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Brief paper

A fixed-point iteration scheme for real-time model predictive control[☆]

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ABSTRACT

A simple model predictive control (MPC) concept for nonlinear systems under input constraints is considered. The presented algorithm takes advantage of an MPC formulation without terminal constraints in order to solve the optimality conditions by a fixed-point iteration scheme that is easy to implement and of algorithmic simplicity. Sufficient conditions for the contraction of the fixed-point iterations are derived. To allow for a real-time implementation within an MPC scheme, a constant number of fixed-point iterations is used in each sampling step and sufficient conditions for asymptotic stability and incremental reduction of the suboptimality are presented.

1. Introduction

Model predictive control (MPC) is a modern control strategy that relies on the solution of an optimal control problem (OCP) to predict the system behavior over a time horizon (De Oliveira Kothare & Morari, 2000; Diehl, Schlöder, Findeisen, Nagy, Allgöwer, 2002; Diehl, Ferreau, & Haverbeke, 2009; Grüne & Pannek, 2011; Mayne, Rawlings, Rao, & Scokaert, 2000; Rawlings & Mayne, 2009). The methodology of MPC is naturally suited to handle constraints and multiple-input systems. However, the iterative solution of the underlying OCP is in general computationally expensive.

An intuitive way to reduce the computational burden in MPC is to use a suboptimal strategy where the underlying OCP is solved approximately in each sampling step. Various suboptimal or real-time MPC approaches for continuous-time or discrete-time systems with different kinds of terminal conditions and assumptions on the underlying optimization algorithm exist in the literature; see e.g DeHaan and Guay (2007), Grüne and Pannek (2010), Graichen and Kugi (2010) Limon, Alamo, Salas, and Camacho (2006), Michalska and Mayne (1993) and Scokaert, Mayne, and Rawlings (1999).

An important aspect for a practical implementation of a real-time capable MPC scheme is to use an efficient algorithm for solving the underlying OCP. For instance, a real-time iteration scheme for discrete-time systems is proposed in Diehl, Findeisen, Allgöwer, Bock, Schlöder (2005), which uses a single-step strategy with a Newton-type iteration scheme and a terminal equality constraint to achieve asymptotic stability. A further approach (Sideris & Bobrow, 2005) that can be applied to MPC solves a sequence of linear quadratic subproblems by means of a Riccati difference equation. The MPC approach in Ohtsuka (2004) for nonlinear continuous-time systems uses a combined continuation/GMRES method that traces the solution of the discretized optimality conditions over the single MPC steps. In Cannon, Liao, and Kouvaritakis (2008), an efficient MPC scheme is considered for linear discrete-time systems based on Pontryagin's Maximum Principle.

In a similar spirit, the goal of this paper is to develop and rigorously investigate an easy-to-implement MPC algorithm with minimal algorithmic complexity. The MPC formulation considers no terminal constraints and instead requires the terminal cost to be a control Lyapunov function (CLF). Pontryagin's Maximum Principle then leads to optimality conditions with separated boundary conditions, which can be efficiently solved with a Picard-like fixed-point iteration scheme that basically consists of two numerical integrations per iteration. Sufficient conditions are derived under which the fixed-point iterations contract towards the optimal solution.

To use the algorithm for real-time MPC, a constant number of fixed-point iterations is considered in each MPC step. The last solution re-initializes the algorithm in the next MPC step. As a sufficient condition it is shown that the fixed-point iterative MPC scheme is asymptotically stable and improving in terms of optimality if the number of iterations per MPC step satisfies a lower bound.

Several norms are used in the following. The Euclidean norm of a vector $p \in \mathbb{R}^q$ is denoted by $\|p\|$. For a time (vector) function

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$p(t) \in \mathbb{R}^q$ defined on $t \in [0, T]$, the supremum norm $\|p\|_{L^\infty} := \max_{[0,T]} \|p(t)\|$ will be used.

2. Problem statement

The results in the paper are derived for nonlinear affine-input systems of the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} g_i(x)u_i =: f(x, u), \qquad x(t_0) = x_0 \tag{1}$$

with state $x \in \mathbb{R}^n$ and input $u = [u_1, \ldots, u_m]^\mathsf{T} \in \mathbb{R}^m$. It is assumed that the functions $f_0, g_i : \mathbb{R}^n \to \mathbb{R}^n$ are twice continuously differentiable in their arguments and that the origin is an equilibrium of the system for $u = 0$, i.e. $f(0, 0) = 0$. The input is subject to the (vector valued) pointwise-in-time box constraints $u(t) \in [u^-, u^+]$.

2.1. Optimal control problem

For a given sampling time Δt and the sampling time points $t_k = t_0 + k\Delta t$, $k \in \mathbb{N}_0^+$, the goal of the MPC strategy is to determine a stabilizing feedback control law for the nonlinear system (1) based on the solution of the optimal control problem (OCP)

$$\min \quad J(x_k, \bar{u}) := V(\bar{x}(T)) + \int_0^T l(\bar{x}(\tau), \bar{u}(\tau))\, \mathrm{d}\tau \tag{2}$$

$$\text{s.t.} \quad \dot{\bar{x}}(\tau) = f(\bar{x}(\tau), \bar{u}(\tau)), \qquad \bar{x}(0) = x_k \tag{3}$$

$$\bar{u}(\tau) \in [u^-, u^+], \quad \tau \in [0, T], \tag{4}$$

where $x_k = x(t_k)$ is the state of the system (1) at time $t = t_k$. The bar denotes internal variables with respect to the internal time $\tau \in [0, T]$ and the horizon length $T \geq \Delta t$. The integral cost function $l : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ is assumed to possess the particular form

$$l(x, u) := l_0(x) + \frac{1}{2} \sum_{i=1}^{m} r_i u_i^2 \tag{5}$$

with the weights $r_i > 0$. Moreover, the function $l_0 : \mathbb{R}^n \to \mathbb{R}_+$ and the terminal cost $V : \mathbb{R}^n \to \mathbb{R}_+$ are assumed to be twice continuously differentiable in their arguments and to satisfy the quadratic bounds $m_l\|x\|^2 \leq l_0(x) \leq M_l\|x\|^2$ and $m_V\|x\|^2 \leq V(x) \leq M_V\|x\|^2$ for some constants $M_l \geq m_l > 0$ and $M_V \geq m_V > 0$.

