Loss of coherence and coherence protection from a graviton bath

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We consider a quantum harmonic oscillator coupled with a graviton bath. We discuss the loss of coherence in the matter sector due to the matter-graviton vertex interaction, which leads to a loss of coherence provided that the matter-wave interferometer can *emit* gravitons allowed by the kinematics, and the most dominant process up to $\sim \sqrt{G}$ and $\sim \mathcal{O}(c^{-2})$. Working in the quantum-field-theory framework, we obtain a master equation by tracing away the gravitational field at the leading order $\sim \mathcal{O}(G)$. We find that the decoherence rate is proportional to the cube of the harmonic trapping frequency and vanishes for a free particle, as expected for a system without a mass quadrupole. Furthermore, our quantum model of graviton emission recovers the known classical formula for gravitational radiation from a classical harmonic oscillator for coherent states with a large occupation number. In addition, we find that the quantum harmonic oscillator eventually settles in a steady state with *a remnant coherence* of the ground and first excited states. In particular, the superposition of number states $\frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$ never decoheres.

I. INTRODUCTION

One of the most striking consequences of General relativity is undoubtedly given by gravitational waves [1]. Such waves propagate through spacetime itself – are part of it – and interact with all matter making it a universal feature of all experiments. The gravitational waves produced by small objects are however hindered by the smallness of gravitational coupling, whilst the gravitational waves produced by large astronomical bodies become attenuated by the large distances to the Earth. Nevertheless, a hundred years from the prediction of gravitational waves [2, 3] the detection of gravitational waves was announced [4].

The feeble strain induced by the passing of gravitational waves has been detected in an optomechanical setup employing suspended mirrors [4]. Whilst most quantum effects remain suppressed at such scales it has been shown that tiny quantum correlations between the phase of light and the position of the mirrors in the Advanced LIGO detectors imprint a non-negligible signal [5]. Furthermore, there is substantial progress towards reaching the motional ground state of $\sim$ 10kg large mirrors where quantum effects become prominent [6, 7].

Although a purely classical treatment of the gravitational field still suffices to explain all of the current experimental data it is nonetheless interesting to ask what would be the quantum signature of gravitational waves and several different theoretical approaches have been considered [8–19]. Recent theoretical works have also investigated the possibility of detecting stochastic graviton noise in the context of gravitational wave observatories [20–22].

However, discerning between classical models of gravity from the quantum version will necessarily require testing coherent features of gravity which cannot be mimicked by any classical noise source. Such a proposal has been devised by considering the two nearby masses – close enough that they interact gravitationally but far enough apart that all other channels of interaction are strongly suppressed – which can entangle only if the gravitational field exhibits bonafide quantum features [23, 24], see also [25] [1] The underlying mechanism for the quantum entanglement of masses (QGEM) the protocol has been analyzed within perturbative quantum gravity [26–30] and the framework of the Arnowitt–Desse–Meissner (ADM) approach [31], as well as in the path integral approach [32], and for the massive graviton [33]. Recent developments include an optomechanical proposal for testing the quantum light-bending interaction [34], a quantum test of the weak equivalence principle [35], and test whether gravity acts as a quantum entity when measured [36].

It is thus interesting to ask whether quantized gravitational waves could also induce coherent effects in quantum systems, which would be difficult to explain using a classical theory of gravity.

In this work, we consider a quantum harmonic oscillator coupled to quantized gravitational waves in the context of perturbative quantum gravity. We first review the results of classical quadrupole radiation emitted by a classical harmonic oscillator (Sec. II). We then obtain the matter-graviton coupling in the laboratory frame of the quantum harmonic oscillator using Fermi normal coordinates (Sec. III). By tracing away the graviton, assumed to be in the vacuum state, we obtain a simple master equation of the Lindblad form [37, 38] for the quantum harmonic oscillator (Sec. IV). The obtained dynamics have some important features. The total energy of the quantum harmonic oscillator and of the emitted gravitons is conserved (Sec. IV A). The decoherence rate scales with the cube of the trapping harmonic frequency, which vanishes for a free particle as expected for a system without

[1] The results of [23] were first reported in a conference talk in Bangalore [24].

a mass quadrupole (Sec. IV B). For coherent states with large occupation numbers, we recover exactly the predictions for a classical linear quandrupole (Sec. IV C). For small occupation numbers, the classical and quantum predictions begin to differ, with the quantum harmonic oscillator retaining some coherence (Sec. IV D). In particular, the quantum harmonic oscillator settles in a partially coherent combination of the ground and first excited states, which is a distinct quantum signature of graviton emission. We conclude by providing order of magnitude estimates.

II. CLASSICAL QUADRUPOLE RADIATION

We begin by briefly summarizing the main features of classical gravitational radiation. We recall that gravitational radiation is sourced by a time-dependent mass quadrupole. In this work, we are primarily interested in the motion along a spatial axis where the simplest mass quadrupole is given by a coupled two-particle system [39] (see Fig. 1). For concreteness we consider two masses, m_1 and m_2, coupled by a quadratic potential:

$$H_{\text{two-particle}} = \frac{p^{2\ (1)}}{2m_1} + \frac{p^{2\ (2)}}{2m_2} + \frac{k}{2}(x^{(1)} - x^{(2)})^2, \quad (1)$$

where k is the spring constant, and $x^{(1)}$ ($p^{(1)}$) and $x^{(2)}$ ($p^{(2)}$) denote the position (momenta) of particle 1 and 2, respectively. It is useful to introduce the center-of-mass coordinates:

$$x \equiv x^{(1)} - x^{(2)}, \qquad p \equiv p^{(1)} + p^{(2)}, \quad (2)$$

$$x_{\text{cm}} \equiv \frac{m_1 x^{(1)} + m_2 x^{(2)}}{m_1 + m_2},\ p_{\text{cm}} \equiv \frac{m_2 p^{(1)} - m_1 p^{(2)}}{m_1 + m_2}, \quad (3)$$

as well as the reduced and total mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad (4)$$

$$M = m_1 + m_2, \quad (5)$$

respectively. Using the center-of-mass quantities we find that the potential in Eq. (1) reduces to

$$H_{\text{two-particle}} = \frac{p_{\text{cm}}^2}{2M} + \frac{p^2}{2\mu} + \frac{\mu\omega_{\text{m}}^2}{2}x^2, \quad (6)$$

where we have defined the harmonic frequency $\omega_{\text{m}}^2 \equiv k/\mu$.

