Quantum Field Theory of Fermion Mixing

M. BLASONE* AND G. VITIELLO†

*Dipartimento di Fisica dell'Università and INFN, Gruppo Collegato, Salerno,
I-84100 Salerno, Italy*

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The fermion mixing transformations are studied in the quantum field theory framework. In particular neutrino mixing is considered and the Fock space of definite flavor states is shown to be unitarily inequivalent to the Fock space of definite mass states. The flavor oscillation formula is computed for two and three flavors mixing and the oscillation amplitude is found to be momentum dependent, a result which may be subject to experimental test. The flavor vacuum state exhibits the structure of $SU(2)$ generalized coherent state.   © 1995 Academic Press, Inc.

1. INTRODUCTION

Mixing transformations of fermion fields play a crucial rôle in high energy physics. The original Cabibbo mixing of d and s quarks and its extension to the Kobayashi-Maskawa three flavors mixing are essential ingredients in the Standard Model phenomenology [1]. On the other hand, although clear experimental evidence is still missing, it is widely believed that neutrino mixing transformations are the basic tool for further understanding of neutrino phenomenology as well as of solar physics [2].

In contrast with the large body of succesful modelling and phenomenological computations (especially for Cabibbo-Kobayashi-Maskawa (CKM) quark mixing, the neutrino mixing still waiting for a conclusive experimental evidence), the quantum field theoretical analysis of the mixing transformations has not been pushed much deeply, as far as we know. The purpose of the present paper is indeed the study of the quantum field theory (QFT) framework of the fermion mixing transformations, thus focusing our attention more on the theoretical structure of fermion mixing than on its phenomenological features.

As we will see, our study is far from being purely academic since, by clarifying the theoretical framework of fermion mixing, we will obtain some results which are also interesting to phenomenlology and therefore to the real life of experiments.

* E-mail: blasone@vaxsa.csied.unisa.it.
† E-mail: vitiello@vaxsa.csied.unisa.it.

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In particular, to be definite, we will focus our attention on neutrino mixing transformations and our analysis will lead to some modifications of the neutrino oscillation formulas, which in fact may be subject to experimental test.

The paper is organized as follows. In Section 2 we study the generator of the Pontecorvo neutrino mixing transformations (two flavors mixing for Dirac fields). We show that in the Lehmann-Symanzik-Zimmermann (LZS) formalism of QFT [3] the Fock space of the flavor states is unitarily inequivalent to the Fock space of the mass eigenstates in the infinite volume limit. The flavor states are obtained as condensate of massive neutrino-antinuetrino pairs and exhibit the structure of $SU(2)$ coherent states [4]. In Section 3 we exhibit the condensation density as a function of the mixing angle, of the momentum and of the neutrino masses. In Section 4 we derive neutrino oscillations whose amplitude turns out to be momentum and mass dependent. This is a novel feature with respect to conventional analysis and may be subject to experimental test. In some sense, from the point of view of phenomenology, this is the most interesting result. Nevertheless, the condensate and coherent state structure of the vacuum is by itself a novel and theoretically interesting feature emerging from our analysis. In Section 5 we extend our considerations to three flavors mixing and show how the transformation matrix is obtained in terms of the QFT generators introduced in Section 2. We also obtain the three flavors oscillation formula. Finally, Section 6 is devoted to the discussion on the consequences of the theoretical results on the experiments and to the conclusions. Although the group theoretical analysis is conceptually simple, specific computations are sometime lengthy and, for the reader convenience, we confine mathematical details to the Appendices.

2. The Vaccum Structure for Fermion Mixing

For definitiveness we consider the Pontecorvo mixing relations [5], although the following discussion applies to any Dirac fields.

The mixing relations are:

$$v_e(x) = v_1(x) \cos \theta + v_2(x) \sin \theta$$
$$v_\mu(x) = -v_1(x) \sin \theta + v_2(x) \cos \theta, \tag{2.1}$$

where $v_e(x)$ and $v_\mu(x)$ are the (Dirac) netrino fields with definite flavors. $v_1(x)$ and $v_2(x)$ are the (free) neutrino fields with definite masses m_1 and m_2, respectively. The fields $v_1(x)$ and $v_2(x)$ are written as

$$v_i(x) = \frac{1}{\sqrt{V}} \sum_{k,r} [u_{k,i}^r \alpha_{k,e}^r e^{i\bar{k}\cdot\bar{x}} + v_{k,i}^r \beta_{k,i}^{r\dagger} e^{-i\bar{k}\cdot\bar{x}}], \qquad i = 1, 2. \tag{2.2}$$

In the following, for simplicity, we will omit the vector notation for $\bar{k}$ and use the same symbol k to denote both $\bar{k}$ and its modulus k. $\alpha_{k,i}^r$ and $\beta_{k,i}^r$, $i = 1, 2$, $r = 1, 2$

are the annihilation operators for the vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$: $\alpha^r_{k,i} |0\rangle_{12} = \beta^r_{k,i} |0\rangle_{12} = 0$. In Eq. (2.2) we have included the time dependence in the wave functions. In the following this dependence will be omitted when no misunderstanding arises. The anticommutation relations are

$$\{v^\alpha_i(x), v^{\beta\dagger}_j(y)\}_{t=t'} = \delta^3(x, y)\,\delta_{\alpha\beta}\,\delta_{ij}, \qquad \alpha, \beta = 1, ..., 4, \tag{2.3}$$

and

$$\{\alpha^r_{k,i}, \alpha^{s\dagger}_{q,j}\} = \delta_{kq}\,\delta_{rs}\,\delta_{ij}; \qquad \{\beta^r_{k,i}, \beta^{s\dagger}_{q,j}\} = \delta_{kq}\,\delta_{rs}\,\delta_{ij}, \qquad i, j = 1, 2. \tag{2.4}$$

All other anticommutators zero. The orthonoromality and completeness relations are:

$$\sum_\alpha u^{r\alpha*}_{k,i} u^{s\alpha}_{k,i} = \sum_\alpha v^{r\alpha*}_{k,i} v^{s\alpha}_{k,i} = \delta_{rs}, \qquad \sum_\alpha u^{r\alpha*}_{k,i} v^{s\alpha}_{-k,i} = \sum_\alpha v^{r\alpha*}_{-k,i} u^{s\alpha}_{i,k} = 0,$$

$$\sum_r (u^{r\alpha*}_{k,i} u^{r\beta}_{k,i} + v^{r\alpha*}_{-k,i} v^{r\beta}_{-k,i}) = \delta_{\alpha\beta}. \tag{2.5}$$

Eqs. (2.1) (or the ones obtained by inverting them) relate the respective hamiltonians $H_{1,2}$ (we consider only the mass terms) and $H_{e,\mu}$[5]

$$H_{1,2} = m_1\,\bar{v}_1 v_1 + m_2\,\bar{v}_2 v_2 \tag{2.6}$$

$$H_{e,\mu} = m_{ee}\,\bar{v}_e v_e + m_{\mu\mu}\,\bar{v}_\mu v_\mu + m_{e\mu}(\bar{v}_e v_\mu + \bar{v}_\mu v_e), \tag{2.7}$$

where $m_{ee} = m_1 \cos^2\theta + m_2 \sin^2\theta$, $m_{\mu\mu} = m_1 \sin^2\theta + m_2 \cos^2\theta$ and $m_{e\mu} = (m_2 - m_1)\sin\theta\cos\theta$.

In QFT the basic dynamics, i.e. the Lagrangian and the resulting field equations, is given in terms of Heisenberg (or interacting) fields. The physical observables are expressed in terms of asymptotic in- (or out-) fields, also called physical or free fields. In the LSZ formalism of QFT [3, 6, 7], the free fields, say for definitiveness the in-fields, are obtained by weak limit of the Heisenberg fields for time $t \to -\infty$. The meaning of the weak limit is that the realization of the basic dynamics in terms of the in-fields is not unique so that the limit for $t \to -\infty$ (or $t \to +\infty$ for the out-fields) is representation dependent. Typical examples are the ones of spontaneously broken symmetry theories, where the same set of Heisenberg field equations describes the normal (symmetric) phase as well as the symmetry broken phase. The representation dependence of the asymptotic limit arises from the existence in QFT of infinitely many unitarily non-equivalent representations of the canonical (anti-)commutation relations [6, 7]. Of course, since observables are described in terms of asymptotic fields, unitarily inequivalent representations describe different, i.e. physically inequivalent, phases. It is therefore of crucial importance, in order to get physically meaningful results, to investigate with much care the mapping among Heisenberg or interacting fields and free fields. Such a mapping is usually called the Haag expansion or the dynamical map [6, 7]. Only in a very rude and naive

approximation we may assume that interacting fields and free fields share the same vacuum state and the same Fock space representation.

We stress that the above remarks apply to QFT, namely to systems with infinite number of degrees of freedom. In quantum mechanics, where finite volume systems are considered, the von Neumann theorem ensures that the representations of the canonical commutation relations are each other unitarily equivalent and no problem arises with uniqueness of the asymptotic limit. In QFT, however, the von Neumann theorem does not hold and much more careful attention is required when considering any mapping among interacting and free fields [6, 7].

With this warnings, mixing relations such as the relations (2.1) deserve a careful analysis. It is in fact our purpose to investigate the structure of the Fock spaces $\mathscr{H}_{1,2}$ and $\mathscr{H}_{e,\mu}$ relative to $v_1(x)$, $v_2(x)$ and $v_e(x)$, $v_\mu(x)$, respectively. In particular we want to study the relation among these spaces in the infinite volume limit. We expect that $\mathscr{H}_{1,2}$ and $\mathscr{H}_{e,\mu}$ become orthogonal in such a limit, since they represent the Hilbert spaces for free and interacting fields, respectively [6, 7]. In the following, as usual, we will perform all computations at finite volume V and only at the end we will put $V \to \infty$.

Our first step is the study of the generator of Eqs. (2.1) and of the underlying group theoretical structure.

Eqs. (2.1) can be put in the form

$$v_e^\alpha(x) = G^{-1}(\theta)\, v_1^\alpha(x)\, G(\theta)$$
$$v_\mu^\alpha(x) = G^{-1}(\theta)\, v_2^\alpha(x)\, G(\theta), \tag{2.8}$$

where $G(\theta)$ is given by

$$G(\theta) = \exp\left[\theta \int d^3x(v_1^\dagger(x)\, v_2(x) - v_2^\dagger(x)\, v_1(x))\right], \tag{2.9}$$

and is (at finite volume) an unitary operator: $G^{-1}(\theta) = G(-\theta) = G^\dagger(\theta)$. We indeed observe that, from eqs. (2.8), $d^2 v_e^\alpha/d\theta^2 = -v_e^\alpha$, $d^2 v_\mu^\alpha/d\theta^2 = -v_\mu^\alpha$. By using the initial conditions $v_e^\alpha|_{\theta=0} = v_1^\alpha$, $dv_e^\alpha/d\theta|_{\theta=0} = v_2^\alpha$ and $v_\mu^\alpha|_{\theta=0} = v_2^\alpha$, $dv_\mu^\alpha/d\theta|_{\theta=0} = -v_1^\alpha$, we see that $G(\theta)$ generates Eqs. (2.1). Of course, $G(\theta)$ preserves the canonical anticommutation relations (2.3).

By introducing the operators

$$S_+ \equiv \int d^3x\, v_1^\dagger(x)\, v_2(x), \qquad S_- \equiv \int d^3x\, v_2^\dagger(x)\, v_1(x) = (S_+)^\dagger, \tag{2.10}$$

$G(\theta)$ can be written as

$$G(\theta) = \exp[\theta(S_+ - S_-)]. \tag{2.11}$$

It is easy to verify that, introducing S_3 and the total charge S_0 as

$$S_3 \equiv \tfrac{1}{2} \int d^3x(v_1^\dagger(x)\, v_1(x) - v_2^\dagger(x)\, v_2(x)),\tag{2.12}$$

$$S_0 \equiv \tfrac{1}{2} \int d^3x(v_1^\dagger(x)\, v_1(x) + v_2^\dagger(x)\, v_2(x)),\tag{2.13}$$

the $su(2)$ algebra is closed:

$$[S_+, S_-] = 2S_3, \qquad [S_3, S_\pm] = \pm S_\pm, \qquad [S_0, S_3] = [S_0, S_\pm] = 0.\tag{2.14}$$

