Classical and Quantum Gravity

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Abstract. In this paper we discuss the Casimir effect in a small cavity at rest in the weak gravitational field of a massive, non-rotating source. We propose a new approach, based upon Schwinger's effective action method, showing that the gravitational interaction induces a small correction in the vacuum energy density, in full agreement with the result we obtained in a previous work [19], following a standard field mode decomposition technique. The present result reinforces the belief that gravity can indeed be effective in modifying the vacuum energy of a quantum field confined to the cavity, against the recent claim appeared in [44] that a weak gravitational field has no influence on the Casimir energy.

PACS numbers: 04.20.-q, 04.62.+v

Submitted to: *Class. Quantum Grav.*
original version: 01 Jul, 2019
revised version: 25 Sept, 2019

1. Introduction

The Casimir effect[1, 2] is one of the most beautiful and interesting manifestations of the vacuum energy. The effect was predicted in 1948 by H. Casimir, who formulated a theory in which a tiny attractive force between two neutral conducting plates arises as a consequence of the altered vacuum expectation value of the electromagnetic field between the plates. Predictions of such force were subsequently extended to real conductors and dielectrics, also taking into account more general geometries. Early experiments provided a qualitative support for the attractive force. We recall the works of Sparnaay [3], Israelachvili and Tabor [4] and van Blokland and Overbeek [5]. The era of high precision tests started in 1997, with the experimental work by Lamoreaux [6] who employed a sphere-plane geometry. Mohideen and collaborators [7], studied the effect by means of atomic force microscopy in 1998. Bressi and collaborators [8] measured the effect between two parallel plates (2002), while Decca and collaborators [9] carried on the measurement of the Casimir force between dissimilar metals in 2003.

Roughly speaking, the Casimir effect arises from deformations of the stationary modes of a quantum field confined to a small cavity. Such deformations are typically induced by some material boundaries (in the simplest case, a couple of reflecting plates) which change the trivial space topology into the $R^2 \times [0, L]$ topology (with L being the plate separation).

A deep theoretical analysis of Casimir effect in flat spacetime has been extensively carried on through the years (see, e.g., [10, 11, 12, 13]), showing that changes in field vacuum energy can be related to some symmetry breaking. In the naive case, the broken symmetry is the translational one, due to the cavity plates or other boundaries with different geometries. An almost complete ad exhaustive review on the topic can be found, e.g., in [14, 15, 16].

Casimir effect has been deeply investigated also in a curved spacetime background, considering the distortion in the vacuum energy in presence of gravito-inertial effects, both in the weak [17, 18, 19, 20, 21, 22, 23, 24, 25, 26] and the strong regime [27, 28, 29, 30, 31]. A possible experimental setup devoted to the measurement of the Casimir force in the Schwarzschild metric of the galactic centre has been proposed in [32], while Casimir effect in extended theories of gravity has been explored in [33, 34, 35, 36].

Concerning the long-standing issue whether vacuum fluctuations gravitate or not [20, 37, 38, 39, 40], we also recall the works by Caldwell [41], Cerdonio and Rovelli [42] and Bengochea et al. [43], where a critical analysis of such issue is carried on in connection with the Cosmological Constant Problem.

Analysis of the Casimir effect, both in the weak (gravitoelectromagnetic) [19, 23] and in the strong field (Kerr, Schwarzschild) [28, 30] regime leads to a quite general result, namely the vacuum energy suffers a small reduction in its absolute value in presence of a gravitational field.

However, in a recent paper [44] it has been suggested that Casimir energy doesn't suffer any change in presence of a weak, static gravitational field. Such a null result doesn't agree with [19]. Also, it appears somewhat striking (as the Authors themselves admit in [44]), hence requiring a deeper investigation. We would also point out the relevance of the present issue, considering the efforts carried on in order to develop highly sensitive experimental setups, devoted to the experimental detection of possible interaction between the vacuum energy and the gravitational field (see, e.g., the *Archimedes Experiment*, [45, 46]).

Given the above issue, the main purpose of the present paper is to reconsider in further detail the possible influence of the weak gravitational field, due to a static source M, on the Casimir energy of a quantum field enclosed in a small cavity at rest with respect to M. In order to avoid possible drawbacks related to the machinery employed in [19] and [44], we will not use the somewhat standard field mode decomposition. Instead, we will analyze the problem following a quite different approach (see, e.g., [23]), based on the Schwinger formalism for the effective action [47, 48, 49, 50, 51].