We make two well-known observations. On one hand, we note that the center-of-mass remains uncoupled and is thus following a completely free motion (i.e., in the general relativistic language the center-of-mass position x_{cm} follows a geodesic). On the other hand, the relative motion is subject to a quadratic potential which can give rise to a time-dependent linear quadrupole moment (and hence can act as a source of gravitational radiation). In particular, we consider the following relative motion (in the first instance neglecting energy dissipation mechanisms):

$$x = l\cos(\omega_m t). \quad (7)$$

The corresponding quadrupole moment tensor is given by

$$D_{ij} = \int \rho(\boldsymbol{x}';\boldsymbol{x}) \left(3x_i' x_j' - r'^2\delta_{ij}\right) d\boldsymbol{x}', \quad (8)$$

where $\rho(\boldsymbol{x})$ is the mass density, $\boldsymbol{x} = (x_1, x_2, x_3)$, i, j denote the spatial components, and $r^2 = \sum_{i=1}^3 x_i^2$. Inserting in Eq. (8) the mass density

$$\rho(\boldsymbol{x}) = \mu\delta(x_1' - x)\delta(x_2')\delta(x_3'), \quad (9)$$

where x is given in Eq. (7), we readily find the following non-vanishing elements:

$$D_{11} = D_{22} = -\frac{1}{2}D_{33} = -\mu l^2 \cos^2(\omega_m t). \quad (10)$$

In particular, the linear quadrupole moment gives rise to gravitational radiation of type "+". The average energy carried away by gravitational waves is given by [39, 40]:

$$\dot{E} = -\frac{16GI^2\omega_{\text{m}}^6}{15c^2}, \quad (11)$$

where $I = ml^2$ is the moment of inertia (for rotations orthogonal to the x-axis passing through the center-of-mass $\boldsymbol{x}_{\text{cm}}$). Thus as the two-particle system is oscillating it will slowly lose energy – the amplitude of the relative motion, l, will decay, while the center-of-mass motion will remain completely unaffected. Any quantum model of quantized gravitational waves should recover the behaviour of classical quadrupole radiation when the state of the harmonic oscillator can be modeled as approximately classical (i.e., with a coherent state).

III. LINEARISED QUANTUM GRAVITY

In this section, our working hypothesis is linearised quantum gravity, for a review see [41]. We further assume matter to be non-relativistic, i.e. slowly moving. We find the dominant interaction between matter and graviton in the Fermi normal coordinates (FNC) which does not have any remnant gauge freedom [1, 42] and is commonly used to describe laboratory experiments [43].

A. Fermi normal coordinates

For simplicity, we will assume that the relevant motion of the particle is along the x-axis and consider the FNC coordinates, $x^\mu = (-ct, x, y, z)$, of an ideal observer following a geodesic trajectory (the situation of an observer following a generic time-like curve can be analyzed

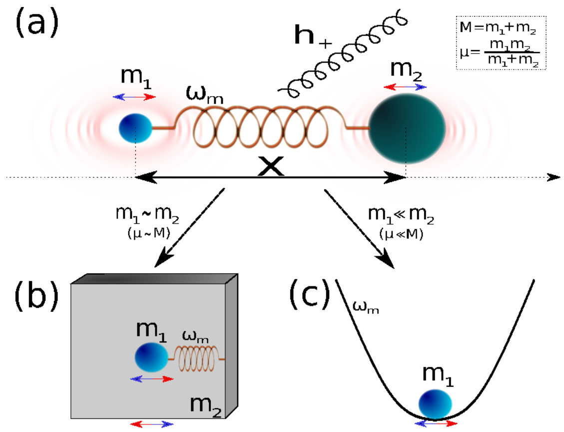

Figure 1. Graphical illustration of linear quadrupole radiation. (a) The linear quadrupole is generated by the relative motion of two masses, m_1 and m_2, which are coupled with coupling $\omega_{\rm m}$. The center-of-mass motion (mass M) is unperturbed by the emission of gravitational waves (which are of type "+"), while the relative motion (mass μ) slowly decays as its energy is converted to gravitational waves and radiated away. The two-particle problem can be always mapped to the problem of a harmonically trapped reduced mass $\mu=\frac{m_1m_2}{m_1+m_2}$ with coupling $\sim\mu x^2$, where x is relative distance between the two masses. (b) Case $m_1\sim m_2$. A physical realization consists of a trapped particle (i.e., mass m_1) inside a box containing the apparatus to generate a harmonic trap (i.e., mass m_2). The recoil of the system/box is on equal footing as $m_1\sim m_2$. (c) Case $m_1\ll m_2$. The problem reduces to the motion of the lighter mass m_1 in a harmonic trap with frequency $\omega_{\rm m}$. The recoil of the heavier mass m_2 is negligible with respect to the recoil of the lighter mass m_1 (which can be approximately identified with the reduced mass μ) .

in a similar fashion without affecting the final results). We start from the general relativistic point-particle Lagrangian:

$$L=-mc^2\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}, \tag{12}$$

where m is the mass of the system, c is the speed of light, and $g_{\mu\nu}$ is the metric expressed in FNC coordinates. In particular, we write the metric as $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, $\eta_{\mu\nu}$ is the Minkowski metric, and $h_{\mu\nu}$ the spacetime curvature perturbation near the geodesic up to order $\mathcal{O}(x^2)$ [44]. Assuming that matter is moving slowly, the dominant contribution to the dynamics will be given by [45]:

$$g_{00}=-(1+\frac{1}{2c^2}\frac{\partial h_{11}^2}{\partial t^2}|_{t=0}x^2), \tag{13}$$

where $\ddot{h}_{11}=2c^2R_{0101}$ is the "+" component of the gravitational waves usually discussed in the transverse-traceless (TT) coordinates, and R is the Riemann tensor (here $\ddot{h}_{11}\equiv\ddot{h}_{11}(t,0)$ denotes only a number evaluated on the reference FNC curve).

For completeness, let us sketch on how to derive Eq. (14). One can decompose the graviational field into plane waves $\sim e^{+ikx}$ and Taylor expand then up to order $\mathcal{O}(x^2)$:

$$h_{11}(t,x)\sim h(t,0)+\frac{\partial h(t,x)}{\partial x}|_{x=0}ikx-\frac{1}{2}\frac{\partial^2h(t,x)}{\partial x^2}|_{x=0}k^2x^2, \tag{14}$$

where $k=\omega_k/c$ and ω_k is the angular frequency of the gravitational field mode. The first term on the RHS of Eq. (14) is a constant and can be omitted, the second term $\sim kx$ vanishes in FNC coordinates (i.e., linear potentials $\sim x$ vanish by choosing an inertial reference frame), while the last term can be rewritten as

$$h_{11}(t,x)\sim-\frac{1}{2c^2}\frac{\partial^2h(t,x)}{\partial t^2}|_{t=0}x^2, \tag{15}$$

where we have used $\omega_k=kc$. For a rigorous derivation we refer the reader to [45].

Using Eqs. (12) and (13) we then readily find the interaction Lagrangian between graviton and matter degrees of freedom:

$$L_{\rm int}=-\frac{m}{4}\frac{\partial h_{11}^2}{\partial t^2}|_{t=0}x^2. \tag{16}$$

The quadratic coupling in Eq. (16) has been derived by assuming $k\Delta x\ll1$, where Δx is the charachteristic size of the system (implicitly assumed in the expansion in Eq. (14)). The latter condition can be rewritten also as $\omega_k\Delta x/c\ll1$, where we have used $\omega_k=kc$. Such a condition can be however rewritten entirely in terms of the properties of the system. In particular, we will find that gravitons couple to the matter system only if $\omega_k\sim2\omega_{\rm m}$, where $\omega_{\rm m}$ is the frequency of the harmonic oscillator (such a condition will naturally emerge from energy conservation of the total matter+graviton system). Our analysis will thus be valid if the state of the quantum harmonic oscillators satisfies $\omega_{\rm m}\Delta x\ll c$ (i.e., non-relativistic matter, with the trapped system moving much more slowly than the speed of light).

B. Harmonic oscillator coupled to gravitational waves

For the two-particle system we have the following interaction Lagrangian

$$L_{\rm int}=-\frac{1}{4}\ddot{h}_{11}\left[m_1(x^{(1)})^2+m_2(x^{(2)})^2\right], \tag{17}$$

where $x^{(1)}(x^{(2)})$ is the position of particle 1 (particle 2). We transform Eq. (17) using the center-of-mass coordinates introduced in Eqs. (2) and (3) to find:

$$L_{\rm int}=-\frac{1}{4}\ddot{h}_{11}\left[Mx_{\rm cm}^2+\mu x^2\right]), \tag{18}$$

where M (μ) is the total (reduced) mass.