Using Eq. (2.2) we can expand S_+, S_-, S_3 and S_0 as follows:

$$S_+ \equiv \sum_k S_+^k = \sum_k \sum_{r,s} (u_{k,1}^{r*} u_{k,2}^{s} \alpha_{k,1}^{r\dagger} \alpha_{k,2}^{s} + v_{-k,1}^{r*} u_{k,2}^{s} \beta_{-k,1}^{r} \alpha_{k,2}^{s}$$
$$+ u_{k,1}^{r*} v_{-k,2}^{s} \alpha_{k,1}^{r\dagger} \beta_{-k,2}^{s\dagger} + v_{-k,1}^{r*} v_{-k,2}^{s} \beta_{-k,1}^{r} \beta_{-k,2}^{s\dagger}),\tag{2.15}$$

$$S_- \equiv \sum_k S_-^k = \sum_k \sum_{r,s} (u_{k,2}^{r*} u_{k,1}^{s} \alpha_{k,2}^{r\dagger} \alpha_{k,1}^{s} + v_{-k,2}^{r*} u_{k,1}^{s} \beta_{-k,2}^{r} \alpha_{k,1}^{s}$$
$$+ u_{k,2}^{r*} v_{-k,1}^{s} \alpha_{k,2}^{r\dagger} \beta_{-k,1}^{s\dagger} + v_{-k,2}^{r*} v_{-k,1}^{s} \beta_{-k,2}^{r} \beta_{-k,1}^{s\dagger}),\tag{2.16}$$

$$S_3 \equiv \sum_k S_3^k = \tfrac{1}{2} \sum_{k,r} (\alpha_{k,1}^{r\dagger} \alpha_{k,1}^{r} - \beta_{-k,1}^{r\dagger} \beta_{-k,1}^{r} - \alpha_{k,2}^{r\dagger} \alpha_{k,2}^{r} + \beta_{-k,2}^{r\dagger} \beta_{-k,2}^{r}),\tag{2.17}$$

$$S_0 \equiv \sum_k S_0^k = \tfrac{1}{2} \sum_{k,r} (\alpha_{k,1}^{r\dagger} \alpha_{k,1}^{r} - \beta_{-k,1}^{r\dagger} \beta_{-k,1}^{r} + \alpha_{k,2}^{r\dagger} \alpha_{k,2}^{r} - \beta_{-k,2}^{r\dagger} \beta_{-k,2}^{r}).\tag{2.18}$$

It is interesting to observe that the operational structure of Eqs. (2.15) and (2.16) is the one of the rotation generator and of the Bogoliubov generator. These structures will be exploited in the following (cf. Section 3 and Appendix D). Using these expansions it is easy to show that the following relations hold:

$$[S_+^k, S_-^k] = 2S_3^k, \qquad [S_3^k, S_\pm^k] = \pm S_\pm^k, \qquad [S_0^k, S_3^k] = [S_0^k, S_\pm^k] = 0,\tag{2.19}$$

$$[S_\pm^k, S_\pm^p] = [S_3^k, S_\pm^p] = [S_3^k, S_3^p] = 0, \qquad k \neq p.\tag{2.20}$$

This means that the original $su(2)$ algebra given in Eqs. (2.14) splits into k disjoint $su_k(2)$ algebras, given by Eqs. (2.19), i.e. we have the group structure $\otimes_k SU_k(2)$.

To establish the relation between $\mathcal{H}_{1,2}$ and $\mathcal{H}_{e,\mu}$ we consider the generic matrix element $_{1,2}\langle a| v_1^\alpha(x) |b\rangle_{1,2}$ (a similar argument holds for $v_2^\alpha(x)$), where $|a\rangle_{1,2}$ is the generic element of $\mathcal{H}_{1,2}$. Using the inverse of the first of the (2.8), we obtain

$$_{1,2}\langle a| G(\theta)\, v_e^\alpha(x)\, G^{-1}(\theta) |b\rangle_{1,2} = {}_{1,2}\langle a| v_1^\alpha(x) |b\rangle_{1,2}.\tag{2.21}$$

Since the operator field v_e is defined on the Hilbert space $\mathcal{H}_{e,\mu}$, Eq. (2.21) shows that $G^{-1}(\theta)\,|a\rangle_{1,2}$ is a vector of $\mathcal{H}_{e,\mu}$, so $G^{-1}(\theta)$ maps $\mathcal{H}_{1,2}$ to $\mathcal{H}_{e,\mu}$: $G^{-1}(\theta): \mathcal{H}_{1,2} \mapsto \mathcal{H}_{e,\mu}$. In particular for the vacuum $|0\rangle_{1,2}$ we have (at finite volume V)

$$|0\rangle_{e,\mu} = G^{-1}(\theta)\,|0\rangle_{1,2}. \tag{2.22}$$

$|0\rangle_{e,\mu}$ is the vacuum for $\mathcal{H}_{e,\mu}$. In fact, from Eqs. (2.8) we obtain the positive frequency operators, i.e. the annihilators, relative to the fields $v_e(x)$ and $v_\mu(x)$ as

$$u^{r\alpha}_{k,e}\,\tilde{\alpha}^r_{k,e} = G^{-1}(\theta)\,u^{r\alpha}_{k,1}\,\alpha^r_{k,1}\,G(\theta), \tag{2.23a}$$

$$u^{r\alpha}_{k,\mu}\,\tilde{\alpha}^r_{k,\mu} = G^{-1}(\theta)\,u^{r\alpha}_{k,2}\,\alpha^r_{k,2}\,G(\theta), \tag{2.23b}$$

$$v^{r\alpha*}_{k,e}\,\tilde{\beta}^r_{k,e} = G^{-1}(\theta)\,v^{r\alpha*}_{k,1}\,\beta^r_{k,1}\,G(\theta), \tag{2.23c}$$

$$v^{r\alpha*}_{k,\mu}\,\tilde{\beta}^r_{k,\mu} = G^{-1}(\theta)\,v^{r\alpha*}_{k,2}\,\beta^r_{k,2}\,G(\theta). \tag{2.23d}$$

Eqs. (2.23) are obtained by using the linearity of operator $G(\theta)$. It is a trivial matter to check that these operators do effectively annihilate $|0\rangle_{e,\mu}$.

Furthermore, for the vacuum state $|0\rangle_{e,\mu}$ the conditions

$$\int d^3x\,v^\dagger_e(x)\,v_e(x)\,|0\rangle_{e,\mu} = 0, \qquad \int d^3x\,v^\dagger_\mu(x)\,v_\mu(x)\,|0\rangle_{e,\mu} = 0 \tag{2.24}$$

hold, as can be verified using Eqs. (2.8) and (2.22), or the definitions (2.23).

In Section 3 we will explicitly compute Eqs. (2.23), thus giving the dynamical map of the flavor operators in terms of the mass operators.

We observe that $G^{-1}(\theta) = \exp[\theta(S_- - S_+)]$ is just the generator for generalized coherent states of $SU(2)$: the flavor vacuum state is therefore an $SU(2)$ coherent state. Let us obtain the explicit expression for $|0\rangle_{e,\mu}$ and investigate the infinite volume limit of Eq. (2.22).

Using the Gaussian decomposition, $G^{-1}(\theta)$ can be written as [4]

$$\exp[\theta(S_- - S_+)] = \exp(-\tan\theta S_+)\exp(-2\ln\cos\theta S_3)\exp(\tan\theta S_-), \tag{2.25}$$

where $0 \leqslant \theta < \pi/2$. Eq. (2.22) then becomes

$$|0\rangle_{e,\mu} = \prod_k \exp(-\tan\theta S^k_+)\exp(-2\ln\cos\theta S^k_3)\exp(\tan\theta S^k_-)\,|0\rangle_{1,2}. \tag{2.26}$$

The right hand side of Eq. (2.26) may be computed by using the relations

$$S^k_3\,|0\rangle_{1,2} = 0, \qquad S^k_\pm\,|0\rangle_{1,2} \neq 0, \qquad (S^k_\pm)^2\,|0\rangle_{1,2} \neq 0, \qquad (S^k_\pm)^3\,|0\rangle_{1,2} = 0, \tag{2.27}$$

and other useful relations which are given in Appendix A. The final expression for $|0\rangle_{e,\mu}$ in terms of $S^k_\pm$ and S^k_3 is:

$$|0\rangle_{e,\mu} = \prod_k |0\rangle^k_{e,\mu} = \prod_k [1 + \sin\theta\cos\theta(S^k_- - S^k_+) + \tfrac{1}{2}\sin^2\theta\cos^2\theta((S^k_-)^2 + (S^k_+)^2)$$

$$- \sin^2\theta S^k_+ S^k_- + \tfrac{1}{2}\sin^3\theta\cos\theta(S^k_-(S^k_+)^2 - S^k_+(S^k_-)^2)$$

$$+ \tfrac{1}{4}\sin^4\theta(S^k_+)^2(S^k_-)^2] \, |0\rangle_{1,2}.\tag{2.28}$$

The state $|0\rangle_{e,\mu}$ is normalized to 1 (see Eq. (2.22)). Eq. (2.28) and Eqs. (2.15) and (2.16) exhibit the rich coherent state structure of $|0\rangle_{e,\mu}$.

Let us now compute ${}_{1,2}\langle 0|\rangle_{e,\mu}$. We obtain

$${}_{1,2}\langle 0|\rangle_{e,\mu} = \prod_k (1 - \sin^2\theta \; {}_{1,2}\langle 0| S^k_+ S^k_- |0\rangle_{1,2}$$

$$+ \tfrac{1}{4}\sin^4\theta \; {}_{1,2}\langle 0| (S^k_+)^2 (S^k_-)^2 |0\rangle_{1,2}),\tag{2.29}$$

where (see Appendix B)

$${}_{1,2}\langle 0| S^k_+ S^k_- |0\rangle_{1,2}$$

$$= {}_{1,2}\langle 0| \sum_{\sigma,\tau}\sum_{r,s} (v^{\sigma *}_{-k,1} u^\tau_{k,2})(u^{s *}_{k,2} v^r_{-k,1}) \beta^\sigma_{-k,1}\alpha^\tau_{k,2}\alpha^{s\dagger}_{k,2}\beta^{r\dagger}_{-k,1} |0\rangle_{1,2}$$

$$= \sum_{r,s} |v^{r *}_{-k,1} u^s_{k,2}|^2 \equiv Z_k.\tag{2.30}$$

In a similar way we find

$${}_{1,2}\langle 0| (S^k_+)^2 (S^k_-)^2 |0\rangle_{1,2} = Z^2_k.\tag{2.31}$$

Explicitly, Z_k is given by

$$Z_k = \frac{k^2[(\omega_{k,2}+m_2) - (\omega_{k,1}+m_1)]^2}{2\omega_{k,1}\omega_{k,2}(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)},\tag{2.32}$$

where $\omega_{k,i} = \sqrt{k^2 + m^2_i}$. The function Z_k depends on k only through its modulus and it is always in the interval $[0, 1[$. It has its maximum at $k = \sqrt{m_1 m_2}$ which tends asymptotically to 1 when $(m_2 - m_1)^2/m_1 m_2 \to \infty$; also, $Z_k \to 0$ when $k \to \infty$.

In conclusion we have

$${}_{1,2}\langle 0|0\rangle_{e,\mu} = \prod_k (1 - \tfrac{1}{2}Z_k \sin^2\theta)^2 \equiv \prod_k \Gamma(k)$$

$$= \prod_k e^{\ln\Gamma(k)} = e^{\sum_k \ln\Gamma(k)}.\tag{2.33}$$

By using the customary continuous limit relation $\sum_k \to (V/(2\pi)^3) \int d^3k$, in the infinite volume limit we obtain

$$\lim_{V \to \infty} {}_{1,2}\langle 0|0 \rangle_{e,\mu} = \lim_{V \to \infty} e^{(V/(2\pi)^3) \int d^3k \ln \Gamma(k)} = 0 \qquad (2.34)$$

since $\Gamma(k) < 1$ for any value of k and of the parameters m_1 and m_2.

Notice that (2.34) shows that the orthogonality between $|0\rangle_{e,\mu}$ and $|0\rangle_{1,2}$ is due to the infrared contributions which are taken in care by the infinite volume limit and therefore high momentum contributions do not influence the result (for this reason here we do not need to consider the regularization problem of UV divergence of the integral of $\ln \Gamma(k)$).

Of course, this orthogonality disappears when $\theta = 0$ and/or when $m_1 = m_2$ (because in this case $Z_k = 0$ for any k and no mixing occurs in Pontecorvo theory).

Eq. (2.34) expresses the unitary inequivalence in the infinite volume limit of the flavor and the mass representations and shows the absolutely non-trivial nature of the mixing transformations (2.1). In other words, the mixing transformations induce a physically nontrivial structure in the flavor vacuum which indeed turns out to be an $SU(2)$ generalized coherent state. In Section 4 we will see how such a vacuum structure may lead to phenomenological consequences in the neutrino oscillations, which possibly may be experimentally tested. From Eq. (2.34) we also see that Eq. (2.22) is a purely formal expression which only holds at finite volume.

We thus realize the limit of validity of the approximation usually adopted when the mass vacuum state (representation for definite mass operators) is identified with the vacuum for the flavor operators. We point out that even at finite volume the vacua identification is actually an approximation since the flavor vacuum is an $SU(2)$ generalized coherent state. In such an approximation, the coherent state structure and many physical features are missed.

3. THE NUMBER OPERATOR, THE DYNAMICAL MAP, AND THE MASS SECTORS

We now calculate the number of the particles condensed in the state $|0\rangle_{e,\mu}$.