The paper is organized as follows. In section 2 we introduce the scenario, namely a small Casimir cavity, at rest in the field of a weak, non-rotating gravitational source M. The cavity encloses a massless scalar field, minimally coupled to the gravitational field of M. We then expand the background metric $g_{\mu\nu}$ in the local frame of a static observer, at rest with respect to the cavity. In section 3 we employ such expansion, obtaining the Klein-Gordon equation for the quantum field confined to the cavity, as well as the corresponding proper-time hamiltonian. In section 4 we follow the Schwinger formalism, writing the effective action W for the quantum field in the cavity and subsequently expanding W in powers of a small gravitational parameter $\gamma = M/R^2$ ($\gamma = GM/(cR)^2$, in SI units). After a trace evaluation and a Riemann

ζ-function regularization, we obtain the Casimir energy density in presence of weak gravity. The main result is that the Casimir energy *does indeed suffer a non-null distortion* due to the gravitational field. Moreover, such correction agrees with that found in [19], in spite of the quite different approach. In Section 5 we briefly comment the results, also discussing the possible causes which could have led in [44] to rather different conclusions.

Throughout the paper, unless otherwise specified, use has been made of natural units: $c = 1$, $\hbar = 1$, $G = 1$. Greek indices take values from 0 to 3; latin ones take values from 1 to 3. The metric signature is -2, with determinant g.

2. The scenario

We are interested in the influence of gravity on a *rigid* Casimir cavity, made of two parallel, perfectly reflecting plates, each of area A, placed at a distance L each other. The rigidity assumption (reasonable in the limit of a weak gravitational field) means that the size and the shape of the cavity cannot be changed by external forces, gravitational or not. As customary, we assume $L \ll \sqrt{A}$. We take as a gravitational source a static, non-rotating mass M. Consider a rectangular set of *general* coordinates $\{t, x, y, z\}$ having the origin at the centre O of the mass M. In the weak field approximation, adopting the Lorenz (harmonic) gauge, the line element reads

$$ds^2 = \big(1 + 2\Phi(r)\big)dt^2 - \big(1 - 2\Phi(r)\big)\delta_{ij}dx^i dx^j, \tag{2.1}$$

where $\Phi(r)$ is the newtonian potential

$$\Phi(r) = -\frac{M}{r}, \qquad r = (x^2 + y^2 + z^2)^{1/2}. \tag{2.2}$$

Exploiting the spherical symmetry, we assume the Casimir plates orthogonal to the $z-$axis (the latter taken along the radial direction), at a fixed distance from the gravitational source M, so that R and $R + L$ represent the radial coordinates of the midpoints of the plates, which we will conventionally call the *lower* and the *upper* plate, respectively. We also assume that the plate extension is small with respect to R, so that the following inequalities hold

$$L \ll \sqrt{A} \ll R. \tag{2.3}$$

Next, we introduce a further reference frame, defining a set of *local* coordinates $\{\overline{t}, \overline{x}, \overline{y}, \overline{z}\}$, adapted to an observer at rest with the cavity. For definiteness, we fix the origin $\overline{O}$ of the local frame at the midpoint of the lower plate, so that $\overline{O} = \{0, 0, R\}$ in general spatial cordinates (see Fig. 1). In the weak field limit, the tetrad field adapted to the observer is readily obtained from (2.1)

$$\begin{aligned} e_{\overline{t}} &= \big(1 - \Phi(r)\big)\partial_t, \\ e_{\overline{x}} &= \big(1 + \Phi(r)\big)\partial_x, \\ e_{\overline{y}} &= \big(1 + \Phi(r)\big)\partial_y, \\ e_{\overline{z}} &= \big(1 + \Phi(r)\big)\partial_z. \end{aligned} \tag{2.4}$$

In order to describe the influence of gravity inside the cavity, we need to expand the spacetime metric around the origin $\overline{O}$ of the local frame. This can be done observing that [52]

$$g_{\mu\nu}(\vec{x}) = g_{\mu\nu}(\vec{x}_0) + \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)_{\vec{x}_0} e^\lambda_{\overline{\alpha}}(\vec{x}_0)\, x^{\overline{\alpha}} + O(x^{\overline{\alpha}})^2, \tag{2.5}$$

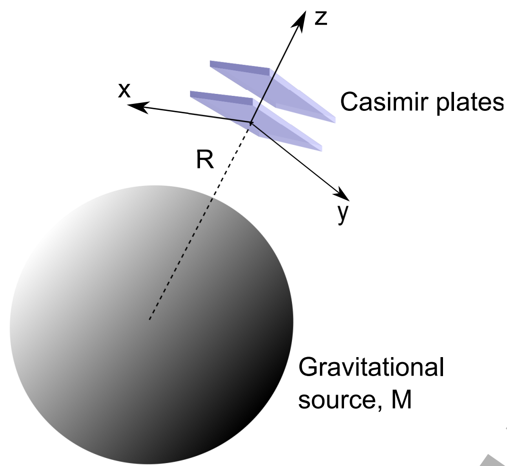

Figure 1. Schematic picture of a static Casimir cavity in the weak gravitational field of a non-rotating massive source. An observer, at rest with the cavity, performs local measurements of the vacuum energy inside the cavity.

where $\vec{x}_0 = (0, 0, R)$ and $\{x^{\overline{\alpha}}\} \equiv \{\overline{t}, \overline{x}, \overline{y}, \overline{z}\}$. Recalling that $dx^\mu = e^{\mu}_{\overline{\alpha}} dx^{\overline{\alpha}}$, we get

