

and in the case of $\beta = \sqrt{S(1-h)}$,

$$\varepsilon_h = 2[(1-h^2)(1-\gamma)]^{1/2}, \tag{23}$$

$$\tanh\Theta = \frac{(1+h)(1-\gamma) - 4(1-h)}{6-2h-(1+h)(1+\gamma)}. \tag{24}$$

From Eqs. (21) and (23), it is seen that $\varepsilon_h = 0$ at $h = 1$, indicating that $h_c = 1$ is indeed a critical point. In the study of this critical point, in the symmetric phase with $h > 1$, one should take $\beta = 0$, while in the broken phase with $h < 1$, one should take $\beta = \sqrt{S(1-h)}$ [52].

B. Relation between eigenfunctions of different H_h

In order to find the relation between eigenstates of the Hamiltonian with different values of the parameter h, we write the Hamiltonian in Eq. (18) in terms of (rescaled) position and momentum operators, namely,

$$Q = \frac{1}{\sqrt{2\varepsilon_h}}(c^\dagger + c), \quad P = i\sqrt{\frac{\varepsilon_h}{2}}(c^\dagger - c). \tag{25}$$

The result is

$$H_h = \tfrac{1}{2}\left\{\varepsilon_h^2 Q^2 + P^2\right\}, \tag{26}$$

where again a constant term has been omitted. Then, as given in standard textbooks, in the Q representation, the eigenfunction corresponding to an eigenenergy $n\varepsilon_h$ ($n = 0,1,2,\ldots$) has the following expression,

$$\Psi_{n,h}(Q) = \sqrt{\frac{1}{2^n n!}}\left(\frac{\varepsilon_h}{\pi}\right)^{1/4}\exp\left\{-\frac{\varepsilon_h}{2}Q^2\right\}H_n(\sqrt{\varepsilon_h}Q), \tag{27}$$

where H_n indicates Hermitian polynomials.

Later, when computing the fidelity and the LE, we will need to use the overlap between the ground state of $H_{h'}$ and the nth eigenstate of H_h, which we denote by C_n,

$$C_n = \langle\Psi_{0,h'}|\Psi_{n,h}\rangle. \tag{28}$$

Here and hereafter, we use the prime to indicate quantities related to a value h' of the controlling parameter. To compute C_n, making use of Eqs. (16) and (17), we write the position and momentum operators in terms of the bosonic mode $(b, b^\dagger)$,

$$Q = \frac{1}{\sqrt{2(\Delta - 2\Gamma)}}(b^\dagger + b), \quad P = i\sqrt{\frac{\Delta - 2\Gamma}{2}}(b^\dagger - b). \tag{29}$$

From Eq. (8), it is seen that the two bosonic modes $b^\dagger$ and $b'^\dagger$ satisfy the following relation,

$$b'^\dagger = b^\dagger + \delta\beta, \tag{30}$$

where $\delta\beta = \beta - \beta'$. Then, it is straightforward to find the following relation between the position operators corresponding to h and h',

$$Q' = \kappa(Q + \zeta), \tag{31}$$

where

$$\kappa = \sqrt{\frac{\Delta - 2\Gamma}{\Delta' - 2\Gamma'}}, \tag{32}$$

$$\zeta = \sqrt{\frac{2}{\Delta - 2\Gamma}}\delta\beta. \tag{33}$$

Making use of Eq. (27) for the parameter value h' and Eq. (31), it is found that the eigenstate $|\Psi_{n,h'}\rangle$ of $H_{h'}$ has the following expression in the Q representation,

$$\Psi_{n,h'}(Q) = \sqrt{\frac{1}{2^n n!}}\left(\frac{\kappa^2\varepsilon_{h'}}{\pi}\right)^{1/4}\exp\left[-\frac{\kappa^2\varepsilon_{h'}}{2}(Q+\zeta)^2\right] \times H_n[\kappa\sqrt{\varepsilon_{h'}}(Q+\zeta)]. \tag{34}$$

