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Class: Machine Learning

Preliminaries

Instructor: Matteo Leonetti

Derivatives

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What is the derivative of

$$f(x) = 2x$$

$$f(x) = 2x + x^2$$

Derivatives

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What is the derivative of

$f(x) = 2x$

$f'(x) = 2$

$f(x) = 2x + x^2$

$f'(x) = 2 + 2x$

Partial Derivatives

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With functions of more than one variable, we can compute the derivative with respect to each variable, treating the others as constants:

$$f(x,y) = 2x + 3y \qquad \frac{\partial f(x,y)}{\partial x} = 2 \qquad \frac{\partial f(x,y)}{\partial y} = 3$$

What are the partial derivatives of:

$$f(x,y,z) = 2x^2 + xy + yz^2$$

Partial Derivatives

With functions of more than one variable, we can compute the derivative with respect of each variable, treating the others as constants:

$$f(x,y)=2x+3y \qquad \frac{\partial f(x,y)}{\partial x}=2 \qquad \frac{\partial f(x,y)}{\partial y}=3$$

What are the partial derivatives of:

$$f(x,y,z)=2x^2+xy+yz^2$$

$$\frac{\partial f(x,y,z)}{\partial x}=4x+y \qquad \frac{\partial f(x,y,z)}{\partial y}=x+z^2 \qquad \frac{\partial f(x,y,z)}{\partial z}=2yz$$

Partial Derivatives

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The vector of partial derivatives is called the **gradient**:

$$f(x, y, z) = 2x^2 + xy + yz^2$$

$$\nabla f(x, y, z) = \begin{bmatrix} \dfrac{\partial f(x, y, z)}{\partial x} \\ \dfrac{\partial f(x, y, z)}{\partial y} \\ \dfrac{\partial f(x, y, z)}{\partial z} \end{bmatrix} = \begin{bmatrix} 4x + y \\ x + z^2 \\ 2yz \end{bmatrix}$$

More on calculus

Watch this video series!

https://www.youtube.com/playlist?list=PLZHQObOWTQDMsr9K-rj53DwVRMYO3t5Yr

On Partial derivatives:

https://youtu.be/AXqhWeUEtQU

Gradient:

https://youtu.be/tIpKfDc295M

Vectors

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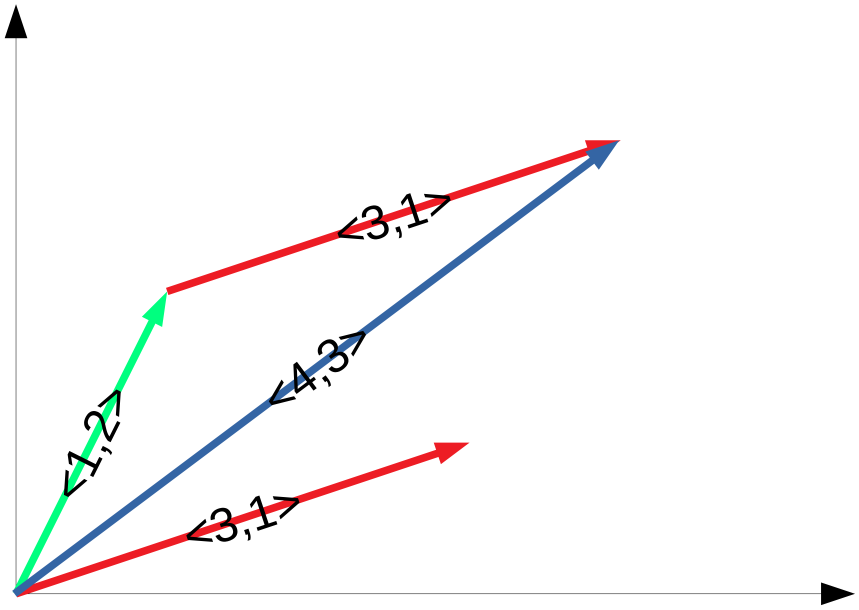
Sum of two vectors: $\langle u_1, u_2, \cdots u_n \rangle + \langle v_1, v_2, \cdots, v_n \rangle = \langle u_1 + v_1, u_2 + v_2, \cdots u_n + v_n \rangle$

