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Abstract. We argue that only by taking into account the quantum properties of the bodies that form the reference frames, physical quantum operators can be defined in quantum gravity. The theory of general relativity coupled to matter introduced in a companion paper is considered. Its formal canonical quantization yields two surprising results: the diffeomorphism constraint can be exactly solved; and the Hamiltonian constraint reduces, in the context of a well defined approximation, to a Schrödinger evolution equation. By using the solutions of the quantum constraints of vacuum general relativity recently obtained in the loop representation, and in the context of a 'realistic' local material reference system, we define a quantum gravitational theory in which the constraints can be solved, the only remaining equation is a regularized Schrödinger equation which expresses the dynamics in the internal clocks, and a class of gauge-invariant physical observables is explicitly displayed.

1. Quantization of the reference systems

In a companion paper [1], we have argued that in order to get ('local') gauge-invariant observables in a generally covariant theory (namely physical observables that have vanishing Poisson brackets with the constraints), one has also to include in the system the dynamics of the objects that form the reference system.

In this paper we study the consequences of this fact on quantum theory. We do that by studying the formal canonical quantization [2, 3] of the theory of general relativity plus matter which was introduced in the previous paper.

In the canonical quantization of the gauge theories one imposes the quantum constraint equations on the quantum states' space H [4]: the states that solve this equations are the physical states of the system. Let H_{Ph} be the linear space of the physical states. Then, one defines the observables on H_{Ph} in terms of the basic operators defined on H. In order that an operator $\hat{O}_{\mathrm{Ph}}$ on H be well defined on the subspace H_{Ph}, it has to commute with the constraints. By taking the $\hbar \longmapsto 0$ limit of the commutator, it follows that $\hat{O}_{\mathrm{Ph}}$ should correspond to a classical observable O_{Ph} with vanishing Poisson brackets with the constraints, namely to a gauge-invariant observable.

Thus, in order to build the quantum theory we cannot avoid the problem of finding the gauge-invariant observables. Since, as argued in the previous paper, gauge-invariant observables (in the specified sense) can be defined only by taking into account the gravitational dynamics of the material bodies that form the reference systems, we are forced to conclude that a consistent quantization of a general covariant theory like general relativity can only be accomplished by also considering the quantum properties of the reference systems.

Section 2 contains a preliminary discussion on the possibility of avoiding the quantization of the reference systems. Then, in section 3, the quantization of the reference systems is studied by building the (formal) quantum theory of the model introduced in the companion paper.

We get two surprising results. The first is that the diffeomorphism constraint can be exactly solved in closed form. This is related to the fact that the 3-diffeo-invariant observables of the model are explicitly known. The second, and most remarkable, result is that the Hamiltonian constraint gives rise to a Schrödinger evolution equation in the approximation considered in the previous paper.

Then we discuss a second, more 'realistic', model in which the material reference system is localized in a finite region of space. In this model we may make use, for the vacuum region, of the solutions of the quantum constraint equations of vacuum general relativity which were recently obtained by making use of Ashtekar's variables [5, 6] in the loop representation [7]. By using these results, the theory is reduced to a system in which the basic constraint equations have been solved, the only remaining equation being a Schrödinger evolution equation in the local internal clock time. A class of local gauge-invariant physical observables is displayed. We suggest that in this way the problem of the absence of physical observables in canonical quantum gravity can be overcome.

2. Which theory should we quantize in order to get a quantum theory of the gravitational field?

With reference to the previous paper [1], we may say that if we want to quantize a general covariant theory like general relativity, we have essentially three choices.

(a) Consider pure general relativity in the non-local interpretation.

(b) Consider pure general relativity in the local interpretation.

(c) Consider a general relativity + matter system in which the matter is used as a reference system.

The choice (a) is, in a sense, more fundamental; but it leads to difficulties. As we repeatedly advertised, no gauge-invariant observable is known in pure (compact space) general relativity†. This is a sad situation, since in the vacuum case solutions of the quantum constraints have been found, and therefore at least one sector of H_{Ph} has being defined [7]. One has H_{Ph}, but not even a single observable quantity that can be defined on it. There are interesting linear operators well defined on H_{Ph}. But in quantum mechanics it is not enough to have operators: one needs their interpretation. The interpretation is provided by relating the operators to classical

† The asymptotically flat case allows the definition of some gauge-invariant observables at infinity, like the ADM mass and momenta. There are some doubts that these essentially non-local observables could be implemented in the quantum theory [8, 9]. Our general philosophy here is that since every observable has to be local (in a suitable sense), fundamental physics cannot drastically depend on the boundary conditions. To explore the opposite point of view, work is in progress on the the asymptotically flat case [10].

observables, namely to measurement procedures. Thus, these operators are useless until a way to interpret them has been found†.