Making use of Eqs. (27) and (34), with the help of the software *Mathematica*, we have computed the overlap C_n, getting

$$C_n = \xi^{1/2}\sqrt{\frac{\lambda^n}{2^n n!}}\exp\left\{-\frac{\kappa^2\varepsilon_h\varepsilon_{h'}}{2(\varepsilon_h + \kappa^2\varepsilon_{h'})}\zeta^2\right\}H_n(\mu), \tag{35}$$

where $H_n(\mu)$ are also Hermitian polynomials and

$$\xi = \frac{2\kappa\sqrt{\varepsilon_h\varepsilon_{h'}}}{\varepsilon_h + \kappa^2\varepsilon_{h'}}, \quad \lambda = \frac{\varepsilon_h - \kappa^2\varepsilon_{h'}}{\varepsilon_h + \kappa^2\varepsilon_{h'}}, \quad \mu = \frac{\kappa\varepsilon_h\sqrt{\varepsilon_{h'}}}{\sqrt{\varepsilon_h^2 - \kappa^4\varepsilon_{h'}^{\prime 2}}}\zeta. \tag{36}$$

III. DECAY OF THE FIDELITY

In this section, making use of results obtained in the previous section, we derive an analytical expression of the fidelity, which is valid in both phases of the LMG model, and discuss its properties. In the symmetric phase, our result reduces to those given in Refs. [31,36]; in the broken phase, our result predicts an N dependence of the fidelity, which was observed numerically previously [31].

The fidelity given by the overlap of the ground states of two neighboring Hamiltonians H_h and $H_{h'}$ is often studied at QPTs, namely,

$$F = |\langle\Psi_{0,h}|\Psi_{0,h'}\rangle|. \tag{37}$$

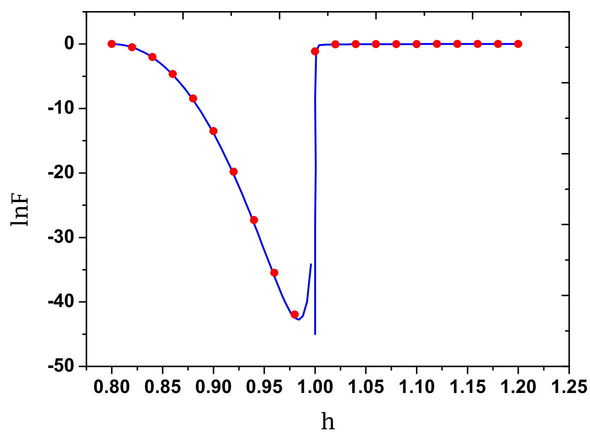

FIG. 1. (Color online) Variation of the fidelity (in the logarithm scale) with a controlling parameter h. Solid circles represent direct numerical simulations and curves are predictions of Eq. (38). The critical point is at $h_c = 1$. Parameters: $N = 2^{12}$, $\gamma = 0.5$, $h' = 1.2$ for $h > 1$ and $h' = 0.8$ for $h < 1$.

It is just $|C_0|$ given by Eq. (35). After some algebra, it is found that

$$F = F_1 F_2, \tag{38}$$

where

$$F_1 = \left[1 - \tanh^2\left(\frac{\Theta' - \Theta}{2}\right)\right]^{1/4}, \tag{39}$$

$$F_2 = \exp\left\{-\frac{(\delta\beta)^2}{e^{\Theta'} + e^{\Theta}}\right\}. \tag{40}$$

As seen in Fig. 1, predictions of this analytical result have good agreement with direct numerical simulations.

The fidelity has a different N dependence in the two phases. In the symmetric phase, $\delta\beta = 0$ ($\beta = \beta' = 0$), hence, $F_2 = 1$ and the fidelity reduces to F_1 in Eq. (39), already given in Ref. [36]. Clearly, F_1 is N independent. Its deviation from the exact fidelity is of the scale $1/\sqrt{N}$. On the other hand, in the broken phase,

$$\delta\beta = \sqrt{N}[\sqrt{(1-h)/2} - \sqrt{(1-h')/2}], \tag{41}$$

and as a result, $\ln F_2 \propto -N$ [see Eq. (40)]. This implies that, for large N, the logarithm of the fidelity is linear in the system's size N, a phenomenon observed numerically in Ref. [31].