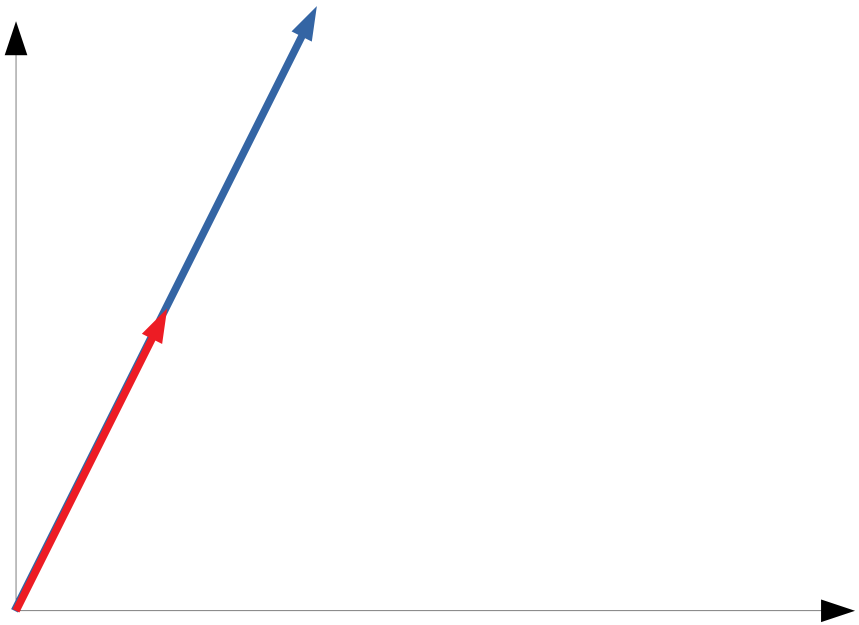
Example: $\langle 1, 2 \rangle + \langle 3, 1 \rangle = \langle 4, 3 \rangle$



Vectors

Multiplication of vector by a constant: $c \cdot \langle v_1, v_2, \cdots, v_n \rangle = \langle cv_1, cv_2, \cdots, cv_n \rangle$

Example: $2 \cdot \langle 1, 2 \rangle = \langle 2, 4 \rangle$



If the constant is different from 1, the length of the vector changes!

Vectors

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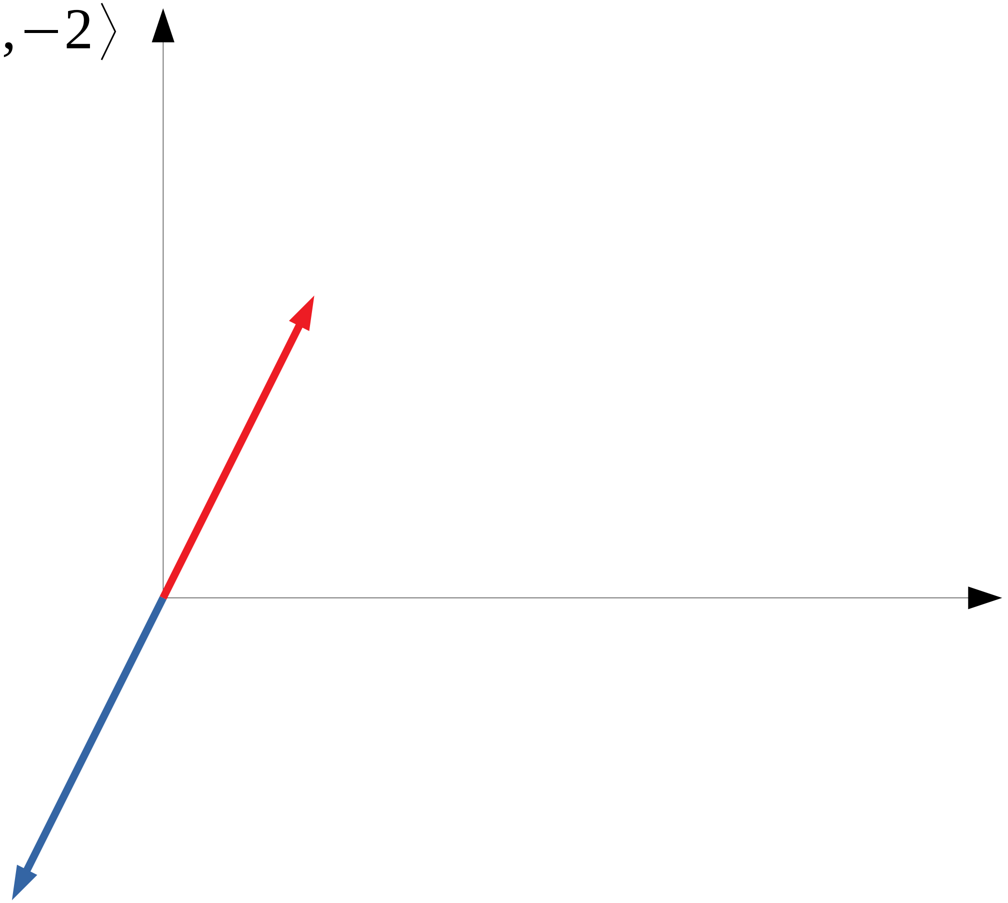
If the constant is negative, the direction changes:

