Geometry of TVS's

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Prerequisite Knowledge

2.1 Set Theory

2.1.1 Functions

Definition 2.1.1 (**Relation**)**.** Let X and Y be nonempty sets. We say that R is a **Relation** from X to Y if $R \subset X \times Y$. If $(a, b) \in R$, then we write aRb. If $X = Y$ then we call R a **Relation** on X.

Definition 2.1.2 (**Function**)**.** Let X and Y be nonempty sets. Let $f \subset X \times Y$ such that for each $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in f$. Then we say that f is a **Function** with **Domain** X and **Codomain** Y and we write $f : X \to Y$. If $(x, y) \in f$, then we write $f(x) = y$. We may also call f a **Map** or a **Mapping**. If $A \subset X$ and $B \subset Y$, then we denote

$$f(A) = \{f(x) \in Y : x \in A\} \quad f^{-1}(B) = \{x \in X : f(x) \in B\}$$

We call $f(A)$ the **Image** of A under f and we call $f^{-1}(B)$ the **Preimage** of B under f. We call $f(X)$ the **Range** of f. When the domain of a function is understood, we may also refer to an unnamed map f by writing, $x \to f(x)$. If $\mathcal{A} \subset 2^X$ and $\mathcal{B} \subset 2^Y$, then we write

$$f(\mathcal{A}) = \{f(A) \in 2^Y : A \in \mathcal{A}\} \quad f^{-1}(\mathcal{B}) = \{f^{-1}(B) \in 2^X : B \in \mathcal{B}\}$$

Definition 2.1.3 (**Identity Map**)**.** Let X be a nonempty set. We define

$$\Delta(X) = \{(x, x) \in X \times X : x \in X\}$$

and we call $\Delta(X)$ the **Diagonal** of $X \times X$ or the **Identity Map** of X. When viewing $\Delta(X)$ as a function, we may denote $\Delta(X) = I_X$

Definition 2.1.4 (**Insertion Function**)**.** Let $A \subset B$ and define $f : A \to B$ by $f(x) = x$. We call f the **Insertion Function** of A into B.

Definition 2.1.5 (**Restriction**)**.** Let X and Y be nonempty sets. Let R be a **Relation** from X to Y. Let $A \subset X$. We define $R|_A = R \cap (A \times Y)$. We call $R|_A$ the **Restriction** of the **Relation** R to the set A.

Definition 2.1.6 (**Extension**)**.** Let X and Y be nonempty sets. Let $g : X \to Y$. Let f be a **Restriction** of g. Then we call g an **Extension** of f.

Definition 2.1.7 (**Inverse**)**.** Let X and Y be nonempty sets. Let R be a **Relation** from X to Y. We define

$$R^{-1} = \{(y, x) \in Y \times X : (x, y) \in R\}$$

We call R^{-1} the **Inverse** of R.

Definition 2.1.8 (**Injection**)**.** Let X and Y be nonempty sets. Let $f : X \to Y$. We say that f is an **Injection**, or that f is **Injective** if for all $x, y \in X$, if $x \neq y$, then $f(x) \neq f(y)$.

Definition 2.1.9 (**Surjection**)**.** Let X and Y be nonempty sets. Let $f : X \to Y$. Suppose that for each $y \in Y$, there exists an $x \in X$ such that $f(x) = y$. Then we say that f is a **Surjection** onto Y, and we call f **Surjective** onto Y.

Definition 2.1.10 (**Bijection**)**.** Let X and Y be nonempty sets and let $f : X \to Y$ be **Surjective** and **Injective**. Then we say that f is **Bijective** and we say that f is a **Bijection**.

Definition 2.1.11 (**Composition**)**.** Let X, Y, and Z be nonempty sets. Let R be a **Relation** from X into Y. Let S be a **Relation** from Y into Z. We define

$$S \circ R = \{(x, z) \in X \times Z : (\exists y \in Y)(xRy \ \ and \ \ ySz)\}$$

We call $S \circ R$ the **Composition** of S with R.

2.1.2 Cardinality

Definition 2.1.12 (Cardinality)**.** We define $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$. We define $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Let $n \in \mathbb{Z}^+$. We define $N_n = \mathbb{Z}^+ \cap [1, n]$. Let X be a nonempty set. Let $f : X \to N_n$ be a **Bijection**. Then, we say that X has **Cardinality** n and we write **Card** $(X) = n$. We say that the **Cardinality** of the empty set is 0 and we write **Card** $(\emptyset) = 0$. More generally, if there exists a **Bijection** between two sets Y and Z, then we write **Card** $(Y) =$ **Card** (Z) and we say that they have the same **Cardinalities**. Define $X_0 = \mathbb{N}$ and for $k \in \mathbb{N}$, define $X_{k+1} = 2^{X_k}$. Then for $k \in \mathbb{N}$, we define $\aleph_k =$ **Card** (X_k). If **Card** $(X) \in \mathbb{N}$, then we say that X is **Finite**. If **Card** $(Z) \in \mathbb{N}$ or **Card** $(Z) = \aleph_0$, then we say that Z is **Denumerable**. If **Card** $(Y) = \aleph_0$, then we say that Y is **Countable**. If **Card** $(W) = \aleph_k$ for $k \geq 1$, then we say that W is **Uncountable**. If **Card** $(V) = \aleph_j$ for $j \in \mathbb{N}$, then we say that V is **Infinite**.

Proposition 2.1.13 (Binary to Finite)**.** Let X be a nonempty set. The following are true.

(i) If X is closed under binary intersections, then X is closed under finite intersections.

(ii) If X is closed under binary unions, then X is closed under finite unions.

Proof of 2.1.13. i. We use induction. Let M be the set of positive integers n for which X is closed under intersections of n sets. The intersection of a single set equals that set, so $1 \in M$.

$2 \in M$ by direct application of the assumption of 2.1.13. i. Let $m \in M$. Let $\{x_i\}_{i=1}^{m+1} \subset 2^X$. Then

$$\bigcap_{i=1}^{m+1} x_i = \left(\bigcap_{i=1}^{m} x_i\right) \cap x_{m+1} \in X$$

so $m + 1 \in M$. Hence $M = \mathbb{Z}^+$ and 2.1.13. i is proven. □

Proof of 2.1.13. ii. We use induction. Let M be the set of positive integers n for which X is closed under unions of n sets. The union of a single set equals that set, so $1 \in M$. $2 \in M$ by direct application of the assumption of 2.1.13. ii. Let $m \in M$. Let $\{x_i\}_{i=1}^{m+1} \subset 2^X$. Then

$$\bigcup_{i=1}^{m+1} x_i = \left(\bigcup_{i=1}^{m} x_i\right) \cup x_{m+1} \in X$$

so $m + 1 \in M$. Hence $M = \mathbb{Z}^+$ and 2.1.13. ii is proven. □

Definition 2.1.14 (**Nested**)**.** Let F and G be nonempty sets. We say that **Nested**(F, G) holds if, for each $g \in G$, there exists $f \in F$ such that $f \subset g$.

2.1.3 Covers, Partitions

Definition 2.1.15 (**Disjoint**)**.** Let X and Y be sets such that $X \cap Y = \emptyset$. Then we say that X and Y are **Disjoint**. Let $F = \{X_\alpha\}_{\alpha \in A}$ be a collection of sets such that for each $\alpha, \beta \in A$ with $\alpha \neq \beta$, X_α and X_β are **Disjoint**. Then we say that F is **Disjoint**.

Definition 2.1.16 (**Cover**, **Subcover**)**.** Let X be a set and let $Y = \{Y_\alpha\}_{\alpha \in A}$ be a collection of sets such that

$$X \subset \bigcup_{\alpha \in A} Y_\alpha$$

Then we say Y is a **Cover** of X, we say Y **Covers** X, and we say X is **Covered By** Y. In the context of talking about a **Cover**, if every member of a **Cover** posses a certain property then we may say that the **Cover** has that property. An exception to this is that when talking about the **Cardinality** or **Disjointedness** of a **Cover**. In such cases we are saying that the **Cover** itself is **Disjoint** or of a particular **Cardinality**, not that the constituent sets each have that **Cardinality** or are themselves **Disjoint** collections. If $Z \subset Y$ **Covers** X, then we call Z a **Subcover** of Y.

Definition 2.1.17 (Partition)**.** Let X be a nonempty set. Let $Y \subset 2^X$ be a **Disjoint Cover** of X. Then we call Y a **Partition** of X.

2.1.4 Infinite Cartesian Product

Definition 2.1.18 (**Cartesian Product**)**.** Let A be a nonempty set. For each $\alpha \in A$, let X_α be a nonempty set. Define

$$\prod_{\alpha \in A} X_\alpha = \left\{ f : A \to \bigcup_{\alpha \in A} X_\alpha \in 2^{A \times \bigcup_{\alpha \in A} X_\alpha} : (\forall \alpha \in A)(f(\alpha) \in X_\alpha) \right\}$$

We call this the **Cartesian Product** of $\{X_\alpha\}_{\alpha\in A}$. For each $\alpha \in A$, we define

$$\pi_\alpha : \prod_{\gamma\in A} X_\gamma \to X_\alpha \qquad\qquad \pi_\alpha(f) = f(\alpha)$$

We call π_α the $\alpha-$**Projection Map**.

Definition 2.1.19 (**Diagonal**)**.** Let X and A be nonempty sets. For each $x \in X$, define $f_x : A \to X$ by $f_x(\alpha) = x$ for each $\alpha \in A$. We define

$$\Delta_A(X) = \left\{ f_x \in \prod_{\alpha\in A} X : x \in X \right\}$$

We call this the **Diagonal** of X with respect to A, or, when A is understood, the **Diagonal** of X.

Definition 2.1.20 (**Function Product**)**.** Let A be a nonempty set. For each $\alpha \in A$, let $f_\alpha : X_\alpha \to Y_\alpha$. Define

$$f : \prod_{\alpha\in A} X_\alpha \to \prod_{\alpha\in A} Y_\alpha$$

by

$$f\left(\{x_\alpha\}_{\alpha\in A}\right) = \{f_\alpha(x_\alpha)\}_{\alpha\in A}$$

Then we call f the **Function Product** of $\{f_\alpha\}_{\alpha\in A}$ and we denote $\prod\limits_{\alpha\in A} f_\alpha := f$. In the case where $\{X_\alpha\}_{\alpha\in A} = \{X_1, \cdots, X_n\}$, we may denote $f = f_1 \times f_2 \times \cdot \times f_n$.

2.1.5 Relations and Orderings

Definition 2.1.21 (**Reflexive**)**.** Let X be a nonempty set. Let R be a **Relation** on X. We say that R is **Reflexive** with respect to X and we say that R posseses **Reflexivity** with respect to X if $I_X \subset R$. When X is understood, we may simply say that R is **Reflexive** or that R posesses **Reflexivity**.

Definition 2.1.22 (**Transitive**)**.** Let X be a nonempty set. Let R be a **Relation** on X. We say that R is **Transitive** and we say that R posseses **Transitivity** if $R \circ R \subset R$.

Definition 2.1.23 (**Preorder**)**.** Let X be a nonempty set. Let R be a **Transitive**, **Reflexive Relation** on X. Then we call R a **Preorder** on X, we call R a **Preordering** of X, and we call (X, R) a **Preordered Set**.

Definition 2.1.24 (**Comparable**)**.** Let (X, R) be a **Preordered Set**. We say that $x, y \in X$ are **Comparable** and that they possess **Comparability** if xRy or yRx.

Definition 2.1.25 (**Symmetric**)**.** Let X be a nonempty set. Let R be a **Relation** on X. We say that R is **Symmetric** and we say that R posseses **Symmetry** if $R = R^{-1}$.

Proposition 2.1.26. Let X be a nonempty set. Let R be a **Relation** on X. The following are true.

(i) $R \cap R^{-1}$ is **Symmetric**.

(ii) $R \cup R^{-1}$ is **Symmetric**.

Proof of 2.1.26 i. If $R \cap R^{-1} = \emptyset$ then it is trivally **Symmetric**. Suppose $R \cap R^{-1} \neq \emptyset$ and let $(x, y) \in R \cap R^{-1}$. Then $(x, y) \in R$, implying by 2.1.25 that $(y, x) \in R^{-1}$. Also this implies $(x, y) \in R^{-1}$ so by 2.1.25 we have $(y, x) \in (R^{-1})^{-1} = R$. Hence $(y, x) \in R \cap R^{-1}$, so **Symmetry** is verified. □

Proof of 2.1.26 ii. Let $(x, y) \in R \cup R^{-1}$. Then either $(x, y) \in R$ or $(x, y) \in R^{-1}$. In the former case, $(y, x) \in R^{-1} \subset R \cup R^{-1}$. In the latter case, $(y, x) \in R \subset R \cup R^{-1}$. Hence in either case $(y, x) \in R \cup R^{-1}$ and **Symmetry** is verified. □

Definition 2.1.27 (**Anti-Symmetric**)**.** Let X be a nonempty set. Let R be a **Relation** on X. We say that R is **Anti-Symmetric** and we say that R posseses **Anti-Symmetry** if $R \cap R^{-1} \subset I_X$.

Definition 2.1.28 (**Maximal Element**)**.** Let X be a nonempty set. Let R be an **Relation** on X. Let $Y \subset X$. Let $a \in Y$. We say that a is a **Maximal Element** of Y and we say that a is a **Maximum** of Y if for every $b \in Y$, if aRb, then $a = b$. The Plural of **Maximum** is a **Maxima**, and we represent the set of **Maxima** of Y with respect to the relation R with $\textbf{Maxima}_R(Y)$. If R is understood, we represent the set of **Maxima** of Y with $\textbf{Maxima}(Y)$.

Definition 2.1.29 (**Minimal Element**)**.** Let X be a nonempty set. Let R be an **Relation** on X. Let $Y \subset X$. Let $a \in Y$. We say that a is a **Minimal Element** of Y and we say that a is a **Minimum** of Y if for every $b \in Y$, if bRa, then we have $a = b$. The Plural of **Minimum** is **Minima**, and we represent the set of **Minima** of Y with respect to the relation R with $\textbf{Minima}_R(Y)$. If R is understood, we represent the set of **Minima** of Y with $\textbf{Minima}(Y)$.

Definition 2.1.30 (**Upper Bound**)**.** Let X be a nonempty set. Let R be a **Relation** on X. Let $Y \subset X$. Let $a \in X$. We say that a is an **Upper Bound** for Y if for every $x \in Y$, we have xRa. If a is an **Upper Bound** for Y, then we say that the set Y is **Bounded From Above** by a. We denote the set of **Upper Bounds** of Y with respect to the relation R with $\textbf{UpperBound}_R(Y)$. When R is understood, we denote this set with $\textbf{UpperBound}(Y)$.

Definition 2.1.31 (**Lower Bound**)**.** Let X be a nonempty set. Let R be a **Relation** on X. Let $Y \subset X$. Let $a \in X$. We say that a is an **Lower Bound** for Y if for every $x \in Y$, we have aRx. If a is an **Lower Bound** for Y, then we say that the set Y is **Bounded From Below** by a. We denote the set of **Lower Bounds** of Y with respect to the relation R with $\textbf{LowerBound}_R(Y)$. When R is understood, we denote this set with $\textbf{LowerBound}(Y)$.

Definition 2.1.32 (**Least Upper Bound**)**.** Let X be a nonempty set. Let R be a **Relation** on X. Let $Y \subset X$. Let $a \in X$. We say that a is a **Least Upper Bound** of Y if $a \in \textbf{Minima}(\textbf{UpperBound}(Y))$. We denote the set of **Least Upper Bounds** for Y with $\textbf{LUB}(Y)$. If $b \in \textbf{LUB}(Y)$, then we call b a **Supremum** of Y. The plural of **Supremum** is **Suprema**. If $\textbf{LUB}(Y) = \{c\}$, then we write $c = \textbf{Sup}(Y)$.

Definition 2.1.33 (**Greatest Lower Bound**)**.** Let X be a nonempty set. Let R be a **Relation** on X. Let $Y \subset X$. Let $a \in X$. We say that a is a **Greatest Lower Bound** of Y if $a \in$ **Maxima**(**LowerBound**(Y)). We denote the set of **Greatest Lower Bounds** for Y with **GLB**(Y). If $b \in$ **GLB**(Y), then we call b an **Infimum** of Y. The plural of **Infimum** is **Infima**. If **GLB**$(Y) = \{c\}$, then we write $c =$ **Inf**(Y).

Definition 2.1.34 (**Equivalence Relation**)**.** A **Symmetric Preorder** on a nonempty set X is called an **Equivalence Relation** on X.

Definition 2.1.35 (**Partial Order**)**.** Let X be a nonempty set. Let $\leq$ be an **Anti-Symmetric Preorder** on X. Then we say that $\leq$ is a **Partial Order** on X, we say that $\leq$ is a **Partial Ordering** of X, and we refer to the pair $(X, \leq)$ as a **Partially Ordered Set**.

Definition 2.1.36 (**Total Order**)**.** Let (X, R) be a **Partially Ordered Set** in which every pair of elements is **Comparable**. Then we call R a **Total Order** on X and we call (X, R) a **Totally Ordered Set**.

Definition 2.1.37 (**Chain**)**.** Let $(X, \leq)$ be a **Partially Ordered Set**. Let $A \subset X$ such that $(A, \leq|_A)$ is a **Totally Ordered Set**. Then we call A a **Chain** in X.

Definition 2.1.38 (**Direction**)**.** Let X be a nonempty set. Let $\leq$ be a **Preorder** on X. If every pair of elements in X has an **Upper Bound** with respect to $\leq$, then we call $\leq$ is a **Direction** on X, we call $\leq$ is a **Directing** of X, and we call $(X, \leq)$ is a **Directed Set**.

Definition 2.1.39 (**Section** of a **Directed Set**)**.** Let $(X, \leq)$ be a **Directed Set**. Let $x \in X$. We define

$$S(x, \leq) = \{y \in X : x \leq y\}$$

We call $S(x, \leq)$ the **Section** of $\leq$ corresponding to $x \in X$.

Definition 2.1.40 (**Lattice**, **Join**, **Meet**)**.** Let $(X, \leq)$ be a **Partially Ordered Set** such that, for every $x, y \in X$, the set $\{x, y\}$ has and unique **Supremum** and a unique **Infimum**. Then we call $(X, \leq)$ **Lattice**. Furthermore, we call **Sup**$\{x, y\}$ the **Join** of x and y and we call **Inf**$\{x, y\}$ the **Meet** of x and y. If every nonempty subset of X has both a **Supremum** and **Infimum** then we call $(X, \leq)$ a **Complete Lattice**.

Definition 2.1.41 (**Sequence**)**.** Let X be a set. A **Sequence** in X is a **Function** $f : \mathbb{N} \to X$. If f is a **Sequence** in X and $f(n) = x_n$ for $n \in \mathbb{N}$, then we may refer to $\{x_n\}_{n\in\mathbb{N}}$ as the **Sequence** itself.

2.1.6 Equivalence Relations

Definition 2.1.42 (**Equivalence Class**)**.** Let X be a nonempty set. Let $\cong$ be an **Equivalence Relation** defined on X. Let $x \in X$. We define the set $[x]_\cong$ by

$$[x]_\cong = \{y \in X : y \cong x\}$$

We call $[x]_\cong$ the **Equivalence Class** of x in $(X, \cong)$.

Proposition 2.1.43 (Equivalence Classes Partition)**.** Let X be a nonempty set. Let $\cong$ be an **Equivalence Relation** defined on X. Let $x, y \in X$. The following statements are equivalent.

(i) $[x]_\cong \cap [y]_\cong \neq \emptyset$

(ii) $x \cong y$

(iii) $[x]_\cong = [y]_\cong$

(iv) $[x]_\cong \subset [y]_\cong$.

2.1.43. i $\implies$ *2.1.43. ii*. Suppose $[x]_\cong \cap [y]_\cong \neq \emptyset$. Then there exists $z \in [x]_\cong \cap [y]_\cong$. Hence $z \cong x$ and $z \cong y$. By **Symmetry**, $x \cong z$. By **Transitivity**, $x \cong y$. □

2.1.43. ii $\implies$ *2.1.43. iv*. Let $x \cong y$ and let $z \in [x]_\cong$. Then $z \cong x \cong y$, so $z \cong y$ and $z \in [y]_\cong$. Since z was arbitrary, we're done. □

2.1.43. ii $\implies$ *2.1.43. iii*. Let $x \cong y$. By 2.1.43. *ii* $\implies$ 2.1.43. *iv*, $[x]_\cong \subset [y]_\cong$. Also, by **Symmetry**, $y \cong x$, so $[y]_\cong \subset [y]_\cong$. Hence, equality holds. □

2.1.43. iv $\implies$ *2.1.43. i*. Let $[y]_\cong \subset [x]_\cong$. Then $y \in [y]_\cong = [y]_\cong \cap [x]_\cong$. Hence 2.1.43. i holds. □

2.1.43. iii $\implies$ *2.1.43. iv*. Obvious. □

Definition 2.1.44 (Quotient Set)**.** Let X be a nonempty set. Let $\cong$ be an **Equivalence Relation** defined on X. We define the set $X/\cong$ by

$$X/\cong = \{[x]_\cong : x \in X\}$$

We call $X/\cong$ the **Quotient Set** of X under the relation $\cong$.

Remark 2.1.45 (**Quotient Set** forms a **Partition**)**.** 2.1.43, paired with the fact that $x \in [x]_\cong$, implies that $X/\cong$ is a **Partition** of X.

Definition 2.1.46 (**Quotient Map**)**.** Let X be a nonempty set. Let $\cong$ be an **Equivalence Relation** on X. Let $X/\cong$ be the **Quotient Set** of X with respect to the relation $\cong$. Define $T : X \to X/\cong$ by setting, for each $x \in X$,

$$T(x) = [x]$$

We call T the **Quotient Map** of X under $\cong$. If $A \subset X$ and $T^{-1}(T(A)) = A$, then we call A a **Fiber** under $\cong$. If $\cong$ is understood, we call A a **Fiber**.

Proposition 2.1.47 (**Quotient Map** is **Surjective**)**.** Let X be a nonempty set. Let $\cong$ be an **Equivalence Relation** on X. Let $T : X \to X/\cong$ be the **Quotient Map** of X under the **Relation** $\cong$. Then T is a **Surjection**.

Proof. Let $K \in X/\cong$. Then for some $x \in X$, $K = [x]$. Then $T(x) = K$. Since K was arbitrary, we are done. □

2.1.7 Nets

Definition 2.1.48 (**Net**)**.** A **Net** is a **Function** mapping from a **Directed Set** $(A, \leq)$ into another set X. If $f : A \to X$ is a **Net** such that for $\alpha \in A$ we have $f(\alpha) = x_\alpha$, then we may use the notation $\{x_\alpha\}_{\alpha \in A} \subset X$.

Definition 2.1.49 (**Section** of a **Net**)**.** Let X be a nonempty set. Let $(A, \leq)$ be a **Directed Set** and let $\sigma = \{x_\alpha\}_{\alpha \in A}$ be a **Net** in X. Let $\gamma \in A$. Let $S(\gamma, \leq)$ be the **Section** of $\leq$ corresponding to γ. We define

$$\{x_\alpha : \alpha \in S(\gamma, \leq)\}$$

to be the **Section** of x_γ in σ.

Proposition 2.1.50 (Net Section)**.** Let X be a nonempty set. and let $\{x_\alpha\}_{\alpha \in A}$ be a **Net** in X. Let $\beta \leq \gamma \in A$. For $\alpha \in A$, let $S(\alpha)$ denote the **Section** of x_α in $\{x_\alpha\}_{\alpha \in A}$. Then $S(\gamma) \subset S(\beta)$.

Proof. Let $y \in S(\gamma)$. Then $y = x_\tau$ for some $\tau \in A$ satisfying $\beta \leq \gamma \leq \tau$. Hence, $y = x_\tau \in S(\beta)$. □

Definition 2.1.51 (**Inductive Order**)**.** Let $(X, \leq)$ be a **Partially Ordered Set**. We say that $\leq$ is an **Inductive Order** on X and we say that X is **Inductively Ordered** by $\leq$ if each **Chain** in X has an **Upper Bound** in X. we also call $(X, \leq)$ an **Inductively Ordered** set in this circumstance.

Theorem 2.1.52 (Zorns Lemma)**.** An **Inductively Ordered** set has a **Maximum**.

Remark 2.1.53. 2.1.52 Is equivalent to the axiom of choice

2.2 Filters

2.2.1 Filter Basics

Definition 2.2.1 (**Filter**)**.** Let X be a nonempty set. Let $\mathcal{F} \subset 2^X$ satisfy the following.

(i) $\mathcal{F} \neq \emptyset$.

(ii) $\emptyset \notin \mathcal{F}$

(iii) If $G_1 \in \mathcal{F}$ and $G_1 \subset G_2 \subset X$, then $G_2 \in \mathcal{F}$.

(iv) If $\{G_1, G_2\} \subset \mathcal{F}$, then $G_1 \cap G_2 \in \mathcal{F}$.

Then we call $\mathcal{F}$ a **Filter** on X.

Proposition 2.2.2. Let X be a nonempty set. Let $\mathcal{F}$ be a **Filter** on X. Let $B \subset X$. The following are true.

(i) $X \in \mathcal{F}$.

(ii) $\mathcal{F}$ is closed under finite intersections.

(iii) The intersection of a collection of **Filters** on X is a **Filter** on X.

(iv) $\mathcal{F}_B$ defined by $\mathcal{F}_B = \{U \cap B : U \in \mathcal{F}\}$ satisfies 2.2.1. i, 2.2.1. iii with respect to B, and 2.2.1. iv. Therefore, $\mathcal{F}_B$ is a **Filter** on B if and only if $\emptyset \notin \mathcal{F}_B$.

Proof of 2.2.2 i. By 2.2.1. i, $\mathcal{F}$ contains a nonempty set A. Since $A \subset X \subset X$, by 2.2.1. iii, $X \in \mathcal{F}$, so 2.2.2 i is proven. □

Proof of 2.2.2 ii. This result is a direct application of 2.2.1. iv paired with 2.1.13. i. □

Proof of 2.2.2 iii. Let $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ be a collection of **Filters** on X. Define $\mathcal{F} = \bigcap_{\alpha \in A} \mathcal{F}_\alpha$. By 2.2.2 i, for each $\alpha \in A$, $X \in \mathcal{F}_\alpha$, so $X \in \mathcal{F}$. Hence $\mathcal{F}$ satisfies 2.2.1. *i*. Furthermore, by 2.2.1. *ii*, for each $\alpha \in A$, $\emptyset \notin \mathcal{F}_\alpha$, so $\emptyset \notin \mathcal{F}$. Therefore $\mathcal{F}$ satisfies 2.2.1. ii. Let $G_1 \in \mathcal{F}$ and let $G_1 \subset G_2 \subset X$. Then, for each $\alpha \in A$, $G_1 \in \mathcal{F}_\alpha$, so $G_2 \in \mathcal{F}_\alpha$ for each $\alpha \in A$, so $G_2 \in \mathcal{F}$. Thus, $\mathcal{F}$ satisfies 2.2.1. iii. Finally, let $\{G_1, G_2\} \subset \mathcal{F}$. Then for each $\alpha \in A$, $\{G_1, G_2\} \subset \mathcal{F}_\alpha$, implying by 2.2.1. *iv* that $G_1 \cap G_2 \in \mathcal{F}_\alpha$, so $G_1 \cap G_2 \in \mathcal{F}$, implying $\mathcal{F}$ satisfies 2.2.1. *iv*. □

Proof of 2.2.2 iv. By 2.2.2 i, $X \in \mathcal{F}$, so $B = X \cap B \in \mathcal{F}_B$. Hence $\mathcal{F}_B$ satisfies 2.2.1. i. If $G_1 \in \mathcal{F}_B$, then there is an $H_1 \in \mathcal{F}$ with $G_1 = H_1 \cap B$. If $G_1 \subset G_2 \subset B$, then $H_1 \cap B \subset G_2$ and $H_1 \cap (X \setminus B) \subset (X \setminus B)$, so $H_1 \subset G_2 \cup (X \setminus B)$, which implies $G_2 \cup (X \setminus B) \in \mathcal{F}$ by 2.2.1. iii. By construction, then, $G_2 = (G_2 \cup (X \setminus B)) \cap B \in \mathcal{F}_B$. Hence $\mathcal{F}_B$ satisfies 2.2.1. *iii*. Next, if $\{G_1, G_2\} \subset \mathcal{F}_B$, then there are $H_1, H_2 \in \mathcal{F}$ with $G_i = H_i \cap B$. Since $\mathcal{F}$ satisfies 2.2.1. *iv*, $H_1 \cap H_2 \in \mathcal{F}$. This implies by construction that $B \cap (H_1 \cap H_2) \in \mathcal{F}_B$, but $B \cap (H_1 \cap H_2) = G_1 \cap G_2$, so $\mathcal{F}_B$ satisfies 2.2.1. iv. □

Definition 2.2.3 (**Coarser**, **Finer**)**.** Let X be a nonempty set. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be **Filters** on X such that $\mathcal{F}_1 \subset \mathcal{F}_2$. Then we say that $\mathcal{F}_1$ is **Coarser** than $\mathcal{F}_2$ and we say that $\mathcal{F}_2$ is **Finer** than $\mathcal{F}_1$. Let A be a collection of **Filters** on X. Let $\mathcal{F}_2 \in A$ be **Finer** than every element of A. Then we say that $\mathcal{F}_2$ is the **Finest** element of A. Let $\mathcal{F}_3 \in A$ be **Coarser** than every element of A. Then we say that $\mathcal{F}_3$ is the **Coarsest** element in A. **Filter Fineness** defines a **Partial Ordering** on the collection of **Filters** on X, where $\mathcal{F}_1 \leq \mathcal{F}_2$ if $\mathcal{F}_2$ is a **Finer Filter** than $\mathcal{F}_1$. A **Maximum** of the **Filter Fineness** relation is called an **Ultrafilter** on X.

Proposition 2.2.4. Let X be a nonempty set. Let $A \subset 2^X$ be nonempty. Then there is a **Filter** on X which contains A if and only if any **Finite** intersection of elements of A is nonempty. Furthermore, if there is a **Filter** containing A, then there is a **Coarsest Filter** containing A, and it is given by the collection of all subsets of X which contain some finite intersection of elements of A.

Proof. The given condition is necessary by a combination of 2.2.2 i and 2.2.1. ii. For sufficiency, let K be the collection of finite intersections of elements of A. Define

$$\mathcal{K} = \{F \cup Y : F \in K \wedge Y \subset X\}$$

Then $A \subset \mathcal{K}$. Since $A \neq \emptyset$, $\mathcal{K} \neq \emptyset$, so $\mathcal{K}$ satisfies 2.2.1. i. Since **Finite** intersections of elements of A are nonempty, $\emptyset \notin K$, implying $\emptyset \notin \mathcal{K}$, so 2.2.1. *ii* holds. Let $P \in \mathcal{K}$ and let $P \subset Q \subset X$. Then there exists $L_P \in K$ and $Y_P \subset X$ such that $P = L_P \cup Y_P$, and $Q = L_P \cup (Y_P \cup Q) \in \mathcal{K}$, so 2.2.1. iii holds for $\mathcal{K}$. Finally, let $G_1, G_2 \in \mathcal{K}$. Then $G_i = U_i \cup P_i$ for $U_i \in K$ and $P_i \subset X$. By definition of $\mathcal{K}$, there are a $\{G_1^j\}_{j=1}^{n_1} \subset A$ and $\{G_2^j\}_{j=1}^{n_2} \subset A$ with $U_i = \bigcap_{j=1}^{n_i} G_i^j$. Clearly $U_1 \cap U_2$, being a finite intersection of elements of A, is an element of K. Furthermore,

$$\begin{aligned} U_1 \cap U_2 &\subset (U_1 \cap U_2) \cup ((U_1 \cap P_2) \cup (U_2 \cap P_1) \cup (P_1 \cap P_2)) \\ &= (U_1 \cup P_1) \cap (U_2 \cup P_2) \\ &= G_1 \cap G_2 \end{aligned}$$

so since $\mathcal{K}$ satisfies 2.2.1. *iii*, $G_1 \cap G_2 \in \mathcal{K}$, so 2.2.1. *iv* applies for $\mathcal{K}$, so $\mathcal{K}$ is a **Filter**. By 2.2.2 ii, any **Filter** containing A would contain K. By 2.2.1. *iii*, any **Filter** containing A would contain $\mathcal{K}$, so any **Filter** on X containing A would be **Finer** than $\mathcal{K}$, and so $\mathcal{K}$ is the **Coarsest Filter** containing A. Hence, 2.2.4 is proven. □

Proposition 2.2.5 (Filter Order Facts)**.** Let X be a nonempty set. The following are true.

(i) Let $\mathcal{F}$ be a **Filter** on X and let $A \subset X$. Then there is a **Filter** containing A on X which is **Finer** than $\mathcal{F}$ if and only if $A \cap U \neq \emptyset$ for each $U \in \mathcal{F}$.

(ii) Let $K = \{\mathcal{F}_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of **Filters** on X. There exists a **Filter** on X which is **Finer** than each $\mathcal{F}_\alpha$ if and only if, for every **Finite** subset $\{\mathcal{F}_{\alpha_i}\}_{i=1}^n \subset \mathcal{K}$, for each $\{U_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{F}_{\alpha_i}$, $\bigcap_{i=1}^n U_i \neq \emptyset$.

(iii) The union of a **Chain** of **Filters** on X is a **Filter** on X which is **Finer** than each element of the **Chain**.

(iv) The intersection of a **Chain** of **Filters** on X is a **Filter** on X which is **Coarser** than each element of the **Chain**.

(v) **Filter Fineness** is an **Inductive Order** on the collection of **Filters** on X.

(vi) **Filter Coarseness** is an **Inductive Order** on the collection of **Filters** on X.

Proof of 2.2.5. i. By 2.2.2 ii the collection of **Finite** intersections of elements of $\mathcal{K} := \{A\} \cup \mathcal{F}$ is given by

$$\mathcal{L} := \mathcal{F} \cup \{A \cap F : F \in \mathcal{F}\}$$

By 2.2.4 there is a **Filter** containing $\mathcal{K}$ if and only if $\emptyset \notin \mathcal{L}$. □

Proof of 2.2.5. ii. By 2.2.2 ii, each $\mathcal{F}_\alpha$ is closed under finite intersections. Therefore, the collection of all **Finite** intersections of elements of

$$\bigcup_{\alpha \in A} \mathcal{F}_\alpha$$

is given by the collection intersections single element each of a **Finite** collection of $\mathcal{F}_\alpha' s$. An application of 2.2.4 finishes the result. □

Proof of 2.2.5. iii. Let $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ be a **Chain** of **Filters** on X and let

$$\mathcal{F} = \bigcup_{\alpha \in A} \mathcal{F}_\alpha$$

Since each $\mathcal{F}_\alpha \neq \emptyset$, $\mathcal{F} \neq \emptyset$, so $\mathcal{F}$ satisfies 2.2.1. i. Furthermore, $\emptyset \notin \mathcal{F}_\alpha$ for any $\alpha \in A$, so $\mathcal{F}$ satisfies 2.2.1. ii Let $G \in \mathcal{F}$ and let $G \subset H \subset X$. Then for some $\alpha \in A$, $G \in \mathcal{F}_\alpha$. Since $\mathcal{F}_\alpha$ is a **Filter**, by 2.2.1. iii, $H \in \mathcal{F}_\alpha \subset \mathcal{F}$, so $H \in \mathcal{F}$. Hence $\mathcal{F}$ satisfies 2.2.1. *iii* Finally, let $G_1, G_2 \in \mathcal{F}$. Then there are $\alpha_1, \alpha_2 \in A$ such that $G_i \in \mathcal{F}_{\alpha_i}$ for $i \in \{1, 2\}$. Either $\mathcal{F}_{\alpha_1} \subset \mathcal{F}_{\alpha_2}$ or $\mathcal{F}_{\alpha_2} \subset \mathcal{F}_{\alpha_1}$, so without loss of generality, let $\mathcal{F}_{\alpha_1} \subset \mathcal{F}_{\alpha_2}$. Then $G_1 \in \mathcal{F}_{\alpha_2}$, implying $G_1 \cap G_2 \in \mathcal{F}_{\alpha_2} \subset \mathcal{F}$, so $G_1 \cap G_2 \in \mathcal{F}$, so $\mathcal{F}$ satisfies 2.2.1. iv Hence $\mathcal{F}$ is a **Filter** on X which is **Finer** than each $\mathcal{F}_\alpha$. $\square$

Proof of 2.2.5. iv. Define

$$\mathcal{F} = \bigcap_{\alpha \in A} \mathcal{F}_\alpha$$

$\emptyset \notin \mathcal{F}_\alpha$ for any $\alpha \in A$, so $\emptyset \notin \mathcal{F}$, implying $\mathcal{F}$ satisfies 2.2.1. *ii*. Furthermore, $X \in \mathcal{F}_\alpha$ for each $\alpha \in A$, so $X \in \mathcal{F}$, implying $\mathcal{F}$ satisfies 2.2.1. *i*. If $G \in \mathcal{F}$ and $G \subset H \subset X$, then for every $\alpha \in A$, $G \in \mathcal{F}_\alpha$. This implies, that by 2.2.1. iii, that for every $\alpha \in A$, $H \in \mathcal{F}_\alpha$. Hence $H \in \mathcal{F}$, so $\mathcal{F}$ satisfies 2.2.1. iii. Finally, let $G_1, G_2 \in \mathcal{F}$. Then $G_1, G_2 \in \mathcal{F}_\alpha$ for every $\alpha \in A$, so for every $\alpha \in A$, by 2.2.1. iv, $G_1 \cap G_2 \in \mathcal{F}_\alpha$ for every $\alpha \in A$, hence $G_1 \cap G_2 \in \mathcal{F}$. Hence, $\mathcal{F}$ is a **Filter** on X. Also, $\mathcal{F} \subset \mathcal{F}_\alpha$ for every $\alpha \in A$, and so is **Coarser** than every $\mathcal{F}_\alpha$. $\square$

Proof of 2.2.5. v. Direct application of 2.2.5. iii to the collection of all **Chains** of **Filters** in X. $\square$

Proof of 2.2.5. vi. Direct application of 2.2.5. iv to the collection of all **Chains** of **Filters** on X. $\square$

2.2.2 Filter Base

Definition 2.2.6 (**Subbasis**)**.** Let X be a nonempty set. Let $A \subset X$ such that

(i) $\emptyset \neq A$.

(ii) Finite intersections of elements of A are nonempty.

Let K denote the collection of finite intersections of elements of A. Define

$$\mathcal{K} = \{U \cup P : U \in K \ \ and \ \ P \subset X\}$$

We call $\mathcal{K}$ the **Filter** on X **Generated By** A. and we call A a **Subbasis** for G. By 2.2.4, $\mathcal{K}$ is in fact a **Filter**, and is the **Coarsest Filter** on X containing A.

Definition 2.2.7 (**Filter Base**)**.** Let X be a nonempty set. Let $\mathcal{B} \subset 2^X$ such that

(i) $\emptyset \neq \mathcal{B}$.

(ii) $\emptyset \notin \mathcal{B}$.

(iii) If $\mathcal{B}_{Int}$ is the collection of binary intersections of elements of $\mathcal{B}$, then Then **Nested**$(\mathcal{B}, \mathcal{B}_{Int})$ holds.

Then we call $\mathcal{B}$ a **Filter Base** on X. By 2.2.8, the **Filter Generated By** a **Filter Base** A is given by $\{U \subset X : (\exists Y \in A)(Y \subset U)\}$. If A, B are **Filter Bases** on X and they **Generate** the same **Filter**, then we call them **Equivalent**.

Proposition 2.2.8. Let X be a nonempty set. Let $A \subset 2^X$. and define

$$\mathcal{U} = \{U \subset X : (\exists a \in A)(a \subset U)\}$$

The following are equivalent.

1. A is a **Filter Base** on X.
2. $\mathcal{U}$ is a **Filter** on X.

1 $\implies$ *2.* Supose A is a **Filter Base** on X. By 2.2.7. *ii*, $\emptyset \notin A$, so $\emptyset \notin \mathcal{U}$, implying that $\mathcal{U}$ satisfies 2.2.1. ii. Also, by 2.2.7. i $\emptyset \neq A \subset \mathcal{U}$, so $\mathcal{U}$ satisfies 2.2.1. i. It is obvious that $\mathcal{U}$ satisfies 2.2.1. iii. Finally, if $G_1, G_2 \in \mathcal{U}$, then there exists $U_1, U_2 \in A$ such that $A_i \subset G_i$. By 2.2.7. iii, there is $B \in A$ satisfying $B \subset U_1 \cap U_2 \subset G_1 \cap G_2$, so $G_1 \cap G_2 \in \mathcal{U}$, implying $\mathcal{U}$ is a **Filter** on X. □

1 $\impliedby$ *2.* If $A = \emptyset$, then $\mathcal{U} = \emptyset$, so A failing 2.2.7. *i* implies $\mathcal{U}$ fails 2.2.1. i. If $\emptyset \in A$, then $\emptyset \in \mathcal{U}$, so A failing 2.2.7. *ii* implies $\mathcal{U}$ fails 2.2.1. ii. Finally, if $\mathcal{A}$ fails 2.2.7. iii, then we can find $B, C \in A$ such that $B \cap C \notin \mathcal{U}$, implying $\mathcal{U}$ fails 2.2.1. *iv*. Hence necessity has been proven. □

Remark 2.2.9. If A is a **Filter Base** on X, then $\mathcal{U}$ defined in 2.2.8 is the **Filter Generated By** A.

Proposition 2.2.10 (Filter Base Facts)**.** Let X be a nonempty set. Let $\mathcal{F}$ and $\mathcal{G}$ be **Filters** on X. Let F be a **Filter Base** for $\mathcal{F}$ and let G be a **Filter Base** for $\mathcal{G}$. The following are true.

(i) The collection of **Finite** intersections of a **Subbasis** A for $\mathcal{F}$ forms a **Filter Base** for $\mathcal{F}$.

(ii) $B \subset \mathcal{F}$ is a **Filter Base** for $\mathcal{F}$ if and only if **Nested**$(B, \mathcal{F})$ holds.

(iii) $\mathcal{F}$ is **Finer** than $\mathcal{G}$ if and only if **Nested**(F, G) holds.

(iv) F is **Equivalent** to G if and only if **Nested**(F, G) and **Nested**(G, F) both hold.

(v) $\mathcal{F}$ is a **Filter Base** for $\mathcal{F}$.

Proof of 2.2.10. i. Define $\mathcal{B}$ to be the collection of all finite intersections of elements of A. By 2.2.6. ii, $\emptyset \notin \mathcal{B}$, so $\mathcal{B}$ satisfies 2.2.7. ii. By 2.2.6. i, $\emptyset \neq A \subset \mathcal{B}$, so $\mathcal{B}$ satisfies 2.2.7. i. Since $\emptyset \notin \mathcal{B}$ is closed under finite intersections, if $U, V \in \mathcal{B}$, then $\emptyset \neq U \cap V \in \mathcal{B}$, so $\mathcal{B}$ can be seen to satisfy 2.2.7. *iii*. Hence $\mathcal{B}$ is a **Filter Base**. Since $A \subset \mathcal{B}$, $\mathcal{B}$ is a **Filter Base** for a **Finer Filter** than $\mathcal{F}$. However, since $A \subset \mathcal{F}$, by 2.2.2 ii, $\mathcal{B} \subset \mathcal{F}$. Hence $\mathcal{F}$ is the **Filter Generated By** $\mathcal{B}$. □

Proof of 2.2.10. ii. ($\Longleftarrow$). Let $\mathcal{G}$ denote the **Filter Generated By** B. Then since $B \subset \mathcal{F}$, $\mathcal{G} \subset \mathcal{F}$. If for each $Y \in \mathcal{F}$ there exists $b \in B$ with $b \subset Y$, then

$$\mathcal{F} \subset \{U \subset X : (\exists b \in B)(b \subset U)\} \subset \mathcal{G} \subset \mathcal{F} \tag{2.1}$$

so that $\mathcal{F} = \mathcal{G}$ and $\{U \subset X : (\exists b \in B)(b \subset U)\} = \mathcal{F}$ is a **Filter** on X. Hence, by 2.2.8, B is a **Filter Base** for $\mathcal{F}$.