From Eq. (18) it would thus appear that the center-of-mass will be more strongly affected by the coupling to the gravitational field than the relative motion (as we always have $M > \mu$) – we will show that this is *not* the case. In particular, we will find that only the relative degree of freedom can give rise to graviton emission, whilst leaving undisturbed the center-of-mass motion – in complete analogy to classical quadrupole radiation. Indeed, the quadrupole is linked to the harmonic trap $\sim \omega_{\mathrm{m}}^2 x^2$, while the center-of-mass motion is unconstrained and can thus only follow a geodesic (i.e., it is in free-fall) and as such cannot radiate. Hence we will first focus on the coupling of the relative motion, x, to the gravitational field. The decoupling of the center-of-mass from gravitational waves can then be ascertained from the formulae of the relative motion by taking the limit $\omega_{\mathrm{m}}^2 \to 0$, and replacing the reduced mass μ with the total mass M (in this limit equations for the center-of-mass and relative motion become of the same form).

C. Matter-graviton Hamiltonian

We now obtain the leading order coupling between gravitons and harmonically trapped quantum matter. We consider the gravitational field expanded in plane waves [15, 46]:

$$\hat{h}_{ij}(t,\boldsymbol{x}) = \int d\boldsymbol{k} \sqrt{\frac{G\hbar}{\pi^2 c^2 \omega_k}} \hat{g}_{\boldsymbol{k},\lambda} \mathsf{e}_{ij}^{\lambda}(\boldsymbol{n}) e^{-i(\omega_k t - \boldsymbol{k}\cdot\boldsymbol{x})} + \text{H.c.}, \tag{19}$$

where G is the Newton's constant, $\omega_k = kc$, $k = \|\boldsymbol{k}\|$, $\boldsymbol{n} = \boldsymbol{k}/\|\boldsymbol{k}\|$, and $\hat{g}_{\boldsymbol{k},\lambda}$ is the annihilation operator. In Eq. (19) we also implicitly assume the summation over the polarizations, $\sum_\lambda$, where $\mathsf{e}_{jk}^{\lambda}$ denote the basis tensors for the two polarizations, $\lambda = 1, 2$. The basis tensors satisfy the completeness relation:

$$\sum_\lambda \mathsf{e}_{ij}^{\lambda}(\boldsymbol{n}) \mathsf{e}_{kl}^{\lambda}(\boldsymbol{n}) = P_{ik}P_{jl} + P_{il}P_{jk} - P_{ij}P_{kl}, \tag{20}$$

where $P_{ij} \equiv P_{ij}(\boldsymbol{n}) = \delta_{ij} - \boldsymbol{n}_i \boldsymbol{n}_j$. From Eq. (16) and (19) we however see that only $\mathsf{e}_{11}^{\lambda}(\boldsymbol{n})$ is relevant for the matter-wave system. For later convenience we write the integral:

$$\int d\boldsymbol{n}\, P_{11}(\boldsymbol{n}) P_{11}(\boldsymbol{n}) = \frac{32\pi}{15}. \tag{21}$$

As we will see this latter expression quantifies the average effect (on the x-axis motional state of the harmonic oscillator) of the gravitons emitted in arbitrary directions.

We can readily write also the corresponding kinetic term for the massless graviton field:

$$H_{\text{grav}} = \int d\boldsymbol{k}\, \hbar \omega_k g_{\boldsymbol{k},\lambda}^{\dagger} \hat{g}_{\boldsymbol{k},\lambda}. \tag{22}$$

In addition, we assume that the matter degree of freedom is harmonically trapped and described by a simple Hamiltonian

$$H_{\mathrm{m}} = \frac{\hat{p}^2}{2m} + \frac{m\omega_{\mathrm{m}}^2}{2} \hat{x}^2, \tag{23}$$

where ω_{m} is the harmonic frequency. As we will see it is convenient to introduce the (adimensional) amplitude quadrature,

$$\hat{X} = \hat{b} + \hat{b}^{\dagger}, \tag{24}$$

which is related to the position observable as $\hat{x} = \delta_{\text{mzpf}} \hat{X}$, and the matter-zero-point-fluctuations are given by

$$\delta_{\text{mzpf}} = \sqrt{\frac{\hbar}{2m\omega_{\mathrm{m}}}}. \tag{25}$$

Furthermore, we introduce the (adimensional) phase quadrature

$$\hat{P} = i(\hat{b}^{\dagger} - \hat{b}) \tag{26}$$

which is related to the physical momentum observable as $\hat{p} = \sqrt{\frac{\hbar m \omega_{\mathrm{m}}}{2}} \hat{P}$. In particular, we can rewrite Eq. (23) in the standard notation:

$$\hat{H}_{\mathrm{m}} = \frac{\hbar\omega_{\mathrm{m}}}{4}(\hat{X}^2 + \hat{P}^2) = \hbar\omega_{\mathrm{m}} \hat{b}^{\dagger}\hat{b}. \tag{27}$$

The interaction Hamiltonian can be derived from the interaction Lagrangian in Eq. (16), and using Eq. (19) we find:

$$H_{\text{int}} = \sum_\lambda \int d\boldsymbol{k}\, \mathcal{G}_{\boldsymbol{k}}^{\lambda} \hat{g}_{\boldsymbol{k},\lambda} \hat{X}^2 + \text{H.c.}, \tag{28}$$

where the coupling is given by

$$\mathcal{G}_{\boldsymbol{k}}^{\lambda} = \sqrt{\frac{G\hbar^3 \omega_k^3}{64\pi^2 c^2 \omega_{\mathrm{m}}^2}} \mathsf{e}_{11}^{\lambda}(\boldsymbol{n}). \tag{29}$$

Importantly, we note that the coupling $\mathcal{G}_{\boldsymbol{k}}^{\lambda}$ in Eq. (29) does not depend on the mass of the matter system, but only on graviton and matter-wave frequencies, $\omega_{\boldsymbol{k}}$ and ω_m, respectively – as such, the effect on the matter system in the mesoscopic is precisely the same as, say, on atomic systems. Of course, for a given value of the position amplitude $\hat{X}$, since $\hat{x} \sim \hat{X}/\sqrt{m}$, the lighter system will have a larger physical position in comparison to the heavier one. Similarly, if we would express the harmonic frequency, ω_{m}, in terms of the spring contant, $K = \omega_{\mathrm{m}}^2/m$, we would again find a dependency of the coupling on the mass of the system m.

In summary, the total Hamiltonian is now given by:

$$\hat{H}_{\text{tot}} = \hat{H}_{\text{grav}} + \hat{H}_{\mathrm{m}} + \hat{H}_{\text{int}}, \tag{30}$$

where $\hat{H}_{\text{grav}}$, $\hat{H}_{\mathrm{m}}$, and $\hat{H}_{\text{int}}$ are given in Eqs. (22), (27), and (28).