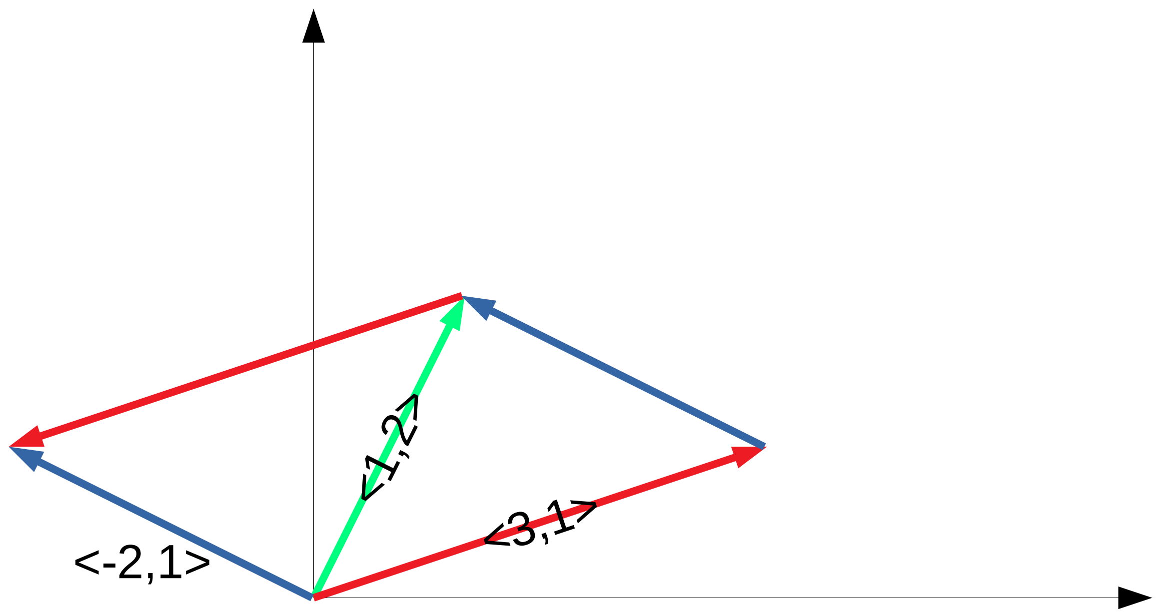
Example: $-1 \cdot \langle 1,2 \rangle = \langle -1,-2 \rangle$



Vectors

With sum and multiplication by a constant we can subtract two vectors:

Example: $\langle 1,2\rangle + -1\cdot\langle 3,1\rangle = \langle -2,1\rangle$



Dot product

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The dot product of two vectors is:

$$\langle u_1, u_2, \cdots, u_n \rangle \cdot \langle v_1, v_2, \cdots, v_n \rangle = u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n = \sum_{i=1}^{n} u_i \cdot v_i$$

