Graviton noise:the Heisenberg picture

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Abstract

We study the geodesic deviation equation for a quantum particle in a linearized quantum gravitational field. Particle's Heisenberg equations of motion are treated as stochastic equations with a quantum noise. We explore the stochastic equation beyond its local approximation as a differential equation. We discuss the squeezed states resulting from an inflationary evolution. We calculate the noise in the thermal and squeezed states .

1 Introduction

For a long time an interaction of a massive particle with a gravitational wave has been treated in analogy with an interaction of a charged particle with an electromagnetic wave. The electromagnetic wave can be represented as a stream of photons. The quantum description of light and its interaction with charged particles has been well-developed and experimentally verified in quantum optics [1][2][3]. The analogous quantum effects of gravity as proportional to the small Newton constant G would be much harder for detection [4][5][6][7] . Quantum gravitational processes in an interaction of macroscopic masses do sum up to the Newtonian force [8][9] in a similar way as an exchange of photons sums up to the Coulomb force between charged particles. However, it seems to be difficult to detect the corresponding quantum exchange phenomena (for some earlier work on decoherence in particle-graviton interaction see [10][11][12][13][14][15][16]). The detection of gravitational waves [17] has been achieved owing to the immense gravitational energies emitted during the merger of black holes. In order to detect quantum gravitational phenomena from distant events comparable energies in such phenomena should be involved. It has been suggested by Parikh, Wilczek and Zahariade [18][19][20] (see also [21][22]) that an interaction with gravitons will produce (besides the classical force) a quantum noise which can be measured in gravitational wave detectors. In refs.[18][19][20] [23][24] the evolution of the particle density matrix and the scattering probability in an environment of gravitons have been calculated. In the semiclassical limit by means of the Feynman-Vernon method [25] a non-linear stochastic differential equation with a noise from gravitons has been derived (it is a stochastic perturbation of the equation of geodesic deviation by the gravitational radiation reaction [26][27]).

In this paper we treat in detail the particle-graviton equations of geodesic deviation as a quantum equation in the Heisenberg picture. In quantum field theory such an approach is based on Feldman-Yang equations [28]. The Heisenberg equations of motion can be solved perturbatively. Then, the correlation functions can be calculated. For particles in an environment of an infinite number of gravitational modes in a Gaussian state the environment is producing an external Gaussian quantum noise [29] to the particle system. In such a way a system of stochastic equations is obtained. Quantum mechanics with dissipation [30][31] and linearized QED [32][33]have been studied in this way. We show that in the limit $\hbar \to 0$ and for small geodesic deviations the stochastic differential equation coincides with the one derived by means of the Feynman-Vernon method in refs.[20] [21][23]. We investigate the Heisenberg geodesic deviation equation beyond the above mentioned approximations. We demonstrate that the radiation reaction force does not depend on the state of the environment but the noise does. We discuss the squeezed states resulting from an inflationary evolution. We calculate the noise in thermal and squeezed states. We show that the noise can be large in high temperature and in the squeezed states produced during inflation [34]. If we can detect primordial gravitational waves then we may hope to detect the noise of the squeezed states. The study of the geodesic deviation equation beyond the approximations considered in [20][21][23] may be relevant for the identification of the (quantum) gravitational source.

The plan of the paper is the following. In secs.1-3 we review the model of quantum geodesic deviation. In sec.4 we study the geodesic deviation in a quantum thermal background calculating the dependence of noise and backreaction on particle's space-time location. In sec.5 we compute the quantum noise in a squeezed state formed during inflation. In sec.6 we summarize the results. In the Appendix we give some details of the calculations of the non-local backreaction.

2 Geodesic deviation

The equation of geodesic deviation describes a relative motion of two particles along the neighboring geodesics. When q^μ is the vector connecting the corresponding points of adjacent geodesics x^μ, then the acceleration of q^μ is [26]

$$\frac{d^2q^\mu}{dt^2} = R^\mu_{\nu\sigma\rho}U^\nu U^\sigma q^\rho, \tag{1}$$

where $U^\mu = \frac{dx^\mu}{dt}$ and $R^\mu_{\nu\sigma\rho}$ is the Riemanian curvature tensor. Eq.(1) can be considered as an evolution equation for one particle of mass m_0 in the falling frame of the other particle [20][21]. We consider eq.(1) on the Minkowski background $g_{\mu\nu} = \eta_{\mu\nu} + \lambda h_{\mu\nu}$ (where $\eta_{\mu\nu}$ is the flat Minkowski metric, $\lambda^2 = 8\pi G$, $\mu, \nu = 0, 1, 2, 3$) in the non-relativistic approximation. Then, $q = (t, \mathbf{q})$ and $U = (1, \mathbf{0})$. In a linear approximation to the Riemannian tensor and in the transverse-traceless gauge of h eq.(1) reads [26]$(j, r = 1, 2, 3)$

$$\frac{d^2 q^j}{dt^2} = \frac{1}{2}\lambda \frac{d^2 h^{jr}}{dt^2} q^r. \tag{2}$$