IV. DEPHASING DUE TO GRAVITON BATH – QFT MODEL

In this section, we will obtain the dynamics of the matter system when coupled to the quantum field model of the gravitational field (see the previous Sec. III). We will refer to the developed model as the QFT model (the gravitational field will be considered from the QFT point of view, while the matter system will be modeled in the first quantization). The interaction between graviton and a matter wave system will give rise to a vertex diagram – the interaction vertex conserves the energy-momentum and the coupling strength are given by $\sqrt{G}$. We will assume that the graviton field is in the ground state, i.e. without any excitations (i.e., an initially empty bath):

$$\langle \hat{g}^{\dagger}_{\boldsymbol{k},\lambda}\hat{g}_{\boldsymbol{k}',\lambda'}\rangle = 0, \tag{31}$$

$$\langle \hat{g}_{\boldsymbol{k},\lambda}\hat{g}^{\dagger}_{\boldsymbol{k}',\lambda'}\rangle = \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{\lambda,\lambda'}, \tag{32}$$

and $\langle \hat{g}^{\dagger}_{\boldsymbol{k},\lambda}\hat{g}^{\dagger}_{\boldsymbol{k}',\lambda'}\rangle = \langle \hat{g}_{\boldsymbol{k},\lambda}\hat{g}_{\boldsymbol{k}',\lambda'}\rangle = 0$.

We construct the quantum master equation for the matter-wave system – by tracing out the gravitational field – closely following the generic derivation from [47] (see Chapter 3.3). We denote the total statistical operator of the problem as $\hat{\rho}^{(\mathrm{tot})}$ (the matter-wave system and the gravitational field), and by $\hat{\rho}$ ($\hat{\rho}^{(\mathrm{g})}$) the reduced statistical operator for the matter-wave system (the gravitational field), obtained by tracing away from the gravitational field (the matter-wave system). The von-Neumann equation can be expressed in the interaction picture as:

$$\frac{d}{dt}\hat{\rho}^{(\mathrm{tot})}_t = -\frac{i}{\hbar}[H^{(\mathrm{int})}_t, \hat{\rho}^{(\mathrm{tot})}_t], \tag{33}$$

where

$$\hat{H}^{(\mathrm{int})}_t = \sum_{\lambda}\int d\boldsymbol{k}\, \mathcal{G}^{\lambda}_{\boldsymbol{k}}\hat{g}_{\boldsymbol{k},\lambda}e^{-i\omega_{\boldsymbol{k}}t}\,\hat{X}^2_t + \mathrm{H.c.} \tag{34}$$

is the interaction Hamiltonian from Eq. (28) transformed into the interaction picture. The amplitude quadrature (the adimensional position observable) in the interaction picture is given by:

$$\hat{X}_t = \hat{b}e^{-i\omega_{\mathrm{m}}t} + \hat{b}^{\dagger}e^{i\omega_{\mathrm{m}}t}. \tag{35}$$

The dynamics in Eq. (33) can be formally solved:

$$\hat{\rho}^{(\mathrm{tot})}_t = \hat{\rho}^{(\mathrm{tot})}_0 - \frac{i}{\hbar}\int_0^t ds\,[\hat{H}^{(\mathrm{int})}_s, \hat{\rho}^{(\mathrm{tot})}_s]. \tag{36}$$

By then inserting Eq. (36) into Eq. (33), and tracing over the bath (the gravitational field), we obtain:

$$\frac{d}{dt}\hat{\rho}_t = -\frac{1}{\hbar^2}\int_0^t ds\,\mathrm{tr_g}[\hat{H}^{(\mathrm{int})}_t, [\hat{H}^{(\mathrm{int})}_s, \hat{\rho}^{(\mathrm{tot})}_s]], \tag{37}$$

where the first-order term, $\mathrm{tr_g}[H^{(\mathrm{int})}_t, \hat{\rho}^{(\mathrm{tot})}_0]$, vanished as $\langle \hat{g}_{\boldsymbol{k},\lambda}\rangle = \langle \hat{g}^{\dagger}_{\boldsymbol{k},\lambda}\rangle = 0$. On the other hand, the second-order term on the right-hand side of Eq. (37) is non-zero – as it depends on the value of the vacuum fluctuations, $\langle \hat{g}_{\boldsymbol{k},\lambda}\hat{g}^{\dagger}_{\boldsymbol{k}',\lambda'}\rangle$, defined in Eq. (32). Eq. (37) is, however, still a formal (exact) relation, containing the net effect of all Feynman diagrams with any number of vertices. We will now discuss the approximations that will lead to the more familiar Lindblad form of the quantum master equation – describing the effect of the dominant tree-level Feynman diagram contributions – exploiting the weakness of the coupling $\sim \sqrt{G}$.

We first impose the Born approximation, $\hat{\rho}^{(\mathrm{tot})}_s \approx \hat{\rho}_s \otimes \hat{\rho}^{(\mathrm{g})}$, on the right hand-side of Eq. (37). Importantly, the Born approximation precludes from the analysis any entanglement between the matter-wave system, and the gravitational field as we are explicitly assuming a factorizable state. Furthermore, the state of the graviton bath, $\hat{\rho}^{(\mathrm{g})} \equiv \hat{\rho}^{(\mathrm{g})}_0$, is always the same as far as the system is concerned – here we are assuming that the interaction between the system and the gravitational field is weak, with negligible effect on the latter.

We next want to make Eq. (37) local in time (i.e., such that the dynamics will depend only on the state $\hat{\rho}_t$ at time t, but not on the state at earlier times) and independent of the choice of the initial time. To this end, we first formally solve the von Neumann equation to connect the state at time s with the state at time t to find:

$$\hat{\rho}^{(\mathrm{tot})}_s = \hat{\rho}^{(\mathrm{tot})}_t + \frac{i}{\hbar}\int_s^t ds'\,[\hat{H}^{(\mathrm{int})}_{s'}, \hat{\rho}^{(\mathrm{tot})}_{s'}], \tag{38}$$

i.e., similarly, as we have done in Eq. (36). We then insert Eq. (38) into Eq. (37) to find:

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_t = &-\frac{1}{\hbar^2}\int_0^t ds\,\underbrace{\mathrm{tr_g}[\hat{H}^{(\mathrm{int})}_t, [\hat{H}^{(\mathrm{int})}_s, \hat{\rho}^{(\mathrm{tot})}_t]]}_{\sim G\ \text{terms}}\\ &-\frac{i}{\hbar^3}\int_0^t ds\int_s^t ds'\,\underbrace{\mathrm{tr_g}[\hat{H}^{(\mathrm{int})}_t, [\hat{H}^{(\mathrm{int})}_s, [\hat{H}^{(\mathrm{int})}_{s'}, \hat{\rho}^{(\mathrm{tot})}_{s'}]]]}_{\sim G^{3/2}\ \text{and higher order terms}},\end{aligned} \tag{39}$$

where we have explicitly separated the dominant terms $\sim G$ in the first line from the higher order terms in the second line (each Hamiltonian operator introduces a vertex $\sim \sqrt{G}$). In the following we will truncate the dynamics at the dominant order $\sim G$ which is justified by the weakness of the graviton-matter coupling $\sim \sqrt{G}$ (see Eqs. (29) and (34)). In this way, we find an equation that is local in time (i.e., it depends only on the state $\hat{\rho}_t$ at time t, but not on the state at earlier times). In addition, we change the integration variable $s \to t-s$ and extend the integration limit to infinity, i.e., $\int_0^t \to \int_0^\infty$, to find:

$$\frac{d}{dt}\hat{\rho}_t = -\frac{1}{\hbar^2}\int_0^\infty ds\,\mathrm{tr_g}[H^{(\mathrm{int})}_t, [H^{(\mathrm{int})}_{t-s}, \hat{\rho}^{(\mathrm{tot})}_t]]. \tag{40}$$

Extending the integration limit to infinity is allowed if the integrand decays sufficiently fast for values of s different from t – such an assumption is valid when the decay time of the graviton bath the correlation function is much faster than the time scale over which the state of the system changes appreciably. This makes the dynamics in Eq. (40) independent of the choice for the initial time (compare with Eq. (37)), which is a sensible requirement for non-relativistic matter coupled to the gravitational field. The steps in Eqs. (39) and (40) is equivalent to taking the Markov approximation commonly performed in analogous electromagnetic calculations (see Refs. [47–50] for more details).