Let us consider, for example, the $\alpha_{k,1}$ particles. As usual we define the number operator as $N^k_{\alpha_1} \equiv \sum_r \alpha^{r\dagger}_{k,1} \alpha^r_{k,1}$ and use the fact that $N^k_{\alpha_1}$ commutes with S^p_3 and with $S^p_\pm$ for $p \neq k$. Then we can write

$$_{e,\mu}\langle 0| N^k_{\alpha_1} |0\rangle_{e,\mu} = {}^k_{e,\mu}\langle 0| N^k_{\alpha_1} |0\rangle^k_{e,\mu}. \qquad (3.1)$$

$|0\rangle^k_{e,\mu}$ has been introduced in Eq. (2.28). From Eq. (3.1) and the relations given in Appendix C we obtain:

$$_{e,\mu}\langle 0| N^k_{\alpha_1} |0\rangle_{e,\mu} = Z_k \sin^2 \theta. \qquad (3.2)$$

The same result is obtained for the number operators $N^k_{\alpha_2}$, $N^k_{\beta_1}$, $N^k_{\beta_2}$:

$$_{e,\mu}\langle 0| \, N^k_{\sigma_i} \, |0\rangle_{e,\mu} = Z_k \sin^2 \theta, \qquad \sigma = \alpha, \beta, \qquad i = 1, 2. \tag{3.3}$$

Eq. (3.2) gives the condensation density of the flavor vacuum state as a function of the mixing angle θ, of the masses m_1 and m_2, and of the momentum modulus k. This last feature is particularly interesting since, as we will see, the vacuum acts as a "momentum (or spectrum) analyzer" when time-evolution and flavor oscillations are considered.

We remark that the Eq. (3.2) (and (3.3)) clearly shows that the flavor vacuum $|0\rangle_{e,\mu}$ is not annihilated by the operators $\alpha^r_{k,i}$, $\beta^r_{k,i}$, $i = 1, 2$. This is in contrast with the usual treatment where flavor vacuum $|0\rangle_{e,\mu}$ is identified with the mass vacuum $|0\rangle_{1,2}$.

Notice that, due to the behaviour of Z_k fo high k, expectation values of $N^k_{\sigma_i}$ are zero for high k (the same is true for any operator O^k_i, $i = 1, 2$, for which is $_{1,2}\langle 0| \, O^k_i \, |0\rangle_{1,2} = 0$).

In order to explicitly exhibit the dynamical map, Eqs. (2.23), it is convenient to redefine the operational parts of the fields $v_e(x)$ and $v_\mu(x)$ as (cf. Eqs. (2.23)) $u^{r,s}_{k,1} \alpha^r_{k,e} \equiv u^{r,s}_{k,e} \tilde{\alpha}^r_{k,e}$, etc., so that we can write:

$$\alpha^r_{k,e} \equiv G^{-1}(\theta) \, \alpha^r_{k,1} \, G(\theta), \tag{3.4a}$$

$$\alpha^r_{k,\mu} \equiv G^{-1}(\theta) \, \alpha^r_{k,2} \, G(\theta), \tag{3.4b}$$

$$\beta^r_{k,e} \equiv G^{-1}(\theta) \, \beta^r_{k,1} \, G(\theta), \tag{3.4c}$$

$$\beta^r_{k,\mu} \equiv G^{-1}(\theta) \, \beta^r_{k,2} \, G(\theta). \tag{3.4d}$$

We observe that $\alpha^r_{k,l}$ and $\beta^r_{k,l}$, $l = e, \mu$, depend on time through the time dependence of $G(\theta)$. We obtain

$$\alpha^r_{k,e} = \cos\theta\alpha^r_{k,1} + \sin\theta \sum_s ((u^{r*}_{k,1} u^s_{k,2}) \, \alpha^s_{k,2} + (u^{r*}_{k,1} v^s_{-k,2}) \, \beta^{s\dagger}_{-k,2}) \tag{3.5a}$$

$$\alpha^r_{k,\mu} = \cos\theta\alpha^r_{k,2} - \sin\theta \sum_s ((u^{r*}_{k,2} u^s_{k,1}) \, \alpha^s_{k,1} + (u^{r*}_{k,2} v^s_{-k,1}) \, \beta^{s\dagger}_{k,1}) \tag{3.5b}$$

$$\beta^r_{-k,e} = \cos\theta\beta^r_{-k,1} + \sin\theta \sum_s ((v^{s*}_{-k,2} v^r_{-k,1}) \, \beta^s_{-k,2} + (u^{s*}_{k,2} v^r_{-k,1}) \, \alpha^{s\dagger}_{k,2}) \tag{3.5c}$$

$$\beta^r_{-k,\mu} = \cos\theta\beta^r_{-k,2} - \sin\theta \sum_s ((v^{s*}_{-k,1} v^r_{-k,2}) \, \beta^s_{-k,1} + (u^{s*}_{k,1} v^r_{-k,2}) \, \alpha^{s\dagger}_{k,1}) \tag{3.5d}$$

Without loss of generality, we can choose the reference frame such that $k = (0, 0, |k|)$. This implies that only the products of wave functions with $r = s$ will survive (see Appendix B). Eqs. (3.5) then assume the simpler form

$$\alpha^r_{k,e} = \cos\theta\alpha^r_{k,1} + \sin\theta(U^*_k \alpha^r_{k,2} + \epsilon^r V_k \beta^{r\dagger}_{-k,2}) \tag{3.6a}$$

$$\alpha^r_{k,\mu} = \cos\theta\alpha^r_{k,2} - \sin\theta(U_k \alpha^r_{k,1} - \epsilon^r V_k \beta^{r\dagger}_{-k,1}) \tag{3.6b}$$

$$\beta^r_{-k,\,c} = \cos\theta\beta^r_{\ k,\,1} + \sin\theta(U^*_k\beta^r_{\ k,\,2} - \varepsilon^r V_k \alpha^{r\dagger}_{k,\,2}) \qquad (3.6c)$$

$$\beta^r_{-k,\,\mu} = \cos\theta\beta^r_{-k,\,2} - \sin\theta(U_k\beta^r_{-k,\,1} + \varepsilon^r V_k \alpha^{r\dagger}_{k,\,1}) \qquad (3.6d)$$

with $\varepsilon^r = (-1)^r$ and

$$U_k \equiv (u^{r*}_{k,\,2} u^r_{k,\,1}) = (v^{r*}_{-k,\,1} v^r_{-k,\,2}) \qquad (3.7a)$$

$$V_k \equiv \varepsilon^r(u^{r*}_{k,\,1} v^r_{-k,\,2}) = -\varepsilon^r(u^{r*}_{k,\,2} v^r_{-k,\,1}), \qquad (3.7b)$$

where the time dependence of U_k and V_k has been omitted. We have

$$V_k = |V_k|\, e^{i(\omega_{k,\,2}+\omega_{k,\,1})t}, \qquad U_k = |U_k|\, e^{i(\omega_{k,\,2}-\omega_{k,\,1})t} \qquad (3.8)$$

$$|U_k| = \left(\frac{\omega_{k,\,1}+m_1}{2\omega_{k,\,1}}\right)^{1/2}\left(\frac{\omega_{k,\,2}+m_2}{2\omega_{k,\,2}}\right)^{1/2}\left(1+\frac{k^2}{(\omega_{k,\,1}+m_1)(\omega_{k,\,2}+m_2)}\right) \qquad (3.9a)$$

$$|V_k| = \left(\frac{\omega_{k,\,1}+m_1}{2\omega_{k,\,1}}\right)^{1/2}\left(\frac{\omega_{k,\,2}+m_2}{2\omega_{k,\,2}}\right)^{1/2}\left(\frac{k}{(\omega_{k,\,2}+m_2)}-\frac{k}{(\omega_{k,\,1}+m_1)}\right) \qquad (3.9b)$$

$$|U_k|^2 + |V_k|^2 = 1, \qquad |V_k|^2 = \frac{1}{2}Z^k. \qquad (3.10)$$

For notational simplicity in the following we put $\omega_i \equiv \omega_{k,\,i}$.

It is also interesting to exhibit the explicit expression of $|0\rangle^k_{c,\,\mu}$ in the reference frame for which $k = (0, 0, |k|)$ (see Appendix D):

$$|0\rangle^k_{c,\,\mu} = \prod_r [(1 - \sin^2\theta\,|V_k|^2) - \varepsilon^r \sin\theta\cos\theta\, V_k(A^r_k + B^r_k)$$

$$+ \varepsilon^r \sin^2\theta V_k(U^*_k C^r_k - U_k D^r_k) + \sin^2\theta V^2_k A^r_k B^r_k]\,|0\rangle_{1,\,2} \qquad (3.11)$$

with

$$A^r_k \equiv \alpha^{r\dagger}_{k,\,1}\beta^{r\dagger}_{-k,\,2}, \qquad B^r_k \equiv \alpha^{r\dagger}_{k,\,2}\beta^{r\dagger}_{-k,\,1}, \qquad C^r_k \equiv \alpha^{r\dagger}_{k,\,1}\beta^{r\dagger}_{-k,\,1}, \qquad D^r_k \equiv \alpha^{r\dagger}_{k,\,2}\beta^{r\dagger}_{-k,\,2}. \qquad (3.12)$$

We observe that Eqs. (3.6) can be obtained by a rotation and by a subsequent Bolgoliubov transformation. To see this is convenient to put (cf. Eq. (3.10))

$$|U_k| \equiv \cos\Theta_k, \qquad |V_k| \equiv \sin\Theta_k, \qquad 0 \leqslant \Theta_k < \frac{\pi}{4} \qquad (3.13)$$

and

$$e^{i(\omega_1-\omega_2)t} \equiv e^{i\psi}, \qquad e^{2i\omega_1 t} \equiv e^{i\phi_1}, \qquad e^{2i\omega_2 t} \equiv e^{i\phi_2} \qquad (3.14)$$

so that Eqs. (3.6) are rewritten as

$$\alpha^r_{k,c} = B_2^{-1} R^{-1} \alpha^r_{k,1} R B_2 \tag{3.15a}$$

$$\beta^r_{-k,c} = B_2^{-1} R^{-1} \beta^r_{-k,1} R B_2 \tag{3.15b}$$

$$\alpha^r_{k,\mu} = B_1^{-1} R^{-1} \alpha^r_{k,2} R B_1 \tag{3.15c}$$

$$\beta^r_{-k,\mu} = B_1^{-1} R^{-1} \beta^r_{-k,2} R B_1, \tag{3.15d}$$

where

$$R = \exp\left\{ \theta \sum_{k,r} \left[(\alpha^{r\dagger}_{k,1} \alpha^r_{k,2} + \beta^{r\dagger}_{-k,1} \beta^r_{-k,2}) e^{i\psi} \right.\right.$$

$$\left.\left. - (\alpha^{r\dagger}_{k,2} \alpha^r_{k,1} + \beta^{r\dagger}_{-k,2} \beta^r_{k,1}) e^{-i\psi} \right] \right\} \tag{3.16}$$

$$B_1 = \exp\left\{ -\sum_{k,r} \Theta_k \varepsilon^r [\alpha^r_{k,1} \beta^r_{-k,1} e^{-i\phi_1} - \beta^{r\dagger}_{-k,1} \alpha^{r\dagger}_{k,1} e^{i\phi_1}] \right\} \tag{3.17}$$

$$B_2 = \exp\left\{ \sum_{k,r} \Theta_k \varepsilon^r [\alpha^r_{k,2} \beta^r_{-k,2} e^{-i\phi_2} - \beta^{r\dagger}_{-k,2} \alpha^{r\dagger}_{k,2} e^{i\phi_2}] \right\}. \tag{3.18}$$

By use of these relations and noting that $R |0\rangle_{1,2} = |0\rangle_{1,2}$, we can "separate" the sectors $\{|0(\Theta)\rangle_1\}$ and $\{|0(\Theta)\rangle_2\}$ out of the full representation space $\{|0\rangle_{e,\mu}\}$: $\{|0(\Theta)\rangle_1\} \otimes \{|0(\Theta)\rangle_2\} \subset \{|0\rangle_{e,\mu}\}$.

The states $|0(\Theta)\rangle_1$ and $|0(\Theta)\rangle_2$ are respectively obtained as

$$|0(\Theta)\rangle_1 \equiv B_1^{-1}(\Theta) |0\rangle_1 = \prod_{k,r} (\cos \Theta_k + \varepsilon^r e^{i\phi_1} \sin \Theta_k \beta^{r\dagger}_{-k,1} \alpha^{r\dagger}_{k,1}) |0\rangle_1 \tag{3.19}$$

$$|0(\Theta)\rangle_2 \equiv B_2^{-1}(\Theta) |0\rangle_2 = \prod_{k,r} (\cos \Theta_k - \varepsilon^r e^{i\phi_2} \sin \Theta_k \beta^{r\dagger}_{-k,2} \alpha^{r\dagger}_{k,2}) |0\rangle_2. \tag{3.20}$$

In the Appendix E we comment on the condensate structure of the sectors $\{|0(\Theta)\rangle_i\}$, $i = 1, 2$.

Finally, we observe that $|(\Theta)\rangle_i$ can be written as

$$|0(\Theta)\rangle_i = \exp\left(-\frac{S_{\alpha_i}}{2} \right) |I_i\rangle = \exp\left(-\frac{S_{\beta_i}}{2} \right) |I_i\rangle \tag{3.21}$$

with $|I_i\rangle \equiv \exp(\sum_{k,r} (-1)^{i+1} \varepsilon^r e^{i\phi_i} \beta^{r\dagger}_{-k,i} \alpha^{r\dagger}_{k,i}) |0\rangle_{1,2}$, and

$$S_{\alpha_i} = -\sum_{k,r} (\alpha^{r\dagger}_{k,i} \alpha^r_{k,i} \log \sin^2 \Theta_k + \alpha^r_{k,i} \alpha^{r\dagger}_{k,i} \log \cos^2 \Theta_k), \qquad i = 1, 2. \tag{3.22}$$

A similar expression holds for S_{β_i}. It is known that S_{α_i} (or S_{β_i}) can be interpreted as the entropy function associated to the vacuum codensate [7].