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \simeq \left[\eta_{\overline{\beta}\overline{\gamma}} + \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)_{\vec{x}_0} e^{\lambda}_{\overline{\alpha}}(\vec{x}_0) e^{\mu}_{\overline{\beta}}(\vec{x}_0) e^{\nu}_{\overline{\gamma}}(\vec{x}_0)\, x^{\overline{\alpha}}\right] dx^{\overline{\beta}} dx^{\overline{\gamma}}. \tag{2.6}$$

The quantity in square brackets in (2.6) represents the expansion of the metric tensor in the local frame of the observer. Let us call it $\overline{g}_{\overline{\beta}\overline{\gamma}}$. We can now evaluate explicitly $\overline{g}_{\overline{\beta}\overline{\gamma}}$ using (2.1) and (2.4) in (2.6). We have, to the desired order

$$\begin{aligned}\overline{g}_{\overline{t}\overline{t}} &= \eta_{\overline{t}\overline{t}} + \left(\frac{\partial g_{tt}}{\partial z}\right)_{\vec{x}_0} e^{z}_{\overline{z}}(\vec{x}_0) e^{t}_{\overline{t}}(\vec{x}_0) e^{t}_{\overline{t}}(\vec{x}_0)\, \overline{z} \\ &= 1 + \frac{2M}{R^2}\big(1 + \Phi(R)\big)\big(1 - 2\Phi(R)\big)^2\, \overline{z} \\ &= 1 + \frac{2M}{R^2}\overline{z} + O(M/R)^3. \end{aligned} \tag{2.7}$$

In a similar way we obtain

$$\overline{g}_{\overline{x}\overline{x}} = \overline{g}_{\overline{y}\overline{y}} = \overline{g}_{\overline{z}\overline{z}} = -1 + \frac{2M}{R^2}\overline{z} + O(M/R)^3, \tag{2.8}$$

all the other $\overline{g}_{\overline{\alpha}\overline{\beta}}$ being vanishing. The resulting metric in the observer's local frame reads

$$ds^2 = (1 + 2\gamma\overline{z})d\overline{t}^2 - (1 - 2\gamma\overline{z})\delta_{ij} d\overline{x}^i d\overline{x}^j, \tag{2.9}$$

where we have put for short

$$\gamma = \frac{M}{R^2}, \tag{2.10}$$

or, in SI units

$$\gamma = \frac{GM}{c^2 R^2}. \tag{2.11}$$

We point out that the metric (2.9) is the same we obtained in [19] by means of an expansion of the potential $\Phi(r)$ around $\vec{x}_0 = (0, 0, R)$, apart from the lack of a *constant* contribution $\Phi(R)$. It has been already proved [19, 44] that $\Phi(R)$ does not contribute to any gravitational correction to the Casimir energy. Thus, we have recovered a rather obvious result, namely *any constant gravitational potential term can be gauged away, hence giving no physical contribution.* Such result is indeed quite general and holds true in presence of any *static*, spherically symmetric gravitational field, also in the strong field limit, as shown in Appendix. An immediate consequence is that changes in the Casimir energy density (if any) are expected to be related to *tidal* effects in the background gravitational field.

3. Massless scalar field

In this section we will focus on a quantum field confined to the Casimir cavity. Recall that the cavity is at rest at a (coordinate) radial distance R from the centre of a massive, non-rotating, weak gravitational source (see Fig. 1). For sake of simplicity we will consider a massless scalar field, obeying the Klein-Gordon equation [53]

$$\frac{1}{\sqrt{-\overline{g}(\overline{x})}}\partial_{\overline{\mu}}\left[\sqrt{-\overline{g}(\overline{x})}\,\overline{g}^{\mu\nu}\partial_{\overline{\nu}}\psi(\overline{x})\right] + \xi\overline{\mathcal{R}}(\overline{x})\psi(\overline{x}) = 0, \tag{3.1}$$

where $\overline{\mathcal{R}}(\overline{x})$ is the scalar curvature. We also assume the minimal coupling condition $\xi = 0$ ‡, so that the Klein-Gordon equation in the local frame reads (in what follows we will suppress the overbars referring to the local coordinates, simply writing $x^{\overline{\alpha}} \to x^{\alpha}$, being understood that we are from now on working in the observer's local frame)

$$\frac{1}{1-2\gamma z}\partial_{\alpha}\left[(1-2\gamma z)g^{\alpha\beta}\partial_{\beta}\right]\psi(x) = 0. \tag{3.2}$$

