Friedrich-Alexander-Universität
Faculty of Engineering

# A Fixed-Point Iteration Scheme for Sensitivity-Based Distributed Optimal Control


**Maximilian Pierer von Esch | 19.03.2024**
**GAMM | Magdeburg**

Deutsche Forschungsgemeinschaft
Funded by the German Research Foundation (DFG), Project number 3870/6-1

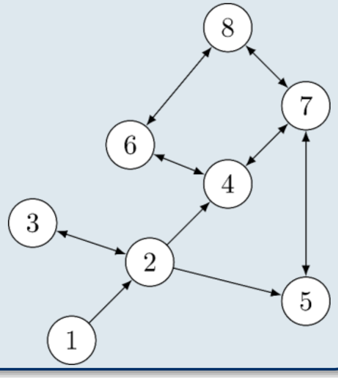
# Motivation

## Examples of distributed systems

- Swarm-based coordination problems
- Modular automation problems (e.g. building control, etc.)
- Supply networks (e.g. smart grids, water distribution systems, etc.)

## Distributed model predictive control (DMPC)

- For large-scale systems: high computational effort for MPC
- Cooperative approach
- Decomposition in subproblems
- Distributed optimization




## Advantages

- Scalability w.r.t. number of subproblems
- Modularity and flexibility
- Plug and play
- No single point of failure

## Challenges

- Design of distributed optimization algorithms
- Convergence/Stability
- Real-time capable implementation
- Communication aspects
- Experimental validation

# Agenda

## Content

- **Problem statement**
- **Sensitivities for neighbor-affine systems**
- **A sensitivity-based algorithm for distributed optimal control**
- **Solution of local optimal control problems via fixed-point iterations**
- **Numerical evaluation**
- **Summary and outlook**

# Agenda

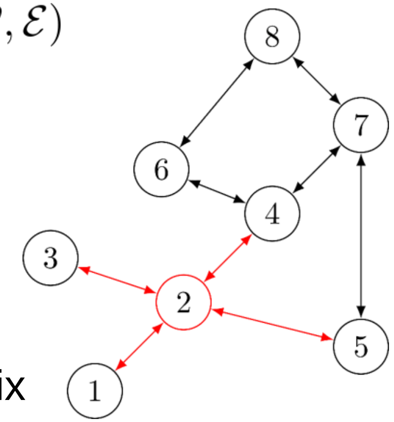
## Content

- **Problem statement**
- Sensitivities for neighbor-affine systems
- A sensitivity-based algorithm for distributed optimal control
- Solution of local optimal control problems via fixed-point iterations
- Numerical evaluation
- Summary and outlook

# System Description

## Distributed systems

- Graph-based description: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- Node represent agents: $i \in \mathcal{V}$
- Edges define couplings
- Neighbors $j \in \mathcal{V}$ are connected via edges
- Neighborhood: $\mathcal{N}_2 = \{1, 3, 4, 5\}$
- Connectedness: Adjacency matrix




- Neighbor- and input-affine agent dynamics

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}_{ii}(\boldsymbol{x}_i) + \boldsymbol{G}_i(\boldsymbol{x}_i)\boldsymbol{u}_i + \sum_{j \in \mathcal{N}_i} \boldsymbol{f}_{ij}(\boldsymbol{x}_i, \boldsymbol{x}_j)$$

$$=: \boldsymbol{f}_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i,0}$$

## Optimal control problem (OCP)

- Central OCP

$$\begin{aligned} \min_{\boldsymbol{u}} \quad & \sum_{i \in \mathcal{V}} J_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}) \\ \text{s.t.} \quad & \dot{\boldsymbol{x}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i,0}, \quad i \in \mathcal{V} \\ & \boldsymbol{u}_i \in [\boldsymbol{u}_i^-, \boldsymbol{u}_i^+], \quad i \in \mathcal{V} \end{aligned}$$

- Costs

$$J_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}) := V_i(\boldsymbol{x}_i(T)) + \int_0^T l_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i})\,\mathrm{d}t$$