In the linearized gravity the Lagrangian describing the geodesic deviation is quadratic in h (we set the velocity of light $c = 1$) [20][21]

$$\begin{aligned}&\int d\mathbf{x}\mathcal{L} = \tfrac{1}{8}\int d\mathbf{k} h^*_\alpha(\mathbf{k}, t)((\tfrac{d}{dt})^2 + k^2) h_\alpha(\mathbf{k}, t) + \tfrac{1}{2} m_0 \tfrac{dq_r}{dt}\tfrac{dq_r}{dt}\\ &-\tfrac{1}{4} m_0 \lambda (2\pi)^{-\frac{3}{2}} \int d\mathbf{k} \exp(i\mathbf{k}\mathbf{q}) \tfrac{d^2 h_{rl}}{dt^2}(\mathbf{k}) q_r q_l.\end{aligned} \tag{3}$$

In eq.(3) h_{rl} is decomposed in the amplitudes h_α (where $\alpha = +, \times$ in the linear polarization) by means of the polarization tensors e^α_{rl} [35]

$$h_{rl}(\mathbf{x}, t) = (2\pi)^{-\frac{3}{2}} \int d\mathbf{k}(h_\alpha(\mathbf{k}) e^\alpha_{rl} \exp(-i\mathbf{k}\mathbf{x}) + h^*_\alpha(\mathbf{k}) e^\alpha_{rl} \exp(i\mathbf{k}\mathbf{x})). \tag{4}$$

The gravitational Hamiltonian $H = H_+ + H_\times$ has the same form as for two independent scalar fields h_α

$$H = \frac{1}{2}\int d\mathbf{k}\Big(-\frac{\delta}{\delta h^\alpha(\mathbf{k})}\frac{\delta}{\delta h^\alpha(-\mathbf{k})} + k^2 h^\alpha(\mathbf{k}) h^\alpha(-\mathbf{k})\Big). \tag{5}$$

We look for Gaussian states of gravitons which are solutions of the Schrödinger equation (with the Hamiltonian (5))

$$\psi^g_t(h) = A \exp\Big(\frac{i}{2\hbar} h\Gamma(t) h + \frac{i}{\hbar} J_t h\Big), \tag{6}$$

where Γ is an operator defined by a bilinear form $\Gamma(t, \mathbf{x}-\mathbf{y})$. Inserting ψ^g_t in the Schrödinger equation with the Hamiltonian (5) we obtain equations for A, Γ, J (in Fourier space)

$$i\hbar\partial_t \ln A = \tfrac{1}{2}\int d\mathbf{k} J(\mathbf{k}) J(-\mathbf{k}) - \tfrac{i\hbar}{2}\delta(\mathbf{0})\int d\mathbf{k}\Gamma(\mathbf{k}), \tag{7}$$

where $\Gamma(\mathbf{k})$ is the Fourier transform of $\Gamma(\mathbf{x})$,

$$\partial_t J = -\Gamma J, \tag{8}$$

$$\partial_t \Gamma = -\Gamma^2 - k^2. \tag{9}$$

The term $\delta(\mathbf{0})$ in the normalization factor(7) results from an infinite sum of vacuum oscillator energies. It could be made finite by a regularization of the

Hamiltonian (5) but this is irrelevant for calculations of the expectation values (because the normalization factor cancels). If we define

$$u(t) = \exp(\int^t ds\Gamma_s), \tag{10}$$

then $\Gamma_t(\mathbf{k}) = u^{-1}\partial_t u$, where u satisfies the equation

$$(\partial_t^2 + k^2)u(k) = 0. \tag{11}$$

Eq.(11) has the general solution

$$u(t) = \sigma\cos(kt) + \delta\sin(kt). \tag{12}$$

Another form of the solution is (used by Guth and Pi [36])

$$u = \cos(kt + \alpha - i\gamma), \tag{13}$$

where the parameters σ, δ, α and γ may depend on k.Then,

$$\begin{aligned} &\Gamma = -k\tan(kt + \alpha - i\gamma) = k\Big(i\sinh(2\gamma) - \sin(kt+\alpha)\cos(kt+\alpha)\Big) \\ &\times\Big(\cosh(2\gamma) + \tfrac{1}{2}\cos(2kt + 2\alpha)\Big)^{-1}. \end{aligned} \tag{14}$$

The covariance of the probability density $|\psi_t^g|^2$ is equal to the inverse of

$$-i(\Gamma(t) - \Gamma^*(t)) = 2k\sinh(2\gamma)\Big(\cosh(2\gamma) + \frac{1}{2}\cos(2kt + 2\alpha)\Big)^{-1}. \tag{15}$$

The fluctuations of h are large if γ is small. It follows from eq.(13) that during the evolution described by the Hamiltonian (5) only the phase is changing $\alpha \rightarrow \alpha + kt$. We can express α and δ in eq.(12) by α and γ of eq.(13). The form (13) of the solution of eq.(11) is useful in order to express the squeezing by a small value of γ.