($\Longrightarrow$). If B is a **Filter Base** for $\mathcal{F}$, then by 2.2.8, $\mathcal{F} = \{Y \subset X : (\exists b \in B)(b \subset Y)\}$ so the desired property holds □

Proof of 2.2.10. iii. Let $\mathcal{F}$ be finer than $\mathcal{G}$. Then by applying 2.2.8

$$G \subset \mathcal{G} \subset \mathcal{F} = \{U \subset X : (\exists f \in F)(f \subset U)\}$$

which is the desired result in one direction. The other direction is by applying applying 2.2.8 to claim $\mathcal{G} \subset \{U \subset X : (\exists f \in F)(f \subset U)\}$. □

Proof of 2.2.10. iv. This is a result of two applications of 2.2.10. iii, one in each direction. □

Proof of 2.2.10. v. Define $\mathcal{U} = \{U \subset X : (\exists f \in \mathcal{F}) \pitchfork (f \subset U)\}$. By construction $\mathcal{F} \subset \mathcal{U}$. By 2.2.1. iii, $\mathcal{U} \subset \mathcal{F}$. Hence, by 2.2.8, $\mathcal{U} = \mathcal{F}$ is a **Filter Base** on X. Clearly $\mathcal{F} \subset \mathcal{F}$ and **Nested**$(\mathcal{F}, \mathcal{F})$ hold, so we can apply 2.2.10. ii so see that $\mathcal{F}$ is a **Filter Base** for $\mathcal{F}$. □

Proposition 2.2.11 (**Net Sections** form a **Filter Base**)**.** Let X be a nonempty set. Let $\sigma = \{x_\alpha\}_{\alpha \in A}$ be a **Net** in X. For each $\alpha \in A$, denote with $S(\sigma, \alpha)$ the **Section** of x_α in σ. Define $\mathcal{B} = \{S(\sigma, \alpha) : \alpha \in A\}$ For each $\alpha \in A$, let $S(\alpha, \leq)$ denote the **Section** of $\leq$ corresponding to α. Then $\mathcal{B}$ is a **Filter Base** on X.

Proof. Since σ is a **Net**, $(A, \leq)$ is a **Directed Set**, implying that $(A, \leq)$ is a **Preordered Set**. Hence, $\leq$ is **Reflexive** so that if $\alpha \in A$, then $x_\alpha \in S(\sigma, \alpha)$. Hence, $\emptyset \notin \mathcal{B}$, so $\mathcal{B}$ satisfies 2.2.7. ii. Furthermore, since $(A, \leq)$ is a **Preordered Set**, A is nonempty, so $\emptyset \neq \mathcal{B}$, implying $\mathcal{B}$ satisfies 2.2.7. i. Finally, let $U, V \in \mathcal{B}$. Then we can find $u, v \in A$ such that $U = S(\sigma, U)$, $V = S(\sigma, V)$. Since A is a **Directed Set**, there exists $w \in A$ with $u \leq w$ and $v \leq w$. Hence by 2.1.50, $S(\sigma, w) \subset S(\sigma, u) \cap S(\sigma, v)$. Since $S(\sigma, w) \in \mathcal{B}$, $\mathcal{B}$ satisfies 2.2.7. iii, and we're done. □

Definition 2.2.12 (**Section Filter**)**.** Let X be a nonempty set. Let $\sigma = \{x_\alpha\}_{\alpha \in A}$ be a **Net** in X. For each $\alpha \in A$, let $S(\sigma, \alpha)$ denote the **Section** of x_α in σ. Define $\mathcal{B}$ by

$$\mathcal{B} = \{S(\sigma, \alpha) : \alpha \in A\}$$

By **??**, $\mathcal{B}$ is a **Filter Base** on X. We call the **Filter Generated By** $\mathcal{B}$ the **Section Filter** of σ. We call the **Section Filter** of the identity **Net** in A the **Section Filter** of A. We denote the **Section Filter** of A with $\mathcal{F}_A$.

2.2.3 Ultrafilters

Definition 2.2.13 (**Ultrafilter**)**.** Let X be a nonempty set. An **Ultrafilter** on X is a **Maximum** of the relation of **Filter Fineness** on X.

Remark 2.2.14 (**Ultrafilter** Existence)**.** Let X be a nonempty set. Let $\mathcal{F}$ be a **Filter** on X. By 2.2.5. iii By 2.2.5. v), **Filter Fineness** an **Inductive Order** on the set of **Filters** of X and it is also an **Inductive Order** on the set of **Filters** which are **Finer** than $\mathcal{F}$. Hence by 2.1.52, $\mathcal{F}$ is contained in an **Ultrafilter** on X. A **Filter Base** for an **Ultrafilter** is called an **Ultrafilter Base**.

Proposition 2.2.15 (Ultrafilter Facts)**.** Suppose the following

(I) X is a nonempty set.

(II) $\mathcal{F}$ is an **Ultrafilter** on X.

(III) $\mathcal{G}$ is a **Filter** on X.

(IV) $\mathcal{K}$ is a **Subbasis** on X.

(V) $\mathcal{M} = \mathcal{K} \cup \{K \subset X : X \setminus K \in \mathcal{K}\}$.

Then the following are true

(i) If $\{A, B\} \subset 2^X$ and $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

(ii) If $\{A_i\}_{i=1}^n \subset 2^X$ such that $\bigcup\limits_{i=1}^{n} A_i \in \mathcal{F}$, then for some $j \in \{1, \cdots, n\}$, $A_j \in \mathcal{F}$.

(iii) If $\mathcal{M} = 2^X$, then $\mathcal{K}$ is an **Ultrafilter** on X.

(iv) $\mathcal{G}$ is the intersection of all **Ultrafilters** on X which contain $\mathcal{G}$.

Proof of 2.2.15. i. We use contradiction. Suppose $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$, but $A \cup B \in \mathcal{F}$. Define $\mathcal{T} = \{G \in 2^X : A \cup G \in \mathcal{F}\}$. Then $B \in \mathcal{T}$, so $\mathcal{T}$ satisfies 2.2.1. i. Furthermore, if $G_1 \in \mathcal{T}$ and $G_1 \subset G_2 \subset X$, then by 2.2.1. iii, $A \cup G_1 \subset A \cup G_2 \in \mathcal{F}$. Hence $G_2 \in \mathcal{T}$ so $\mathcal{T}$ satisfies 2.2.1. iii. Let $G_3, G_4 \in \mathcal{T}$. Then,

$$A \cup (G_3 \cap G_4) = (A \cup G_3) \cap (A \cup G_4) \in \mathcal{F}$$

so that $G_3 \cap G_4 \in \mathcal{T}$ and therefore $\mathcal{T}$ satisfies 2.2.1. iv. Finally since $A \notin \mathcal{F}$, $\emptyset \notin \mathcal{T}$, so $\mathcal{T}$ satisfeis 2.2.1. *ii*, and therefore $\mathcal{T}$ is a **Filter** on X. Trivially, $\mathcal{F} \subset \mathcal{T}$ but since $B \in \mathcal{T} \setminus \mathcal{F}$, this contradicts 2.2.15. II Hence the result holds. $\square$

Proof of 2.2.15. ii. We use induction on n. Obviously the result holds for $n = 1$ and by 2.2.15. i, the result also holds for $n = 2$. Suppose the result holds for $n = k$ Let $\{A_i\}_{i=1}^{k+1} \subset 2^X$ such that $\bigcup\limits_{i=1}^{k+1} A_i \in \mathcal{F}$. Then since the result holds for $n = 2$, either $A_{k+1} \in \mathcal{F}$ or $\bigcup\limits_{i=1}^{k} A_i \in \mathcal{F}$. Since the result holds for $n = k$, either $A_{k+1} \in \mathcal{F}$ or $A_i \in \mathcal{F}$ for $i \in \{1, \cdots, k\}$. Hence the result holds for $n = k + 1$. Hence the result holds in general. $\square$

Proof of 2.2.15. iii. I first prove that $\mathcal{M} = 2^X$, paired with 2.2.15. I, 2.2.15. IV, and 2.2.15. V implies $\mathcal{K}$ is a **Filter** on X. By 2.2.15. I, $\mathcal{M} \neq \emptyset$. By 2.2.15. V, then $\mathcal{K} \neq \emptyset$, so $\mathcal{K}$ satisfies 2.2.1. i. By 2.2.6. ii, $\emptyset \notin \mathcal{K}$, so $\mathcal{K}$ satisfies 2.2.1. ii. Let $G_1 \in \mathcal{K}$ and let $G_1 \subset G_2 \subset X$. Then $G_1 \cap (X \setminus G_2) = \emptyset$, which by 2.2.6. ii implies $X \setminus G_2 \notin \mathcal{K}$. Since $\mathcal{M} = 2^X$, we conclude $G_2 \in \mathcal{K}$, so $\mathcal{K}$ satisfies 2.2.1. iii. Finally, let $G_1, G_2 \in \mathcal{K}$. By assumption, either $G_1 \cap G_2 \in \mathcal{K}$ or $X \setminus (G_1 \cap G_2) \in \mathcal{K}$. If $X \setminus (G_1 \cap_G 2) \in \mathcal{K}$, then by 2.2.6. ii, $G_1 \cap G_2 \cap (X \setminus (G_1 \cap G_2)) \neq \emptyset$, a contradiction. Hence, $G_1 \cap G_2 \in \mathcal{K}$, so that 2.2.1. iv is satisfied by $\mathcal{K}$. Hence $\mathcal{K}$ is a **Filter** on X. By 2.2.14, there is an **Ultrafilter** $\mathcal{L}$ containing $\mathcal{K}$. If $\mathcal{K}$ is not an **Ultrafilter**, then $\exists B \in \mathcal{L} \setminus \mathcal{K}$. Since $\mathcal{M} = 2^X$, $X \setminus B \in \mathcal{K} \subset \mathcal{L}$, implying $\emptyset = B \cap (X \setminus B) \in \mathcal{L}$, contradicting 2.2.1. ii, thus $\mathcal{K}$ is an **Ultrafilter**. □

Proof of 2.2.15. iv. Let $\{\mathcal{P}_\alpha\}_{\alpha \in A}$ be the collection of all **Ultrafilters** on X containing $\mathcal{G}$. Define $\mathcal{P} = \bigcap\limits_{\alpha \in A} \mathcal{P}_\alpha$. I must schow $\mathcal{P} = \mathcal{G}$. By 2.2.2 iii, $\mathcal{P}$ is a **Filter** on X, and by construction $\mathcal{G} \subset \mathcal{P}$. Let $B \in 2^X \setminus \mathcal{G}$. Then, by 2.2.1. iii, there is no $G \in \mathcal{G}$ with $G \subset A$. Hence, for each $G \in \mathcal{G}$, $G \cap (X \setminus A) \neq \emptyset$. Therefore, we can apply 2.2.5. i to claim that there is a **Filter** $\mathcal{G}_1$ on X which is **Finer** than $\mathcal{G}$ satisfying $X \setminus A \in \mathcal{G}_1$. By 2.2.14, there is an $\alpha \in A$ such that $X \setminus A \in \mathcal{G}_1 \subset \mathcal{P}_\alpha$, so $X \setminus A \in \mathcal{P}$. Since $\mathcal{P}$ satisfies 2.2.1. ii and 2.2.1. iv, $X \setminus A \in \mathcal{P}$ implies $A \notin \mathcal{P}$ and so $\mathcal{P} \subset \mathcal{G}$. This completes the proof. □

2.2.4 Induced Filters

Definition 2.2.16 (**Induced**)**.** Let X be a nonempty set. Let $\mathcal{F}$ be a **Filter** on X. Let $A \subset X$. Define $\mathcal{F}_A := \{U \cap A : U \in \mathcal{F}\}$. Suppose $\emptyset \notin \mathcal{F}_A$. Then by 2.2.2 iv $\mathcal{F}_A$ is a **Filter** on A which we call the **Filter Induced** by $\mathcal{F}$.

Proposition 2.2.17 (**Induced Filter** facts)**.** Let X be a nonempty set. Let $\mathcal{F}$ be a **Filter** on X. Let $\mathcal{B}$ be a **Filter Base** for $\mathcal{F}$. Let $\mathcal{G}$ be an **Ultrafilter** on X. Let $A \subset X$. Define

$$\mathcal{F}_A = \{A \cap U : U \in \mathcal{F}\}$$
$$\mathcal{B}_A = \{A \cap U : U \in \mathcal{B}\}$$
$$\mathcal{G}_A = \{A \cap U : U \in \mathcal{G}\}$$

The following are true

(i) If $\mathcal{F}_A$ is a **Filter** on A, then $\mathcal{B}_A$ is a **Filter Base** for $\mathcal{F}_A$.

(ii) $\mathcal{G}_A$ is a **Filter** on A if and only if $A \in \mathcal{G}$. In this case, $\mathcal{G}_A$ is an **Ultrafilter** on A.

Proof of 2.2.17. i. Let $U \in \mathcal{F}_\alpha$. Then there exists $V \in \mathcal{F}$ such that $U = A \cap V$. Since $\mathcal{B}$ is a **Filter Base** for $\mathcal{F}$, by 2.2.10. ii, there exists $B \in \mathcal{B}$ satisfying $B \subset V$. Then $B \cap A \subset A \cap V = U$. But $B \cap A \in \mathcal{B}_A$, so since $\mathcal{B}_A \subset \mathcal{F}_A$, we can the apply 2.2.10. *ii* to claim that $\mathcal{B}_A$ is a **Filter Base** for $\mathcal{F}_A$. □

Proof of 2.2.17. ii. Even if $\mathcal{G}$ was merely a **Filter** on X, by 2.2.1. ii and 2.2.2 ii $A \in \mathcal{G}$ is sufficient to guarantee that $\mathcal{G}_A$ is a **Filter** on A. Now suppose $\mathcal{G}_A$ is a **Filter** on A Then $A \cap U \neq \emptyset$ for $U \in \mathcal{G}$. Since $\mathcal{G}$ is an **Ultrafilter** on X, we can apply maximality with 2.2.5. i, to see that $A \in \mathcal{G}$. Finally, if $P \subset A$ satisfies $P \notin \mathcal{G}_A$, then $P \notin \mathcal{G}$, Since $\mathcal{G}$ is an **Ultrafilter**, by 2.2.5. i, $P \cap U = \emptyset$ for some $U \in \mathcal{G}$. This implies $P \cap (U \cap A) = \emptyset$, and since $U \cap A \in \mathcal{G}_A$, we can apply 2.2.5. i to conclude that there is no **Filter Finer** than $\mathcal{G}_A$ on A which contains P. Since $P \subset A$ was arbitrary, $\mathcal{G}_A$ is an **Ultrafilter** on A.

□

2.2.5 Direct and Inverse Images of a Filter Base

Proposition 2.2.18 (Direct Filter Image)**.** Suppose the following.

1. X and Y are nonempty sets.
2. $f : X \to Y$ is **Surjective**.
3. For $i \in \{1, 2\}$, $\mathcal{B}_i$ is a **Filter Base** for a **Filter** $\mathcal{F}_i$ on X.
4. $\mathcal{F}_2$ is **Finer** than $\mathcal{F}_1$.
5. $\mathcal{K}$ is an **Ultrafilter Base** on X.

Then the following are true.

(i) $f(\mathcal{F}_1)$ is a **Filter** on Y.

(ii) $f(\mathcal{B}_1)$ is a **Filter Base** for $f(\mathcal{F}_1)$.

(iii) $f(\mathcal{F}_2)$ is a **Finer** than $f(\mathcal{F}_1)$.

(iv) $f(\mathcal{K})$ is an **Ultrafilter Base** on Y.

Proof of 2.2.18. i. Since $\emptyset \notin \mathcal{F}_1$, $\emptyset \notin f(\mathcal{F}_1)$, so $f(\mathcal{F}_1)$ satisfies 2.2.1. ii. Since $\emptyset \neq \mathcal{F}_1$, $f(\mathcal{F}_1) \neq \emptyset$, so $f(\mathcal{F}_1)$ satisfies 2.2.1. *i*. Let $G_1 \in f(\mathcal{F}_1)$. The there exists $U \in \mathcal{F}_1$ such that $f(U) = G_1$. Let $G_1 \subset G_2 \subset Y$. Then, $U \subset f^{-1}(G_2)$, which by 2.2.1. iii implies $f^{-1}(G_2) \in \mathcal{F}_1$. Then, since f is **Surjective**, $G_2 = f\left(f^{-1}\left(G_2\right)\right) \in f(\mathcal{F}_1)$, so $f(\mathcal{F}_1)$ satisfies 2.2.1. iii. Finally, if $G_1, G_2 \in f(\mathcal{F}_1)$, then there are $K_1, K_2 \in \mathcal{F}_1$ with $f(K_i) = G_i$ for $i \in \{1, 2\}$. By 2.2.1. iv, $K_1 \cap K_2 \in \mathcal{F}_1$. Also, $f(K_1 \cap K_2) \subset f(K_1) \cap f(K_2)$, so by 2.2.1. iii, $f(K_1) \cap f(K_2) \in f(\mathcal{F}_1)$. Hence $f(\mathcal{F}_1)$ satisfies 2.2.1. *iv* and is therefore a **Filter** on Y. □

Proof of 2.2.18. ii. By 2.2.7. i, $\emptyset \neq \mathcal{B}_1$, so $\emptyset \neq f(\mathcal{B}_1)$, and thus $f(\mathcal{B}_1)$ satisfies 2.2.7. i. By 2.2.7. ii, $\emptyset \notin \mathcal{B}_1$, so $\emptyset \notin f(\mathcal{B}_1)$, implying $\mathcal{B}_1$ satisfies 2.2.7. ii. Finally, let $U_1, U_2 \in f(\mathcal{B}_1)$. Then there exists $V_i \in \mathcal{B}_1$ with $f(V_i) = U_i$. Then, since $\mathcal{B}_1$ satisfies 2.2.7. iii, there exists $V \in \mathcal{B}_1$ such that $V \subset V_1 \cap V_2$, so $f(V) \subset f(V_1) \cap f(V_2)$. Also, $f(V) \in f(\mathcal{B}_1)$, so $f(\mathcal{B}_1)$ satisfies 2.2.7. iii. Hence, $f(\mathcal{B}_1)$ is a **Filter Base** on Y. Now, if $V \in f(\mathcal{F}_1)$, then by definition, there exists $U \in \mathcal{F}_1$ with $f(U) = V$. By 2.2.10. ii, there exists a $b \in \mathcal{B}_1$ with $b \subset U$. This implies $f(b) \subset f(U) = V$, but $f(b) \in f(\mathcal{B}_1)$. Furthermore, since $\mathcal{B}_1 \subset \mathcal{F}_1$, $f(\mathcal{B}_1) \subset f(\mathcal{F}_1)$, so we can apply 2.2.10. ii to claim that $f(\mathcal{B}_1)$ is a **Filter Base** for $f(\mathcal{F}_1)$. □

Proof of 2.2.18. iii. If $\mathcal{F}_1 \subset \mathcal{F}_2$ then $f(\mathcal{F}_1) \subset f(\mathcal{F}_2)$. An invocation of 2.2.18. ii finishes the result. □

Proof of 2.2.18. iv. Let $\mathcal{G}$ denote the **Ultrafilter** for which $\mathcal{K}$ is an **Ultrafilter Base**. Let $U \subset Y$. Since f is **Surjective**,

$$X = f^{-1}(U) \cup \left(X \setminus f^{-1}(U)\right) = f^{-1}(U) \cup f^{-1}\left(f(X) \setminus U\right) = f^{-1}(U) \cup f^{-1}(Y \setminus U)$$

and by 2.2.2 i, we have $f^{-1}(U) \cup f^{-1}(Y \setminus U) \in \mathcal{G}$. Since $\mathcal{G}$ is an **Ultrafilter**, by 2.2.15. i, either $f^{-1}(U) \in \mathcal{G}$ or $f^{-1}(Y \setminus U) \in \mathcal{G}$. This implies either $U \in f(\mathcal{G})$ or $Y \setminus U \in f(\mathcal{G})$. Since U is arbitrary, by 2.2.15. iii, $f(\mathcal{G})$ is an **Ultrafilter** on X. Finally, by 2.2.18. ii, $f(\mathcal{K})$ is a **Filter Base** for $f(\mathcal{G})$, so the result holds. □

Proposition 2.2.19 (Inverse Filter Image)**.** Suppose the following

1. X and Y are nonempty sets.
2. $f : X \to Y$.
3. $\mathcal{B}$ is a **Filter Base** for a **Filter** $\mathcal{F}$ on Y.
4. $\mathcal{F}_{f(X)} = \{U \cap f(X) : U \in \mathcal{F}\}$.
5. $\mathcal{G}$ is a **Filter** on Y.
6. $\mathcal{G}_{f(X)} = \{U \cap f(X) : U \in \mathcal{G}\}$.

Then the following are true

(i) $f^{-1}(\mathcal{B})$ is a **Filter Base** on X if and only if $\emptyset \notin f^{-1}(\mathcal{B})$.

(ii) If $f^{-1}(\mathcal{B})$ is a **Filter Base** on X, then $f\left(f^{-1}(\mathcal{B})\right)$ is a **Filter Base** for a **Filter** on Y which is **Finer** than $\mathcal{F}$.

(iii) If $f\left(f^{-1}(\mathcal{B})\right)$ is a **Filter Base** for $\mathcal{G}$, then $\mathcal{G}_{f(X)} = \mathcal{F}_{f(X)}$.

Proof of 2.2.19. i. Necessity of $\emptyset \notin f^{-1}(\mathcal{B})$ is obvious by 2.2.7. ii. For sufficiency, suppose $\emptyset \notin f^{-1}(\mathcal{B})$. Then $f^{-1}(\mathcal{B})$ satisfies 2.2.7. ii trivially. Furthermore, by 2.2.7. *i*, $\mathcal{B} \neq \emptyset$, so $f^{-1}(\mathcal{B}) \neq \emptyset$, so $f^{-1}(\mathcal{B})$ satisfies 2.2.7. i. Finally, let $U_1, U_2 \in f^{-1}(\mathcal{B})$. Then there exist $V_1, V_2 \in \mathcal{B}$ such that $U_i = f^{-1}(V_i)$. By 2.2.7. iii, there exists $W \in \mathcal{B}$ such that $W \subset V_1 \cap V_2$, and $f^{-1}(W) \in f^{-1}(\mathcal{B})$. Clearly,

$$f^{-1}(W) \subset f^{-1}(V_1 \cap V_2) = f^{-1}(V_1) \cap f^{-1}(V_2) = U_1 \cap U_2$$

Hence $f^{-1}(\mathcal{B})$ satisfies 2.2.7. iii □

Proof of 2.2.19. ii. If $f^{-1}(\mathcal{B})$ is a **Filter Base** on X, then we can leverage 2.2.18. ii to claim that $f\left(f^{-1}(\mathcal{B})\right)$ is a **Filter Base** for a **Filter** on $f(X)$, and therefore also a **Filter Base** for a **Filter** on Y. In particular, $f\left(f^{-1}(\mathcal{B})\right)$ is a **Filter Base**. Furthermore, if $b \in \mathcal{B}$, then $f(f^{-1}(b)) \in f(f^{-1}(\mathcal{B}))$ and $f(f^{-1}(b)) = f(X) \cap b \subset b$, so by 2.2.10. iii, $f(f^{-1}(\mathcal{B}))$ is a **Filter Base** for a **Finer Filter** than $\mathcal{F}$. □

Proof of 2.2.19. iii. If $f\left(f^{-1}(\mathcal{B})\right)$ is a **Filter Base** for $\mathcal{G}$, then $\emptyset \notin f^{-1}(\mathcal{B})$. By 2.2.19. i, we then conclude $f^{-1}(\mathcal{B})$ is a **Filter Base** on X. That allows us to leverage 2.2.19. ii to conclude that $\mathcal{F} \leq \mathcal{G}$. Furthermore,

$$f\left(f^{-1}(\mathcal{B})\right) = \{f(X) \cap B : B \in \mathcal{B}\} \subset scF_{f(X)}$$

so that $\mathcal{G}_{f(X)} \leq \mathcal{F}_{f(X)}$. Since the inclusion goes both ways, equality holds. □

2.2.6 Filter Products

Definition 2.2.20 (Product Filter). Suppose the following

1. A is a nonempty set.
2. $\{X_\alpha\}_{\alpha \in A}$ is a collection of nonemptyset sets.
3. For each $\alpha \in A$, $\mathcal{F}_\alpha$ is a **Filter** on X_α.
4. For each $\gamma \in A$, $\pi_\gamma : \prod_{\alpha \in A} X_\alpha \to X_\gamma$ represents the **Projection Map**.

Then we define the **Filter** on $\prod_{\alpha \in A} X_\alpha$ **Generated By**

$$\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$$

to be the **Product Filter** on $\prod_{\alpha \in A} (X_\alpha, \mathcal{F}_\alpha)$.

Proposition 2.2.21. Suppose the following

1. A is a nonempty set.
2. $\{X_\alpha\}_{\alpha \in A}$ is a collection of nonemptyset sets.
3. For each $\alpha \in A$, $\mathcal{B}_\alpha$ is a **Filter Base** for a **Filter** $\mathcal{F}_\alpha$ on X_α.
4. For each $\alpha \in A$, $\mathcal{G}_\alpha$ is a **Subbasis** which **Generates** $\mathcal{F}_\alpha$.
5. For each $\gamma \in A$, $\pi_\gamma : \prod_{\alpha \in A} X_\alpha \to X_\gamma$ represents the $\gamma-$**Projection Map**.
6. $\mathcal{G} := \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{G}_\alpha)$.
7. $\mathcal{B}$ is the collection of finite intersections of elements of $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}_\alpha)$.
8. $\mathcal{F}$ is the **Product Filter** on $\prod_{\alpha \in A} (X_\alpha, \mathcal{F}_\alpha)$.

Then the following are true

(i) $\mathcal{F}$ is well defined.

(ii) $\mathcal{G}$ is a **Subbasis** for $\mathcal{F}$.

(iii) $\mathcal{B}$ is a **Filter Base** for $\mathcal{F}$.

(iv) $\mathcal{F}$ is the **Coarsest Filter** on $\prod_{\alpha \in A} X_\alpha$ such that for every $\alpha \in A$, $\pi_\alpha(\mathcal{F}) = \mathcal{F}_\alpha$.

Proof of 2.2.21. i. Since $A \neq \emptyset$, there exists a $\gamma \in A$. By 2.2.1. i, $\mathcal{F}_\gamma \neq \emptyset$. Hence

$$\emptyset \neq \pi_\gamma^{-1}(\mathcal{F}_\gamma) \subset \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$$

so that $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ satisfies 2.2.6. i. Additionally, let $\{U_i\}_{i=1}^n \subset \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$. Then for each $1 \leq i \leq n$, there exists an $\alpha_i \in A$ and a $V_i \in \mathcal{F}_{\alpha_i}$ such that $U_i = \pi_{\alpha_i}^{-1}(V_i)$. For $i, j \in \{1, \cdots, n\}$, define $i \equiv j$ if and only if $\alpha_i = \alpha_j$. It is clear that $\equiv$ is an **Equivalence Relation** on $\{1, \cdots, n\}$. Let $k \in \{1, \cdots, n\}/\equiv$. Then

$$\begin{aligned} \bigcap_{p \in k} U_p &= \bigcap_{p \in k} \pi_{\alpha_p}^{-1}(V_{\alpha_p}) \\ &= \bigcap_{p \in k} \pi_{\alpha_{min(k)}}^{-1}(V_p) \\ &= \pi_{\alpha_{min(k)}}^{-1}\left(\bigcap_{p \in k} V_p\right) \end{aligned}$$

By 2.2.2 ii, $\{V_p\}_{p \in k} \subset \mathcal{F}_{\alpha_{min(k)}}$, and also $\bigcap_{p \in k} V_p \in \mathcal{F}_{\alpha_{min(k)}}$, so by 2.2.1. ii, $\bigcap_{p \in k} V_p \neq \emptyset$. For each $k \in \{1, \cdots, n\}/\equiv$, let $x_k \in \pi_{\alpha_{min(k)}}^{-1}\left(\bigcap_{p \in k} V_p\right)$. Partition $A = A' \cup (A \setminus A')$ where $A' = \{\alpha_{min(k)} | k \in \{1, \cdots, n\}/\equiv\}$. For each $\alpha \in A \setminus A'$, Define $F_\alpha = X_\alpha$. For each $\alpha_{min(k)} \in A'$, define $F_{\alpha_{min(k)}} = \{x_k\}$. Then by the axiom of choice,

$$\emptyset \neq \prod_{\alpha \in A} F_\alpha \subset \bigcap_{i=1}^n U_i$$

so that $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ satisfies 2.2.6. ii. Hence $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ is a **Subbasis** on X, and the **Filter** that it **Generates** is $\mathcal{F}$. □

Proof of 2.2.21. iii. Since

$$\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}_\alpha) \subset \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$$

and by 2.2.21. i, $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}_\alpha)$ generates a **Filter** $\mathcal{F}_0$ on $\prod_{\alpha \in A} X_\alpha$. In Particular, the above equation implies $\mathcal{F}_0 \subset \mathcal{F}$. Let $\alpha \in A$. By 2.2.19. i, both $\pi_\alpha^{-1}(\mathcal{F}_\alpha)$ and $\pi_\alpha^{-1}(\mathcal{B}_\alpha)$ are **Filter Bases** for a **Filter** on the product set. Let $U \in \pi_\alpha^{-1}(\mathcal{F}_\alpha)$. Then, by 2.2.10. ii, there is a $b \in \mathcal{B}_\alpha$ such that $b \subset U$. Hence $f^{-1}(b) \subset f^{-1}(U)$. Which implies by 2.2.10. iii that $\pi_\alpha^{-1}(\mathcal{B}_\alpha)$ is a **Filter Base** for a **Finer Filter** than $\pi_\alpha^{-1}(\mathcal{F}_\alpha)$. This implies that any **Filter** containing $\pi_\alpha^{-1}(\mathcal{B}_\alpha)$ also contains $\pi_\alpha^{-1}(\mathcal{F}_\alpha)$. Since, for each $\alpha \in A$, $\pi_\alpha^{-1}(\mathcal{B}_\alpha) \subset \mathcal{F}_0$, for each $\alpha \in A$, $\pi_\alpha^{-1}(\mathcal{F}_\alpha) \subset \mathcal{F}_0$. Hence $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha) \subset \mathcal{F}_0$, so by 2.2.6, $\mathcal{F} \subset \mathcal{F}_0$. Hence $\mathcal{F} = \mathcal{F}_0$. Also, by 2.2.10. i, since $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}_\alpha)$ is by construction a **Subbasis** for $\mathcal{F}_0$, Since $\mathcal{B}$ is the collection of **Finite** intersections of elements of $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}_\alpha)$, $\mathcal{B}$ is a **Filter Base** for $\mathcal{F}_0 = \mathcal{F}$. □

Proof of 2.2.21. ii. For each $\alpha \in A$, let $\mathcal{K}_\alpha$ the collection of **Finite** intersections of elements of $\mathcal{G}_\alpha$. By 2.2.10. i, for each $\alpha \in A$, $\mathcal{K}_\alpha$ is a **Filter Base** for $\mathcal{F}_\alpha$. An application of 2.2.21. iii implies that $\mathcal{F}$ is the **Filter** generated by $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{K}_\alpha)$. But each

$$\pi_\alpha^{-1}(\mathcal{K}_\alpha) = \left\{ \bigcap_{i=1}^{n} U_\alpha | U_\alpha \in \pi_\alpha^{-1}(\mathcal{G}_\alpha) \right\}$$

So the collection of **Finite** intersections of elements of $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{K}_\alpha)$ equals the collection of finite intersections of elements of $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{G}_\alpha)$, they generate the same **Filter**. □

Proof of 2.2.21. iv. I first show that $\mathcal{F}$ has the described property, and then show that it is the **Coarsest** on the product set with the property. Fix $\gamma \in A$. Clearly, by **Surjectivity**, $\mathcal{F}_\gamma = \pi_\gamma\left(\pi_\gamma^{-1}(\mathcal{F}_\gamma)\right) \subset \pi_\gamma(\mathcal{F})$. Now, let $U \in \pi_\gamma(\mathcal{F})$. Then $U = \pi_\gamma(V)$ for some $V \in \mathcal{F}$. Since $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ is a **Subbasis** for $\mathcal{F}$, by 2.2.10. i and 2.2.10. ii, there is a colleciton $\{U_i\}_{i=1}^n \subset \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ such that $\bigcap_{i=1}^{n} U_i \subset V$. Let each $U_i \in \pi_{\alpha_i}^{-1}(\mathcal{F}_{\alpha_i})$. If, for all $i \in \{1, \cdots, n\}$, $\alpha_i \neq \gamma$ then $U = X_\gamma \in \mathcal{F}_\alpha$. Otherwise, let V' be the intersection of all $U_i's$ such that $\alpha_i = \gamma$. Then $V' \in \pi_\gamma^{-1}(\mathcal{F}_\gamma)$ and $V' \subset V$. Hence $\pi_\gamma(V') \in \mathcal{F}_\gamma$ and $\pi_\gamma(V') \subset \pi_\gamma(V) = U$, so by 2.2.1. iii, $U \in \mathcal{F}_\gamma$, implying $\pi_\gamma(\mathcal{F}) \subset \mathcal{F}_\gamma$. Since the inclusion goes both ways, $\pi_\gamma(\mathcal{F}) = \mathcal{F}_\gamma$. To see that $\mathcal{F}$ is the **Coarsest** with this filter, not that if $\mathcal{H}$ is a **Filter** on the product set such that $\pi_\gamma(\mathcal{H}) = \mathcal{F}_\gamma$, then

$$\pi_\gamma^{-1}(\mathcal{F}_\gamma) = \pi_\gamma^{-1}\left(\pi_\gamma(\mathcal{H})\right) \subset \mathcal{H}$$

Hence if this occurs for every γ then $\mathcal{F} \subset \mathcal{H}$, exactly what we are trying to show.

□

2.3 Topological Spaces

2.3.1 Open Sets, Closed Sets, and Neighborhoods

Definition 2.3.1 (Topological Space)**.** Let X be a nonempty set and let $\mathcal{T} \subset 2^X$ such that

(i) $X \in \mathcal{T}$.

(ii) $\emptyset \in \mathcal{T}$.

(iii) $\mathcal{T}$ is closed under arbitrary unions.

(iv) $\mathcal{T}$ is closed under finite intersections.

Then we call $\mathcal{T}$ a **Topology** on X and we call $(X, \mathcal{T})$ a **Topological Space**.

Definition 2.3.2 (**Discrete Topology**, **Indiscrete Topology**)**.** Let X be a nonempty set. We call $\{X, \emptyset\}$ the **Indiscrete Topology** on X and we call 2^X the **Discrete Topology** on X.

Definition 2.3.3 (**Open**, **Closed**)**.** Let $(X, \mathcal{T})$ be a **Topological Space**, and let $A \in \mathcal{T}$. Then we say that A is **Open** in $(X, \mathcal{T})$ and we say that $X \setminus A$ is **Closed** in $(X, \mathcal{T})$. When confusion is unlikely we say that A is **Open** in X or in $\mathcal{T}$, or that A is **Open** with no qualification.

Definition 2.3.4 (**Compact**)**.** We say that a **Topological Space** $(X, \mathcal{T})$ is **Compact** if every **Open Cover** for X has a **Finite Subcover**.

Definition 2.3.5 (REMOVE)**.**

Definition 2.3.6 (**Coarse**, **Fine**)**.** Let X be a nonempty set. Let $\mathcal{T}_1, \mathcal{T}_2$ be **Topologies** on X such that $\mathcal{T}_1 \subset \mathcal{T}_2$. In this case, we say that $\mathcal{T}_1$ is more **Coarse** than $\mathcal{T}_2$, we say that $\mathcal{T}_1$ is **Coarser** than $\mathcal{T}_2$, we say that $\mathcal{T}_2$ is more **Fine** than $\mathcal{T}_1$, we say that $\mathcal{T}_2$ is **Finer** than $\mathcal{T}_1$, we write $\mathcal{T}_1 \leq_{TopFine(X)} \mathcal{T}_2$, and we write $\mathcal{T}_2 \leq_{TopCoarse(X)} \mathcal{T}_1$.

Let K denote the collection of all **Topologies** on X. Let $A \subset K$. If one exists, an **Upper Bound** of A with respect to $\leq_{TopFine(X)}$ which is a member of A is called the **Finest** topology in A. If one exists, a **Upper Bound** of A with respect to $\leq_{TopCoarse(X)}$ which is a member of A is called the **Coarsest** topology in A. **Fineness** defines a **Partial Order** on the set of **Topologies** of X. The intersection of any collection of **Topologies** on X is a **Topology** on X, so that $\leq_{TopCoarse(X)}$ is a **Direction** on X.

Definition 2.3.7 (**Neighborhood**, **Neighborhood Filter**)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. Let $A \subset B \subset C \subset X$ and let B be **Open** in $(X, \mathcal{T})$. Then we call C a **Neighborhood** of A in $(X, \mathcal{T})$. If $x \in X$, then we call a **Neighborhood** of $\{x\}$ a **Neighborhood** of x. We denote the collection of all **Neighborhoods** of $x \in X$ with $\mathcal{U}_{\mathcal{T}}(x)$. By 2.3.8, $\mathcal{U}_{\mathcal{T}}(x)$ is a **Filter** on X, and se we call this the **Neighborhood Filter** of $\mathcal{T}$ at x or simply the **Neighborhood Filter** of x.

Proposition 2.3.8 (**Neighborhood Filter** is a **Filter**)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. For each $x \in X$, let $\mathcal{U}_{\mathcal{T}}(x)$ denote the **Neighborhood Filter** of x. The following are true

(i) $\mathcal{U}_{\mathcal{T}}$ (x) is a **Filter** on X.

(ii) For each $U \in \mathcal{U}_{\mathcal{T}}(x)$, $x \in U$.

(iii) Let $x, y \in X$. Then, if $U \in \mathcal{U}_{\mathcal{T}}(x)$, then there exists $V \in \mathcal{U}_{\mathcal{T}}(x)$ such that for each $y \in V$, $U \in \mathcal{U}_{\mathcal{T}}(y)$.

Proof of 2.3.8 i. Clearly $x \in X \subset X \subset X \subset X \in \mathcal{T}$, so $X \in \mathcal{U}_{\mathcal{T}}(x)$. Thus $\mathcal{U}_{\mathcal{T}}(x)$ satisfies 2.2.1. i. Also, since $x \notin \emptyset$, $\emptyset \notin \mathcal{U}_{\mathcal{T}}(x)$. Hence $\mathcal{U}_{\mathcal{T}}(x)$ satisfies 2.2.1. ii. If $\{G_1, G_2\} \subset \mathcal{U}_{\mathcal{T}}(x)$, then there are **Open** U_i with $x \in U_i \subset G_i$. For these U_i, $x \in U_1 \cap U_2 \subset U_1 \cap U_2 \subset G_1 \cap G_2$ and $U_1 \cap U_2 \in \mathcal{T}$. Hence, $\mathcal{U}_{\mathcal{T}}$ satifies 2.2.1. iv. It is obvious that $\mathcal{U}_{\mathcal{T}}(x)$ satisfies 2.2.1. iii. ☐

Proof of 2.3.8 ii. Painfully Obvious ☐

Proof of 2.3.8 iii. Let $U \in \mathcal{U}_{\mathcal{T}}(x)$. Then there exists **Open** V with $x \in V \subset U$. Since V is **Open**, $V \in \mathcal{U}_{\mathcal{T}}(x)$. Let $y \in V$. Then, $y \in V \subset V \subset U$, so $U \in \mathcal{U}_{\mathcal{T}}(y)$. Hence 2.3.8 *iii* is satisfied. ☐

Proposition 2.3.9 (**Topology** from **Neighborhood Filters**)**.** Let X be a nonempty set. For each $x \in X$, let $\mathcal{U}(x) \subset 2^X$ such that each $\mathcal{U}(x)$ satisfies 2.2.1. i, 2.2.1. iii, 2.2.1. iv, and 2.3.8 ii. Further assume that the collection $\{\mathcal{U}(x) : x \in X\}$ satisfies 2.3.8 iii. Then there exists a unique topology $\mathcal{T}$ on X such that for each $x \in X$, $\mathcal{U}(x)$ is the **Neighborhood Filter** for $\mathcal{T}$ at x.

Proof. Define $\mathcal{T} = \{U \subset X : (\forall x \in U)(U \in \mathcal{U}_x)\}$. I first show that $\mathcal{T}$ is a **Topology** on X. Clearly, $\emptyset \in \mathcal{T}$. By 2.2.1. iii, for each $x \in X$, there exists $B \subset X$ such that $B \in \mathcal{U}_x$. Hence, by 2.2.1. iii, $X \in \mathcal{U}_x$. Hence, $X \in \mathcal{T}$. Let $U_1, U_2 \in \mathcal{T}$. Let $x \in U_1 \cap U_2$. Then $x \in U_1$‘ and $x \in U_2$, so $U_1 \in \mathcal{U}_x$ and $U_2 \in \mathcal{U}_x$. By 2.2.1. iv, $U_1 \cap U_2 \in \mathcal{U}_x$. Hence $U_1 \cap U_2 \in \mathcal{T}$. By 2.1.13. i, $\mathcal{T}$ is closed under finite intersections. Let $\{U_\alpha\}_{\alpha \in A} \subset \mathcal{T}$. Let $x \in \bigcup_{\alpha \in A} U_\alpha$. Then for some $\beta \in A$, $x \in U_\beta$. Hence $U_\beta \in \mathcal{U}_x$. Since $U_\beta \subset \bigcup_{\alpha \in A} U_\alpha$, by 2.2.1. iii, $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$. Hence $\mathcal{T}$ is a **Topology** on X.

I now show that $\mathcal{T}$ has the required properties. Let $x \in X$. I must show $\mathcal{U}_x = \mathcal{U}_{\mathcal{T}}(x)$. Let $N \in \mathcal{U}_{\mathcal{T}}(x)$. Then there exists $U \in \mathcal{T}$ with $x \in U \subset N$. Since $x \in U \in \mathcal{T}$, $U \in \mathcal{U}_x$. Since $U \subset N$, $N \in \mathcal{U}_x$. Hence $\mathcal{U}_{\mathcal{T}}(x) \subset \mathcal{U}_x$. Now let $U \in \mathcal{U}_x$. Then by 2.3.8 iii there exists $V \in \mathcal{U}_x$ such that for every $y \in V$, $U \in \mathcal{U}_y$. Let $y \in V$. Then $\exists \tilde{U} \in \mathcal{U}_y$ such that for every $z \in \tilde{U}$, $V \in \mathcal{U}_{\mathcal{T}}(z)$. Since $y \in \tilde{U}$, $V \in \mathcal{U}_y$. Since $y \in V$ was arbitrary, $V \in \mathcal{T}$. Hence $x \in V \subset U \in \mathcal{U}_{\mathcal{T}}(x)$. Thus $\mathcal{U}_x \subset \mathcal{U}_{\mathcal{T}}(x)$.