It is assumed that an optimal solution $\bar{u}_k^*(\tau) := \bar{u}^*(\tau; x_k)$, $\bar{x}_k^*(\tau) := \bar{x}^*(\tau; x_k, \bar{u}_k^*)$, $\tau \in [0, T]$ of the OCP (2)–(4) exists for all $x_k \in \mathbb{R}^n$ and that the optimal cost $J^*(x_k) := J(x_k, \bar{u}_k^*)$ is twice continuously differentiable.

2.2. Optimal MPC strategy

MPC strategies usually assume that the optimal solution of (2)–(4) is exactly known. On the sampling interval $[t_k, t_{k+1})$, the first part of the optimal control $\bar{u}_k^*(\tau)$ is used as control input for the system (1), which represents a nonlinear "sampled" control law of the form

$$u(t_k + \tau) = \bar{u}_k^*(\tau) =: \kappa(\bar{x}_k^*(\tau); x_k), \quad \tau \in [0, \Delta t) \tag{6}$$

with $\kappa(0; x_k) = 0$. In the next MPC step t_{k+1}, the OCP (2)–(4) is solved again with $\bar{x}(0) = x_{k+1}$ that – in the nominal case – is determined by $x_{k+1} = \bar{x}_k^*(\Delta t)$.

To show the stability of the MPC scheme without terminal constraints, it is often assumed that the terminal cost $V(x)$ represents a control Lyapunov function (CLF) on an invariant set Ω_β containing the origin.

Assumption 1. There exists a feedback law $u = q(x) \in [u^-, u^+]$ and a non-empty compact set $\Omega_\beta = \{x \in \mathbb{R}^n : V(x) \leq \beta\}$ containing the origin such that $\forall x \in \Omega_\beta$ the CLF inequality $\dot{V}(x, q(x)) + l(x, q(x)) \leq 0$ with $\dot{V} := \frac{\partial V}{\partial x} f$ holds.

In order to proceed, consider the following level set of the optimal cost that will be used in Assumption 2 as well as throughout the paper:

$$\Gamma_\alpha = \{x \in \mathbb{R}^n : J^*(x) \leq \alpha\}, \quad \alpha := \beta\left(1 + \frac{m_l}{M_V}T\right). \tag{7}$$

Assumption 2. For all $x_0 \in \Gamma_\alpha$ and (piecewise continuous) $u(t) \in [u^-, u^+]$, $t \in [0, T]$, the solution of (1) is bounded, i.e. $\exists$ a compact set $X \subset \mathbb{R}^n$, s.t. $x(t) \in X\ \forall t \in [0, T]$. Moreover, the optimal feedback κ in (6) is Lipschitz over X.

A consequence of Assumption 2 is that there exist constants $m_J, M_J > 0$ such that

$$m_J\|x_k\|^2 \leq J^*(x_k) \leq M_J\|x_k\|^2 \quad \forall x_k \in \Gamma_\alpha. \tag{8}$$

The following stability results can now be derived for the optimal MPC law (6) Graichen and Kugi (2010), Jadbabaie, Yu, and Hauser (2001) and Limon et al. (2006).

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Then, for all $x_0 \in \Gamma_\alpha$, the optimal cost in step $x_{k+1} = \bar{x}_k^*(\Delta t)$ decreases according to*

$$J^*(\bar{x}_k^*(\Delta t)) \leq J^*(x_k) - \int_0^{\Delta t} l(\bar{x}_k^*(\tau), \bar{u}_k^*(\tau))\, \mathrm{d}\tau \tag{9}$$

and the origin of the closed-loop system under the optimal control law (6) is exponentially stable.

3. OCP solution by fixed-point iteration

This section presents an optimization algorithm for the numerical solution of OCP (2)–(4) within a real-time MPC framework. The algorithm is based on a fixed-point iteration scheme that takes advantage of the OCP formulation without terminal constraints.

3.1. A fixed-point iteration scheme

The algorithm is based on Pontryagin's Maximum Principle (PMP) to determine a solution for OCP (2)–(4). With the Hamiltonian

$$H(x, \lambda, u) = l(x, u) + \lambda^\mathsf{T} f(x, u) \tag{10}$$

and the adjoint state $\lambda \in \mathbb{R}^n$, the canonical boundary value problem (BVP) reads

$$\dot{x} = f(x, u), \qquad x(0) = x_k \tag{11}$$

$$\dot{\lambda} = -H_x(x, \lambda, u), \qquad \lambda(T) = V_x(x(T)). \tag{12}$$

The functions H_x and V_x denote the partial derivatives of H and V with respect to x. Given the optimal control $\bar{u}_k^*(\tau)$, $\tau \in [0, T]$ for OCP (2)–(4), the PMP states that there exists an adjoint trajectory $\bar{\lambda}_k^*(\tau)$, $\tau \in [0, T]$ such that (i) $\bar{x}_k^*(\tau)$ and $\bar{\lambda}_k^*(\tau)$ satisfy the canonical BVP (11), (12) and (ii) $\bar{u}_k^*(\tau)$ minimizes the Hamiltonian

$$\min_{u \in [u^-, u^+]} H(\bar{x}_k^*(\tau), \bar{\lambda}_k^*(\tau), u) \quad \forall \tau \in [0, T]. \tag{13}$$