To the present order of approximation, we can recast the above equation in a more suitable form

$$\partial_t^2\psi - (1+4\gamma z)\nabla^2\psi = 0. \tag{3.3}$$

We can write (3.3) as§

$$(\Box + \hat{V})\psi = 0 \tag{3.4}$$

where $\Box = \eta^{\alpha\beta}\partial_{\alpha}\partial_{\beta}$ is the flat d'Alembertian in the observer's local frame and

$$\hat{V} = -4\gamma z\nabla^2, \tag{3.5}$$

is a small perturbation (because of the smallness of γ) depending on the tidal effects induced by the gravitational field inside the cavity. From (3.4) the proper-time Hamiltonian $\hat{H}$ can be read out as

$$\hat{H} = \hat{H}_0 + \hat{V}, \tag{3.6}$$

where

$$\hat{H}_0 = \partial_t^2 - \nabla^2. \tag{3.7}$$

‡ Notice that non-minimal coupling ($\xi \neq 0$) is expected to give negligible contribution, since at the present order of approximation we may assume $\overline{\mathcal{R}}(\overline{x}) = 0$ as in a vacuum. Nevertheless, non-null values of ξ could be of some interest when exploring the Casimir effect in extended theories of gravity (see, e.g., [33, 34, 35, 36]).

§ Throughout the text a caret will label *operator* quantities.

4. The Effective Action

In this section we will adopt the Schwinger's formalism [48, 49, 50, 51] in order to evaluate the effective action W for the scalar field inside the Casimir cavity. Following [54], we start writing down the effective action as

$$W = \lim_{\nu\to 0} W(\nu), \tag{4.1}$$

where

$$W(\nu) = -\frac{i}{2}\int_0^\infty ds\, s^{\nu-1} \mathrm{Tr}\, e^{-is\hat{H}}, \tag{4.2}$$

and the limit $\nu \to 0$ has to be taken at the end of calculations. In (4.2) the trace

$$\mathrm{Tr}\, e^{-is\hat{H}} = \sum \int d^4x \langle x|e^{-is\hat{H}}|x\rangle, \tag{4.3}$$

has to be evaluated all over the continuous as well the discrete degrees of freedom, including those of spacetime. Let us define

$$\hat{K}_s = e^{-is\hat{H}}, \qquad \hat{K}_s^{(0)} = e^{-is\hat{H}_0}, \tag{4.4}$$

then

$$W(\nu) = -\frac{i}{2}\int_0^\infty ds\, s^{\nu-1} \mathrm{Tr}\hat{K}_s. \tag{4.5}$$

Formally

$$i\partial_s \hat{K}_s = \hat{H}\hat{K}_s, \tag{4.6}$$

whose solution is

$$\hat{K}_s = \hat{K}_s^{(0)} - i\int_0^s ds'\, \hat{K}_{s-s'}^{(0)} \hat{V}\hat{K}_{s'}. \tag{4.7}$$

Hereafter we will work up to the first order in the small parameter γ. At the present order of approximation, we may replace $\hat{K}_{s'}$ with $\hat{K}_{s'}^{(0)}$. Then (4.7) reads

$$\hat{K}_s = \hat{K}_s^{(0)} + \delta\hat{K}_s, \tag{4.8}$$

where

$$\begin{aligned}\delta\hat{K}_s &= -i\int_0^s ds'\, e^{-i(s-s')\hat{H}_0}\hat{V}e^{-is'\hat{H}_0} \\ &= -i\int_0^s ds' \left(e^{-i(s-s')\hat{H}_0}\hat{V}e^{i(s-s')\hat{H}_0}\right)e^{-is\hat{H}_0}.\end{aligned} \tag{4.9}$$

Recalling that, for any operators $\hat{A}$ and $\hat{B}$

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \cdots, \tag{4.10}$$

and noticing that

$$[\hat{H}_0, \hat{V}] = 8\gamma\partial_z\nabla^2, \tag{4.11}$$

and

$$[\hat{H}_0, [\hat{H}_0, [\cdots[\hat{H}_0, \hat{V}\underbrace{]\cdots]]]}_{n>1} = 0, \tag{4.12}$$

we can rewrite (4.8) as follows

$$\hat{K}_s = \hat{K}_s^{(0)} + \left[4i\gamma s(z + is\partial_z)\nabla^2\right]e^{-is\hat{H}_0}. \tag{4.13}$$

Consequently, from (4.5) we have the splitting

$$W(\nu) = W_0(\nu) + \delta W(\nu), \tag{4.14}$$

where

$$W_0(\nu) = -\frac{i}{2}\int_0^\infty ds\, s^{\nu-1}\mathrm{Tr}\hat{K}_s^{(0)}, \tag{4.15}$$

and

$$\delta W(\nu) = -\frac{i}{2}\int_0^\infty ds\, s^{\nu-1}\mathrm{Tr}\left[4i\gamma s(z + is\partial_z)\nabla^2 e^{-is\hat{H}_0}\right]. \tag{4.16}$$

In what follows we will use (4.15) and (4.16) to deduce both the flat spacetime Casimir energy density and the corresponding gravitational correction.