- Neighbor-affine integral costs

$$l_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}) := l_{ii}(\boldsymbol{x}_i) + \frac{1}{2}\boldsymbol{u}_i^{\mathrm{T}} \boldsymbol{R}_i \boldsymbol{u}_i + \sum_{j \in \mathcal{N}_i} l_{ij}(\boldsymbol{x}_i, \boldsymbol{x}_j)$$

# Agenda

## Content

- Problem statement
- **Sensitivities for neighbor-affine systems**
- A sensitivity-based algorithm for distributed optimal control
- Solution of local optimal control problems via fixed-point iterations
- Numerical evaluation
- Summary and Outlook

# Sensitivities

$$\boldsymbol{f}_{ij}(\boldsymbol{x}_i, \boldsymbol{x}_j),\ l_{ij}(\boldsymbol{x}_i, \boldsymbol{x}_j)$$

**Agent $i$**

$$\begin{aligned} \min_{\boldsymbol{u}_i} \quad & J_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}) \\ \text{s.t.} \quad & \dot{\boldsymbol{x}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}) \\ & \boldsymbol{x}_i(0) = \boldsymbol{x}_{i,0} \\ & \boldsymbol{u}_i \in [\boldsymbol{u}_i^-, \boldsymbol{u}_i^+] \end{aligned}$$

Sensitivities

**Agent $j$**

$$\begin{aligned} \min_{\boldsymbol{u}_j} \quad & J_j(\boldsymbol{x}_j, \boldsymbol{u}_j, \boldsymbol{x}_{\mathcal{N}_j}) \\ \text{s.t.} \quad & \dot{\boldsymbol{x}}_j = \boldsymbol{f}_j(\boldsymbol{x}_j, \boldsymbol{u}_j, \boldsymbol{x}_{\mathcal{N}_j}) \\ & \boldsymbol{x}_j(0) = \boldsymbol{x}_{j,0} \\ & \boldsymbol{u}_j \in [\boldsymbol{u}_j^-, \boldsymbol{u}_j^+] \end{aligned}$$

$$\boldsymbol{f}_{ji}(\boldsymbol{x}_j, \boldsymbol{x}_i),\ l_{ji}(\boldsymbol{x}_j, \boldsymbol{x}_i)$$

What change in the global costs does a change in the states of Agent $i$ induce via the local costs of Agent $j$?

# Sensitivities

- Definition as the Gâteaux directional derivative

$$\delta J_j(\boldsymbol{x}_j, \boldsymbol{u}_j, \boldsymbol{x}_{\mathcal{N}_j})(\delta \boldsymbol{x}_i) = \frac{\mathrm{d} J_j(\boldsymbol{x}_j, \boldsymbol{u}_j, \boldsymbol{x}_{\mathcal{N}_j} + \epsilon\, \delta \bar{\boldsymbol{x}}_i)}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}$$

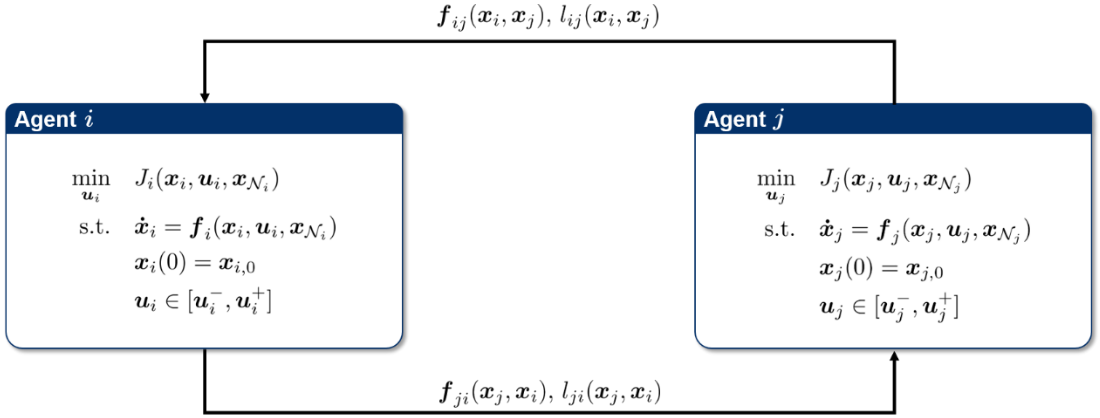
- With direction vector $\delta \bar{\boldsymbol{x}}_i = [\boldsymbol{0}^\mathrm{T} \dots \boldsymbol{0}^\mathrm{T}\ \delta \boldsymbol{x}_i^\mathrm{T}\ \boldsymbol{0}^\mathrm{T} \dots \boldsymbol{0}^\mathrm{T}]^\mathrm{T}$
- Compact and efficient computation