I now show that the $\mathcal{T}$ is the unique **Topology** with this property. Let $\mathcal{T}_1$ be another **Topology** on X such that for each $x \in X$, $\mathcal{U}_{\mathcal{T}}(x) = \mathcal{U}_x = \mathcal{U}_{\mathcal{T}_1}(x)$. Let $U \in \mathcal{T}$. Let $x \in U$. Then $\exists U_x \in \mathcal{U}_{\mathcal{T}}(x) = \mathcal{U}_{\mathcal{T}_1}(x)$ such that $U_x \subset U$. Since $x \in U_x \in \mathcal{U}_{\mathcal{T}_1}(x)$, there exists $x \in V_x \in \mathcal{T}_1$ such that $V_x \subset U_x$. Hence $U = \bigcup_{x \in U} V_x \in \mathcal{T}_1$. Hence $\mathcal{T} \subset \mathcal{T}_1$. The proof for the other direction is identical. Hence uniqueness holds. □

Definition 2.3.10 (Relation of Equal **Neighborhood Filters**)**.** Let $(Z, \mathcal{T}_Z)$ be a **Topological Space** Define the relation $\cong \subset Z \times Z$ by setting, for $x, y \in Z$,

$$x \cong y \iff \mathcal{U}_{\mathcal{T}_Z}(x) = \mathcal{U}_{\mathcal{T}_Z}(y)$$

We call $\cong$ the **Relation Of Equal Neighborhood Filters**$(Z, \mathcal{T}_Z)$.

Proposition 2.3.11 (**Relation Of Equal Neighborhood Filters** is **Equivalence Relation**)**.** The **Relation Of Equal Neighborhood Filters** $\cong$ on a **Topological Space** $(Z, \mathcal{T}_Z)$ forms an **Equivalence Relation** on Z.

Proof. Let $x \in (Z, \mathcal{T}_Z)$. Then $\mathcal{U}_{\mathcal{T}_Z}(x) = \mathcal{U}_{\mathcal{T}_Z}(x)$, so $x \cong x$. Thus $\cong$ is **Reflexive**. Let $x, y \in (Z, \mathcal{T}_Z)$. Suppose $x \cong y$. Then $\mathcal{U}_{\mathcal{T}_Z}(x) = \mathcal{U}_{\mathcal{T}_Z}(y)$, so trivially $\mathcal{U}_{\mathcal{T}_Z}(y) = \mathcal{U}_{\mathcal{T}_Z}(x)$, and thus $y \cong x$. Hence, $\cong$ is **Symmetric** Let $x, y, z \in (Z, \mathcal{T}_Z)$. Let $x \cong y$ and $y \cong z$. Then, $\mathcal{U}_{\mathcal{T}_Z}(x) = \mathcal{U}_{\mathcal{T}_Z}(y) = \mathcal{U}_{\mathcal{T}_Z}(z)$ so that $x \cong z$. Thus $\cong$ is **Transitive**. Since $\cong$ is **Reflexive**, **Symmetric**, and **Transitive**, it is an **Equivalence Relation**. □

Definition 2.3.12 (**Accumulation Point**, **Closure**, **Interior**, **Boundary**)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. Let $A \subset X$. We define the following.

1. $A' = \{x \in X : (\forall U \in \mathcal{U}_{\mathcal{T}}(A))((U \setminus A) \cap \{x\} \neq \emptyset)\}$

2. $\overline{A} = A \cup A'$
3. $\partial(A) = \overline{A} \cap \overline{X \setminus A}$
4. $\mathring{A} = A \setminus \overline{X \setminus A}$

We call an element of A' an **Accumulation Point** of A. We call $\overline{A}$ the **Closure** of A. We call $\mathring{A}$ the **Interior** of A. We call $\partial(A)$ the **Boundary** of A.

Definition 2.3.13 (**SubBasis**)**.** Let X be a nonempty set. Let $B \subset 2^X$. We denote the **Coarsest Topology** on X containing B with $\mathcal{T}_X(B)$. We say that B is a **SubBasis** for $\mathcal{T}_X(B)$ and we call $\mathcal{T}_X(B)$ the **Topology** on X **Generated By** B.

Proposition 2.3.14 (Characterization Of Generated Topology)**.** Let X be a nonempty set. Let $F \subset 2^X$. Let K be the collection of finite intersections of elements of F. Let $\mathcal{K}$ be the collection of unions of elements of K. Define

$$\mathcal{T}_{Prop} = \mathcal{K} \cup \{X, \emptyset\}$$

Then $\mathcal{T}_X(F) = \mathcal{T}_{Prop}$.

Proof. We first show that $\mathcal{T}_{Prop}$ is a **Topology** on X. To prove closure under arbitrary unions, Let $B \neq \emptyset$ and $\{B_\beta\}_{\beta \in B} \subset \mathcal{T}_{Prop}$ Then for each $\beta \in B$, we can find A_β such that for each $\alpha_\beta \in A_\beta$, there is an $N_{\alpha_\beta} \in \mathbb{N}$ such that for each $i \in \{1, \cdots, N_{\alpha_\beta}\}$, $U_{i,\alpha_\beta} \in F$ and

$$B_\beta = \bigcup_{\alpha \in A_\beta} \bigcap_{i=1}^{N_{\alpha_\beta}} U_{i,\alpha_\beta}$$

Hence, we can write

$$\begin{aligned}\bigcup_{\beta \in B} B_\beta &= \bigcup_{\beta \in B} \bigcup_{\alpha_\beta \in A_\beta} \bigcap_{i=1}^{N_{\alpha_\beta}} U_{i,\alpha_\beta} \\ &= \bigcup_{\alpha_\beta \in \bigcup\limits_{\beta \in B} A_\beta} \bigcap_{i=1}^{N_{\alpha_\beta}} U_{i,\alpha_\beta} \in \mathcal{T}_{Prop}\end{aligned}$$

To prove closure under finite intersections, let $N \in \mathbb{N}$ and $\{B_j\}_{j=1}^N \subset \mathcal{T}_{Prop}$. Then for each $j \in \{1, \cdots, N\}$, there is an A_j such that for each $\alpha_j \in A_j$, there is an $N_{\alpha_j} \in \mathbb{N}$ such that for each $i \in \{1, \cdots, N_{\alpha_j}\}$, $U_{i,\alpha_j} \in F$ and

$$\begin{aligned}\bigcap_{j=1}^{N} B_j &= \bigcap_{j=1}^{N} \bigcup_{\alpha_j \in A_j} \bigcap_{i=1}^{N_{\alpha_j}} U_{\alpha_j,i} \\ &= \bigcup_{\{\alpha_j\}_{j=1}^N \in \prod\limits_{j \in \{1,\cdots,N\}} A_j} \left(\bigcap_{j=1}^{N} \bigcap_{i=1}^{N_{\alpha_j}} U_{\alpha_j,i} \right) \in \mathcal{T}_{Prop}\end{aligned}$$

By construction, $X \in \mathcal{T}_{Prop}$ and $\emptyset \in \mathcal{T}_{Prop}$, so $\mathcal{T}_{Prop}$ is in fact a **Topology** on X. By taking the union over the intersection of a single element, we have $F \subset \mathcal{T}_{Prop}$, so that $\mathcal{T}_X(F) \subset \mathcal{T}_{Prop}$. Furthermore, $\mathcal{T}_X(F)$ is closed under finite intersections and arbitrary unions so that it must contain $\mathcal{T}_{Prop}$. Hence, equality holds.

□

Definition 2.3.15 (**Basis**)**.** Let $(X, \mathcal{T})$ be a **Topological Space** and let $B \subset \mathcal{T}$ such that each element of $\mathcal{T}$ can be written as a union of elements of B. Then we call B a **Basis** for $\mathcal{T}$.

Proposition 2.3.16. Let $(X, \mathcal{T})$ be a **Topological Space** and let $\mathcal{G} \subset \mathcal{T}$ such that $\emptyset \in \mathcal{G}$. The following conditions are equivalent

(i) For every **Open** U, for every $x \in U$, there exists an $G_x \in \mathcal{G}$ such that $x \in G_x \subset U$.

(ii) $\mathcal{G}$ is a **Basis** for $\mathcal{T}$.

2.3.16 i $\implies$ *2.3.16 ii* . Let $U \in \mathcal{T}$. Then we can write $U = \bigcup_{x \in U} G_x$, implying that $\mathcal{G}$ is a **Basis**. □

2.3.16 i $\impliedby$ *2.3.16 ii* . Let U be **Open**, then since $\mathcal{G}$ is a **Basis**, there is a $\{G_\alpha\}_{\alpha \in A} \subset \mathcal{G}$ such that $U = \bigcup_{\alpha \in A} G_\alpha$. Hence, if $x \in U$, then $x \in G_\alpha$ for some $\alpha \in A$, and obviously $G_\alpha \subset U$, so 2.3.16 ii holds and we're done. □

Proposition 2.3.17 (Basis Of Generated Topology)**.** Let X be a nonempty set. Let $B \subset 2^X$ Then B is a **Basis** for $\mathcal{T}_X(B)$ if and only if each of the following hold

(i) For each $x \in X$, there exists $U \in B$ such that $x \in U$.

(ii) $\emptyset \in B$.

(iii) For each $U, V \in B$ and for each $x \in U \cap V$, there is a $W \in B$ satisfying $x \in W \subset U \cap V$.

Proof of $\impliedby$. Suppose 2.3.17 i, 2.3.17 ii, and 2.3.17 iii hold. Let $\mathcal{K}$ denote the collection of arbitrary unions of **Finite** intersections of elements of B. By 2.3.17 i, $X \in \mathcal{K}$. By 2.3.17 ii, $\emptyset \in \mathcal{K}$. Hence, by 2.3.14, $\mathcal{T}_X(B) = \mathcal{K}$. Hence, to show that B is a **Basis** for $\mathcal{T}_X(B)$, it is sufficient to show that B any **Finite** intersection of elements of B can be written as a union of elements of B. I prove in the case of binary intersections, but the inductive step has a similar proof to that of the binary case. Hence, let $U, V \in B$ with $U \cap V \neq \emptyset$. Then for each $x \in U \cap V$, by 2.3.17 iii, there exists a $W_x \in B$ such that $x \in W_x \subset U \cap V$. Hence, we can write

$$U \cap V \subset \bigcup_{x \in U \cap V} W_x \subset U \cap V$$

showing that **Finite** intersections of elements of B can be written as unions of elements of B. Hence B is a **Basis** for $\mathcal{T}_X(B)$. □

Proof of $\implies$. Let B be a **Basis** for $\mathcal{T}_X(B)$, Since $X \in \mathcal{T}_X(B)$, 2.3.17 i holds. Since $\emptyset \in \mathcal{T}_X(B)$, 2.3.17 ii holds. Let $U, V \in B$. Let $x \in U \cap V$. Then $U \cap V \in \mathcal{T}_X(B)$. Since B is a **Basis** for $\mathcal{T}_X(B)$, there exists $\{B_\alpha\}_{\alpha \in A} \subset B$ such that $U \cap V = \bigcup_{\alpha \in A} B_\alpha$. Hence, for some $\gamma \in A$, $x \in B_\gamma \subset U \cap V$. Hence 2.3.17 iii holds. □

Definition 2.3.18 (**Fundamental System Of Neighborhoods**)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. Let $x \in X$. Let $\mathcal{U}_{\mathcal{T}}(x)$ denote the **Neighborhood Filter** of $\mathcal{T}$ at x. We say that $\mathcal{K}$ is a **Fundamental System Of Neighborhoods** for X at x if

(i) $\mathcal{K} \subset \mathcal{U}_{\mathcal{T}}(x)$.

(ii) For each $U \in \mathcal{U}_{\mathcal{T}}(x)$, there exists $V \in \mathcal{K}$ such that $V \subset U$.

It is clear that $\mathcal{U}_{\mathcal{T}}(x)$ is a **Fundamental System Of Neighborhoods** for X at x.

Definition 2.3.19 (**Neighborhood Basis**)**.** Let $(X, \mathcal{T})$ be a **Topological Space** and let $x \in X$. Let $F \subset \mathcal{T}$ such that **Nested** $(U, \mathcal{U}_{\mathcal{T}}(x))$. Further, let $x \in G$ for each $G \in F$. Then we call F a **Neighborhood Basis** for $\mathcal{T}$ at x.

Proposition 2.3.20. REMOVE

2.3.2 Continuous Functions

Definition 2.3.21 (**Continuity At** a point)**.** Let $(X, \mathcal{T}_X)$ be a **Topological Space**. Let $(Y, \mathcal{T}_Y)$ be a **Topological Space**. Let $f : X \to Y$. Let $x_0 \in X$. Let $\mathcal{U}_{\mathcal{T}_X}(x_0)$ be the **Neighborhood Filter** of $\mathcal{T}_X$ at x_0. Let $\mathcal{U}_{\mathcal{T}_Y}(f(x_0))$ be the **Neighborhood Filter** of $\mathcal{T}_Y$ at $f(x_0)$. We say that f is **Continuous At** x_0, and we say that f posesses **Continuity At** x_0, if **Nested**$(\mathcal{U}_{\mathcal{T}_X}(x_0), f^{-1}\left(\mathcal{U}_{\mathcal{T}_Y}(f(x_0))\right))$ holds.

Proposition 2.3.22. Let $(X, \mathcal{T}_X)$ be a **Topological Space**. Let $(Y, \mathcal{T}_Y)$ be a **Topological Space**. Let $A \subset X$. Let $x_0 \in \overline{A}$. Let $f : X \to Y$ be **Continuous At** x_0. Then $f(x_0) \in \overline{f(A)}$.

Proof. If $x_0 \in A$, then $f(x_0) \in f(A) \subset \overline{f(A)}$. Otherwise, $x_0 \in A'$ Let V be a **Neighborhood** of $f(x_0)$. Then $f^{-1}(V) \in f^{-1}\left(\mathcal{U}_{\mathcal{T}}(f(x_0))\right)$. By **Continuity At** x_0 there exists $U \in \mathcal{U}_{\mathcal{T}}(x_0)$ such that $U \subset f^{-1}(V)$. Since $x_0 \in A'$, $U \cap A \neq \emptyset$, so $A \cap f^{-1}(V) \neq \emptyset$. Thus $V \cap f(A) \neq \emptyset$. Since $V \in \mathcal{U}_{\mathcal{T}}(f(x_0))$ was arbitrary, $f(x_0) \in \overline{A}$.

□

Definition 2.3.23 (Function **Continuity**)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be **Topological Spaces**. We say that a function $f : X \to Y$ is **Continuous** and that it exhibits **Continuity** with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ if $f^{-1}(\mathcal{T}_Y) \subset \mathcal{T}_X$. We may make the **Topologies** explicit by writing $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, in which case we just say that f is **Continuous** or that f posesses **Continuity**.

Proposition 2.3.24 (global **Continuity** iff **Continuity At** each point)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be **Topological Spaces**. Let $f : X \to Y$. The following are equivalent:

(i) f is **Continuous**.

(ii) f is **Continuous At** each point in X.

(iii) For each $A \subset X$, $f\left(\overline{A}\right) \subset \overline{f(A)}$

(iv) For each **Closed** $B \subset Y$, $f^{-1}(B)$ is **Closed** in X.

Proof of 2.3.24. *i* $\implies$ 2.3.24. *ii*. Let $x_0 \in X$. Let $\mathcal{U}$ denote the **Neighborhood Filter** of $\mathcal{T}_Y$ at $f(x_0)$. Let $\mathcal{V}$ denote the **Neighborhood Filter** of $\mathcal{T}_X$ at x_0. Let $U \in \mathcal{U}$. Then by 2.3.7 there exists **Open** $\tilde{U} \in \mathcal{U}$ such that $f(x_0) \in \tilde{U} \subset U$. By the assumption 2.3.24. i, $f^{-1}\left(\tilde{U}\right) \in \mathcal{T}_X$. Also, since $f(x_0) \in \tilde{U}$, $x_0 \in f^{-1}\left(\tilde{U}\right)$. Hence, $f^{-1}\left(\tilde{U}\right) \in \mathcal{V}$. Since $f^{-1}\left(\tilde{U}\right) \subset f^{-1}(U)$, we conclude **Nested**$(\mathcal{V}, f^{-1}(\mathcal{U}))$. Hence f is **Continuous At** x_0. Since $x_0 \in X$ was arbitrary, we are done. □

Proof of 2.3.24. *ii* $\implies$ 2.3.24. *iii*. This result is a direct application of 2.3.22. □

Proof of 2.3.24. *iii* $\implies$ 2.3.24. *iv*. Let K be **Closed** in Y. Then $\overline{K} = K$, so

$$f\left(\overline{f^{-1}(K)}\right) \subset \overline{f\left(f^{-1}(K)\right)} \subset \overline{K} = K$$

This implies $\overline{f^{-1}(K)} \subset f^{-1}(K)$, so $f^{-1}(K)$ is **Closed**. □

Proof of 2.3.24. *iv* $\implies$ 2.3.24. *i*. Let $U \subset Y$ be **Open**. Then $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is **Closed** so $f^{-1}(U)$ is **Open**. □

Definition 2.3.25 (**Homeomorphism**)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be **Topological Spaces**. Let $f : X \to Y$ such be a **Continuous Bijection** such that $f^{-1} : Y \to X$ is also **Continuous**. Then we say that f is a **Homeomorphism** from X to Y and we say that X and Y are **Homeomorphic** and we say that f operates **Homeomorphically**.

Definition 2.3.26 (**Weak Topology**)**.** Let X be a nonempty set. Let A be a nonempty set. For each $\alpha \in A$, let $(Y_\alpha, \mathcal{T}_\alpha)$ be a **Topological Space**, and let $\phi_\alpha : X \to (Y_\alpha, \mathcal{T}_\alpha)$. Let $\mathcal{T}$ be the **Coarsest** possible **Topology** on X such that for each $\alpha \in A$, $\phi_\alpha : (X, \mathcal{T}) \to (Y_\alpha, \mathcal{T}_\alpha)$ is **Continuous**. We call $\mathcal{T}$ the **Weak Topology** on X induced by $\{\phi_\alpha\}_{\alpha \in A}$.

Definition 2.3.27 (**Inductive Topology**)**.** Let X be a nonempty set. Let A be a nonempty set. For each $\alpha \in A$, let $(Y_\alpha, \mathcal{T}_\alpha)$ be a **Topological Space**. Furthermore, for each $\alpha \in A$, let $\phi_\alpha : (Y, \mathcal{T}_\alpha) \to X$. Let $\mathcal{T}$ be the **Finest** topology on X for which each ϕ_α is **Continuous**. We call $\mathcal{T}$ the **Inductive Topology** on X induced by $\{\phi_\alpha\}_{\alpha \in A}$.

Definition 2.3.28 (**Open**)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be **Topological Spaces**. We say that $f : X \to Y$ is **Open** if $f(U)$ is **Open** in $(Y, \mathcal{T}_Y)$ for every **Open** $U \in (X, \mathcal{T}_X)$.

Definition 2.3.29 (**Closed**)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be **Topological Spaces**. We say that $f : X \to Y$ is **Closed** if $f(K)$ is **Closed** in $(Y, \mathcal{T}_Y)$ for every **Closed** $K \in (X, \mathcal{T}_X)$.

2.3.3 Subspaces And Quotient Spaces

Definition 2.3.30 (**Subspace Topology**)**.** Let $(X, \mathcal{T}_X)$ be a **Topological Space**. Let $Y \subset X$. The **Subspace Topology** of Y relative to $(X, \mathcal{T}_X)$ is defined to be the **Weak Topology** on Y generated by the **Insertion Function** $f : Y \to X$. We denote the **Subspace Topology** with $\mathcal{T}_Y$. We call $(Y, \mathcal{T}_Y)$ the **Subspace Topological Space**. Unless otherwise specified, when referring to a subset of a **Topological Space**, we consider that subset as being a **Topological Space** which is endowed with the **Subspace Topology**, and when we say that a subset of a **Topological Space** has a particular (Topological) property which has thus far only been defined for a **Topological Space**, we mean that the **Subspace Topological Space** has that property.

Definition 2.3.31 (Quotient Space Topology)**.** Let $(Z, \mathcal{T}_Z)$ be a topological space. Let $\cong$ be the **Relation Of Equal Neighborhood Filters**$(Z, \mathcal{T}_Z)$. Let T be the **Quotient Map** of Z under the relation $\cong$. Define $\mathcal{T}_{Z/\cong}$ by

$$\mathcal{T}_{Z/\cong} = \left\{ \bigcup_{x \in U} \{T(x)\} \in 2^{Z/\cong} : U \in \mathcal{T}_Z \right\}$$

By 2.3.32, $\mathcal{T}_{Z/\cong}$ is a topology on $Z/\cong$. We call $\mathcal{T}_{Z/\cong}$ the **Quotient Topology** and we call $\left(Z/\cong, \mathcal{T}_{Z/\cong}\right)$ the **Quotient Topological Space** of $(Z, \mathcal{T}_Z)$.

Proposition 2.3.32 (Quotient Space Topology)**.** Let $(Z, \mathcal{T}_Z)$ be a **Topological Space** with **Quotient Topological Space** $\left(Z/\cong, \mathcal{T}_{Z/\cong}\right)$ and **Quotient Map** T. Then the following are true.

(i) $\mathcal{T}_{Z/\cong}$ is a **Topology** on $Z/\cong$.

(ii) $T : (Z, \mathcal{T}_Z) \to (Z/\cong, \mathcal{T}_{Z/\cong})$ is **Continuous**.

(iii) If U is **Open** (**Closed**) in $(Z, \mathcal{T}_Z)$, then $T(U)$ and $T(Z \setminus U)$ **Partition** $Z/\cong$.

(iv) If U is **Open** in $(Z, \mathcal{T}_Z)$, then U is a **Fiber**.

(v) If K is **Closed** in $(Z, \mathcal{T}_Z)$, then K is a **Fiber**.

(vi) $T : (Z, \mathcal{T}_Z) \to (Z/\cong, \mathcal{T}_{Z/\cong})$ is **Open**.

(vii) $T : (Z, \mathcal{T}_Z) \to (Z/\cong, \mathcal{T}_{Z/\cong})$ is **Closed**.

(viii) $(Z, \mathcal{T}_Z)$ is a **Compact** space if and only if $(Z/\cong, \mathcal{T}_{Z/\cong})$ is a **Compact** space.

(ix) If $\mathcal{B}$ is a **Basis** for $\mathcal{T}_z$, then $\{T(U) : U \in \mathcal{B}\}$ is a **Basis** for $\mathcal{T}_{Z/\cong}$.

(x) If T is **Injective**, then it is a **Homeomorphism**.

Proof of 2.3.32. i. Since $\emptyset \in \mathcal{T}_Z$, we have

$$\emptyset = \bigcup_{x \in \emptyset} \{Tx\} \in \mathcal{T}_{Z/\cong}$$

Since $Z \in \mathcal{T}_Z$, and by 2.1.45,

$$Z/\cong = \bigcup_{x \in Z} \{[x]\} = \bigcup_{x \in Z} \{T(x)\} \in \mathcal{T}_{Z/\cong}$$

Let $\{U_\alpha : \alpha \in A\} \subset \mathcal{T}_{Z/\cong}$. For each $\alpha \in A$, there exists $B_\alpha \in \mathcal{T}_Z$ such that we have

$$U_\alpha = \bigcup_{x \in B_\alpha} \{Tx\}$$

Since $\bigcup_{\alpha \in A} B_\alpha \in \mathcal{T}_Z$, we have

$$\bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} \bigcup_{x \in U_\alpha} \{T(x)\} = \bigcup_{x \in \bigcup\limits_{\alpha \in A} B_\alpha} \{T(x)\} \in \mathcal{T}_{Z/\cong}$$

Let $\{U_i\}_{i=1}^n \subset \mathcal{T}_{Z/\cong}$. For each $i \in \{1, ..., n\}$, there exists $B_i \in \mathcal{T}_Z$ such that

$$U_i = \bigcup_{x \in B_i} \{T(x)\}$$

Suppose

$$[x_0] \in \bigcap_{i=1}^n \bigcup_{x \in B_i} \{T(x)\}$$

Then for each $i \in \{1, ..., n\}$, there is a $y_i \in B_i$ such that $y_i \cong x_0$. Since each B_i is **Open**, the definition of $\cong$ implies that $x_0 \in B_i$ for every i. Hence,

$$x_0 \in \bigcap_{i=1}^n B_i$$

Implying

$$[x_0] \in \bigcup_{x \in \bigcap\limits_{i=1}^n B_i} \{[x]\}$$

Hence,

$$\bigcap_{i=1}^n \bigcup_{x \in B_i} \{T(x)\} \subset \bigcup_{x \in \bigcap\limits_{i=1}^n B_i} \{[x]\}$$

Furthermore, since the reverse inclusion is obvious, and since $\bigcap_{i=1}^n B_i \in \mathcal{T}_Z$, we have

$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n \bigcup_{x \in B_i} \{T(x)\} = \bigcup_{x \in \bigcap\limits_{i=1}^n B_i} \{T(x)\} \in \mathcal{T}_{Z/\cong}$$

□

Proof of 2.3.32. ii. Let $V \in \mathcal{T}_{Z/\cong}$. Let $x_0 \in T^{-1}(V)$. Then $[x_0] \in V$. By definition, there is a $U \in \mathcal{T}_Z$ such that

$$T(U) \subset \bigcup_{x \in U} \{T(x)\} = V$$

Hence there is a $y_0 \in U$ such that

$$[x_0] \in T(y_0) = \{[y_0]\}$$

Therefore, $x \cong y$. The definition of the **Relation Of Equal Neighborhood Filters** implies $\mathcal{U}(x_0) = \mathcal{U}(y_0)$. Hence, $x_0 \in U \subset T^{-1}(V)$. □

Proof of 2.3.32. iii. Let K be **Closed** in $(Z, \mathcal{T}_Z)$. Then to each point $x_0 \in Z \setminus K$ there corresponds a **Neighborhood** U_{x_0} of x_0 **Disjoint** from K. Hence $y_0 \ncong x_0$ for any $y_0 \in K$, Hence $T(K)$ is **Disjoint** from $T(Z \setminus K)$. This fact, paired with 2.1.47, implies $T(Z \setminus K)$ and T(K) is a **Partition** of $Z/\cong$. □

Proof of 2.3.32. iv. Let $U \in \mathcal{T}_Z$. The nontrivial direction to prove is $T^{-1}(T(U)) \subset U$. Let $y \in T^{-1}(T(U))$. Then $[y] = Ty \in T(U)$. Hence, $[y] = T(x) = [x]$ for some $x \in U$. Since $y \cong x$ and $x \in U \in \mathcal{U}_{\mathcal{T}_Z}(x)$, we have $U \in \mathcal{U}_{\mathcal{T}_Z}(y)$. Hence $y \in U$. Since y was arbitrary, $T^{-1}(T(U)) \subset U$, and equality is obvious because the other direction of inclusion is trivial. □

Proof of 2.3.32. v. Let K be **Closed** in $(Z, \mathcal{T}_Z)$. 2.3.32. iii implies $Z/\cong$ is partitioned by $T(K)$ and $T(Z \setminus K)$. Hence, $T(K) = T(Z) \setminus T(Z \setminus K)$

By 2.3.32. iv,

$$\begin{aligned}
T^{-1}(T(K)) &= T^{-1}(T(Z) \setminus T(Z \setminus K)) \\
&= T^{-1}(Z/\cong \setminus T(Z \setminus K)) \\
&= T^{-1}(Z/\cong) \setminus T^{-1}(T(Z \setminus K)) \\
&= Z \setminus (Z \setminus K) \\
&= K
\end{aligned}$$

□

Proof of 2.3.32. vi. Let $U \in \mathcal{T}_Z$. Then by definition of the **Quotient Topology**

$$TU = \bigcup_{x \in U} \{T(x)\} \in \mathcal{T}_{Z/\cong}$$

which is open. □

Proof of 2.3.32. vii. Let K be **Closed** in $(Z, \mathcal{T}_Z)$. Then $Z \setminus K \in \mathcal{T}_Z$. By 2.3.32. iii and 2.3.32. v, we know $T(K) = Z/\cong \setminus T(Z \setminus K)$ and also that $T(Z \setminus K) \in \mathcal{T}_{Z/\cong}$. Hence $T(K)$ is **Closed** in $(Z/\cong, \mathcal{T}_{Z/\cong})$. □

Proof of 2.3.32. viii. Let $(Z, \mathcal{T}_Z)$ be **Compact**. Let $\{U_\alpha\}_{\alpha \in A}$ be an **Open** covering of $(Z/\cong, \mathcal{T}_{Z/\cong})$. Then $\{T^{-1}(U_\alpha) : \alpha \in A\}$ is an **Open** covering of $(Z, \mathcal{T}_Z)$. **Compactness** of $(Z, \mathcal{T}_Z)$ guarantees the existence of a finite subcovering $\{T^{-1}(U_{\alpha_i})\}_{i=1}^n$. Hence $\{U_{\alpha_i}\}_{i=1}^n = \{TT^{-1}(U_{\alpha_i})\}_{i=1}^n$ is an **Open Cover** of $(Z/\cong, \mathcal{T}_{Z/\cong})$. Hence, the **Compactness** of $(Z/\cong, \mathcal{T}_{Z/\cong})$ is verified.

Now, suppose $(Z/\cong, \mathcal{T}_{Z/\cong})$ is **Compact**. Let $\{V_\beta : \beta \in B\}$ be an **Open Cover** of $(Z, \mathcal{T}_Z)$. Since T is an **Open** mapping, $\{T(V_\beta) : \beta \in B\}$ is an **Open Cover** of $(Z/\cong, \mathcal{T}_{Z/\cong})$ which by **Compactness** has a **Finite Subcover** $\{T(V_{\beta_i})\}_{i=1}^n$. By 2.3.32. iv, $\{V_{\beta_i}\}_{i=1}^n = \{T^{-1}(T(V_{\beta_i}))\}_{i=1}^n$ is then an **Open Subcover** of $(Z, \mathcal{T}_Z)$. □

Proof of 2.3.32. ix. Let $\mathcal{B}$ be a basis for $\mathcal{T}_z$ and let $V \in \mathcal{T}_{Z/\cong}$. Then $T^{-1}(Z) \in \mathcal{T}_Z$, and so there is a subcollection $\{U_\alpha\}_{\alpha \in A} \subset \mathcal{B}$ such that $T^{-1}(V) = \bigcup_{\alpha \in A} U_\alpha$. Hence,

$$\begin{aligned} V &= T(T^{-1}(V)) \\ &= T\left(\bigcup_{\alpha \in A} U_\alpha\right) \\ &= \bigcup_{\alpha \in A} T(U_\alpha) \end{aligned}$$

□

Proof of 2.3.32. x. If T is **Injective**, then since it is **Continuous** by 2.3.32. ii, open by 2.3.32. vi, and **Surjective** by 2.1.47, it is a **Homeomorphism**. □

2.3.4 Product Spaces

Definition 2.3.33 (**Product Topology**)**.** Let A be a nonempty set. For each $\alpha \in A$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a **Topological Space**. For each $\alpha \in A$, let $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \to (X_\alpha, \mathcal{T}_\alpha)$ denote the $\alpha-$**Projection Map**. We call the **Weak Topology** on $\prod_{\alpha \in A} X_\alpha$ induced by $\{\pi_\alpha\}_{\alpha \in A}$ the **Product Topology**.

2.3.5 Convergence of Filters

Remark 2.3.34. By 2.2.10. v, a **Filter** $\mathcal{F}$ is a **Filter Base** for itself, and any **Filter Base** is a **Subbasis** for the **Filter** it generates. For this reason, when stating definitions or results about **Filters**, we will prefer to assume that an object is a **Subbasis** if possible, only resorting to assuming something is a **Filter Base** or **Filter** when necessary, and we will prefer to show that things are **Filters**.

Definition 2.3.35 (**Convergence**)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. Let $\mathcal{F}$ be a **Filter** on X. Let $\mathcal{B}$ be a **Filter Base** for $\mathcal{F}$. Let $\mathcal{U}_\mathcal{T}$ (x) denote the **Neighborhood Filter** of $\mathcal{T}$ at x. Let $\mathcal{F}$ be **Finer** than $\mathcal{U}_\mathcal{T}$ (x). Then we say the following:

1. x is a **Limit** of $\mathcal{B}$.
2. $\mathcal{B}$ **Converges** to x.
3. $\mathcal{B}$ is **Convergent** to x.
4. $\mathcal{B}$ posesses **Convergence** to x.

Proposition 2.3.36 (**Convergence** Facts)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. Let $x \in X$. Let $\mathcal{U}_\mathcal{T}$ (x) denote the **Neighborhood Filter** for $\mathcal{T}$ at x. Let $\mathcal{F}$ be a **Filter** on X. Let $\mathcal{B}$ be a **Filter Base** for $\mathcal{F}$. The following are true.

(i) x is a **Limit** of $\mathcal{F}$ if and only if x is a **Limit** of $\mathcal{B}$.

(ii) x is a **Limit** of $\mathcal{B}$ if and only if, for some **Fundamental System Of Neighborhoods** $\mathcal{U}$ of x, **Nested**$(\mathcal{B},\mathcal{U})$ holds.

(iii) x is a **Limit** of $\mathcal{B}$ if and only if, for every **Fundamental System Of Neighborhoods** $\mathcal{U}$ of x, **Nested**$(\mathcal{B},\mathcal{U})$ holds.

(iv) If x is a **Limit** of $\mathcal{F}$ and $\mathcal{G}$ is **Finer** than $\mathcal{F}$ then x is a **Limit** of $\mathcal{G}$.

(v) If x is a **Limit** of $\mathcal{F}$ in $(X,\mathcal{T})$, and $\mathcal{T}_1$ is a **Coarser Topology** on X, then x is a **Limit** of $\mathcal{F}$ in $(X,\mathcal{T}_1)$.

(vi) Let $\{\mathcal{F}_\alpha\}_{\alpha\in A}$ be a collection of **Filters** on X each of which have x as a **Limit**. Then $\bigcap_{\alpha\in A}\mathcal{F}_\alpha$ has x as a **Limit**.

(vii) x is a **Limit** of $\mathcal{F}$ if and only if x is a **Limit** of every **Ultrafilter** which is **Finer** than $\mathcal{F}$.

Proof of 2.3.36. i. This is a direct result of 2.2.10. v. □

Proof of 2.3.36. ii. ($\Longrightarrow$) Let x be a **Limit** for $\mathcal{B}$. By 2.3.35, $\mathcal{U}_{\mathcal{T}}(x)\subset\mathcal{F}$. Hence, **Nested**$(\mathcal{F},\mathcal{U}_{\mathcal{T}}(x))$ holds. Since **Nested** is **Transitive** and by 2.2.10. ii, **Nested** $(\mathcal{B},\mathcal{U}_{\mathcal{T}}(x))$ holds. Since $\mathcal{U}_{\mathcal{T}}(x)$ is a **Fundamental System Of Neighborhoods** for x, this direction of the proof is complete.

($\Longleftarrow$) Let $\tilde{\mathcal{U}}$ be a **Fundamental System Of Neighborhoods** for x such that **Nested** $\left(\mathcal{B},\tilde{\mathcal{U}}\right)$ holds. Then by 2.3.18 ii, **Nested** $\left(\tilde{\mathcal{U}},\mathcal{U}_{\mathcal{T}}(x)\right)$. Since **Nested** is **Transitive** we conclude **Nested** $(\mathcal{B},\mathcal{U}_{\mathcal{T}}(x))$. Since $\mathcal{B}\subset\mathcal{F}$, **Nested** $(\mathcal{F},\mathcal{U}_{\mathcal{T}}(x))$ holds. By 2.2.10. iii, we conclude $\mathcal{U}_{\mathcal{T}}(x)\le\mathcal{F}$, and so x is a **Limit** of $\mathcal{B}$ by 2.3.35. □

Proof of 2.3.36. iii. ($\Longleftarrow$) This is a trivial consequence of the ($\Longleftarrow$) direction of 2.3.36. ii.

($\Longrightarrow$) Let x be a **Limit** of $\mathcal{B}$. Then $\mathcal{U}_{\mathcal{T}}(x)\subset\mathcal{F}$. Let $\mathcal{U}$ be a **Fundamental System Of Neighborhoods** For X at x. By 2.3.18 i, $\mathcal{U}\subset\mathcal{U}_{\mathcal{T}}(x)$, so $\mathcal{U}\subset\mathcal{U}_{\mathcal{T}}(x)\subset\mathcal{F}$. Hence **Nested** $(\mathcal{F},\mathcal{U})$ holds. Since **Nested** $(\mathcal{B},\mathcal{F})$ holds and since **Nested** is a **Transitive**, **Nested** $(\mathcal{B},\mathcal{U})$ holds. □

Proof of 2.3.36. iv. This is clear since **Filter Fineness** is a **Partial Ordering** and is therefore **Transitive**. □

Proof of 2.3.36. v. If $\mathcal{T}_1$ is **Coarser** than $\mathcal{T}$, then $\mathcal{U}_{\mathcal{T}_1}(x)\subset\mathcal{U}_{\mathcal{T}}(x)$. If x is a **Limit** of $\mathcal{F}$ in $(X,\mathcal{T})$, then $\mathcal{U}_{\mathcal{T}}(x)\subset\mathcal{F}$. Hence, $\mathcal{U}_{\mathcal{T}_1}(x)\subset\mathcal{F}$, so x is a **Limit** of $\mathcal{F}$ in $(X,\mathcal{T}_1)$. □

Proof of 2.3.36. vi. Let $U\in\mathcal{U}_{\mathcal{T}}(x)$. Then for each $\alpha\in A$, $U\in\mathcal{F}_\alpha$. Hence $U\in\bigcap_{\alpha\in A}\mathcal{F}_\alpha$. Hence $\mathcal{U}_{\mathcal{T}}(x)\subset\bigcap_{\alpha\in A}\mathcal{F}_\alpha$. so x is a **Limit** of $\bigcap_{\alpha\in A}\mathcal{F}_\alpha$. □

Proof of 2.3.36. vii. ($\Longrightarrow$) This direction is a direct consequence of 2.3.36. iv.

($\Longleftarrow$) By 2.2.15. iv, $\mathcal{F}$ is the intersection of all **Ultrafilters** on X which contain $\mathcal{F}$. By 2.3.36. vi, if these all have x as a **Limit**, then $\mathcal{F}$ must also have x as a **Limit**. □

Definition 2.3.37 (**Cluster Point**)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. Let $x \in X$. Let $\mathcal{B}$ be a **Filter Base** in X. We say that x is a **Cluster Point** of $\mathcal{B}$ if

$$x \in \bigcap_{U \in \mathcal{B}} \overline{U}$$

Proposition 2.3.38 (**Cluster Point**)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. Let $\mathcal{B}$ and $\mathcal{D}$ be **Equivalent Filter Bases**. Let $\mathcal{F}$ be the **Filter Generated By** $\mathcal{B}$. Let $\mathcal{E}$ be a **Filter Base** on X which generates a **Filter** which is **Coarser** than $\mathcal{F}$. Let $\mathcal{G}$ be an **Ultrafilter** in X. Let $x \in X$. The following are true.

(i) The set of **Cluster Points** of $\mathcal{B}$ is **Closed** in $(X, \mathcal{T})$.

(ii) If x is a **Limit** of $\mathcal{B}$, then x is a **Cluster Point** of $\mathcal{B}$.

(iii) If x is a **Cluster Point** of $\mathcal{B}$, then x is a **Cluster Point** of $\mathcal{E}$.

(iv) x is a **Cluster Point** of $\mathcal{B}$ if and only if x is a **Cluster Point** of $\mathcal{D}$.

(v) x is a **Cluster Point** of $\mathcal{B}$ if and only if x is a **Cluster Point** of $\mathcal{F}$, viewed as a **Filter Base**.

(vi) Let $\mathcal{U}$ be a **Fundamental System Of Neighborhoods** for $\mathcal{T}$ at x. Then x is a **Cluster Point** for $\mathcal{B}$ if and only if for each $U \in \mathcal{U}$ and for each $B \in \mathcal{B}$, $U \cap B \neq \emptyset$.

(vii) x is a **Cluster Point** for $\mathcal{F}$ if and only if there is a **Filter Finer** than $\mathcal{F}$ for which x is a **Limit**.

(viii) x is a **Cluster Point** of $\mathcal{G}$ if and only if x is a **Limit** of $\mathcal{G}$.

(ix) If x is a **Cluster Point** for $\mathcal{B}$ in $(X, \mathcal{T})$, and if $\mathcal{T}_1$ is a **Coarser Topology** on X than $\mathcal{T}$, then x is a **Cluster Point** for $\mathcal{B}$ in $(X, \mathcal{T}_1)$.

Proof of 2.3.38. i. By 2.3.37, the set of **Cluster Points** of $\mathcal{B}$ is an intersection of **Closed** sets and is therfore itself **Closed**. □

Proof of 2.3.38. iii. Since $\mathcal{B}$ generates a **Filter** which is **Finer** than that generated by $\mathcal{E}$, **Nested** $(\mathcal{B}, \mathcal{E})$ holds. Let x be a **Cluster Point** of $\mathcal{B}$. Let $E \in \mathcal{E}$. Then there exists $B \in \mathcal{B}$ such that $B \subset E$. Since x is a **Cluster Point** of $\mathcal{B}$, $x \in \overline{B} \subset \overline{E}$. Since $E \in \mathcal{E}$ was arbitrary, $x \in \bigcap_{E \in \mathcal{E}} \overline{E}$ and is therefore a **Cluster Point** of $\mathcal{E}$. □

Proof of 2.3.38. iv. This is a direct result of two applications of 2.3.38. iii. □

Proof of 2.3.38. v. Since $\mathcal{F}$, viewed as a **Filter Base** is **Equivalent** to $\mathcal{B}$, this result follows from a direct application of 2.3.38. iv. □

Proof of 2.3.38. vi. ($\Longrightarrow$) Let $U \in \mathcal{U}$. Then $U \in \mathcal{U}_{\mathcal{T}}(x)$. Let x be a **Cluster Point** for $\mathcal{B}$. Let $B \in \mathcal{B}$. Then $x \in \overline{B}$. Since $U \in \mathcal{U}_{\mathcal{T}}(x)$, $U \cap B \neq \emptyset$.