3 Heisenberg equations of motion

The equation for the gravitational field resulting from the Lagrangian (3) is

$$\frac{d^2h^{rl}(\mathbf{k})}{dt^2} + k^2h^{rl}(\mathbf{k}) = (2\pi)^{-\frac{3}{2}}\lambda f^{rl}\exp(-i\mathbf{k}\mathbf{q}(t)), \tag{16}$$

where

$$f^{rl} = \frac{m_0}{2}\frac{d^2}{dt^2}q^r q^l. \tag{17}$$

The solution of eq.(16) in the transverse-traceless gauge for $t \geq t_0$ (we assume that when $t \leq t_0$ then h_{rl} is a free wave) is

$$\begin{array}{l} h_{rl}(\mathbf{k}) = h^w_{rl} + h^f_{rl} + h^I_{rl} \equiv h^w_{rl} + e^\alpha_{rl}(h^\alpha_0 \cos(kt) + k^{-1}\Pi^\alpha_0 \sin(kt)) + \\ \lambda(2\pi)^{-\frac{3}{2}}\Lambda_{rl;mn}\int_{t_0}^t k^{-1}\sin(k(t-t'))f_{mn}(t')\exp(-i\mathbf{k}\mathbf{q}(t'))dt', \end{array} \tag{18}$$

here Λ projects to the transverse-traceless gauge

$$\begin{array}{l} 2\Lambda_{ij;mn}(\frac{\mathbf{k}}{k}) = 2e^\alpha_{ij}e^\alpha_{mn} = (\delta_{im} - k^{-2}k_ik_m)(\delta_{jn} - k^{-2}k_jk_n) \\ +(\delta_{in} - k^{-2}k_ik_n)(\delta_{jm} - k^{-2}k_jk_m) - \frac{2}{3}(\delta_{ij} - k^{-2}k_ik_j)(\delta_{nm} - k^{-2}k_nk_m). \end{array} \tag{19}$$

h^w_{rl} describes the classical wave (possibly one mode of it) and

$$h^f_{rl}(t,\mathbf{x}) = (2\pi)^{-\frac{3}{2}}\int d\mathbf{k}\exp(i\mathbf{k}\mathbf{x})h^f_{rl}(\mathbf{k},t)$$

is a superposition of quantum free modes decomposed in eq.(18) into h^α_0 and Π^α_0 as quantum initial conditions. h^I_{rl} is the gravitational field created by the particle motion. h^f_{rl} (as well as h^w_{rl}) satisfy the homogeneous equation

$$\frac{d^2h^f_{rl}(\mathbf{k})}{dt^2} + k^2h^f_{rl}(\mathbf{k}) = 0.$$

The equation of motion for the coordinate $\mathbf{q}$ is

$$\begin{array}{l} \frac{d^2q_r}{dt^2} = \frac{\lambda}{2}(2\pi)^{-\frac{3}{2}}\int d\mathbf{k}\exp(i\mathbf{k}\mathbf{q}(t))(\frac{d^2h^w_{rl}(\mathbf{k})}{dt^2} + \frac{d^2h^f_{rl}(\mathbf{k})}{dt^2})q^l(t) \\ +\frac{\lambda}{2}(2\pi)^{-\frac{3}{2}}\int d\mathbf{k}\exp(i\mathbf{k}\mathbf{q}(t))\frac{d^2h^I_{rl}(\mathbf{k})}{dt^2}q^l(t). \end{array} \tag{20}$$