$$\delta J_j(\boldsymbol{x}_j, \boldsymbol{u}_j, \boldsymbol{x}_{\mathcal{N}_j})(\delta \boldsymbol{x}_i) = \frac{\mathrm{d} J_j(\boldsymbol{x}_j, \boldsymbol{u}_j, \boldsymbol{x}_{\mathcal{N}_j} + \epsilon\, \delta \bar{\boldsymbol{x}}_i)}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}$$

$$\vdots$$

$$\begin{aligned} \delta J_j(\boldsymbol{x}_j, \boldsymbol{u}_j, \boldsymbol{x}_{\mathcal{N}_j})(\delta \boldsymbol{x}_i) &= \int_0^T \left(\partial_{\boldsymbol{x}_i} l_{ji}(\boldsymbol{x}_j, \boldsymbol{x}_i) + (\partial_{\boldsymbol{x}_i} \boldsymbol{f}_{ji}(\boldsymbol{x}_j, \boldsymbol{x}_i))^\mathrm{T} \boldsymbol{\lambda}_j\right)^\mathrm{T} \delta \boldsymbol{x}_i \,\mathrm{d}t \\ &= \int_0^T \boldsymbol{g}_{ji}^\mathrm{T}\, \delta \boldsymbol{x}_i \,\mathrm{d}t \end{aligned}$$

- Adjoint state $\boldsymbol{\lambda}_j$ of neighbor necessary




Resulting local problem:

$$\begin{aligned} \min_{\boldsymbol{u}_i}\quad & \bar{J}_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}) := J_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}) \\ & \quad + \sum_{j \in \mathcal{N}_i} \delta \bar{J}_j(\boldsymbol{x}_j^{k-1}, \boldsymbol{u}_j^{k-1}, \boldsymbol{x}_{\mathcal{N}_j}^{k-1}) \boxed{(\boldsymbol{x}_i - \boldsymbol{x}_i^{k-1})} \\ \text{s.t.}\quad & \dot{\boldsymbol{x}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i,0} \\ & \boldsymbol{u}_i \in [\boldsymbol{u}_i^-, \boldsymbol{u}_i^+] \end{aligned}$$

# Agenda

## Content

- Problem statement
- Sensitivities for neighbor-affine systems
- **A sensitivity-based algorithm for distributed optimal control**
- Solution of local optimal control problems via fixed-point iterations
- Numerical evaluation
- Summary and outlook

# Algorithm

## Sensitivity-based distributed optimal control

- Initialize $(\boldsymbol{x}_i^0, \boldsymbol{\lambda}_i^0)$ and send to neighbors $j \in \mathcal{N}_i$ , set $k=1$, choose $k_{\max}$
  - Local gradient calculation for all $j \in \mathcal{N}_i$

$$\boldsymbol{g}_{ji}^{k-1} = \partial_{\boldsymbol{x}_i} l_{ji}(\boldsymbol{x}_j^{k-1}, \boldsymbol{x}_i^{k-1}) + (\partial_{\boldsymbol{x}_i} \boldsymbol{f}_{ji}(\boldsymbol{x}_j^{k-1}, \boldsymbol{x}_i^{k-1}))^{\mathrm{T}} \boldsymbol{\lambda}_j^{k-1}$$