Furthermore, the fidelity has different scaling behaviors on the two sides of the critical point $h_c = 1$. On one hand, in the symmetric phase, as shown in Ref. [36], in the thermodynamic limit, the fidelity is in fact a function of the ratio

$$\eta = (h-1)/(h'-1), \tag{42}$$

namely,

$$F = F_1 \simeq G_1(\eta) \equiv \frac{\sqrt{2}\sqrt[8]{\eta}}{\sqrt{\sqrt{\eta}+1}}. \tag{43}$$

Clearly, $F \neq 0$ for $0 < \eta < \infty$ and $F \to 0$ when η goes to either 0 or ∞ (cf. Fig. 1 for an illustration of the case of $\eta \to 0^+$). On the other hand, in the broken phase and in the neighborhood of the critical point, making use of Eq. (24) and the relation $e^{\Theta} = \sqrt{(1+\tanh\Theta)/(1-\tanh\Theta)}$, it is easy to find that $e^{\Theta} \simeq \sqrt{(1-\gamma)/(2\epsilon)}$, where $\epsilon = 1 - h$. Substituting this result into Eq. (40), we get

$$\ln F_2 \simeq -N(\epsilon)^{3/2} G_2(\eta), \tag{44}$$

where

$$G_2(\eta) = \frac{(\sqrt{\eta}-1)^2}{\eta(\sqrt{\eta}+1)\sqrt{2(1-\gamma)}}. \tag{45}$$

Combining this result with the expression of F_1 in Eq. (43), it is seen that the fidelity has the following scaling behavior in the broken phase,

$$\ln F \simeq -N(\epsilon)^{3/2} G_2(\eta) + \ln G_1(\eta). \tag{46}$$

Clearly, the fidelity is a function of both η and ϵ in the broken phase.

IV. DECAY OF THE QUANTUM LE

The quantum LE [53] has been studied extensively in recent years [35,54–65]. It gives a measure of the instability of quantum motion under small perturbation, defined as the overlap of the time evolution of the same initial state $|\Psi(t_0)\rangle$ under two slightly different Hamiltonians H and $H' = H + \epsilon_p V$. Explicitly, with $t_0 = 0$, it is defined by $M(t) = |m(t)|^2$, where $m(t)$ is the LE amplitude,

$$m(t) = \langle\Psi(0)|\exp(iHt)\exp(-iH't)|\Psi(0)\rangle, \tag{47}$$

where and hereafter we set the Planck constant $\hbar$ unit.

A. Analytical expression of the LE

In this section, making use of results given in Sec. II, we derive an analytical expression of the LE. In the study of the LE at QPTs, $|\Psi(0)\rangle$ is usually taken as the ground state of a Hamiltonian, say, $|\Psi_{0,h'}\rangle$ of $H_{h'}$, and the system evolves under another Hamiltonian, say, H_h. Then, one has

$$M(t) = \left|\sum_{n=0}^{N} |C_n|^2 e^{in\varepsilon_h t}\right|^2, \tag{48}$$

where Eq. (28) has been used. Substituting Eq. (35) into Eq. (48) and making use of the following relation,

$$\sum_{n=0} \frac{\nu^n}{2^n n!} H_n(u) H_n(v)$$
$$= (1-\nu^2)^{-1/2} \times \exp\left\{\frac{2uv\nu - (u^2+v^2)\nu^2}{1-\nu^2}\right\},$$

we obtain

$$m(t) = \xi \exp\left\{-\frac{\kappa^2 \varepsilon_h \varepsilon_{h'}}{(\varepsilon_h + \kappa^2 \varepsilon_{h'})}\zeta^2\right\}(1 - \lambda^2 e^{2i\varepsilon_h t})^{-1/2}$$
$$\times \exp\left\{\frac{2\lambda\mu^2 e^{i\varepsilon_h t}}{1+\lambda e^{i\varepsilon_h t}}\right\}. \tag{49}$$

After some algebra, we get the following expression of the LE,

$$M(t) = M_1(t) M_2(t), \tag{50}$$

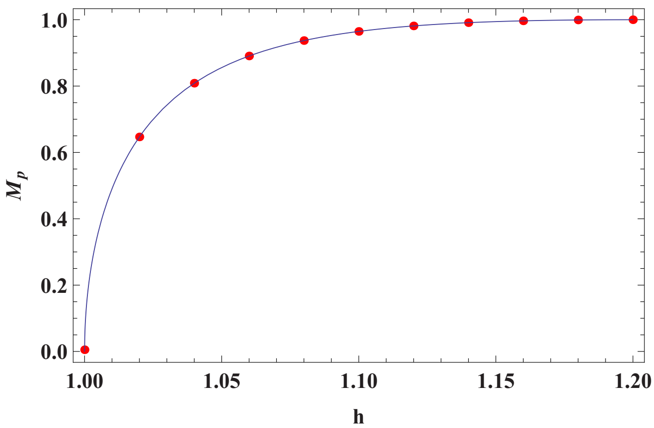

FIG. 2. (Color online) Variation of M_p, the minimum value of the LE, with the parameter h in the symmetric phase. Circles represent results of numerical simulations and the solid line is given by the analytical prediction in Eq. (53). Parameters: $h' = 1.2$ and $N = 2^{12}$.