The first step in deriving the algorithm consists in finding the optimal control function

$$u = h(x, \lambda) = \begin{bmatrix} h_1(x, \lambda) \\ \vdots \\ h_m(x, \lambda) \end{bmatrix} \tag{14}$$

that minimizes the Hamiltonian (10). With the affine-input system (1) and the integral cost (5), the minimization of $H(x, \lambda, u)$ can be split into the subproblems

$$\min_{u_i \in [u_i^-, u_i^+]} \left\{ \frac{1}{2} r_i u_i^2 + \lambda^\mathsf{T} g_i(x) u_i \right\} \tag{15}$$

for which the minimizing control function is given by

$$u_i = h_i(x, \lambda) := \begin{cases} u_i^0 & \text{if } u_i^0 \in (u_i^-, u_i^+) \\ u_i^- & \text{if } u_i^0 \le u_i^- \\ u_i^+ & \text{if } u_i^0 \ge u_i^+ \end{cases} \tag{16}$$

with the unconstrained minimizer $u_i^0 = -\frac{1}{r_i}\lambda^\mathsf{T} g_i(x)$.[2] Inserting (14) into the canonical BVP (11), (12) gives

$$\dot{x} = F(x, \lambda), \qquad x(0) = x_k \tag{17}$$

$$\dot{\lambda} = G(x, \lambda), \qquad \lambda(T) = V_x(x(T)) \tag{18}$$

with the new functions $F(x, \lambda) := f(x, h(x, \lambda))$ and $G(x, \lambda) := -H_x(x, \lambda, h(x, \lambda))$. The BVP (17), (18) has n initial conditions for the state x and n terminal conditions for the adjoint state λ. This feature can be exploited by solving (17), (18) in a fixed-point iteration scheme, for which the following assumption is required.

Assumption 3. For any continuous trajectory $x(t) \in X$, $t \in [0, T]$, (18) has a bounded solution, i.e. $\exists$ a compact set X_λ s.t. $\lambda(t) \in X_\lambda$ $\forall t \in [0, T]$.

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(1) Initialization:

- choose $\bar{\lambda}_k^{(0)}(\tau) \in X_\lambda, \ \tau \in [0, T]$

(2) Fixed-point iteration $(1 \le j \le r)$

- compute $\bar{x}_k^{(j)}(\tau), \ \tau \in [0, T]$ by forward integration of

$$\dot{\bar{x}}_k^{(j)}(\tau) = F(\bar{x}_k^{(j)}(\tau), \bar{\lambda}_k^{(j-1)}(\tau)) \tag{19}$$

  with initial condition $\bar{x}_k^{(j)}(0) = x_k$
- compute $\bar{\lambda}_k^{(j)}(\tau), \ \tau \in [0, T]$ by backward integration of

$$\dot{\bar{\lambda}}_k^{(j)}(\tau) = G(\bar{x}_k^{(j)}(\tau), \bar{\lambda}_k^{(j)}(\tau)) \tag{20}$$

  with terminal condition $\bar{\lambda}_k^{(j)}(T) = V_x(\bar{x}_k^{(j)}(T))$
- Stop if a suitable convergence criterion is fulfilled or if $j = r$. Otherwise set $j \leftarrow j + 1$ and return to (2).

The fixed-point iteration is similar to a Picard iteration and is algorithmically simple and easy to implement, as it basically consists of two numerical integrations per iteration. In practice, the integration of (19) and (20) requires a discretization of the time interval $[0, T]$, where the number of discretization points affects both the numerical accuracy as well as the computation time (Graichen, Kiefer, & Kugi, 2009).

Lemma 1. *Suppose that Assumptions 2 and 3 hold. Then, there exists an upper bound on the horizon length $T_{\max} > 0$ such that for $T < T_{\max}$, the fixed-point iteration scheme is contracting with some ratio $p \in (0, 1)$, i.e.*

$$\|\Delta\bar{x}_k^{(j)}\|_{L^\infty} \le p\, \|\Delta\bar{x}_k^{(j-1)}\|_{L^\infty}, \quad j = 2, 3, \ldots \tag{21}$$

$$\|\Delta\bar{\lambda}_k^{(j)}\|_{L^\infty} \le p\, \|\Delta\bar{\lambda}_k^{(j-1)}\|_{L^\infty}, \quad j = 1, 2, \ldots \tag{22}$$

with $\Delta\bar{x}_k^{(j)} := \bar{x}_k^{(j)} - \bar{x}_k^*$ and $\Delta\bar{\lambda}_k^{(j)} := \bar{\lambda}_k^{(j)} - \bar{\lambda}_k^*$.

Proof. The proof relies on several Lipschitz estimates. Based on the twice continuous differentiability of the functions l_0, f_0 and g_i it can be shown that h as well as F and G are Lipschitz continuous, i.e. $\exists$ constants $L_h, L_F, L_G > 0$ s.t.

$$\|h(x, \lambda) - h(y, \mu)\| \le L_h(\|x - y\| + \|\lambda - \mu\|) \tag{23}$$

$$\|F(x, \lambda) - F(y, \mu)\| \le L_F(\|x - y\| + \|\lambda - \mu\|) \tag{24}$$

$$\|G(x, \lambda) - G(y, \mu)\| \le L_G(\|x - y\| + \|\lambda - \mu\|) \tag{25}$$

for all $x, y \in X$ and $\lambda, \mu \in X_\lambda$. A first bound on $\|\Delta\bar{x}_k^{(j)}\|_{L^\infty}$ follows from Assumption 2 and Gronwall's inequality

$$\begin{aligned} \|\Delta\bar{x}_k^{(j)}(\tau)\| &\le L_F \int_0^\tau \|\Delta\bar{x}_k^{(j)}(\tau)\| + \|\Delta\bar{\lambda}_k^{(j-1)}(\tau)\|\, \mathrm{d}s \\ &\le L_F T e^{L_F \tau} \|\Delta\bar{\lambda}_k^{(j-1)}\|_{L^\infty} \end{aligned}$$

which shows that

$$\|\Delta\bar{x}_k^{(j)}\|_{L^\infty} \le L_F T e^{L_F T} \|\Delta\bar{\lambda}_k^{(j-1)}\|_{L^\infty}. \tag{26}$$