5. Casimir energy density: the flat spacetime case

This section can be regarded as a propedeutic one, being devoted to a review of the mathematical machinery [48, 54] leading to the evaluation of the Casimir energy density in the flat spacetime case ($\gamma = 0$, $\delta W = 0$). Consider the trace of $\hat{K}_s^{(0)}$ in (4.15)

$$Tr\hat{K}_s^{(0)} = \sum_n \int d^4x d^2p_\perp d\omega |\langle x|\xi\rangle|^2 e^{-is(-\omega^2+p_\perp^2+(n\pi/L)^2)}, \tag{5.1}$$

where $|\xi\rangle = |p_\perp, \omega, n\rangle$, $n \in N$, are the eigenstates of $\hat{H}_0$, satisfying

$$\begin{aligned}\langle\xi|\xi'\rangle &= \delta(\omega-\omega')\delta^{(2)}(p_\perp - p'_\perp)\delta_{n\,n'},\\ \langle x|\xi\rangle^* = \langle\xi|x\rangle &= \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{2}{L}}e^{i\omega t}e^{-ip_\perp\cdot x_\perp}\sin\left(\frac{n\pi z}{L}\right).\end{aligned} \tag{5.2}$$

In (5.1) and (5.2) $\vec{x} = (x_\perp, z)$, meanwhile $x_\perp = (x, y)$ represents the transverse coordinates inside the cavity. Similary, $\vec{p} = (p_\perp, p^z)$, $p_\perp = (p^x, p^y)$. Also, $p_t = p^t = \omega$ (remember the -2 metric signature). Notice that, from (5.2), $\langle\xi|x\rangle \equiv \psi(x)$ represents the normalized field modes, satisfying the usual Dirichlet boundary conditions inside the cavity in a flat spacetime background

$$\psi(t, x_\perp, z = 0) = \psi(t, x_\perp, z = L) = 0. \tag{5.3}$$

Integrating out the spacetime degrees of freedom x, y, z and t we get

$$Tr\hat{K}_s^{(0)} = \frac{AT}{(2\pi)^3}\sum_n \int d^2p_\perp d\omega\, e^{-is(-\omega^2+p_\perp^2+(n\pi/L)^2)}, \tag{5.4}$$

with A being the area of the plates and T the finite duration of the measurement [48]. We now integrate over the transverse momentum $p_\perp$ as well as over ω, thus obtaining

$$Tr\hat{K}_s^{(0)} = \frac{AT}{(2\pi)^3}\frac{\pi^{3/2}}{i\sqrt{-i}s^{3/2}}\sum_n e^{-is(n\pi/L)^2}. \tag{5.5}$$

We finally evaluate the zero-order contribution to the effective action substituting (5.5) in (4.15). We have

$$W_0(\nu) = -\frac{\pi^{3/2}}{2\sqrt{-i}}\frac{AT}{(2\pi)^3}\int_0^\infty ds\, s^{\nu-3/2-1}\sum_n e^{-is(n\pi/L)^2}. \tag{5.6}$$

Putting $u = is(n\pi/L)^2$ and performing a Wick rotation we get [54, 55]

$$\begin{aligned} W_0(\nu) =& -\frac{AT}{16\pi^{3/2}}(-i)^{\nu-2}\left(\frac{L}{\pi}\right)^{2\nu-3}\sum_n\left(\frac{1}{n}\right)^{2\nu-3}\int_0^\infty du^{\nu-3/2-1}e^{-u} \\ =& -\frac{AT}{16\pi^{3/2}}(-i)^{\nu-2}\left(\frac{L}{\pi}\right)^{2\nu-3}\Gamma(\nu-3/2)\zeta(2\nu-3), \end{aligned} \tag{5.7}$$

where the Euler $\Gamma-$function $\Gamma(z) = \int_0^\infty dt\, t^{z-1}e^{-t}$ and the Riemann $\zeta-$function $\zeta(z) = \sum_n 1/n^z$ have been introduced. Taking the $\nu \to 0$ limit, and recalling the well-known values

$$\Gamma(-3/2) = \frac{4\sqrt{\pi}}{3}, \qquad \Gamma(-1/2) = -2\sqrt{\pi}, \qquad \zeta(-3) = \frac{1}{120}, \tag{5.8}$$

obtained by means of analytic continuation of $\Gamma(z)$ and $\zeta(z)$ [56], we finally have

$$W_0 = \frac{AT\pi^2}{1440L^3}, \tag{5.9}$$

from which we get the energy density

$$\epsilon^{(0)}_{Cas} = -\frac{1}{AT}\frac{\partial W_0}{\partial T} = -\frac{\pi^2}{1440L^4}, \tag{5.10}$$

or, in SI units

$$\epsilon^{(0)}_{Cas} = -\frac{1}{AT}\frac{\partial W_0}{\partial T} = -\frac{\hbar c\pi^2}{1440L^4}. \tag{5.11}$$

In the limit $\gamma \to 0$ (i.e., flat spacetime), $\delta W = 0$ [see (4.16)], so that $W = W_0$. We have then recovered the well-known result (5.10) for the Casimir energy density in a flat spacetime.