($\Longleftarrow$) Let $B \in \mathcal{B}$. Let $U \in \mathcal{U}_{\mathcal{T}}(x)$. Since $\mathcal{U}$ is a **Fundamental System Of Neighborhoods** for X at x, there exists $V \in \mathcal{U}$ with $x \in V \subset U$. By assumption, $V \cap B \neq \emptyset$. Hence $U \cap B \neq \emptyset$. Since $U \in \mathcal{U}_{\mathcal{T}}(x)$ was arbitrary, $x \in \overline{B}$. Since $B \in \mathcal{B}$ was arbitrary, x is a **Cluster Point** of x. □

Proof of 2.3.38. ii. Let x be a **Limit** of $\mathcal{B}$. Let $B \in \mathcal{B}$. Let $U \in \mathcal{U}_{\mathcal{T}}(x)$. By 2.3.36. iii, there exists $V \in \mathcal{B}$ such that $V \subset U$. By 2.2.7. iii, there exists a nonempty $W \in \mathcal{B}$, such that $W \subset B \cap V$. Since $W \cap U \neq \emptyset$, $B \cap U \neq \emptyset$. Since $\mathcal{U}_{\mathcal{T}}(x)$ is a **Fundamental System Of Neighborhoods** for X at x, since $U \in \mathcal{U}_{\mathcal{T}}(x)$ was arbitrary, and since $B \in \mathcal{B}$ was arbitrary, we can apply 2.3.38. vi to conclude x is a **Cluster Point** of $\mathcal{B}$. □

Proof of 2.3.38. vii. ($\Longrightarrow$) Define $A = \mathcal{F} \cup \mathcal{U}_{\mathcal{T}}(x)$. Let x be a **Cluster Point** of $\mathcal{F}$. Then, since $\mathcal{U}_{\mathcal{T}}(x)$ is a **Fundamental System Of Neighborhoods** for X at x, by 2.3.38. vi and 2.1.13. i, finite intersections of elements of A are nonempty. Hence, by 2.2.4, there is a **Filter** $\mathcal{G}$ on X which contains A. By construction, $\mathcal{G}$ is **Finer** than $\mathcal{F}$ and $\mathcal{U}_{\mathcal{T}}(x) \leq \mathcal{G}$, so x is a **Limit** of $\mathcal{G}$.

($\Longleftarrow$) Let $\mathcal{G}$ be a **Finer** than $\mathcal{F}$ and let x be a **Limit** for $\mathcal{G}$. By 2.3.38. ii, x is a **Cluster Point** for $\mathcal{G}$. By 2.3.38. iii, since $\mathcal{F}$ is **Coarser** than $\mathcal{G}$, x is a **Cluster Point** of $\mathcal{F}$. □

Proof of 2.3.38. viii. ($\Longleftarrow$) This direction is a direct consequence of 2.3.38. ii.

($\Longrightarrow$) this direction is a direct consequence of 2.3.38. vii combined with the fact that an **Ultrafilter** is not properly contained in any **Filter**. □

Proof of 2.3.38. ix. If $\mathcal{T}_1$ is **Coarser** than $\mathcal{T}$, than for any $B \in \mathcal{B}$, $\overline{B}_{\mathcal{T}} \subset \overline{B}_{\mathcal{T}_1}$, implying

$$\bigcap_{B \in \mathcal{B}} \overline{B}_{\mathcal{T}} \subset \bigcap_{B \in \mathcal{B}} \overline{B}_{\mathcal{T}_1}$$

□

Definition 2.3.39 (**Limit**)**.** Let X be a nonempty set and Y be a **Topological Space**. Let $\mathcal{B}$ be a **Filter Base** in X and let $f : X \to Y$. Let $y \in Y$. If y is a **Limit** of $f(\mathcal{B})$, then we say that y is a **Limit** of f with respect to $\mathcal{B}$ and we write $y \in \lim_{\mathcal{B}} f$ or we may write $y \in \lim\limits_{x,\mathcal{B}} f(x)$. If $\lim_{\mathcal{B}} f$ contains just a single point then we will, as an abuse of notation, write $y = \lim_{\mathcal{B}} f$ and $y = \lim\limits_{x,\mathcal{B}} f(x)$. If y is a **Cluster Point** $f(\mathcal{B})$ then we say that y is a **Cluster Point** of f with respect to $\mathcal{B}$.

Definition 2.3.40 (**Limit**)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. Let $\sigma = \{x_\alpha\}_{\alpha \in A} \subset X$ be a **Net** in X. We say that $x \in X$ is a **Limit** of σ if x is a **Limit** of σ with respect to the **Section Filter** on A. We say that $x \in X$ is a **Cluster Point** of σ if x is a **Cluster Point** of σ with respect to the **Section Filter** on A.

Proposition 2.3.41 (**Limit**)**.** Let X be a nonempty set. Let $(Y, \mathcal{T})$ be a **Topological Space**. Let $f : X \to Y$. Let $\mathcal{F}$ be a **Filter** in X. Let $\mathcal{B}$ be a **Filter Base** for $\mathcal{F}$. Let $y \in Y$. The following are true

(i) y is a **Limit** of f with respect to $\mathcal{B}$ if and only if, for each **Neighborhood** V of y, there exists $M \in \mathcal{B}$ with $f(M) \subset V$.

(ii) y is a **Limit** of f with respect to $\mathcal{B}$ if and only if for each **Neighborhood** V of y, $f^{-1}(V) \in \mathcal{F}$.

(iii) y is a **Cluster Point** of f with respect to $\mathcal{B}$ if and only if for each **Neighborhood** V of y and each $M \in \mathcal{B}$, $f(M) \cap V \neq \emptyset$.

(iv) If y is a **Limit** of f with respect to $\mathcal{B}$ in $(Y, \mathcal{T})$ and $\mathcal{T}_1$ is a **Coarser Topology** on Y than $\mathcal{T}$, then y is a **Limit** of f with respect to $\mathcal{B}$ in $(Y, \mathcal{T}_1)$.

(v) If y is a **Cluster Point** of f with respect to $\mathcal{B}$ in $(Y, \mathcal{T})$ and $\mathcal{T}_1$ is a **Coarser Topology** on Y than $\mathcal{T}$, then y is a **Cluster Point** of f with respect to $\mathcal{B}$ in $(Y, \mathcal{T}_1)$.

(vi) If y is a **Limit** of f with respect to $\mathcal{B}$ and $\mathcal{F}_1$ is a **Finer Filter** on X than $\mathcal{F}$, then y is a **Limit** of f with respect to $\mathcal{F}_1$.

(vii) If y is a **Cluster Point** of f with respect to $\mathcal{B}$ and $\mathcal{F}_1$ is a **Coarser Filter** on X than $\mathcal{F}$, then y is a **Cluster Point** of f with respect to $\mathcal{F}_1$.

(viii) y is a **Cluster Point** of f with repsect to $\mathcal{B}$ if and only if there is a **Filter** $\mathcal{G}$ on X which is **Finer** than $\mathcal{F}$ such that y is a **Limit** of f with respect to $\mathcal{G}$.

(ix) The set of **Cluster Points** of f with respect to $\mathcal{B}$ is **Closed** in Y and may be empty.

Proof of 2.3.41. i. This is obvious from an application of 2.3.39 and 2.3.36. ii. ☐

Proof of 2.3.41. ii. This is a direct application of 2.3.41. i. ☐

Proof of 2.3.41. iii. This result comes from an application of 2.3.38. vi. ☐

Proof of 2.3.41. iv. This is a direct consequence of 2.3.36. v. ☐

Proof of 2.3.41. v. This is a direct consequence of 2.3.38. ix. ☐

Proof of 2.3.41. vi. This is a direct consequence of 2.3.36. iv. ☐

Proof of 2.3.41. vii. This is a direct consequence of 2.3.38. iii. ☐

Proof of 2.3.41. viii. This result falls from a direct application of 2.3.38. vii paired with 2.2.19. ☐

Proof of 2.3.41. ix. This is obvious. ☐

Definition 2.3.42 (**Limit** of a **Function** at a point)**.** Let $(X, \mathcal{T}_X)$ be a **Topological Space**. Let $(Y, \mathcal{T}_Y)$ be a **Topological Space**. Let $f : X \to Y$. Let $a \in X$. Let $\mathcal{B}$ be the **Neighborhood Filter** of X at a. Let y be a **Limit** of f with respect to $\mathcal{B}$. Then instead of the standard notation

$$y \in \lim_{x, \mathcal{B}} f(x)$$

we write

$$y \in \lim_{x \to a} f(x)$$

and we say that y is a **Limit** of f at a. If y is a **Cluster Point** of f with respect to $\mathcal{B}$, then we say that y is a **Cluster Point** of f at a.

Proposition 2.3.43. For $i \in \{0, 1\}$, let $(X_i, \mathcal{T}_i)$ be **Topological Spaces**. Let $f : X_0 \to X_1$. Let $x_0 \in X_0$. The following are true.

(i) f is **Continuous At** x_0 if and only if $f(x_0) \in \lim_{x\to x_0} f(x)$.

(ii) If f is **Continuous** at x_0, then for every **Filter Base** $\mathcal{B}$ in X which **Converges** to x_0, we have $f(\mathcal{B})$ **Converges** to $f(x_0)$.

(iii) If, for every **Ultrafilter** $\mathcal{U}$ on X which **Converges** to a , we have $f(\mathcal{U})$ converges to $f(a)$, then f is **Continuous** at a.

(iv) Let X_2 be a nonempty set. Let $\mathcal{F}$ be a **Filter** on X_2. Let $f_1 : X_2 \to X_0$. Let $x_0 \in \lim_{x,\mathcal{F}} g(x)$. If f is **Continuous At** x_0, then $f(x_0) \in \lim_{x,\mathcal{F}} f \circ g(x)$.

Proof of 2.3.43 i. ($\Longrightarrow$) Let f be **Continuous** at x_0. Then **Nested** $(\mathcal{U}_{\mathcal{T}_0}(x_0), f^{-1}(\mathcal{U}_{\mathcal{T}_1}(f(x_0))))$ holds. Let $V \in \mathcal{U}_{\mathcal{T}_1}(f(x_0))$. Then there exists $U \in \mathcal{U}_{\mathcal{T}_0}(x_0)$ such that $U \subset f^{-1}(V)$. Hence $f(U) \subset V$. Since $\mathcal{U}_{\mathcal{T}_1}(f(x_0))$ was arbitrary, **Nested** $(f(\mathcal{U}_{\mathcal{T}_0}(x_0)), \mathcal{U}_{\mathcal{T}_1}(f(x_0)))$ holds, so $f(x_0) \in \lim_{x\to x_0} f(x)$.

($\Longleftarrow$) Let $f(x_0) \in \lim_{x\to x_0} f(x_0)$. Then **Nested** $(f(\mathcal{U}_{\mathcal{T}_0}(x_0)), \mathcal{U}_{\mathcal{T}_1}(f(x_0)))$ holds. Let $U \in f^{-1}(\mathcal{U}_{\mathcal{T}_1}(f(x_0)))$. Then there is a $\tilde{U} \in \mathcal{U}_{\mathcal{T}_1}(f(x_0))$ such that $U = f^{-1}\left(\tilde{U}\right)$. By nesting there exists $V \in f(\mathcal{U}_{\mathcal{T}_0}(x_0))$ such that $V \subset \tilde{U}$. By definition there exists $\tilde{V} \in \mathcal{U}_{\mathcal{T}_0}(x_0)$ such that $V = f\left(\tilde{V}\right)$. Then we have $f\left(\tilde{V}\right) = VB \subset \tilde{U}$, so $\tilde{V} \subset f^{-1}\left(\tilde{U}\right) = U$. Since $U \in f^{-1}(\mathcal{U}_{\mathcal{T}_1}(f(x_0)))$ was arbitrary and $V \in \mathcal{U}_{\mathcal{T}_0}(x_0)$, we conclude **Nested** $(\mathcal{U}_{\mathcal{T}_0}(x_0), f^{-1}(\mathcal{U}_{\mathcal{T}_1}(f(x_0))))$ holds. Thus f is **Continuous** at x_0. □

Proof of 2.3.43 ii. Let f be **Continuous** at x_0. Then **Nested** $(f(\mathcal{U}_{\mathcal{T}_0}(x_0)), \mathcal{U}_{\mathcal{T}_1}(f(x_0)))$ holds. Let $\mathcal{B}$ be a **Filter Base** in X. Let x_0 be a **Limit** of $\mathcal{B}$. Then **Nested** $(\mathcal{B}, \mathcal{U}_{\mathcal{T}_0}(x_0))$ holds. Hence **Nested** $(f(\mathcal{B}), f(\mathcal{U}_{\mathcal{T}_0}(x_0)))$ holds. By **Transitivity**, we conclude

$$\textbf{Nested}\,(f(\mathcal{B}), \mathcal{U}_{\mathcal{T}_1}(f(x_0)))$$

so x_0 is a **Limit** of $f(\mathcal{B})$. □

Proof of 2.3.43 iii. Suppose f is not **Continuous** at x_0. Then it is not the case that **Nested** $(f(\mathcal{U}_{\mathcal{T}_0}(x_0)), \mathcal{U}_{\mathcal{T}_1}(f(x_0)))$ holds. Hence, for some $V \in \mathcal{U}_{\mathcal{T}_1}(f(x_0))$, and for each $U \in \mathcal{U}_{\mathcal{T}_0}(x_0)$, $f(U) \cap (X_1 \setminus V) \neq \emptyset$. Hence, by 2.2.4, $\{f^{-1}(X_1 \setminus V)\} \cup \mathcal{U}_{\mathcal{T}_0}(x_0)$ is contianed within a **Filter** $\mathcal{G}$ on X_1. Since $\mathcal{U}_{\mathcal{T}_0 0}(x_0) \subset \mathcal{G}$, x_0 is a **Limit** of $\mathcal{G}$. Furthermore, since $V \in \mathcal{U}_{\mathcal{T}_1}(f(x_0))$, and since $f(f^{-1}(X_1 \setminus V)) \in f(\mathcal{G})$, by 2.3.38. vi, since $V \cap f(f^{-1}(X_1 \setminus V)) \subset V \cap (X_1 \setminus V) = \emptyset$, we conclude $f(x_0)$ is not a **Cluster Point** of $f(\mathcal{G})$. By 2.3.38. vii, there is no **Filter** on X_1 which is **Finer** than $f(\mathcal{G})$ which has $f(x_0)$ as a **Limit**. By 2.2.13, there is an **Ultrafilter** $\tilde{\mathcal{G}}$ on X containing $\mathcal{G}$. By 2.3.36. iv, Since x_0 is a **Limit** of $\mathcal{G}$, x_0 is a **Limit** of $\tilde{\mathcal{G}}$. But by the above argumentation, $f(x_0)$ is not a **Limit** of $f\left(\tilde{\mathcal{G}}\right)$. Thus we have, under the assumption that f is not **Continuous** at x_0, constructed an **Ultrafilter** $\tilde{\mathcal{G}}$ such that x_0 is a **Limit** of $\tilde{\mathcal{G}}$ but $f(x_0)$ is not a **Limit** of $f\left(\tilde{\mathcal{G}}\right)$. □

Proof of 2.3.43 iv. This falls from several applications of 2.3.43 ii Let $U \in \mathcal{U}_{\mathcal{T}_1}(f(x_0))$. Since f is **Continuous** at x_0, there exists $V \in \mathcal{U}_{\mathcal{T}_0}(x_0)$ such that $f(V) \subset U$. Since x_0 is a **Limit** of $g(\mathcal{F})$, there exists $F \in \mathcal{F}$ such that $g(F) \subset V$. Hence $(f \circ g)(F) \subset U$. Since $U \in \mathcal{U}_{\mathcal{T}_1}(f(x_0))$ was arbitrary, we conclude **Nested** $((f \circ g)(\mathcal{F}), \mathcal{U}_{\mathcal{T}_1}(f(x_0)))$. Hence $f(x_0) \in \lim\limits_{x,\mathcal{F}} (f \circ g)(x)$. □

2.3.6 Separation Axioms

Definition 2.3.44 (Separation Axioms)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. We define the following.

(i) We say X is **Hausdorff**, or **T2** if distinct points in X have **Disjoint Neighborhoods**.

(ii) We say that X is **Pseudo-Hausdorff** if, for each $x_0, x_1 \in X$, either x_0 and x_1 have **Disjoint Neighborhoods** or $\mathcal{U}_{\mathcal{T}}(x_0) = \mathcal{U}_{\mathcal{T}}(x_1)$.

Remark 2.3.45. It is clear that any **Hausdorff** space is **Pseudo-Hausdorff**, and that a space is **Pseudo-Hausdorff** if and only if its quotient under the **Relation Of Equal Neighborhood Filters** is **Hausdorff**.

Proposition 2.3.46 (**Hausdorff** Characterizations)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. The following are equivalent.

(i) X is **Hausdorff**.

(ii) For all $x \in X$, if $\mathcal{U}_{\mathcal{T}}(x)$ is the **Neighborhood Filter** of X at x, then

$$\bigcap_{U \in \mathcal{U}_{\mathcal{T}}(x)} \overline{U} = \{x\}$$

(iii) $\Delta(X)$ is **Closed** in $X \times X$.

(iv) For any index set A, $\Delta_A(X)$ is **Closed** in $\prod\limits_{\alpha \in A} X$.

(v) A **Filter** $\mathcal{F}$ in X has at most one **Limit**.

(vi) If a **Filter** $\mathcal{F}$ in X **Converges**, say $\mathcal{F} \to x$, then x is that **Filter**'s only **Cluster Point**.

Proof of 2.3.46 i $\implies$ 2.3.46 ii. Clearly, $\{x\} \subset \bigcap\limits_{U \in \mathcal{U}_{\mathcal{T}}(x)} \overline{U}$. For the other direction, let $y \neq x$. Then, since X is **Hausdorff**, there exist $U_y \in \mathcal{U}_{\mathcal{T}}(y)$ and $U_x \in \mathcal{U}_{\mathcal{T}}(x)$ such that $U_y \cap U_x = \emptyset$. Hence, $y \notin \overline{U_x}$. Hence, $y \notin \bigcap\limits_{U \in \mathcal{U}_{\mathcal{T}}(x)} \overline{U}$. □

Proof of 2.3.46 ii $\implies$ 2.3.46 iii. Let $(x, y) \in (X \times X) \setminus \Delta(X)$. Then $y \neq x$, so $y \notin \bigcap\limits_{U \in \mathcal{U}_{\mathcal{T}}(x)} \overline{U}$. Hence, there exists $U_x \in \mathcal{U}_{\mathcal{T}}(x)$ such that $y \notin \overline{U_x}$. Hence, there exists $U_y \in \mathcal{U}_{\mathcal{T}}(y)$ such that $U_y \cap U_x = \emptyset$. Then $U_x \cap U_y$ is a **Neighborhood** of (x, y) in $X \times X$ which is **Disjoint** from $\Delta(X)$. Thus $U_x \times U_y \subset (X \times X) \setminus \Delta(X)$, so that $(X \times X) \setminus \Delta(X)$ is **Open**, □

Proof of 2.3.46 iii $\implies$ 2.3.46 iv. Let $\{x_\alpha\}_{\alpha\in A} \in \left(\prod_{\alpha\in A} X\right) \setminus \Delta_A(X)$. Then there exists $\alpha, \beta \in A$ such that $x_\alpha \neq x_\beta$. This implies $(x_\alpha, x_\beta) \in (X \times X) \setminus \Delta(X)$. Hence there is an $X \times X$-**Neighborhood** U of (x, y) such that $U \cap \Delta(X) = \emptyset$. There exists $U_\alpha \in \mathcal{U}_\mathcal{T}(x_\alpha)$ and $U_\beta \in \mathcal{U}_\mathcal{T}(x_\beta)$ such that $(x_\alpha, x_\beta) \in U_\alpha \times U_\beta \subset \Delta(X)$. Then $\pi_\alpha^{-1}(U_\alpha) \cap \pi_\beta^{-1}(U_\beta)$ is a **Neighborhood** of $\{x_\alpha\}_{\alpha\in A}$ which is disjoint from $\Delta_A(X)$. Since $\{x_\alpha\}_{\alpha\in A} \in \left(\prod_{\alpha\in A} X\right) \setminus \Delta_A(X)$ was arbitrary, we conclude i$\left(\prod_{\alpha\in A} X\right) \setminus \Delta_A(X)$ is **Open**. □

Proof of 2.3.46 iv $\implies$ 2.3.46 iii. Trivial and obvious. □

Proof of 2.3.46 iii $\implies$ 2.3.46 i. Let $x, y \in X$ with $x \neq y$. Then $(x, y) \in X \times X \setminus \Delta(X)$. Since $\Delta(X)$ is **Closed**, there is an open $U \in \mathcal{U}_{\mathcal{T}_{X\times X}}((x, y))$ such that $U \cap \Delta(X) = \emptyset$. There exists $U_x \in \mathcal{U}_\mathcal{T}(x)$ and $U_y \in \mathcal{U}_\mathcal{T}(y)$ such that $U_x \times U_y \subset U \subset (X \times X) \setminus \Delta(X)$. Hence, $U_x \cap U_y = \emptyset$. Since $x \neq y$ were arbitrary, X is **Hausdorff**. □

Proof of 2.3.46 i $\implies$ 2.3.46 vi. Let X be **Hausdorff**. Let $\mathcal{F}$ be a **Filter** in X. Let x_0 be a **Limit** of x_0. Let $y \in X$ with $y \neq x_0$. Then there exists $U_y \in \mathcal{U}_\mathcal{T}(y)$ and $U_x \in \mathcal{U}_\mathcal{T}(x)$ with $U_x \cap U_y = \emptyset$. Since x is a **Limit** of $\mathcal{F}$, for some $F \in \mathcal{F}$, $F \subset U_x$. Hence $F \cap U_y = \emptyset$. By 2.3.38. vi, y is not a **Cluster Point** of $\mathcal{F}$, □

Proof of 2.3.46 vi $\implies$ 2.3.46 v. This result is a direct application of 2.3.38. ii. □

Proof of 2.3.46 v $\implies$ 2.3.46 i. Let $x, y \in X$ with $x \neq y$ and suppose x and y don't have any **Disjoint Neighborhoods**. Then by 2.2.4, there exists a **Filter** on X containing $\mathcal{U}_\mathcal{T}(x) \cup \mathcal{U}_\mathcal{T}(y)$, which would have both x as a **Limit** and y as a **Limit**. □

2.3.7 Compactness

Proposition 2.3.47 (**Compact** Image)**.** Let X be a **Compact Topological Space**. Let Y be a **Topological Space**. Let $f : X \to Y$ be **Continuous**. Then $f(X)$ is **Compact**.

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be an **Open Cover** of $f(X)$. Since f is **Continuous**, $\{f^{-1}(U_\alpha)\}_{\alpha\in A}$ is a **Open Cover** of X. Since X is **Compact**, there is a **Finite Subcover** $\{f^{-1}(U_{\alpha_i})\}_{i=1}^n$. Since $X \subset \bigcup_{i=1}^n f^{-1}(U_{\alpha_i})$, $f(X) \subset \bigcup_{i=1}^n U_{\alpha_i}$. Hence $f(X)$ is **Compact**. □

Proposition 2.3.48 (Compact Filter)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. The following are equivalent

(i) X is **Compact**.

(ii) Each **Ultrafilter** in X has a **Limit**.

Proof of 2.3.48 ii implies 2.3.48 i. I prove the converse. Let X be non-**Compact**. Then there is an **Open Cover** for X, $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ with no **Finite** Subcover. Define

$$F = \{X \setminus U_\alpha\}_{\alpha\in A}$$

Since $\mathcal{U}$ has no **Finite Subcover**, **Finite** intersections of elements of F are nonempty. Hence, by 2.2.4 and 2.2.14 there is an **Ultrafilter** G on X which contains F. Let $y \in X$. Since $\mathcal{U}$ is a **Cover** for X, for some $\alpha \in A$, $y \in U_\alpha$. Since $U_\alpha \in \mathcal{T}$, $U_\alpha \in \mathcal{U}_{\mathcal{T}}(y)$. Since $U_\alpha \cap (X \setminus U_\alpha) = \emptyset$, by 2.3.38. vi, y is not a **Cluster Point** of G. By 2.3.38. viii, y is not a **Limit** of G. Since $y \in X$ was arbitrary, G has no **Limits** in X. □

Proof of 2.3.48 i implies 2.3.48 ii. Let X be **Compact**. Let G be an **Ultrafilter** on X. Suppose, for sake of contradiction, that G has no **Limit**. Then, for each $x \in X$, there is a $U_x \in \mathcal{U}_{\mathcal{T}}(x)$ such that $U_x \notin G$. Hence, for each $x \in X$, we can find $x \in V_x \in \mathcal{T}$ such that $V_x \notin G$. Then $\mathcal{V} = \{V_x\}_{x \in X}$ is an **Open Cover** for X. Since X is **Compact**, $\mathcal{V}$ has a **Finite Subcover** $\{V_{x_i}\}_{i=1}^n$. Since $X = \bigcup_{i=1}^{n} V_{x_i}$ and $X \in G$, by 2.2.15. i, for some i, $1 \le i \le n$, $V_{x_i} \in G$, a contradiction. Hence G has a **Limit**. □

Theorem 2.3.49 (Tychonoff)**.** Let A be a nonempty set. For each $\alpha \in A$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a **Topological Space**. Define $X = \prod_{\alpha \in A} X_\alpha$. Endow X with the **Product Topology**. The following are equivalent.

(i) X is **Compact**.

(ii) Each X_α is **Compact**.

Proof of 2.3.49 ii implies 2.3.49 i. Let each X_α be **Compact**. Let $\mathcal{G}$ be an **Ultrafilter** on X. Since each $\pi_\alpha : X \to X_\alpha$ is **Surjective**, by 2.2.18. iv, for each $\alpha \in A$, $\pi_\alpha(\mathcal{G})$ is an **Ultrafilter Base** for an **Ultrafilter** $\mathcal{G}_\alpha$ on X_α. By 2.3.48, for each $\alpha \in A$, $\mathcal{G}_\alpha$ has a **Limit** $x_\alpha \in X_\alpha$. Define $x = \{x_\alpha\}_{\alpha \in A} \in X$. I claim that x is a **Limit** for $\mathcal{G}$. To see this, let $U \in \mathcal{U}_{\mathcal{T}}(x)$. Then there exists $\{\alpha_i\}_{i=1}^n \subset A$ and $U_{\alpha_i} \in \mathcal{U}_{\mathcal{T}_{\alpha_i}}(x_{\alpha_i})$ such that $x \in \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \subset U$. Since $\mathcal{G}_{\alpha_i}$ has x_{α_i} as a **Limit**, fore each $1 \le i \le n$, there exists $V_i \in \mathcal{G}_{\alpha_i}$ such that $V_i \subset U_{\alpha_i}$. Since $\pi_{\alpha_i}(\mathcal{G})$ is an **Ultrafilter Base** for $\mathcal{G}_{\alpha_i}$, for each $1 \le i \le n$, there exists $W_i \in \mathcal{G}$ such that $\pi_{\alpha_i}(W_i) \subset V_i \subset U_{\alpha_i}$. Hence, for each $1 \le i \le n$, $W_i \subset \pi_{\alpha_i}^{-1}(U_{\alpha_i})$. Furthermore, since $\pi_{\alpha_i}(W_i) \in \mathcal{G}_\alpha$, $x_{\alpha_i} \in \pi_{\alpha_i}(W_i)$. This implies $x \in \bigcap_{i=1}^{n} W_i$. Thus we have

$$x \in \bigcap_{i=1}^{n} W_i \subset \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \subset U$$

Since $\mathcal{G}$ is a **Filter**, $\bigcap_{i=1}^{n} W_i \in \mathcal{G}$, so x is a **Limit** of $\mathcal{G}$. Hence, by 2.3.48, X is **Compact**.

□

Proof of 2.3.49 i implies 2.3.49 ii. Let X be **Compact**. For each $\alpha \in A$, let π_α be the $\alpha -$ **Projection Map**. By 2.3.33, each π_α is **Continuous**. Hence, by 2.3.47, for each $\alpha \in A$, $X_\alpha = \pi_\alpha(X)$ is **Compact**. □

Proposition 2.3.50. Let X be a **Compact Topological Space**. Let $F \subset X$ be **Closed**. Then F is **Compact**.

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be an **Open Cover** for F. Then $\{U_\alpha\}_{\alpha\in A} \cup \{X \setminus F\}$ is a **Open Cover** for X. Since X is **Compact** there is $\{\alpha_i\}_{i=1}^n \subset A$ such that $\{U_{\alpha_i}\}_{i=1}^n \cup \{X \setminus F\}$ is an **Open Cover** for X. Since $(X \setminus F) \cap F = \emptyset$, $\{U_{\alpha_i}\}_{i=1}^n$ is an **Open Cover** for F. Hence F is **Compact**. □

Proposition 2.3.51. Let X be a **Pseudo-Hausdorff Topological Space**. Let $\cong$ denote the **Relation Of Equal Neighborhood Filters** on X. Let $\pi : X \to X/\cong$ denote the **Quotient Map**. Let $K \subset X$ be a **Compact Fiber** under $\cong$. Then K is **Closed**.

Proof. Let $x_0 \in X \setminus K$. Then since K is a **Fiber**. $\pi(x_0) \in (X/\cong) \setminus \pi(K)$. Since $X/\cong$ is **Hausdorff**, for each $x \in K$, there is an $X/\cong$ **Neighborhood** U_x of $\pi(x_0)$ and and $X/\cong$ **Neighborhood** V_x of x such that $U \cap V = \emptyset$. Then $\{V_x\}_{x\in\pi(K)}$ is an **Open Cover** of $\pi(K)$. Since π is continuous, $\pi(K)$ is **Compact**. Hence, there is an $\{x_i\}_{i=1}^n \subset \pi(K)$ such that $K \subset \bigcup_{i=1}^n V_{x_i}$. Define $U = \bigcap_{i=1}^n U_{x_i}$. Then $\pi(x_0) \in U$ and $U \cap V = \emptyset$. Since $\pi(K) \subset V$, $U \cap \pi(K) = \emptyset$. Since $\pi(x_0) \in U$, $x_0 \in \pi^{-1}(U)$, and $K \cap \pi^{-1}(U) = \emptyset$, so $x \in X \setminus \overline{K}$. Hence $K = \overline{K}$, so K is **Compact**. □

Corollary 2.3.52 (Compact Closed)**.** A **Compact** subset of a **Hausdorff** space is **Closed**.

Proof. Since every subset of a **Hausdorff** space is a **Fiber**, we can apply 2.3.51 to get this result. □

Proposition 2.3.53 (Compact Contained)**.** Let X be a nonempty set. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be **Topologies** on X. Let $\cong_i$ denote the **Relation Of Equal Neighborhood Filters** on $\mathcal{T}_i$. Let $\cong_1 = \cong_2$. Let $\mathcal{T}_1$ be **Pseudo-Hausdorff**. Let $\mathcal{T}_2$ be **Compact**. Let $\mathcal{T}_1 \subset \mathcal{T}_2$. Then $\mathcal{T}_1 = \mathcal{T}_2$.

Proof. Let $F \subset X$ be **Closed** in $\mathcal{T}_2$. By 2.3.50, F, endowed with its $\mathcal{T}_2$ subspace **Topology**, is **Compact**. Since $\mathcal{T}_1 \subset \mathcal{T}_2$, F, endowed with its $\mathcal{T}_1$ subspace **Topology**, is **Compact**. Since F is $\mathcal{T}_2 -$ **Closed**, by 2.3.32, F is a **Fiber** under $\cong_2$. Hence F is a **Fiber** under $\cong_1$. Hence, we can apply 2.3.51 to claim that F is **Closed** in $\mathcal{T}_1$. Since an arbitrary $\mathcal{T}_2 -$ **Closed** set is $\mathcal{T}_1 -$ **Closed**, $\mathcal{T}_2 \subset \mathcal{T}_1$. By assumption, the reverse inclusion holds, so equality is verified. □

Proposition 2.3.54. Let X be a **Pseudo-Hausdorff Topological Space**. Let $K \subset X$ be **Compact**. Let $\cong$ denote the **Relation Of Equal Neighborhood Filters** in X. Then $\pi(K)$ is **Closed**, $\overline{K} = \pi^{-1}(\pi(K))$, and $\overline{K}$ is **Compact**.

Proof. Since K is **Compact**, $\pi(K)$ is **Compact**. Since $X/\cong$ is **Hausdorff**, we can apply 2.3.52 to see that $\pi(K)$ is **Closed**. Furthermore, by continuity of π, $\pi^{-1}(\pi(K))$ is a **Closed** set containing K. Also, by 2.3.32. v, $\overline{K}$ is a **Fiber**. Thus, we have

$$\overline{K} \subset \pi^{-1}(\pi(K)) \subset \pi^{-1}\left(\pi\left(\overline{K}\right)\right) = \overline{K}$$

Hence $\overline{K} = \pi^{-1}(\pi(K))$.

Finally, let $\{U_\alpha\}_{\alpha\in A}$ be a **Open Cover** of $\overline{K}$. Then $\{\pi(U_\alpha)\}_{\alpha\in A}$ is a **Open Cover** of $\pi(\overline{K}) = \pi(K)$. Since $\pi(K)$ is **Compact**, there is a finite $\{\alpha_i\}_{i=1}^n \subset A$ such that $\{\pi(V_{\alpha_i})\}_{i=1}^n$ covers $\pi(K) = \pi\left(\overline{K}\right)$. Hence $\{V_{\alpha_i}\}_{i=1}^n$ covers $\overline{K}$ so $\overline{K}$ is **Compact**.

□

2.3.8 Countability Axioms

Definition 2.3.55 (First Countable). Let $(x, \mathcal{T})$ be a **Topological Space**. We say that X is **First Countable** if for each $x \in X$, there is a **Countable Neighborhood Basis** for $\mathcal{T}$ at X.

Definition 2.3.56 (Second Countable). A **Topological Space** which permits a **Countable Basis** is called **Second Countable**

Definition 2.3.57 (Dense). Let $(X, \mathcal{T})$ be a **Topological Space** and let $A \subset X$. We say that A is **Dense** in X if $\overline{A} = X$.

Definition 2.3.58 (Separable). We say that a **Topological Space** which permits a **Countable Dense** subset is **Separable**.

Definition 2.3.59 (Lindelof). A **Topological Space** in which every **Open Cover** permits a **Countable Subcover** is called a **Lindelof** space.

2.4 Algebraic Structures

2.4.1 Binary Operations and Magmas

Definition 2.4.1. REMOVE

Definition 2.4.2 (Commutative). Let X and Y be nonempty sets. We say that a map $f : X \times X \to Y$ is **Commutative** if for each $x_0, x_1 \in X$, $f(x_0, x_1) = f(x_1, x_0)$.

Definition 2.4.3 (Operation, Unary Operation, Binary Operation). Let X be a nonempty set. Let $n \in \mathbb{Z}^+$. We call a mapping

$$T : X^n \to X$$

an n-ary **Operation** on X. If $n = 1$ then we call T a **Unary Operation** on X. If $n = 2$, then we call T a **Binary Operation** on X. If T is a **Binary Operation** on X, we sometimes use the notation

$$xTy = T(x, y)$$

Definition 2.4.4 (Magma). Let X be a nonempty set. Let $T : X \times X \to X$ be a **Binary Operation** on X. We call the pair (X, T) a **Magma**. When T is clear, we call X a **Magma**. If T is **Commutative**, then we call (X, T) a **Commutative Magma**. In general, this naming convention is used for any algebraic structure defined on a set via a **Binary Operation** with particular properties. Now let $(M, +)$ be a **Magma**, let $x \in M$, and let $A, B \subset M$. Let $\mathcal{U}, \mathcal{V} \subset 2^M$. We define the following:

$$x + A = \{x + y : y \in A\}$$

$$A + x = \{y + x : y \in A\}$$
$$A + B = \{a + b : a \in A \quad and \quad b \in B\}$$
$$\mathcal{U} + \mathcal{V} = \{U + V : (U \in \mathcal{U})(V \in \mathcal{V})\}$$
$$\mathcal{U} + A = \{U + A : U \in \mathcal{U}\}$$
$$A + \mathcal{U} = \{A + U : U \in \mathcal{U}\}$$
$$\mathcal{U} + x = \{U + x : U \in \mathcal{U}\}$$
$$x + \mathcal{U} = \{x + U : U \in \mathcal{U}\}$$

Definition 2.4.5 (**Magma Homomorphism**)**.** Let $(X, \oplus_X)$ and $(Y, \oplus_Y)$ be **Magmas**. Let $T : X \to Y$ satisfy, for each $x_1, x_2 \in X$.

$$T(x_1 \oplus_X x_2) = T(x_1) \oplus_Y T(x_2)$$

Then we call T a **Magma Homomorphism**. We represent the collection of **Magma Homomorphisms** from $(X, \oplus_X)$ to $(Y, \oplus_Y)$ with $H_{\mathbf{Magma}}((X, \oplus_X), (Y, \oplus_Y))$, or, when $\oplus_X$ and $\oplus_Y$ are clear, $H_{\mathbf{Magma}}(X, Y)$. A **Magma Homomorphism** is called **Additive** and posseses the property **Additivity**.

Definition 2.4.6 (**Left Identity Element**, **Right Identity Element**)**.** Let (X, L) and (X, R) be **Magmas**. Let $l, r \in X$ such that for every $x \in X$ we have

$$lLx = x$$
$$xRr = x$$

In such a scenario, we say that l is a **Left Identity Element** of (X, L), and we say that r is a **Right Identity Element** of (X, R).

Definition 2.4.7 (**Identity Element**)**.** Let $(X, \oplus)$ be a **Magma**. Let $e \in X$ be both a **Left Identity Element** and a **Right Identity Element** of $\oplus$. Then, we say that e is an **Identity Element** of $(X, \oplus)$.

Definition 2.4.8 (**Unital Magma**)**.** Let $(X, \oplus)$ be a **Magma** with **Identity Element** e. Then we call $(X, \oplus, e)$ a **Unital Magma**.

Definition 2.4.9 (**Unital Magma Homomorphism**)**.** Let $(X, \oplus_X, e_X)$ and $(Y, \oplus_Y, e_Y)$ be **Unital Magmas** and $T : X \to Y$ be a **Magma Homomorphism** such that $T(e_X) = e_Y$. Then we call T a **Unital Magma Homomorphism**. We represent the set of **Unital Magma Homomorphisms** between X and Y with $H_{\mathbf{UMagma}}(X, Y)$. If T is a **Unital Magma Homomorphism**, then we call $T^{-1}(e_Y)$ the **Kernel** of T. We denote the **Kernel** of T with $Kernel(T)$.

Definition 2.4.10 (**Left Inverse**, **Right Inverse**)**.** Let $(X, \oplus, e)$ be a **Unital Magma**. Let $l, r \in X$ such that

$$l \oplus r = e$$

Then we say that l is a **Left Inverse** of r in $(X, \oplus, e)$ and we say that r is a **Right Inverse** of l in $(x, \oplus, e)$. Furthermore, we say that r is **Left Invertible** in $(X, \oplus, e)$ and that l is **Right Invertible** in $(X, \oplus, e)$.

Definition 2.4.11 (**Inverse**)**.** Let $(X, \oplus, e)$ be a **Unital Magma**. Let $x, y \in X$ such that x is a **Left Inverse** of y and x is a **Right Inverse** of y. Then, we say that x is an **Inverse** of y in $(X, \oplus, e)$, we say y an **Invertible** element of $(X, \oplus, e)$, and we write $x = y^{-1}$.

Definition 2.4.12 (**Associative**)**.** Let T be a **Binary Operation** on a set X. We say that T is **Associative** and we say that T posses **Associativity** if for each $x, y, z \in X$, we have

$$T(x, T(y, z)) = T(T(x, y), z)$$

Definition 2.4.13 (**Consistent**)**.** Let $(X, \oplus)$ be a **Magma** and let R be a **Relation** on X such that for each $\{x_0, x_1, y_0, y_1\} \subset X$, if $x_0 R x_1$ and $y_0 R y_1$, then

$$(x_0 \oplus y_0) R (x_1 \oplus y_1)$$

Then we say that R is **Consistent** with $(X, \oplus)$, and we say that R posesses **Consistency** with respect to $(X, \oplus)$. A **Congruence** on M is an **Equivalence Relation** which is **Consistent** with M.

Definition 2.4.14 (**Quotient Magma**)**.** Let $(M, \cdot)$ be a **Magma**. Let $\cong$ be a **Congruence** on M. Define $\boxdot : M/\cong \times M/\cong \to M/\cong$ by setting, for each $x, y \in M$,

$$[x] \boxdot [y] = [x \cdot y]$$

By 2.4.15, $(M/\cong, \boxdot)$ is a **Magma**, which we call the **Quotient Magma** of M relative to $\cong$.

Proposition 2.4.15 (**Quotient Magma**)**.** Let $(M, \cdot)$ be a **Magma**. Let $\cong$ be a **Congruence** on M. Let $(M/\cong, \boxdot)$ be the **Quotient Magma**. The following are true.

(i) The **Quotient Magma** is well defined. That is $\boxdot$, as it is defined in 2.4.14, is a well defined **Binary Operation** on $M/\cong$.

(ii) If $\cdot$ is **Associative**, then $\boxdot$ is **Associative**.

(iii) If $\cdot$ is **Commutative**, then $\boxdot$ is **Commutative**.

(iv) If $e \in M$ is a **Identity Element** of $(M, \cdot)$, then $[e]$ is an **Identity Element** of $(M/\cong, \boxdot)$.

(v) If $x \in M$ is an **Invertible** element of M, then $[x]$ is an **Invertible** of $M/\cong$, and $[x]^{-1} = [x^{-1}]$.

(vi) The **Quotient Map** $T : M \to M/\cong$ defined by $T(x) = [x]$ is a **Magma Homomorphism**.

Proof of 2.4.15 i. Let $x_0, x_1, y_0, y_1 \in M$ such that $[x_0] = [x_1]$ and $[y_0] = [y_1]$. Then $x_0 \cong x_1$ and $y_0 \cong y_1$, so since $\cong$ is **Consistent**, $x_0 \cdot y_0 \cong x_1 \cdot y_1$. Hence, $[x_0] \boxdot [y_0] = [x_0 \cdot y_0] = [x_1 \cdot y_1] = [x_1] \boxdot [y_1]$. $\square$

Proof of 2.4.15 ii. Let M be **Associative**. Let $x, y, z \in M$. Then,

$$([x] \boxdot [y]) \boxdot [z] = ([x \cdot y]) \boxdot [z] = [(x \cdot y) \cdot z] = [x \cdot (y \cdot z)] = [x] \boxdot [y \cdot z] = [x] \boxdot ([y] \boxdot [z])$$

□

Proof of 2.4.15 iii. Let $\cdot$ be **Commutative**. Let $x, y \in M$. Then,

$$[x] \boxdot [y] = [x \cdot y] = [y \cdot x] = [y] \boxdot [x]$$

□

Proof of 2.4.15 iv. Let e be an **Identity Element** of M. Let $x \in M$. Then,

$$[e] \boxdot [x] = [e \cdot x] = [x] = [x \cdot e] = [x] \boxdot [e]$$

□

Proof of 2.4.15 v. Let x be an **Invertible** of M. Then,

$$[x] \boxdot [x^{-1}] = [x \cdot x^{-1}] = [e]$$

and

$$[x^{-1}] \boxdot [x] = [x^{-1} \cdot x] = [e]$$

These two equations, paired with 2.4.15 iv implies $[x]^{-1} = [x^{-1}]$. □

Proof of 2.4.15 vi. Let $x, y \in M$. Then,

$$T(x \cdot y) = [x \cdot y] = [x] \boxdot [y] = T(x) \boxdot T(y)$$

□

Proposition 2.4.16 (**Magma Homomorphism**)**.** Let M_1 and M_2 be **Magmas**. Let $T : M_1 \to M_2$ be a **Magma Homomorphism**. Define $\cong \subset M_1 \times M_1$ by $x \cong y$ if and only if $T(x) = T(y)$. The following are true.

(i) $\cong$ is a **Congruence** on M_1.

(ii) Let $Q : M_1 \to M_1/\cong$ denote the **Quotient Map**. Define $\tilde{T} : M_1/\cong \to Range(T)$ by setting, for each $x \in M_1/\cong$, $\tilde{T}([x]) = T(x)$. Then $\tilde{T}$ is a well defined **Bijective Magma Homomorphism**, $T = \tilde{T} \circ Q$, and $Q = \tilde{T}^{-1} \circ T$.