Next, consider the integral form of (20) in reverse time

$$\hat{\lambda}_k^{(j)}(\tau) = V_x(\hat{x}_k^{(j)}(0)) - \int_0^\tau G(\hat{x}_k^{(j)}(s), \hat{\lambda}_k^{(j)}(s))\, \mathrm{d}s$$

with $\hat{x}_k^{(j)}(\tau) := \bar{x}_k^{(j)}(T - \tau)$ and $\hat{\lambda}_k^{(j)}(\tau) := \bar{\lambda}_k^{(j)}(T - \tau)$. A bound on $\Delta\hat{\lambda}_k^{(j)} := \hat{\lambda}_k^{(j)} - \hat{\lambda}_k^*$ then follows from (25) and a Lipschitz estimate for V_x with constant $L_V > 0$

$$\begin{aligned} \|\Delta\hat{\lambda}_k^{(j)}(\tau)\| &\le L_V \|\Delta\hat{x}_k^{(j)}(0)\| + L_G T \|\Delta\hat{x}_k^{(j)}\|_{L^\infty} \\ &\quad + L_G \int_0^\tau \|\Delta\hat{\lambda}_k^{(j)}(s)\|\, \mathrm{d}s \end{aligned}$$

with $\|\Delta\hat{x}_k^{(j)}(\tau)\| = \|\Delta\bar{x}_k^{(j)}(T - \tau)\|$. Using Gronwall's inequality and taking the L^∞-norm on both sides one obtains in the original variables

$$\|\Delta\bar{\lambda}_k^{(j)}\|_{L^\infty} \le (L_V + L_G T) e^{L_G T} \|\Delta\bar{x}_k^{(j)}\|_{L^\infty}. \tag{27}$$

The combination of (26) and (27) directly proves (21), (22) with $p := L_F T e^{(L_F + L_G)T}(L_V + L_G T)$. Hence, there exists a maximum horizon length $T_{\max} = T_{\max}(L_F, L_G, L_V)$ such that for all $T < T_{\max}$ the contraction ratio satisfies $p < 1$. $\square$

Remark 1. The maximum horizon length $T_{\max}$ for contraction can often be further enlarged by damping the Picard iterations. Thus, (19) and (20) become

$$\begin{aligned} \dot{\hat{x}}_k^{(j)}(\tau) &= F(\hat{x}_k^{(j)}(\tau), \bar{\lambda}_k^{(j-1)}(\tau)), \qquad \hat{x}_k^{(j)}(0) = x_k \\ \dot{\hat{\lambda}}_k^{(j)}(\tau) &= G(\bar{x}_k^{(j)}(\tau), \hat{\lambda}_k^{(j)}(\tau)), \qquad \hat{\lambda}_k^{(j)}(T) = V_x(\bar{x}_k^{(j)}(T)) \end{aligned}$$

with the intermediate trajectories $\hat{x}_k^{(j)}(\tau)$ and $\hat{\lambda}_k^{(j)}(\tau)$. The actual iterates are given by

$$\begin{aligned} \bar{x}_k^{(j)}(\tau) &= \varepsilon \hat{x}_k^{(j)}(\tau) + (1 - \varepsilon)\bar{x}_k^{(j-1)}(\tau) \\ \bar{\lambda}_k^{(j)}(\tau) &= \varepsilon \hat{\lambda}_k^{(j)}(\tau) + (1 - \varepsilon)\bar{\lambda}_k^{(j-1)}(\tau), \quad \tau \in [0, T] \end{aligned}$$

with the damping factor $\varepsilon \in (0, 1]$.

[2] The derivation of (14) is not necessarily restricted to the form of the cost function (5), but could, e.g., also be derived for a separable cost with strongly convex control part.

4. Real-time model predictive control

In order to implement the algorithm within a real-time MPC framework, a fixed number of fixed-point iterations is considered in each MPC step. This strategy naturally leads to a suboptimal OCP solution. The investigation of stability therefore requires to additionally consider the evolution of the optimization error as a measure for the suboptimality in each MPC step.

4.1. Implementation of fixed-point iteration scheme

Ideally, the algorithm is stopped if a convergence criterion is satisfied. However, for a real-time implementation, it is more appropriate to use a constant number of fixed-point iterations, r, in each MPC step. This leads to the suboptimal trajectories $\bar{x}_k^{(r)}(\tau)$, $\bar{\lambda}_k^{(r)}(\tau)$, $\tau \in [0, T]$ and the suboptimal control following from (14)

$$\bar{u}_k^{(r)}(\tau) = h(\bar{x}_k^{(r)}(\tau), \bar{\lambda}_k^{(r)}(\tau)), \quad \tau \in [0, T]. \tag{28}$$

The first part of the predicted control trajectory $\bar{u}_k^{(r)}(\tau)$ is used as control input

$$u(t_k + \tau) = \bar{u}_k^{(r)}(\tau), \quad \tau \in [0, \Delta t). \tag{29}$$

In the nominal case, the system (1) is steered to the next point $x_{k+1} = \bar{x}_k^{(r)}(\Delta t)$ and the algorithm is re-initialized with the previous adjoint state trajectory

$$\bar{\lambda}_{k+1}^{(0)}(\tau) := \bar{\lambda}_k^{(r)}(\tau), \quad \tau \in [0, T]. \tag{30}$$

Note that the MPC implementation of the fixed-point algorithm implicitly assumes that the value of $T_{\max}$ complies with the sampling time, i.e. $T_{\max} \geq \Delta t$.

4.2. Stability results

Compared to the optimal feedback law (6), the control (29) represents a suboptimal feedback. A measure for the suboptimality in each step k is the norm

$$\|\Delta\bar{x}_k^{(r)}\|_{L^\infty} = \|\bar{x}_k^{(r)} - \bar{x}_k^*\|_{L^\infty}$$

between the suboptimal and the optimal predicted state trajectories. The next lemma reveals the interdependency of the optimal cost $J^*(x_k)$ and $\|\Delta\bar{x}_k^{(r)}\|_{L^\infty}$.