6. Casimir energy density: the gravitationally induced contribution

We will now move to employ the above technique to the evaluation of the gravitational correction to the Casimir energy density, due to the gravitational field. Consider the second term in the expansion (4.14). Using (4.16) and (5.2) we get

$$\begin{aligned} \delta W(\nu) = -\frac{i}{2}\int_0^\infty ds\, s^{\nu-1}(4i\gamma s)\int d^4x\sum_n\int d^2p_\perp d\omega|\langle x|\xi\rangle|^2(z+is\partial_z) \\ \times\left(-p_\perp^2-(n\pi/L)^2\right)\sin^2\left(n\pi z/L\right)e^{-is(-\omega^2+p_\perp^2+(n\pi/L)^2)}. \end{aligned} \tag{6.1}$$

Performing the $dxdydt$ as well as the $d^2p_\perp d\omega$ integrations we have (notice that $\partial_z \to -ip_z = ip^z$)

$$\begin{aligned} \delta W(\nu) = -\frac{4\gamma AT\sqrt{\pi}}{(2\pi)^3L\sqrt{-i}}\int_0^\infty ds\, s^{\nu+1/2-1}\int_0^L dz\sum_n \sin^2(n\pi z/L)e^{-is(n\pi/L)^2} \\ \times\left[-\frac{\pi z}{s^2}+\frac{\pi z}{is}\left(\frac{n\pi}{L}\right)^2+\frac{\pi}{s}\left(\frac{n\pi}{L}\right)-\frac{\pi}{i}\left(\frac{n\pi}{L}\right)^3\right]. \end{aligned} \tag{6.2}$$

After a dz integration we obtain

$$\delta W(\nu) = -\frac{4\gamma AT\sqrt{\pi}}{(2\pi)^3 L\sqrt{-i}} \sum_n \int_0^\infty ds\, e^{-is(n\pi/L)^2} \Big[-\frac{\pi L^2}{4}\Big(s^{\nu-3/2-1} + i\Big(\frac{n\pi}{L}\Big)^2 s^{\nu-1/2-1}\Big) + \frac{\pi L}{2}\Big(\frac{n\pi}{L}\Big)\Big(s^{\nu-1/2-1} + i\Big(\frac{n\pi}{L}\Big)^2 s^{\nu+1/2-1}\Big)\Big]. \tag{6.3}$$

Putting $u = is(n\pi/L)^2$ as in the above section we get, also taking the limit $\nu \to 0$ [55, 56]

$$\delta W = \frac{4\gamma AT\sqrt{\pi}}{(2\pi)^3 L\sqrt{-i}}\Big(\frac{\pi^4}{4L}\Big)\Big[(-i)^{-3/2}\Gamma(-3/2)\zeta(-3) - (-i)^{1/2}\Gamma(-1/2)\zeta(-3)\Big]. \tag{6.4}$$

Notice that the two terms in the square brackets in (6.4) come from the first row of (6.3). The terms in the second row of (6.3) give no contribution, being proportional to $\zeta(-2) = 0$. Using again (5.8), we finally find the gravitational correction to the effective action

$$\delta W = \frac{\gamma AT\pi^2}{1440L^2}. \tag{6.5}$$

From (5.9) and (6.5) we get the full effective action W in presence of a weak gravitational field

$$W = \frac{AT\pi^2}{1440L^3} + \frac{\gamma AT\pi^2}{1440L^2} = \frac{AT\pi^2}{1440L^3}(1+\gamma L). \tag{6.6}$$

The corresponding Casimir energy density is then

$$\epsilon_{Cas} = -\frac{1}{AL}\frac{\partial}{\partial T}W = -\frac{\pi^2}{1440L^4}(1+\gamma L). \tag{6.7}$$

Being in presence of a (slightly) curved spacetime background, we have now to express L in terms of its *proper* value [19]

$$L_p = \int_0^L dz\sqrt{g^\Sigma_{zz}} = \int_0^L dz\sqrt{-g_{zz}} = L\Big(1 - \frac{1}{2}\gamma L\Big), \tag{6.8}$$

where $g^\Sigma_{\alpha\beta} = -g_{\alpha\beta} + g_{t\alpha}g_{t\beta}/g_{tt}$, derived from (2.9), is the induced metric [57] on the hypersurface Σ defining the cavity volume. Upon inversion of (6.8) we get, to the required order of approximation

$$L \simeq \Big(1 + \frac{1}{2}\gamma L_p\Big)L_p. \tag{6.9}$$

Using (6.9) in (6.7) finally yields

$$\epsilon_{Cas} = -\frac{\pi^2}{1440L_p^4}(1-\gamma L_p) = \epsilon^{(0)}_{Cas}(1-\gamma L_p), \tag{6.10}$$

or, in SI units

$$\epsilon_{Cas} = -\frac{\hbar c\pi^2}{1440L_p^4}(1-\gamma L_p). \tag{6.11}$$

Quite interestingly, (6.10) agrees with the result obtained in [19], although through a rather different approach. This reinforces the belief that *a gravitational field does indeed modify the vacuum energy inside a Casimir cavity.*

7. Concluding remarks

In this paper we have proposed a novel approach to the analysis of the influence of a weak gravitational field on the Casimir energy for a massless scalar field confined to a small cavity, placed at rest nearby a massive gravitational source. At variance from what claimed in a recent work [44], the present approach - based upon the Schwinger formalism for the effective action - allowed us to prove the existence of a non-null gravitational correction to the Casimir energy density.