  - Solve local OCP

$$\begin{aligned} \min_{\boldsymbol{u}_i} \quad & \bar{J}_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}) := J_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}) \\ & + \sum_{j \in \mathcal{N}_i} \delta \bar{J}_j(\boldsymbol{x}_j^{k-1}, \boldsymbol{u}_j^{k-1}, \boldsymbol{x}_{\mathcal{N}_j}^{k-1})(\boldsymbol{x}_i - \boldsymbol{x}_i^{k-1}) \\ \text{s.t.} \quad & \dot{\boldsymbol{x}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i,0} \\ & \boldsymbol{u}_i \in [\boldsymbol{u}_i^-, \boldsymbol{u}_i^+] \end{aligned}$$

  - Send trajectories $(\boldsymbol{x}_i^k, \boldsymbol{\lambda}_i^k)$ to all neighbors $j \in \mathcal{N}_i$
  - Quit, if $k = k_{\max}$ , else

$k \leftarrow k+1$

# Agenda

## Content

- Problem statement
- Sensitivities for neighbor-affine systems
- A sensitivity-based algorithm for distributed optimal control
- **Solution of local optimal control problems via fixed-point iterations**
- Numerical evaluation
- Summary and outlook

# Solution of Local OCP

## Optimality Conditions

- Local Hamiltonian

$$H_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{\lambda}_i) := l_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}) + \boldsymbol{\lambda}_i^{\mathrm{T}} \boldsymbol{f}_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1})$$
$$\boxed{+ \sum_{j \in \mathcal{N}_i} (\partial_{\boldsymbol{x}_i} l_{ji}^{k-1} + (\partial_{\boldsymbol{x}_i} \boldsymbol{f}_{ji}^{k-1})^{\mathrm{T}} \boldsymbol{\lambda}_j^{k-1})(\boldsymbol{x}_i - \boldsymbol{x}_i^{k-1})}$$

- Canonical boundary value problem

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}), \qquad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i,0}$$
$$\dot{\boldsymbol{\lambda}}_i = -\partial_{\boldsymbol{x}_i} H_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{\lambda}_i), \quad \boldsymbol{\lambda}_i(T) = \partial_{\boldsymbol{x}_i} V_i(\boldsymbol{x}_i(T))$$

- Minimization of Hamiltonian

$$\min_{\boldsymbol{u}_i \in [\boldsymbol{u}_i^-, \boldsymbol{u}_i^+]} H_i(\boldsymbol{x}_i(t), \boldsymbol{u}_i, \boldsymbol{\lambda}_i(t)), \quad t \in [0, T]$$

$$\begin{aligned} \min_{\boldsymbol{u}_i} \quad & \bar{J}_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}) := J_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}) \\ & \boxed{+ \sum_{j \in \mathcal{N}_i} \delta \bar{J}_j(\boldsymbol{x}_j^{k-1}, \boldsymbol{u}_j^{k-1}, \boldsymbol{x}_{\mathcal{N}_j}^{k-1})(\boldsymbol{x}_i - \boldsymbol{x}_i^{k-1})} \\ \text{s.t.} \quad & \dot{\boldsymbol{x}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i,0} \\ & \boldsymbol{u}_i \in [\boldsymbol{u}_i^-, \boldsymbol{u}_i^+] \end{aligned}$$

## Optimality conditions

- Analytical solution via projection

$$\boldsymbol{u}_i = \boldsymbol{h}_i(\boldsymbol{x}_i, \boldsymbol{\lambda}_i) := \begin{cases} \boldsymbol{u}_i^- & \text{if } \hat{\boldsymbol{u}}_i \leq \boldsymbol{u}_i^- \\ \boldsymbol{u}_i^+ & \text{if } \hat{\boldsymbol{u}}_i \geq \boldsymbol{u}_i^+ \\ \hat{\boldsymbol{u}}_i & \text{if } \hat{\boldsymbol{u}}_i \in (\boldsymbol{u}_i^-, \boldsymbol{u}_i^+) \end{cases} \qquad \hat{\boldsymbol{u}}_i = \boldsymbol{R}_i^{-1} \boldsymbol{G}_i(\boldsymbol{x}_i)^{\mathrm{T}} \boldsymbol{\lambda}_i$$