In summary, applying the approximations from the previous two paragraphs to Eq. (37), we find the following Markovian master equation:

$$\frac{d}{dt}\hat{\rho}_t = -\frac{1}{\hbar^2}\int_0^\infty ds\, \mathrm{tr}_{\mathrm{g}}[H_t^{(\mathrm{int})},[H_{t-s}^{(\mathrm{int})},\hat{\rho}_t\otimes\hat{\rho}^{(\mathrm{g})}]]. \quad (41)$$

We then proceed by inserting the interaction Hamiltonian from Eq. (34) into Eq. (41) to eventually find

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_t = &-\frac{1}{\hbar^2}\int_0^\infty ds \int d\boldsymbol{k}\int d\boldsymbol{k}' \\ &\times \langle \hat{g}_{\boldsymbol{k},\lambda}\hat{g}^\dagger_{\boldsymbol{k}',\lambda'}\rangle\, \mathcal{G}^\lambda_{\boldsymbol{k}}\mathcal{G}^{\lambda'}_{\boldsymbol{k}'}\, e^{-i(\omega_k-\omega_{k'})t} \\ &\times \Big\{e^{-i\omega_{k'}s}\hat{X}_t^2\hat{X}_{t-s}^2\hat{\rho}_t - e^{i\omega_{k'}s}\hat{X}_t^2\hat{\rho}_t\hat{X}_{t-s}^2 \\ &\quad e^{i\omega_k s}\hat{\rho}_t\hat{X}_{t-s}^2\hat{X}_t^2 - e^{-i\omega_k s}\hat{X}_{t-s}^2\hat{\rho}_t\hat{X}_t^2\Big\}, \end{aligned} \quad (42)$$

where we have already used the fact that there are no excitations of the gravitational field, $\langle \hat{g}^\dagger_{\boldsymbol{k},\lambda}\hat{g}_{\boldsymbol{k}',\lambda'}\rangle = 0$ (see Eq. (31)).

We now insert the non-zero value for the vacuum fluctuations, $\langle \hat{g}_{\boldsymbol{k},\lambda}\hat{g}^\dagger_{\boldsymbol{k}',\lambda'}\rangle \sim \delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{\lambda,\lambda'}$ (see Eq. (32)):

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_t = &-\frac{1}{\hbar^2}\int_0^\infty ds \int d\boldsymbol{k}\sum_\lambda(\mathcal{G}^\lambda_{\boldsymbol{k}})^2 \\ &\times \Big\{e^{-i\omega_k s}\hat{X}_t^2\hat{X}_{t-s}^2\hat{\rho}_t - e^{i\omega_k s}\hat{X}_t^2\hat{\rho}_t\hat{X}_{t-s}^2 \\ &\quad e^{i\omega_k s}\hat{\rho}_t\hat{X}_{t-s}^2\hat{X}_t^2 - e^{-i\omega_k s}\hat{X}_{t-s}^2\hat{\rho}_t\hat{X}_t^2\Big\}, \end{aligned} \quad (43)$$

and inserting the expression for the coupling $\mathcal{G}^\lambda_{\boldsymbol{k}}$ from Eq. (29), to obtain:

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_t = &-\int_0^\infty ds \int d\boldsymbol{k}\,\frac{G\hbar\omega_k^3}{64\pi^2c^2\omega_m^2}\sum_\lambda \mathsf{e}^\lambda_{11}(\boldsymbol{k})\mathsf{e}^\lambda_{11}(\boldsymbol{k}) \\ &\times \Big\{e^{-i\omega_k s}\hat{X}_t^2\hat{X}_{t-s}^2\hat{\rho}_t - e^{i\omega_k s}\hat{X}_t^2\hat{\rho}_t\hat{X}_{t-s}^2 \\ &\quad e^{i\omega_k s}\hat{\rho}_t\hat{X}_{t-s}^2\hat{X}_t^2 - e^{-i\omega_k s}\hat{X}_{t-s}^2\hat{\rho}_t\hat{X}_t^2\Big\}. \end{aligned} \quad (44)$$

The summation can be evaluated using the completeness relation from Eq. (20) and the relation in Eq. (21) – we then integrate over the solid angle by first expressing the integration measure as $d\boldsymbol{k} = k^2dkd\boldsymbol{n} = \frac{\omega_k^2}{c^3}d\omega_k d\boldsymbol{n}$, where $k = \|\boldsymbol{k}\|$ and $\boldsymbol{n} = \boldsymbol{k}/\|\boldsymbol{k}\|$, we obtain:

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_t = &-\int_0^\infty ds\int_0^\infty d\omega_k\,\frac{G\hbar\omega_k^5}{30\pi c^5\omega_{\mathrm{m}}^2}. \\ &\times \Big\{e^{-i\omega_k s}\hat{X}_t^2\hat{X}_{t-s}^2\hat{\rho}_t - e^{i\omega_k s}\hat{X}_t^2\hat{\rho}_t\hat{X}_{t-s}^2 \\ &\quad e^{i\omega_k s}\hat{\rho}_t\hat{X}_{t-s}^2\hat{X}_t^2 - e^{-i\omega_k s}\hat{X}_{t-s}^2\hat{\rho}_t\hat{X}_t^2\Big\}, \end{aligned} \quad (45)$$

We now finally insert the position amplitude observable from Eq. (35), and apply the rotating wave approximation, i.e. we keep terms with equal number of $\hat{b}$ and $\hat{b}^\dagger$, and neglect the other fast rotating terms which typically give only a small correction, see [51]:

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_t = &-\int_0^\infty d\omega_k\,\frac{G\hbar\omega_k^5}{30\pi c^5\omega_{\mathrm{m}}^2}\int_0^\infty ds \\ &e^{-i(\omega_k-2\omega_{\mathrm{m}})s}\left(\hat{b}^{\dagger 2}\hat{b}^2\hat{\rho}_t - \hat{b}^2\hat{\rho}_t\hat{b}^{\dagger 2}\right) \\ &+ e^{i(\omega_k-2\omega_{\mathrm{m}})s}\left(\hat{\rho}_t\hat{b}^{\dagger 2}\hat{b}^2 - \hat{b}^2\hat{\rho}_t\hat{b}^{\dagger 2}\right) \end{aligned} \quad (46)$$

where we have kept only the non-zero contribution, $\sim \delta(\omega_s-\omega_k)$, while we have omitted the other contributions. Specifically, the terms $\sim \delta(\omega_s+\omega_k)$ are zero as the graviton cannot have negative frequency, while the terms $\sim \omega_k^5\delta(\omega_k)$ vanish. We then finally integrate overall possible out-going graviton frequencies, ω_k, using the fact that $\int_0^\infty ds\, e^{-i(\omega_k-2\omega_{\mathrm{m}})s} = \pi\delta(\omega_k-2\omega_{\mathrm{m}})$. Eventually, we find a simple Lindblad equation (in Schrödinger picture):

$$\frac{d}{dt}\hat{\rho}_t = \gamma_{\mathrm{grav}}\left(\hat{b}^2\hat{\rho}_t\hat{b}^{\dagger 2} - \frac{1}{2}\{\hat{b}^{\dagger 2}\hat{b}^2,\hat{\rho}_t\}\right), \quad (47)$$

where $\{\cdot\,,\cdot\}$ denotes the anti-commutator, and the emission rate is given by[2]

$$\gamma_{\mathrm{grav}} = \frac{32}{15}t_{\mathrm{Pl}}^2\omega_{\mathrm{m}}^3. \quad (48)$$