Choice (b) implies some subtleties. Indeed, when read in the local interpretation, general relativity is not a gauge theory in which the undetermined degrees of freedom are non-observable gauges, but it is an approximate theory in which the undetermined degrees of freedom just represent some physical degrees of freedom over which we do not have control [1]. This has some consequences. To understand these consequences let us consider a simple model. Consider two particles, x and y on a line. The first one represents a physical degree of freedom the dynamics of which we know, the second one a degree of freedom the dynamics of which we do not know. The first one represents the true gravitational degrees of freedom, the second one the reference system degrees of freedom. Let the equations of motion be

$$\ddot{x} = 0 \qquad \ddot{y} = f(t) \tag{1}$$

where $f(t)$ is an arbitrary external force, unknown, which drives y. These equations can be obtained by the Hamiltonian

$$H = \frac{1}{2m_x}p_x^2 + \frac{1}{2m_y}p_y^2 + N(t)p_y \tag{2}$$

where $\dot{N}(t) = f(t)$. Since the equations of motion do not contain m_y, we can take the $m_y \longmapsto \infty$ limit in the Hamiltonian and get

$$H = \frac{1}{2m_x}p_x^2 + N(t)p_y \tag{3}$$

which, indeed, again gives the same evolution for $y(t)$. Note that $N(t)$ must not be considered as a Lagrange multiplier, but just as an unknown arbitrary function. Let us consider the quantization. A standard Schrödinger quantization of the Hamiltonian (2) gives a wavefunction $\psi \in L_2[R^2]$, which evolves according to a standard Schrödinger equation. The solution of this equation can be obtained from the transformation properties of the wavefunction under arbitrary coordinate transformations (indeed the evolution in y is the one of a free particle in an arbitrary reference system). It is a superposition of waves

$$\psi(x,y,t) = \int \mathrm{d}^3k \int \mathrm{d}^3p\, f(k,p)\, \psi_{k,p}(x,y,t) \tag{4}$$

$$\psi_{k,p}(x,y,t) = \exp\left[\frac{\mathrm{i}}{\hbar}\left(kx + \frac{k^2}{2m_x}t\right)\right] \exp\left[\frac{\mathrm{i}}{\hbar}\left((y - s(t))p + \frac{p^2}{2m_y}t\right)\right] \tag{5}$$

where $\dot{s}(t) = N(t)$. The important point to notice is that the y dependence is not trivial and contains information even if $s(t)$ is not known. For instance, suppose at $t = 0$ the wavefunction was concentrated in $y = 0$; then the centre of the wavepacket at later times depends on $s(t)$, but once this is known, the spread of the wavepacket is uniquely determined. Note that, consistent with our interpretation, the wavefunctional

† A possible alternative way out may be given by studying the interpretation of these operators in terms of the gauge fixing provided by the linearization. Work is in progress in this direction [11].

contains information on both x and y, both of which can be measured. For every state ψ, the outcome of a measurement of x and y can be obtained by the standard rules.

Suppose we decide that, since y is totally undetermined, we treat it in the quantization as we usually treat gauge degrees of freedom. This can be achieved by considering the Hamiltonian (3), and treating N as a Lagrange multiplier. We obtain a gauge theory with the constraint $p_y = 0$. By imposing it on the states we would obtain

$$\psi(x, y, t) = \psi(x, t). \tag{6}$$

Then the entire information on y is lost, and the theory does not describe y anymore. Thus this treatment is inconsistent with our interpretation of the y degree of freedom: if we interpret a local invariance of the equation as a lack of knowledge rather than a gauge invariance, then the Dirac quantization procedure is not viable.

In general relativity, the role of $N(t)$ (and $s(t)$) is played by the lapse function. To impose the constraints on the wavefunction amounts to requiring that the wavefunction does not depend at all on the (arbitrary) reference system variables. Thus, the standard Dirac constraint quantization is totally incompatible with the quantization of the theory in the local interpretation, which is the interpretation in which the local invariance of the Einstein equation is interpreted as a lack of knowledge of the motion of the reference system rather than as a genuine gauge invariance. If we want to adhere to the local interpretation, not only are we dealing with an approximate theory (as shown in [1]), but also we have to rethink the quantization procedure.

The last choice (c) is the one to which we adhere in this paper, and which is analysed in the next sections.

3. Quantization of the gravity + reference system theory

In this section, we study the formal canonical quantization [2, 3] of the system defined in the companion paper. We refer to that paper for the notation. We recall here that the theory is defined, in the canonical framework, in terms of the phase space coordinates $g_{ab}(x), p^{ab}(x), X^a(y), p_a(y), T(y), P(x)$ by the constraints

$$\begin{aligned} H(f) &= H^{\mathrm{ADM}}(f) - \int \mathrm{d}^3y\ f^a(X(y))\, p_a(y) \\ H(f) &= H^{\mathrm{ADM}}(f) - \int \mathrm{d}^3y\ f(X(y))\sqrt{c^2m^2 + p^2(y) + (\omega^2/c^2)(P(y))^2}. \end{aligned} \tag{7}$$

g_{ab} and p^{ab} are the standard space metric and its conjugate momentum (related to the extrinsic curvature of the space). The vectors y label the infinitesimal particles of a 'fluid' that form the material reference system: $X^a(y)$ is the position of the particle y and $p^a(y)$ the related momentum. $T(y)$ is the value of a (clock) variable attached to the particle y and $P(y)$ its momentum.