Quite interestingly, the present result perfectly agrees with our early calculations [19], although the followed approaches are quite different. This reinforces the belief that *tidal* effects, due to the gravitational background, are indeed responsible of changes in the Casimir energy, as measured by an observer at rest with the cavity.

In that respect, the rather unexpected null result obtained in [44] is still to be understood. In our opinion, there are two main points in [44] deserving some comment.

The first one concerns the criticism to the approximation technique used in [19] to get the second order ($O(M/R)^2$) gravitational correction to the Casimir energy. Indeed, in [44] the approach followed in [19] is shown to give a wrong result (namely, a non-null contribution to the vacuum energy) when applied to the first-order ($O(M/R)$) correction; it is consequently suggested that *also* the second-order correction is likely to be affected by a similar drawback. As shown at the end of section 2 (as well as in [19] and [44]) and in the Appendix, assumption of (almost) constant gravitational potentials $g_{\mu\nu}$'s inside the cavity *does lead* indeed to a null gravitational contribution to the Casimir energy. However, we point out that the calculations performed in the Appendix did not assume any constraint on the *constant* part of the gravitational potentials ($\alpha(\overline{\rho})$ and $C(\overline{\rho})$). Differently stated, while a coordinate-dependent term in an expansion of the gravitational potential inside the cavity can be considered reasonably *small* (because of the cavity smallness), the *constant* part of the potential (evaluated at the observer's location, e.g., at the lower cavity plate - see section 2) could indeed be *large*. As an example, the measurement could be performed nearby the horizon of a (Schwarzschild) black hole. So, on general grounds, the criticism raised in [44] seems not appropriate, since the approach followed in [19] to evaluate the second-order corrections to the Casimir energy couldn't be employed as well to check the first-order contribution.

Our second point is about the null result obtained in [44]. It could be likely that such a different outcome has to do with the approximation used in finding the normalized field modes. Actually, the field modes found in [44] are *almost* (i.e., not *exactly*) satisfying the Klein-Gordon scalar product - see (7.7) in Appendix.

While being rather convinced of the influence of gravity upon vacuum energy, we would like to point out that our actual knowledge of the Casimir effect in presence of gravity is still plagued by several unresolved issues. As an example, to the best of our knowledge, till now the finiteness of the Casimir plates has not been adequately taken into account, although in the literature there are contributions dealing with edge and/or finite-size effects [58, 59, 60]. This issue has also to do with the weak limit form (2.1) of the employed isotropic Schwarzschild metric. In that respect, we believe that more care should be devoted to a complete 3D analysis of the Casimir effect in presence of a gravitational field.

Also, an open issue concerns the meaning of the energy measurement inside the cavity. More properly, a physical measurement could involve the detection of the tiny force between the plates, or the weighting of the Casimir apparatus, as in the experimental proposal [46].

Apart from the above difficulties, the present method seems nevertheless a powerful and promising tool. As an example, it offers the advantage of not requiring the exact form of the normalized field modes (at least as far as an expansion of the effective action W around the flat spacetime solution can be considered a physically satisfying approximation). Furthermore, it could (in principle) be employed in a non-perturbative approach, hence representing a possible starting point for a more exhaustive investigation of the role of gravity, also in the strong gravitational field regime.

Let us conclude pointing out that the various, sometime contrasting results about the present topic strongly suggest that a full, deeper understanding of the mechanism governing the interaction between gravity and quantum fields is still awaiting to be reached.

Acknowledgments

We would like to thank the Referees, whose comments and suggestions were helpful to improve the present manuscript.

Appendix

The metric of a general static, spherically symmetric spacetime can be written in isotropic coordinates as follows

$$ds^2 = g_{\mu\nu}(\rho)dx^\mu dx^\nu = \alpha^2(\rho)dt^2 - C^2(\rho)d\vec{x}\cdot d\vec{x}, \tag{7.1}$$

where $\rho = (d\vec{x}\cdot d\vec{x})^{1/2}$. Also $\sqrt{-g} = \alpha C^3$. Consider a static observer $\mathbf{u} = \alpha^{-1}\partial_\mathbf{t}$ at rest w.r.t. a small Casimir cavity, made of two parallel, perfectly reflecting plates, each of area A, at a (coordinate) distance L each other. The cavity is kept at a fixed (coordinate) radial distance $\overline{\rho}$ from the symmetry centre of the gravitational source, and oriented in space as described in section 2. Let $V_p = \int_V dxdydz\sqrt{g^\Sigma}$ be the *proper* volume of the cavity, with $g^\Sigma_{\mu\nu}$ being the induced metric on the hypersurface Σ, representing the cavity volume. Given a massless, scalar field ψ, confined to the cavity and minimally coupled to the background gravitational field, the corresponding Klein-Gordon equation reads

$$\partial_\mu\big[\sqrt{-g}g^{\mu\nu}\partial_\nu\big]\psi = 0. \tag{7.2}$$

Taking into account the smallness of the Casimir cavity, we expand the metric around the point $\rho = \overline{\rho}$, thus writing

$$g_{\mu\nu}(\rho) = g_{\mu\nu}(\overline{\rho}) + \delta g_{\mu\nu}(\rho), \tag{7.3}$$