Proof of 2.4.16 i. It is obvious that T is an **Equivalence Relation**. What remains to show is that $\cong$ is **Consistent** with M. Suppose $x_0 \cong x_1$ and $y_0 \cong y_1$. Then since T is a **Magma Homomorphism**,

$$T(x_0 y_0) = T(x_0)T(y_0) = T(x_1)T(y_1) = T(x_1 y_1)$$

Hence $x_0 y_0 \cong x_1 y_1$. □

Proof of 2.4.16 ii. We first show that T is well defined. Let $[x] = [y]$. We must show that $\tilde{T}([x]) = \tilde{T}([y])$. Since $[x] = [y]$, $x \cong y$. Hence, $T(x) = T(y)$, which gives us $\tilde{T}([x]) = T(x) = T(y) = \tilde{T}([y])$.

Suppose $\tilde{T}([x]) = \tilde{T}([y])$. Then $T(x) = T(y)$, so $x \cong y$, so $[x] = [y]$. Hence $\tilde{T}$ is **Injective**.

Let $y \in Range(T)$. Then there exists $x \in M_1$ such that $T(x) = y$. Hence $\tilde{T}([x]) = T(x) = y$. Thus, $\tilde{T}$ is **Surjective**.

Now let $x, y \in M$. Then, $\tilde{T}([x][y]) = \tilde{T}([xy]) = T(xy) = T(x)T(y) = \tilde{T}([x])\tilde{T}([y])$. Thus $\tilde{T}$ is a **Magma Homomorphism**.

Finally, if $x \in M_1$, then $\tilde{T} \circ Q(x) = \tilde{T}([x]) = T(x)$, so $T = \tilde{T} \circ Q$. The final equation falls from the **Bijectivity** of $\tilde{T}$. □

Definition 2.4.17 (**Semigroup**)**.** An **Associative Magma** is called a **Semigroup**.

Definition 2.4.18 (**Monoid**)**.** A **Monoid** is a **Unital Magma** which is also a **Semigroup**.

Definition 2.4.19 (**Partially Ordered Magma**, **Totally Ordered Magma**, **Directed Magma**)**.** Let $(X, \oplus)$ be a **Magma**. Let T be a **Total Ordering** on X which is **Consistent** with $(X, \oplus)$. Let P be a **Partial Ordering** on X which is **Consistent** with $(X, \oplus)$. Let D be a **Directing** on X which is **Consistent** with $(X, \oplus)$. We call $(X, \oplus, T)$ a **Totally Ordered Magma**. we call $(X, \oplus, P)$ a **Partially Ordered Magma**, and we call $(X, \oplus, D)$ a **Directed Magma**.

2.4.2 Groups

Definition 2.4.20 (**Group**)**.** Let $(X, \oplus, e)$ be a **Monoid** such that each $x \in X$ is **Invertible**. Then we call $(X, \oplus, e)$ a **Group**. Out of respect, we sometimes call a **Commutative Group** an **Abelian Group**.

Definition 2.4.21 (**Group Inverse Operator**)**.** Let $(X, \oplus, e)$ be a group. We denote with ${T^{-1}}_G$ the function defined as follows: ${T^{-1}}_G : X \to X$,

$$ {T^{-1}}_G(x) = x^{-1} $$

We call ${T^{-1}}_G$ the **Group Inverse Operator** of $(X, \oplus, e)$.

Definition 2.4.22 (**Translation Operator**)**.** Let G be a **Magma**. Let $g \in G$. We define ${T^R}_g : G \to G$ and ${T^L}_g : G \to G$ by setting, for each $x \in G$,

$$ {T^R}_g(x) = xg $$

$$ {T^L}_g(x) = gx $$

We call ${T^R}_g$ the **Right Translation** of G by g, and we call ${T^L}_g$ the **Left Translation** of G by g. If the operation of G is **Commutative**, then we define $\mathbf{T}_g = {T^R}_g = {T^L}_g$ which we call **Translation** of G by g.

Definition 2.4.23 (**Symmetric**)**.** Let G be a group and $A \subset G$. We define

$$A^{-1} = \{a^{-1} : a \in A\}$$

If $A = A^{-1}$ then we say that A is **Symmetric**.

Definition 2.4.24. Let G be a **Group**. Let $H \subset G$ be **Symmetric**. Furthermore, suppose $HH \subset H$. Then we call H a **Subgroup** of G. If K is a **Subgroup** of G and for each $x \in G$, $xKx^{-1} = K$, then we call K a **Normal Subgroup** of G.

Proposition 2.4.25. Let G be a **Group** and $K \subset G$. The following are equivalent.

(i) K is a **Normal Subgroup** of G.

(ii) There exists a **Congruence** $\cong$ on G such that $H = [e]$.

(iii) There exists a unique **Congruence** $\cong$ on G such that $H = [e]$.

(iv) There exists a group H and a **Magma Homomorphism** $T : G \to H$ such that $K = T^{-1}(n)$ where n is the **Identity Element** of H. .

Proof of 2.4.25 i implies 2.4.25 ii. Let K be a **Normal Subgroup** of G. For $x, y \in G$, define $x \cong y$ if and only if $xy^{-1} \in K$. Since $e \in K$, if $x \in K$, then $x^{-1} \in K^{-1} = K$, so $e = xx^{-1} \in KK^{-1} = KK \subset K$, so $x \cong x$. If $x \cong y$, then $yx^{-1} = (xy^{-1})^{-1} \in K^{-1} = K$. Hence $y \cong x$. Finally, if $x \cong y$ and $y \cong z$ then $xz^{-1} = (xy^{-1})(yz^{-1}) \in KK \subset K$, so $x \cong z$. Hence $\cong$ is an **Equivalence Relation**. Now let $x_0 \cong x_1$ and $y_0 \cong y_1$. Then $x_0y_0(x_1y_1)^{-1} = x_0y_0y_1^{-1}x_1^{-1} \subset xKx^{-1} \subset K$, so $x_0y_0 \cong x_1y_1$. Hence $\cong$ is a **Congruence** on G.

Now let $x \in [e]$. Then $x \cong e$, so $xe^{-1} = x \in K$. Hence $[e] \subset K$. If instead $y \in K$, then $ye^{-1} = ye = y \in K$, so $y \cong e$, implying $y \in [e]$. Hence $K = [e]$. □

Proof of 2.4.25 ii implies 2.4.25 iii. Let $\cong_1$ and $\cong_2$ be **Congruences** on G such that $H = [e]_1 = [e]_2$. Let $x \cong_1 y$. Then $[x]_1 = [xy^{-1}y]_1 = [xy^{-1}]_1[y]_1 = [xy^{-1}][x]_1$. Hence $[xy^{-1}]_1 = [e]_1$, so $xy^{-1} \cong_1 e$. Hence $xy^{-1} \cong_2 e$. Thus, $[x]_2 = [xy^{-1}y]_2 = [xy^{-1}]_2[y]_2 = [e]_2[y]_2 = [y]_2$. Thus $x \cong_2 y$. Hence $\cong_1 \subset \cong_2$. By a similar argument, $\cong_2 \subset \cong_1$, There is at most 1 **Congruence** on G such that $[e] = K$ for any subset K of G. □

Proof of 2.4.25 iii implies 2.4.25 iv. Let $\cong$ be a **Congruence** on G for which $[e] = K$. By 2.4.15 ii, 2.4.15 iv, and 2.4.15 v the **Quotient Magma** $G/\cong$ is a **Group**. By 2.4.15 vi, $T : G \to G/\cong$ defined by $T(x) = [x]$ is a **Magma Homomorphism**. Furthermore, by 2.4.15 iv, $[e]$ is the **Identity Element** of $G/\cong$. Hence, because $K = [e] = \{x \in G : x \cong e\} = \{x \in G : [x] = [e]\} = \{x \in G : T(x) = [e]\} = T^{-1}([e])$ □

Proof of 2.4.25 iv implies 2.4.25 i. It suffices to show that $K = T^{-1}(e)$ is a **Normal Subgroup** of G. Let $x, y \in K$. Then $T(xy) = T(x)T(y) = ee = e$, so $xy \in K$, so $KK \subset K$. Furthermore, fi $x \in K$, then $e = T(e) = T(xx^{-1}) = T(x)T(x^{-1}) = eT(x^{-1}) = T(x^{-1})$, so $x^{-1} \in K$. Hence K is **Symmetric**. Now, let $y \in K$ and let $x \in G$. Then $T(xyx^{-1}) = T(x)T(y)T(x^{-1}) = T(x)T(x^{-1}) = T(xx^{-1}) = T(e) = e$. Hence K is a **Normal Subgroup**. □

Remark 2.4.26 (**Quotient Group**)**.** Let G be a **Group**. Let $\cong$ be a **Congruence** on G. Then by 2.4.25 there is a **Normal Subgroup** K of G such that $x \cong y$ if and only if $xy^{-1} \in K$. For this reason, we denote $X/\cong$ with X/K. Furthermore, by 2.4.15 ii, 2.4.15 iv, and 2.4.15 v, X/K is a **Group** whose **Identity Element** is $[e]$. We call this **Group** the **Quotient Group** of G corresponding to K.

Proposition 2.4.27 (**Commutative Subgroups** are **Normal Subgroups**)**.** Any **Subgroup** of a **Commutative Group** is a **Normal Subgroup**.

Proof. Let K be a **Subgroup** of a **Commutative Group** G. Let $x \in K$ and $y \in G$. Then $yxy^{-1} = yy^{-1}x = x \in K$. □

Theorem 2.4.28 (Group Isomorphism Theorem)**.** Let G_1 and G_2 be **Groups**. Let $T : G_1 \to G_2$ be a **Magma Homomorphism**. Let $K = Kernel(T) := T^{-1}(e_2)$. Let $Q : G_1 \to G_1/K$ be the **Quotient Map**. Define $\tilde{T} : G_1/Kernel(T) \to Range(T)$ by $\tilde{T}([x]) = T(x)$. Then $\tilde{T}$ is a group isomorphism, $T = \tilde{T} \circ Q$, and $Q = \tilde{T}^{-1} \circ T$.

Proof. This is a direct consequence of 2.4.16, 2.4.25, and 2.4.26. □

2.4.3 Rings and Fields

Definition 2.4.29 (**Ring**)**.** Let R be a nonempty set. Let $(R, +, 0)$ be a **Commutative Group**. Let $(R \setminus \{0\}, \cdot, 1)$ be a **Monoid**. Suppose that for each $a, b, c \in R$ we have

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

Then we call $(R, +, \cdot, 0, 1)$ a **Ring**. When the context is clear, we call R a **Ring**. If $\cdot$ is **Commutative**, then we call R a **Commutative Ring**. If $(R \setminus \{0\}, \cdot, 1)$ is a **Commutative Group** then we call R a **Field**.

2.4.4 Vector Space

Definition 2.4.30 (**Vector Space**)**.** Let $(F, +_F, \cdot_F, 0_F, 1_F)$ be a **Field**. Let $(V, +, 0)$ be a **Commutative Group**. Let $\cdot : \mathcal{F} \times V \to V$ satisfy, for each $\alpha, \beta \in F$ and for each $x, y \in V$,

(i) $\alpha \cdot (\beta \cdot x) = (\alpha \cdot_F \beta) \cdot x$.

(ii) $(\alpha +_F \beta) \cdot x = (\alpha \cdot x) + (\beta \cdot x)$.

(iii) $\alpha \cdot (x + y) = (\alpha \cdot x) + (\alpha \cdot y)$.

(iv) $1 \cdot x = x$

Then we call V a **Vector Space** over F. Let $\alpha \in F$. Let $x \in V$. Let $A \subset F$. Let $B \subset V$. Then we define

$$\alpha B = \{\alpha \cdot b : b \in B\}$$

$$Ax = \{a \cdot x : a \in A\}$$

$$AB = \bigcup_{a \in A} aB$$

Definition 2.4.31 (**Scalar Homogeneous**)**.** Let V and U be **Vector Spaces** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $p : V \to U$ such that for each $\alpha \in \mathbb{F}$ and each $x \in V$,

$$p(\alpha x) = \alpha p(x)$$

Then we say p is **Scalar Homogeneous** and p posesses **Scalar Homogeneity**.

Definition 2.4.32 (**Absolutely Scalar Homogeneous**)**.** Let V and U be a **Vector Spaces** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $p : V \to U$ such that for each $\alpha \in \mathbb{F}$ and each $x \in V$,

$$p(\alpha x) = |\alpha|\, p(x)$$

Then we say p is **Absolutely Scalar Homogeneous** and we say p posesses **Absolute Scalar Homogeneity**.

Remark 2.4.33 (**Scalar Homogeneous** or **Absolutely Scalar Homogeneous** operator at 0 is 0)**.** If V is a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, then for each $x \in V$, $0x = 0$. Hence, if p is an **Absolutely Scalar Homogeneous** operator on V, then for any $x \in V$

$$p(0) = p(0x) = |0|p(x) = 0p(x) = 0$$

If instead p is **Scalar Homogeneous** operator on V, then we have

$$p(0) = p(0x) = 0p(x) = 0$$

that is, in either case, $p(0) = 0$.

Definition 2.4.34 (**Subadditive**)**.** Let G be a **Magma** and $(H, \leq)$ be a **Partially Ordered Magma**. We call a mapping $p : G \to H$ **Subadditive** if, for every $x, y \in G$,

$$p(xy) \leq p(x)p(y)$$

Under these circumstances, we may also say that p posesses **Subadditivity**.

Definition 2.4.35 (**Linear**)**.** Let V, U be **Vector Spaces** over a **Field** $\mathbb{F}$. Let $T : V \to U$ a **Scalar Homogeneous Magma Homomorphism**. Then we say T is **Linear**.

Definition 2.4.36 (**Linear Combination**)**.** Let $\mathcal{F}$ be a **Field**. Let V be a **Vector Space** over $\mathcal{F}$. Let $K \subset V$. Let $\{x_i\}_{i=1}^n \subset K$. Let $\{\alpha_i\}_{i=1}^n \subset \mathcal{F}$. Define

$$z = \sum_{i=1}^{n} \alpha_i x_i$$

We call z a **Linear Combination** of elements of K with coefficients $\{\alpha_i\}_{i=1}^n$.

Definition 2.4.37 (**Span**)**.** Let V be a **Vector Space**. Let $K \subset V$. We define $span(K)$ to be the collection of all **Linear Combinations** of elements of K. We call this set the **Span** of K.

Definition 2.4.38 (Linearly Independent). Let V be a **Vector Space** over a **Field** $\mathcal{K}$. Let $A \subset V$. We say that A is **Linearly Independent** if, for any finite subset $\{x_i\}_{i=1}^n \subset A$, the only solution to

$$\sum_{i=1}^{n} \beta_i x_i = 0 \qquad\qquad \{\beta_i\}_{i=1}^n \subset \mathcal{K}$$

is $\beta_i = 0$ for $1 \leq i \leq n$. In such a scenario, we may also say that A posesses **Linear Independence**. A subset of a **Vector Space** which is not **Linearly Independent** is said to be **Linearly Dependent** and to posess **Linear Dependence**.

Definition 2.4.39 (Hamel Basis). Let V be a **Vector Space**. Let $K \subset V$. Suppose $span(K) = V$ and K is **Linearly Independent**. Then we call K a **Hamel Basis** for V.

Definition 2.4.40 (Kroneker-δ). Let $(\mathcal{R}, +, \cdot, 0_{\mathcal{R}}, 1_{\mathcal{R}})$. be a **Ring**. Define $\delta : \mathbb{N} \times \mathbb{N} \to \mathcal{R}$ by

$$\delta(i,j) = \begin{cases} 1_{\mathcal{R}} & i = j \\ 0_{\mathcal{R}} & i \neq j \end{cases}$$

We call δ the **Kroneker-δ** function over $\mathcal{R}$. When $\mathcal{R}$ is clear, we call δ the **Kroneker-δ** function. We denote $\delta(i,j) = \delta_{i,j}$.

Definition 2.4.41 (Standard Basis). Let $(\mathcal{F}, +, \cdot, 0, 1)$ be a **Field**. Define $\oplus : \mathcal{F}^n \times \mathcal{F}^n \to \mathcal{F}^n$ by

$$\{x_i\}_{i=1}^n \oplus \{y_i\}_{i=1}^n = \{x_i + y_i\}_{i=1}^n$$

Define $\odot : \mathcal{F} \times \mathcal{F}^n \to \mathcal{F}^n$ by

$$\alpha \odot \{x_i\}_{i=1}^n = \{\alpha x_i\}_{i=1}^n$$

Then $\mathcal{F}^n$ is a **Vector Space** over $\mathcal{F}$. For $1 \leq j \leq n$, define $e_i = \{\delta_{i,j}\}_{j=1}^n$, where $\delta_{i,j}$ is the **Kroneker-δ** function over $\mathcal{F}$. Then $\{e_i\}_{i=1}^n$ is a **Hamel Basis** for $\mathcal{F}^n$. We call $\{e_i\}_{i=1}^n$ the **Standard Basis** for $\mathcal{F}^n$.

Definition 2.4.42 (Vector Subspace). Let V be a **Vector Space** over a **Field** $\mathcal{F}$. Let $U \subset V$ such that $span(U) \subset U$. Then we call U a **Vector Subspace** of V.

Remark 2.4.43 (Vector Subspace). Let V be a **Vector Space** over a **Field** $\mathcal{F}$. Let $U \subset V$ be a **Vector Subspace**. Then $U + U \subset U$. Furthermore, since for each $x \in V$, $-x = (-1)x$, $-U \subset U$. Hence U is a **Subgroup** of V. Furthermore, since B is **Commutative**, by 2.4.27, U is a **Normal Subgroup** of V, so the **Quotient Group** V/U is well defined. As shown in the next item, there is a natural way to turn V/U into a **Vector Space** itself.

Definition 2.4.44 (Quotient Vector Space). Let $(V, +, \cdot)$ be a **Vector Space** over a **Field** $\mathcal{F}$. let $U \subset V$ be a **Vector Subspace**. Denote the corresponding **Quotient Group** with $(V/U, \oplus)$. Define $\odot : \mathcal{F} \times V/U \to V/U$ by setting, for $x \in U$ and $\alpha \in \mathcal{F}$,

$$\alpha \odot [x] = [\alpha \cdot x]$$

By 2.4.45, $(V/U, \oplus, \odot)$ is a **Vector Space** over $\mathcal{F}$.

Proposition 2.4.45 (**Quotient Vector Space**)**.** Let V be a **Vector Space**. Let $U \subset V$ be a **Vector Subspace**. Let $(U/V, \oplus, \odot)$ denote the **Quotient Vector Space** as defined in 2.4.44. The following are true.

(i) $\odot$ is well defined and makes V/U into a **Vector Space** over $\mathcal{F}$.

(ii) Let $Q : V \to V/U$ denote the **Quotient Map**. Then Q is **Linear**.

Proof of 2.4.45 i. We first show that $\odot$ is well defined. Suppose $[x] \cong [y]$ and let $\alpha \in \mathcal{F}$. We must show that $[\alpha x] = [\alpha y]$. Since $[x] = [y]$, $x \cong y$, so $x - y \in U$. Hence $\alpha x - \alpha y = \alpha(x - y) \in \alpha U \subset U$. Hence $[\alpha x] = [\alpha y]$, so $\alpha \odot [x] = [\alpha \cdot x] = [\alpha \cdot y] = \alpha \odot [y]$. thus $\odot$ is well defined. By 2.4.26 and 2.4.15 iii, all that remains is to show that V/U satisfies 2.4.30 i, 2.4.30 ii, 2.4.30 iii, and 2.4.30 iv. Let $\alpha, \beta \in \mathcal{F}$ Let $x, y \in V$. For 2.4.30 i, we have

$$\alpha \odot (\beta \odot [x]) = \alpha \odot [\beta \cdot x] = [\alpha \cdot (\beta \cdot x)] = [(\alpha\beta) \cdot x] = (\alpha\beta) \odot [x]$$

For 2.4.30 ii, we have

$$(\alpha + \beta) \odot [x] = [(\alpha + \beta) \cdot x] = [(\alpha \cdot x) + (\beta \cdot x)] = [\alpha \cdot x] \oplus [\beta \cdot x] = (\alpha \odot [x]) \oplus (\beta \odot [x])$$

For 2.4.30 iii, we have

$$\alpha \odot ([x] \oplus [y]) = \alpha \odot [x+y] = [\alpha \cdot (x + y)] = [(\alpha \cdot x) + (\alpha \cdot y)] = [\alpha \cdot x] \oplus [\alpha \cdot y] = (\alpha \odot [x]) + (\alpha \odot [y])$$

For 2.4.30 iv, we have

$$1 \odot [x] = [1 \cdot x] = [x]$$

which concludes this proof. □

Proof of 2.4.45 ii. By 2.4.15 vi, Q is a **Magma Homomorphism**. Hence it is sufficient to show that Q is **Scalar Homogeneous**. Let $\alpha \in \mathcal{F}$ and let $x \in V$. Then $Q(\alpha x) = [\alpha x] = \alpha \odot [x] = \alpha Q(x)$, finishing the proof. □

Proposition 2.4.46 (**Linear** operator quotient)**.** Let $\mathcal{F}$ be a **Field**. Let V_1 and V_2 be **Vector Spaces** over $\mathcal{F}$. Let $T : V_1 \to V_2$ be **Linear**. Define $\tilde{T} : V_1/Kernel(T) \to Range(T)$ by $\tilde{T}([x]) = T(x)$. The following are true

1. $\tilde{T}$ is a **Bijective Linear** operator.

2. $T = \tilde{T} \circ Q$.

Proof. By 2.4.28, $\tilde{T}$ is a **Bijective Magma Homomorphism** and $T = \tilde{T} \circ Q$. Furthermore, if $[x] \in V_1/Kernel(T)$ and $\alpha \in \mathcal{F}$, then

$$\tilde{T}(\alpha[x]) = \tilde{T}([\alpha x]) = T(\alpha x) = \alpha T(x) = \alpha \tilde{T}([x])$$

so $\tilde{T}$ is also **Scalar Homogeneous** and is therefore **Linear**, finishing the proof. □

Definition 2.4.47 (**Space of Linear Operators**)**.** Let U and V be **Vector Spaces** over the same **Field** $\mathbb{F}$. We denote with $L(U, V)$ the set of **Linear** operators $T : U \to V$. We refer to $L(U, V)$ as the **Space of Linear Operators** from U to V. We endow $L(U, V)$ with the operations of pointwise addition and pointwise scalar multiplication, which makes $L(U, V)$ into a **Vector Space**.

Definition 2.4.48 (**Algebraic Dual**)**.** Let V be a **Vector Space** over a **Field** $\mathbb{F}$. We define $V' = L(V, \mathbb{F})$. We call V' the **Algebraic Dual** of V. If $x^* \in V'$, then we call x^* a **Linear Functional**.

Proposition 2.4.49 (**Linearly Independent Linear Functionals**)**.** Let V be a **Vector Space** over a **Field** $\mathbb{F}$. Let $\{x_i^*\}_{i=1}^n \subset V'$ be **Linearly Independent**. Let $\{e_i\}_{i=1}^n$ be the **Standard Basis** for $\mathbb{F}^n$. Define $S : V \to \mathbb{F}^n$ by $S(v) = \sum_{i=1}^n \langle v, x_i^* \rangle e_i$ Then S is **Surjective**.

Proof. Supose otherwise. Then there exists $0 \neq c \in \mathbb{F}^n$, represented $c = \sum_{i=1}^n c_i e_i$ such that $c \perp Range(S)$. Hence, for every $x \in X$,

$$\begin{aligned} 0 &= \left\langle \sum_{i=1}^n \langle x, x_i^* \rangle e_i, c \right\rangle \\ &= \left\langle \sum_{i=1}^n \langle x, x_i^* \rangle e_i, \sum_{i=1}^n c_i e_i \right\rangle \\ &= \sum_{i=1}^n \langle x, x_i^* \rangle \overline{c_i} \\ &= \left\langle x, \sum_{i=1}^n \overline{c_i} x_i^* \right\rangle \end{aligned}$$

This implies that $\sum_{i=1}^n \overline{c_i} x_i^* = 0$, a contradiction. $\square$

Proposition 2.4.50 (Functional Kernel Intersection)**.** Let V be a **Vector Space** over a **Field** $\mathbb{F}$. Let $\{v_i\}_{i=1}^n \subset V'$. Let $v \in V'$. Suppose $\bigcap_{i=1}^n Kernel(v_i) \subset Kernel(v)$. Then $v \in span(v_1, \cdots, v_n)$.

Proof. Without loss of generality we let $\{v_i\}_{i=1}^n$ be **Linearly Independent**. Let $\{e_i\}_{i=1}^n$ be the **Standard Basis** of $\mathbb{F}^n$. Define $S : V \to \mathbb{F}^n$ by $S(x) = \sum_{i=1}^n \langle x, v_i \rangle e_i$. By 2.4.49, S is **Surjective**. Let $Q : V \to V/Kernel(S)$ be the **Quotient Map**. By 2.4.46, $\tilde{S} : V/Kernel(S) \to \mathbb{F}^n$ defined by $\tilde{S}([x]) = S(x)$ is a **Linear Bijection** which satisfies $S = \tilde{S} \circ Q$ and $Q = \tilde{S}^{-1} \circ S$. Since $Kernel(S) = \bigcap_{i=1}^n Kernel(v_i) \subset Kernel(v)$, there exists $\tilde{v} : X/Kernel(S) \to \mathbb{F}$ such that $v = \tilde{v} \circ Q$. Also, $\tilde{v} \circ \tilde{S}^{-1} : \mathbb{F}^n \to \mathbb{F}$ is **Linear**, so there are $\{\sigma_i\}_{i=1}^n \subset \mathbb{F}$ such that if $\sum_{i=1}^n x_i e_i \in \mathbb{F}^n$, we have

$$\tilde{v} \circ \tilde{S}^{-1} \left(\sum_{i=1}^n x_i e_i \right) = \sum_{i=1}^n x_i \left(\tilde{v} \circ \tilde{S}^{-1} \right) (e_i) = \sum_{i=1}^n x_i \sigma_i$$

Hence, for $x \in V$, we have

$$\begin{aligned}
\langle x, v\rangle &= \tilde{v} \circ Q(x) \\
&= \tilde{v} \circ \tilde{S}^{-1} \circ S(x) \\
&= \left(\tilde{v} \circ \tilde{S}^{-1}\right)\left(\sum_{i=1}^{n} \langle x, v_i\rangle e_i\right) \\
&= \sum_{i=1}^{n} \langle x, v_i\rangle \sigma_i \\
&= \left\langle x, \sum_{i=1}^{n} \sigma_i v_i \right\rangle
\end{aligned}$$

Hence $v \in span(v_1, \cdots, v_n)$. □

Definition 2.4.51 (**Balanced**)**.** Let V be a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $S \subset V$. We call S a **Balanced Set** and we say that S is **Balanced** if for each $\alpha \in \mathbb{F}$ with $|\alpha| \leq 1$ we have $\alpha S \subset S$.

Definition 2.4.52 (**Absorbing**)**.** Let V be a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $A, B \subset V$. We say that A **Absorbs** B if there exists a $c > 0$ such that $B \subset cA$. In such a scenario, A is also said to **Absorb** B, and we say that B is **Absorbed** by A. If A **Absorbs** every singleton in V, then we call A an **Absorbing Set** or we say that A is **Absorbing**.

Definition 2.4.53 (**Scaling Operator**)**.** Let V be a **Vector Space** over a **Field** $\mathbb{F}$. Let $\alpha \in \mathbb{F}$. We define $M_\alpha : V \to V$ by setting, for each $x \in V$, $M_\alpha(x) = \alpha x$. We call M_α the **Scaling Operator** of V by α.

Proposition 2.4.54 (**Scaling Operator**)**.** Let V be a **Vector Space** over a **Field** $\mathbb{F}$. Let $\alpha, \beta \in \mathbb{F}$. Then $M_\alpha \circ M_\beta = M_{\alpha*\beta}$.

Proof. Let $v \in V$. Then

$$\begin{aligned}
M_\alpha \circ M_\beta v &= M_\alpha(\beta * v) \\
&= \alpha * (\beta * v) \\
&= (\alpha * \beta) * v \\
&= M_{\alpha*\beta} v
\end{aligned}$$

□

Definition 2.4.55 (**Interval**)**.** Let V be a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $x, y \in V$. We define the following sets:

$$\begin{aligned}
[x, y] &= \{tx + (1-t)y : t \in [0, 1]\} \\
[x, y) &= \{tx + (1-t)y : t \in [0, 1)\} \\
(x, y] &= \{tx + (1-t)y : t \in (0, 1]\} \\
(x, y) &= \{tx + (1-t)y : t \in (0, 1)\}
\end{aligned}$$

We refer to any of these sets as **Intervals** in V. Even in the absence of a topological structure, we use the following language:

1. $[x, y]$ is called a **Closed Interval**.
2. (x, y) is called an **Open Interval**.
3. $(x, y]$ and $[x, y)$ are called **Half-Open Intervals** or **Half-Closed Intervals**.

Definition 2.4.56 (**Convex**)**.** Let V be a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $K \subset V$. We say that K is **Convex** if for every pair $x, y \in K$, we have $[x, y] \subset K$.

2.5 Pseudometrics

Definition 2.5.1 (**Triangle Inequality**)**.** Let X be a nonempty set and $(Y, +, \leq)$ be a **Totally Ordered Magma**. We say that a map $f : X \times X \to Y$ satisfies the **Triangle Inequality** if for each $x_0, x_1, x_2 \in X$, we have

$$f(x_0, x_2) \leq f(x_0, x_1) + f(x_1, x_2)$$

Definition 2.5.2 (**Pseudometric**)**.** Let X be a nonempty set. Let $d : X \times X \to [0, \infty)$ be **Commutative**, satisfy the **Triangle Inequality**, and for each $x \in X$,

$$d(x, x) = 0$$

Then we call d a **Pseudometric** on X and we call (X, d) a **Pseudometric Space**.

Definition 2.5.3 (**Metric**)**.** Let (X, d) be a **Pseudometric Space**. If d has the property that for $x, y \in X$, if $x \neq y$, then

$$d(x, y) \neq 0$$

Then we call d a **Metric** on X and we call (X, d) a **Metric Space**

Definition 2.5.4 (**Isometry**)**.** For $i \in \{0, 1\}$, let (X_i, d_i) be **Pseudometric Spaces**. Let $f : X_0 \to X_1$ satisfy, for each $x, y \in X_0$,

$$d_0(x, y) = d_1 \left(f(x_0), f(x_1)\right)$$

Then we call f an **Isometry** between X_0 and $Range(f)$, we say that (X_0, d_0) and (X_1, d_1) are **Isometric**, and we say that f operates **Isometrically**.

Definition 2.5.5 (Pseudometric Cauchy Sequence)**.** Let (X, d) be a **Pseudometric Space**. We say that a sequence $\{x_i\}_{i \in \mathbb{N}}$ is a **Pseudometric Cauchy Sequence** if, for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each pair $m, n \in \mathbb{N}$ such that $m > N$ and $n > N$, we have

$$d(x_m, x_n) < \epsilon$$

Definition 2.5.6 (**Pseudometric-Convergence**)**.** Let (X, d) be a **Pseudometric Space**. Let $\{x_i\}_{i \in \mathbb{N}}$ be a **Sequence** in (X, d). Let $x_0 \in X$. We say that $\{x_i\}_{i \in \mathbb{N}}$ exhibits **Pseudometric-Convergence** to x_0 in (X, d), or we say that $\{x_i\}_{i \in \mathbb{N}}$ **Pseudometric-Converges** to x_0 in (X, d), or we say that $\{x_i\}_{i \in \mathbb{N}}$ is **Pseudometrically-Convergent** to $x_0 \in X$ if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for every $n > N$, we have

$$d(x_0, x_n) < \epsilon$$

Proposition 2.5.7 (Convergent Implies Cauchy)**.** Let (X, d) be a **Pseudometric Space**. Let $\{x_i\}_{i\in\mathbb{N}}$ be a **Pseudometrically-Convergent Sequence**. Then $\{x_i\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence**.

Proof. Since $\{x_i\}$ converges, let $x_i \to x$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $n > N$, we have $d(x_i, x) < \frac{\epsilon}{2}$. For this N, if $m, n > N$, then we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \epsilon$$

and so the sequence is a **Pseudometric Cauchy Sequence**, as advertised. □

Definition 2.5.8 (Uniformly Cauchy)**.** Let A be a nonempty set. For each $\alpha \in A$, let (X_α, d_α) be a **Pseudometric Space**. For each $\alpha \in A$, let $\phi_\alpha := \{x_i^\alpha\}_{i\in\mathbb{N}}$ be a **Sequence** in X_α. We say that the collection $\{\phi_\alpha\}_{\alpha\in A}$ is **Uniformly Cauchy** if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for each pair $m, n \in N$ such that $m > N$ and $n > N$, and for each $\alpha \in A$, we have

$$d_\alpha\left(x_n^\alpha, x_m^\alpha\right) < \epsilon$$

Definition 2.5.9 (**Uniform Convergence**)**.** Let A be a nonempty set. For each $\alpha \in A$, let (X_α, d_α) be a **Pseudometric Space** and let $\phi_\alpha := \{x_i^\alpha\}_{i\in\mathbb{N}} \subset X_\alpha$ be a **Sequence** in X_α. We say that the collection $\{\phi_\alpha\}_{\alpha\in A}$ is **Uniformly Convergent** to $\{x_\alpha\}_{\alpha\in A} \in \prod_{\alpha\in A} X_\alpha$ if for each $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for each $n > N$, and for every $\alpha \in A$, we have

$$d_\alpha(x_i^\alpha, x_\alpha) < \epsilon$$

In this scenario, we may equivalently say that $\{\phi_\alpha\}$ demonstrates **Uniform Convergence** to $\{x_\alpha\}_{\alpha\in A}$ or that it **Converges Uniformly** to $\{x_\alpha\}_{\alpha\in A}$. When we mention **Uniform Convergence** without specifying the limit, we are only claiming that one exists.

Proposition 2.5.10 (Uniform Cauchy and Pointwise Convergence implies Uniform Convergence)**.** Let A be a nonempty set. For each $\alpha \in A$, let (X_α, d_α) be a **Pseudometric Space** and let $\phi_\alpha := \{x_i^\alpha\}_{i\in\mathbb{N}} \subset X_\alpha$ be a **Sequence**. Suppose the collection $\{\phi_\alpha\}_{\alpha\in A}$ is **Uniformly Cauchy** and that each ϕ_α is **Pseudometrically-Convergent**, say $x_i^\alpha \to x_\alpha$. Then $\{\phi_\alpha\}_{\alpha\in A}$ is **Uniformly Convergent** to $\{x_\alpha\}_{\alpha\in A}$.

Proof. Let $\epsilon > 0$. Then, since $\{\phi_\alpha\}_{\alpha\in A}$ is **Uniformly Cauchy**, there is an $N \in \mathbb{N}$ such that for $m, n > N$, we have $d_\alpha(x_n^\alpha, x_m^\alpha) < \frac{\epsilon}{2}$. Since each ϕ_α **Pseudometric-Converges** to x_α, there are $N_\alpha \in \mathbb{N}$. such that for any $n_\alpha > N_\alpha$, we have $d_\alpha(x_{n_\alpha}^\alpha, x_\alpha) < \frac{\epsilon}{2}$. For each $\alpha \in A$, define $M_\alpha = max(N+1, N_\alpha+1)$. Let $n > N$. Then, for any $\alpha \in A$, we have.

$$\begin{aligned}
d_\alpha(x_n^\alpha, x_\alpha) &\leq d_\alpha(x_n^\alpha, x_{M_\alpha}^\alpha) + d_\alpha(x_{M_\alpha}, x_\alpha) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

completing the proof. □

Definition 2.5.11 (Pseudometric Complete)**.** We say that a **Pseudometric Space** (X, d) is **Pseudometric-Complete** if each **Pseudometric Cauchy Sequence** sequence in (X, d) **Pseudometric-Converges** to a limit in X.

In the case that d is a **Metric**, then being **Pseudometric-Complete** is equivalent to being **Complete** in the classical sense, so we will commonly refer to a **Pseudometric Space** which is **Pseudometric-Complete** as simply being **Complete**.

Definition 2.5.12 (Pseudometric Ball)**.** Let (X, d) be a **Pseudometric Space**. For each $x_0 \in X$ and each $\epsilon > 0$, we define the following.

1. $B_d(x_0, \epsilon) := \{y \in X : d(x_0, y) < \epsilon\}$ denotes the **Open Ball** about x_0 with radius ϵ.
2. $\overline{B_d}(x_0, \epsilon) := \{y \in X : d(x_0, y) \leq \epsilon\}$ denotes the **Closed Ball** about x_0 with radius ϵ.
3. We call the **Open Ball** about x_0 with radius 1 the **Open Unit Ball** about x_0.
4. We call the **Closed Ball** about x_0 with radius 1 the **Closed Unit Ball** about x_0.

Definition 2.5.13 (**Pseudometric Topology**)**.** Let (X, d) be a **Pseudometric Space**, and let $\mathcal{B}$ be the set of **Open Ball**'s in (X, d). By 2.5.14, $\mathcal{B}$, with the addition of $\emptyset$, is the **Basis** for a unique **Topology** $\mathcal{T}_d$ on X. We call $\mathcal{T}_d$ the **Pseudometric Topology** induced by d on X.

Proposition 2.5.14 (**Pseudometric Topology**)**.** Let (X, d) be a **Pseudometric Space** and let $\mathcal{B}$ be the set of **Open Ball**'s in (X, d), along with $\emptyset$. Let $\mathcal{T}_d$ denote $\mathcal{T}_X(\mathcal{B})$. The following are true.

(i) $\mathcal{B}$ is a **Basis** for $\mathcal{T}_d$.

(ii) The **Pseudometric Topology** is **First Countable**.

Proof of 2.5.14 i. Let $U, V \in \mathcal{B}$. Let $x \in U \cap V$. Then there exists $x_u, x_v \in X$ and $\epsilon_u, \epsilon_v > 0$ such that $U = B_d(x_u, \epsilon_u)$ and $V = B_d(x_v, \epsilon_v)$. Define $\delta = min\left(\frac{\epsilon_u - d(x,x_u)}{2}, \frac{\epsilon_v - d(x,x_v)}{2}\right)$. Let $y \in B(x; \delta)$. Then

$$d(y, x_u) \leq d(y, x) + d(x, x_u) \leq \delta + d(x, x_u) \leq \frac{\epsilon_u}{2} + \frac{d(x, x_u)}{2} < \epsilon_u$$

Similarly, $d(y, x_v) < \epsilon_v$, so $x \in B_d(x; \delta) \subset U \cap V$. By 2.3.17, the result holds. □

Proof of 2.5.14 ii. Let $x_0 \in X$. I claim that

$$\mathcal{B}_{x_0} := \left\{B_d\left(x_0; \frac{1}{n}\right) : n \in \mathbb{N}\right\}$$

is a **Neighborhood Basis** for $(X, \mathcal{T}_d)$ at x_0. Let $U \in \mathcal{U}_{\mathcal{T}_d}(x)$ be **Open** in $\mathcal{T}_d$. Since $\mathcal{B}$ is a **Basis** for $\mathcal{T}_d$, for some $y_0 \in X$ and $\epsilon > 0$, $x_0 \in B_d(y_0; \epsilon) \subset U$. Let $\delta = d(x_0, y_0)$. Then $\epsilon - \delta > 0$. Define

$$n = \left\lceil \frac{1}{\epsilon - \delta} \right\rceil$$

Then we have

$$B_d\left(x_0; \frac{1}{n}\right) \subset B_d(x_0 : \epsilon - \delta) \subset B(y_0; \epsilon) \subset U$$

□

Definition 2.5.15 (Relation Of Zero Distance)**.** Let (X, d) be a **Pseudometric Space**. Define the relation $\cong_d$ on $X \times X$ by setting, for $x, y \in X$,

$$x \cong_d y \iff d(x, y) = 0$$

We call $\cong_d$ the **Relation Of Zero Distance** on (X, d).

Proposition 2.5.16 (Relation Of Zero Distance is the Relation Of Equal Neighborhood Filters)**.** Let (X, d) be a **Pseudometric Space**. Let $\cong_{\mathcal{T}_d}$ be the **Relation Of Equal Neighborhood Filters** $(X, \mathcal{T}_d)$. Let $\cong_d$ be the **Relation Of Zero Distance** on (X, d). Then $\cong_{\mathcal{T}_d} = \cong_d$.

Proof. Let $x, y \in X$ and suppose $x_0 \cong_d y_0$. Let $U \in \mathcal{U}_{\mathcal{T}_d}(x_0)$. Then for some $\epsilon > 0$, $x_0 \in B(x_0; \epsilon) \subset U$. Since $x_0 \cong_d y_0$, $d(x_0, y_0) = 0$, so $y_0 \in B(x_0; \epsilon) \subset U$. Hence $U \in \mathcal{U}_{\mathcal{T}_d}(y_0)$. The arbitrary nature of $U \in \mathcal{U}_{\mathcal{T}_d}(x_0)$ implies $\mathcal{U}_{\mathcal{T}_d}(x_0) \subset \mathcal{U}_{\mathcal{T}_d}(y_0)$. A reverse construction would just as easily show the reverse inclusion, so we conclude that $x_0 \cong_{\mathcal{T}_d} y_0$. Now suppose $x_0 \cong_{\mathcal{T}_d} y$. Then for each $n \in \mathbb{N}$, $y_0 \in B_d\left(x_0; \frac{1}{n}\right)$. Hence $d(x_0, y_0) < \frac{1}{n}$ for each $n \in \mathbb{Z}^+$, and so $d(x_0, y_0) = 0$ and $x_0 \cong_d y_0$. □

Definition 2.5.17 (**Metric Induced By The Pseudometric**)**.** Let (X, d) be a **Pseudometric Space**, and let $\cong$ be the **Relation Of Zero Distance**, which by 2.5.16 is also the **Relation Of Equal Neighborhood Filters** on $(X, \mathcal{T}_d)$. Define $\tilde{d} : X/\cong \to [0, \infty)$ by

$$\tilde{d}\left([x], [y]\right) = d(x, y)$$

By 2.5.18, $\tilde{d}$ is well defined and is in fact a **Metric** on $X/\cong$, so we call $\tilde{d}$ the **Metric Induced By The Pseudometric** d on X, or we call it the **Pseudometric Induced Metric** of (X, d).

Proposition 2.5.18 (Metric Space Induced By Pseudometric Space)**.** Let (X, d) be a **Pseudometric Space**, $\cong$ the **Relation Of Zero Distance** on (X, d) and $\tilde{d}$ be defined as in 2.5.17. Let $(X/\cong, \mathcal{T}_{X/\cong})$ be the **Quotient Topological Space** with **Quotient Map** T, and let $(X/\cong, \mathcal{T}_{\tilde{d}})$ be the **Topological Space** induced by the **Metric Space** $(X/\cong, \tilde{d})$. The following are true.

(i) $\tilde{d}$ is a well defined **Metric** on $X/\cong$.

(ii) $\mathcal{T}_{X/\cong} = \mathcal{T}_{\tilde{d}}$

(iii) T is an **Isometric Surjection** of (X, d) onto $(X/\cong, \tilde{d})$

(iv) (X, d) is a **Metric Space** if and only if T is **Injective**.