Following DeWitt, we introduce on the configuration space $\{g_{ab}(x), X^a(y), T(y)\}$ a wavefunctional

$$\Psi[g, X, T] \tag{8}$$

and we define the quantum theory by imposing on Ψ the quantum constraints

$$\hat{H}_\mu(x)\Psi[g, X, T] = 0 \tag{9}$$

(the hat indicates operators) obtained from the constraints (7) by the replacements

$$\begin{aligned} p^{ab}(x) &\longmapsto \hat{p}^{ab}(x) = -\mathrm{i}\hbar\frac{\delta}{\delta g_{ab}(x)} \\ p_a(y) &\longmapsto \hat{p}_a(y) = -\mathrm{i}\hbar\frac{\delta}{\delta X^a(y)} \\ P(y) &\longmapsto \hat{P}(y) = -\mathrm{i}\hbar\frac{\delta}{\delta T(y)}. \end{aligned} \tag{10}$$

We are not interested here in problems related to the measure that defines the Hilbert structure, or the ordering of the Hamiltonian constraint. The momentum constraint is ordered so that it generates three-dimensional diffeomorphisms. The surprise is that these equations are not completely intractable. Indeed, the momentum constraint can be completely and exactly solved. This is related to the fact that we know explicitly the classical Diff3 invariant observables. Indeed, we define, on the configuration space, the observable $\tilde{g}_{ab}(y)$, which is a functional of g and X,

$$\tilde{g}_{ab}[g,X](y) \equiv \frac{\partial X^c(y)}{\partial y^a}\frac{\partial X^d(y)}{\partial y^b}g_{cd}(X(y)). \tag{11}$$

Then a direct calculation shows that every state of the form

$$\Psi[g,X,T] = \Psi[\tilde{g},T] \tag{12}$$

solves the momentum constraint. This follows, of course, from the fact that $\tilde{g}$ and T coordinatize the Diff3 invariant configuration space.

Now, consider the approximation of the full theory considered in section 4. In this approximation the Hamiltonian constraint becomes

$$\hat{H}_0^{\mathrm{ADM}}(f)\Psi[\tilde{g},T] - \mathrm{i}\hbar\frac{\omega}{c}\int \mathrm{d}^3y f(X(y))\frac{\delta}{\delta T(y)}\Psi[\tilde{g},T] = 0. \tag{13}$$

From this equation it follows that

$$-\mathrm{i}\hbar\frac{\delta}{\delta T(y)}\Psi[\tilde{g},T] = \frac{c}{\omega}\tilde{\hat{H}}_0^{\mathrm{ADM}}(y)\Psi[\tilde{g},T]. \tag{14}$$

where the tilde over the ADM Hamiltonian constraints indicates that it operates on the $\tilde{g}$ variable rather than on the g variable.

Equation (14) is a crucial result. It is a Schrödinger equation in the clock time $T(y)$. More precisely it is a Schwinger–Tomonaga *multi-fingered time evolution equation*. It fixes the value of $\Psi[\tilde{g},T]$ everywhere if the value of $\Psi[\tilde{g},0]$ is known. Given the discussion in section 4 of [1], it is clear how we can interpret this result. We can identify the $\tilde{g}$ variable in this section with the initial value ($T = 0$) of the $\tilde{g}$ physical observable defined in [1]. $\tilde{g}_{ab}(y)$ at $T(y) = 0$ can be seen as a complete set of gauge-invariant configuration observables, and $\Psi[\tilde{g},0]$ contains all the information on the state and can be seen as an initial state. Then the Schrödinger equation (14) gives the relation with alternative descriptions, in terms of $\tilde{g}$ at different times. This can be read as an evolution equation in the clock's time.

In particular, we may decide to describe the system by sets of measurements performed in correspondence with the sets of spacetime points in which the clocks have the same value $T(y) = T$. Then we can write, without loosing information,

$$\Psi[\tilde{g}_{ab}(y)](T) = \Psi[\tilde{g}_{ab}(y), T] \tag{15}$$

which satisfies the standard Schrödinger equation

$$-\mathrm{i}\hbar\frac{\partial}{\partial T}\,\Psi[\tilde{g}](T) = \left(\frac{c}{\omega}\int \mathrm{d}^3y\;\hat{\tilde{H}}_0^{\mathrm{ADM}}(y)\right)\Psi[\tilde{g}](T). \tag{16}$$

We think, but it has still to be checked, that any other additional matter term in the theory (for instance electromagnetic or Yang–Mills fields) would appear in this Schrödinger equation in the correct form. This would be an interesting test of the ideas proposed in this paper†.