We insert the gravitational field from eq.(18) into eq.(20). Then,

$$\frac{d^2q_r}{dt^2} = \frac{\lambda}{2}\frac{d^2h^w_{rl}(\mathbf{q})}{dt^2}q^l(t) + F_r(\mathbf{q},t) + N_{rl}(t,\mathbf{q})q^l, \tag{21}$$

where F is a non-linear interaction (the backreaction discussed in the next section) resulting from the interaction of the mass m_0 with its own gravitational field. The noise $N_{rl}(t,\mathbf{q})$ is expressed by the initial values of the canonical (free) variables $h^\alpha(\mathbf{k})$ and $\Pi^\alpha(\mathbf{k})$

$$\begin{array}{l} N^{rl}(t,\mathbf{q}) = \int d\mathbf{k}N^{rl}(\mathbf{k},\mathbf{q},t) \\ = -\frac{\lambda}{2}(2\pi)^{-\frac{3}{2}}\int d\mathbf{k}\exp(i\mathbf{k}\mathbf{q}(t))k^2e^\alpha_{rl}(h^\alpha_0\cos(kt) + k^{-1}\Pi^\alpha_0\sin(kt)). \end{array} \tag{22}$$

4 Stochastic equations in the thermal environment

Let us first consider $\mathbf{q} \simeq 0$ in eq.(22)(then we do not need to care about the non-commutativity of $\mathbf{q}(t)$). We denote $N_{rl}(t) \equiv N_{rl}(t,\mathbf{0})$. Then

$$N^{rl}(t)=\int d\mathbf{k}N^{rl}(k,\mathbf{0},t)=-\frac{\lambda}{2}(2\pi)^{-\frac{3}{2}}\int d\mathbf{k}k^2e^{\alpha}_{rl}(h^{\alpha}_0\cos(kt)+k^{-1}\Pi^{\alpha}_0\sin(kt)). \tag{23}$$

Let us assume that $h^{\alpha}(\mathbf{k})$ and $\Pi^{\alpha}(\mathbf{k})$ are distributed according to the classical Gibbs distribution

$$d\Pi^{\alpha}_0 dh^{\alpha}_0 \exp\Big(-\frac{\beta}{2}\int d\mathbf{k}(|\Pi_{\alpha}|^2+k^2|h_{\alpha}|^2)\Big). \tag{24}$$

here $\beta=\frac{1}{k_BT}$ where T is the temperature and k_B is the Boltzmann constant. Then,

$$<\overline{h}^{\alpha}_0(\mathbf{k})h^{\alpha}_0(\mathbf{k}')>=\beta^{-1}k^{-2}\delta(\mathbf{k}-\mathbf{k}'), \tag{25}$$

and

$$<\overline{\Pi}^{\alpha}_0(\mathbf{k})\Pi^{\alpha}_0(\mathbf{k}')=\beta^{-1}\delta(\mathbf{k}-\mathbf{k}'). \tag{26}$$

The noise (23) has the correlations calculated in [23]

$$\Big\langle N^{rl}(t)N^{mn}(t')\Big\rangle=<\Lambda_{rl:mn}>\frac{\lambda^2}{4\pi}\beta^{-1}\partial_t^2\partial_{t'}^2\delta(t-t'), \tag{27}$$

where the angular average of Λ over $\mathbf{k}k^{-1}$ is

$$\frac{1}{4\pi}<\Lambda_{ij;mn}>=\frac{1}{5}(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})-\frac{2}{15}\delta_{ij}\delta_{nm}. \tag{28}$$

When $\mathbf{q}$ is taken into account (but non-commutativity of $\mathbf{q}$'s is ignored)

$$\begin{aligned}&\Big\langle\overline{N}^{rl}(\mathbf{q},t)N^{mn}(\mathbf{q},t')\Big\rangle\\&=\frac{\lambda^2}{4}\beta^{-1}(2\pi)^{-3}\int d\mathbf{k}k^2\Lambda_{rl:mn}\cos(k(t-t'))\exp(i\mathbf{kq}(t')-i\mathbf{kq}(t)).\end{aligned} \tag{29}$$

In eq.(29) we can calculate the angular average over $k^{-1}\mathbf{k}$ using the formula

$$\int d\Omega\Lambda^{rl;mn}(\mathbf{k}k^{-1})\exp(i\mathbf{kx})=4\pi\Lambda^{rl;mn}(k^{-1}\nabla_{\mathbf{x}})(k|\mathbf{x}|)^{-1}\sin(k|\mathbf{x}|) \tag{30}$$

where $d\Omega$ is an average over the spherical angle .

We can express the integral (29) in configuration space performing the k-integral of trigonometric functions using the result (28) for the average $<\Lambda_{rl:mn}>$ (19) over the spherical angle and differentiating the formula

$$\int_0^{\infty}dkk^{-1}\sin(ku)=\frac{\pi}{2}\epsilon(u) \tag{31}$$

over u where $\epsilon(u)$ is an antisymmetric function such that $\epsilon(u)=1$ for $u>0$. The result of calculations gives a long formula but the main outcome is that noise is concentrated on the light cone.