(v) $(X/\cong, \tilde{d})$ is complete if and only if (X, d) is **Pseudometric-Complete**.

Proof of 2.5.18 i. First we show that $\tilde{d}$ is well defined as a function, that is, that if $x_0, y_0 \in X$ and $x_1 \cong x_0$ and $y_1 \cong y_0$, then we should have

$$\tilde{d}\left([x_0], [y_0]\right) = \tilde{d}\left([x_1], [y_1]\right)$$

This is easy, as

$$\begin{aligned} d(x_0, y_0) &\leq d(x_0, x_1) + d(x_1, y_1) + d(y_1, y_0) \\ &= d(x_1, y_1) \\ &\leq d(x_1, x_0) + d(x_0, y_0) + d(y_0, y_1) \\ &= d(x_0, y_0) \end{aligned}$$

Nonnegativity falls directly from the nonnegativity of d. Proving that $\tilde{d}$ is **Commutative** is equally trivial

$$\tilde{d}\left([x], [y]\right) = d(x, y) = d(y, x) = \tilde{d}\left([y], [x]\right)$$

Proving that $\tilde{d}$ satisfies the **Triangle Inequality** is similarly simple, letting $x_0, y_0, z_0 \in X$, we have

$$\begin{aligned} \tilde{d}\left([x_0], [z_0]\right) &= d(x_0, z_0) \\ &\leq d(x_0, y_0) + d(y_0, z_0) \\ &= \tilde{d}\left([x_0], [y_0]\right) + \tilde{d}\left([y_0], [z_0]\right) \end{aligned}$$

All that remains is to show positivity on nonequal arguements. Let $x_0, y_0 \in X$ such that $[x_0] \neq [y_0]$. Then $x_0 \ncong y_0$. Hence

$$\tilde{d}\left([x_0], [y_0]\right) = d(x_0, y_0) \neq 0$$

□

Proof of 2.5.18 ii. By 2.3.32. ix, $\mathcal{B}_{\cong} := \{T(B_d(x;\epsilon)) : x \in X, \epsilon > 0\}$ is a **Basis** for $\mathcal{T}_{X/\cong}$. By definition, $\mathcal{B}_{\tilde{d}} := \{B_{\tilde{d}}([x]\,;\epsilon) : x \in X, \epsilon > 0\}$ is a **Basis** for $\mathcal{T}_{\tilde{x}}$.

I claim that for each $x \in X$ and $\epsilon > 0$,

$$T\left(B_d(x;\epsilon)\right) = B_{\tilde{d}}\left([x]\,;\epsilon\right) \tag{2.2}$$

To see this, suppose $\tilde{y} \in T\left(B_d(x;\epsilon)\right)$. Then $\tilde{y} = T(y)$ for some $y \in B_d(x;\epsilon)$. Hence

$$\begin{aligned} \tilde{d}(\tilde{y}, [x]) &= \tilde{d}(T(y), [x]) \\ &= \tilde{d}([y], [x]) \\ &= d(y, x) \\ &< \epsilon \end{aligned}$$

Hence $\tilde{y} \in B_d([x]\,;\epsilon)$, and so

$$T\left(B_d(x;\epsilon)\right) \subset B_{\tilde{d}}\left([x]\,;\epsilon\right) \tag{2.3}$$

Suppose $[y] \in B_{\tilde{d}}([x]; \epsilon)$. Then $d(x,y) = \tilde{d}()\,[x], [y]) < \epsilon$, so $y \in B_d(x;\epsilon)$. Hence $[y] = T(y) \in T(B_d(x;\epsilon))$, so the reverse inclusion also holds, and so the above claim holds. This, paired with the fact that

$$\{[x] : x \in X\} = X/\cong$$

finishes the result.

□

Proof of 2.5.18 iii. Let $x, y \in X$. Then,

$$d(x,y) = \tilde{d}([x], [y]) = \tilde{d}(T(x), T(y))$$

T is **Surjective** by 2.1.47. □

Proof of . Let T be **Injective**. Let $x, y \in X$ with $x \neq y$. Then $T(x) \neq T(y)$, so by 2.5.18 i and 2.5.18 iii, $0 < \tilde{d}(T(x), T(y)) = d(x,y)$. Hence d is a **Metric**.

Now suppose d is a **Metric**. Let $x, y \in X$ with $x \neq y$. Then $d(x,y) > 0$. Hence by 2.5.18 iii, $\tilde{d}(T(x), T(y)) = d(x,y) > 0$. Hence $T(x) \neq T(y)$. □

Proof of 2.5.18 v. Let (X,d) be **Pseudometric-Complete**. Let $\{[x_i]\}_{i\in\mathbb{N}} \subset (X/\cong, \tilde{d})$ be a **Pseudometric Cauchy Sequence**. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $m, n > N$, we have

$$d(x_m, x_n) = \tilde{d}(Tx_m, Ty_m) = \tilde{d}([x_m], [x_n]) < \epsilon$$

So the **Sequence** $\{x_i\}_{i\in\mathbb{N}} \subset (X,d)$ is **Pseudometric Cauchy Sequence**. Since (X,d) is **Pseudometric-Complete**, this **Sequence** has a limit, say $x_i \to x \in (X,d)$. But, we have $[x_i] = Tx_i \to Tx = [x]$, so $\{[x_i]\}$ is convergent, and since that **Sequence** was arbitrary, $(X/\cong, \tilde{d})$ is **Pseudometric-Complete**.

Let $(X/\cong, \tilde{d})$ be **Pseudometric-Complete**. Let $\{x_i\} \subset X$ be a **Pseudometric Cauchy Sequence**. Let $\epsilon > 0$. Then there exist $N \in \mathbb{N}$ such that for $m, n > N$, we have

$$\tilde{d}([x_m], [x_n]) = \tilde{d}(Tx_m, Tx_n) = d(x_m, x_n) < \epsilon$$

so that $\{[x_i]\}_{i\in\mathbb{N}}$ is also a **Pseudometric Cauchy Sequence**. Since $(X/\cong, \tilde{d})$ is **Pseudometric-Complete**, this **Sequence** has a limit, say $[x_i] \to y \in X/\cong$. Since T is **Surjective**, for some $x \in X$, $Tx \in y$, and so

$$d(x, x_i) = \tilde{d}(Tx, Tx_i) = \tilde{d}(y, [x_i]) \to 0$$

meaning $x_i \to x$ and we are done.

□

Remark 2.5.19 (**Metric Space** Correspondence)**.** Note that in the case of a **Metric Space** the condition of being **Pseudometric-Complete** is equivalent to the condition being Complete, and A **Sequence** is a **Pseudometric Cauchy Sequence** if and only if it is a Cauchy **Sequence**.

Definition 2.5.20 (**Pseudometrizable**)**.** Let $(X, \mathcal{T})$ be a **Topological Space**. Let $\tilde{d}$ be a **Pseudometric** on X.

1. We say that $\mathcal{T}$ and $\tilde{d}$ are **Compatible** if $\mathcal{T}$ is the **Pseudometric Topology** on $(X, \tilde{d})$.

2. We say that $(X, \mathcal{T})$ is **Pseudometrizable** if there exists a **Pseudometric** d on X which is **Compatible** with d.

3. We say that $(X, \mathcal{T})$ is **Metrizable** if there exists a **Metric** d on X which is **Compatible** with d.

Proposition 2.5.21 (**Pseudometrizable** Prequotient)**.** Let $(X, \mathcal{T}_X)$ be a **Topological Space** with **Relation Of Equal Neighborhood Filters** $\cong$, and with **Quotient Topological Space** $\left(X/\cong, \mathcal{T}_{X/\cong}\right)$ and **Quotient Map** T. Let $\left(X/\cong, \mathcal{T}_{X/\cong}\right)$ be **Pseudometrizable** with **Pseudometric** $\tilde{d}$. The following hold.

(i) Define $d : X^2 \to [0, \infty)$ by $d(x, y) = \tilde{d}\,([x], [y])$. Then $\tilde{d}$ is a **Pseudometric** on X which is **Compatible** with $\mathcal{T}_X$.

(ii) $\tilde{d}$ is a **Metric** $(X/\cong, \mathcal{T}_{X/\cong})$.

(iii) If T is **Injective**, then d as defined above is a **Metric** on X.

Proof Of 2.5.21 i. We first prove d to be a **Pseudometric** on X. First, observe that if $x, y \in X$, then $d(x, y) = \tilde{d}([x], [y]) \in [0, \infty)$ so that d is well defined. Also, $d(x, y) = \tilde{d}([x], [y]) = \tilde{d}([y], [x]) = d(y, x)$, so d is **Commutative**. Furthermore,

$$\begin{aligned} d(x, z) &= \tilde{d}([x], [z]) \\ &\leq \tilde{d}([x], [y]) + \tilde{d}([y], [z]) \\ &= d(x, y) + d(y, z) \end{aligned}$$

so d satisfies the **Triangle Inequality**. Lastly, $d(x, x) = \tilde{d}([x], [x]) = 0$, and so d is a **Pseudometric** on X.

Let $\mathcal{T}_d$ denote the **Pseudometric Topology** on (X, d). What remains to show is that $\mathcal{T}_X = \mathcal{T}_d$. Since $d(x, y) = \tilde{d}([x], [y]) = \tilde{d}(Tx, Ty)$, T is an **Isometry**. Let $x \in U \in \mathcal{T}_X$. Then $[x] \in T(U) \in \mathcal{T}_{X/\cong}$. Hence, there is an $\epsilon > 0$ such that $B_{\tilde{d}}([x], \epsilon) \subset T(U)$. By 2.3.32. iv, $T^{-1}(B_{\tilde{d}}([x], \epsilon) \subset T^{-1}(T(U)) = U$. Furthermore, by 2.3.32. ii, $T^{-1}(B_{\tilde{d}}([x], \epsilon) \in \mathcal{T}_X$. Since T is an **Isometry** $B_d(x, \epsilon) = T^{-1}(B_{\tilde{d}}([x], \epsilon) \subset U$. Thus we have found an **Open Ball** contained in U containing an arbitrary point of U. Hence, $\mathcal{T}_X \subset \mathcal{T}_d$. As part of the preceeding arguement we also showed that an arbitrary d-**Open Ball** was in $\mathcal{T}_X$, so $\mathcal{T}_d \subset \mathcal{T}_X$, and so equality holds and we're done. □

Proof of 2.5.21 ii. Let $x, y \in X$ wtih $[x] \neq [y]$. Then $x \ncong y$. By 2.5.16, $x \ncong_d y$. Hence $\tilde{d}([x], [y]) = d(x, y) > 0$. □

Proof of 2.5.21 iii. Let T be **Injective**, and suppose $x, y \in X$ with $x \neq y$. Then $[x] = Tx \neq Ty = [y]$, so by 2.5.21 ii, $d(x, y) = \tilde{d}([x], [y]) > 0$.

□

2.6 Topological Algebra

2.6.1 Topological Groups

Definition 2.6.1 (**Topological Group**)**.** Let $(G, +, e)$ be a **Group**. Let $\mathcal{T}$ be a **Topology** on G such that $+ : G \times G \to G$ and $T^{-1}{}_G$ are both **Continuous** with respect to the **Product Topology**. Then we call $(G, \mathcal{T})$ a **Topological Group**.

Proposition 2.6.2 (Group Invariance)**.** Let $(G, \mathcal{T})$ be a **Topological Group**. Let $x \in G$. Then $T^R{}_x$, $T^L{}_x$, and $T^{-1}{}_G$ are **Homeomorphisms** of G with itself.

Proof. Let $x \in G$. Let $y \in G$. Let $V \in \mathcal{T}$ with $T^L{}_x(y) \in V$. Then $xy \in V$. That is, $(x, y) \in *^{-1}(V)$ By **Continuity** of $*$, there are $U_1, U_2 \in \mathcal{T}$, $x \in U_1$, $y \in U_2$ such that $U_1U_2 \subset V$. Since $T^L{}_x(U_2) = xU_2 \subset U_1U_2 \subset V$, $U_2 \subset L_x^{-1}(V)$. Since V was arbitrary, $T^L{}_x$ is **Continuous At** y. Since y was arbitrary, $T^L{}_x$ is **Continuous**. Similarly, $T^L{}_{x^{-1}}$ is **Continuous**. Since $\left(T^L{}_x\right)^{-1} = T^L{}_{x^{-1}}$, $T^L{}_x$ has a **Continuous** inverse and so is a **Homeomorphism**. The proof for $T^R{}_x$ is virtually unchanged. Since $\left(T^{-1}{}_G\right)^{-1} = T^{-1}$ and $T^{-1}{}_G$ is **Continuous**, it is a **Homeomorphism**. □

Definition 2.6.3 (**Local Basis**)**.** Let $(G, \mathcal{T})$ be a **Topological Group** with **Identity Element** e. We call a **Neighborhood Basis** of $\mathcal{T}$ about e a **Local Basis** for $(G, \mathcal{T})$.

Proposition 2.6.4 (Symmetric Contained)**.** Let $(G, \mathcal{T})$ be a **Topological Group** with **Identity Element** e. Let $e \in U \in \mathcal{T}$. Then there exists an **Open Symmetric** V containing e such that $VV \subset U$.

Proof. Since $ee = e \in U$, and the **Group** operation is is **Continuous**, there are **Open** V_1 and V_2 with $e \in V_1$, $e \in V_2$ such that $V_1V_2 \subset U$. Define $V = V_1 \cap V_2 \cap V_1^{-1} \cap V_2^{-1}$. Since $T^{-1}{}_G$ is a **Homeomorphism** V_1^{-1} and V_2^{-1} are **Open**. Hence V is **Open**. Also, $e \in V$, so $\emptyset \neq V$ and V is clearly **Symmetric**. Furthermore $VV \subset V_1V_2 \subset U$. concluding the proof. □

Proposition 2.6.5 (Local Basis Symmetric)**.** It is clear by 2.6.4 that any **Topological Group** contains a **Local Basis** consisting of only **Symmetric** sets, and for any **Local Basis** of a **Topological Group**, we can find a **Local Basis** of the same **Cardinality** which is **Symmetric**.

Proposition 2.6.6 (Group Separation)**.** Let $(G, \mathcal{T})$ be a **Topological Group** with **Identity Element** e. Let $K \subset G$ be **Compact**. Let $C \subset G$ be **Closed**. Let $K \cap C = \emptyset$. Then there exists an **Symmetric Open** V containing e such that $VKV \cap VCV = \emptyset$.

Proof. Let $y \in K$. Then $y \in G \setminus C$. Since C is **Closed**, there is an **Open Symmetric** $\tilde{V}_y$ containing e such that $\tilde{V}_y\{y\} \cap C = \emptyset$. By 3 applications of 2.6.4 there exists an **Open Symmetric** V_y containing e such that $V_y^8 \subset \tilde{V}_y$. Then $V_y^8\{y\} \cap C = \emptyset$. Then, since by 2.6.2, $T^L{}_y$ is a **Homeomorphism**, there is an **Open** U_y containing e such that $yU_y \subset V_y y$. By 2.6.4 we can without loss of generality assume U_y is **Symmetric**. Hence $V_y^4\{y\}U_y^4 \cap C = \emptyset$. Define $W_y = U_y \cap V_y$. Then W_y is **Open**, **Symmetric**, $e \in W_y$, and $W_y^4\{y\}W_y^4 \cap C = \emptyset$. Since W_y is **Symmetric**, $\left(W_y^2\{y\}W_y^2\right) \cap \left(W_y^2 C W_y^2\right) = \emptyset$.

Thus we have, for each $x \in K$, a **Open**, **Symmetric** W_x containing e such that $(W_x^2\{x\}W_x^2) \cap (W_x^2 C W_x^2) = \emptyset$. Since $K \subset \bigcup_{x \in K} W_x\{x\}W_x$, we can find a $\{x_i\}_{i=1}^n \subset K$ such that $K \subset \bigcup_{i=1}^n W_{x_i}\{x_i\}W_{x_i}$. Define $W = \bigcap_{i=1}^n W_{x_i}$. Then

$$\begin{aligned}
WKW &\subset W\left(\bigcup_{i=1}^n W_{x_i}\{x_i\}W_{x_i}\right)W \\
&= \bigcup_{i=1}^n (WW_{x_i}\{x_i\}W_{x_i}W) \\
&\subset \bigcup_{i=1}^n (W_{x_i}^2\{x_i\}W_{x_i}^2) \\
&\subset G \setminus \left(\bigcap_{i=1}^n W_{x_i}^2 C W_{x_i}^2\right) \\
&\subset G \setminus \left(\bigcap_{i=1}^n W_{x_i} C W_{x_i}\right) \\
&\subset G \setminus \left(\left(\bigcap_{i=1}^n W_{x_i}\right) C \left(\bigcap_{i=1}^n W_{x_i}\right)\right) \\
&= G \setminus (WCW)
\end{aligned}$$

□

Proposition 2.6.7 (**Closed** Product)**.** Let $(G, \mathcal{T})$ be a **Topological Group** with **Identity Element** e. Let $C \subset G$ be **Closed**. Let $K \subset G$ be **Compact**. Then CK is **Closed** and KC is **Closed**.

Proof. Let $x \notin CK$. Then $(C^{-1}x) \cap K = \emptyset$. By 2.6.2, $C^{-1}x$ is **Closed**. Hence, by 2.6.6, there exists an **Open** **Symmetric** V contining e such that $(C^{-1}xV) \cap (KV) = \emptyset$. Hence $(xVV^{-1}) \cap (CK) = \emptyset$, so $x \notin \overline{CK}$, so $CK = \overline{CK}$. Hence CK is **Closed**. The proof for KC is virtually identical. □

Proposition 2.6.8 (Local Base Nesting)**.** Let G be a **Topological Group** with **Identity Element** e. Let $\mathcal{B}$ be a **Local Basis** for G. Let $b_0 \in \mathcal{B}$. Then there exists $b_1 \in \mathcal{B}$ such that $\overline{b_1} \subset b_0$.

Proof. Let $b_0 \in B$. Then $C = G \setminus b_0$ is **Closed** and $K = \{e\}$ is **Compact**. Furthermore $e \in b_0$, so $K \cap C = \emptyset$, so by 2.6.6, there is an **Open** V such that

$$V \cap ((G \setminus b_0)V) = (eV) \cap ((G \setminus b_0)V) = \emptyset$$

If $x \notin b_0$, then $xV \subset (G \setminus b_0)V$ is **Disjoint** from V, so $x \notin \overline{V}$. Hence $\overline{V} \subset b_0$. Since $\mathcal{B}$ is a **Local Basis**, there is a $b_1 \in \mathcal{B}$ with $b_1 \subset V$. Hence $\overline{b_1} \subset \overline{V} \subset b_0$. □

Proposition 2.6.9 (Closure Characterization)**.** Let G be a **Topological Group**. Let $\mathcal{B} = \{V_\alpha\}_{\alpha\in B}$ be a **Local Basis** for G. Let $A \subset X$. Then

$$\overline{A} = \bigcap_{\alpha\in B} (A + V_\alpha)$$

Proof. It is clear that by 2.6.5, we can, without loss of generality, assume that $\mathcal{B}$ consists of **Symmetric** sets. Let $x \in \overline{A}$. This is true if and only if, for each $\alpha \in B$, $(x + V_\alpha) \cap A = \emptyset$. Since $\mathcal{B}$ consists of **Symmetric** sets, this is true if and only if, for each $\alpha \in B$, $\{x\} \cap (A + V_\alpha) = \{x\} \cap (A - V_\alpha) \neq \emptyset$. This is equivalent to saying that $x \in A + V_\alpha$ for every $\alpha \in B$. This is then equivalent to

$$x \in \bigcap_{\alpha\in B} (A + V_\alpha)$$

Since every step involved here was a logical equivalence,

$$\overline{A} = \bigcap_{\alpha\in B} (A + V_\alpha)$$

□

Proposition 2.6.10 (Sum Closure)**.** Let $(G, \mathcal{T})$ be a **Topological Group** with **Identity Element** e and let $A_0, A_1 \subset G$. Then

$$\overline{A_0} + \overline{A_1} \subset \overline{A_0 + A_1} \tag{2.4}$$

Furthermore, if there exists a **Compact** K such that either $A_0 \subset K$ or $A_1 \subset K$, then

$$\overline{A_0} + \overline{A_1} = \overline{A_0 + A_1}$$

Proof. Let $a_i \in \overline{A_i}$ for $i \in \{0, 1\}$. Let $e \in V \in \mathcal{T}$. By 2.6.4 there is a **Symmetric** V' satisfying $e \in V' \in \mathcal{T}$ and $V' + V' \subset V$. By 2.6.2, there exists a **Symmetric Open** $\tilde{V}$ containing e such that $\tilde{V}a_1 \subset a_1V'$ and $\tilde{V} \subset V'$. Since $a_i \in \overline{A_i}$, $A_i \cap \left(a_i + \tilde{V}\right) \neq \emptyset$. Hence, for $i \in \{1, 2\}$, there exists $x_i \in A_i \cap \left(a_i + \tilde{V}\right)$. Hence,

$$\begin{aligned} x_0 + x_1 &\in \left(A_0 \cap (a_0 + \tilde{V})\right) + \left(A_1 \cap (a_1 + \tilde{V})\right) \\ &\subset (A_0 + A_1) \cap \left(a_0 + \tilde{V} + a_1 + \tilde{V}\right) \\ &\subset (A_0 + A_1) \cap (a_0 + a_1 + V' + V') \\ &\subset (A_0 + A_1) \cap (a_0 + a_1 + V) \neq \emptyset \end{aligned}$$

Since $e \in V \in \mathcal{T}$ was arbitrary, $a_0 + a_1 \in \overline{A_0 + A_1}$, Hence

$$\overline{A_0} + \overline{A_1} \subset \overline{A_0 + A_1}$$

Now, if there is a compact K such that $A_0 \subset K$, then $\overline{A_0} \subset \overline{K}$. Furthermore, by 2.3.54, $\overline{K}$ is **Compact**. Hence, by 2.3.50, $\overline{A_0}$ is **Compact**. Hence, by 2.6.7 $\overline{A_0} + \overline{A_1}$ is **Closed**. Thus, we have

$$\begin{aligned}\overline{A_0 + A_1} &\subset \overline{\overline{A_0} + \overline{A_1}} \\ &= \overline{A_0} + \overline{A_1}\end{aligned}$$

So equality holds. If instead $A_1 \subset K$, the argument is identical.

□

Proposition 2.6.11 (Birkhoff-Kakutani)**.** Let $(G, \mathcal{T})$ be a **Topological Group** with **Identity Element** e. Let $\mathcal{B} = \{\mathcal{V}_i\}_{i\in\mathbb{N}}$ be a **Local Basis** for G. Then there exists a **Pseudometric** d on G with the following properties.

1. $\mathcal{T}$ is the **Pseudometric Topology** on G induced by d.
2. For each $x, y, z \in G$, $d(zx, zy) = d(x, y)$.

Proof. By 2.6.4 G has a **Local Basis** $\{V_i\}_{i\in\mathbb{N}}$ consisting of **Symmetric** subsets satisfying, for each $n \in \mathbb{N}$, $V_{n+1}V_{n+1} \subset V_n$. For each $r \in \mathbb{Q} \cap [0, 1)$, let $C(n, r) \in \{0, 1\}$ be the n^{th} bit of $r's$ finite expansion. That is, for $r \in \mathbb{Q}$, let

$$r = \sum_{n\in\mathbb{N}} \frac{C(n, r)}{2^n}$$

Let P be the set of $r \in \mathbb{Q} \cap [0, 1)$ for which $C(n, r)$ is nonzero for only finitely many r. For each $r \in P$, let Γ_r denote the $n \in \mathbb{N}$ for which $C(n, r) \neq 0$. Since G is not assumed to be **Commutative**, define, for a **Finite** set $K = \{x_k\}_{k=1}^m \subset \mathbb{N}$ in which $x_1 < x_2 < \cdots < x_m$,

$$\prod_{n\in K} U_n = U_{x_1}U_{x_2}U_{x_3}\cdots U_{x_m}$$

Now, define $A : P \cup (1, \infty) \to \mathcal{T}$ by

$$A(r) = \begin{cases} \prod_{n\in\Gamma_r} V_n & r < 1 \\ G & r \geq 1 \end{cases}$$

I first claim that if $r \leq s$, then $A(r) \subset A(s)$. To prove this, let $r \leq s$. Define Γ_r and Γ_s as above. Then there exists a $k \in \mathbb{N}$ such that $\Gamma_r \cap [0, k] \subset \Gamma_s$ and $k + 1 \in \Gamma_s \setminus \Gamma_r$. Define $N = [k + 2, \infty) \cap \Gamma_r$. Then since $V_{n+1}V_{n+1} \subset V_n$ for all n, it is clear that

$$\prod_{n\in N} V_n \subset V_{k+1}$$

Hence,

$$\begin{aligned}
A(r) &= \left(\prod_{n\in\Gamma_r\cap[0,k]} V_n\right)\left(\prod_{n\in N} V_n\right)\\
&\subset \left(\prod_{n\in\Gamma_r\cap[0,k]} V_n\right) V_{k+1}\\
&\subset \left(\prod_{n\in\Gamma_s\cap[0,k]} V_n\right) V_{k+1}\\
&\subset A(s)
\end{aligned}$$

I now make a second claim: that if $r \in P \cap (0,1)$ and if $n > \max\{k \in \mathbb{Z}^+ : C(rk,r) = 1\}$ then $A(r)A\left(\frac{1}{2^n}\right) = A\left(r + \frac{1}{2^n}\right)$. This is clear because, for all $k \in \mathbb{Z}^+$,

$$C(k,r) + C\left(k, \frac{1}{2^n}\right) = C\left(k, r + \frac{1}{2^n}\right)$$

Thus we have

$$A(r)A\left(\frac{1}{2^n}\right) = \left(\prod_{k\in\Gamma_r} U_k\right) U_n = \prod_{k\in\Gamma_{r+\frac{1}{2^n}}} U_k = A\left(r+\frac{1}{2^n}\right)$$

I now claim that for every $n \in \mathbb{Z}^+$, for every $r \in P$, we have

$$A(r)A\left(\frac{1}{2^n}\right) \subset A\left(r + \frac{3}{2^n}\right)$$

By the first and second claims, it is sufficient to prove in the case $n \leq \max\{n \in \mathbb{Z}^+ : C(n,r) = 1\}$. Now, Define $\Gamma_r^- = \Gamma_r \cap [0,n)$ and $\Gamma_r^+ = \Gamma_r \cap [n,\infty)$. Then by assumption $\Gamma_r^+ \neq \emptyset$. Define

$$r_1 = \frac{1}{2^{n-1}} - \sum_{j\in\Gamma_r^+} \frac{1}{2^j}$$

Then $r_1 > 0$ and Define

$$r_2 = r + r_1$$

Then $\max\{k \in \mathbb{Z}^+ : C(r_2,k) \neq 0 \leq n-1\}$. Hence, by claim 2, $A(r_2)A\left(\frac{1}{2^n}\right) = A\left(r_2 + \frac{1}{2^n}\right)$. Also $r < r_2 < r + \frac{1}{2^{n-1}}$. By this observation, paired with claim 01, we have

$$\begin{aligned}
A(r)A\left(\frac{1}{2^n}\right) &\subset A(r_2)A\left(\frac{1}{2^n}\right)\\
&= A\left(r_2 + \frac{1}{2^n}\right)\\
&\subset A\left(r + \frac{1}{2^{n-1}} + \frac{1}{2^n}\right)\\
&\subset A\left(r + \frac{3}{2^n}\right)
\end{aligned}$$

Define, for $x \in G$, $\tilde{d}(x) = \inf\{r \in [0, \infty) : x \in A(r)\}$. Then $\tilde{d}(x) \leq 1$ for all $x \in G$. Define, for $x, y \in G$,

$$d(x, y) = \sup_{h \in G} \left\{ \left| \tilde{d}(hx) - \tilde{d}(hy) \right| \right\}$$

Since $\tilde{d}$ is bounded, d is well defined. d is clearly **Commutative**, satisfies the **Triangle Inequality**, and satisfies $d(x, x) = 0$, so d is a **Pseudometric** on G. That d is left invariant is equally clear.

What remains to show is that the **Pseudometric Topology** $\mathcal{T}_d$ generated by d on G Since d is left invariant, it suffices to show that for each $\epsilon > 0$, there is an n such that $V_n \subset B(e; \epsilon)$, and for each $k \in \mathbb{N}$, there is a $\delta > 0$ such that $B(e; \delta) \subset V_k$.

Showing $\mathcal{T} \subset \mathcal{T}_d$ is easy. Given $n \in \mathbb{N}$,

$$B\left(e, \frac{1}{2^{n+1}}\right) \subset A\left(\frac{1}{2^n}\right) = U_n$$

For the other direction, let $\epsilon > 0$. Let $n \in \mathbb{N}$ such that $\frac{3}{2^n} < \epsilon$. Let $u \in U_n = A\left(\frac{1}{2^n}\right)$. Let $z \in G$ and let r be any positive number such that $z \in A(r)$. Then

$$zu \in A(r)A\left(\frac{1}{2^n}\right) \subset A\left(r + \frac{3}{2^n}\right)$$

Hence,

$$\tilde{d}(zu) \leq r + \frac{3}{2^n}$$

The nature of r implies

$$\tilde{d}(zu) \leq \inf\{r \in (0, \infty) : z \in A(r)\} + \frac{3}{2^n} = \tilde{d}(z) + \frac{3}{2^n} \tag{2.5}$$

Now let s be any positive number such that $zu \in A(s)$. Then

$$z \in A(s)u^{-1} \subset A(s)U_n^{-1} = A(s)U_n \subset A\left(s + \frac{3}{2^n}\right)$$

Hence

$$\tilde{d}(z) \leq s + \frac{3}{2^n}$$

Similar to the above, the nature of s implies

$$\tilde{d}(z) \leq \tilde{d}(zu) + \frac{3}{2^n} \tag{2.6}$$

By 2.6 and 2.5, and the arbitrary nature of z, $d(u, e) \leq \frac{3}{2^n} < \epsilon$. Since $u \in U_n$ was arbitrary, $U_n \subset B(e; \epsilon)$. □

Proposition 2.6.12 (**Normal Subgroup**)**.** Let G be a **Topological Group** with **Identity Element** e. Then $\overline{\{e\}}$ is a **Normal Subgroup** of G.

Proof. Let $x \in \overline{\{e\}}$ and $y \in \overline{\{e\}}$. Then $y^{-1} \in \overline{\{e\}}$. Hence $xy^{-1} \in \overline{\{e\}} \cdot \overline{\{e\}} = \overline{\{e\}\{e\}} = \overline{\{e\}}$ so $\overline{\{e\}}$ is a **Subgroup** of G. Let $h \in \overline{\{0\}}$ and let $x \in G$. Let $V \in \mathcal{U}_{\mathcal{T}}(0)$. Then there exists **Open** $U \in \mathcal{U}_{\mathcal{T}}(0)$ such that $xU \subset Vx$. Since $h \in \overline{\{e\}}$, $e \in Uh$. Hence, $e = xx^{-1} = xex^{-1} \in xUhx^{-1} \subset Vxhx^{-1}$ Hence $xhx^{-1} \in \overline{\{e\}}$, so $x\overline{\{e\}}x^{-1} \subset \overline{\{e\}}$. In otherwords, $\overline{\{e\}}$ is a **Normal Subgroup** of G. □

Definition 2.6.13 (**Topology of Uniform Convergence**)**.** Let X be a nonempty set. Let $(Y, \mathcal{T}_Y)$ be a **Topological Group**. Let $\mathcal{B}$ be a **Local Basis** for Y. Let $\mathcal{F}$ be a **Subgroup** of the set of functions $T : X \to Y$. Suppose $\emptyset \neq \mathcal{G} \subset 2^X$ such that $(\mathcal{G}, \subset)$ is a **Directed Set**. For each $x \subset X$ and $y \subset Y$, and define $M(x, y) = \{f \in \mathcal{F} : f(x) \subset y\}$.

$$\mathcal{B}_{\mathcal{F}} = \{M(x, y) : x \in \mathcal{B} \ \ and \ \ y \in \mathcal{B}\}$$

For each $f \in \mathcal{F}$, define

$$\mathcal{B}(f) = f + \mathcal{B}_{\mathcal{F}}$$

By 2.6.14, there is a unique left-invariant **Topology** $\mathcal{T}$ on $\mathcal{F}$ such that for each $f \in \mathcal{F}$, $\mathcal{B}(f)$ is a **Filter Base** for the **Neighborhood Filter** of $\mathcal{T}$ at f. We call $\mathcal{T}$ the **Topology of Uniform Convergence** induced by $\mathcal{G}$ on $\mathcal{F}$.

Proposition 2.6.14 (**Topology of Uniform Convergence** is a **Topology**)**.** Let X be a nonempty set. Let $(Y, \mathcal{T}_Y)$ be a **Topological Group**. Let $\mathcal{F}$ be a **Subgroup** of the set of functions $T : X \to Y$. Let $\emptyset \neq \mathcal{G} \subset 2^X$ such that $(\mathcal{G}, \subset)$ is a **Directed Set**. Then the **Topology of Uniform Convergence** is a left-invariant **Topology** on $\mathcal{F}$.

Proof. Existence and uniqueness will be proven by a use of 2.3.9 to the collection $\{\mathcal{U}(f)\}_{f \in \mathcal{F}}$. From there, left-invariance is clear, as $\mathcal{U}(f) = f * \mathcal{U}(e)$. Let $f \in \mathcal{F}$. I first show that $\mathcal{B}(f)$ is a **Filter Base** for a **Filter** on $\mathcal{F}$, every element of which contains f. Since $\mathcal{F}$ is a **Subgroup**, it contains the function $g(x) = e$ for each $x \in X$. We have $g \in M(x, y)$ for each $x \in \mathcal{G}$ and $y \in \mathcal{B}$. Hence $f = f * g \in f * M(x, y)$ for each $x \in \mathcal{G}$ and $y \in \mathcal{B}$. Hence $\mathcal{B}(f)$ satisfies 2.2.7. ii. Furthermore, since $\mathcal{G} \neq \emptyset$, $\mathcal{B}(f)$ satisfies 2.2.7. i. Let $U_1, U_2 \in \mathcal{G}$ and let $V_1, V_2 \in \mathcal{B}$. Since $(\mathcal{G}, \subset)$ is a **Directed Set**, there exists $U \in \mathcal{G}$ such that $U_1 \cup U_2 \subset U$. Since $\mathcal{B}$ is a **Local Basis** for $\mathcal{T}_y$, there exists $V \in \mathcal{B}$ such that $V \subset V_1 \cap V_2$. Then $(f * M(U, V)) \subset (f * M(U_1, V_1)) \cap (f * M(U_2, V_2))$. Hence $\mathcal{B}(f)$ satisfies 2.2.7. iii, and so $\mathcal{F}$ is a **Filter Base** for a **Filter** on f. Hence $\mathcal{B}(f)$ is a **Filter Base** for a **Filter** $\mathcal{U}(f)$ on $\mathcal{F}$. Now, let $f_1 \in \mathcal{F}$. Let $U_1 \in \mathcal{G}$ and let $V_1 \in \mathcal{B}$. By 2.6.4, there exists $V \in \mathcal{B}$ such that $VV \subset V_1$. Let $y \in fM(U_1, V)$. Then

$$yM(U_1, V) \subset fM(U_1, V)M(U_1, V) \subset fM(U_1, VV) \subset fM(U_1, V_1)$$

Since $yM(U_1, V) \in \mathcal{B}(y)$, $yM(U_1, V) \in \mathcal{U}(y)$, and so $fM(U_1, V_1) \in \mathcal{U}(y)$. Hence $\{\mathcal{U}(f)\}_{f \in \mathcal{F}}$ satisfies 2.3.8 iii. By 2.3.9, there is a unique **Topology** $\mathcal{T}$ on $\mathcal{F}$ sucht that for each $f \in \mathcal{F}$, $\mathcal{U}(f)$ is the **Neighborhood Filter** for $\mathcal{T}$ at f. □

Proposition 2.6.15 (**Topology of Uniform Convergence** is Compatible)**.** Let X be a nonempty set. Let $(Y, \mathcal{T}_Y)$ be a **Commutative Topological Group** with **Identity Element** e. Let $\mathcal{F}$ be a **Subgroup** of the set of functions $T : X \to Y$. Let $\emptyset \neq \mathcal{G} \subset 2^X$ such that $(\mathcal{G}, \subset)$ is a **Directed Set**. Let $\mathcal{T}$ denote the **Topology of Uniform Convergence** on $\mathcal{F}$ determined by $\mathcal{G}$. Then $(\mathcal{F}, \mathcal{T})$ is a **Topological Group**.

Proof. Let $f, g \in \mathcal{F}$. Let $U \in \mathcal{U}_{fg}$. Then there exists $U_1 \in \mathcal{G}$ and $V \in \mathcal{T}_Y$ such that $fgM(U_1, V) \subset U$. By 2.6.4 there exists **Open** $\tilde{V} \subset V$ such that $e \in \tilde{V}$ and $\tilde{V}\tilde{V} \subset V$. Then, since $\cdot$ is **Commutative**,

$$\begin{aligned}\left(fM(U_1,\tilde{V})\right)\left(gM(U_1,\tilde{V})\right) &= fgM(U_1,\tilde{V})M(U_1,\tilde{V}) \\ &\subset fgM(U_1,\tilde{V}\tilde{V}) \\ &\subset fgM(U_1,V) \\ &\subset U\end{aligned}$$

Hence the **Group Operation** of $\mathcal{F}$ is **Continuous At** (f, g). Since $f, g \in \mathcal{F}$ were arbitrary, the **Group Operation** of $\mathcal{F}$ is **Continuous** with respect to $\mathcal{T}$. I now show that ${T^{-1}}_{\mathcal{F}}$ is **Continuous** with respect to $\mathcal{T}$. Let $f \in \mathcal{F}$. Let $U \in \mathcal{U}_{f^{-1}}$. Then there exists $U_1 \in \mathcal{G}$ and $V \in \mathcal{T}_Y$ containing e such that $f^{-1}M(U_1, V) \subset U$. Let $g \in fM(U_1, V^{-1})$. Then $g = fh$ for some $h \in M(U_1, V^{-1})$. Hence, if $x \in U$, then

$$g^{-1}(x) = (g(x))^{-1} = (f(x)h(x))^{-1} = h^{-1}(x)f^{-1}(x) \in \left(V^{-1}\right)^{-1} f^{-1}(x) = f^{-1}(x)V$$

Hence $g^{-1} \in f^{-1}M(U_1, V) \subset U$. Hence ${T^{-1}}_{\mathcal{F}}(fM(U_1, V^{-1})) \subset f^{-1}M(U_1, V)$. This concludes the proof. ☐

2.6.2 Topological Vector Spaces

Definition 2.6.16 (**Compatible**)**.** Let $(V, +, \cdot, 0)$ be a **Vector Space** over $\mathbb{F}$ and $\mathcal{T}$ be a **Topology** on V such that $(V, +, \mathcal{T})$ is a **Topological Group** and $\cdot : \mathbb{F} \times V \to V$ is **Continuous**. Then we say that $\mathcal{T}$ is **Compatible** with $(V, +, \cdot, 0)$. When $+$ and $\cdot$ are obvious, we say that $\mathcal{T}$ is **Compatible** with V.

Definition 2.6.17 (**Topological Vector Space**)**.** Let $(V, +, \cdot, 0)$ be a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\mathcal{T}$ be a **Topology** on V which is **Compatible** with $(V, +, \cdot, 0)$. Then we call $(V, \mathcal{T})$ a **Topological Vector Space**.

Definition 2.6.18 (**Locally Convex**)**.** We say that a **Topological Vector Space** $(X, \mathcal{T})$ is **Locally Convex** if $(X, \mathcal{T})$ has a **Local Basis** consisting only of **Convex** sets. A **Locally Convex** space is said to posess **Local Convexity**.

Proposition 2.6.19 (Existence of **Balanced Neighborhood Basis** of 0 in a **Topological Vector Space**)**.** Let $(X, \mathcal{T})$ be a **Topological Vector Space** over a **Field** $\mathbb{F}$. The following are true.

(i) If $U \in \mathcal{U}_{\mathcal{T}}(0)$, then there is a **Balanced** , **Open** $V \subset U$ such that $V \in \mathcal{U}_{\mathcal{T}}(0)$.

(ii) There exists a **Neighborhood Basis** about $0 \in X$ for $\mathcal{T}$ consisting entirely of **Balanced** sets.

(iii) If $U \in \mathcal{U}_{\mathcal{T}}(0)$ is **Convex** , then there is a **Convex** **Balanced** , **Open** $V \subset U$ such that $V \in \mathcal{U}_{\mathcal{T}}(0)$.

(iv) If $(X, \mathcal{T})$ is **Locally Convex**, then there exists a **Neighborhood Basis** about $0 \in X$ for $\mathcal{T}$ consisting entirely of **Balanced Convex** sets.

Proof of 2.6.19 i. Since scalar multiplication is **Continuous At** 0, there is an **Open** disk $V \subset \mathbb{F}$ and an **Open** $W \subset X$ with $0 \in W$ such that $VW \subset U$. VW is clearly balanced. □

Proof of 2.6.19 ii. Let $\{U_\alpha\}_{\alpha \in A}$ be a **Local Basis** for $\mathcal{T}$. Then, by 2.6.19 i, for each $\alpha \in A$, there is a $W_\alpha \subset U_\alpha$ such that W_α is **Balanced Set** and $0 \in W_\alpha$. Clearly $\{W_\alpha\}_{\alpha \in A}$ forms a **Local Basis** for X. □

Proof of 2.6.19 iii. By 2.6.19 i, there is a **Balanced Set Open** $W \subset U$. Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Then $\alpha^{-1}W = \subset W \subset U$. Hence $W \subset \alpha U$, so if we define $A = \bigcap_{|\alpha|=1} \alpha U$, then $0 \in W \subset A$. Thus $0\overset{\circ}{A}$. For this reason, it suffices to show $\overset{\circ}{A}$ is **Balanced Set** . It then suffices to show A is **Balanced Set** . Let $\alpha \in \mathbb{Z}$ with $|\alpha| \leq 1$. Then $\alpha = r\beta$ for some $\beta \in \mathbb{C}$ with $|\beta| = 1$ and $r \in [0, 1]$. Since αU is **Convex** and contains 0, $r\alpha U \subset \alpha U$. Hence,

$$r\beta A = r\beta \bigcap_{|\alpha|=1} \alpha U = \bigcap_{|\alpha|=1} r\alpha U \subset \bigcap_{|\alpha|=1} \alpha U = A$$

Thus A is **Balanced Set** . □

Proof of 2.6.19 iv. This is a similar arguement to that of 2.6.19 ii. □

Definition 2.6.20 (TVS Bounded Set)**.** Let $(V, \mathcal{T})$ be a **Topological Vector Space**. Let $A \subset V$. We say that A is **TVS-Bounded** with respect to $\mathcal{T}$, or when confusion is unlikely we simply say that A is **TVS-Bounded** if for every $U \in \mathcal{U}_{\mathcal{T}}(0)$, there exists an $\alpha \in \mathbb{F}$, $\alpha > 0$, such that $A \subset \alpha U$.