4. Neutrino Oscillations

We are now ready to study the flavor oscillations. In order to compare our result with the conventional one [1, 2, 5] we first reproduce the usual oscillation formula.

In the original Pontecorvo and collaborators treatment [5], the vacuum state for definite flavor neutrinos is identified with the vacuum state for definite mass neutrinos: $|0\rangle_{e,\mu} = |0\rangle_{1,2} \equiv |0\rangle$. As we have shown in the previous Section, such an identification is not possible in QFT; however, it is allowed at finite volume where no problem of unitary inequivalence arises in the choice of the Hilbert space. As already observed, even at finite volume, the vacua identification is only an approximation. For shortness we refer to such an identification simply as to the finite volume approximation, meaning by that the approximation which is allowed at finite volume.

The number operators relative to electronic and muonic neutrinos are

$$N^{k,r}_{\alpha_e} = \alpha^{r\dagger}_{k,e} \alpha^r_{k,e} = \cos^2 \theta \alpha^{r\dagger}_{k,1} \alpha^r_{k,1} + \sin^2 \theta \alpha^{r\dagger}_{k,2} \alpha^r_{k,2}$$

$$+ \sin \theta \cos \theta (\alpha^{r\dagger}_{k,1} \alpha^r_{k,2} + \alpha^{r\dagger}_{k,2} \alpha^r_{k,1}), \tag{4.1}$$

$$N^{k,r}_{\alpha_\mu} = \alpha^{r\dagger}_{k,\mu} \alpha^r_{k,\mu} = \cos^2 \theta \alpha^{r\dagger}_{k,2} \alpha^r_{k,2} + \sin^2 \theta \alpha^{r\dagger}_{k,1} \alpha^r_{k,1}$$

$$- \sin \theta \cos \theta (\alpha^{r\dagger}_{k,1} \alpha^r_{k,2} + \alpha^{r\dagger}_{k,2} \alpha^r_{k,1}), \tag{4.2}$$

and, obviously

$$\langle 0| N^{k,r}_{\alpha_e} |0\rangle = \langle 0| N^{k,r}_{\alpha_\mu} |0\rangle = 0. \tag{4.3}$$

The one electronic neutrino state is of the form

$$|\alpha^r_{k,e}\rangle = \cos \theta |\alpha^r_{k,1}\rangle + \sin \theta |\alpha^r_{k,2}\rangle. \tag{4.4}$$

where $|\alpha^r_{k,e}\rangle \equiv \alpha^{r\dagger}_{k,e} |0\rangle$, $|\alpha^r_{k,1}\rangle \equiv \alpha^{r\dagger}_{k,1} |0\rangle$, $|\alpha^r_{k,2}\rangle \equiv \alpha^{r\dagger}_{k,2} |0\rangle$. The time evoluted of this state is controlled by the time evolution of $|\alpha^r_{k,1}\rangle$ and $|\alpha^r_{k,2}\rangle$

$$|\alpha^r_{k,e}(t)\rangle = e^{iH_{1,2}t} |\alpha^r_{k,e}\rangle = \cos \theta e^{-i\omega_1 t} |\alpha^r_{k,1}\rangle + \sin \theta e^{-i\omega_2 t} |\alpha^r_{k,2}\rangle. \tag{4.5}$$

We have

$$\langle \alpha^r_{k,e}| N^{k,r}_{\alpha_e} |\alpha^r_{k,e}\rangle = 1 \tag{4.6}$$

and

$$\langle \alpha^r_{k,e}(t)| N^{k,r}_{\alpha_e} |\alpha^r_{k,e}(t)\rangle = \cos^4 \theta + \sin^4 \theta + \sin^2 \theta \cos^2 \theta (e^{-i(\omega_2 - \omega_1)t} + e^{+i(\omega_2 - \omega_1)t})$$

$$= 1 - \sin^2 2\theta \sin^2 \left(\frac{\Delta\omega}{2} t \right). \tag{4.7}$$

The number of α_e particles therefore oscillates in time with a frequency given by the difference $\Delta\omega$ in the energies of the physical components α_1 and α_2. This oscillation is a flavor oscillation since we have at the same time

$$\langle \alpha^r_{k,e}(t)| \, N^{k,r}_{\alpha_\mu} \, |\alpha^r_{k,e}(t)\rangle = \sin^2 2\theta \sin^2 \left(\frac{\Delta\omega}{2} t\right) \tag{4.8}$$

so that

$$\langle \alpha^r_{k,e}(t)| \, N^{k,r}_{\alpha_e} \, |\alpha^r_{k,e}(t)\rangle + \langle \alpha^r_{k,e}(t)| \, N^{k,r}_{\alpha_\mu} \, |\alpha^r_{k,e}(t)\rangle = 1. \tag{4.9}$$

Notice that the traditional derivation of Eq. (4.7) is the same as the one presented above since $\langle \alpha^r_{k,e}(t)| \, N^{k,r}_{\alpha_e} \, |\alpha^r_{k,e}(t)\rangle = |\langle \alpha^r_{k,e}|\alpha^r_{k,e}(t)\rangle|^2$, as can be seen from Eqs. (4.4) and (4.5). Eqs. (4.7)–(4.9) are the well known results.

Let us now go to the QFT framework.

We have seen that $|0\rangle_{1,2}$ and $|0\rangle_{e,\mu}$ are orthogonal in the infinite volume limit. We choose to work in the physical (mass) representation $|0\rangle_{1,2}$ (same conclusions are of course reached by working, with due changes, in the flavor representation $|0\rangle_{e,\mu}$) in order to follow the time evolution of the physical components $|\alpha^r_{k,1}\rangle$ and $|\alpha^r_{k,2}\rangle$.

The number operators are now (cf. Eqs. (3.6))

$$N^{k,r}_{\alpha_e} = \alpha^{r\dagger}_{k,e}\alpha^r_{k,e} = \cos^2\theta \alpha^{r\dagger}_{k,1}\alpha^r_{k,1} + \sin^2\theta \, |U_k|^2 \alpha^{r\dagger}_{k,2}\alpha^r_{k,2} + \sin^2\theta |V_k|^2 \beta^r_{-k,2}\beta^{r\dagger}_{-k,2}$$

$$+ \sin\theta\cos\theta(U^*_k \alpha^{r\dagger}_{k,1}\alpha^r_{k,2} + U_k \alpha^{r\dagger}_{k,2}\alpha^r_{k,1} + \varepsilon^r V^*_k \beta^r_{-k,2}\alpha^r_{k,1}$$

$$+ \varepsilon^r V_k \alpha^{r\dagger}_{k,1}\beta^{r\dagger}_{-k,2}) + \varepsilon^r \sin^2\theta(V_k U_k \alpha^{r\dagger}_{k,2}\beta^{r\dagger}_{-k,2} + V^*_k U^*_k \beta^r_{-k,2}\alpha^r_{k,2}) \tag{4.10}$$

$$N^{k,r}_{\alpha_\mu} = \alpha^{r\dagger}_{k,\mu}\alpha^r_{k,\mu} = \cos^2\theta \alpha^{r\dagger}_{k,2}\alpha^r_{k,2} + \sin^2\theta \, |U_k|^2 \alpha^{r\dagger}_{k,1}\alpha^r_{k,1} + \sin^2\theta \, |V_k|^2 \beta^r_{k,1}\beta^{r\dagger}_{-k,1}$$

$$- \sin\theta\cos\theta(U^*_k\alpha^{r\dagger}_{k,1}\alpha^r_{k,2} + U_k\alpha^{r\dagger}_{k,2}\alpha^r_{k,1} - \varepsilon^r V^*_k \beta^r_{-k,1}\alpha^r_{k,2}$$

$$- \varepsilon^r V_k \alpha^{r\dagger}_{k,2}\beta^{r\dagger}_{-k,1}) - \varepsilon^r \sin^2\theta(V_k U^*_k \alpha^{r\dagger}_{k,1}\beta^{r\dagger}_{-k,1} + V^*_k U_k \beta^r_{-k,1}\alpha^r_{k,1}). \tag{4.11}$$

The one electronic neutrino state is given by

$$|\alpha^r_{k,e}\rangle \equiv \alpha^{r\dagger}_{k,e} \, |0\rangle_{1,2} = \cos\theta \, |\alpha^r_{k,1}\rangle + \sin\theta \, U_k \, |\alpha^r_{k,2}\rangle \tag{4.12}$$

with $|\alpha^r_{k,i}\rangle \equiv \alpha^{r\dagger}_{k,i} \, |0\rangle_{1,2}$, $i = 1, 2$. The action of the flavor operator $\alpha^{r\dagger}_{k,e}$ is defined on $|0\rangle_{1,2}$ through the mapping (3.6) (see also Eqs. (3.4)). Note that, as it must be,

$$\langle \alpha^r_{k,e} \, | \, \alpha^r_{k,e}\rangle = 1 - \sin^2\theta|V_k|^2 = 1 - {}_{1,2}\langle 0| \, \alpha^{r\dagger}_{k,e}\alpha^r_{k,e} \, |0\rangle_{1,2}, \tag{4.13}$$

since $\alpha^r_{k,e} \, |0\rangle_{1,2} \neq 0$.

The time evolution of the state $|\alpha^r_{k,e}\rangle$ is given by

$$|\alpha^r_{k,e}(t)\rangle = e^{-iH_{1,2}t} \, |\alpha^r_{k,e}\rangle = \cos\theta e^{-i\omega_1 t} \, |\alpha^r_{k,1}\rangle + \sin\theta U_k e^{-i\omega_2 t} \, |\alpha^r_{k,2}\rangle. \tag{4.14}$$

We have

$$\langle \alpha_{k,c}^r | N_{\alpha_c}^{k,r} | \alpha_{k,c}^r \rangle = 1 - \sin^2 \theta \, |V_k|^2 = 1 - {}_{1,2}\langle 0| N_{\alpha_c}^{k,r} |0\rangle_{1,2} \qquad (4.15)$$

so that

$$\langle \alpha_{k,c}^r | N_{\alpha_c}^{k,r} | \alpha_{k,c}^r \rangle + {}_{1,2}\langle 0| N_{\alpha_c}^{k,r} |0\rangle_{1,2} = 1. \qquad (4.16)$$

We note that the expectation value of $N_{\alpha_c}^{k,r}$ in the vacuum $|0\rangle_{1,2}$ provides an essential contribution to the normalization Eq. (4.16) (see also Eq. (4.13)).

It is also interesting to observe that the term ${}_{1,2}\langle 0| N_{\alpha_c}^{k,r} |0\rangle_{1,2}$ plays the rôle of zero point contribution when considering the energy contribution of $\alpha_{k,c}^r$ particles.

Note that we also have

$$_{1,2}\langle 0| N_{\sigma_1}^{k,r} |0\rangle_{1,2} = \sin^2 \theta \, |V_k|^2, \qquad \sigma = \alpha, \beta, \qquad l = e, \mu, \qquad (4.17)$$

and

$$\langle \alpha_{k,c}^r | N_{\alpha_\mu}^{k,r} | \alpha_{k,c}^r \rangle = \sin^2 \theta \, |V_k|^2 \, (1 - \sin^2 \theta \, |V_k|^2). \qquad (4.18)$$

Eqs. (4.17) show that the condensate structure of $|0\rangle_{1,2}$ in terms of definite flavor fields and of course are analogous to Eqs. (3.3) which show the condensate structure of $|0\rangle_{e,\mu}$ in terms of definite mass fields.