The quantum Bose-Einstein distribution is (for general discussion of quantum noise see [29][37][38])

$$< (h^{\alpha}(\mathbf{k}))^{+}h^{\alpha}(\mathbf{k}') >= \frac{1}{2}\hbar k^{-1}\coth(\frac{1}{2}\hbar\beta k))\delta(\mathbf{k}-\mathbf{k}') \tag{32}$$

and

$$< (\Pi^{\alpha}(\mathbf{k}))^{+}\Pi^{\alpha}(\mathbf{k}') >= \frac{1}{2}\hbar k\coth(\frac{1}{2}\beta k))\delta(\mathbf{k}-\mathbf{k}'). \tag{33}$$

Then, for the noise we obtain

$$\begin{aligned}&\frac{1}{2}\Big\langle N^{mn}(\mathbf{q},t')N^{rl}(\mathbf{q},t)+N^{rl}(\mathbf{q},t)N^{mn}(\mathbf{q},t')\Big\rangle\\&=\frac{\lambda^2}{4}(2\pi)^{-3}\int d\mathbf{k})k^4\Lambda_{rl:mn}\frac{1}{2}\hbar k^{-1}\coth(\frac{1}{2}\hbar\beta k))\cos(k(t-t'))\exp(i\mathbf{k}\mathbf{q}(t')-i\mathbf{k}\mathbf{q}(t)).\end{aligned} \tag{34}$$

In the quantum case the operators $N^{rl}(t,\mathbf{q})$ do not commute

$$\begin{aligned}&[N^{rl}(t,\mathbf{q}),N^{mn}(t',\mathbf{q}')]\\&=i\hbar\frac{\lambda^2}{4}(2\pi)^{-3}\int d\mathbf{k}k^3\Lambda^{rl:mn}\sin(k(t-t'))\exp(i\mathbf{k}\mathbf{q}(t)-i\mathbf{k}\mathbf{q}(t')).\end{aligned} \tag{35}$$

From the uncertainty relation we can conclude that the noise cannot be arbitrarily small what is bringing difficulties in the precision of measurements [37][38] but according to [18][20] this is just the large noise which is interesting for investigation in gravitation wave experiments.

The non-linear force resulting from the graviton environment is

$$\begin{aligned}&F^r=\frac{1}{2}\lambda^2(2\pi)^{-3}\int d\mathbf{k}\exp(i\mathbf{k}\mathbf{q}(t))q^l(t)\Lambda_{rl;mn}\\&\partial_t^2\int_{t_0}^t k^{-1}\sin(k(t-t'))\exp(-i\mathbf{k}\mathbf{q}(t'))f_{mn}(t')dt'\\&=\frac{1}{2}\lambda^2(2\pi)^{-3}\int d\mathbf{k}\exp(i\mathbf{k}\mathbf{q}(t))q^l(t)\\&\Lambda_{rl;mn}(\int_{t_0}^t\partial_t\cos(k(t-t'))\exp(-i\mathbf{k}\mathbf{q}(t')f_{mn}(t')dt'+f_{mn}(t)\exp(-i\mathbf{k}\mathbf{q}(t)).\end{aligned} \tag{36}$$

We write the last factor as

$$\begin{aligned}&(\int_{t_0}^t\partial_t\cos(k(t-t'))\exp(-i\mathbf{k}\mathbf{q}(t'))f_{mn}(t')dt'+f_{mn}(t)\exp(-i\mathbf{k}\mathbf{q}(t))\\&=(-\int_{t_0}^t\partial_{t'}\cos(k(t-t'))\exp(-i\mathbf{k}\mathbf{q}(t'))f_{mn}(t')dt'+f^{mn}(t)\exp(-i\mathbf{k}\mathbf{q}(t)\\&=\int_{t_0}^t\cos(k(t-t'))\Big(dt'\partial_{t'}\Big(f_{mn}(t')\exp(-i\mathbf{k}\mathbf{q}(t'))\Big)\\&+\cos(k(t-t_0))f_{mn}(t_0)\exp(-i\mathbf{k}\mathbf{q}(t_0)))\Big)\\&=\int_{t_0}^t\cos(k(t-t'))\partial_{t'}\Big(f_{mn}(t')\exp(-i\mathbf{k}\mathbf{q}(t'))dt'\Big)\\&=\int_{t_0}^t dt'\cos(k(t-t'))\exp(-i\mathbf{k}\mathbf{q}(t'))dt'(\partial_{t'}f_{mn}(t')-i\mathbf{k}\frac{d\mathbf{q}}{dt'}),\end{aligned} \tag{37}$$

where we assumed $f_{mn}(t_0)=0$. In the approximation $\exp(-i\mathbf{k}\mathbf{q}(t'))\simeq 1$ the integral (37) is

$$\begin{aligned}&\int_{t_0}^t\int dkk^2\cos(k(t-t'))\partial_{t'}f_{mn}(t')dt'\\&=-2\pi\int_{t_0}^t\partial_{t'}^2\delta(t-t')\partial_{t'}f_{mn}(t')dt'=2\pi\partial_t^3 f_{mn}(t).\end{aligned} \tag{38}$$