Assume now *negligible metric gradients* inside the cavity ($\left|\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right| \sim \mathrm{Max}\{|\delta g_{\mu\nu}|\}/L \ll 1/L$). Then we may approximate (7.2) as follows

$$g^{\mu\nu}(\overline{\rho})\partial_\mu\partial_\nu\psi = \alpha^{-2}(\overline{\rho})\partial_t^2\psi - C^{-2}(\overline{\rho})\nabla^2\psi = 0. \tag{7.4}$$

Also, the proper volume of the Casimir cavity reads $\overline{V}_p = LAC^3(\overline{\rho})$. Imposing the usual Dirichlet boundary conditions at the plates, we readily obtain the field normal mode solutions

$$\psi(x) = N_n e^{-i\omega_n t}e^{ik_\perp\cdot x_\perp}\sin\big(n\pi z/L\big), \tag{7.5}$$

where $x_\perp = (x, y)$ and $k_\perp = (k_x, k_y)$ are the transverse coordinates and field momenta inside the cavity. Using the above modes in (7.4) we find the corresponding eigenfrequencies

$$\omega_n^2(k_\perp) = \frac{\alpha(\overline{\rho})^2}{C^2(\overline{\rho})}\left(k_\perp^2 + (n\pi/L)^2\right). \tag{7.6}$$

Normalizing the eigenmodes according to the Klein-Gordon scalar product

$$\langle\psi_m, \psi_n\rangle = i\int_\Sigma d\Sigma\sqrt{g^\Sigma}u^\mu\left[(\partial_\mu\psi_m)\psi_n^* - \psi_m\partial\psi_n^*\right], \tag{7.7}$$

we get

$$N_n^2 = \frac{\alpha(\overline{\rho})}{4\pi^2 L\omega_n C^3(\overline{\rho})}. \tag{7.8}$$

We now evaluate the (vacuum) Casimir energy density (w.r.t. the static observer $\mathbf{u}$)

$$\begin{aligned}\epsilon_{Cas} &= \frac{1}{\overline{V}_p}\int dxdydz\sqrt{g^\Sigma}u^\mu u^\nu\langle 0|T_{\mu\nu}|0\rangle \\ &= \frac{1}{\overline{V}_p}\int dxdydz\sqrt{g^\Sigma}\alpha^{-2}\sum_n\int d^2k_\perp T_{tt},\end{aligned} \tag{7.9}$$

where

$$T_{tt} = \partial_t\psi_n^*\partial\psi_n - \frac{1}{2}g_{tt}\left[g^{\mu\nu}\partial_\mu\psi_n^*\partial_\nu\psi_n\right]. \tag{7.10}$$

Performing the $dxdydz$ integrations we get, after some algebra

$$\epsilon_{Cas} = \frac{1}{8\pi^2 L\alpha(\overline{\rho})C^3(\overline{\rho})}\sum_n\int d^2k_\perp\omega_n^2(k_\perp). \tag{7.11}$$

Using (7.6) we rewrite the above result as

$$\epsilon_{Cas} = C^{-4}(\overline{\rho})\left[\frac{1}{8\pi^2 L}\sum_n\int d^2k_\perp\left(k_\perp^2 + \frac{n^2\pi^2}{L^2}\right)^{1/2}\right]. \tag{7.12}$$

We immediately recognize (see, e.g., [14]) that the quantity in square brackets is the *flat spacetime value* (5.10) of the Casimir energy density, hence

$$\epsilon_{Cas} = -C^{-4}(\overline{\rho})\frac{\pi^2}{1440L^4}. \tag{7.13}$$

As a last step, we express the result in term of the *proper* plate separation L_p, where

$$L_p = \int_0^L dxC(\overline{\rho}) = LC(\overline{\rho}), \tag{7.14}$$

thus obtaining

$$\epsilon_{Cas} = -\frac{\pi^2}{1440L_p^4}. \tag{7.15}$$

The above result shows that *if* the gravitational gradients inside the cavity are negligible (hence likely to be discarded), the Casimir energy density *does not suffer any gravitational distortion*, although the (almost constant) gravitational potentials are not zero. In other words, in order to observe changes in the vacuum energy *tidal* gravitational effects (related to coordinate-dependent contributions in the gravitational potentials) are required.

We point out that the presence of constant gravitational potentials does not modify the flat spacetime value of Casimir energy, *also if the involved potentials are not small*. Actually, *no smallness assumption* on the *constant* part of the gravitational potentials has been made in the above derivation.

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