We also have

$$\langle \alpha_{k,c}^r(t)| N_{\alpha_c}^{k,r} | \alpha_{k,c}^r(t) \rangle = \cos^4 \theta + |U_k|^2 \sin^4 \theta + |V_k|^2 \sin^2 \theta \cos^2 \theta$$

$$+ |U_k|^2 \sin^2 \theta \cos^2 \theta (e^{\,i(\omega_2 - \omega_1)t} + e^{\,+i(\omega_2 - \omega_1)t})$$

$$= (1 - \sin^2 \theta \, |V_k|^2) - |U_k|^2 \sin^2 2\theta \sin^2 \left(\frac{\Delta\omega}{2}t\right). \qquad (4.19)$$

This result reproduces the one obtained in the finite volume approximation (cf. Eq. (4.7)) when $|U_k| \to 1$ (and $|V_k| \to 0$). The fraction of $\alpha_{k,\mu}^r$ particles in the same state is

$$\langle \alpha_{k,c}^r(t)| N_{\alpha_c}^{k,r} | \alpha_{k,c}^r(t) \rangle = |U_k|^2 \sin^2 2\theta \sin^2 \left(\frac{\Delta\omega}{2}t\right)$$

$$+ \sin^2 \theta \, |V_k|^2 \, (1 - \sin^2 \theta \, |V_k|^2), \qquad (4.20)$$

where we recognize the contribution from the $\alpha_{k,\mu}^r$ condensate in the state $|\alpha_{k,c}^r\rangle$ (cf. Eq. (4.18)). Eq. (4.19) is to be compared with the approximated one (4.7). Note that

$$\langle \alpha_{k,c}^r(t)| N_{\alpha_c}^{k,r} | \alpha_{k,c}^r(t) \rangle + \langle \alpha_{k,c}^r(t)| N_{\alpha_\mu}^{k,r} | \alpha_{k,c}^r(t) \rangle$$

$$= \langle \alpha_{k,c}^r | N_{\alpha_c}^{k,r} | \alpha_{k,c}^r \rangle + \langle \alpha_{k,c}^r | N_{\alpha_\mu}^{k,r} | \alpha_{k,c}^r \rangle. \qquad (4.21)$$

The normalization relation is written as

$$\langle \alpha^r_{k,e}(t)|\, N^{k,r}_{\alpha_e}\, |\alpha^r_{k,e}(t)\rangle +\, _{1,2}\langle 0|\, N^{k,r}_{\alpha_e}\, |0\rangle_{1,2}$$

$$+ \langle \alpha^r_{k,e}(t)|\, N^{k,r}_{\alpha_\mu}\, |\alpha^r_{k,e}(t)\rangle - \langle \alpha^r_{k,e}|\, N^{k,r}_{\alpha_\mu}\, |\alpha^r_{k,e}\rangle = 1. \qquad (4.22)$$

We may also use $V_k \equiv \langle \alpha^r_{k,e}|\alpha^r_{k,e}\rangle$ as normalization factor and write Eqs. (4.15), (4.18), (4.19), (4.20) and the normalization relation as

$$\frac{1}{V_k}\, \langle \alpha^r_{k,e}|\, N^{k,r}_{\alpha_e}\, |\alpha^r_{k,e}\rangle = 1 \qquad (4.23)$$

$$\frac{1}{V_k}\, \langle \alpha^r_{k,e}|\, N^{k,r}_{\alpha_\mu}\, |\alpha^r_{k,e}\rangle = \sin^2 \theta\, |V_k|^2 \qquad (4.24)$$

$$\frac{1}{V_k}\, \langle \alpha^r_{k,e}(t)|\, N^{k,r}_{\alpha_e}\, |\alpha^r_{k,e}(t)\rangle = 1 - R_k \sin^2 2\theta \sin^2\left(\frac{\Delta\omega}{2}t\right) \qquad (4.25)$$

$$\frac{1}{V_k}\, \langle \alpha^r_{k,e}(t)|\, N^{k,r}_{\alpha_\mu}\, |\alpha^r_{k,e}(t)\rangle = R_k \sin^2 2\theta \sin^2\left(\frac{\Delta\omega}{2}t\right)$$

$$+ \frac{1}{V_k}\, \langle \alpha^r_{k,e}|\, N^{k,r}_{\alpha_\mu}\, |\alpha^r_{k,e}\rangle \qquad (4.26)$$

$$\frac{1}{V_k}\, \langle \alpha^r_{k,e}(t)|\, N^{k,r}_{\alpha_e}\, |\alpha^r_{k,e}(t)\rangle + \frac{1}{V_k}\, \langle \alpha^r_{k,e}(t)|\, N^{k,r}_{\alpha_\mu}\, |\alpha^r_{k,e}(t)\rangle$$

$$- \frac{1}{V_k}\, \langle \alpha^r_{k,e}|\, N^{k,r}_{\alpha_\mu}\, |\alpha^r_{k,e}\rangle = 1. \qquad (4.27)$$

respectively, with $R_k \equiv |U_k|^2/.\ V_k = (1 - |V_k|^2)/(1 - \sin^2 \theta\, |V_k|^2)$.

In conclusion, Eqs. (4.19) and (4.20) exhibit the corrections to the flavor oscillations coming from the condensate contributions.

The conventional (approximate) results (4.7) and (4.8) are obtained when the condensate contributions are missing (in the $|V_k| \to 0$ limit).

We stress that the limit $k \to \infty$ of Eqs. (4.19) and (4.20) gives an exact result which, although coincides with Eqs. (4.7) and (4.8), is not the outcome of the usual finite volume approximation.

Notice that the fraction of $\alpha^r_{k,e}$ particles which is condensed into the vacuum $|0\rangle_{1,2}$ is "frozen", i.e. does not oscillate in time, as it is easily seen by noting that $e^{-iH_{1,2}t}|0\rangle_{1,2} = |0\rangle_{1,2}$ (cf. also Eq. (4.15)).

Since the correction factor is related to the vacuum condensate, we see that the vacuum acts as a "momentum (or spectrum) analyzer" for the oscillating neutrinos: neutrinos with $k \gg \sqrt{m_1 m_2}$ oscillate with a larger amplitude than neutrinos with $k \simeq \sqrt{m_1 m_2}$, due to the vacuum structure. Such a vacuum spectral analysis effect may sum up to other effects (such as MSW effect [8] in the matter) in depressing or enhancing neutrino oscillations.

It is an interesting question to ask if an experimental test may show the dependence of the flavor oscillation amplitude on the momentum exhibited in the above results. We discuss such a question in Section 6.

The above scheme is easily generalized to the oscillations in the matter. As well known [1, 2, 8], the two flavors oscillations picture is modified due to the different interaction of v_e and v_μ with the electrons of the medium. There are two contributions to this interaction: the first one, coming from neutral current, amounts to $-G_F n_n/\sqrt{2}$ and it is present for both v_e and v_μ; the second one, coming from charged current, is given by $\sqrt{2}\,G_F n_e$ and it is present only for v_e. Here n_e and n_n are the electron and neutron densities, respectively. This produces a difference in the effective masses of v_e and v_μ, which can be expressed in terms of new free fields $\tilde{v}_1$ and $\tilde{v}_2$ with new masses $\tilde{m}_1$ and $\tilde{m}_2$ and a mixing angle $\tilde{\theta}$.

The mixing relations (2.1) are thus rewritten as

$$v_e(x) = \tilde{v}_1(x) \cos \tilde{\theta} + \tilde{v}_2(x) \sin \tilde{\theta}$$
$$v_\mu(x) = -\tilde{v}_1(x) \sin \tilde{\theta} + \tilde{v}_2(x) \cos \tilde{\theta}. \tag{4.28}$$

The tilde quantities are calculated by diagonalizing the hamiltonian of v_e and v_μ matter, which is

$$H^{\text{matter}}_{e,\mu} = H^{\text{vacuum}}_{e,\mu} - G_F n_n/\sqrt{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sqrt{2}\,G_F n_e \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.29}$$

The matter and the vacuum parameters are related as [1, 2, 8]

$$\sin^2 2\tilde{\theta} \simeq \sin^2 2\theta \left(\frac{\Delta m^2}{\Delta \tilde{m}^2} \right) \tag{4.30}$$

$$\Delta \tilde{m}^2 = [(D - \Delta m^2 \cos 2\theta)^2 + (\Delta m^2 \sin 2\theta)^2]^{1/2} \tag{4.31}$$

and

$$\tilde{m}^2_{1,2} = \tfrac{1}{2}(m_1^2 + m_2^2 + D \mp \Delta \tilde{m}^2), \tag{4.32}$$

where $D = 2\sqrt{2}\,G_F n_e k$.

When $D = 2\sqrt{2}\,G_F n_e^{crit}k = \Delta m^2 \cos 2\theta$ there is resonance and $\sin^2 2\tilde{\theta}$ goes to unity (MSW effect).

In our scheme it is possible to treat oscillations in matter starting with mixing relations (4.28) and repeating all the procedure described above. So we can use all the results obtained in our treatment simply by substituting θ, m_1, m_2 with the corresponding tilde quantities. In particular, the oscillation formula becomes in the matter:

$$\langle \alpha^r_{k,e}(t) | N^{k,r}_{\alpha_e} | \alpha^r_{k,e}(t) \rangle = (1 - \sin^2 \tilde{\theta}\,|\tilde{V}_k|^2) - |\tilde{U}_k|^2 \sin^2 2\tilde{\theta} \sin^2 \left(\frac{\Delta \tilde{\omega}}{2} t \right). \tag{4.33}$$

5. Three Flavors Fermion Mixing

The extension of our discussion to three flavors is complicated by the proliferation of terms in the explicit computation of the quantities of interest. However, it is possible to extract some results from the structure of the annihilators, without explicitly writing the expression for the vacuum state.

Among the various possible parametrizations of the three fields mixing matrix, we choose to work with

$$
M = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}
\tag{5.1}
$$

with $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$, since it is the familiar parametrization of CKM matrix [1].

To generate the M matrix, we define

$$
G_{12}(\theta_{12}) = \exp(\theta_{12}L_{12}), \qquad G_{23}(\theta_{23}) = \exp(\theta_{23}L_{23}), \qquad G_{13}(\theta_{13}) = \exp(\theta_{13}L_{13}), \tag{5.2}
$$

where

$$
L_{12} \equiv \int d^3x(v_1^\dagger(x)\,v_2(x) - v_2^\dagger(x)\,v_1(x)) \tag{5.3a}
$$

$$
L_{23} \equiv \int d^3x(v_2^\dagger(x)\,v_3(x) - v_3^\dagger(x)\,v_2(x)) \tag{5.3b}
$$

$$
L_{13} \equiv \int d^3x(e^{i\delta}v_1^\dagger(x)\,v_3(x) - e^{-i\delta}v_3^\dagger(x)\,v_1(x)) \tag{5.3c}
$$

so that

$$
v_e^\chi(x) = G_{12}^{-1}G_{12}^{-1}G_{23}^{-1}v_1^\chi(x)\,G_{23}G_{13}G_{12} \tag{5.4a}
$$

$$
v_\mu^\chi(x) = G_{12}^{-1}G_{13}^{-1}G_{23}^{-1}v_2^\chi(x)\,G_{23}G_{13}G_{12} \tag{5.4b}
$$

$$
v_\tau^\chi(x) = G_{12}^{-1}G_{13}^{-1}G_{23}^{-1}v_3^\chi(x)\,G_{23}G_{13}G_{12}. \tag{5.4c}
$$

The matrix M is indeed obtained by using the following relations:

$$
[v_1^\chi(x), L_{12}] = v_2^\chi(x), \qquad [v_1^\chi(x), L_{23}] = 0, \qquad [v_1^\chi(x), L_{13}] = e^{i\delta}v_3^\chi(x) \tag{5.5a}
$$

$$
[v_2^\chi(x), L_{12}] = -v_1^\chi(x), \qquad [v_2^\chi(x), L_{23}] = v_3^\chi(x), \qquad [v_2^\chi(x), L_{13}] = 0 \tag{5.5b}
$$

$$
[v_3^\chi(x), L_{12}] = 0, \qquad [v_3^\chi(x), L_{23}] = -v_2^\chi(x), \qquad [v_3^\chi(x), L_{13}] = -e^{-i\delta}v_1^\chi(x). \tag{5.5c}
$$

Notice that the phase δ is unavoidable for three fields mixing, while it can be incorporated in the fields definition for two fields mixing.

The vacuum in the flavor representation is

$$|0\rangle_{e\mu\tau} = G_{12}^{-1} G_{13}^{-1} G_{23}^{-1} |0\rangle_{123}. \tag{5.6}$$

We do not give here the explicit form of this state, which is very complicated and is a combination of all possible couples $\alpha_{k,i}^{r\dagger} \beta_{k,j}^{r\dagger}$ with $i, j = 1, 2, 3$. Nevertheless, we can obtain physical informations from the structure of the annihilators $\alpha_{k,l}^{r}$, $\beta_{k,l}^{r}$ ($l = e, \mu, \tau$). In the reference fame $k = (0, 0, |k|)$ we obtain (see Appendix F):

$$\alpha_{k,e}^{r} = c_{12} c_{13} \alpha_{k,1}^{r} + s_{12} c_{13} (U_{12}^{k*} \alpha_{k,2}^{r} + \varepsilon^{r} V_{12}^{k} \beta_{-k,2}^{r\dagger})$$
$$+ e^{i\delta} s_{13} (U_{13}^{k*} \alpha_{k,3}^{r} + \varepsilon^{r} V_{13}^{k} \beta_{-k,3}^{r\dagger}), \tag{5.7a}$$

$$\alpha_{k,\mu}^{r} = (c_{12} c_{23} - e^{i\delta} s_{12} s_{23} s_{13}) \alpha_{k,2}^{r} - (s_{12} c_{23} + e^{-i\delta} c_{12} s_{23} s_{13})$$
$$\times (U_{12}^{k} \alpha_{k,1}^{r} - \varepsilon^{r} V_{12}^{k} \beta_{-k,1}^{r\dagger}) + s_{23} c_{13} (U_{23}^{k*} \alpha_{k,3}^{r} + \varepsilon^{r} V_{23}^{k} \beta_{-k,3}^{r\dagger}), \tag{5.7b}$$

$$\alpha_{k,\tau}^{r} = c_{23} c_{13} \alpha_{k,3}^{r} - (c_{12} s_{23} + e^{i\delta} s_{12} c_{23} s_{13})(U_{23}^{k} \alpha_{k,2}^{r} - \varepsilon^{r} V_{23}^{k} \beta_{-k,2}^{r\dagger})$$
$$+ (s_{12} s_{23} - e^{-i\delta} c_{12} c_{23} s_{13})(U_{13}^{k} \alpha_{k,1}^{r} - \varepsilon^{r} V_{13}^{k} \beta_{-k,1}^{r\dagger}), \tag{5.7c}$$

$$\beta_{-k,e}^{r} = c_{12} c_{13} \beta_{-k,1}^{r} + s_{12} c_{13} (U_{12}^{k*} \beta_{-k,2}^{r} - \varepsilon^{r} V_{12}^{k} \alpha_{k,2}^{r\dagger})$$
$$+ e^{-i\delta} s_{13} (U_{13}^{k*} \beta_{-k,3}^{r} - \varepsilon^{r} V_{13}^{k} \alpha_{k,3}^{r\dagger}), \tag{5.7d}$$

$$\beta_{-k,\mu}^{r} = (c_{12} c_{23} - e^{i\delta} s_{12} s_{23} s_{13}) \beta_{-k,2}^{r} - (s_{12} c_{23} + e^{i\delta} c_{12} s_{23} s_{13})$$
$$\times (U_{12}^{k} \beta_{-k,1}^{r} + \varepsilon^{r} V_{12}^{k} \alpha_{k,1}^{r\dagger}) + s_{23} c_{13} (U_{23}^{k*} \beta_{-k,3}^{r} - \varepsilon^{r} V_{23}^{k} \alpha_{k,3}^{r\dagger}), \tag{5.7e}$$

$$\beta_{-k,\tau}^{r} = c_{23} c_{13} \beta_{-k,3}^{r} - (c_{12} s_{23} + e^{i\delta} s_{12} c_{23} s_{13})(U_{23}^{k} \beta_{-k,2}^{r} + \varepsilon^{r} V_{23}^{k} \alpha_{k,2}^{r\dagger})$$
$$+ (s_{12} s_{23} - e^{i\delta} c_{12} c_{23} s_{13})(U_{13}^{k} \beta_{-k,1}^{r} + \varepsilon^{r} V_{13}^{k} \alpha_{k,1}^{r\dagger}). \tag{5.7f}$$