Definition 2.6.21 (**Bounded Linear Operator**)**.** Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. For $i \in \{0, 1\}$, let $(V_i, \mathcal{T}_i)$ be a **Topological Vector Spaces** over $\mathbb{F}$. We say that a **Linear** operator $T : (V_1, \mathcal{T}_1) \to (V_2, \mathcal{T}_2)$ is a **Bounded Linear Operator** if for each $U \in V_1$ with U **TVS-Bounded** with respect to $\mathcal{T}_0$, TU is **TVS-Bounded** with respect to $\mathcal{T}_1$.

Definition 2.6.22 (**Space Of Continuous Linear Operators**)**.** Let $(U, \mathcal{T}_U)$ and $(V, \mathcal{T}_V)$ each be a **Topological Vector Space** over the same **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $L(U, V)$ denote the **Space of Linear Operators** from U to V. We denote with $CL((U, \mathcal{T}_U), (V, \mathcal{T}_V))$ the subset of $L(U, V)$ consisting only of the **Continuous** operators. When $\mathcal{T}_U$ and $\mathcal{T}_V$ are understood, we may denote $CL((U, \mathcal{T}_U), (V, \mathcal{T}_V)) = CL(U, V)$

Remark 2.6.23 (**Space Of Continuous Linear Operators** is a **Vector Subspace**)**.** Let $(U, \mathcal{T}_U)$ and $(V, \mathcal{T}_V)$ each be a **Topological Vector Space** over the same **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $L(U, V)$ denote the **Space of Linear Operators** from U to V. Let $CL(U, V)$ denote the **Space Of Continuous Linear Operators** from U to V. Then $CL(U, V)$ is a **Vector Subspace** of $L(U, V)$.

Proposition 2.6.24 (**Topology of Uniform Convergence** is a Tvs)**.** Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let Y be a **Topological Vector Space** over $\mathbb{F}$. Let X be a nonempty set. Let $\mathcal{F}$ be a **Vector Subspace** of the set of maps $T : X \to Y$. Let $\mathcal{G} \subset X$. Let $\mathcal{T}$ denote the **Topology of Uniform Convergence** on $\mathcal{F}$ induced by $\mathcal{G}$. Then $(\mathcal{F}, \mathcal{T})$ is a **Topological Vector Space** over $\mathbb{F}$ if and only if, for each $U \in \mathcal{G}$, for each $f \in \mathcal{F}$, the set $f(U)$ is **TVS-Bounded** in Y.

Proof. Suppose $(\mathcal{F}, \mathcal{T})$ is a **Topological Vector Space**. Then $\cdot : \mathbb{F} \times \mathcal{F} \to \mathcal{F}$ is **Continuous**. Let $f \in \mathcal{F}$. Let $U \in \mathcal{G}$. Let $V \in \mathcal{T}_Y$. Since $\cdot$ is **Continuous**, $T : \mathbb{F} \to \mathcal{F}$ defined by $T(\alpha) = \alpha f$ is **Continuous**. Hence $T^{-1}(M(U,V))$ contains some $\alpha_0 > 0$. Hence $alpha_0 f \in M(U,V)$. That is $\alpha_0 f(U) \subset V$. That is, $f(U) \subset \frac{1}{\alpha_0} V$, so $f(U)$ is absorbed by V. Since $V \in \mathcal{T}_Y$ is an arbitrary **Open** set containing 0, $f(U)$ is **TVS-Bounded**. Since $U \in \mathcal{G}$ is arbitrary, this direciton of the theorem is proven.

Now let each $f(U)$ be **TVS-Bounded**. Let $\alpha \in \mathbb{F}$. Let $f \in \mathcal{F}$. Let $U \in \mathcal{G}$. Let $V \in \mathcal{T}_Y$ such that $0 \in V$. By 2.6.4 there exists $\tilde{V} \in \mathcal{T}_Y$ such that $0 \in \tilde{V}$ and $\tilde{V} + \tilde{V} + \tilde{V} + \tilde{V} \subset V$. Let $V' \in \mathcal{T}_Y$ such that $\alpha V' \subset \tilde{V}$. Let V'' be a **Balanced Open** subset of Y containing 0 such that $V'' \subset V' \cap \tilde{V}$. Since $f(U)$ is **TVS-Bounded** there exists $\epsilon_0 > 0$ such that

$$\epsilon_0 f(U) \subset V'' \qquad |\epsilon_0| < |\alpha|$$

Set $\Gamma = B_{\mathbb{C}}(0; \epsilon_0)$. Set $\Omega = M(U, V'')$. Let $h \in \Gamma(f + \Omega)$. Then there exists $\gamma \in \mathbb{C}$ with $|\gamma| < \epsilon_0$ and there exists $g \in M(U, V'')$ such that $h = (\alpha + \gamma)(f + g)$. Let $x \in U$. Then,

$$\begin{aligned}
h(x) &= (\alpha + \epsilon)(f + g)(x) \\
&= \alpha f(x) + \gamma f(x) + \alpha g(x) + \gamma g(x) \\
&\in \alpha f(x) + \gamma f(U) + \alpha g(U) + \gamma g(U) \\
&= \alpha f(x) + \frac{\gamma}{\epsilon_0} \epsilon_0 f(U) + \alpha g(U) + \gamma g(U) \\
&\subset \alpha f(x) + \frac{\gamma}{\epsilon_0} V'' + \alpha V'' + \gamma V'' \\
&\subset \alpha f(x) + V'' + \alpha V'' + \alpha V'' \\
&\subset \alpha f(x) + \tilde{V} + \tilde{V} + \tilde{V} \\
&\subset \alpha f(x) + V
\end{aligned}$$

Hence, $h \in \alpha f + M(U,V)$. Hence, $\Gamma\Omega \subset \alpha f + M(U,V)$. Since $U \in \mathcal{G}$ and $V \in \mathcal{T}_Y$ containing 0 were arbitrary, $\cdot$ is **Continuous At** (α, f). Since $\alpha \in \mathbb{F}$ and $f \in \mathcal{F}$ were arbitrary, $\cdot$ is **Continuous**. By 2.6.15, $(\mathcal{F}, \mathcal{T})$ is a **Topological Group**, so **Continuity** of $\cdot$ is enough to guarantee that $(\mathcal{F}, \mathcal{T})$ is a **Topological Vector Space**.

□

2.7 Seminormed Spaces

Definition 2.7.1 (Seminorm)**.** Let V be a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. We say that a map $||\cdot|| : V \to [0, \infty)$ is a **Seminorm** on V if it is both **Subadditive** and **Absolutely Scalar Homogeneous**. In this case, we refer to $(V, ||\cdot||)$ as a **Seminormed**

Space. We say that $||\cdot||$ is **Non-Degenerate** if there is at least one $v \in V$ with $||v|| > 0$. We say that $||\cdot||$ is **Degenerate** if it is not **Non-Degenerate**. We may also refer to the **Seminormed Space** $(V, ||\cdot||)$ as being **Degenerate** or **Non-Degenerate**.

Definition 2.7.2 (Norm)**.** Let $(V, ||\cdot||)$ be a **Seminormed Space**. such that for every $x \in V$, $x \neq 0$ implies $||x|| \neq 0$. Then we call $||\cdot||$ as a **Norm** on V and we call $(V, ||\cdot||)$ a **Normed Space**.

Proposition 2.7.3 (Subadditive Operator On a Group Induces a Metric)**.** Let $(G, +, e)$ be a **Group** and let $(H, +, \leq)$ be a **Totally Ordered Magma**. Let $p : G \to H$ be **Subadditive**. Define $d : G \times G \to H$ by setting, for each $x, y \in G$, $d(x, y) = p(xy^{-1})$. Then d satisfies the **Triangle Inequality**.

Proof. let $x, y, z \in G$. Then

$$\begin{aligned} d(x, z) &= p(xz^{-1}) \\ &= p(xez^{-1}) \\ &= p(xy^{-1}yz^{-1}) \\ &\leq p(xy^{-1})p(yz^{-1}) \\ &= d(x, y)d(y, z) \end{aligned}$$

completing the proof. □

Definition 2.7.4 (**Seminorm Topology**)**.** Let $(X, ||\cdot||)$ be a **Seminormed Space**. define $d_{||\cdot||} : V \times V \to [0, \infty)$ by setting, for $x, y \in X$,

$$d_{||\cdot||}(x, y) = ||x - y||$$

Observe the following:

1. 2.4.33 guarantees that $d_{||\cdot||}(x, x) = 0$ for $x \in X$.

2. 2.7.3 guarantees that d satisfies the **Triangle Inequality**.

3. d is **Commutative**, as we have

$$d(x, y)_{||\cdot||} = ||x - y|| = |-1|\,||x - y|| = ||y - x|| = d(y, x)$$

Hence, $d_{||\cdot||}$ is a **Pseudometric** on X, which we call the **Pseudometric induced by the Seminorm** on X. We refer to $(X, d_{||\cdot||})$ as the **Pseudometric Space induced by the Seminormed Space** $(X, ||\cdot||$. We refer to the **Pseudometric Topology** induced by $d_{||\cdot||}$ as the **Seminorm Topology** induced by $||\cdot||$, and unless otherwise specified, when we reference $(X, ||\cdot||)$, we consider it to be endowed with this topology.

Definition 2.7.5 (**Complete Seminormed Space**, **Banach Space**)**.** Let $(X, ||\cdot||)$ be a **Seminormed Space**. Let d be the **Pseudometric** induced on X by $||\cdot||$. If (X, d) is **Pseudometric-Complete**, then we call $(X, ||\cdot||)$ a **Complete Seminormed Space**. If $(X, ||\cdot||)$ is a **Normed Space**, then under these same circumstances we call $(X, ||\cdot||)$ a **Complete Normed Space**. A **Complete Normed Space** is called a **Banach Space**.

Definition 2.7.6 (**Seminorm Kernel**)**.** Let $(V, ||\cdot||)$ be a **Seminormed Space**. Define the set $\mathcal{K}^{\text{ernel}}{}_{(V,||\cdot||)}$ by

$$\mathcal{K}^{\text{ernel}}{}_{(B,||\cdot||)} = \{x \in V | \, ||x|| = 0\}$$

We call this set the **Seminorm Kernel** of the space $\mathcal{K}^{\text{ernel}}{}_{(V,||\cdot||)}$. When confusion is unlikely, we may denote this set with $\mathcal{K}^{\text{ernel}}$, $\mathcal{K}^{\text{ernel}}{}_V$, or even $\mathcal{K}^{\text{ernel}}{}_{||\cdot||}$, or we may just refer to it as the **Seminorm Kernel**, the **Seminorm Kernel** of V, or the **Seminorm Kernel** of $||\cdot||$.

Proposition 2.7.7 (Seminorm Kernel is a vector Subspace)**.** Let $(X, ||\cdot||)$ be a **Seminormed Space** over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ with corresponding **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$. Then $\mathcal{K}^{\text{ernel}} = \overline{\{0\}}$, and in particular, the following hold.

(i) $\mathcal{K}^{\text{ernel}} = \overline{\{0\}}$, and is therefore **Closed**.

(ii) $\mathcal{K}^{\text{ernel}}$ is a **Vector Subspace** of X.

(iii) $\mathcal{K}^{\text{ernel}} = X$ if and only if X is **Degenerate**.

Proof of 2.7.7 i. Let $x \in X \setminus \mathcal{K}^{\text{ernel}}$. Then $||x|| > 0$. Hence $B\left(x, \frac{||x||}{2}\right)$ is a **Neighborhood** of x **Disjoint** from $\mathcal{K}^{\text{ernel}}$. Hence $x \in X \setminus \overline{\{0\}}$. That is, $\overline{\{0\}} \subset \mathcal{K}^{\text{ernel}}$. Now let $x \in X \setminus \overline{\{0\}}$. Then there is $U \in \mathcal{U}_{\mathcal{T}}(x)$ not containing 0. Then there exists $\delta > 0$ with $B(x;\delta) \subset U$. This implies $||x|| \geq \delta > 0$, so $x \notin \mathcal{K}^{\text{ernel}}$. Hence $\mathcal{K}^{\text{ernel}} \subset \overline{\{0\}}$. □

Proof of 2.7.7 ii. By 2.7.7 i and 2.6.12, $\mathcal{K}^{\text{ernel}}$ is a **Normal Subgroup** of X. Let $x \in \mathcal{K}^{\text{ernel}}$. Let $\alpha \in \mathbb{F}$. Then $||\alpha x|| = |\alpha| \, ||x|| = 0$, so $\alpha x \in \mathcal{K}^{\text{ernel}}$. Hence $\mathcal{K}^{\text{ernel}}$ is a **Vector Subspace** of X. □

Proof of 2.7.7 iii. Trivial. □

Proposition 2.7.8 (Equivalence Mod Kernel is Pseudometric Equivalence)**.** Let $(X, ||\cdot||)$ be a **Seminormed Space**. with **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$. Let d denote the **Pseudometric induced by the Seminorm**. Let $\cong_d$ denote the **Relation Of Zero Distance** with respect to d. Then $\cong_{\mathcal{K}^{\text{ernel}}} = \cong_d$.

Proof. Let $x, y \in X$ and let $x \cong_{\mathcal{K}^{\text{ernel}}} y$. Then, since $x - y \in \mathcal{K}^{\text{ernel}}$, Then $d(x, y) := ||x - y|| = 0$, so $x \cong_d y$. Hence $\cong_{\mathcal{K}^{\text{ernel}}} \subset \cong_d$ Now let $x, y \in X$ with $x \cong_d y$. Then $||x - y|| = d(x, y) = 0$, so $x - y \in \mathcal{K}^{\text{ernel}}$, and therefore $x \cong_{\mathcal{K}^{\text{ernel}}} y$. Hence, $\cong_d \subset \cong_{\mathcal{K}^{\text{ernel}}}$. Since inclusion goes both directions, $\cong_{\mathcal{K}^{\text{ernel}}} = \cong_d$. □

Definition 2.7.9 (Quotient Space Mod Kernel)**.** Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $(X, ||\cdot||)$ be a **Seminormed Space** over $\mathbb{F}$. Denote the **Seminorm Kernel** of $(X, ||\cdot||)$ with $\mathcal{K}^{\text{ernel}}$. By 2.7.7 ii and 2.7.7 i, $\mathcal{K}^{\text{ernel}}$ is a **Closed Vector Subspace** of X. We endow $X/\mathcal{K}^{\text{ernel}}$ with the **Quotient Vector Space** structure, which we call the **Seminorm Kernel Quotient Vector Space** of the **Seminormed Space** $(X, ||\cdot||)$.

Definition 2.7.10 (Quotient Norm Space)**.** Let $(X, ||\cdot||)$ be a **Seminormed Space** with **Pseudometric induced by the Seminorm** d, **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$, and **Seminorm Kernel Quotient Vector Space** $X/\mathcal{K}^{\text{ernel}}$. Let $\tilde{d} : X/\mathcal{K}^{\text{ernel}} \times X/\mathcal{K}^{\text{ernel}} \to [0, \infty)$ be the **Metric Induced By The Pseudometric**.

Define $||\cdot||_{\mathcal{K}^{\text{ernel}}} : X/\mathcal{K}^{\text{ernel}} \to [0,\infty)$ by setting, for $x \in X$,

$$||[x]||_{\mathcal{K}^{\text{ernel}}} = \tilde{d}([x],[0])$$

By 2.7.11, $(X/\mathcal{K}^{\text{ernel}}, ||\cdot||_{\mathcal{K}^{\text{ernel}}})$ is a **Normed Space** which we call the **Quotient Normed Space** of $(X, ||\cdot||)$, and we call $||\cdot||_{\mathcal{K}^{\text{ernel}}}$ the **Quotient Norm**. Whenever we refer to $X/\mathcal{K}^{\text{ernel}}$, unless otherwise specified, we endow it with this **Norm** and the **Topology** generated by this **Norm**. Furthermore, whenever we consider $X/\mathcal{K}^{\text{ernel}}$, unless otherwise specified, we consider it as possesing the **Topology** generated by the **Norm** $||\cdot||_{\mathcal{K}^{\text{ernel}}}$.

Proposition 2.7.11 (Quotient Normed Space)**.** Let $(X, ||\cdot||)$ be a **Seminormed Space** with **Pseudometric induced by the Seminorm** d, **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$, and **Seminorm Kernel Quotient Vector Space** $X/\mathcal{K}^{\text{ernel}}$. Let $\tilde{d} : X/\mathcal{K}^{\text{ernel}} \times X/\mathcal{K}^{\text{ernel}} \to [0,\infty)$ be the **Metric Induced By The Pseudometric**. Let $T : X \to X/\mathcal{K}^{\text{ernel}}$ denote the **Quotient Map** of X into $X/\mathcal{K}^{\text{ernel}}$ (Recalling that the **Relation Of Equal Neighborhood Filters** equals the **Relation Of Zero Distance** equals the relation of **Equivalence MOD-$\mathcal{K}^{\text{ernel}}$**), so they would all produce the same quotient map) Let $||\cdot||_{\mathcal{K}^{\text{ernel}}}$ denote the **Quotient Norm**.

The following are true.

(i) $||\cdot||_{\mathcal{K}^{\text{ernel}}}$ is a **Norm** on $X/\mathcal{K}^{\text{ernel}}$.

(ii) $\tilde{d}$ is the **Pseudometric induced by the Seminorm** $||\cdot||_{\mathcal{K}^{\text{ernel}}}$, and thus they produce the same **Topology**.

(iii) T has all of the properties described in 2.3.32.

(iv) T is **Linear**.

(v) T is **Surjective**.

(vi) T is an **Isometry**.

(vii) T is **Injective** if and only if $||\cdot||$ is a **Norm**.

Proof of 2.7.11 i. First, note that $Range(||\cdot||_{\mathcal{K}^{\text{ernel}}}) \subset Range(\tilde{d}) \subset [0,\infty)$, so that $||\cdot||_{\mathcal{K}^{\text{ernel}}}$ has the correct **Domain** and **Codomain**. For **Subadditivity**, let $[x],[y] \in X/\mathcal{K}^{\text{ernel}}$. Then

$$\begin{aligned}
||[x]+[y]||_{\mathcal{K}^{\text{ernel}}} &= ||[x+y]||_{\mathcal{K}^{\text{ernel}}} \\
&= \tilde{d}\left([x+y],[0]\right) \\
&= d(x+y,0) \\
&= ||x+y|| \\
&\leq ||x|| + ||y|| \\
&= d(x,0) + d(y,0) \\
&= \tilde{d}\left([x],[0]\right) + \tilde{d}\left([y],[0]\right) \\
&= ||[x]||_{\mathcal{K}^{\text{ernel}}} + ||[y]||_{\mathcal{K}^{\text{ernel}}}
\end{aligned}$$

For **Absolute Scalar Homogeneity**, let $\alpha \in \mathbb{F}$ and $[x] \in X/\mathcal{K}^{\text{ernel}}$. Then,

$$\begin{aligned}
||[\alpha x]||_{\mathcal{K}^{\text{ernel}}} &= \tilde{d}\left([\alpha x], [0]\right) \\
&= d(\alpha x, 0) \\
&= ||\alpha x|| \\
&= |\alpha|\,||x|| \\
&= |\alpha|\,||[x]||_{\mathcal{K}^{\text{ernel}}}
\end{aligned}$$

Finally, suppose $[x] \neq 0$. Then, since the additive identity of $X/\mathcal{K}^{\text{ernel}}$ is $\mathcal{K}^{\text{ernel}}$, $x \notin \mathcal{K}^{\text{ernel}}$. Hence $||[x]||_{\mathcal{K}^{\text{ernel}}} = \tilde{d}([x], 0) = d(x, 0) = ||x|| > 0$.

□

Proof of 2.7.11 ii. Let D denote the **Pseudometric induced by the Seminorm** $||\cdot||_{\mathcal{K}^{\text{ernel}}}$. Then, for $[x], [y] \in X/\mathcal{K}^{\text{ernel}}$,

$$\begin{aligned}
\tilde{d}([x], [y]) &= d(x, y) \\
&= ||x - y|| \\
&= ||x - y - 0|| \\
&= d(x - y, 0) \\
&= \tilde{d}([x - y], 0) \\
&= ||[x - y]||_{\mathcal{K}^{\text{ernel}}} \\
&= ||[x] - [y]||_{\mathcal{K}^{\text{ernel}}} \\
&= D\left([x], [y]\right)
\end{aligned}$$

Since these two **Pseudometric**'s are equal, they produce the same **Topology**. Furthermore, by applying 2.5.18, we see that the **Topology** generated by $||\cdot||_{\mathcal{K}^{\text{ernel}}}$ is also the **Quotient Topology** on $X/\mathcal{K}^{\text{ernel}}$. □

Proof of 2.7.11 iii. T is the topological **Quotient Map** and the **Norm Topology** is the **Quotient Topology**, so the assumptions of 2.3.32 are satisfied. □

Proof of 2.7.11 iv. This is a direct consequence of 2.4.45 ii. □

Proof of 2.7.11 v. This is a direct consequence of 2.1.47. □

Proof of 2.7.11 vi. This is a direct consequence of 2.5.18 iii. □

Proof of 2.7.11 vii. This is a direct consequence of 2.5.18. □

Remark 2.7.12 (Quotient Normed Space)**.** If $(X, ||\cdot||_X)$ is a **Normed Space** then by parts 2.7.11 iv, 2.7.11 v, 2.7.11 vi, and 2.7.11 vii, $T : X \to \mathcal{K}^{\text{ernel}}{}_X$ is an isomorphism of **Normed Spaces** satisfying $Tx = \{x\}$. For this reason, as an abuse of notation, later in this document, I may not distinguish between the quotient $X/\mathcal{K}^{\text{ernel}}{}_X$ and the space X if X is a **Normed Space**, and similarly, I may not distinguish between $x \in X$ and $\{x\} \in X/\mathcal{K}^{\text{ernel}}{}_X$.

Proposition 2.7.13. Let $(X, ||\cdot||)$ be a **Seminormed Space** with **Quotient Normed Space** $(X/\mathcal{K}^{\text{ernel}}, ||\cdot||_{\mathcal{K}^{\text{ernel}}})$.

Then X is **Pseudometric-Complete** if and only if $X/\mathcal{K}^{\text{ernel}}$ is **Complete**.

Proof. Let X be **Pseudometric-Complete**. Let $\{[x_i]\}_{i\in\mathbb{N}} \subset X/\mathcal{K}^{\text{ernel}}$ be a **Pseudometric Cauchy Sequence**. Let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for $m, n > N$ we have

$$||[x_m - x_n]||_{\mathcal{K}^{\text{ernel}}} < \epsilon \tag{2.7}$$

For this N, we have

$$||x_m - x_n|| = ||[x_m - x_n]||_{\mathcal{K}^{\text{ernel}}} < \epsilon \tag{2.8}$$

so that $\{x_i\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence**. Since X is **Pseudometric-Complete**, there is a $x \in X$ such that $||x_i - x|| \to 0$, but since T is an **Isometry**,

$$||[x] - [x_i]|| = ||[x_i - x]||_{\mathcal{K}^{\text{ernel}}} \to 0 \tag{2.9}$$

and so $[x_i] \to [x]$. so that $X/\mathcal{K}^{\text{ernel}}$ is **Complete**.

Now suppose instead that $X/\mathcal{K}^{\text{ernel}}$ is **Complete** and suppose $\{x_i\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence** in X. Since $||[x_i - x_j]||_{\mathcal{K}^{\text{ernel}}} = ||x_i - x_j||$, $\{[x_i]\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence** in $X/\mathcal{K}^{\text{ernel}}$, which therefore has a limit $y \in X/\mathcal{K}^{\text{ernel}}$. Since T is **Surjective**, $y = [x]$ for some $x \in X$, and it is easy to see that $x_i \to x$ so that X is **Pseudometric-Complete**.

□

Definition 2.7.14 (Space of Continuous Linear Operators From a Seminormed Space into a Normed Space)**.** Let $(X, ||\cdot||_X)$ be a **Non-Degenerate Seminormed Space**. Let $(Y, ||\cdot||_Y)$ be a **Seminormed Space**. We denote with $BL\left((X, ||\cdot||_X), (Y, ||\cdot||_Y)\right)$ the collection of **Continuous Linear** operators $T : (X, ||\cdot||_X) \to (Y, ||\cdot||_Y)$. When the **Topologies** on X and Y are understood, we denote this set with $BL\left(X, Y\right)$. We refer to $BL\left(X, Y\right)$ as the **Space of Bounded Linear Operators** from $(X, ||\cdot||_X)$ to $(Y, ||\cdot||_Y)$, or when $||\cdot||_X$ and $||\cdot||_Y$ are understood, from X to Y.

We endow $BL\left(X, Y\right)$ with the algebraic operations of pointwise scalar multiplication and pointwise addition, making $BL\left(X, Y\right)$ a vector space.

We define $||\cdot|| : BL(X, Y) \to [0, \infty)$ by defining, for $T \in BL(X, Y)$

$$||T|| = \sup_{||x||_X \neq 0} \frac{||Tx||_Y}{||x||_X}$$

As will be proven in 2.7.15, $||\cdot||$ is a **Seminorm** on $BL(X, Y)$, which we refer to as the **Operator Seminorm** on $BL(X, Y)$. induced by the **Seminorm** $||\cdot||_X$ on X and the **Seminorm** $||\cdot||_Y$ on Y.

In the case that $||\cdot||_Y$ is a **Norm**, rather than just a **Seminorm**, by 2.7.15 , $||\cdot||$ is a **Norm** on $BL(X, Y)$, which we instead call the **Operator Norm**.

Proposition 2.7.15 (Space of Bounded Linear Operators On Seminormed Spaces)**.** Let $(X, ||\cdot||_X)$ be a **Seminormed Space**. Let $(Y, ||\cdot||_Y)$ be a **Seminormed Space**. Let $BL(X, Y)$ denote the **Space of Bounded Linear Operators** from X to Y. Let $||\cdot||$ denote the **Operator Seminorm**.

The following are true.

(i) $||\cdot||$ is a well-defined **Seminorm** on $BL(X,Y)$.

(ii) If $||\cdot||_Y$ is a **Norm**, then so is $||\cdot||$.

(iii) If $T \in BL(X,Y)$ and $\alpha \in (0,\infty)$, then $||T|| = \sup\limits_{||x||_X=\alpha} \frac{||Tx||_Y}{||x||_X}$.

(iv) If $T \in BL(X,Y)$ and $\alpha \in (0,\infty)$, , then $||T|| = \sup\limits_{0<||x||_X\leq\alpha} \frac{||Tx||_Y}{||x||_X} = \sup\limits_{0<||x||_X<\alpha} \frac{||Tx||_Y}{||x||_X}$.

(v) If $T \in BL(X,Y)$ and $x \in X$, then $||Tx||_Y \leq ||T||\,||x||_X$.

(vi) $S : X \to Y$ is linear , $S(\mathcal{K}^{\text{ernel}}{}_X) \subset \mathcal{K}^{\text{ernel}}{}_Y$, and $\sup\limits_{||x||_X\neq 0} \frac{||Sx||_Y}{||x||_X} < \infty$, if and only if $S \in BL(X,Y)$.

(vii) A **Sequence** $\{T_i\}_{i\in\mathbb{N}} \subset BL(X,Y)$ is a **Pseudometric Cauchy Sequence** if and only if there exists an $\alpha > 0$ such that the collection of sequences $\{\{T_ix\}_{i\in\mathbb{N}} : x \in B_X(0;\alpha)\}$ is **Uniformly Cauchy** if and only if for every $\beta > 0$, the collection of sequences $\{\{T_ix\}_{i\in\mathbb{N}} : x \in B_X(0;\beta)\}$ is is **Uniformly Cauchy**.

(viii) If $T_i \to T$ with respect to $||\cdot||$, then $T_ix \to Tx$ with respect to $||\cdot||_Y$ for each $x \in X$

(ix) A sequence $\{T_i\}_{i\in\mathbb{N}} \subset BL(X,Y)$ converges with respect to $||\cdot||$ if and only if it is a **Pseudometric Cauchy Sequence** and for each x_α in some Hamel basis $\{x_\alpha\}_{\alpha\in A} \subset X$, the sequence $\{T_ix_\alpha\}_{\alpha\in A}$ converges with respect to $||\cdot||_Y$.

(x) Let X be **Non-Degenerate**. Then $BL(X,Y)$ is complete if Y is complete.

(xi) If $BL(X,Y)$ is **Non-Degenerate** then Y is **Non-Degenerate**.

(xii) If $S : X \to Y$ is **Linear**, then $S \in BL(X,Y)$ if and only if there exists a constant $c \in (0,\infty)$ such that for every $x \in X$. $||Tx|| \leq c\,||x||$.

(xiii) If $T \in BL(X,Y)$, and $A = \{c \in (0,\infty) : ||Tx|| \leq c\,||x||\,(\forall x \in X)\}$, then $||T|| = \inf(A)$.

Proof of 2.7.15 i. Since X is **Non-Degenerate**, there exists an $x \in X$ with $||x||_X \neq 0$, so for each $T \in BL(X,Y)$, the set that the supremum is being taken over is nonempty. Also, it is clear that $Range(||\cdot||) \subset [0,\infty)$,

For **Subadditivity**, let $T_i \in BL(X,Y)$ for $i \in \{0,1\}$. and $x \in X$ with $||x|| > 0$. Then, since $||\cdot||_Y$ is **Subadditive**,

$$\frac{||(T_0+T_1)x||_Y}{||x||_X} \leq \frac{||T_0x||_Y}{||x||_X} + \frac{||T_1x||_Y}{||x||_X}$$

Since this is true for each x with $||x||_X \neq 0$, taking the supremum of each side yields

$$\begin{aligned}\sup_{||x||_X \neq 0}\left(\frac{||(T_0+T_1)x||_Y}{||x||_X}\right) &\leq \sup_{||x||_X \neq 0}\left(\frac{||T_0x||_Y}{||x||_X}+\frac{||T_1x||_Y}{||x||_X}\right)\\ &\leq \sup_{||x||_X \neq 0}\left(\frac{||T_0x||_Y}{||x||_X}\right)+\sup_{||x||_X \neq 0}\left(\frac{||T_1x||_Y}{||x||_X}\right)\end{aligned}$$

Hence, $||T_0 + T_1|| \leq ||T_0|| + ||T_1||$ so that $||\cdot||$ is **Subadditive**. For **Absolute Scalar Homogeneity**, let $T \in BL(X, Y)$, $\alpha \in \mathbb{F}$, and $x \in X$ with $||x||_X \neq 0$. Then

$$\frac{||(\alpha T)x||_Y}{||x||_X} = \frac{||\alpha(Tx)||_Y}{||x||_X} = |\alpha| \frac{||Tx||_Y}{||x||_X}$$

Hence taking the supremum finishes the proof. □

Proof of 2.7.15 ii. Let $T \neq 0 \in BL(X, Y)$. Then for some $x \in X$, $Tx \neq 0$. Then Tx has a neighborhood U disjoint from 0_Y, Hence $x \in T^{-1}(U)$ but not $0_X \in T^{-1}(U)$, since $T0_X = 0_Y$. Since U is a neighborhood of x disjoint from 0, there is an $\epsilon > 0$ such that $0_X \subset \complement U \subset \complement \overline{B_X}(x; \epsilon)$, and therefore $||x||_X > \epsilon$. Since $||x||_X > 0$, it is ranged over in the supremum defining $||T||$, and so

$$0 < \frac{||Tx||_Y}{||x||_X} \leq \sup_{||x||_X \neq 0} \frac{||Tx||_X}{||x||_X} = ||T|| \tag{2.10}$$

□

Proof of 2.7.15 iii. Let $\alpha \in (0, \infty)$ Let $T \in BL(X, Y)$. Then, there is a sequence $\{x_i\} \subset X$ with each $||x_i||_X \neq 0$ such that

$$\frac{||Tx_i||_Y}{||x_i||_X} \to ||T|| \tag{2.11}$$

For each $i \in \mathbb{N}$, define $y_i = \alpha x_i / ||x_i||_X$. then each $||y_i|| = \alpha$, and by **Absolute Scalar Homogeneity** of T, we have

$$\frac{||Ty_i||_Y}{||y_i||_X} = \frac{||Tx_i||_Y}{||x_i||_X} \to ||T|| \tag{2.12}$$

, completing the proof. □

Proof of 2.7.15 iv. If we define, for $T \in BL(X, Y)$, $f(T) = \sup_{0<normx_X\leq\alpha} \frac{||Tx||_Y}{||x||_X}$, then since $||\cdot||^{-1}((0, \alpha)) \subset ||\cdot||^{-1}((0, \infty))$, we have $f(T) \leq ||T||$ and since $||\cdot||^{-1}(\{\alpha\}) \subset ||\cdot||^{-1}((0, \alpha))$, we have $||T|| \leq f(T)$. proving the first equality. The second is found by applying the same arguement to $\alpha/2$ and realizing that $(0, \alpha/2] \subset (0, \alpha)$. □

Proof of 2.7.15 v. Let $T \in BL(X, Y)$ and $x \in X$. If $||Tx||_Y \neq 0$, then $B_Y(Tx, \frac{||Tx||_Y}{2})$ is a neighborhood of Tx disjoint from 0. Continuity of T impliese x then has a neighborhood disjoint from $0 \in T^{-1}(0)$, implying that $||x||_X \neq 0$.

Hence if $||x||_X = 0$, then we know $||Tx||_Y = 0$, so that the relation

$$||Tx||_Y \leq ||T|| \, ||x||_X \tag{2.13}$$

If $||x||_X \neq 0$, then by definition of supremum,

$$\frac{||Tx||_Y}{||x||_X} \leq ||T||$$

so that $||Tx||_Y \leq ||T|| \, ||x||_X$. □

Proof of 2.7.15 vi. I assume the first 3 conditions and show that $S \in BL(X, Y)$. It is necessary and sufficient to show that S is continuous. Let $F = \sup_{||x||_X \neq 0} \frac{||Sx||_Y}{||x||_X}$. If $F = 0$, then $S(X) \subset \mathcal{K}^{\text{ernel}}{}_Y$. Every neighborhood of every point in $\mathcal{K}^{\text{ernel}}{}_Y$ contains $\mathcal{K}^{\text{ernel}}{}_Y$, so in that case continuity holds. Suppose $F \neq 0$. By translation invariance of the topology, it is sufficient to consider neighborhoods of $0_Y \in Y$. Let $\epsilon > 0$. Define $V = B_X\left(0; \frac{\epsilon}{F}\right)$. Let $x_0 \in V$. If $||x_0||_X = 0$, then $S(x_0) \in S(\mathcal{K}^{\text{ernel}}{}_X) \subset \mathcal{K}^{\text{ernel}}{}_Y \subset B_Y(0; \epsilon)$. If $||x_0||_X \neq 0$, then $||Sx||_Y \leq F \, ||x||_X < \epsilon$, so $s(x_0) \in B_Y(0; \epsilon)$. Hence $S\left(B_X\left(0; \frac{\epsilon}{F}\right)\right) \subset B_Y(0; \epsilon)$. so S is continuous,and this direction fo the proof is complete.

Suppose conversely that $S \in BL(X, Y)$. Then S is **Linear** by definition , and the supremum expression is finite by 2.7.15 i of this result. Since S is **Linear**, $S0_X = 0_Y$. Since S is **Continuous**,

$$\begin{aligned} S(\mathcal{K}^{\text{ernel}}{}_X) &= S\left(\overline{\{0_X\}}\right) \\ &\subset \overline{S(\{0_X\})} \\ &= \overline{\{0_Y\}} \\ &= \mathcal{K}^{\text{ernel}}{}_Y \end{aligned}$$

□

Proof of 2.7.15 vii. $(3 \implies 2)$ is trivial, as is $(2 \implies 3)$.

I now prove $(1 \implies 3)$. Let $\{T_i\}_{i \in \mathbb{N}}$ be a **Pseudometric Cauchy Sequence**. Let $\beta > 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $m, n > N$,

$$||T_n - T_m|| < \frac{\epsilon}{\beta}$$

Let $x \in B_X(0; \beta)$. Then

$$\begin{aligned} ||T_m x - T_n x||_Y &= ||(T_m - T_n)x||_Y \\ &\leq ||T_m - T_n|| \, ||x||_X \\ &< \epsilon \end{aligned}$$

Since $x \in B_X(0;\beta)$ was arbitrary, $\{\{T_i x\}_{i\in\mathbb{N}} : x \in B_X(0;\beta)\}$ is **Uniformly Cauchy**.

I now prove ($3 \implies 1$). Let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for $m, n > N$, for each $x \in B_X(0;2)$,

$$||T_m x - T_n x|| < \epsilon$$

In particular, if $||x|| = 1$, then

$$\frac{||(T_m - T_n)x||_Y}{||x||_X} = ||(T_m - T_n)x||_Y < \epsilon \tag{2.14}$$

Hence, by taking the supremum over such x and applying 2.7.15 iii, $||T_m - T_n|| < \epsilon$. □

Proof of 2.7.15 viii. Let $T_i \to T$. Let $x \in X$. If $x \in \mathcal{K}^{\text{ernel}}{}_X$, then $T_i(x) \in \mathcal{K}^{\text{ernel}}{}_Y$ for $i \in \mathbb{N}$ and $T_x \in \mathcal{K}^{\text{ernel}}{}_Y$, so convergence is obvious. Suppose $||x||_X > 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $n > N$, $||T_n - T|| < \frac{\epsilon}{||x||_X}$. For such n,

$$||T_i x - Tx||_Y \leq ||T_i - T|| \, ||x||_X < \epsilon$$

□

Proof of 2.7.15 ix. ($\implies$) Suppose $T_i \to T$. Then, by 2.5.7, $\{T_i\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence**. An application of 2.7.15 viii implies the pointwise convergence on a hamel basis.

($\impliedby$) Let $\{x_\alpha\}_{\alpha\in A}$ be a Hamel basis for X. Let $T_i x_\alpha \to y_\alpha$ for $\alpha \in A$. Define $T : X \to Y$ by setting, for $x \in X$, for any $\{\alpha_i\}_{i=1}^n \subset A$ $\{\beta_i\}_{i=1}^n \subset \mathbb{F}$,

$$T\left(\sum_{i=1}^n \beta_i x_{\alpha_i}\right) = \sum_{i=1}^n \beta_i y_{\alpha_i} \tag{2.15}$$

The uniqueness of a hamel basis representation implies that T is well defined. It is clear also that T is linear, and that $T\left(\mathcal{K}^{\text{ernel}}{}_X\right) \subset \mathcal{K}^{\text{ernel}}{}_Y$.

Let $x \in X$. Then we can find a unique representation, $x = \sum_{j=1}^n \beta_j x_{\alpha_j}$ where $x_{\alpha_j} \in A$ and $\beta_j \in \mathbb{F}$ for every j. For each $j \in \{1, ..., n\}$, there is an N_j such that if $n_j > N_j$, the

$$\left|\left|T_{n_j} x_{\alpha_j} - y_{\alpha_j}\right|\right| < \frac{\epsilon}{n(|\beta_j| + 1)} \tag{2.16}$$

Let $N = max\{N_j\}_{j=1}^n$. Let $m > N$. Then, we have

$$\begin{aligned}
||T_m x - Tx|| &= \left|\left|T_m\left(\sum_{j=1}^n \beta_j x_{\alpha_j}\right) - T\left(\sum_{j=1}^n \beta_j x_{\alpha_j}\right)\right|\right| \\
&= \left|\left|\sum_{j=1}^n \beta_j \left(T_m x_{\alpha_j} - Tx_{\alpha_j}\right)\right|\right| \\
&= \sum_{j=1}^n |\beta_j| \left|\left|T_m x_{\alpha_j} - y_{\alpha_j}\right|\right| \\
&< \epsilon
\end{aligned}$$

Since $m > N$ was arbitrtary , $T_i x \to Tx$ for $x \in X$.

Since $T_i x \to Tx$ for $x \in B_X(0;2)$, and since 2.7.15 vii, paried with the assumption that $\{T_i\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence**, $\{\{T_i x\}_{i\in\mathbb{N}}\}_{x\in B_X(0;2)}$ is **Uniformly Convergent** to $\{Tx\}_{x\in B_X(0;2)}$.

Let $\epsilon > 0$. By **Uniform Convergence**, there is an $N \in \mathbb{N}$ such that for $n > N$, $x \in B_X(0;2)$, we have

$$||T_n x - Tx|| < \epsilon \tag{2.17}$$

In particular, if $||x|| = 1$,

$$\frac{||(T_n - T)x||_Y}{||x||_X} < \epsilon \tag{2.18}$$

Implying first by 2.7.15 vi that $T \in BL(X,Y)$, and second that Hence $T_i \to T$ with respect to $||\cdot||$. $\square$

Proof of 2.7.15 x. Let Y be Pseudometric Complete. and let X be **Non-Degenerate**. Let $\{T_\alpha\}_{\alpha\in A} \subset BL(X,Y)$ be a **Pseudometric Cauchy Sequence**. Let $\{x_\alpha\}_{\alpha\in A}$ be a Hamel basis for X. Let $\alpha \in A$. If $x_\alpha \in \mathcal{K}^{\text{ernel}}{}_X$, then $T_i x_\alpha \in \mathcal{K}^{\text{ernel}}{}_Y$ for $i \in \mathbb{N}$, and so $T_i x_\alpha \to 0$. Otherwise, $||x_\alpha||_X > 0$, so for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n > N$, we have $||T_m - T_n|| < \frac{\epsilon}{||x_\alpha||}$. For such m and n,

$$\begin{aligned}
\frac{||T_m x_\alpha - T_n x_\alpha||_Y}{||x_\alpha||_X} &= \frac{||(T_m - T_n)\, x_\alpha||_Y}{||x_\alpha||_X} \\
&\leq ||T_n - T_m|| \\
&< \frac{\epsilon}{||x_\alpha||}
\end{aligned}$$

By multiplyging by $||x_\alpha||_X$, we see that $\{T_i x_\alpha\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence**. Since Y is **Complete**, these sequences converge, say $T_i x_\alpha \to y_\alpha$. This allows us apply 2.7.15 ix to claim that $\{T_i\}_{i\in\mathbb{N}}$ converges in $||\cdot||$. $\square$

Proof of 11. If $BL(X,Y)$ is nondegenerate, then for some $T \in BL(X,Y)$, for some $x \in X$, $||Tx||_Y \neq 0$. $\square$

Proof of 2.7.15 xii. Let $S \in BL(X,Y)$. Let $c = ||S|| + 1$. Then, if $||x|| \neq 0$, we have

$$\frac{||Sx||}{||x||} \leq \sup_{||x||\neq 0} \frac{||Sx||}{||x||} = ||S|| = c \tag{2.19}$$

so multiplying by $||x||$ gives $||Sx|| \leq c\,||x||$. If $||x|| = 0$, then $x \in \mathcal{K}^{\text{ernel}}{}_X$. Hence, by 2.7.15 vi, $Sx \in \mathcal{K}^{\text{ernel}}{}_X$, so

$$||Sx|| = 0 = c0 = c\,||Sx|| \tag{2.20}$$

finishing this direction of the proof.

Suppose instead, now, that there was a constant $c \in (0,1)$ such that for each $x \in X$, we had $||Sx|| \leq c\,||x||$. Then, if $||x|| = 0$, we must have $||Sx|| \leq c\,||x|| = 0$, and so $S(\mathcal{K}^{\text{ernel}}{}_X) \subset \mathcal{K}^{\text{ernel}}{}_Y$. Furthermore, if $||x|| \neq 0$, then we can divide by $||x||$ to get

$$\frac{||Sx||}{||x||} \leq c \tag{2.21}$$

Taking the supremum over all x with $||x|| \neq 0$ gives us the desired result. □

Proof of 2.7.15 xiii. By 2.7.15 v, $||T||$ is one possible value for c, so $||T|| \in A$. That is, $inf(A) \leq ||T||$. Furthermore, if $\epsilon > 0$, then there is an $x \in X$ with

$$\frac{||Tx||}{||x||} > ||T|| - \epsilon \tag{2.22}$$

implying $||Tx|| > (||T|| - \epsilon)\,||x||$ so that $||T|| - \epsilon \notin A$. Hence $||T|| - \epsilon < inf(A)$. Since this is true for every positive epsilon, $||T|| < inf(A)$. □

Remark 2.7.16 (Converses)**.** The converses of 2.7.15 x and 2.7.15 xi are true, but their proof relies upon the Hahn Banach theorem for Seminormed spaces, and so their official claim is delayed until later in the narrative.