In the approximate form of the Hamiltonian constraint, we dropped the original absolute value. As shown in [1], it is consistent to restrict the classical theory to the sector $P > 0$. This amounts to assuming that the direction in which the coordinate-time flows is not observable. Thus, the classical theory that we are quantizing is supplemented by the $P > 0$ condition, and we may drop the absolute value. However, we should then bring this condition to the quantum theory. In the quantum theory, this becomes a positive frequency condition for the solutions of the Schrödinger equation. In turn, this implies that we have to restrict the physical Hilbert space to the states on which the Hamiltonian is positive. While this procedure is certainly viable (for the same reason as why we may select the $p^0 > 0$ sector of a quantum free particle), nevertheless, as pointed out by Kuchař [12], it raises certain issues about the quantum observability even of the classically gauge-invariant observables that we have found in the companion paper. The translations to the quantum theory of constraints expressed by inequalities has been studied by Isham [13]. The reason for which the classical gauge-invariant observables may give problems in the quantum theory is exemplified by the standard example of the quantization of a free particle on R^+: while the momentum of the particle is a good classical observable, it cannot be translated in a genuine self-adjoint operator, essentially because the eigenfunctions of the derivative operator cannot stay within a half-line. Although these problems are certainly serious, they will not be addressed in this paper. We leave them for a future, more rigorous, investigation.

We have shown that the coupled general relativity + matter theory defined in [1] can be formally quantized in the canonical framework, obtaining the two following results. The quantum momentum constraint equation can be exactly solved; the quantum Hamiltonian constraint gives rise, in the approximation defined in section 4 of [1], to a Schrödinger equation in the clock time T. All the quantities that appear in the final theory are gauge invariant.

4. A 'realistic' model of a local material reference system, the 'rocket'

Two features of the model presented in the last section are particularly disturbing. Firstly, free particles are not particularly nice objects in view of our essentially field theoretical understanding of fundamental physics.

† This issue has been raised by A Ashtekar.

Secondly, and more importantly, the matter that we couple to general relativity as a physical reference system should represent the physical laboratory and objects that the experimenter uses for his or her measurements. The model in which the laboratory fills up the entire space is therefore quite unrealistic. For instance, it would kill all cosmological and astronomical physics, which is based on vacuum Einstein equations. In this section we partially improve the model in regard to the second criticism. We leave the first problem open.

In the philosophy of the present papers, there is nothing like *the* correct model of a reference system. Indeed, different concrete physical situations, in which different matter is used as the reference system, would require to be described by different models.

At the opposite extreme of filling the entire space with our laboratory, we could consider the simplest reference system given by a single particle. Attractive as this picture may be, it is not viable for a technical reason that will be discussed later. Since one particle is not enough, we take a small cloud of them. These will represent, for instance, the atoms of a rocket† sent through the solar system to measure gravitational fields (or maybe towards a black hole to measure quantum gravitational effects!). Including into the picture the electromagnetic interactions among these atoms, which structure the materials, would complicate, but not substantially modify the general picture.

We label these small clouds of particles by a parameter y which now belongs to a finite region of R^3_y. We denote this finite region by $\mathcal{R}$ (for rocket): $\mathcal{R} = \{y \in R^3_y; |y| \leq 1\}$. Our basic variables are now

$$g_{ab}(x) \qquad X^a(y) \qquad T(y) \qquad y \in \mathcal{R}. \tag{17}$$

The Lagrangian and Hamiltonian and constraints are the same as in the previous model, with the difference that the domain of integration of all the y integrals is $\mathcal{R}$. Let us analyse this model. The point is that there are two regions: the rocket and the outside. We can solve the diffeomorphism invariance as we did previously only partially, namely only inside the rocket. We define the metric on the rocket by

$$\check{g}_{ab}(y) \equiv \frac{\partial X^c(y)}{\partial y^a}\frac{\partial X^d(y)}{\partial y^b} g_{cd}(X(y)). \tag{18}$$

We decompose the constraints as follows

$$\begin{aligned} \check{H}_\mu(y) &= H_\mu(X(y)) \\ H^{\rm vac}_\mu(x) &= H_\mu(x) - \int_{\mathcal{R}} \mathrm{d}^3y\ \delta^3(x, X(y))\ \check{H}_\mu(y) \end{aligned} \tag{19}$$

(the superscript vac is for vacuum). The set of constraints $\check{H}_\mu(y), H^{\rm vac}_\mu(x)$ is equivalent to the original set. Similarly we define

$$g^{\rm vac}_{ab}(x) = g_{ab}(x) - \int_{\mathcal{R}} \mathrm{d}^3y\ \delta^3(x, X(y))\ \check{g}_{ab}(y) \tag{20}$$

† A peaceful rocket.