- Modified boundary value problem

$$\dot{\boldsymbol{x}}_i = \boldsymbol{F}_i(\boldsymbol{x}_i, \boldsymbol{\lambda}_i, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}), \qquad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i,0}$$
$$\dot{\boldsymbol{\lambda}}_i = \boldsymbol{H}_i(\boldsymbol{x}_i, \boldsymbol{\lambda}_i, \boldsymbol{x}_i^{k-1}, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}, \boldsymbol{\lambda}_{\mathcal{N}_i}^{k-1}), \quad \boldsymbol{\lambda}_i(T) = \partial_{\boldsymbol{x}_i} V_i(\boldsymbol{x}_i(T))$$

# Solution via Local Fixed-Point Iterations

## Solution of the local OCP via fixed-point iterations

- Warm-start with $\boldsymbol{\lambda}_i^{0|k} = \boldsymbol{\lambda}_i^{k-1}$, set $j=1$, choose $j_{\max}$
  - Compute $\boldsymbol{x}_i^{j|k}(t)$ via forward integration of

$$\dot{\boldsymbol{x}}_i^{j|k} = \boldsymbol{F}_i(\boldsymbol{x}_i^{j|k}, \boldsymbol{\lambda}_i^{j-1|k}, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}), \quad \boldsymbol{x}_i^{j|k}(0) = \boldsymbol{x}_{i,0}$$

  - Compute $\boldsymbol{\lambda}_i^{j|k}(t)$ via backward integration of

$$\dot{\boldsymbol{\lambda}}_i^{j|k} = \boldsymbol{H}_i(\boldsymbol{x}_i^{j|k}, \boldsymbol{\lambda}_i^{j|k}, \boldsymbol{x}_i^{k-1}, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}, \boldsymbol{\lambda}_{\mathcal{N}_i}^{k-1}), \quad \boldsymbol{\lambda}_i^{j|k}(T) = \partial_{\boldsymbol{x}_i} V_i(\boldsymbol{x}_i^{j|k}(T))$$

  - Quit, if $j = j_{\max}$, else

$j \leftarrow j+1$

- Compute $\boldsymbol{u}_i^k = \boldsymbol{h}_i(\boldsymbol{x}_i^{j_{\max}|k}, \boldsymbol{\lambda}_i^{j_{\max}|k})$ and set $\boldsymbol{x}_i^k = \boldsymbol{x}_i^{j_{\max}|k}$, $\boldsymbol{\lambda}_i^k = \boldsymbol{\lambda}_i^{j_{\max}|k}$
- Return to Step 3 of higher-level sensitivity-based algorithm

# Combined Algorithm

## Sensitivity-based distributed optimal control based on fixed-point iterations

- Initialize $(\boldsymbol{x}_i^0, \boldsymbol{\lambda}_i^0)$ an send to neighbors $j \in \mathcal{N}_i$, set $k = 1$, choose $k_{\max}$
  - Local gradient calculation for all $j \in \mathcal{N}_i$

$$\boldsymbol{g}_{ji}^{k-1} = \partial_{\boldsymbol{x}_i} l_{ji}(\boldsymbol{x}_j^{k-1}, \boldsymbol{x}_i^{k-1}) + (\partial_{\boldsymbol{x}_i} \boldsymbol{f}_{ji}(\boldsymbol{x}_j^{k-1}, \boldsymbol{x}_i^{k-1}))^{\mathrm{T}} \boldsymbol{\lambda}_j^{k-1}$$

  - Solve local OCP
    - Warm-start with $\boldsymbol{\lambda}_i^{0|k} = \boldsymbol{\lambda}_i^{k-1}$, set $j = 1$, choose $j_{\max}$
      - Compute $\boldsymbol{x}_i^{j|k}(t)$ via forward integration of

$$\dot{\boldsymbol{x}}_i^{j|k} = \boldsymbol{F}_i(\boldsymbol{x}_i^{j|k}, \boldsymbol{\lambda}_i^{j-1|k}, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}), \quad \boldsymbol{x}_i^{j|k}(0) = \boldsymbol{x}_{i,0}$$