Lemma 2. *Suppose that Assumption 2 holds. Then, there exist constants $0 < a \leq 1$ and $b, c > 0$ such that $\forall x_k \in \Gamma_\alpha$ the optimal cost at the next point $x_{k+1} = \bar{x}_k^{(r)}(\Delta t)$ satisfies*

$$J^*(\bar{x}_k^{(r)}(\Delta t)) \leq (1-a)J^*(x_k) + b\sqrt{J^*(x_k)}\,\|\Delta\bar{x}_k^{(r)}\|_{L^\infty} + c\|\Delta\bar{x}_k^{(r)}\|_{L^\infty}^2. \tag{31}$$

Proof. The proof re-uses the results of Theorem 1 to derive (31). To this end, the optimal cost at point $x_{k+1} = \bar{x}_k^{(r)}(\Delta t)$ is related to $x_{k+1}^* = \bar{x}_k^*(\Delta t)$ by the equivalent reformulation with $\Delta x := x_{k+1} - x_{k+1}^* = \Delta\bar{x}_k^{(r)}(\Delta t)$:

$$\begin{aligned} J^*(x_{k+1}) &= J^*(x_{k+1}^*) + \int_0^1 \nabla J^*(x_{k+1}^* + s\Delta x)\Delta x\,\mathrm{d}s \\ &= J^*(x_{k+1}^*) + \int_0^1 \Bigg[\nabla J^*(x_{k+1}^*) \\ &\quad + \int_0^s \nabla^2 J^*(x_{k+1}^* + s_2\Delta x)\Delta x\,\mathrm{d}s_2\Bigg]\Delta x\,\mathrm{d}s \\ &\leq J^*(x_{k+1}^*) + B\|x_{k+1}^*\|\,\|\Delta\bar{x}_k^{(r)}\|_{L^\infty} + \frac{1}{2}B\|\Delta\bar{x}_k^{(r)}\|_{L^\infty}^2. \end{aligned}$$

The last line follows from $\|\Delta x\| \leq \|\Delta\bar{x}_k^{(r)}\|_{L^\infty}$ and $J^*(x) \in \mathcal{C}^2$, which implies that there exists a constant $B > 0$ s.t. $\|\nabla J^*(x)\| \leq B\|x\|$ and $\|\nabla^2 J^*(x)\| \leq B$ $\forall x \in \Gamma_\alpha$. The optimal MPC case (9) together with (A.3) as well as (8) and (A.1) for $\|x_{k+1}^*\| = \|\bar{x}_k^*(\Delta t)\|$ finally prove (31) with $0 < a \leq 1$ in (A.3) and $b, c > 0$. □

The relation (31) indicates that the overall MPC performance is limited by $\|\Delta\bar{x}_k^{(r)}\|_{L^\infty}$ as a measure for the suboptimality in step k. A more rigorous investigation requires a closer look at the behavior of $\|\Delta\bar{x}_k^{(r)}\|_{L^\infty}$.

Lemma 3. *Suppose that Assumption 2 holds and that $T < T_{\max}$ (see Lemma 1). Then, there exist constants $d, e > 0$ such that $\forall x_k \in \Gamma_\alpha$*

$$\|\Delta\bar{x}_{k+1}^{(r)}\|_{L^\infty} \leq p^{r-1}(1+d)\|\Delta\bar{x}_k^{(r)}\|_{L^\infty} + p^{r-1}e\sqrt{J^*(x_k)}. \tag{32}$$

Proof. Starting with $\bar{\lambda}_{k+1}^{(0)}(\tau)$ in MPC step $k+1$ and applying r iterations leads to

$$\|\Delta\bar{x}_{k+1}^{(r)}\|_{L^\infty} \leq p^{r-1}\|\Delta\bar{x}_{k+1}^{(1)}\|_{L^\infty}. \tag{33}$$

The norm $\|\Delta\bar{x}_{k+1}^{(1)}\|_{L^\infty} = \|\bar{x}_{k+1}^{(1)} - \bar{x}_{k+1}^*\|_{L^\infty}$ can be bounded from above by expanding it into three parts

$$\|\Delta\bar{x}_{k+1}^{(1)}\|_{L^\infty} \leq \|\bar{x}_{k+1}^{(1)} - \bar{x}_k^{(r)}\|_{L^\infty} + \|\Delta\bar{x}_k^{(r)}\|_{L^\infty} + \|\bar{x}_k^* - \bar{x}_{k+1}^*\|_{L^\infty}. \tag{34}$$

The first and the third term in (34) can be replaced by the estimates (A.4)–(A.7) in the Appendix which involve $\|\Delta\bar{x}_k^{(r)}\|_{L^\infty}$ and the current point $\|x_k\|$ – or via (8) – the optimal cost $J^*(x_k)$. This shows that there exist constants $d, e > 0$ such that (34) can be written as

$$\|\Delta\bar{x}_{k+1}^{(1)}\|_{L^\infty} = (1+d)\|\Delta\bar{x}_k^{(r)}\|_{L^\infty} + e\sqrt{J^*(x_k)}. \tag{35}$$

Using (35) together with (33) finally proves (32). □

The first right-hand side term in (32) is the contraction term for the optimization error, which is slowed down by the factor $(1+d)$. Moreover, the reduction of $\|\Delta\bar{x}_k^{(r)}\|_{L^\infty}$ is counteracted by the current optimal cost $J^*(x_k)$.

The overall stability of the MPC scheme cannot be seen at first glance due to the (nonlinear) coupling of the two inequalities (31) and (32), although it is clear that a larger number of iterations r directly enhances the overall convergence ratio $p^{r-1} \in (0, 1)$. This is stated more precisely in the following theorem.