From Eqs. (5.7) we observe that, in contrast with the case of two flavors mixing, the condensation densities are now different for different flavors (cf. Eq. (4.17)):

$$_{123}\langle 0| N_{\alpha_e}^{k,r} |0\rangle_{123} = {}_{123}\langle 0| N_{\beta_e}^{k,r} |0\rangle_{123} = s_{12}^{2} c_{13}^{2} |V_{12}^{k}|^{2} + s_{13}^{2} |V_{13}^{k}|^{2}, \tag{5.8a}$$

$$_{123}\langle 0| N_{\alpha_\mu}^{k,r} |0\rangle_{123} = {}_{123}\langle 0| N_{\beta_\mu}^{k,r} |0\rangle_{123} = |s_{12} c_{23} + e^{-i\delta} c_{12} s_{23} s_{13}|^{2}$$
$$\times |V_{12}^{k}|^{2} + s_{23}^{2} c_{13}^{2} |V_{23}^{k}|^{2}, \tag{5.8b}$$

$$_{123}\langle 0| N_{\alpha_\tau}^{k,r} |0\rangle_{123} = {}_{123}\langle 0| N_{\beta_\tau}^{k,r} |0\rangle_{123} = |c_{12} s_{23} + e^{-i\delta} s_{12} c_{23} s_{13}|^{2} |V_{23}^{k}|^{2}$$
$$+ |s_{12} s_{23} - e^{-i\delta} c_{12} c_{23} s_{13}|^{2} |V_{13}^{k}|^{2}. \tag{5.8c}$$

To study the three flavors neutrino oscillations, we observe that, in the finite volume approxiation

$$|\alpha_{k,e}^{r}\rangle = c_{12} c_{13} |\alpha_{k,1}^{r}\rangle + s_{12} c_{13} |\alpha_{k,2}^{r}\rangle + e^{i\delta} s_{13} |\alpha_{k,3}^{r}\rangle. \tag{5.9}$$

The time evolution of this state is given by

$$|\alpha_{k,e}^r(t)\rangle = c_{12}c_{13}e^{-i\omega_1 t}|\alpha_{k,1}^r\rangle + s_{12}c_{13}e^{-i\omega_2 t}|\alpha_{k,2}^r\rangle + e^{i\delta}s_{13}e^{-i\omega_3 t}|\alpha_{k,3}^r\rangle. \quad (5.10)$$

The number operator is

$$N_{\alpha_e}^{k,r} = \alpha_{k,e}^{r\dagger}\alpha_{k,e}^r = c_{12}^2 c_{13}^2 \alpha_{k,1}^{r\dagger}\alpha_{k,1}^r + c_{12}s_{12}c_{13}^2 \alpha_{k,1}^{r\dagger}\alpha_{k,2}^r + e^{i\delta}c_{12}c_{13}s_{13}\alpha_{k,1}^{r\dagger}\alpha_{k,3}^r$$

$$+ c_{12}s_{12}c_{13}^2 \alpha_{k,2}^{r\dagger}\alpha_{k,1}^r + s_{12}^2 c_{13}^2 \alpha_{k,2}^{r\dagger}\alpha_{k,2}^r + e^{i\delta}s_{12}c_{13}s_{13}\alpha_{k,2}^{r\dagger}\alpha_{k,3}^r$$

$$+ e^{-i\delta}c_{12}c_{13}s_{13}\alpha_{k,3}^{r\dagger}\alpha_{k,1}^r + e^{-i\delta}s_{12}c_{13}s_{13}\alpha_{k,3}^{r\dagger}\alpha_{k,2}^r + s_{13}^2 \alpha_{k,3}^{r\dagger}\alpha_{k,3}^r \quad (5.11)$$

and thus

$$\langle \alpha_{k,e}^r(t)| N_{\alpha_e}^{k,r}|\alpha_{k,e}^r(t)\rangle = c_{12}^4 c_{13}^4 + s_{12}^4 c_{13}^4 + s_{13}^4 + c_{12}^2 s_{12}^2 c_{13}^4 (e^{-i(\omega_2-\omega_1)t}$$

$$+ e^{+i(\omega_2-\omega_1)t}) + c_{12}^2 c_{13}^2 s_{13}^2 (e^{-i(\omega_3-\omega_1)t} + e^{+i(\omega_3-\omega_1)t})$$

$$+ s_{12}^2 c_{13}^2 s_{13}^2 (e^{-i(\omega_3-\omega_2)t} + e^{i(\omega_3-\omega_2)t}),$$

i.e.,

$$\langle \alpha_{k,e}^r(t)| N_{\alpha_e}^{k,r}|\alpha_{k,e}^r(t)\rangle = 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\left(\frac{\omega_2-\omega_1}{2}t\right)$$

$$- \cos^2\theta_{12}\sin^2 2\theta_{13}\sin^2\left(\frac{\omega_3-\omega_1}{2}t\right)$$

$$- \sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\left(\frac{\omega_3-\omega_2}{2}t\right). \quad (5.12)$$

On the other hand, the QFT computations give

$$\langle \alpha_{k,e}^r| N_{\alpha_e}^{k,r}|\alpha_{k,e}^r\rangle = 1 - s_{12}^2 c_{13}^2 |V_{12}^k|^2 - s_{13}^2 |V_{13}^k|^2, \quad (5.13)$$

$$_{123}\langle 0| N_{\alpha_e}^{k,r}|0\rangle_{123} = s_{12}^2 c_{13}^2 |V_{12}^k|^2 + s_{13}^2 |V_{13}^k|^2, \quad (5.14)$$

and

$$\langle \alpha_{k,e}^r(t)| N_{\alpha_e}^{k,r}|\alpha_{k,e}^r(t)\rangle = c_{12}^4 c_{13}^4 + |U_{12}^k|^2 s_{12}^4 c_{13}^4 + |U_{13}^k|^2 s_{13}^4$$

$$+ |V_{12}^k|^2 c_{12}^2 s_{12}^2 c_{13}^4 + |V_{13}^k|^2 c_{12}^2 c_{13}^2 s_{13}^2$$

$$+ (|U_{12}^k|^2 |V_{13}^k|^2 + |U_{13}^k|^2 |V_{12}^k|^2) s_{12}^2 c_{13}^2 s_{13}^2$$

$$+ |U_{12}^k|^2 c_{12}^2 s_{12}^2 c_{13}^4 (e^{-i(\omega_2-\omega_1)t} + e^{+i(\omega_2-\omega_1)t})$$

$$+ |U_{13}^k|^2 c_{12}^2 c_{13}^2 s_{13}^2 (e^{-i(\omega_3-\omega_1)t} + e^{+i(\omega_3-\omega_1)t})$$

$$+ |U_{12}^k|^2 |U_{13}^k|^2 s_{12}^2 c_{13}^2 s_{13}^2 (e^{-i(\omega_3-\omega_2)t} + e^{+i(\omega_3-\omega_2)t}),$$

namely,

$$\langle \alpha^r_{k,e}(t) | N^{k,r}_{\alpha_e} | \alpha^r_{k,e}(t) \rangle = (1 - \sin^2 \theta_{12} \cos^2 \theta_{13} |V^k_{12}|^2 - \sin^2 \theta_{13} |V^k_{13}|^2)$$

$$- |U^k_{12}|^2 \cos^4 \theta_{13} \sin^2 2\theta_{12} \sin^2 \left(\frac{\omega_2 - \omega_1}{2} t \right)$$

$$- |U^k_{13}|^2 \cos^2 \theta_{12} \sin^2 2\theta_{13} \sin^2 \left(\frac{\omega_3 - \omega_1}{2} t \right)$$

$$- |U^k_{12}|^2 |U^k_{13}|^2 \sin^2 \theta_{12} \sin^2 2\theta_{13} \sin^2 \left(\frac{\omega_3 - \omega_2}{2} t \right), \quad (5.15)$$

which is to be compared with Eq. (5.12).

6. EXPERIMENTAL IMPLICATIONS AND CONCLUSIONS

We now discuss if and to which extent the theoretical analysis presented in the previous Sections bears any effect to the experimental observations.

By considering the Pontecorvo mixing transformations for neutrino Dirac fields [5], we have shown that in the LSZ formalism of QFT [3, 6, 7] the Fock space of definite flavor states turns out to be unitarily inequivalent in the infinite volume limit to the Fock space of definite mass states. The flavor states are obtained as condensate of massive neutrino-antineutrino pairs and exhibit the structure of $SU(2)$ generalized coherent states [4]. The condensation density is computed as a function of the mixing angle, of the momentum modulus, and of the neutrino masses.

The problem is now to estimate in which physcial conditions this non-trivial condensate may produce observable consequences in the experiments.
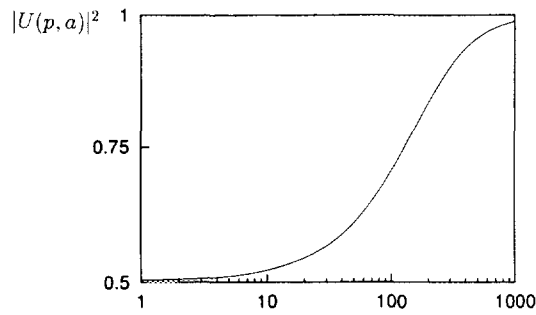

FIG. 1. Plot of $|U(p, a)|^2$ against p with $a = 5 \times 10^4$.

The neutrino oscillation formula has been derived and its amplitude turns out to be momentum dependent. This suggest to us to estimate the deviations of the momentum dependent amplitude from the one of the usual approximation in order to see to which extent experimental consequences may be expected.

Our first remark is that the corrections depend on the modulus k through $|U_k|^2$. Since $|U_k|^2 \to 1$ when $k \to \infty$, these corrections disappear in the infinite momentum limit. However, for small k, the oscillation amplitude is depressed by the factor $|U_k|^2$: the depression factor ranges from 1 to $\frac{1}{2}$ depending on k and on the masses values, according to the behaviour of the $|U_k|^2$ function (see Fig. 1).

We therefore see that the relativistic limit ($k \gg \sqrt{m_1 m_2}$) of the proper QFT treatment reproduces the usual result, as already observed above. There is, however, a range of values of k where the oscillation amplitude is depressed, and where some effect can be possibly observed. One can show that these values depend on the ratio $a = (m_2 - m_1)^2/m_1 m_2$. To plot the function $|U_k|^2$ we have used mass values compatible with the present experimental bounds. It is convenient to rewrite $|U_k|^2$ as

$$|U_k|^2 \equiv |U(p, a)|^2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + a(p/(p^2 + 1))^2}}\right)$$

with

$$p = \frac{k}{\sqrt{m_1 m_2}}, \qquad a = \frac{(\Delta m)^2}{m_1 m_2}, \qquad 0 \leqslant a < +\infty.$$

We find that sensible deviations are obtained by taking, for instance, $m_1 = 5$ eV, $m_2 = 250$ KeV, so that $a \simeq 5 \times 10^4$, and thus $|U_k|^2 \simeq 0.75$ for $k = 146$ KeV, while $|U_k|^2 \simeq 0.95$ for $k = 518$ KeV, etc. We observe that experiments aimed to observe neutrino oscillations by passive large detectors include neutrino momentum as low as few hundreds of KeV [2]. In these low momentum ranges one can thus expect, on the basis of the above discussion, larger deviations from the usual formula and better fit with the present paper results.

One further comment is that by conveniently probing the low momentum range one may obtain indications on the neutrino masses since $|U_k|^2$ has its minimum at $k = \sqrt{m_1 m_2}$.