where

$$M_1(t) = \left[1 + \sinh^2(\delta\Theta)\sin^2(\varepsilon_h t)\right]^{-1/2}, \tag{51}$$

$$M_2(t) = \exp\{-2e^{-\Theta'}(\delta\beta)^2 R(t)\}. \tag{52}$$

Here, $\delta\Theta = \Theta - \Theta'$ and

$$R(t) = \frac{[1-\cos(\varepsilon_h t)][1 - |\tanh(\delta\Theta)|]}{1 + |\tanh(\delta\Theta)|\cos(\varepsilon_h t)}.$$

Note that Eq. (50) is valid in both phases.

We briefly recall the properties of the LE decay in the symmetric phase, which has been studied in Ref. [37]. In this phase, $\delta\beta = 0$ and, hence, the LE is given by $M_1(t)$ in Eq. (51), which is a concise form of a result of Ref. [37]. Clearly, the LE is N independent in this phase and has a period $T = \pi/\varepsilon_h$. Substituting Eq. (21) into this expression, one gets $T = \pi/\sqrt{4(h-\gamma)(h-1)}$, which diverges in the limit $h \to 1$. The minimum value of the LE, denoted by M_p, is given by

$$M_p = [1 + \sinh^2\delta\Theta]^{-1/2}. \tag{53}$$

Making use of Eq. (22), after some algebra, one gets

$$\sinh^2\delta\Theta = \frac{(h-h')^2(1-\gamma)^2}{4(h'-1)(h-1)(h'-\gamma)(h-\gamma)}. \tag{54}$$

It is seen that M_p is also N independent and it approaches 0 when $(h-h')^2/[(h'-1)(h-1)]$ goes to infinity (see Fig. 2 for an illustration). Furthermore, for h sufficiently close to the critical point with h' fixed, the main decaying behavior of the LE in the time region $[0, T/2]$ is a $1/t$ decay [37].

B. LE decay in the broken phase

Now, we discuss the properties of the LE in the broken phase with $\beta \neq 0$. According to Eqs. (51) and (52), the LE is also a periodic function of the time in this phase. The period is $T = 2\pi/\varepsilon_h$. In the neighborhood of the critical point $h_c = 1$, $T \simeq \pi/\sqrt{2(1-\gamma)(1-h)}$, also divergent in the limit $h \to 1$. The minimum value of the LE is now given by

$$M_p = \exp\{-4e^{-\Theta'}(\delta\beta)^2\}. \tag{55}$$

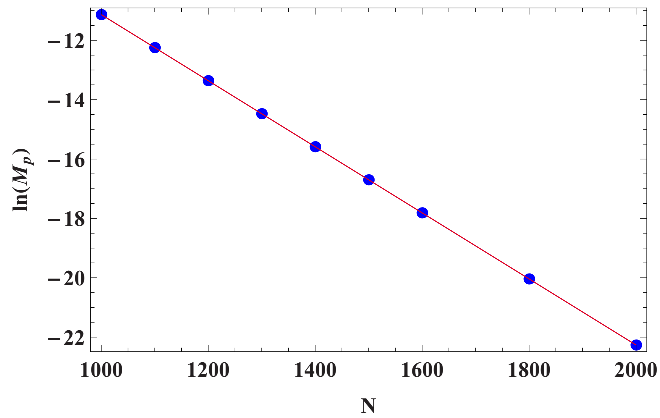

FIG. 3. (Color online) Variation of M_p (in the logarithm scale) with the system's size N in the broken phase. The solid line represents the analytical prediction of Eq. (55). Parameters: $h' = h_c - 0.1$ and $h = h_c - 0.05$.

In contrast to the symmetric phase, the value of M_p is N dependent. In fact, since $\delta\beta^2 \sim N$ [cf. Eq. (41)], M_p scales as $\ln(M_p) \propto -N$ (Fig. 3). It decreases when h approaches the critical point with a fixed h' (Fig. 4).