Theorem 2. *Suppose that Assumptions 1–3 hold. Let the horizon length T be such that the fixed-point iteration scheme is contracting with ratio $p \in (0, 1)$ (see Lemma 1) and let the number of iterations per MPC step satisfy*

$$r > 1 + \log_p\left(\frac{\sqrt{1-\frac{a}{2}}\gamma}{(1+d)\gamma + e}\right), \quad \gamma := \frac{-b + \sqrt{b^2 + 2ac}}{2c}. \tag{36}$$

Then, for all initial states $x_0 \in \Gamma_\alpha$ (i.e. $J^*(x_0) \leq \alpha$) and the first optimization error satisfying $\|\Delta\bar{x}_0^{(1)}\|_{L^\infty} \leq \gamma\sqrt{\alpha}p^{1-r}$, the origin of the closed-loop system resulting from the control (29) is asymptotically stable and the optimization error $\|\Delta\bar{x}_k^{(r)}\|_{L^\infty}$ decays asymptotically.

Proof. The first part of the proof consists in showing by induction that the optimal cost $J^*(x_k)$ and the optimization error $\|\Delta\bar{x}_k^{(r)}\|_{L^\infty}$ decrease according to

$$J^*(x_k) \leq \left(1 - \frac{a}{2}\right)^k \alpha \tag{37}$$

$$\|\Delta\bar{x}_k^{(r)}\|_{L^\infty} \leq \gamma\sqrt{\left(1-\frac{a}{2}\right)^k \alpha}, \quad k \in \mathbb{N}_0^+ \tag{38}$$

with $1 - a/2 < 1$. The relations are satisfied for $k = 0$ (induction start) since $J^*(x_0) \leq \alpha$ and $\|\Delta\bar{x}_0^{(r)}\|_{L^\infty} \leq \gamma\sqrt{\alpha}$ follow from the assumptions in Theorem 2 and the contraction rate (21). Postulating that (37) and (38) hold for step k (induction hypothesis), the estimate (31) with $x_{k+1} = \bar{x}_k^{(r)}(\Delta t)$ and γ from (36) leads to

$$J^*(x_{k+1}) \leq \left(1 - \tfrac{a}{2}\right)^{k+1} \alpha.$$

To obtain an estimate on the optimization error $\|\Delta\bar{x}_{k+1}^{(r)}\|_{L^\infty}$, note that the condition on the iteration number r in (36) can be written as $p^{r-1} < \sqrt{1 - \frac{a}{2}}\gamma/[(1+d)\gamma + e]$. Then, (32) together with the hypotheses (37), (38) show that

$$\|\Delta\bar{x}_{k+1}^{(r)}\|_{L^\infty} < \gamma\sqrt{\left(1 - \frac{a}{2}\right)^{k+1}\alpha},$$

which completes the proof by induction. The relations (37) and (38) imply exponential decay of $J^*(x_k)$ and asymptotic decay of $\|\Delta\bar{x}_k^{(r)}\|_{L^\infty}$ since $1 - a/2 < 1$ is strictly satisfied (remember that $0 < a \leq 1$ in Lemma 2).

To show asymptotic stability in continuous-time, consider the state trajectory of the closed-loop system $x(t) = x(t_0 + k\Delta t + \tau) = \bar{x}_k^{(r)}(\tau)$ with $\tau \in [0, \Delta t)$ and $k \in \mathbb{N}_0^+$. An upper bound on $\|\bar{x}_k^{(r)}(\tau)\|$ follows from applying the reverse triangle inequality to $\|\Delta\bar{x}_k^{(r)}(\tau)\| = \|\bar{x}_k^{(r)}(\tau) - \bar{x}_k^*(\tau)\|$, $\tau \in [0, \Delta t)$:

$$\begin{aligned}\|\bar{x}_k^{(r)}(\tau)\| &\leq \|\Delta\bar{x}_k^{(r)}(\tau)\| + \|\bar{x}_k^*(\tau)\| \\ &\leq \|\Delta\bar{x}_k^{(r)}\|_{L^\infty} + \frac{e^{\hat{L}\tau}}{\sqrt{m_J}}\sqrt{J^*(x_k)} \\ &\leq \left(\gamma + \frac{e^{\hat{L}\tau}}{\sqrt{m_J}}\right)\sqrt{\left(1 - \frac{a}{2}\right)^k \alpha} \end{aligned} \tag{39}$$

by using (A.1). Hence, there exists a time function $\phi(t)$ with $\|x(t)\| \leq \phi(t)$ and $\lim_{t\to\infty}\phi(t) = 0$ for all $x_0 \in \Gamma_\alpha$, which implies that the origin of the closed-loop system is asymptotically stable on the set Γ_α. □

The theorem shows that (under the given assumptions) a sufficient number of fixed-point iterations per MPC step can always be found in order to guarantee asymptotic stability of the closed-loop system. Moreover, the asymptotic decay of the optimization error illustrates the incremental refinement of the fixed-point iteration scheme.

The choice of the horizon length T represents a compromise between the stability of the fixed-point algorithm (see Lemma 1) and the size of the domain of attraction Γ_α as defined in (7). Nevertheless, for all $T > 0$ the domain of attraction Γ_α of the MPC is at least as big as the domain of the CLF law. To show this, consider $x_k \in \Omega_\beta$ and

$$J^*(x_k) \leq V(\bar{x}^q(T)) + \int_0^T l(\bar{x}^q(\tau), \bar{u}^q(\tau))\mathrm{d}\tau, \tag{40}$$

where $\bar{x}^q(\tau)$, $\bar{u}^q(\tau)$ with $\bar{x}^q(0) = x_k$ represent the trajectories under the CLF law $u = q(x)$. Assumption 1 implies that $V(\bar{x}^q(T)) \leq V(\bar{x}^q(0)) - \int_0^T l(\bar{x}^q(\tau), \bar{u}^q(\tau))\mathrm{d}\tau$. Hence, (40) becomes $J^*(x_k) \leq V(x_k) \leq \beta < \alpha$, which shows that $\Omega_\beta \subseteq \Gamma_\alpha$.

5. Conclusions

Suboptimal solution concepts are efficient means in order to utilize model predictive control (MPC) for fast dynamical systems. The algorithm presented in this paper takes advantage of a free endpoint MPC formulation and solves the optimality conditions for the underlying optimal control problem (OCP) with an easy-to-implement fixed-point iteration scheme that consists of two coupled forward and backward integrations. The sufficient conditions for asymptotic stability of the MPC scheme as derived in the paper are in accordance with the general observation that a better control performance can often be achieved for a larger number of iterations. Future research will be spent on the numerical and experimental evaluation of the real-time MPC scheme as well as its comparison with previous results (Graichen, Egretzberger, & Kugi, 2010; Graichen et al., 2009) and other real-time MPC schemes.