Example:

$$\langle 1, 2 \rangle \cdot \langle -1, 3 \rangle = -1 + 6 = 5$$

Norm

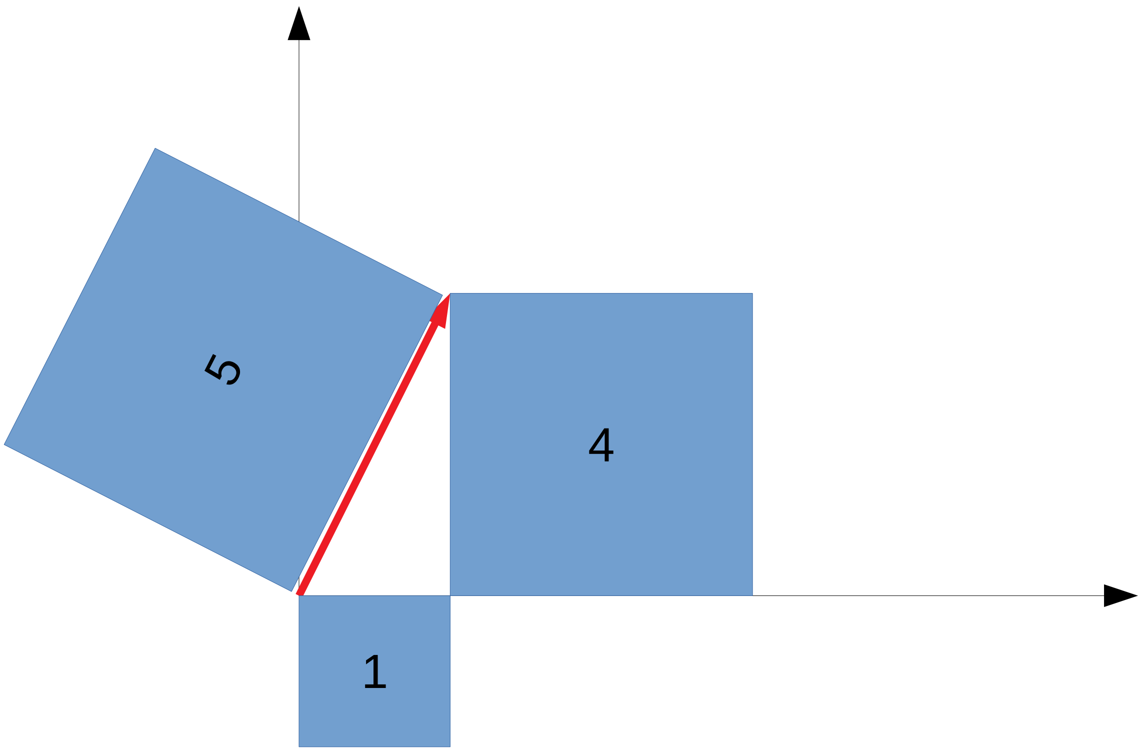
The **norm** of a vector is the square root of the dot product of the vector with itself:

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} = \sqrt{\langle v_1, v_2, \cdots, v_n\rangle \cdot \langle v_1, v_2, \cdots, v_n\rangle} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

Example:

$$\|\langle 1, 2\rangle\| = \sqrt{1+4} = \sqrt{5}$$

But wait, this is the Pythagorean theorem!



The norm of a vector is the "length" of the vector.

Unit-length vector

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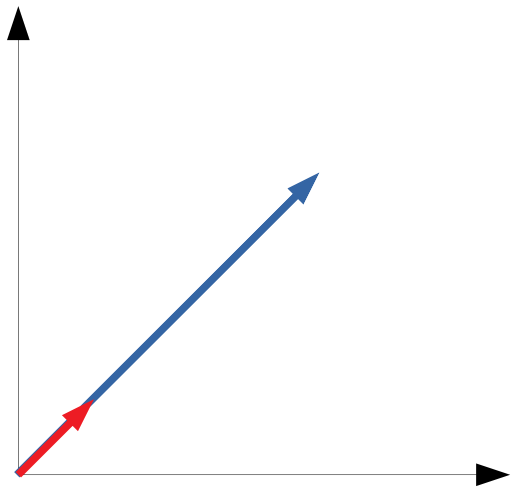
If we divide a vector by its norm, we obtain a vector of the same direction, but length 1:

$$\|\boldsymbol{v}\| = \|\langle 2,2\rangle\| = \sqrt{8}$$

$$\boldsymbol{v}_u = \langle 2,2\rangle \cdot \frac{1}{\sqrt{8}} = \langle \frac{2}{\sqrt{8}}, \frac{2}{\sqrt{8}} \rangle$$