Then, we can write eq.(21) in the form (we omit the classical wave h^w)

$$\frac{d^2q^r}{dt^2}=-\frac{\lambda^2}{5\pi}(\delta_{rm}\delta_{ln}-\frac{1}{3}\delta_{rl}\delta_{mn})q^l\partial_t^3 f^{mn}+N^{rl}(t)q_l. \tag{39}$$

This is the perturbation by noise obtained in [18][20][23] of the gravitational backreaction equation of ref.[27].

If in eq.(36) we take $\exp(-i\mathbf{k}\mathbf{q}(t'))$ into account then we obtain an integro-differential equation (we cannot get rid of the t' integral in eq.(36) but the $\mathbf{k}$ integral can be calculated, see the Appendix). The force is non-local in time, but its exact form may be relevant for an interpretation of particle motion. The gravitational wave detectors have a finite resolution range in frequency as well as in space and time. For this reason it is useful to have expressions for the noise correlations both in the frequency domain and in the space-time .

In the calculations (35) and (36) we have ignored the non-commutativity of $\mathbf{q}(t)$ and $\mathbf{q}(t')$. We have

$$\begin{aligned}&\exp(i\mathbf{k}\mathbf{q}(t))\exp(-i\mathbf{k}\mathbf{q}(t'))=\exp(i\mathbf{k}\mathbf{q}(t))-i\mathbf{k}\mathbf{q}(t'))\\&\times\exp\Big(\frac{1}{2}[\mathbf{k}\mathbf{q}(t)),\mathbf{k}\mathbf{q}(t')]\Big)\Big(1+O(\hbar^2)\Big)\end{aligned} \tag{40}$$

Eq.(39) holds true in the limit $\hbar\to 0$ and with the approximation that $\mathbf{q}\simeq\mathbf{0}$. In [20] the "particle" is macroscopic. Then $\mathbf{q}$ is classical. However, eq.(21) is applicable to quantum particles as well. We could have gravitons interacting with molecules or crystals. The variation of molecule's length or crystal's size could be transmitted to an interferometer. We can allow large q as in eqs.(29) and (36) but such equations are reliable only till the order $\hbar$, because at higher orders of $\hbar$ we would need to solve the Heisenberg equations of motion (21) (it seems possible only in a perturbative expansion in $\hbar$)till the higher order in $\hbar$ and calculate the commutators $\mathbf{q}(t)$ and $\mathbf{q}(t')$ (as in eq.(40)).

5 The noise from the squeezed states of (inflationary) gravitons

In this section we study tensor perturbations (gravitational waves) evolving during the inflationary epoch and subsequently reaching us in a flat Minkowski space. We consider the flat expanding metric

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=dt^2-a^2d\mathbf{x}^2. \tag{41}$$

It is useful to introduce the conformal time τ

$$\tau=\int^t a(s)^{-1}ds.$$

We decompose h_{rl} into polarization components h^α (as in eq.(4) for $a=1$) $h_{rl}=a^{-1}e^\alpha_{rl}h^\alpha$. Then the Hamiltonian is [39]

$$H = \frac{1}{2}\int d\mathbf{x}\Big((\Pi^\alpha)^2 - h^\alpha(\triangle + a''a^{-1})h^\alpha\Big), \tag{42}$$

where $\Pi^\alpha = -i\hbar\frac{\delta}{\delta h^\alpha}$ is the canonical momentum. The solution of the Schrödinger equation

$$i\hbar\partial_\tau\Psi = H\Psi$$

has the Gussian form (6) if Γ satisfies the equation

$$\partial_\tau\Gamma + \Gamma^2 + (k^2 - a''a^{-1})\Gamma = 0. \tag{43}$$

As in eq.(10) we express Γ in the form $\Gamma = u^{-1}\partial_\tau u$. Then, u is a solution of the equation

$$\partial_\tau^2 u + (k^2 - a''a^{-1})u = 0. \tag{44}$$

In the inflationary era described by an exponential expansion $a = \exp(H_0 t)$ (the cosmic time t is related to τ in eqs.(43)-(44) as $\tau = -H_0^{-1}\exp(-H_0 t)$, where H_0 is the Hubble constant) eq.(44) reads

$$\partial_\tau^2 u + (k^2 - 2\tau^{-2})u = 0. \tag{45}$$

The general solution of eq.(45) (analogous to the solution (12) in the Minkowski space) is

$$u = \sigma\cos(k\tau) + \delta\sin(k\tau) - \sigma(k\tau)^{-1}\sin(k\tau) + \delta(k\tau)^{-1}\cos(k\tau). \tag{46}$$