Appendix. Additional bounds

An upper bound on the optimal state trajectory $\bar{x}_k^*(\tau)$ starting at $\bar{x}_k^*(0) = x_k$ can be obtained from Assumption 2 and using Gronwall's inequality:

$$\begin{aligned}\|\bar{x}_k^*(\tau)\| &\leq \|x_k\| + \int_0^\tau \|f(\bar{x}_k^*(s)), \kappa(\bar{x}_k^*(s); x_k)\|\,\mathrm{d}s \\ &\leq \|x_k\| + \hat{L}\int_0^\tau \|\bar{x}_k^*(s)\|\,\mathrm{d}s \leq e^{\hat{L}\tau}\|x_k\| \end{aligned} \tag{A.1}$$

with $\hat{L} := L_f(1 + L_\kappa)$ and a Lipschitz constant $L_f > 0$ for the system dynamics (1). A lower bound on $\|\bar{x}_k^*(\tau)\|$ is obtained using an inverse Gronwall lemma (Gollwitzer, 1969)

$$\begin{aligned}\|\bar{x}_k^*(\tau)\| &\geq \|x_k\| - \int_0^\tau \|f(\bar{x}_k^*(s)), \kappa(\bar{x}_k^*(s); x_k)\|\,\mathrm{d}s \\ &\geq \|x_k\| - \hat{L}\int_0^\tau \|\bar{x}_k^*(s)\|\,\mathrm{d}s \geq e^{-\hat{L}\tau}\|x_k\|. \end{aligned} \tag{A.2}$$

A lower bound on the integral term in (9) follows from $m_l\|x\|^2 \leq l_0(x)$ together with (8) and (A.2)

$$\begin{aligned}&\int_0^{\Delta t} l(\bar{x}_k^*(\tau), \bar{u}_k^*(\tau))\,\mathrm{d}\tau \geq \int_0^{\Delta t} m_l\|\bar{x}_k^*(\tau)\|^2\,\mathrm{d}\tau \\ &\quad\geq aJ^*(x_k), \quad a := \frac{m_l}{2\hat{L}M_J}(1 - e^{-2\hat{L}\Delta t}) \end{aligned} \tag{A.3}$$

with $0 < a \leq 1$. Further estimates are required in the proof of Lemma 3, see (34). Using (30) as well as (19) and (24), the difference between the state trajectory $\bar{x}_k^{(r)}(\tau)$ and $\bar{x}_{k+1}^{(1)}(\tau)$ in the next MPC step is bounded by

$$\begin{aligned}&\|\bar{x}_{k+1}^{(1)}(\tau) - \bar{x}_k^{(r)}(\tau)\| =: \|\xi(\tau)\| \leq \|\bar{x}_k^{(r)}(\Delta t) - x_k\| \\ &\quad + \int_0^\tau \|F(\bar{x}_{k+1}^{(1)}(s), \bar{\lambda}_k^{(r)}(s)) - F(\bar{x}_k^{(r)}(s), \bar{\lambda}_k^{(r)}(s))\|\,\mathrm{d}s \\ &\quad\leq \|\bar{x}_k^{(r)}(\Delta t) - x_k\| + L_F\int_0^\tau \|\xi(s)\|\,\mathrm{d}s. \end{aligned}$$

The L^∞-norm $\|\bar{x}_k^{(r)} - x_k\|_{L^\infty}$ and Gronwall's inequality eventually lead to

$$\|\bar{x}_{k+1}^{(1)} - \bar{x}_k^{(r)}\|_{L^\infty} \leq e^{L_F T}\|\bar{x}_k^{(r)} - x_k\|_{L^\infty} \tag{A.4}$$

with the still unknown norm $\|\bar{x}_k^{(r)} - x_k\|_{L^\infty}$. A second estimate required in (34) concerns the distance between the optimal state trajectories $\bar{x}_k^*(\tau)$ and $\bar{x}_{k+1}^*(\tau)$ in two successive MPC steps that can be estimated as follows

$$\begin{aligned}&\|\bar{x}_{k+1}^*(\tau) - \bar{x}_k^*(\tau)\| =: \|\xi(\tau)\| \leq \|\bar{x}_k^{(r)}(\Delta t) - x_k\| \\ &\quad + \int_0^\tau \|f(\bar{x}_{k+1}^*(s), \bar{u}_{k+1}^*(s)) - f(\bar{x}_k^*(s), \bar{u}_k^*(s))\|\,\mathrm{d}s \\ &\quad\leq \|\bar{x}_k^{(r)}(\Delta t) - x_k\| + \int_0^\tau L_f\|\xi(s)\| \end{aligned}$$

$$+ L_f \|\kappa(\bar{x}^*_{k+1}(s); \bar{x}^{(r)}_k(\Delta t)) - \kappa(\bar{x}^*_k(s); x_k)\| \, \mathrm{d}s,$$
$$\le (1 + L_f L_\kappa)\|\bar{x}^{(r)}_k(\Delta t) - x_k\| + \hat{L}\int_0^\tau \|\xi(s)\| \, \mathrm{d}s,$$

where (6) and Assumption 2 were additionally used. This bound can be further simplified with the L^∞-norm $\|\bar{x}^{(r)}_k - x_k\|_{L^\infty}$ and Gronwall's inequality:

$$\|\bar{x}^*_{k+1} - \bar{x}^*_k\|_{L^\infty} \le (1 + L_f L_\kappa)e^{\hat{L}T}\|\bar{x}^{(r)}_k - x_k\|_{L^\infty}. \tag{A.5}$$