This new vector has norm 1:

$$\|\langle \frac{2}{\sqrt{8}}, \frac{2}{\sqrt{8}} \rangle\| = \frac{4}{8} + \frac{4}{8} = 1$$

Any vector u parallel to v can be written as:

$$\boldsymbol{u} = d\,\boldsymbol{v}_u$$

Where d is the length of u

For instance: $\langle 5,5\rangle = \frac{5\sqrt{8}}{2} \langle \frac{2}{\sqrt{8}}, \frac{2}{\sqrt{8}} \rangle$

What is the norm of <5,5> ?

Orthogonal vectors

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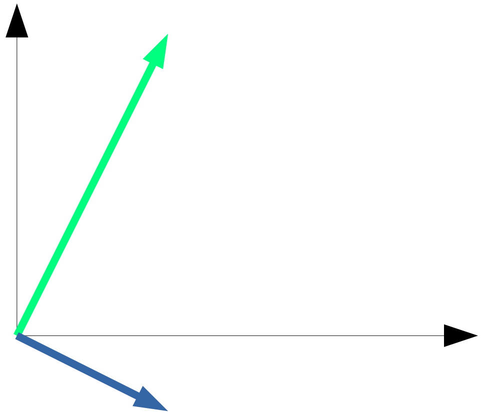
The dot product of two vectors has the following property:

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|u\| \cdot \|v\| \cdot \cos\theta$$

Where θ is the angle between the vectors

It follows that the dot product between two non-null vectors is 0 if and only if the vectors are **orthogonal**

$$\langle 1,2 \rangle \cdot \langle 1, -\frac{1}{2} \rangle = 0$$



More on vectors

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Watch these videos!

Vectors: https://youtu.be/fNk_zzaMoSs

Dot product: https://youtu.be/LyGKycYT2v0

Surfaces

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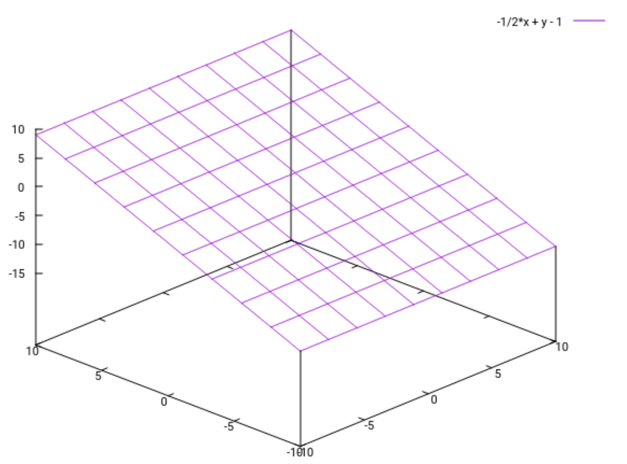
If $f(x_1, x_2, \cdots, x_n)$ is a function of n variables.

The points satisfying the equation: $f(x_{1,} x_{2,} \cdots, x_n) = 0$ lie on a surface in a space of n dimensions.

For example, consider the function: $f(x, y, z) = \frac{1}{2}x - y + z + 1$

The points satisfying the equation $\frac{1}{2}x - y + z + 1 = 0$ lie on a plane:



Surfaces

The points satisfying the inequality $f(x_1, x_2, \cdots, x_n) \geq 0$ lie on one side of the surface

Which side? Let's see on hyperplanes, and in particular in this 2D example, on a straight line:

$$f(x, y) = \frac{1}{2}x - y + 1 \geq 0$$

Let's evaluate the function on some points:

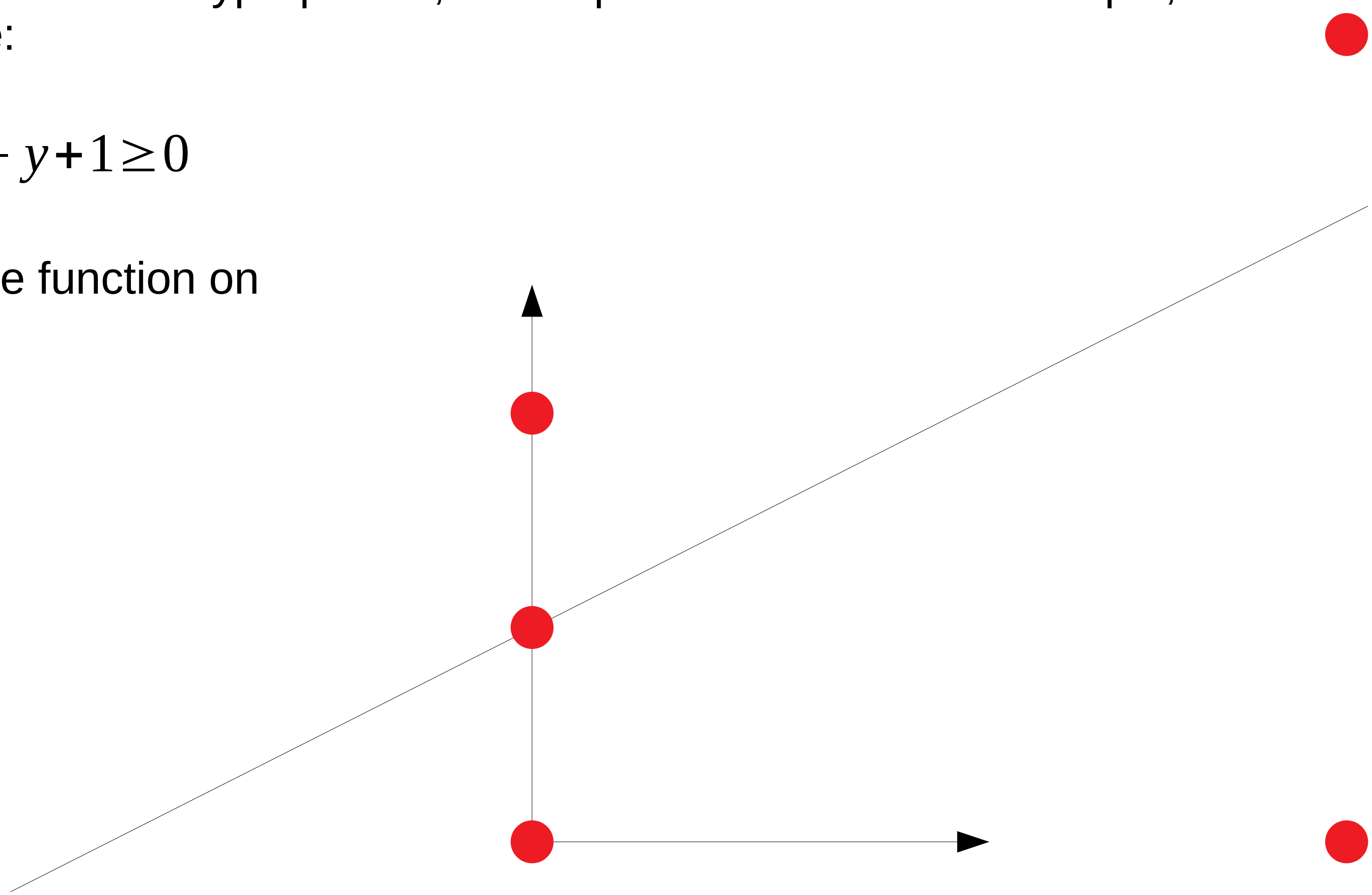
$$f(0,0) = 1$$

$$f(0,1) = 0$$

$$f(0,2) = -1$$

$$f(4,0) = 3$$

$$f(4,4) = -1$$



Surfaces

$$f(x,y)=\frac{1}{2}x-y+1\geq 0$$

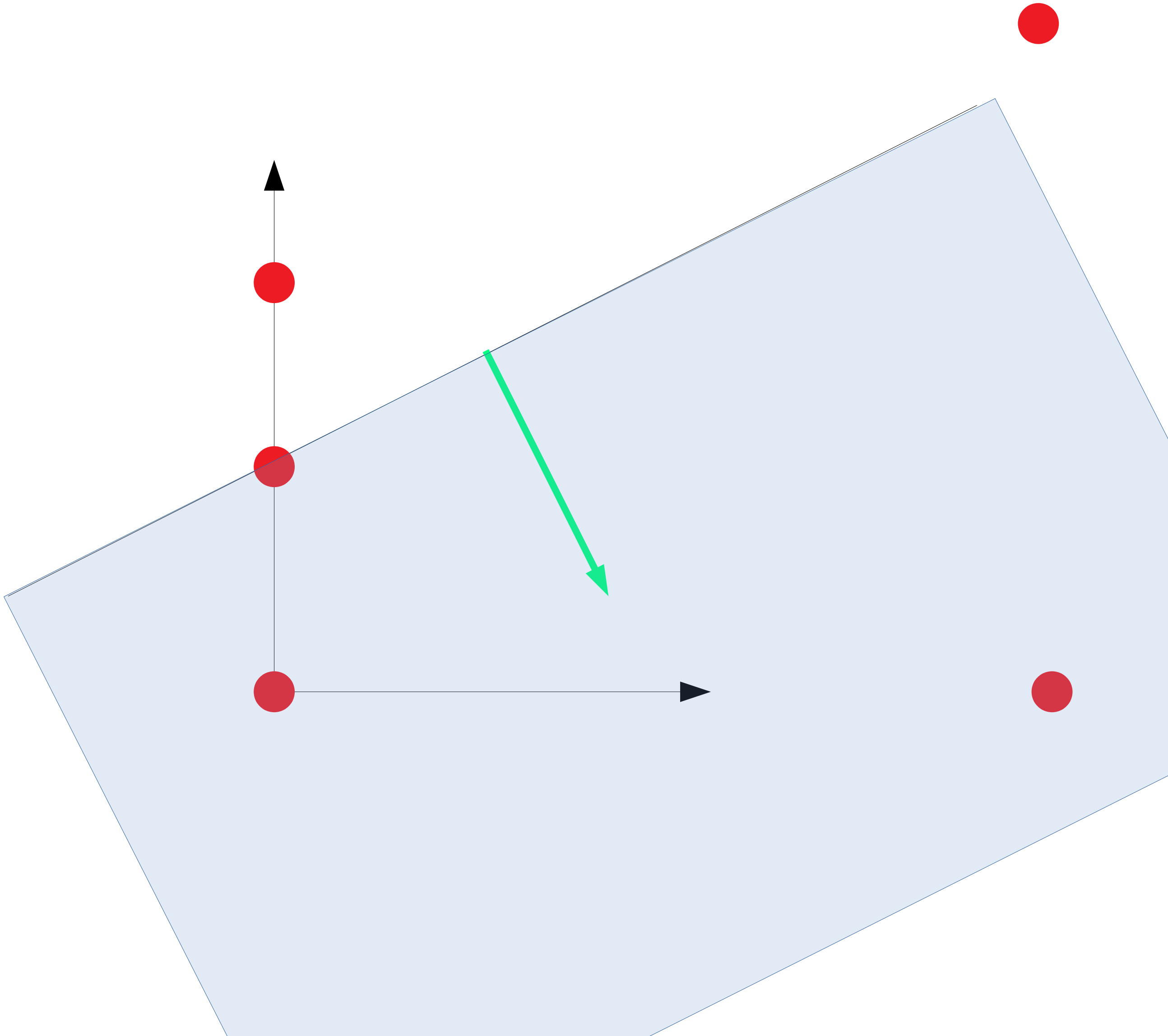
$f(0,0)=1$ $f(4,0)=3$

$f(0,1)=0$

$f(4,4)=-1$

$f(0,2)=-2$

The solution of the inequality is a half-plane, which contains all the points on the same side, with respect to the line, as the vector of parameters multiplied by the variables of the line. The same is true in more dimensions.



Question

Is the inequality $\frac{1}{2}x - y + z + 1 \geq 0$ satisfied by the points above or below the corresponding plane?