γ_{grav} is parameter-free and depends on fundamental constants of nature only through the Planck time, $t_{\mathrm{Pl}} = (G\hbar/c^5)^{1/2}$, and is independent of the mass or any other intrinsic or extrinsic property of the system apart from the frequency ω_{m} (of course, if one would express ω_{m} in terms of the mechanical spring constant, $K = \omega_{\mathrm{m}}^2/m$, then γ_{grav} would depend on the mass of the system as $m^{3/2}$). Here we have implicitly performed the calculation for long-wavelength gravitons (with wavelength λ large compared to the size of the interferometer Δx) as for typical experimental frequencies ω_{m} and superpositions sizes Δx we always have $\lambda = \frac{2\pi c}{\omega_{\mathrm{m}}} \gg \Delta x$. We leave

[2] The decoherence rate γ_{grav} in Eq.(48) depends only on the Planck time, t_{Pl}, and on the frequency of the matter-wave system, ω_{m}. Interestingly, if we set naively the mechanical frequency to the inverse of the Planck time, i.e. $\omega_{\mathrm{m}} \sim t_{\mathrm{Pl}}^{-1}$, then we find that the matter-wave system – even for single phonon occupancies – is expected to lose coherence in the Planck time, $t_{\mathrm{Pl}} \sim 10^{-43}$s.

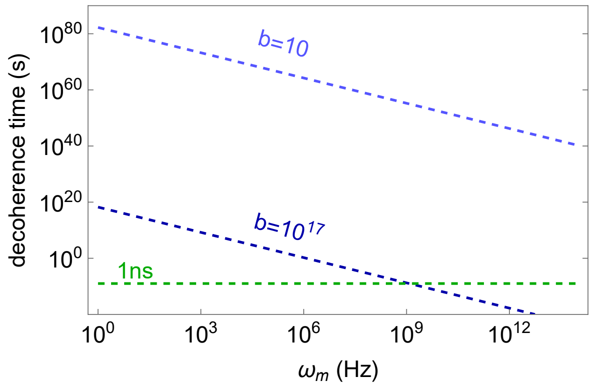

Figure 2. Approximate decoherence time for coherent states $|b\rangle$ as a function of the harmonic frequency ω_{m} (see Eq. (54)). For low number occupation numbers (e.g., $b = 10$) the decoherence time is very long and beyond experimental realities. To significantly reduce the decoherence time we have to consider large occupation numbers (e.g., $b = 10^{17}$). As an example we consider a harmonic oscillator with frequency $\omega_{\mathrm{m}} = 2\pi \times 1\,\mathrm{GHz}$ and the Planck mass $m = \sqrt{\hbar c/G} \approx 2 \times 10^{-8}\,\mathrm{kg}$. The green dashed line denotes a reference threshold decoherence time of 1 ns corresponding to one oscillation time $2\pi\,\omega_{\mathrm{m}}^{-1}$. The trapped Planck mass would have an associated displacement $\delta x = |b|\sqrt{\hbar/(2m\omega_{\mathrm{m}})} \approx 5\,\mathrm{cm}$ and velocity $\delta v = |b|\sqrt{\hbar\omega_{\mathrm{m}}/(2m)} \approx 10\,\mathrm{ms}^{-1}$. We leave feasibility studies for future research.

an analogous calculation for short-wavelength gravitons for future research (we would need to consider higher-order terms in the FNC expansion in Eq. (13)).

The quantum master equation in Eq. (47) is valid for a wide range of particle masses, from the microscopic, e.g. neutrons and atoms, to the mesoscopic scale and beyond, e.g. nanoparticles and micro-sized objects. The rate in Eq. (48) can be used to estimate the gravitational dephasing for any harmonically trapped system. We first define the *bare* gravitational decoherence time as

$$t_{\mathrm{d}} = \gamma_{\mathrm{grav}}^{-1}, \tag{49}$$

i.e., the inverse of the decoherence rate in Eq. (48).

The effective decoherence rate however depends on the initial state of the system. We can estimate the order of magnitude of decoherence by computing the decay of the phonon number:

$$\frac{d}{dt}\langle\hat{n}\rangle_t = \mathrm{tr}\left[\hat{n}\frac{d}{dt}\hat{\rho}_t\right], \tag{50}$$

where $\langle\,\cdot\,\rangle_t = \mathrm{tr}[\,\cdot\,\rho_t]$, and $\hat{n} = \hat{b}^\dagger\hat{b}$. Specifically, inserting the QFT model dynamics in Eq. (47), and the cyclic property of the trace, we readily find:

$$\frac{d}{dt}\langle\hat{n}\rangle_t = \gamma_{\mathrm{grav}}\langle\hat{b}^{\dagger 2}\hat{b}^\dagger\hat{b}\hat{b}^2 - \frac{1}{2}\{\hat{b}^{\dagger 2}\hat{b}^2, \hat{b}^\dagger\hat{b}\}\rangle_t. \tag{51}$$

Using the commutation relation $[\hat{b}, \hat{b}^\dagger] = 1$, we then eventually find:

$$\frac{d}{dt}\langle\hat{n}\rangle_t = -2\gamma_{\mathrm{grav}}\langle\hat{b}^{\dagger 2}\hat{b}^2\rangle_t, \tag{52}$$

where we note the *negative* sign of the expression in the right-hand side (this matches similar results obtained in the optical case [52]). The phonon decay signature in Eq. (52) could be detected in tailored made experiments probing energy dissipation [53] (we leave feasibility studies for future research). To get an order of magnitude estimate for the decoherence of a coherent state with a large occupation number we can replace the operators $\hat{b}$ and $\hat{b}^\dagger$ in Eq. (52) with the corresponding classical observables b and b^*:

$$\frac{d}{dt}\langle\hat{n}\rangle_t = -2\gamma_{\mathrm{grav}}|b|^4. \tag{53}$$

We can thus define the decoherence time for a coherent state $|b\rangle$ as

$$t_{\mathrm{d}}^{(\mathrm{estimated})} = t_{\mathrm{d}}/(2|b|^4), \tag{54}$$

We thus see that the effective decoherence time scales inversely with the square of the occupation number $n = |b|^2$ (see Fig. (2)).