The solution which behaves as a plane wave $\exp(ik\tau)$ for $\tau \to \infty$ results from the choice $\sigma = 1$ and $\delta = i$. Then, at the end τ_e of inflation (for a small $k\tau_e$)

$$\Gamma_e \simeq (ik^3\tau_e^2 - \tau_e^{-1})(1 + k^2\tau_e^2)^{-1}. \tag{47}$$

The squeezing and decoherence during inflation has been discussed earlier in [34][40][41][42](the k^{-3} covariance is characteristic of the exponential expansion [43][44]). We take $\psi_0^e = A\exp(\frac{i}{2\hbar}h^\nu\Gamma_e h^\nu)$ as the initial wave function for our graviton detection experiment. From its derivation it is clear that the non-linear term for the particle geodesic deviation equation (21) does not depend on the graviton wave function (although this term may have the non-local correction in comparison to eq.(39) as discussed in the Appendix). In eq.(21) the noise term (its correlation function) depends on the quantum state of the graviton. We have discussed the evolution of Gaussian wave packets in various epochs in [45]. We concluded that the squeezing described by small $i(\Gamma_e - \Gamma_e^*)$ remains small ($\simeq k^3$) during the radiation and baryonic eras. According to eq.(14) during the Minkowski space evolution only the phase α is changing.

We calculate now the noise correlation functions in the Gaussian state ψ_0^e. For this purpose we may use the expression for the noise (22) in terms of the initial field and the canonical momentum (the noise correlation functions are calculated in a different parametrization of squeezed states in [20][21][46]). Then (if $J=0$) the noise has the correlations

$$
\begin{aligned}
&(\psi_0^e, N^{rl}(t,\mathbf{q})N^{mn}(t',\mathbf{q}')\psi_0^e) = \frac{\lambda^2}{4}(2\pi)^{-3}\int d\mathbf{k}k^4\exp(i\mathbf{k}(\mathbf{q}(t)-\mathbf{q}(t'))\Lambda^{rl;mn} \\
&\Big(i\hbar(\Gamma_e-\Gamma_e^*)^{-1}(\cos(kt)+k^{-1}\Gamma_e\sin(kt))(\cos(kt')+k^{-1}\Gamma_e\sin(kt')) \\
&-i\hbar k^{-1}\Big(\sin(kt)\cos(kt')+\Gamma_e k^{-1}\sin(kt)\sin(kt')\Big)\Big)
\end{aligned}
\tag{48}
$$

If we make the usual assumption that $k\tau_e$ is small and use the approximation $\exp(i\mathbf{kq})\simeq 1$ then $\Gamma_e-\Gamma_e^*\simeq 2ik^3\tau_e^2$. Hence,

$$
\begin{aligned}
&(\psi_0^e, N^{rl}(t,\mathbf{0})N^{mn}(t',\mathbf{0})\psi_0^e) \simeq \\
&\tau_e^{-2}\frac{\lambda^2}{4}(2\pi)^{-3}2\pi < \Lambda^{rl;mn} > \int dk k^3\hbar k^{-3}\cos(kt)\cos(kt') \\
&= \tau_e^{-2}\frac{\lambda^2}{4}(2\pi)^{-3}2\pi^2 < \Lambda^{rl;mn} > \partial_t\partial_{t'} \\
&\times\Big((\partial_t-\partial_{t'})\delta(t-t')+(\partial_t+\partial_{t'})\delta(t+t')\Big)
\end{aligned}
\tag{49}
$$

which can be large for a small value of τ_e.

The aim of the stochastic geodesic deviation equation in refs.[18][20][21] is to describe the motion of the arms of the interferometer. In such a case the classical (more precisely a semi-classical) limit (39) of eq.(21) may be sufficient. However, eq.(21) can describe quantum objects (large molecules, crystals) as well. The interferometers can be sensitive to their change of size. In such a case, besides the noise, the quantum evolution of the coordinate $\mathbf{q}$ may be relevant. It is not simple to solve the non-linear quantum Heisenberg equation (21). One can do it in perturbation expansion in λ and $\hbar$. For the purpose of illustration let us consider an oscillator of frequency ω, i.e., we add to eq.(39) the force $-m_0\omega^2 q^r$. Then, eq.(39) reads

$$
\frac{d^2q^r}{dt^2}+\omega^2q^r = -\frac{\lambda^2}{5\pi}(\delta_{rm}\delta_{ln}-\frac{1}{3}\delta_{rl}\delta_{mn})q^l\partial_t^3 f^{mn}+\lambda\tilde{N}^{rl}(t)q_l. \tag{50}
$$