At this stage of the investigation we may thus conclude that the QFT analysis presented in this paper should not produce any observable implication in the large momentum limit ($k \gg \sqrt{m_1 m_2}$). On the contrary, for small momentum neutrinos one should expect smaller oscillation amplitudes than the ones predicted on the basis of the conventional treatment. Such a conclusion comes from the explicit study of the function $|U_k|^2$.

It is interesting to observe that a "warning signal that non relativistic neutrinos must be handled with special care" has been put forward also in ref. [9], where indeed, although in a scheme conceptually different from the present one, oscillations have been shown to be drastically depressed for non relativistic neutrinos with

different masses. In ref. [9] as well as in ref. [10] further analysis on the wave packet nature of massive neutrinos, on their coherence length and on the wave packet spreading is also made. The spreading of the momentum as well as coherence length and mass degeneracy are also considered in ref. [11], although in a different perspective (where neutrinos are considered as virtual, actually non directly observable particles). In the present paper we have not faced such problems since we have been mostly concerned with the vacuum condensate structure emerging in the physics of fermion field mixing. In ref. [10] it is also presented a QFT discussion for a specific process by considering the rôle of spatially localized source and detector, and also in such a case the usual treatment is recovered in the relativistic limit. It seems therefore, on the basis of these works, including the present one, that interesting new insights in neutrino physics may be obtained by probing the non relativistic domain.

Further analysis of experimental implications are in progress [12].

Let us finally close the paper with the folowing comments.

Although the physically relevant quantities are given by expectation values of the observables, nevertheless it is interesting to observe that the ratio of the amplitudes of the $|\alpha^r_{k,1}\rangle$ and $|\alpha^r_{k,2}\rangle$ components of the state $|\alpha^r_{k,c}(t)\rangle$ is constant in time, as can be seen from Eq. (4.14), and that such a feature persists even in the limit $k \to \infty$ (i.e., $|U_k| \to 1$) where, however, the oscillation formula (4.19) reduces to the usual one (4.7). This is in contrast with the finite volume approximation where "decoherence" between the components $|\alpha^r_{k,1}\rangle$ and $|\alpha^r_{k,2}\rangle$ arises from the phase factor $\exp(-i\,\Delta\omega t)$ (see Eq. (4.5)).

We have shown that our discussion can be extended to the oscillations in matter and the corresponding oscillation formula is obtained.

We have also studied the three flavors mixing and have obtained the corresponding oscillation formula, which also in this case is momentum dependent.

We stress the crucial role played by the existence in QFT of infinitely many unitarily inequivalent representations of the canonical anti-commutation rules.

The problem of constructing the approrpiate Fock space for weak fields, i.e. for mixed (flavor) neutrino fields, has been also studied in ref. [13]. There, however, the conclusion was that it is impossible to construct the Fock space for weak neutrinos, namely, that is impossible to define creation and annihilation flavor operators obeying canonical anticommutation relations. An "approximate" Fock space for weak states was shown to exist only in the extremely relativistic limit and for almost degenerate mass eigenvalues ($\Delta m \simeq 0$). In this respect, the present paper discussion clarifies this situation since the careful analysis in terms of the generator of the mixing transformations shows that the Fock space for flavor fields does exist, that its vacuum is an $SU(2)$ generalized coherent state with neutrino-antineutrino condensate structure and that the Fock space for massive free neutrinos and the one for flavor neutrinos are unitarily inequivalent representations of the canonical anticommutation relations in the infinite volume limit. Moreover, the annihilation and the creation operators for the flavor fields are explicitly constructed and satisfy the canonical anticommutation relations.

We thus realize the limit of validity of the identification of the vacuum state for definite mass neutrinos with the vacuum state for definite flavor neutrinos. We point out that such an identification is actually an approximation since the flavor vacuum has the structure of an $SU(2)$ generalized coherent state, and it is only allowed at finite volume. The vacua identification is meaningless in QFT since the mass and the flavor space are unitarily inequivalent.

Finally, we observe that although our discussion has been focused on the neutrino mixing, nevertheless it can be extended, with due changes, to other fermion mixing transformations, such as the CKM mixing transformations. In this last case, the so called free fields in the LSZ formalism are to be understood as the asympototically free quark fields.

The study of the QFT for mixing of boson fields with different masses is also in progress. In that case preliminary results show [14] that relations analogous to Eqs. (3.2) and (3.6) hold so that we have a non trivial vacuum structure. In the boson case we find $|U_k| = \cosh \sigma_k$ and $|V_k| = \sinh \sigma_k$, with $\sigma_k = 1/2 \log(\omega_{k,1}/\omega_{k,2})$ where $\omega_{k,i}$, $i = 1, 2$, is the boson energy.

Although mixing of fields in QFT is an old subject (see [15] among the oldest papers on flavor mixing) it appears still to be a rich land for research.

APPENDIX A

Using the algebra (2.19) and the relations (2.27), we have

$$S^k_+ S^k_- |0\rangle_{1,2} = S^k_- S^k_+ |0\rangle_{1,2}$$

$$(S^k_+)^2 (S^k_-)^2 |0\rangle_{1,2} = (S^k_-)^2 (S^k_+)^2 |0\rangle_{1,2}$$

$$(S^k_+)^2 S^k_- |0\rangle_{1,2} = S^k_- (S^k_+)^2 |0\rangle_{1,2} + 2S^k_+ |0\rangle_{1,2}$$

$$(S^k_-)^2 S^k_+ |0\rangle_{1,2} = S^k_+ (S^k_-)^2 |0\rangle_{1,2} + 2S^k_- |0\rangle_{1,2}$$

$$(S^k_+)^3 S^k_- |0\rangle_{1,2} = 6(S^k_+)^2 |0\rangle_{1,2}$$

$$(S^k_+)^4 S^k_- |0\rangle_{1,2} = 0$$

$$(S^k_+)^3 (S^k_-)^2 |0\rangle_{1,2} = 6S^k_- (S^k_+)^2 |0\rangle_{1,2}$$

$$(S^k_-)^3 (S^k_+)^2 |0\rangle_{1,2} = 6S^k_+ (S^k_-)^2 |0\rangle_{1,2}$$

$$(S^k_+)^4 (S^k_-)^2 |0\rangle_{1,2} = 24(S^k_+)^2 |0\rangle_{1,2}$$

$$(S^k_+)^5 (S^k_-)^2 |0\rangle_{1,2} = 0$$

$$S^k_+ S^k_- (S^k_+)^2 |0\rangle_{1,2} = 4(S^k_+)^2 |0\rangle_{1,2}$$

$$S^k_- S^k_+ (S^k_-)^2 |0\rangle_{1,2} = 4(S^k_-)^2 |0\rangle_{1,2}$$

$$S^k_3 S^k_- S^k_+ |0\rangle_{1,2} = 0$$

$$S^k_3 (S^k_+)^2 (S^k_-)^2 |0\rangle_{1,2} = 0$$

$$(S_3^k)^n \, S_-^k \, |0\rangle_{1,2} = (-1)^n \, S_-^k \, |0\rangle_{1,2}$$

$$(S_3^k)^n \, (S_-^k)^2 \, |0\rangle_{1,2} = (-2)^n \, (S_-^k)^2 \, |0\rangle_{1,2}$$

$$(S_3^k)^n \, S_+^k \, |0\rangle_{1,2} = S_+^k \, |0\rangle_{1,2}$$

$$(S_3^k)^n \, (S_+^k)^2 \, |0\rangle_{1,2} = 2^n (S_+^k)^2 \, |0\rangle_{1,2}$$

$$S_3^k \, S_-^k \, (S_+^k)^2 \, |0\rangle_{1,2} = S_-^k \, (S_+^k)^2 \, |0\rangle_{1,2}$$

$$S_3^k \, S_+^k \, (S_-^k)^2 \, |0\rangle_{1,2} = -S_+^k \, (S_-^k)^2 \, |0\rangle_{1,2}.$$

Use of the above relations gives Eq. (2.28).

Appendix B

Wave functions and Z_k:

$$u_{k,i}^r(t) = \hat{u}_{k,i}^r \, e^{-i\omega_{k,i}t} = A_i \begin{pmatrix} \xi^r \\[4pt] \dfrac{\bar{\sigma} \cdot \bar{k}}{\omega_{k,i} + m_i} \, \xi^r \end{pmatrix} e^{-i\omega_{k,i}t},$$

$$v_{k,i}^r(t) = \hat{v}_{k,i}^r \, e^{i\omega_{k,i}t} = A_i \begin{pmatrix} \dfrac{\bar{\sigma} \cdot \bar{k}}{\omega_{k,i} + m_i} \, \xi^r \\[4pt] \xi^r \end{pmatrix} e^{i\omega_{k,i}t}$$

$$\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad A_i \equiv \left(\frac{\omega_{k,i} + m_i}{2\omega_{k,i}} \right)^{1/2}, \qquad i = 1, 2, \quad r = 1, 2.$$

$$\hat{v}_{-k,1}^{1\dagger} \, \hat{u}_{k,2}^1 = -\hat{v}_{-k,1}^{2\dagger} \, \hat{u}_{k,2}^2 = A_1 A_2 \left(\frac{-k_3}{\omega_{k,1} + m_1} + \frac{k_3}{\omega_{k,2} + m_2} \right)$$

$$\hat{v}_{-k,1}^{1\dagger} \, \hat{u}_{k,2}^2 = (\hat{v}_{-k,1}^{2\dagger} \, \hat{u}_{k,2}^1)^* = A_1 A_2 \left(\frac{-k_1 + ik_2}{\omega_{k,1} + m_1} + \frac{k_1 - ik_2}{\omega_{k,2} + m_2} \right).$$

Using the above relations Eq.(2.32) is obtained.
Eq. (2.31) follows if one observes that

$$(S_-^k)^2 \, |0\rangle_{1,2} = 2[(u_{k,2}^{1*} v_{-k,1}^2)(u_{k,2}^{2*} v_{-k,1}^1) - (u_{k,2}^{1*} v_{-k,1}^1)(u_{k,2}^{2*} v_{-k,1}^2)]$$

$$\times \alpha_{k,2}^{1\dagger} \beta_{-k,1}^{2\dagger} \alpha_{k,2}^{2\dagger} \beta_{-k,1}^{1\dagger} \, |0\rangle_{1,2}$$

$$= Z_k \, e^{2i(\omega_{k,1} + \omega_{k,2})t} \alpha_{k,2}^{1\dagger} \beta_{-k,1}^{2\dagger} \alpha_{k,2}^{2\dagger} \beta_{-k,1}^{1\dagger} \, |0\rangle_{1,2}.$$

Appendix C

Relationships for the number operator and Eq. (3.2):

$$[N_1^k, S_+^k] = \Sigma_{r,s}((u_{k,1}^{r*} u_{k,2}^s) \alpha_{k,1}^{r\dagger} \alpha_{k,2}^s + (u_{k,1}^{r*} v_{-k,2}^s) \alpha_{k,1}^{r\dagger} \beta_{-k,2}^{s\dagger})$$

$$[N_1^k, S_-^k] = -\Sigma_{r,s}((u_{k,2}^{r*} u_{k,1}^s) \alpha_{k,2}^{r\dagger} \alpha_{k,1}^s + (v_{-k,2}^{r*} u_{k,1}^s) \beta^r_{k,2} \alpha_{k,1}^s)$$

$$[N_1^k, S_-^k] |0\rangle_{1,2} = N_1^k S_-^k |0\rangle_{1,2} = N_1^k (S_-^k)^2 |0\rangle_{1,2} = 0$$

$$[N_1^k, S_-^k] S_-^k |0\rangle_{1,2} = [N_1^k, S_-^k](S_-^k)^2 |0\rangle_{1,2} = [N_1^k, S_+^k](S_+^k)^2 |0\rangle_{1,2} = 0$$

$$[N_1^k, S_+^k] |0\rangle_{1,2} = S_+^k |0\rangle_{1,2}, \qquad [N_1^k, S_+^k] S_+^k |0\rangle_{1,2} = (S_+^k)^2 |0\rangle_{1,2}$$

$$N_1^k S_+^k |0\rangle_{1,2} = S_+^k |0\rangle_{1,2}, \qquad N_1^k (S_+^k)^2 |0\rangle_{1,2} = 2(S_+^k)^2 |0\rangle_{1,2}$$

$$_{1,2}\langle 0| S_-^k N_1^k S_+^k |0\rangle_{1,2} = Z_k$$

$$_{1,2}\langle 0| S_-^k N_1^k S_-^k (S_+^k)^2 |0\rangle_{1,2} = {}_{1,2}\langle 0| (S_-^k)^2 S_+^k N_1^k S_+^k |0\rangle_{1,2} = (Z_k)^2$$

$$_{1,2}\langle 0| S_+^k S_-^k N_1^k S_+^k S_-^k |0\rangle_{1,2} = Z_k + \tfrac{1}{2}(Z_k)^2$$

$$_{1,2}\langle 0| (S_-^k)^2 N_1^k (S_+^k)^2 |0\rangle_{1,2} = 2(Z_k)^2$$

$$_{1,2}\langle 0| (S^k)^2 S_+^k N_1^k S_-^k (S_+^k)^2 |0\rangle_{1,2} = 6(Z_k)^2$$

$$_{1,2}\langle 0| (S_+^k)^2 S^k N_1^k S_+^k (S^k)^2 |0\rangle_{1,2} = 2(Z_k)^2$$

$$_{1,2}\langle 0| (S_+^k)^2 (S^k)^2 N_1^k S_+^k S^k |0\rangle_{1,2} = {}_{1,2}\langle 0| S_+^k S_-^k N_1^k (S_+^k)^2 (S_-^k)^2 |0\rangle_{1,2}$$
$$= 4(Z_k)^2$$

$$_{1,2}\langle 0| (S_+^k)^2 (S_-^k)^2 N_1^k (S_+^k)^2 (S_-^k)^2 |0\rangle_{1,2} = 24(Z_k)^2.$$