which is a metric that agrees with the original metric in vacuum and vanishes where the rocket is. It is straightforward to verify that

$$\begin{aligned}
&\{\tilde{H}(f), \tilde{g}_{ab}(x)\} = 0 \\
&\{H^{\rm vac}(f), \tilde{g}_{ab}(x)\} = 0 \\
&\{\tilde{H}(f), g^{\rm vac}_{ab}(x)\} = 0 \\
&\{H^{\rm vac}(f), g^{\rm vac}_{ab}(x)\} = \delta_f g^{\rm vac}_{ab}(x)
\end{aligned} \tag{21}$$

where $\delta_f g^{\rm vac}_{ab}(x)$ is the standard diffeo transformation generated by the constraints. Because of these equations, the quantum constraint equation

$$\tilde{H}(f)\Psi[g, X, T] = 0 \tag{22}$$

is solved by any wavefunctional of the form

$$\Psi[g, X, T] = \Psi[g^{\rm vac}, \tilde{g}, T]. \tag{23}$$

In order to deal with remaining momentum constraint

$$H^{\rm vac}(f)\Psi[g, X, T] = 0 \tag{24}$$

it is convenient, given any configuration $X^a(y)$, to choose an arbitrary extension $X^a_{\rm ext}(y)$ defined for all the y in R^3_y, and such that, on $\mathcal{R}$, it agrees with X. We denote the complement of the rocket as $\mathcal{S}$ (for sky); $\mathcal{S} = \{y \in R^3_y \mid |y| > 1\}$. We will check at the end that nothing depends on the extension chosen. By using $X_{\rm ext}$ we define

$$H^{\rm ext}_a(y) = H_a(X_{\rm ext}(y)) \tag{25}$$

and

$$g^{\rm ext}_{ab}(y) \equiv \frac{\partial X^c_{\rm ext}(y)}{\partial y^a}\frac{\partial X^d_{\rm ext}(y)}{\partial y^b} g_{cd}(X_{\rm ext}(y)). \tag{26}$$

Then the residual constraint can be written as

$$H_{\rm vac}(f)\ \Psi[g, T] = 0 \qquad {\rm Supp}\, f \subset \mathcal{S}. \tag{27}$$

In this equation the original variable x has completely disappeared, everything is defined on R^3_y. The equation requires that Ψ depends on $g^{\rm ext}$ only modulo diffeomorphisms with support in the sky $\mathcal{S}$ (outside the rocket). This requirement is clearly independent of the particular extension chosen. Note that, outside the rocket, the momentum constraint is given by just the ADM part.

Let us now consider the Hamiltonian constraint. Its vacuum component gives

$$H^{\rm ADM}(y)\Psi[g^{\rm ext}, T] = 0 \qquad y \in \mathcal{S} \tag{28}$$

The rocket term gives, in the usual approximation,

$$-{\rm i}\hbar\frac{\delta}{\delta T(y)}\Psi[g^{\rm ext}, T] = \frac{c}{\omega}H^{\rm ADM}(y)\Psi[g^{\rm ext}, T] \qquad y \in \mathcal{R} \tag{29}$$

Thus, we have a mixed situation: inside the rocket, since there is a reference system, the metric is observable, and there is an evolution in the clock's time. Outside the rocket the wavefunctionals have to be diff invariant and to satisfy the Hamiltonian ADM constraint.

At this point we can put together the strength of our (complementary) understandings of both regions. Indeed, we do know how to solve the vacuum constraints equations. These have been solved, in the framework of Ashtekar's new variables version of general relativity [5], by using the loop representation of the quantum theory [7]. In these works no observable was available. Now the rockets provides some observables. Let us introduce the loop representation machinery.

First, it is clear that everything that has been said in this paper works equally well in Ashtekar's version of general relativity.

We introduce the Ashtekar variables $A_a(x)$ and $\tilde{\sigma}_a(x)$ on the (extended and complexified) phase space of general relativity. We define

$$A_a^{\mathrm{ext}}(x) =\equiv \frac{\partial X^b_{\mathrm{ext}}(y)}{\partial y^a} A_b(X_{\mathrm{ext}}(y)). \tag{30}$$

Then we repeat step by step the previous derivation. We find that our theory is given in the Ashtekar representation by the states $\Psi[A^{\mathrm{ext}},T]$ and by the constraints

$$\begin{aligned}
&\hat{C}_{\mathrm{ext}}(f)\Psi[A^{\mathrm{ext}},T] = 0 && \mathrm{Supp}\, f \subset \mathcal{S} \\
&\hat{C}(y)\Psi[A^{\mathrm{ext}},T] = 0 && y \in \mathcal{S} \\
&-\mathrm{i}\hbar\frac{\delta}{\delta T(y)}\Psi[A^{\mathrm{ext}},T] = \frac{c}{\omega}C(y)\Psi[A^{\mathrm{ext}},T] && y \in \mathcal{R}
\end{aligned} \tag{31}$$

where $C(f)$ and C are the Ashtekar's diffeomorphisms and Hamiltonian constraints [6], expressed in terms of A^{ext}.