Here we write $N^{rl}(t)=\lambda\tilde{N}^{rl}(t)$ where $\tilde{N}^{rl}(t)$ does not depend on λ. We look for perturbative solutions of eq.(50) in an expansion in $\lambda=\sqrt{8\pi G}$. Let q_0^r be the solution of the harmonic oscillator (zeroth order in λ). At the first order in λ

$$
\frac{d^2q_1^r}{dt^2}+\omega^2q_1^r = \tilde{N}^{rl}(t)q_0^l. \tag{51}
$$

At order λ^2 we obtain

$$
\frac{d^2q_2^r}{dt^2}+\omega^2q_2^r = -\frac{64}{15\pi}\omega^4 m_0 q_0^r q_0^n\frac{dq_0^n}{dt}+\tilde{N}^{rl}(t)q_1^l. \tag{52}
$$

Eq.(52) shows the friction term $\gamma^{rn}\frac{dq^n}{dt}$ in Newton equation. The friction matrix γ has the non-zero eigenvalue $\frac{512Gm_0}{15}\mathbf{q}^2\omega^4$. The ratio of the gravitational friction to the electromagnetic one is approximately $\mathbf{q}^2\omega^2m_0Ge^{-2}$ where e is the electric charge and G is the Newton constant. For the electron $m_0Ge^{-2}\simeq 10^{-42}$. Hence, the gravitational friction is unmeasurable unless $\omega^2\mathbf{q}^2$ is large. As pointed out in [18][20] this is the noise in eq.(39) which can be measurable if τ_e is small. The friction is related to the width of the spectral line [1] so the gravitational effect on spectral lines is not observable as discussed from another point of view in [5][6].

6 Summary and conclusions

We have studied in some detail the quantum geodesic deviation equation in the Heisenberg picture beyond the earlier semi-classical approximations. We reveal the dependence of the backreaction force and of the quantum noise on the particle position q when kq is not negligible where k describes the graviton's wave number distribution in a quantum graviton state. For primordial gravitons coming from the inflationary era the main contribution to the graviton's probability distribution comes from small k (then kq is small). However, for thermal gravitons large k may be important (depending on the relevant values of q). The calculations of the quantum noise correlation functions considered in this paper may be useful for a determination of the source of the gravitational waves if they have a quantum origin .

7 Appendix: the non-local backreaction

The force in eq.(36) is

$$\begin{aligned}&F^r = \frac{1}{2}\lambda^2(2\pi)^{-3}q^l(t)\int d\mathbf{k}\exp(i\mathbf{k}\mathbf{q}(t) - i\mathbf{k}\mathbf{q}(t'))\\ &\Lambda_{rl;mn}(\int_{t_0}^t dt'\cos(k(t-t'))(\partial_{t'}f_{mn}(t') - i\mathbf{k}\frac{d\mathbf{q}}{dt'}).\end{aligned} \tag{53}$$

We can perform the angular integral using eq.(30). Then

$$\begin{aligned}&F_I^r = \frac{1}{2}\lambda^2(2\pi)^{-3}q^l(t)4\pi\int_0^\infty dkk^2\int_{t_0}^t dt'\cos(k(t-t'))\partial_{t'}f_{mn}(t')\Lambda_{rl;mn}(\frac{\nabla_{\mathbf{q}}}{k})\\ &\times\Big(k^{-1}\sin(k|\mathbf{q}(t)-\mathbf{q}(t')|)|\mathbf{q}(t)-\mathbf{q}(t')|^{-1}\\ &+\cos(k|\mathbf{q}(t)-\mathbf{q}(t')|)|\mathbf{q}(t)-\mathbf{q}(t')|^{-2}\mathbf{q}(t')(\mathbf{q}(t)-\mathbf{q}(t'))\\ &-k^{-1}\sin(k|\mathbf{q}(t)-\mathbf{q}(t')|)|\mathbf{q}(t)-\mathbf{q}(t')|^{-3}\mathbf{q}(t')(\mathbf{q}(t)-\mathbf{q}(t'))\Big)\end{aligned} \tag{54}$$

Performing the differentiation $\Lambda_{rl;mn}(\frac{\nabla_{\mathbf{q}}}{k})$ we obtain a trigonometric function multiplied by an integer power of k. The k-integral can be calculated using the representation (31) of the ϵ function (and derivatives of it). The integral representation may be useful for a study of non-local effects in the solutions of the quantum geodesic deviation equation.

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