      - Compute $\boldsymbol{\lambda}_i^{j|k}(t)$ via backward integration of

$$\dot{\boldsymbol{\lambda}}_i^{j|k} = \boldsymbol{H}_i(\boldsymbol{x}_i^{j|k}, \boldsymbol{\lambda}_i^{j|k}, \boldsymbol{x}_i^{k-1}, \boldsymbol{x}_{\mathcal{N}_i}^{k-1}, \boldsymbol{\lambda}_{\mathcal{N}_i}^{k-1}), \quad \boldsymbol{\lambda}_i^{j|k}(T) = \partial_{\boldsymbol{x}_i} V_i(\boldsymbol{x}_i^{j|k}(T))$$

      - Quit, if $j = j_{\max}$, else

$j \leftarrow j + 1$

    - Compute $\boldsymbol{u}_i^k = \boldsymbol{h}_i(\boldsymbol{x}_i^{j_{\max}|k}, \boldsymbol{\lambda}_i^{j_{\max}|k})$ and set $\boldsymbol{x}_i^k = \boldsymbol{x}_i^{j_{\max}|k}$, $\boldsymbol{\lambda}_i^k = \boldsymbol{\lambda}_i^{j_{\max}|k}$
    - Return to Step 3 of higher-level sensitivity-based algorithm
  - Send trajectories $(\boldsymbol{x}_i^k, \boldsymbol{\lambda}_i^k)$ to all neighbors $j \in \mathcal{N}_i$
  - Quit, if $k = k_{\max}$, else

$k \leftarrow k + 1$

# Convergence of local fixed-point iterations

**Theorem (Convergence local fixed-point iterations) [1]**

There exists an upper bound on the prediction horizon $\bar{T} > 0$,
such that for any $T < \bar{T}$ there exists a $p \in [0,\ 1)$, such that for all $k$

$$\|\Delta \boldsymbol{x}_i^{j|k}\|_{L_\infty} \leq p \|\Delta \boldsymbol{x}_i^{j-1|k}\|_{L_\infty}, \quad j = 2, 3, \ldots$$

$$\|\Delta \boldsymbol{\lambda}_i^{j|k}\|_{L_\infty} \leq p \|\Delta \boldsymbol{\lambda}_i^{j-1|k}\|_{L_\infty}, \quad j = 1, 2, \ldots$$

with $\Delta \boldsymbol{x}_i^{j|k} := \boldsymbol{x}_i^{j|k} - \boldsymbol{x}_i^{j-1|k}$ and $\Delta \boldsymbol{\lambda}_i^{j|k} := \boldsymbol{\lambda}_i^{j|k} - \boldsymbol{\lambda}_i^{j-1|k}$.

Comments:

- Assumption: continuous differentiability of dynamics and costs
- Maximum prediction horizon depends on system characteristics and costs
- Constructive approach: Often too conservative for design purposes

→ Nevertheless, a sufficiently small prediction horizon for convergence can always be found

[1] Maximilian Pierer von Esch, Andreas Völz and Knut Graichen, “A fixed-point scheme for sensitivity-based distributed optimal control”, IEEE Transactions on Automatic Control, 2023 (submitted)

# Convergence Sensitivity-Based Distributed Optimal Control

**Theorem (Convergence distributed optimal control)[1]**

There exists a maximum prediction horizon $0 < \hat{T} \leq \bar{T}$, such that for any $T < \hat{T}$ there exists a matrix $\boldsymbol{P} \in \mathbb{R}^{2N \times 2N}$ with $\|\boldsymbol{P}\| \in [0,\, 1)$, such that

$$\begin{bmatrix} \|\Delta \boldsymbol{x}_i^k\|_{L_\infty} \\ \|\Delta \boldsymbol{\lambda}_i^k\|_{L_\infty} \end{bmatrix}_{\mathcal{V}} \leq \boldsymbol{P} \begin{bmatrix} \|\Delta \boldsymbol{x}_i^{k-1}\|_{L_\infty} \\ \|\Delta \boldsymbol{\lambda}_i^{k-1}\|_{L_\infty} \end{bmatrix}_{\mathcal{V}}$$

with $\Delta \boldsymbol{x}_i^k := \boldsymbol{x}_i^{1|k+1} - \boldsymbol{x}_i^{j_{\max}|k}$ and $\Delta \boldsymbol{\lambda}_i^k := \boldsymbol{\lambda}_i^{1|k+1} - \boldsymbol{\lambda}_i^{j_{\max}|k}$.