In order to obtain an expression for $\|\bar{x}^{(r)}_k - x_k\|_{L^\infty}$ in (A.4) and (A.5), consider the distance from x_k to $\bar{x}^{(r)}_k(\tau)$

$$\|\bar{x}^{(r)}_k(\tau) - x_k\| =: \|\xi(\tau)\| \le \int_0^\tau \|f(x_k + \xi(s), \bar{u}^{(r)}_k(s))\| \, \mathrm{d}s$$
$$\le L_f(\|x_k\| + \|\bar{u}^{(r)}_k\|_{L^\infty})T + L_f \int_0^\tau \|\xi(s)\| \, \mathrm{d}s.$$

Gronwall's inequality and the L^∞-norm finally yield

$$\|\bar{x}^{(r)}_k - x_k\|_{L^\infty} \le L_f(\|x_k\| + \|\bar{u}^{(r)}_k\|_{L^\infty})Te^{L_f T}. \tag{A.6}$$

The norm $\|\bar{u}^{(r)}_k\|_{L^\infty}$ can be bounded by considering

$$\|\bar{u}^{(r)}_k(\tau)\| \le \|\bar{u}^{(r)}_k(\tau) - \bar{u}^*_k(\tau)\| + \|\bar{u}^*_k(\tau)\|$$
$$\le L_h(\|\Delta\bar{x}^{(r)}_k(\tau)\| + \|\Delta\bar{\lambda}^{(r)}_k(\tau)\|) + L_\kappa\|\bar{x}^*_k(\tau)\|.$$

The second line is based on (14), (23) and (6) with Assumption 2 to derive the term $L_\kappa\|\bar{x}^*_k(\tau)\|$. Finally, using the L^∞-norm on both sides and (27) as well as (A.1) leads to

$$\|\bar{u}^{(r)}_k\|_{L^\infty} \le L_h(1 + (L_V + L_G T)e^{L_G T})\|\Delta\bar{x}^{(r)}_k\|_{L^\infty} + L_\kappa e^{\hat{L}T}\|x_k\|. \tag{A.7}$$

References

Cannon, M., Liao, W., & Kouvaritakis, B. (2008). Efficient MPC optimization using Pontryagin's minimum principle. *International Journal of Robust and Nonlinear Control*, *18*, 831–844.

DeHaan, D., & Guay, M. (2007). A real–time framework for model–predictive control of continuous–time nonlinear systems. *IEEE Transactions on Automatic Control*, *52*(11), 2047–2057.

De Oliveira Kothare, S. L., & Morari, M. (2000). Contractive model predictive control for constrained nonlinear systems. *IEEE Transactions on Automatic Control*, *45*(6), 1053–1071.

Diehl, M., Bock, H. G., Schlöder, J. P., Findeisen, R., Nagy, Z., & Allgöwer, F. (2002). Real-time optimization and nonlinear model predictive control of processes governed by differential–algebraic equations. *Journal of Process Control*, *12*, 577–585.

Diehl, M., Findeisen, R., Allgöwer, F., Bock, H. G., & Schlöder, J. P. (2005). Nominal stability of real–time iteration scheme for nonlinear model predictive control. *IEE Proceedings – Control Theory and Applications*, *152*(3), 296–308.

Diehl, M., Ferreau, H. J., & Haverbeke, N. (2009). Efficient numerical methods for nonlinear MPC and moving horizon estimation. In L. Magni, D. M. Raimondo, & F. Allgöwer (Eds.), *Nonlinear model predictive control* (pp. 391–417).

Gollwitzer, H. E. (1969). A note on a functional inequality. *Proceedings of the American Mathematical Society*, *23*(3), 642–647.

Graichen, K., Egretzberger, M., & Kugi, A. (2010). Suboptimal model predictive control of a laboratory crane. In *Proc. 8th NOLCOS* (pp. 397–402). Italy: Bologna.

Graichen, K., Kiefer, T., & Kugi, A. (2009). Real–time trajectory optimization under input constraints for a flatness–controlled laboratory helicopter. In *European control conference (ECC) 2009* (pp. 2061–2066). Hungary: Budapest.

Graichen, K., & Kugi, A. (2010). Stability and incremental improvement of suboptimal MPC without terminal constraints. *IEEE Transactions on Automatic Control*, *55*(11), 2576–2580.

Grüne, L., & Pannek, J. (2011). *Nonlinear model predictive control: theory and algorithms*. London: Springer.

Grüne, L., & Pannek, J. (2010). Analysis of unconstrained NMPC schemes with incomplete optimization. In *Proc. 8th NOLCOS* (pp. 238–243). Italy: Bologna.

Jadbabaie, A., Yu, J., & Hauser, J. (2001). Unconstrained receding horizon control of nonlinear systems. *IEEE Transactions on Automatic Control*, *46*(5), 776–783.

Limon, D., Alamo, T., Salas, F., & Camacho, E. F. (2006). On the stability of constrained MPC without terminal constraint. *IEEE Transactions on Automatic Control*, *51*(5), 832–836.

Mayne, D. Q., Rawlings, J. B., Rao, C. V., & Scokaert, P. O. M. (2000). Constrained model predictive control: stability and optimality. *Automatica*, *36*(6), 789–814.

Michalska, H., & Mayne, D. Q. (1993). Robust receding horizon control of constrained nonlinear systems. *IEEE Transactions on Automatic Control*, *38*(11), 1623–1633.

Ohtsuka, T. (2004). A continuation/GMRES method for fast computation of nonlinear receding horizon control. *Automatica*, *40*(4), 563–574.

Rawlings, J. B., & Mayne, D. Q. (2009). *Model predictive control: theory and design*. Madison, WI: Nob Hill Publishing.

Scokaert, P. O. M., Mayne, D. Q., & Rawlings, J. B. (1999). Suboptimal model predictive control (feasibility implies stability). *IEEE Transactions on Automatic Control*, *44*(3), 648–654.

Sideris, A., & Bobrow, J. E. (2005). An efficient sequential linear quadratic algorithm for solving nonlinear optimal control problems. *IEEE Transactions on Automatic Control*, *50*, 2043–2047.

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