Then, we can transform from A^{ext} to the loop representation. The loop representation is a representation for quantum general relativity in which the states are expressed as functionals of multiloops†. The transformation from the Ashtekar representation to the loop representation is formally accomplished by means of a functional transform which is an infinite-dimensional analogue of the Fourier transform. Here we want to transform from $\Psi[A^{\mathrm{ext}},T]$ to $\Psi[\alpha,T]$, where α is a single loop, or a multiloop (a set of a finite number of loops). Thus we define

$$\Psi[\alpha,T] = \int[\mathrm{d}A_{\mathrm{ext}}]\,\mathrm{Tr}\,P\,\exp\left(\int_\alpha A_{\mathrm{ext}}\right)\Psi[A_{\mathrm{ext}},T]. \tag{32}$$

This formula allows us to transfer the quantum operators defined in the Ashtekar representation to the $\Psi[\alpha,T]$ space. As described in [7], one may proceed in a more rigorous way, and avoid the use of the ill defined functional integral, by quantizing a suitable algebra of functions of A_{ext}, but at the present level of rigour we do not need here the rigorous version of the theory. Here α is a multiple loop on the space R^3_y,

† The term representation is used here in the sense of Dirac [16], as in the coordinate representation and momentum representation for a finite-dimensional system.

and, we recall, the clock variable $T(y)$ is defined for the y in the rocket $\mathcal{R}$. In this loop representation the constraint equations become

$$\hat{C}^{\text{loop}}(f)\Psi[\alpha,T] = 0 \qquad \text{Supp } f \subset \mathcal{S} \tag{33}$$

$$\hat{C}^{\text{loop}}(y)\Psi[\alpha,T] = 0 \qquad y \in \mathcal{S} \tag{34}$$

$$-\mathrm{i}\hbar\frac{\delta}{\delta T(y)}\Psi[\alpha,T] = \frac{c}{\omega}\, C^{\text{loop}}(y)\Psi[\alpha,T] \qquad y \in \mathcal{R}. \tag{35}$$

The first two of these equations are solved in [7]. Equation (34) is solved by any Ψ which has support only on the loops that have no corners or intersections outside the rocket. The general solution of equation (33) is obtained by requiring that Ψ has the same value for every two multiple loops α and β which can be transformed one into the other by a diffeomorphism with support on the sky. Namely on any two loops α and β which are identical in the rocket, but are just knotted and linked in the same way in the sky. In other words we require that $\Psi[\alpha,T]$ depends on the location of α in the rocket and on the way α is knotted and linked in the sky. Let us assume that Ψ has these properties†. Then, the only remaining equation is equation (35). As we did in section 5, we may, without loosing any information, decide to describe the system in terms of just one time, and we have

$$-\mathrm{i}\hbar\frac{\partial}{\partial T}\,\Psi[\alpha](T) = \frac{c}{\omega}\int_{\mathcal{R}} \mathrm{d}^3y\, C^{\text{loop}}(y)\,\Psi[\alpha](T). \tag{36}$$

In [7] the Hamiltonian loop constraint operator C^{loop} is defined in such a way that its action on its kernel is non-divergent. An alternative definition of the Hamiltonian loop constraint operator is studied by Blencowe in [17]. According to this reference, the operator C^{loop} is a regularized operators defined in the entire loop space. See also the subsection of [7] on the shift operator. By using the Blencowe Hamiltonian constraint C^{loop}, equation (36) is a regularized Schrödinger equation that expresses the quantum gravitational evolution of the physical states in the clock time T. Equation (36) is our main result. By using this equation, concrete calculations in quantum gravity could perhaps be performed.

Note that the value of any $T^n[\alpha]$ observable [7] inside the rocket, at any given time T, is a gauge-invariant quantity. For instance

$$T^o[\alpha](0) \equiv \operatorname{Tr} P\, \exp\left(\int_\alpha A^{\text{ext}}\right) \qquad \alpha : S^1 \longmapsto \mathcal{R} \tag{37}$$

is a (complex) *gauge-invariant* physical observable in quantum gravity. It is related to the change in the internal direction of a left-handed spin particle dragged along a loop defined by the particles of the rocket.

In conclusion, we have defined in this section a theory in which quantum general relativity is coupled to a quantized local reference system, that can be interpreted as a free-falling laboratory (rocket). The constraints of the quantum theory are solved. The physical wavefunction is a function of the clock inside the laboratory and of sets

† There are some subtleties here: to which order in the derivatives at the border of the rocket do the two loops have to agree, in order to be identified? We leave these finer points to a later and more careful analysis of the model.

of loops which may go in and out the laboratory. The wavefunction does not depend on the locations of the loop outside the laboratory, but only on the way they are knotted and linked. It does depend on the location of the loops inside the laboratory. Moreover it has support only on loops that are smooth and non-intersecting outside the laboratory. These quantum states represent physical quantum states of the non-perturbative quantum gravitational theory. To first order in the approximation of big lengths (with respect to c/ω) these states evolve in the laboratory clock time according to the Schrödinger equation (36), in which the Hamiltonian is a regularized operator. The (complex) observables of the theory are given by the $T^n[\alpha](T)$ observables defined in [7], where α lies inside the rocket, and the observable is defined at the clock time T.