Moreover, the iterates $\boldsymbol{u}_i^k(t), \boldsymbol{x}_i^k(t), \boldsymbol{\lambda}_i^k(t)$ fulfill the first-order optimality conditions of the central problem for $k \to \infty$.

Comments:

- The matrix $\boldsymbol{P}$ depends on the topology, coupling strength, prediction horizon, and the number of lower-level iterations $j_{\max}$
- Upper-bounding linear discrete-time system for errors
- Converged iterates fulfill the first-order optimality conditions of the central problem

[1] Maximilian Pierer von Esch, Andreas Völz and Knut Graichen, “A fixed-point scheme for sensitivity-based distributed optimal control”, IEEE Transactions on Automatic Control, 2023 (submitted)

# Agenda

## Content

- Problem statement
- Sensitivities for neighbor-affine systems
- A sensitivity-based algorithm for distributed optimal control
- Solution of local optimal control problems via fixed-point iterations
- **Numerical evaluation**
- Summary and outlook

# Numerical evaluation

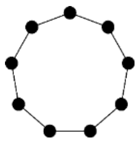
## Nonlinear example system

- Coupled Oscillators:

$$\begin{bmatrix}\dot{x}_{1,i}\\ \dot{x}_{2,i}\end{bmatrix} = \begin{bmatrix} x_{2,i} \\ -\beta x_{1,i}^3 + x_{1,i}u_i + d\sum_{j\in\mathcal{N}_i} x_{1,i}x_{1,j}\end{bmatrix}$$

→ Input- and neighbor-affine dynamics

- Input constraints: $u_i \in [-1, 1]$
- Topology



- Quadratic integral costs

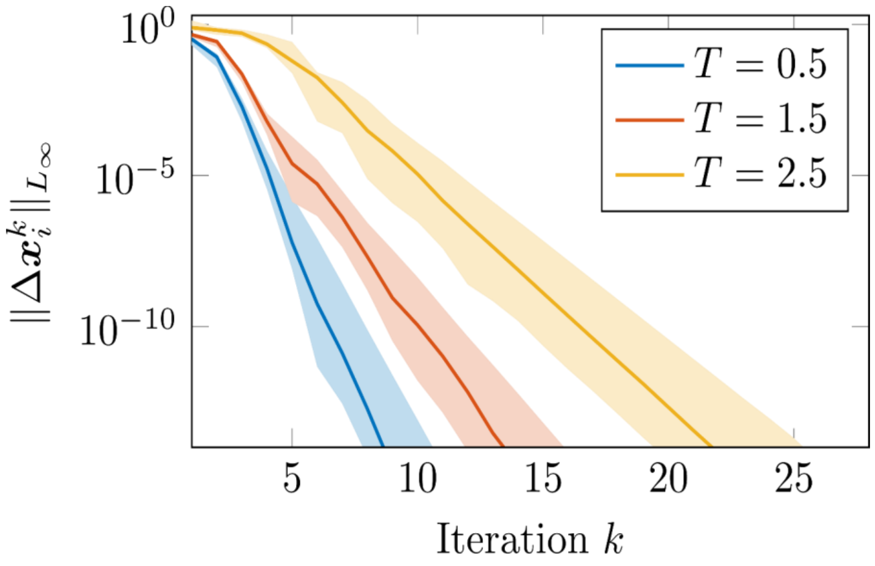
$$l_i(\boldsymbol{x}_i, u_i) = \frac{1}{2}(\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{x}_i + u_i^2)$$

## Convergence analysis

- Random initial conditions

$$\boldsymbol{x}_{i,0} \in [0.75, 0.25] \times [0.75, 0.25]$$

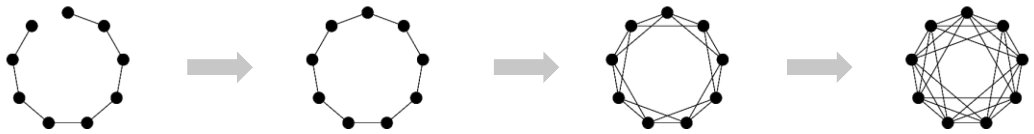
- Constant number of $j_{\max} = 2$ fixed-point iterations