Usually, the term M_2 decays much faster than M_1, hence, the decaying behavior of the LE is mainly determined by $M_2(t)$. An example of the decay of the LE is given in Fig. 5, which shows that our prediction of Eq. (50) works quite well even when the LE decays to values as small as e^{-35}. Initially, the LE has a Gaussian decay, predicted by M_2 in Eq. (52),

$$M(t) \propto \exp\{-k_g t^2\}, \tag{56}$$

where $k_g = e^{-\Theta'}(\delta\beta)^2(\varepsilon_h)^2(1 - |\tanh\delta\Theta|)/(1 + \tanh\delta\Theta|)$. The figure shows that the Gaussian decay is followed by an exponential decay, much faster than the $1/t$ decay in the symmetric phase discussed above.

To understand the exponential decay of the LE discussed above, one may focus on the decaying behavior of the term $M_2(t)$. When $\varepsilon_h t$ is around $\pi/2$, the slope of the function $\cos(\varepsilon_h t)$ is almost a constant and the function has approximately a linear dependence on the time t, and as a result, the LE has approximately an exponential decay,

$$M(t) \propto \exp\{-kt\}, \tag{57}$$

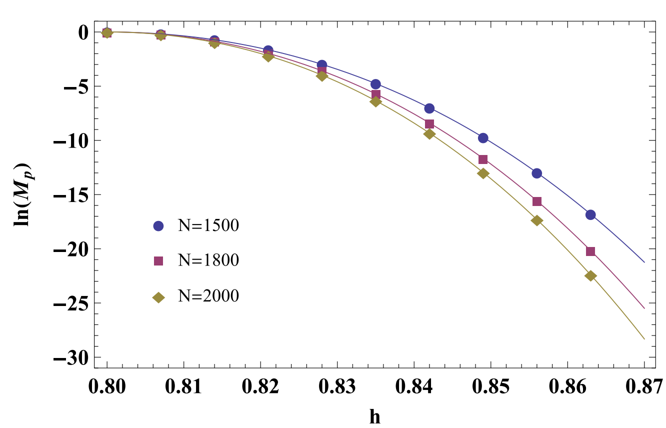

FIG. 4. (Color online) Similar to Fig. 3, for a variation of M_p with h at $h' = 0.8$.

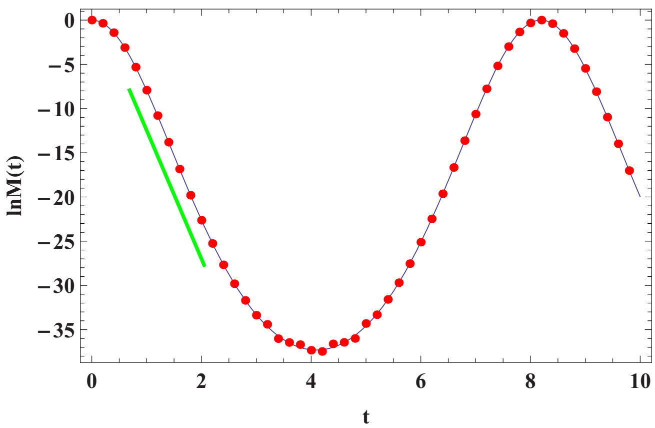

FIG. 5. (Color online) Decay of the LE (circles) in the broken phase (logarithm scale). The solid curve indicates the prediction of Eq. (50) and the (green) short straight line is for guiding eyes. Parameters: $N = 2^{12}$, $h = 0.9$, and $h' = h - 0.06$.

where

$$k = 2e^{-\Theta'}(\delta\beta)^2\varepsilon_h[1 - |\tanh\delta\Theta|]. \tag{58}$$

Obviously, $k \propto N$, since $\delta\beta \propto \sqrt{N}$ (Fig. 6). The dependence of the rate k on the parameter h, with h' and N fixed, is illustrated in Fig. 7. With increasing h, a small deviation is seen in the prediction of Eq. (58) from direct numerical simulations, which should be due to the contribution of M_1. Finally, we give an overall picture for the behavior of the LE in the vicinity of the critical point (Fig. 8). Clearly, the LE has different decaying behaviors on the two sides of the critical point, with decay much faster in the broken phase.