5. Perspectives

In this section we collect some considerations on the previous results and some speculations on the possibility of extending and improving them.

We may note that the approximation in which a Schrödinger equation holds is not a semiclassical approximation. Rather, it is a certain physical regime, defined in the Hamiltonian picture by the fact that physical quantities with a dimension of a length are big with respect to c/ω and in the Lagrangian framework by the fact that the motion of the physical clocks is adequate (is as fast as possible). Therefore this way of recovering of the Schrödinger equation is different from the one discussed, for instance, in [2, 18]. However, it is likely that there is a strict relation between the two points of view.

One can consider higher-order terms in c/ω, and study the terms that have to be added to the Schrödinger equation. By expanding in powers of c/ω, and also keeping the second term, the Hamiltonian constraint is

$$H(y) = H^{\mathrm{ADM}}(y) - \left(\frac{c}{\omega}\right)^{-1} P(y) - \frac{c}{\omega}\,(m^2+p^2)P^{-1} + \mathrm{O}\left(\frac{c^3}{\omega^3}\right). \tag{38}$$

The corresponding quantum equation is (we arbitrarily choose an ordering)

$$\begin{aligned} -\mathrm{i}\hbar\frac{\delta}{\delta T(y)}\,\Psi[\tilde{g},T] &= \frac{c}{\omega}\hat{\tilde{H}}_0^{\mathrm{ADM}}(y)\Psi[\tilde{g},T] \\ &\quad + \mathrm{i}\hbar\frac{c^2}{\omega^2}\left(\frac{\delta}{\delta T(y)}\right)^{-1}\left(m^2+\frac{\delta^2}{\delta X^2(y)}\right)\Psi[\tilde{g},T] + \mathrm{O}\left(\frac{c^4}{\omega^4}\right). \end{aligned} \tag{39}$$

If we represent the states in terms of a uniform clock time $T(y) = T$ and we express everything in terms of the $\tilde{g}$ variable, we have the 'corrected' Schrödinger equation for the wavefunctional $\Psi[\tilde{g}](T)$

$$-\mathrm{i}\hbar\frac{\partial}{\partial T}\,\Psi = \frac{c}{\omega}\,\hat{H}\,\Psi + \mathrm{i}\hbar\frac{c^2}{\omega^2}\,\hat{H}^I\Psi + \mathrm{O}\left(\frac{c^4}{\omega^4}\right). \tag{40}$$

where the Hamiltonian is the integral of the ADM Hamiltonian constraint

$$\hat{H} = \int_{\mathcal{R}} \mathrm{d}^3y\,\hat{\tilde{H}}_0^{\mathrm{ADM}}(y). \tag{41}$$

The c^2/ω^2 term is the quantum gravitational correction to the Schrödinger equation due to the physical nature of the clocks. In our model it is given by

$$\hat{H}^I = \int_{\mathcal{R}} \mathrm{d}^3y \left(m^2 + D_a \frac{\delta}{\delta \tilde{g}_{ab}(y)} \tilde{g}^{bc} D_d \frac{\delta}{\delta \tilde{g}_{cd}(y)} \right). \tag{42}$$

It is very tempting to claim that this c^2/ω^2 term gives indeed physical effects that are, in principle, observable. Note that it is non-local in time. Once more we stress that in spite of this term the probabilistic interpretation of the quantum theory still holds.

A weakness of the model presented in section 3 is the kind of matter chosen to represent the reference system. In our philosophy, as we already said, there is not a model of reference system which is the correct and unique one. Different reference systems correspond to a different laboratory used to perform gravitational experiments. More sophisticated models can be studied.

For instance, one may study what are the consequences of considering the discrete nature of the particles and taking a finite number of particles. Note that in the loop representation there is no observable localized in one point. This is a further reason for which a single-particle reference system would not be enough. It is not clear to us if this impossibility of single points reference systems and observables reflects any underlying physical reason†. An attractive alternative is to take a minimal reference system formed only by a loop of particles ($\mathcal{R} \sim S^1$). This reference system defines a single loop observable (and its powers). By using this reference system, the great advantage is that the physical Hamiltonian is expressed by a one-dimensional integral. Thus the dynamics has the structure of the two-dimensional field theories, which are now pretty well under control.