A. Conservation of total energy

It is important to note that the Feynman vertex diagram $\sim \sqrt{G}$ arising from Eqs. (28) and (29) is energy conserving. Let us consider energy eigenstates of the harmonic oscillator which can be characterized by the energy $\hbar\omega$. The matter-wave systems of initial energy ω_{i} emits an *on-shell* graviton of frequency $2\omega_{\mathrm{m}}$ resulting in a final frequency $\omega_{\mathrm{f}} = \omega_{\mathrm{i}} - 2\omega_{\mathrm{m}}$ for the matter-wave system (with $\omega_{\mathrm{f}} < \omega_{\mathrm{i}}$). The graviton frequency $2\omega_{\mathrm{m}}$ arises from the quadratic position coupling of the matter-wave system, $\sim \hat{x}^2$, to the gravitational field, $\sim \hat{h}_{11}$ (see Eq. (16)). We can also understand why graviton emission is the only possible process at order $\sim \sqrt{G}$ based on physical considerations. As the gravitational field is initially in the lowest energy state (in the ground state) it can only absorb energy from the matter-system (i.e., graviton emission from the matter-wave system) while all other processes are forbidden by energy conservation.

The energy balance can be then summarized as:

$$\underbrace{\hbar\,\omega_{\mathrm{i}} - \hbar\,\omega_{\mathrm{f}}}_{\mathrm{matter}} = \underbrace{2\hbar\omega_{\mathrm{m}} = \hbar\omega_k}_{\mathrm{graviton}}, \tag{55}$$

where the initial (final) energy of the harmonic oscillator is $\hbar\,\omega_{\mathrm{i}}(\hbar\,\omega_{\mathrm{f}})$, and the energy of the emitted *on-shell* graviton is $\hbar\,\omega_k$. Importantly, as $\omega_{\mathrm{m}} > 0$ the matter system loses energy, i.e. $\omega_{\mathrm{f}} < \omega_{\mathrm{i}}$. The energy of the matter subsystem is monotonously decreasing with the energy carried away by the emitted gravitons, but the total energy of the matter-graviton system remains conserved. Importantly, we did not impose energy conservation at any stage, but rather the energy balance in Eq. (55) arises directly from the matter-graviton coupling and the quantum field theory analysis (see how the condition $\pi\delta(\omega_k - 2\omega_{\rm m})$ emerges after Eq. (46)). All other processes at order $\sim \sqrt{G}$ are forbidden by energy conservation.

B. Coherence protection for the center-of-mass of an isolated system

We recall that $\hat{\rho}_t$ in Eq. (47) is the state associated with the relative motion between two masses with reduced mass μ and coupling rate $\omega_{\rm m}$ associated to the mass quadrupole (see Fig. (1)). We first note that when we decouple the two masses, i.e., $\omega_{\rm m} \to 0$, then $\gamma_{\rm grav} \to 0$, then the decoherence rate in Eqs. (48) vanishes. This is not surprising, as the relative motion of two decoupled particles (a system without a mass quadrupole) is no longer coupled to on-shell gravitation (each mass will source a gravitational potential, but this happens via exchange of off-shell gravitons). Importantly, as can be seen from Eqs. (6) and (18) the Hamiltonian for the center-of-mass motion can be formally mapped to the Hamiltonian of the relative motion in the limiting case $\omega_{\rm m} \to 0$ (where the reduced mass μ is replaced by the total mass M). Thus we find the first key result: the coherence of the *center-of-mass* motion of an *isolated* system is decoupled from on-shell gravitons and its coherence will be *completely* protected.

C. Recovering classical gravitational radiation

We can further show that the quantum prediction $\frac{d}{dt}\langle \hat{H}_{\rm m}\rangle_t$ reduces to the classical energy decay from a linear quadrupole when we consider coherent states with large occupation numbers (i.e., classical-like states). In Eq. (53) we have already considered such a case, and by multiplying with $\hbar\omega_{\rm m}$ we immediately find:

$$\frac{d}{dt}\langle \hat{H}_{\rm m}\rangle_t = -2\hbar\omega_{\rm m}\gamma_{\rm grav}|b|^4. \tag{56}$$

In this case, the linear quadrupole moment is generated by the reduced mass μ with the motion given by $x = \mathrm{Re}[le^{i\omega t}]$ – this corresponds to the amplitude $b = le^{i\omega t}/(2\delta_{\rm mzpf})$ where the factor 2 in the denominator is the consequence of the chosen convention $x = b + b^*$, and $\delta_{\rm mzpf} = \sqrt{\frac{\hbar}{2\mu\omega_{\rm m}}}$ is the zero-point motion. In other words, we have :

$$|b| = l\sqrt{\frac{\mu\omega_{\rm m}}{2\hbar}}. \tag{57}$$

We now insert in Eq. (56) the amplitude from Eq. (57), the expression for the emission rate $\gamma_{\rm grav} = \frac{32}{15}t_{\rm Pl}^2\omega_{\rm m}^3$ from Eq. (48), and use the definition of the Planck time $t_{\rm Pl} = \sqrt{G\hbar/c^5}$, to eventually obtain

$$\frac{d}{dt}\langle \hat{H}_{\rm m}\rangle_t = -\frac{16GI^2\omega_{\rm m}^6}{15c^5}, \tag{58}$$

where we have introduced the moment of inertia $I = \mu l^2$. Importantly, Eq. (58) matches exactly the classical linear quadrupole radiation formula in Eq. (11). The obtained QFT model can be thus seen as the quantum counterpart of the classical gravitational theory.

D. Coherence protection for the relative-motion

Let us now discuss the consequences of the QFT model when we have low occupation numbers. The master equation in Eq. (47) can be solved analytically with the steady-state given by [54]:

$$\hat{\rho}_\infty = \lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1| + \lambda_c\left[|0\rangle\langle 1| + |1\rangle\langle 0|\right], \tag{59}$$

where λ_0,λ_1 and λ_{01} depend on the initial matter state $\hat{\rho}_0$, and $|\cdot\rangle$ denotes a number state (see Appendix A).

When discussing a coupling to an empty gravitational bath the naive expectation is that the system will eventually decay to the ground state emitting all of its energy – in line with the energy decay predicted by the classical gravitational radiation formula discussed above. However, as we start approaching the ground state the nature of the two-phonon process $\sim \hat{b}^2$ in Eq. (47) begins to modify the continuous classical picture. Indeed, applying twice the annihilation operator $\hat{b}$ to the number state $|n\rangle$ induces a transition:

$$|n\rangle \to |n-2\rangle, \tag{60}$$

and thus the number states $|1\rangle$, $|0\rangle$ are unable to decay further (as negative occupation numbers are prohibited by energy conservation). Hence, the matter-wave system decays to the state in Eq. (59) where it retains a remnant coherence λ_c.

Let us now consider some basic examples. We first consider a spatial superposition state: $|\psi_0\rangle \sim [|\beta\rangle + |-\beta\rangle]$, where $|\pm\beta\rangle$ denote coherent states, and $\beta \in \mathbb{R}$ can be interpreted as the superposition size. By writing the state on number basis one readily finds that $|\psi_0\rangle = \sum_n \langle 2n|\psi_0\rangle|2n\rangle$ which is expected for a state with even parity. The two-phonon process in Eq. (60) will then eventually lead to a decay of the state $|\psi_0\rangle$ to the ground state $|0\rangle$. We next consider the superposition of number states $|\psi_0\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$. By constructing $\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0|$ we notice we are in the steady state defined in Eq. (59) with $\lambda_0 = \lambda_1 = \lambda_c = 1/2$. Such states do not decay, but rather retain their coherence *indefinitely*.