# Numerical evaluation

## Influence of topology

- Increasing number of neighbors



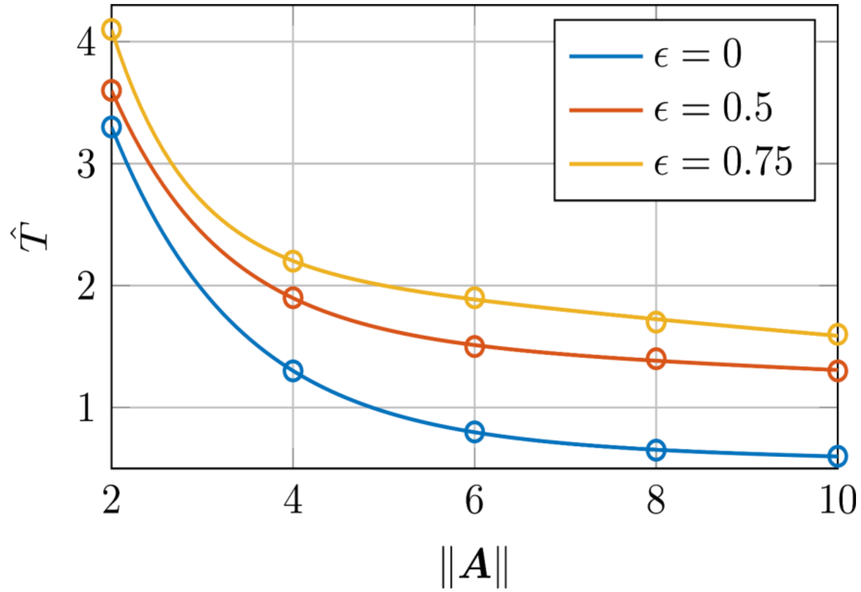
- Norm of adjacency matrix $\|\boldsymbol{A}\|$ as a measure of interconnectedness
- Damping of trajectories

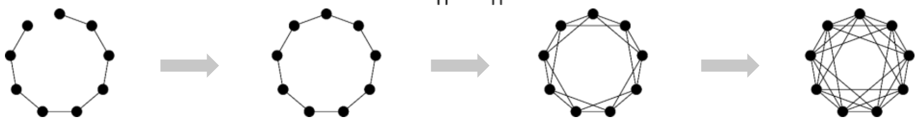
$$\boldsymbol{x}_i^{j|k}(t) \leftarrow (1-\epsilon)\boldsymbol{x}_i^{j|k}(t) + \epsilon\boldsymbol{x}_i^{k-1}(t)\,, \quad t \in [0,T]$$
$$\boldsymbol{\lambda}_i^{j|k}(t) \leftarrow (1-\epsilon)\boldsymbol{\lambda}_i^{j|k}(t) + \epsilon\boldsymbol{\lambda}_i^{k-1}(t)\,, \quad t \in [0,T]$$

- Maximum prediction horizon until divergence

## Impact of topology and damping

- A larger prediction horizon can be chosen with higher damping



# Agenda

## Content

- Problem statement
- Sensitivities for neighbor-affine systems
- A sensitivity-based algorithm for distributed optimal control
- Solution of local optimal control problems via fixed-point iterations
- Numerical evaluation
- **Summary and outlook**

# Summary and outlook

## Conclusions

- Distributed optimization algorithm for solving state-coupled OCPs
- Sensitivities are defined as the Gâteaux derivative and ensure coordination between the agents
- Local computation of sensitivities possible due to neighbor-affine structure
- One neighbor-to-neighbor communication step per iteration and only local computations $\rightarrow$ fully distributed algorithm
- Theoretical convergence guarantees

## Outlook

- Experimental validation
- Asynchronous execution
- Communication aspects
- Transmission losses
- Stability of the higher-level DMPC scheme
- Different system descriptions

# Thank you for your attention!