V. COMPARISON WITH OTHER MODELS

In this section, we compare the LMG model and other models in the decaying behaviors of the fidelity and of the LE. Below, we use λ and λ' to denote two values of the controlling parameter concerned, with $\delta\lambda = \lambda - \lambda'$, and use λ_c to indicate the considered critical point. As discussed in the Introduction, previous investigations show two types of decaying behaviors of the fidelity and of the LE at QPTs. It should be useful

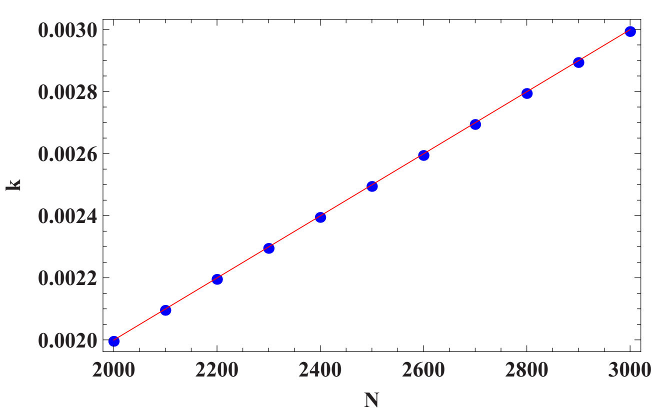

FIG. 6. (Color online) Variation of the decay rate k (circles) of the LE, in the region indicated by the short straight line in Fig. 5, with the system's size N. The solid line represents the analytical predication of Eq. (58). Parameters: $h' = 0.9$ and $h = h' - 0.001$.

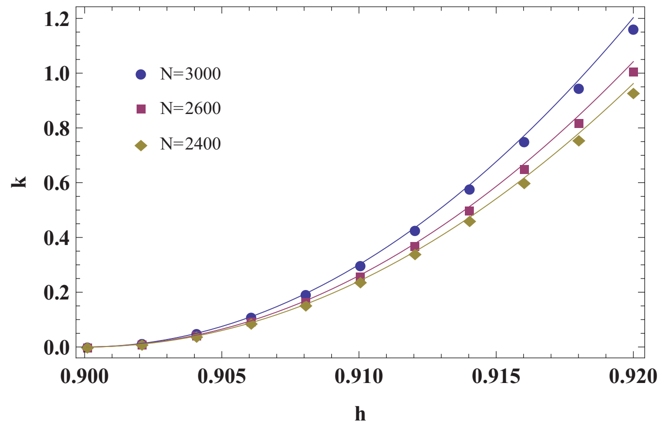

FIG. 7. (Color online) Similar to Fig. 6 for the dependence on the parameter h, with $h' = 0.9$.

to distinguish between two types of systems, which we call type-I and type-II systems, respectively, in what follows. A type-I system possesses a single bosonic zero mode at the critical point, while a type-II system possesses (infinitely) many fermionic (or bosonic) zero modes at the critical point.

Let us first discuss the fidelity. In a type-I system, in the thermodynamic limit, metric quantities such as the fidelity usually depend only on the ratio $\eta = (\lambda - \lambda_c)/(\lambda' - \lambda_c)$ [36]; the Dicke model gives an example showing the validity of this prediction [36]. Meanwhile, scaling arguments suggests that, in systems such as a 1D Ising chain in a transverse field, which belong to the type-II, the fidelity is a function of both the ratio η and the difference $\delta\lambda$, specifically,

$$\ln F(\lambda,\delta) \simeq -N|\delta\lambda|^{d\nu}A(\eta), \tag{59}$$

where N is the particle number and A is some scaling function [29].

Next, for the LE, in a type-I system, when there exists another nonzero bosonic mode, such as in the Dicke model, a semiclassical approach predicts a power-law decay [36,38]. On the other hand, in a type-II system which has many bosonic zero modes, the semiclassical approach predicts an exponential

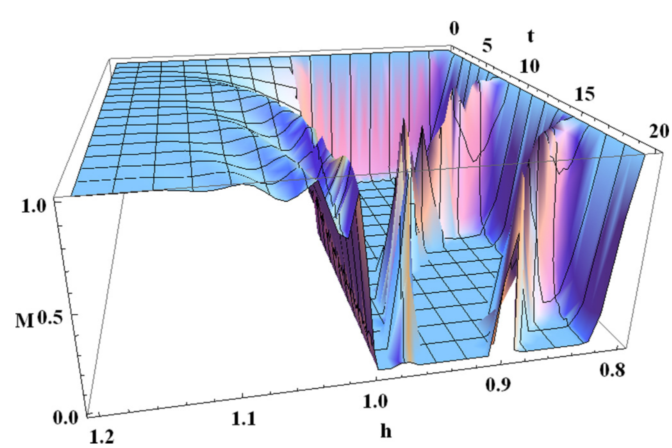

FIG. 8. (Color online) A three-dimensional diagram of the LE, as a function h and t. Parameters: $h' = 1.2$ in the symmetric phase, $h' = 0.8$ in the broken phase, and $N = 2^{12}$.