Given our essentially field theoretical understanding of the world, it would perhaps be more appropriate to couple a field, rather than a continuum of particles, to general relativity. The technical difficulty, then, is related to the possibility of inverting the field, in order to use its value as a coordinate. But maybe this is not the correct thing to do: we should use a second quantized language for the field, and use the second quantization particles for defining the physical points. Would this change our conclusions in an essential way? Would the creation and annihilation processes of a field theory, or the properties of symmetry under particle exchange, modify the picture we gave here (which is essentially defined by a fixed particle number sector of the full theory)?‡

The idea discussed in section 4 is the split of the universe into a rocket and a sky. Diffeomorphism invariance holds in the sky, but is hidden by the matter in the rocket. We owe to Louis Crane the idea that meaningful local observables in quantum gravity could be obtained by splitting the system in two regions. Note that this configuration

† This possibility was suggested by Isham.

‡ Another possibility is to consider the hypothesis that fundamental objects are string-like at high energy. Standard string theory is defined on a fixed background in the target space, and is not diffeomorphism invariant. But it has been repeatedly suggested that the theory represents, in some sense, the low-energy (broken) phase of a theory that is fully diffeomorphism invariant at high energy. In this case, if any definition of physical spacetime points has to be recovered, one has go through something like the analysis of this paper; but the reference system has to be defined in terms of strings. At low energies these would behave like particles, but at high energy they become excited and they are spread in some finite region of Σ [19]. Does this mean that string theory implies that the physical space cannot be *defined* below a certain scale [19, 20]?

can also be interpreted as a configuration for a scattering experiment. Indeed, we may think that the sky is the 'internal' scattering region, and the rocket is the 'external' laboratory region. In this way we can recover the asymptotically flat case without unnaturally breaking the diffeomorphism invariance. The crucial result, in this case, is that the time evolution (and the Schrödinger equation) is defined by the laboratory (the rocket).

Louis Crane has recently observed [21] that the structure that one obtains in the loop representation by performing this splitting is essentially isomorphic with the one defined by the axioms of the topological quantum field theories. More precisely, every state of quantum gravity corresponds to a given topological quantum theory. Since these, in turn, are in correspondence with the class of the rational conformal quantum field theories, these developments suggest that there may be a correspondence between conformal theories and states of quantum gravity in the loop representation. The rocket–sky loop model can be considered as a first step toward the implementation of these ideas.

6. Conclusions

The main thesis of the work described in this paper and in the companion one is that gravitational physics cannot be properly understood unless one takes into account the physical nature and the gravitational interactions of the bodies that form the reference system.

In the classical theory one can always work in an approximation in which the effects and the dynamics of the material reference systems are neglected; but in the quantum theory one has to take into account the quantum properties of the objects that form the reference system. It is in this sense that we have a 'quantized spacetime'.

In the quantum theory, in fact, in order to define the physical gauge-invariant quantum operators and therefore to understand which quantities are subject to quantum fluctuations, one is forced to explicitly include the material reference systems in the system. The key point is that these physical material objects of the reference system add to the system some additional degrees of freedom, which can be gauge transformed to give physical gauge-invariant meaning to the gauge part of the gravitational metric.

We can now answer the specific question that opens the companion paper. It is wrong to consider both the quantum fluctuations of the metric and the quantum fluctuations of the particles used to identify the points. Both the metric and the position of the particle are non-gauge-invariant concepts. The only gauge-invariant quantities are the relative quantities, namely the metric distances between the particles themselves. Since gauge degrees of freedom must not be quantized, and therefore are not subject to quantum fluctuation, *it is not correct to assert that both the metric and the particle position fluctuate*. Indeed there is just one fluctuating object, which is the metric that defines the distance between the particles ($\tilde{g}$). This is a crucial conceptual result of the present work.

The main technical result of the two papers is that the program of coupling matter degrees of freedom, and then gauge fixing by absorbing them in a new gauge-invariant observable metric, can indeed be carried out in a concrete fashion. This is true both in the classical theory, where the program provides reasonable gauge-invariant observables and in the (formal) quantum theory.

In the quantum theory, certain surprising results follow. The first one is that the momentum constraint of the coupled theory can be exactly solved. The second and perhaps most remarkable result is that the Hamiltonian constraint gives rise to a functional equation which reduces, in a certain approximation, to a Schrödinger evolution equation.

This result strongly supports the thesis that the Schrödinger equation describes an approximate regime. It is important to stress here that general quantum mechanics and its standard Copenhagen probabilistic interpretation ***still hold*** also outside the Schrödinger approximation. This is discussed in detail in [14].

The model of reference system used in these papers can certainly be improved in several directions. The rocket–sky local model presented in the section 4 is more realistic than the infinite model of section 3. Other models of reference systems certainly deserve to be studied: this is the only way we see in order to construct realistic gauge-invariant physical observables in quantum gravity.

The rocket–sky model becomes particularly interesting if the results that have been obtained in the loop representation are included in the picture and added to the results of this paper. In the rocket–sky loop theory we have physical states, gauge-invariant physical observables and a regularized Schrödinger equation. Many problems remain open before a complete quantum theory of gravity can be defined. But we think that having a regularized Schrödinger equation, a well defined set of gauge-invariant quantities and a set of physical states may be a new, exciting and encouraging situation in quantum gravity.

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