Eq. (3.2) is then obtained as

$$_{c,\mu}\langle 0| N_{\alpha_1}^k |0\rangle_{c,\mu} = Z_k(\sin^2\theta \cos^2\theta + \sin^4\theta) + (Z_k)^2 \sin^4\theta$$

$$\times (-\cos^2\theta + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta + \tfrac{1}{2}\cos^4\theta + \tfrac{1}{2} + \tfrac{3}{2}\sin^4\theta)$$

$$= Z_k \sin^2\theta.$$

Appendix D

For the calculation of $|0\rangle^k_{c,\mu}$ it is useful to choose $k = (0, 0, |k|)$. In this reference frame the operators S_+^k, S^k, S_3^k are written as

$$S_+^k \equiv \sum_r S_+^{k,r} = \sum_r (U_k^* \alpha_{k,1}^{r\dagger} \alpha_{k,2}^r - \varepsilon^r V_k^* \beta_{-k,1}^r \alpha_{k,2}^r$$

$$+ \varepsilon^r V_k \alpha_{k,1}^{r\dagger} \beta_{-k,2}^{r\dagger} + U_k \beta_{-k,1}^r \beta_{-k,2}^{r\dagger})$$

$$S_-^k \equiv \sum_r S_-^{k,r} = \sum_r (U_k \alpha_{k,2}^{r\dagger} \alpha_{k,1}^r + \varepsilon^r V_k^* \beta_{-k,2}^r \alpha_{k,1}^r$$

$$- \varepsilon^r V_k \alpha_{k,2}^{r\dagger} \beta_{-k,1}^{r\dagger} + U_k^* \beta_{-k,2}^r \beta_{-k,1}^{r\dagger})$$

$$S_3^k \equiv \sum_r S_3^{k,r} = \tfrac{1}{2} \sum_{k,r} (\alpha_{k,1}^{r\dagger} \alpha_{k,1}^r - \beta_{-k,1}^{r\dagger} \beta_{-k,1}^r - \alpha_{k,2}^{r\dagger} \alpha_{k,2}^r + \beta_{k,2}^{r\dagger} \beta_{-k,2}^r),$$

where U_k, V_k have been defined in Eqs. (3.7)–(3.10) and $\varepsilon^r = (-1)^r$. It is easy to show that the $su(2)$ algebra holds for $S_{\pm}^{k,r}$ and $S_3^{k,r}$, which means that the $su_k(2)$ algebra given in Eqs. (2.19) splits into r disjoint $su_{k,r}(2)$ algebras. Using the Gaussian decomposition, $|0\rangle_{e,\mu}^k$ can be written as

$$|0\rangle_{e,\mu}^k = \prod_r \exp(-\tan\theta\, S_+^{k,r})\exp(-2\ln\cos\theta\, S_3^{k,r})\exp(\tan\theta\, S_-^{k,r})\, |0\rangle_{1,2},$$

where $0 \leqslant \theta < \pi/2$. The final expression for $|0\rangle_{e,\mu}^k$ in terms of $S_{\pm}^{k,r}$ and $S_3^{k,r}$ is then

$$|0\rangle_{e,\mu}^k = \prod_r [1 + \sin\theta \cos\theta(S_-^{k,r} - S_+^{k,r}) - \sin^2\theta\, S_+^{k,r} S_-^{k,r}]\, |0\rangle_{1,2},$$

from which we finally obtain Eq. (3.11).

APPENDIX E

If one wants to work with the "mass" sectors $\{|0(\Theta)\rangle_1\}$ and $\{|0(\Theta)\rangle_2\}$ (see Sec. 3), the tensor product formalism must be used, e.g.,

$$(O_1 + O_2)(|0(\Theta)\rangle_1 \otimes |0(\Theta)\rangle_2) \equiv (O_1 \otimes I + I \otimes O_2)(|0(\Theta)\rangle_1 \otimes |0(\Theta)\rangle_2)$$

$$= O_1\, |0(\Theta)\rangle_1 \otimes |0(\Theta)\rangle_2 + |0(\Theta)\rangle_1 \otimes O_2\, |0(\Theta)\rangle_2$$

with O_i, $i = 1, 2$, any product of ν_i neutrino field operators.

For example, we have $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$, and

$$\alpha_{k,1}^r \equiv \alpha_{k,1}^r \otimes I, \qquad \alpha_{k,2}^r \equiv I \otimes \alpha_{k,2}^r,$$

so that

$$\alpha_{k,1}^r \alpha_{k,2}^{r\dagger} = (\alpha_{k,1}^r \otimes I)(I \otimes \alpha_{k,2}^{r\dagger}) = \alpha_{k,1}^r \otimes \alpha_{k,2}^{r\dagger}$$

and

$$R^{-1}\alpha_{k,1}^r R = \cos\theta(\alpha_{k,1}^r \otimes I) + e^{i\psi}\sin\theta(I \otimes \alpha_{k,2}^r)$$

$$B_2^{-1}R^{-1}\alpha_{k,1}^r R B_2 = \cos\theta(\alpha_{k,1}^r \otimes I) + e^{i\psi}\sin\theta(I \otimes \alpha_{k,2}^r(\Theta))$$

with

$$\alpha^r_{k,2}(\Theta) = \cos\Theta_k \ \alpha^r_{k,2} + \varepsilon^r e^{i\phi_2} \sin\Theta_k \beta^{r\dagger}_{-k,2}$$

and

$$B_2^{-1} |0\rangle_{1,2} = |0\rangle_1 \otimes |0(\Theta)\rangle_2.$$

We note that $|0(\Theta)\rangle_i$, $i = 1, 2$, are the vacuum states (i.e. are annihilated by) for the operators $\alpha^r_{k,i}(\Theta) = B(\Theta)_i^{-1} \alpha^r_{k,i} B(\Theta)_i$ and $\beta^r_{k,i}(\Theta) = B(\Theta)_i^{-1} \beta^r_{k,i} B(\Theta)_i$.
We also note that

$$\alpha^r_{k,e}(|0\rangle_1 \otimes |0(\Theta)\rangle_2) = \beta^r_{k,e}(|0\rangle_1 \otimes |0(\Theta)\rangle_2) = 0;$$

$$\alpha^r_{k,\mu}(|0(\Theta)\rangle_1 \otimes |0\rangle_2) = \beta^r_{k,\mu}(|0(\Theta)\rangle_1 \otimes |0\rangle_2) = 0,$$

but

$$\alpha^r_{k,e}(|0(\Theta)\rangle_1 \otimes |0\rangle_2) \neq 0, \qquad \alpha^r_{k,\mu}(|0\rangle_1 \otimes |0(\Theta)\rangle_2) \neq 0,$$

etc. Moreover,

$$_i\langle 0(\Theta)| \ N^{k,r}_{\sigma_i} \ |0(\Theta)\rangle_i = \sin^2\Theta_k, \qquad i = 1, 2,$$

and

$$_2\langle 0(\Theta)| \ N^{k,r}_{\sigma_i} \ |0(\Theta)\rangle_2 = {}_1\langle 0(\Theta)| \ N^{k,r}_{\sigma_2} \ |0(\Theta)\rangle_1 = 0,$$

for $\sigma = \alpha, \beta$ which show the condensate structure of the sectors $\{|0(\Theta)\rangle_i\}$, $i = 1, 2$.

APPENDIX F

Useful relations for three flavors mixing.
We work in the frame $k = (0, 0, |k|)$ and for simplicity we omit the k and the helicity indices.

$$\begin{cases} G_{23}^{-1} \alpha_1 G_{23} = \alpha_1 \\ G_{13}^{-1} \alpha_1 G_{13} = c_{13}\alpha_1 + e^{i\delta}s_{13}(U^*_{13}\alpha_3 + \varepsilon_r V_{13}\beta_3^\dagger) \\ G_{12}^{-1} \alpha_1 G_{12} = c_{12}\alpha_1 + s_{12}(U^*_{12}\alpha_2 + \varepsilon_r V_{12}\beta_2^\dagger) \end{cases}$$

$$\begin{cases} G_{23}^{-1} \alpha_2 G_{23} = c_{23}\alpha_2 + s_{23}(U^*_{23}\alpha_3 + \varepsilon_r V_{23}\beta_3^\dagger) \\ G_{13}^{-1} \alpha_2 G_{12} = \alpha_2 \\ G_{12}^{-1} \alpha_2 G_{12} = c_{12}\alpha_2 - s_{12}(U_{12}\alpha_1 - \varepsilon_r V_{12}\beta_1^\dagger) \end{cases}$$

$$\begin{cases} G_{23}^{-1}\alpha_3 G_{23} = c_{23}\alpha_3 - s_{23}(U_{12}\alpha_2 - \varepsilon_r V_{12}\beta_2^\dagger) \\ G_{13}^{-1}\alpha_3 G_{13} = c_{13}\alpha_3 - e^{-i\delta}s_{13}(U_{13}\alpha_1 - \varepsilon_r V_{13}\beta_1^\dagger) \\ G_{12}^{-1}\alpha_3 G_{12} = \alpha_3 \end{cases}$$

$$\begin{cases} G_{23}^{-1}\beta_1^\dagger G_{23} = \beta_1^\dagger \\ G_{13}^{-1}\beta_1^\dagger G_{13} = c_{13}\beta_1^\dagger + e^{i\delta}s_{13}(U_{13}\beta_3^\dagger - \varepsilon_r V_{13}^*\alpha_3) \\ G_{12}^{-1}\beta_1^\dagger G_{12} = c_{12}\beta_1^\dagger + s_{12}(U_{12}\beta_2^\dagger - \varepsilon_r V_{12}^*\alpha_2) \end{cases}$$

$$\begin{cases} G_{23}^{-1}\beta_2^\dagger G_{23} = c_{23}\beta_2^\dagger + s_{23}(U_{23}\beta_3^\dagger - \varepsilon_r V_{23}^*\alpha_3) \\ G_{13}^{-1}\beta_2^\dagger G_{13} = \beta_2^\dagger \\ G_{12}^{-1}\beta_2^\dagger G_{12} = c_{12}\beta_2^\dagger - s_{12}(U_{12}^*\beta_1^\dagger + \varepsilon_r V_{12}^*\alpha_1) \end{cases}$$

$$\begin{cases} G_{23}^{-1}\beta_3^\dagger G_{23} = c_{23}\beta_3^\dagger - s_{23}(U_{23}^*\beta_2^\dagger + \varepsilon^r V_{23}^*\alpha_2) \\ G_{13}^{-1}\beta_3^\dagger G_{13} = c_{13}\beta_3^\dagger - e^{i\delta}s_{13}(U_{13}^*\beta_1^\dagger + \varepsilon_r V_{13}^*\alpha_1) \\ G_{12}^{-1}\beta_3^\dagger G_{12} = \beta_3^\dagger \end{cases}$$

with

$$V_{ij}^k = |V_{ij}^k| e^{i(\omega_{k,j} + \omega_{k,i})t}, \qquad U_{ij}^k = |U_{ij}^k| e^{i(\omega_{k,j} - \omega_{k,i})t}$$

$$|U_{ij}^k| = \left(\frac{\omega_{k,i} + m_i}{2\omega_{k,i}}\right)^{1/2} \left(\frac{\omega_{k,j} + m_j}{2\omega_{k,j}}\right)^{1/2} \left(1 + \frac{k^2}{(\omega_{k,i} + m_i)(\omega_{k,j} + m_j)}\right)$$

$$|V_{ij}^k| = \left(\frac{\omega_{k,i} + m_i}{2\omega_{k,i}}\right)^{1/2} \left(\frac{\omega_{k,j} + m_j}{2\omega_{k,j}}\right)^{1/2} \left(\frac{k}{(\omega_{k,j} + m_j)} - \frac{k}{(\omega_{k,i} + m_i)}\right),$$

where $i, j = 1, 2, 3$ and $j > i$, and

$$|U_{ij}^k|^2 + |V_{ij}^k|^2 = 1, \qquad i = 1, 2, 3 \qquad j > i$$

$$(V_{23}^k V_{13}^{k*} + U_{23}^{k*} U_{13}^k) = U_{12}^k, \qquad (V_{23}^k U_{13}^{k*} - U_{23}^{k*} V_{13}^k) = -V_{12}^k$$

$$(U_{12}^k U_{23}^k - V_{12}^{k*} V_{23}^k) = U_{13}^k, \qquad (U_{23}^k V_{12}^k + U_{12}^{k*} V_{23}^k) = V_{13}^k$$

$$(V_{12}^{k*} V_{13}^k + U_{12}^{k*} U_{13}^k) = U_{23}^k, \qquad (V_{12}^k U_{13}^k - U_{12}^k V_{13}^k) = -V_{23}^k.$$

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