decay of the LE, namely,

$$M(t) \simeq \exp[-K_s(\delta\lambda)^2 t], \tag{60}$$

where K_s is a quantity mainly determined by the underlying dynamics [38]. In a 1D Ising chain in a transverse field, low-lying fermionic modes can be mapped to bosonic modes, and it has been found that the LE has an exponential decay as given in Eq. (60) [38], with $K_s \propto N$ [39]. The validity of the above predictions in the Dicke model and in the Ising chain has been found on both sides of the considered critical points.

Now, we discuss the decay of the fidelity and of the LE in the LMG model. This model has one bosonic zero mode at the critical point, therefore, it is a type-I system. Let us first discuss the fidelity. In the symmetric phase, the fidelity indeed depends merely on the ratio η [see Eq. (43)], as predicted for a type-I system. However, in the broken phase, near the critical point, the fidelity behaves somewhat as in a type-II system, in which the fidelity depends on both η and $\delta\lambda$ [comparing Eqs. (46) and (59)] [66].

Next, we discuss the LE decay in the LMG model. In the symmetric phase, the LE may have a $1/t$ decay [37], in consistence with the above-discussed prediction for type-I systems, although the LMG model has one bosonic mode only. However, in the broken phase, the main decaying behavior of the LE is approximately an exponential decay [see Eq. (57)], again similar to that predicted for type-II systems. The similarity exists even in some details, that is, (1) when $|h - h'|$ is much smaller than both $(1 - h)$ and $(1 - h')$, the decay rate k is proportional to $(\delta\lambda)^2$, similar to that given in Eq. (60), and (2) $k \propto N$ due to $\delta\beta \propto \sqrt{N}$, similar to K_s in the Ising chain discussed above.

The origin of the above-discussed peculiarity of the broken phase in the LMG model lies in the fact that the parameter β in the bosonic model $b^\dagger = a^\dagger - \beta$, as well as $\delta\beta$, diverges as $\sqrt{N}$ in the thermodynamic limit. In fact, due to this divergence, the treatment given in Ref. [36] for the η scaling of the fidelity is not applicable to the broken phase of the LMG model. Also, since the above-discussed semiclassical approach to the LE is valid only for a small perturbation, it is not applicable to the broken phase with large $\delta\beta$.

VI. CONCLUSION AND DISCUSSION

In this paper, analytical expressions have been derived for the fidelity and for the LE in the broken phase of the LMG model for large particle number N. Making use of these results, decaying behaviors of the fidelity and of the LE have been studied in detail. It has been shown that the LMG model behaves differently on the two sides of its critical point. Specifically, on one hand, in the symmetric phase the decaying behaviors of the fidelity and of the LE are similar to those in models such as the Dicke model, which has a single bosonic zero mode at the critical point and in which the spatial dimension d_s is irrelevant. On the other hand, in the broken phase a similarity is seen with models such as the 1D Ising chain in a transverse field, which has (infinitely) many fermionic zero modes at the critical point and in which d_s is relevant.

There have been many discussions about the similarity between the LMG model and the Dicke model near their critical points [67–71] and also some debates about it [72,73]. We note that all these discussions are for quantities that are given under one Hamiltonian only, such as entanglement and critical exponents. Results given in this paper show that quantities which are determined by two Hamiltonians, such as the fidelity and the LE, have completely different decaying behaviors in the broken phase of the LMG model than in both phases of the Dicke model.

The above results show that, in the classification of QPTs, by making use of the decaying behaviors of the fidelity and of the LE, the situation in the LMG model is more complex than in the other models discussed above. The origin of the peculiar behaviors found in the broken phase of the LMG model should lie in the fact that the difference between neighboring bosonic modes (with fixed values of the controlling parameter) becomes infinitely large in the thermodynamic limit. In the future, it would be of interest to study whether any other models may have peculiar behaviors such as those observed in the LMG model.

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