V. SUMMARY AND DISCUSSION

In this paper, we have developed a quantum field theory (QFT) model to describe the emission of gravitons from a harmonically trapped system. A harmonic oscillator has a linear mass quadrupole which gives rise to graviton emission. If however, the mass quadrupole is abstent, then the system will not emit gravitons. As a result, a free isolated system or the center-of-mass degree of freedom does not decohere via graviton emission but rather retains its coherence indefinitely. Only systems with a mass quadrupole are coupled to on-shell gravitons.

Importantly our analysis recovers the classical gravitational radiation formulae when we consider coherent states with large occupation numbers (i.e., classical-like states). The QFT model predicts a deviation from classical predictions only when we are close to the ground state and the quantized nature of the fields becomes important. In particular, we have found that the state $\sim |0\rangle + |1\rangle$, i.e., a superposition of the ground and first excited states will remain coherent indefinitely. The reason is that the graviton emission process only allows transitions $|n\rangle \to |n-2\rangle$ for the harmonic oscillator, which can be seen as a direct consequence of Einstein's equivalence principle and of the quadrupole nature of gravitational waves. While linear potentials $\sim x$ vanish by choosing an inertial reference frame, the quadratic coupling $\sim x^2$ cannot be canceled by a change of coordinates and indeed it models the interaction with "+" graviational waves. In the quantum domain, the quadratic coupling $\hat{x}^2$ gives rise to the two-phonon transitions $|n\rangle \to |n-2\rangle$, which is the only process allowed by energy-momentum conservation.

It is interesting to estimate the order of magnitude of such effects for the QGEM (quantum gravity induced entanglement of masses) protocol [23]. In the original proposal, there are two quantum masses whose center of mass is separated by a distance d, while their spatial superposition size is assumed to be Δx. In order to obtain an entanglement phase of order one – due to exchange of virtual gravitons – the masses (assumed to be the same in the simplest case) were taken to be $m_1 = m_2 \sim 10^{-14}$kg, $d = 450\ \mu$m, and $\Delta x \sim 250\mu$m. The masses are kept in a well-preserved vacuum at low temperature to eliminate strong sources of decoherence mediated via electromagnetic interactions, and the entire setup is assumed to be in a free fall to minimize the effect of classical noise sources. Assuming that the interferometric loop is completed in a time $t = 1\,$s we can estimate an effective harmonic trap frequency as $\omega_{\mathrm{m}} \sim 2\pi \times 1\,$Hz. We then find that the amplitude is $b = \Delta x/\delta_{\mathrm{mzpf}} \sim 10^{14}$, where we recall that the zero-point motion of the matter system is $\delta_{\mathrm{mzpf}} = \sqrt{\hbar/(2m\omega_{\mathrm{m}})}$. From Fig. 2 we see that the expected decoherence time due to graviton emission is very large and should thus not pose an experimental problem (with $b \sim 10^{14}$ the decoherence time is between the two blue-dashed lines, which at $\omega_{\mathrm{m}} \sim 2\pi \times 1\,$Hz is longer than the age of the universe). In a more rigorous analysis we would need to decompose the interferometric paths in frequency space [55, 56] instead of using an effective harmonic frequency ω_{m} and solve the master equation numerically, but the order of magnitude of the effects should not change.

In Fig. 2 we have tentatively suggested that high-frequency mechanical oscillators could be used for testing graviton emission. The analysis however suggests that one would need to use states with large occupation numbers, where quantum and classical prediction cannot be distinguished experimentally. Indeed, the QFT model reduces to the classical predictions. Ideally, we would like a scheme with a harmonic oscillator near the ground state where quantum effects become more pronounced, but unfortunately there the graviton emission process is very slow. We nonetheless hope that the obtained theoretical results will inspire the development of schemes to test quantum effects related to quantized gravitational waves such as the discovered coherence protection mechanism.

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Appendix A: time evolution with gravitational emission

In this appendix, we provide for completeness the solution of the graviton emission dynamics in Eq. (47). We note that the emission of two phonons into the gravitational field which can be re-interpreted as the *absorption of two phonons by the gravitational field* – the dynamics can be thus formally mapped to the process of two-photon absorption by the optical field. We summarize the exact solution following the presentation of the optical case [54].

1. Exact solution

Following [54] we introduce a normalized time $\tau = 2\gamma_{\mathrm{grav}} t$ (such that when $\tau \sim 0.5$ we expect to see the first prominent effects) and the transformed density matrix

$$\psi_n(\mu, \tau) \equiv \sqrt{\frac{(n+\mu)!}{n!}} \langle n|\hat{\rho}_\tau|n+\mu\rangle. \tag{A1}$$

The solution for the elements with $\mu \neq 1$ is given by

$$\psi_n(\mu,\tau) = \sum_{\substack{k=n \\ (k-n \text{ even})}}^{\infty} \frac{(-1)^{k/2-n/2} 2^n}{n!} \times \frac{\Gamma(k/2+n/2+\sigma)}{\Gamma(\sigma)\Gamma(k/2-n/2+1)} A_k^\sigma e^{-\lambda_k \tau}, \tag{A2}$$

where

$$A_k^\sigma = \frac{(k+\sigma)\Gamma(\sigma)}{2^k \pi^{1/2}} \sum_{\substack{m=k \\ (m-k \text{ even})}}^{\infty} \frac{m!}{(m-k)} \times \frac{m!\Gamma(m/2-k/2+1/2)}{(m-k)!\Gamma(m/2+k/2+\sigma+1)} \psi_m(\mu,0), \quad \text{(A3)}$$

$\sigma = \frac{1}{2}(\mu-1)$ and $\lambda_k = k(k+\mu-1) + \frac{1}{2}\mu(\mu-1)$.

The solution for elements with $\mu = 1$ (with $n > 0$) is given by

$$\psi_n(1,\tau) = \sum_{\substack{k=n \\ (k-n \text{ even})}}^{\infty} \frac{(-1)^{k/2-n/2} 2^{n-1} k}{n!} \times \frac{\Gamma(k/2+n/2)}{\Gamma(k/2-n/2+1)} B_k e^{-k^2\tau} \quad \text{(A4)}$$

where

$$B_k = \sum_{\substack{m=k \\ (m-k \text{ even})}}^{\infty} \frac{m!}{(m/2+k/2)!(m/2-k/2)!} \times \frac{1}{2^{2-\delta(k)}} \psi_m(1,0), \quad \text{(A5)}$$

and $\delta(k) = 0$ ($\delta(k) = 1$) if $k = 0$ ($k > 0$).

We find that the two-phonon process in Eq. (47) induces a non-trivial steady-state:

$$\hat{\rho}_\infty = \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1| + \lambda_c \left[|0\rangle\langle 1| + |1\rangle\langle 0| \right], \quad \text{(A6)}$$

where λ_0,λ_1 and λ_{01} depend on the initial matter state $\hat{\rho}_0$,and $|\cdot\rangle$ denotes here a number state. In particular, we have that λ_0,λ_1 are the sum of the initial even/odd phonon numbers:

$$\lambda_0 = \sum_{n=0}^{\infty} \langle 2n|\hat{\rho}_0|2n\rangle, \; \lambda_1 = \sum_{n=0}^{\infty} \langle 2n+1|\hat{\rho}_0|2n+1\rangle, \quad \text{(A7)}$$

while the steady-state coherence is given by the sum of the initial coherences between neighboring number states:

$$\lambda_c = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \sqrt{\frac{(2n+1)!}{(2n)!}} \langle 2n|\hat{\rho}_0|2n+1\rangle. \quad \text{(A8)}$$

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