Definition 2.7.17 (**Codomain Quotient Map**)**.** Let X and Y be **Seminormed Spaces**. Define $\mathcal{Q}_Y : BL(X,Y) \to BL(X, Y/\mathcal{K}^{\text{ernel}}{}_Y)$ by setting, for each $x \in X$, and for each $T \in BL(X,Y)$,

$$\mathcal{Q}_Y T x = [Tx]$$

We call $\mathcal{Q}_Y$ the **Codomain Quotient Map** of X and Y. Let $T \in BL(X,Y)$. We call $\mathcal{Q}_Y T$ the **Codomain Quotient Operator** of T.

Proposition 2.7.18 (Codomain Quotient Operator)**.** Let X and Y be **Seminormed Spaces** with **Codomain Quotient Map** $\mathcal{Q}_Y$. The following are true.

1. $\mathcal{Q}_Y$ is a well defined continuous linear surjective isometry.

2. If Y is a **Normed Space**, then $\mathcal{Q}_Y$ is invertible with a continuous inverse.

Proof Of 1. Since $Tx \in Y$ for any $x \in X$, $[Tx]_Y$ is defined for any $x \in X$. Furthermore, if $q_y : Y \to Y/\mathcal{K}^{\text{ernel}}$ is the **Quotient Map** of Y under **Equivalence MOD-$\mathcal{K}^{\text{ernel}}$**, then $\mathcal{Q}_Y T = q_y \circ T$. By 2.7.11, $q_y \in BL(Y, Y/\mathcal{K}^{\text{ernel}})$. Hence, $\mathcal{Q}_Y T = q_y \circ T \in BL(X, Y/\mathcal{K}^{\text{ernel}})$. Hence $\mathcal{Q}_Y$ is well defined.

For linearity, let $\alpha \in \mathbb{F}$ and $S, T \in BL(X,Y)$. Let $x \in X$. Then,

$$\begin{aligned}
\mathcal{Q}_Y (\alpha T + S)\, x &= [(\alpha T + S)\, x]_Y \\
&= [\alpha Tx + Sx]_Y \\
&= [\alpha Tx]_Y + [Sx]_Y \\
&= \alpha\, [Tx]_Y + [Sx_Y] \\
&= \alpha \mathcal{Q}_Y Tx + \mathcal{Q}_Y Sx \\
&= (\alpha \mathcal{Q}_Y T + \mathcal{Q}_Y S)\, x
\end{aligned}$$

For being an isometry, let $T \in BL(X,Y)$ and let $x \in X$. Then, since $||[Tx]_Y||_{Y/\mathcal{K}^{\text{ernel}}} = ||Tx||_Y$,

$$\frac{||\mathcal{Q}Tx||_{Y/\mathcal{K}^{\text{ernel}}}}{||x||_X} = \frac{||[Tx]_Y||_{Y/\mathcal{K}^{\text{ernel}}}}{||x||_X} = \frac{||Tx||_Y}{||x||_X}$$

and thus taking the norm over x with $||x||_X \neq 0$ will yield the same result. Hence $||T|| = ||\mathcal{Q}_Y T||$.

For surjectivity, let $\tilde{T} \in BL(X, Y/\mathcal{K}^{\text{ernel}}{}_Y)$. Let $\{x_\alpha\}_{\alpha \in A}$ be a hamel basis for X. For each $\alpha \in A$, let $y_\alpha \in \tilde{T}x_\alpha$. Define $T : X \to Y$ by

$$T\left(\sum_{i=1}^{n} \beta_{\alpha_i} x_{\alpha_i}\right) = \sum_{i=1}^{n} \beta_{\alpha_i} y_{\alpha_i} \tag{2.23}$$

T is obviously linear and has the property $[Tx] = \tilde{T}x$. and since $\tilde{T} \in BL(X, Y/\mathcal{K}^{\text{ernel}})$, $\tilde{T}\mathcal{K}^{\text{ernel}}{}_X \subset \mathcal{K}^{\text{ernel}}{}_{(Y/\mathcal{K}^{\text{ernel}}{}_Y)} = 0$. Hence $T\mathcal{K}^{\text{ernel}}{}_X \subset \mathcal{K}^{\text{ernel}}{}_Y$. Furthermore, if $x \in X$ with $||x||_X \neq 0$, then

$$\frac{||Tx||_Y}{||x||_X} = \frac{||[Tx]_Y||_{Y/\mathcal{K}^{\text{ernel}}}}{||x||_X} = \frac{\left|\left|\tilde{T}x\right|\right|_{Y/\mathcal{K}^{\text{ernel}}}}{||x||_X}$$

Therefore T is bounded. Hence $T \in BL(X,Y)$, and $\mathcal{Q}_Y T = \tilde{T}$. Thus we have surjectivity, and are done. □

Proof Of 2. If Y is a **Normed Space**, a linear isometric homeomorphism by 2.7.11. In particular, in this case, q_y is injective, meaning that if $T, S \in BL(X,Y)$ where $T \neq S$, then $Tx_0 \neq Sx_0$ for some $x_0 \in X$. For this x_0, $q_y T x_0 \neq q_y S x_0$, so $\mathcal{Q}_Y T \neq \mathcal{Q}_Y S$. Therefore $\mathcal{Q}_Y$ is injective, and therefore a bijection. The inverse of an isometry is also an isometry and therefore continuous, finishing this proof. □

Definition 2.7.19 (Quotient Operator)**.** Let X, Y be **Seminormed Spaces** with **Seminorm Kernels** $\mathcal{K}^{\text{ernel}}{}_X$, $\mathcal{K}^{\text{ernel}}{}_Y$. Define $Q : BL(X,Y) \to BL(X/\mathcal{K}^{\text{ernel}}{}_X, Y/\mathcal{K}^{\text{ernel}}{}_Y)$ by setting, for $T \in BL(X,Y)$, for $x \in X$,

$$QT\,[x]_X = [Tx]_Y \tag{2.24}$$

We call Q the **Operator Quotient Map** of X and Y and we call QT the **Quotient Operator** of T.

Proposition 2.7.20 (Quotient Operator)**.** Let X, Y be **Seminormed Spaces** with **Seminorm Kernels** $\mathcal{K}^{\text{ernel}}{}_X$, $\mathcal{K}^{\text{ernel}}{}_Y$ and **Operator Quotient Map** Q. Then Q is a well-defined **Linear Surjective Isometry**.

Proof. We first show that Q is well defined. Let $T \in BL(X,Y)$ and let $x_0, x_1 \in X$ such that $[x_0] = [x_1]$. Then $||x_0 - x_1||_X = 0$, so since T is **Continuous**, $||Tx_0 - Tx_1||_Y = 0$. Hence $Tx_0 \cong Tx_1$, so $[Tx_0] = [Tx_1]$.

For **Linearity**, let $\alpha \in \mathbb{F}$, and let $T, S \in BL(X,Y)$. Let $x \in X$. Then

$$\begin{aligned} Q(\alpha T + S)[x]_X &= [(\alpha T + S)x]_Y \\ &= \alpha [Tx]_Y + [Sx]_Y \\ &= \alpha QT[x]_X + QS[x]_X \\ &= (\alpha QT + QS)[x]_X \end{aligned}$$

Since $x \in X$ was arbitrary, Q is **Linear**.

As for being an **Isometry**, let $T \in BL(X,Y)$ and let $x \in X$. Since $||[x]|| = ||x||$ and $||Tx|| = ||[Tx]||$, we have

$$\begin{aligned} \frac{\left|\left|QT[x]_{X/\mathcal{K}^{\text{ernel}}_X}\right|\right|_{Y/\mathcal{K}^{\text{ernel}}_Y}}{||[x]||_{X/\mathcal{K}^{\text{ernel}}_X}} &= \frac{||[Tx]||_{Y/\mathcal{K}^{\text{ernel}}_Y}}{||[X]||_{X/\mathcal{K}^{\text{ernel}}_X}} \\ &= \frac{||Tx||_Y}{||x||_X} \end{aligned}$$

and so taking the supremum over $||x|| \neq 0$ gives us that this is an **Isometry**.

For surjectivity, let $\tilde{T} \in BL(X/\mathcal{K}^{\text{ernel}}_X, Y/\mathcal{K}^{\text{ernel}}_Y)$. Let $\{x_\alpha\}_{\alpha \in A}$ be a Hamel basis for X. For each $\alpha \in A$, let $y_\alpha \in \tilde{T}[x_\alpha]_X$. Now define

$$T\sum_{i=1}^{n} \beta_i x_{\alpha_i} = \sum_{i=1}^{n} \beta_i y_{\alpha_i} \tag{2.25}$$

Then $T : X \to Y$ is obviously **Linear**, and $Tx \in \tilde{T}[x]_X$ for $x \in X$. Hence,

$$‘\frac{||Tx||_Y}{||x||_X} = \frac{\left|\left|\tilde{T}[x]_X\right|\right|_{Y/\mathcal{K}^{\text{ernel}}_Y}}{||[x]_X||_{X/\mathcal{K}^{\text{ernel}}_X}} \tag{2.26}$$

so T is bounded, and hence $T \in BL(X,Y)$, but that also implies that by definition, $QT = \tilde{T}$, so we have proven surjectivity. ☐

Definition 2.7.21 (Canonical Isomorphism Of The Quotient Space Of Continuous **Linear** Operators)**.** Let X, Y be **Seminormed Spaces** with **Seminorm Kernels** $\mathcal{K}^{\text{ernel}}_X$, $\mathcal{K}^{\text{ernel}}_Y$. Let $\mathcal{K}^{\text{ernel}}$ denote the **Seminorm Kernel** of $BL(X,Y)$. Let Q denote the **Operator Quotient Map** of X and Y. Define $\Theta_{(X,Y)} : BL(X,Y)/\mathcal{K}^{\text{ernel}} \to BL(X/\mathcal{K}^{\text{ernel}}_X, Y/\mathcal{K}^{\text{ernel}}_Y)$ by setting, for each $T \in BL(X,Y)$.

$$\Theta_{(X,Y)}([T]) = QT$$

We call $\Theta_{(X,Y)}$ the **Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators** from X to Y. When X and Y are understood, we may denote the **Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators** simply with Θ. By 2.7.22, $\Theta_{(X,Y)}$ is an isomorphism of **Normed Spaces**. That is, Θ is a **Linear**, **Isometric Homeomorphism**.

Proposition 2.7.22. Let X, Y be **Seminormed Spaces**. Let Θ denote the **Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators** from X to Y. Then Θ is a **Linear**, **Isometric Homeomorphism**.

Proof. By 2.7.11, part 1, $Y/\mathcal{K}^{\text{ernel}}{}_Y$ is a **Normed Space**, Hence by 2.7.15, part 2, $BL(X/\mathcal{K}^{\text{ernel}}{}_X, Y/\mathcal{K}^{\text{ernel}}{}_Y)$ is a **Normed Space**. Similarly, by 2.7.11, part 1, $BL(X,Y)/\mathcal{K}^{\text{ernel}}$ is a **Normed Space**. Hence, it is sufficient to show that Θ is a well-defined **Surjective Linear Isometry**.

For well definedness, let $T, S \in BL(X,Y)$ with $[T] = [S]$. Then, $||T - S|| = 0$, so if $x \in X$, $||Tx - Sx|| = 0$. Hence $Tx \cong Sx$ and since x was arbitrary, $QT = QS$.

Let q denote the **Quotient Map** $q : BL(X,Y) \to BL(X,Y)/\mathcal{K}^{\text{ernel}}$. By parts 4, 5, and 6 of 2.7.11, q is a **Linear Surjective Isometry**. Also, by definition, $\Theta \circ q = Q$. Since Q is **Surjective**, Θ is **Surjective**. Since Q is an **Isometry**, and q is a surjecive **Isometry**, $Theta$ is an **Isometry**. Since Q is **Linear**, and since q is **Surjective** and **Linear**, Θ is **Linear**. □

Definition 2.7.23. REMOVE

Definition 2.7.24 (Dual Space)**.** Let $(X, ||\cdot||)($ be a **Seminormed Space**. We call $BL(X, \mathbb{F})$ the **Topological Dual Space** of $(X, ||\cdot||)$, and we denote $BL(X, \mathbb{F})$ with the symbol X^*. If $x^* \in X^*$, then we use the notational convention of writing, for $x \in X$.

$$\langle x, x^* \rangle := x^*(x)$$

It would also be correct to refer to the **Topological Dual Space** of $(X, ||\cdot||)$ as the **First Topological Dual Space** of X

Remark 2.7.25 (**Topological Dual Space** is a **Normed Space**)**.** *Let X be a **Seminormed Space**. Then, using 2.7.15, since $\mathbb{F}$ is a **Normed Space**, so is X^*.*

Theorem 2.7.26 (**Topological Dual Space** Isomorphism)**.** Let X be a **Seminormed Space**. Define $\Omega : X^* \to \left(X/\mathcal{K}^{\text{ernel}}{}_X\right)^*$ by setting, for $x^* \in X$, and for $x \in X$,

$$\langle x, x^* \rangle = \langle [x], \Omega x^* \rangle$$

Then Ω is a **Linear**, **Bijective**, **Isometry** which operates **Homeomorphically**. That is, X^* and $(X/\mathcal{K}^{\text{ernel}}{}_X)^*$ are isomorphic, and that isomorphism is explicitly given by Ω.

Proof. Consider the following

$$BL(X, \mathbb{F}) \xrightarrow{q} BL(X, \mathbb{F})/\mathcal{K}^{\text{ernel}}{}_{BL(X/\mathbb{F})} \xrightarrow{\Theta} BL(X/\mathcal{K}^{\text{ernel}}{}_X, \mathbb{F}/\mathcal{K}^{\text{ernel}}{}_{\mathbb{F}}) \xrightarrow{\mathcal{Q}_{\mathbb{F}}^{-1}} BL(X/\mathcal{K}^{\text{ernel}}{}_X, \mathbb{F})$$

where q is the **Quotient Map**, which is a **Linear Isometric Homeomorphism** in this case by parts 4, 5, 6, and 7 of of 2.7.11, Θ is the **Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators**, which is a **Linear Isometric Homeomorphism** by 2.7.22 and $\mathcal{Q}_{\mathbb{F}}$ is the **Codomain Quotient Map**. which is in this case a **Linear**, **Isometric Homeomorphism** by 2.7.18

Since $\Omega = \mathcal{Q}_{\mathbb{F}}^{-1} \circ \Theta \circ q$, and since each of the described properties are preserved under composition, Ω is also a **Linear Isometric Homeomorphism**. □

Remark 2.7.27 (**Topological Dual Space** is a **Normed Space**)**.** Let X be a **Seminormed Space**. Since $X/\mathcal{K}^{\text{ernel}}{}_X$ is a **Normed Space**, so is $(X/\mathcal{K}^{\text{ernel}}{}_X)^*$. By 2.7.26, we have a **Linear**, **Isometric Bijection** between X^* and $(X/\mathcal{K}^{\text{ernel}}{}_X)^*$. Hence X^* is a **Normed Space**.

2.7.1 Seminormed Hahn Banach Theorem

Theorem 2.7.28 (Hahn Banach Theorem For Seminormed Spaces)**.** Let $(X, ||\cdot||)$ be a **Seminormed Space**, let $x_i \in X$ for $i \in \{0,1\}$ such that $||x_0 - x_1||_X \neq 0$, and let X^* denote $X's$ **Topological Dual Space**. The following are true.

(i) If $Z \subset X$ is a subspace and $z^* \in Z^*$, then there is an extension x^* of z^*, $x^* \in X^*$ such that
$$||z^*||_{Z^*} = ||x^*||_{X^*}$$

(ii) If $x \in X$, with $||x|| \neq 0$, then there exists an $x^* \in X$ with $||x^*|| = 1$ and $\langle x, x^* \rangle = ||x||_X$.

(iii) If $x \in X$, then
$$||x||_X = \sup_{0 \neq x^* \in X^*} \frac{\langle x, x^* \rangle}{||x^*||}$$

(iv) If Y is a **Non-Degenerate Seminormed Space**, and if $x_0 \in X$, with $||x_0|| \neq 0$, then there exists an $S \in BL(X,Y)$ with $||S|| = 1$ and
$$||Sx_0|| = ||x_0||$$

Proof of 2.7.28 i. For $\alpha \in \{Z, X\}$, let $\Omega_\alpha : \alpha^* \to (\alpha/\mathcal{K}^{\text{ernel}}{}_\alpha)^*$ denote the isomorphism defined in 2.7.26. Let q denote the quotient operator $q : X \to X/\mathcal{K}^{\text{ernel}}$. Define $T : Z/\mathcal{K}^{\text{ernel}}{}_Z \to q(Z)$ bv $T([z]_{\cong_Z}) = [z]_{\cong_X}$. Since Z is endowed with the subspace Topology, T is obviously a **Linear Isometric Homeomorphism**.

Define $\Gamma_Z : (Z/\mathcal{K}^{\text{ernel}}{}_Z)^* \to q(Z)^*$ by setting, for $\phi^* \in (Z/\mathcal{K}^{\text{ernel}}{}_Z)^*$, for $[z]_Z \in Z/\mathcal{K}^{\text{ernel}}{}_Z$,
$$\langle T[z]_Z, \Gamma_Z \phi^* \rangle = \langle [z]_Z, \phi^* \rangle$$
Then Γ_Z is a **Linear** Bijective **Isometry**. Hence $\Gamma_Z \circ \Omega_Z z^* \in q(Z)^*$ with $||\Gamma_Z \circ \Omega_Z z^*||_{q(Z)^*} = ||z^*||_{Z^*}$.

Thus we can apply the Hahn Banach theorem for **Normed Spaces** to claim the existence of $x_q^* \in (X/\mathcal{K}^{\text{ernel}}{}_X)^*$ where x_q^* is an extension of $\Gamma_Z \circ \Omega_Z z^*$ and
$$\left|\left|x_q^*\right|\right|_{(X/\mathcal{K}^{\text{ernel}}{}_X)^*} = ||\Gamma_Z \circ \Omega_Z z^*||_{(q(Z))^*} = ||z^*||_{Z^*}$$
Finally, letting $x^* = \Omega_X^{-1} x_q^*$, we have $x^* \in X^*$, $||x^*||_{X^*} = \left|\left|x_q^*\right|\right|_{(X/\mathcal{K}^{\text{ernel}}{}_X)^*} = ||z^*||_{Z^*}$, and if $z \in Z$, then
$$\begin{aligned}\langle z, x^* \rangle &= \left\langle [z]_X, x_q^* \right\rangle \\ &= \langle [z]_X, \Gamma_Z \circ \Omega_Z z^* \rangle \\ &= \langle [z]_Z, \Omega_Z z^* \rangle \\ &= \langle z, z^* \rangle\end{aligned}$$
□

Proof of 2. Let $Z = span(x)$. Define $z^* \in Z^*$ by $\langle \alpha x, z^* \rangle = \alpha\,||x||$. Then $||z^*|| = 1$. Also, by part 1 of this result, it has an extension $x^* \in X^*$ with $||x^*|| = ||z^*|| = 1$ and $\langle x, x^* \rangle = ||x||$. □

Proof of 3. If $||x|| = 0$, then for every $x^* \in X$, $\langle x, x^* \rangle = 0$. Hence

$$||x||_X = \sup_{0 \neq x^* \in X^*} \frac{\langle x, x^* \rangle}{||x^*||} = \sup_{x^* \in \partial B_{X^*}(0;1)} \frac{\langle x, x^* \rangle}{||x^*||} = 0$$

Otherwise, let $x^* \in X^*$ guaranteed to exist by part 2 which satisfies $||x^*|| = 1$, $\langle x, x^* \rangle = ||x||$. Then

$$\begin{aligned} ||x|| &= \frac{\langle x, x^* \rangle}{||x^*||} \\ &\leq \sup_{x^* \in \partial B_{X^*}(0;1)} \frac{\langle x, x^* \rangle}{||x^*||} \\ &\leq \sup_{0 \neq x^* \in X^*} \frac{\langle x, x^* \rangle}{||x^*||} \end{aligned}$$

The other direction of the inequality falls directly from the definition of the norm on X^*, and is trivial, so we are done. □

Proof of 4. By part 2 of this result, there exists $x_0^* \in X^*$ with $||x_0^*|| = 1$ and $\langle x_0, x_0^* \rangle = ||x_0||$. Since Y is **Non-Degenerate**, there exists $y_0 \in Y$ with $||y_0|| = 1$. Define $T : \mathbb{F} \to Y$ by $T\alpha = \alpha y$. Then $||T|| = ||y|| = 1$. Define $S : X \to Y$ by $S = T \circ x_0^*$. Then $||S|| \leq ||T||\,||x_0^*|| = 1$, and $||Sx_0|| = ||\langle x_0, x_0^* \rangle\, y|| = \langle x_0, x_0^* \rangle = ||x_0||$. Hence $||S|| \geq 1$ and therefore $||S|| = 1$. □

2.7.2 Seminorm Adjoints

Proposition 2.7.29 (Linear Operator Notation)**.** When dealing with mappings of spaces of **Linear** operators into spaces of other **Linear** operators, or even functions in general, notation can get confusing, and presenting such things using ordinary notation without ambiguity can often require a plethora of parenthesis, which hamper readability of an arguement.

For this reason, at points in this document, I sometimes express the image $\beta(\alpha)$ using

$$\langle \alpha, \beta \rangle$$

Where $\beta : X \to Y$ and $\alpha \in X$.

I combine this notation with usual function notation, particularly in cases similar to the following. For $i \in \{0, 1\}$, let X_i, Y_i, Z_i be sets. For $\alpha \in \{X, Y, Z\}$, let F_α be the set of maps $f : \alpha_0 \to \alpha_1$. If $T : F_X \to F_Y$, $y \in Y_0$, and $f \in F_X$, then I would notate

$$\langle y, Tf \rangle$$

rather than $Tf(y)$ or $(T(f)(y))$

Definition 2.7.30 (**Adjoint Operator**)**.** Let X, Y, and Z be **Seminormed Spaces** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. . Let $T \in BL(X,Y)$. We define the operator $T_Z^\times : BL(Y,Z) \to BL(X,Z)$ by setting, for $S \in BL(Y,Z)$ and $x \in X$,

$$\left\langle x, \mathcal{T}_Z^\times S \right\rangle = \langle Tx, S \rangle \tag{2.27}$$

or, equivalently,

$$\mathcal{T}_Z^\times S = S \circ T \tag{2.28}$$

We call $T_Z^\times$ the **Adjoint Operator** of T relative to the space Z, we denote $T_{\mathbb{F}}^\times = T^\times$, and we refer to $T^\times : Y^* \to X^*$ as simply the **Adjoint Operator** of T.

Proposition 2.7.31 (**Adjoint Operator**)**.** Let X, Y, and Z be **Seminormed Spaces** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. . Let $T \in BL(X,Y)$. Let $\mathcal{T} = T_Z^\times$ denote the **Adjoint Operator** of T relative to the space Z. Let Q_y denote the **Quotient Map** The following are true.

(i) $\mathcal{T}$ is Linear.

(ii) If $S \in BL(Y,Z)$, then $\mathcal{T}S \in BL(X,Z)$. (That is, the **Adjoint Operator** is well defined as a concept).

(iii) $\mathcal{T} \in BL\left(BL(Y,Z), BL(X,Z)\right)$.

(iv) $||\mathcal{T}|| = ||T||$

(v) If T is **Surjective**, then $\inf\limits_{||x||=1} ||Tx|| \leq \inf\limits_{||S||_{BL(Y,Z)}=1} ||\mathcal{T}S||$. Also TODO: Weaken T surjectivity condition To Range(T) dense in Y.

(vi) If Range(T) is not dense in Y, then $\inf\limits_{||S||_{BL(Y,Z)}=1} ||\mathcal{T}S|| = 0$

(vii) $\mathcal{T}$ is **Surjective** if and only if T is **Injective** and has **Closed** range in Y.

Proof of 2.7.33 i. Let $S, R \in BL(Y,Z)$, $\alpha \in \mathbb{F}$, and $x \in X$. Then,

$$\begin{aligned}
\langle x, \mathcal{T}(\alpha S + R) \rangle &= \langle Tx, \alpha S + R \rangle \\
&= \alpha \langle Tx, S \rangle + \langle Tx, R \rangle \\
&= \alpha \langle x, \mathcal{T}S \rangle + \langle x, \mathcal{T}R \rangle \qquad\qquad = \langle x, \alpha\mathcal{T}S \rangle + \langle x, \mathcal{T}R \rangle \\
&= \langle x, \alpha\mathcal{T}S + \mathcal{T}R \rangle
\end{aligned}$$

Since $x \in X$ was arbitrary, **Linearity** is verified. ☐

Proof of 2.7.33 ii. Let $S \in BL(Y,Z)$. Then, $\mathcal{T}S = S \circ T$. The composition of continuous operators is continuous, so $\mathcal{T}S$ is continuous. The composition of linear operators is linear, so $\mathcal{T}S$ is linear. This, paired with **Linearity**, implies $\mathcal{T}S \in BL(X,Z)$. ☐

Proof of 2.7.33 iii. Let $S \in BL(Y,Z)$. Then, if $x \in X$

$$||\langle x, \mathcal{T}S \rangle|| = ||\langle Tx, S \rangle|| \leq ||S||\, ||Tx|| \leq ||S||\, ||T||\, ||x||$$

Hence $||\mathcal{T}S|| \leq ||S||\, ||T||$ Since T is linear, and since S was arbitrary, by part 12 of 2.7.15, $\mathcal{T} \in BL\left(BL(Y,Z), BL(X,Z)\right)$. ☐

Proof of 2.7.33 iv. For any $S \in BL(Y,Z)$, $\mathcal{T}S = S \circ T$, so $||\mathcal{T}S|| \leq ||S||\,||T||$. Hence $||\mathcal{T}|| \leq ||T||$. Now let $x_0 \in X$. Then, by part 4 of 2.7.28, there exists $S \in BL(Y,Z)$ with $||S|| = 1$ and $||STx_0|| = ||Tx_0||$. Hence,

$$\begin{aligned} ||Tx_0|| &= ||STx_0|| \\ &= ||(S \circ T)x_0|| \\ &= ||(\mathcal{T}S)x_0|| \\ &\leq ||\mathcal{T}||\,||S||\,||x_0|| \\ &= ||\mathcal{T}||\,||x_0|| \end{aligned}$$

Since $x_0 \in X$ is arbitrary, $||T|| \leq ||\mathcal{T}||$. Since the inequality goes both ways, $||T|| = ||\mathcal{T}||$. □

Proof of 2.7.33 v. Let $\Gamma = \inf\limits_{||x||=1} ||Tx||$, and let $S \in BL(Y,Z)$ with $||S|| = 1$. Then,

$$\{x|\,||Tx|| \leq \Gamma\} \subset B_X(0;1)$$

so

$$\sup_{||x||\leq 1} |\langle Tx, S\rangle| \geq \sup_{||Tx||\leq \Gamma} |\langle Tx, S\rangle|$$

Also, since T is **Surjective** by assumption,

$$\sup_{||Tx||\leq \Gamma} |\langle Tx, S\rangle| = \sup_{||y||\leq \Gamma} |\langle y, S\rangle|$$

From these two we arrive at the inequality

$$\begin{aligned} ||\mathcal{T}S|| &= \sup_{||x||\leq 1} |\langle x, \mathcal{T}S\rangle| \\ &= \sup_{||x||\leq 1} |\langle Tx, S\rangle| \\ &\geq \sup_{||Tx||\leq \Gamma} |\langle Tx, S\rangle| \\ &= \sup_{||y||\leq \Gamma} |\langle y, S\rangle| \\ &= \Gamma \\ &= \inf_{||x||=1} ||Tx|| \end{aligned}$$

Since $S \in \partial B_{BL(Y,Z)}(0;1)$ was arbitrary, we conclude $\inf\limits_{||S||=1} ||\mathcal{T}S|| \geq \inf\limits_{||x||=1} ||Tx||$ □

2.7.3 Higher order Seminorm Duals

Definition 2.7.32 (Higher Order Dual Spaces)**.** Let X be a **Seminormed Space**. From 2.7.24 we know that the **Topological Dual Space** of X, X^*, is also called the **First Topological Dual Space** of X. Building on this, for $n \in \{2,3,4,...,\}$ we call the **First Topological Dual Space** of X^* the 2^{nd} **Topological Dual Space** of X, we call the **First**

Topological Dual Space of the 2^{nd} **Topological Dual Space** of X the 3^{rd} **Topological Dual Space** of X, and in general the **First Topological Dual Space** of the $(n)^{th}$ **Topological Dual Space** of X the $(n+1)^{th}$ **Topological Dual Space** of X.

In general, we denote the $(n)^{th}$ **Topological Dual Space** of X with X^{n*}, though when n is small, we may denote $X^{**} = X^{2*}$, $X^{***} = X^{3*}$, et cetera.

Definition 2.7.33 (Higher Order Dual Space Isomorphism)**.** Let X be a **Seminormed Space** over a **Field** $\mathbb{F}$. Let $\Omega : X^* \to (X/\mathcal{K}^{\text{ernel}}{}_X)^*$ be the **Linear Bijective Isometry** defined in 2.7.26. Define

$$\Omega_1 = \Omega$$

and also define, for $2 \leq n \in \mathbb{N}$, $\Omega_n : X^{n*} \to \left(X/\mathcal{K}^{\text{ernel}}{}_X\right)^{n*}$ by

$$\Omega_n = \left(\Omega_{n-1}^{\times}\right)^{-1}$$

By

it is clear that the adjoint of a **Linear Bijective Isometry** of **Normed Spaces** is also a **Linear Bijective Isometry** of **Normed Spaces**, and so each Ω_n is as well.

Definition 2.7.34 (**Canonical Embedding** of X into X^{**})**.** Let X be a **Seminormed Space**. Define $c_X : X \to X^{**}$ by setting, for each $x^* \in X^*$, for each $x \in X$

$$\langle x^*, c(x)\rangle = \langle x, x^*\rangle$$

We call c_X the **Canonical Embedding** of X into X^*. As normal, if X is understood, we may denote $c_X = c$. If c is **Surjective**, then we say that X is **Reflexive**.

Proposition 2.7.35 (Canonical Embedding)**.** Let X be a **Seminormed Space** and let c denote its **Canonical Embedding**. The following are true.

1. c is well defined
2. c is **Linear**.
3. c is an **Isometry**.
4. c is an **Injection** if and only if X is a **Normed Space**.
5. If $q : X \to X/\mathcal{K}^{\text{ernel}}$ is the **Quotient Map**, $c_{X/\mathcal{K}^{\text{ernel}}}$ is the **Canonical Embedding** of $(X/\mathcal{K}^{\text{ernel}})$ into $(X/\mathcal{K}^{\text{ernel}})^{**}$ and $\Omega_2 : X^{**} \to (X/\mathcal{K}^{\text{ernel}})^{**}$ is the linear bijective **Isometry** defined in 2.7.33, then $c = \Omega_2^{-1} \circ c_{X/\mathcal{K}^{\text{ernel}}} \circ q$. //TODO: COME BACK TO THIS AND PROVE IT ONCE THE ISOS ARE CLEARED UP
6. c_X is **Surjective** if and only if $c_{X/\mathcal{K}^{\text{ernel}}}$ is **Surjective**.
7. X is **Reflexive** if and only if $X/\mathcal{K}^{\text{ernel}}$ is **Reflexive**.

Proof of 1. For any $x \in X$, $c(x)$ as a function is obviously well defined. Hence, I just need to show that, for any $x \in X$, $c(x) \in X^{**}$. That is , I must show that $c(x)$ is continuous and linear.

For linearity, if $x^*, y^* \in X^*$ and $\alpha \in \mathbb{F}$, we have

$$\begin{aligned} \langle \alpha x^* + y^*, c(x) \rangle &= \langle x, \alpha x^* + \alpha y^* \rangle \\ &= \alpha \langle x, x^* \rangle , \langle y, y^* \rangle \\ &= \alpha \langle x^*, c(x) \rangle + \langle y^*, c(x) \rangle \end{aligned}$$

Thus linearity holds.

For continuity, let $x \in X$ and let $x^* \in X^*$.

$$\begin{aligned} |\langle x^*, c(x) \rangle| &= |\langle x, x^* \rangle| \\ &\leq ||x|| \, ||x^*|| \end{aligned}$$

so that $c(x)$ is bounded with $||c(x)|| \leq ||x||$. □

Proof of 2. Let $\alpha \in \mathbb{F}$ and $x, y \in X$. Let $x^* \in X$. Then,

$$\begin{aligned} \langle x^*, c(\alpha x + y) \rangle &= \langle \alpha x + y, x^* \rangle \\ &= \alpha \langle x, x^* \rangle + \langle y, x^* \rangle \\ &= \alpha \langle x^*, c(x) \rangle + \langle x^*, c(y) \rangle \end{aligned}$$

, finishing the proof. □

Proof of 3. Let $x_0 \in X$ and $x^* \in X^*$. Then,

$$\begin{aligned} |\langle x^*, c(x_0) \rangle| &= |\langle x_0, x^* \rangle| \\ &\leq ||x_0|| \, ||x^*|| \end{aligned}$$

so that $||c(x_0)|| \leq ||(|| \, x_0)$. For the other direciton, by 2.7.28 part 2, there exists an $x_0^* \in X^*$ satisfying $||x_0^*|| = 1$ and $\langle x_0, x_0^* \rangle = ||x||$ □

We see that $\langle x_0^*, c(x_0) \rangle = \langle x_0, x_0^* \rangle = ||x|| = ||x_0|| \, ||x_0^*||$ so that $||c(x_0)|| \geq ||x_0||$. Since the inequality goes both ways, $||x_0|| = ||c(x_0)||$, and c is therefore an **Isometry**.

Proof of 4. Let X be a **Normed Space**. Then X^* separates points in X. Let $x \in X$ and $y \in X$ with $x \neq y$. Since X^* separates points in X, there exists $x^* \in X^*$ with $\langle x^*, c(x) \rangle = \langle x, x^* \rangle \neq \langle y, x^* \rangle , \langle x^*, c(y) \rangle$ so that $c(x) \neq c(y)$. Hence c is **Injective**.

Now supose instead that c is **Injective** and let $x, y \in X$ with $||x - y|| = 0$. We find that for any $x^* \in X^*$,

$$\begin{aligned} |\langle x^*, c(x) - c(y) \rangle| &= |\langle x^*, c(x - y) \rangle| \\ &= |\langle x - y, x^* \rangle| \\ &=\leq ||x^*|| \, ||x - y|| \\ = 0 \end{aligned}$$

so that $||c(x) - c(y)|| = 0$. Since X^{**} is a normed space, this implies $c(x) = c(y)$, which through **Injectiveness** implies $x = y$. Hence we have the implication $||x - y|| = 0 \implies x = y$, so that X is a **Normed Space**. □

Proof of 5. Proceeding directly from the definition, we have

$$\begin{aligned}\left\langle x^*, \Omega_2^{-1} \circ c_{X/\mathcal{K}^{\text{ernel}}} \circ q(x)\right\rangle &= \left\langle x^*, \Omega^\times \circ c_{X/\mathcal{K}^{\text{ernel}}} \circ q(x)\right\rangle \\ &= \left\langle \Omega x^*, c_{X/\mathcal{K}^{\text{ernel}}} \circ q(x)\right\rangle \\ &= \langle q(x), \Omega x^*\rangle \\ &= \langle x, x^*\rangle \\ &= \langle x^*, c(x)\rangle\end{aligned}$$

So we are done. □

Proof of 6. Since Ω_2 is a Bijection, by the prior part of this result, c is a surjection if and only if $c_{X/\mathcal{K}^{\text{ernel}}} \circ q$ is a surjection where $q : X \to X/\mathcal{K}^{\text{ernel}}$ is the **Quotient Map**. Since q is a surjection, c is a surjection if and only if$c_{X/\mathcal{K}^{\text{ernel}}}$ is a surjection. □

Proof of 7. This is a direct restatement of Part 06 of this result. □

Definition 2.7.36 (Weak Topologies Relating To Seminormed and Normed Spaces)**.** *latex*
$\mathfrak{weak}$
$\mathfrak{weak}^*$

Similar to in the context of a normed space, if X is a seminormed space, we define the weak topology on X to be the topology on X generated by X^*, and the *weak** topology on X^* to be the topology generated by $c(X)$. Before moving on to the classical theory revamped, I present on more useful result about weak topologies of seminormed spaces.

Proposition 2.7.37 (Weak Quotients)**.** *Let X be a seminormed space and $\{Y_\alpha\}_{\alpha\in A}$ be a collection of topological spaces. For each $\alpha \in A$ let $\phi_\alpha : X \to Y_\alpha$ have the property that for every $x, y \in X$, for every $\alpha \in A$, $||x - y|| = 0 \implies \phi_\alpha(x) = \phi_\alpha(y)$. For each $\alpha \in A$, define $\tilde{\phi}_\alpha : X/\,||\cdot||^{-1}\{0\} \to Y_\alpha$ by $\tilde{\phi}_\alpha[x] = \phi_\alpha x$. Let $\mathcal{T}_w$ denote the weak topology on X induced by $\{\phi_\alpha\}_{\alpha\in A}$, and $\mathcal{T}_{\tilde{w}}$ denote the weak topology on $X/\,||\cdot||^{-1}\{0\}$ induced by $\{\tilde{\phi}_\alpha\}_{\alpha\in A}$. Then*

$$(X, \mathcal{T}_w)/\,||\cdot||^{-1}\{0\} = (X/\,||\cdot||^{-1}\{0\}, \mathcal{T}_{\tilde{w}}) \tag{2.29}$$

Proof. □

Finally, before we move on, recall that if X, Y are Topological vector spaces, we can topologize the set of continuous linear operators from X to Y, denoted $BL(X, Y)$ by saying that $\{T_\alpha\}_{\alpha\in A} \subset BL(X, Y)$ converges to $T \in BL(X, Y)$ if there is a neighborhood U of 0 in X such that $T_\alpha x \to Tx$ uniformly for $x \in U$.

2.8 Classical Results With A Twist

By **??** and 2.7.37, many of the classical theorems relating a normed space and its duals still hold in the context of a seminormed space without too much alteration of the proofs. Since the author has not seen these results presented in this context, they are presented with proof below.

2.8.1 Helly

In this subsection, we develop Helly's theorem in the context of a seminormed space, which will serve as valuable lemma throughout this document. Its location here is due to the fact that is is a generalization of a lemma commonly used to prove the Goldstine Theorem.

Theorem 2.8.1 (Helly's Theorem)**.** Let $(X, ||\cdot||)$ be a **Seminormed Space**. Let $M > 0$. Let $\{\alpha_i\}_{i=1}^n \subset \mathbb{C}$, Let $\{x_i^*\}_{i=1}^n \subset X^*$. Then the following are equivalent.

(i) For each $\epsilon > 0$, there is an $x_\epsilon \in X$ such that $||x_\epsilon|| < M + \epsilon$ and $\langle x_\epsilon, x_i^* \rangle = \alpha_i$ for $1 \leq i \leq n$.

(ii) For every $\epsilon > 0$, there is an $x_\epsilon \in X$ such that $||x_\epsilon|| \leq M$ and $|\langle x_\epsilon, x_i^* \rangle - \alpha_i| < \epsilon$ for $1 \leq i \leq n$.

(iii) For each $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$,
$$\left| \sum_{i=1}^n \beta_i \alpha_i \right| \leq M \left|\left| \sum_{i=1}^n \beta_i x_i^* \right|\right| \tag{2.30}$$

2.8.1 i $\implies$ *2.8.1 ii*. Let $\epsilon > 0$. Define $K = \sup\limits_{i=1}^{n} ||x_i^*||$. Define $\epsilon_2 = \frac{\epsilon}{2K}$. Then by 2.8.1 i, there exists an $\tilde{x} \in X$ satisfying
$$||\tilde{x}|| < M + \epsilon_2 \tag{2.31}$$
and
$$\langle \tilde{x}, x_i^* \rangle = \alpha_i \qquad (1 \leq i \leq n) \tag{2.32}$$
Define $x_0 = \frac{M}{M+\epsilon_2}\tilde{x}$. Then by 2.31
$$||x_0|| = \frac{M}{M+\epsilon_2} ||\tilde{x}|| < \frac{M}{M+\epsilon_2} (M + \epsilon_2) = M$$
and by 2.32
$$||\tilde{x} - x_0|| = \left|\left| \tilde{x} - \frac{M}{M+\epsilon_2}\tilde{x} \right|\right| = \left| 1 - \frac{M}{M+\epsilon_2} \right| ||\tilde{x}|| = \left| \frac{\epsilon_2}{M+\epsilon_2} \right| ||\tilde{x}|| < \epsilon_2 \tag{2.33}$$
Hence, if $1 \leq i \leq n$, then by 2.32 and 2.33 and since $||x_i^*|| \leq K$,
$$\begin{aligned} |\langle x_0, x_i^* \rangle - \alpha_i| &\leq |\langle x_0 - \tilde{x}, x_i^* \rangle| + |\langle \tilde{x}, x_i^* \rangle - \alpha_i| \\ &= |\langle x_0 - \tilde{x}, x_i^* \rangle| \\ &\leq K\epsilon_2 \\ &\leq \frac{\epsilon}{2} < \epsilon \end{aligned}$$
Hence, 2.8.1 ii holds with x_0 in place of x_ϵ. □

2.8.1 ii $\implies$ *2.8.1 iii*. Let $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$. Let $\epsilon > 0$. Then by 2.8.1 ii, there exists $x_0 \in X$ satisfying

$$||x_0|| \leq M \tag{2.34}$$

and

$$|\langle x_0, x_i^* \rangle - \alpha_i| < \epsilon \qquad (1 \leq i \leq n) \tag{2.35}$$

Then

$$\begin{aligned}
\left| \sum_{i=1}^n \beta_i \alpha_i \right| &= \left| \sum_{i=1}^n \beta_i \left(\alpha_i - \langle x_0, x_i^* \rangle + \langle x_0, x_i^* \rangle \right) \right| \\
&\leq \left| \sum_{i=1}^n \beta_i \left(\alpha_i - \langle x_0, x_i^* \rangle \right) \right| + \left| \sum_{i=1}^n \beta_i \langle x_0, x_i^* \rangle \right| \\
&< \epsilon \sum_{i=1}^n |\beta_i| + \left| \left\langle x_0, \sum_{i=1}^n \beta_i x_i^* \right\rangle \right| \\
&\leq \epsilon \sum_{i=1}^n |\beta_i| + M \left|\left| \sum_{i=1}^n \beta_i x_i^* \right|\right|
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude

$$\left| \sum_{i=1}^n \beta_i \alpha_i \right| \leq M \left|\left| \sum_{i=1}^n \beta_i x_i^* \right|\right|$$

Since $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$ was arbitrary, 2.8.1 iii holds. $\square$

2.8.1 iii $\implies$ *2.8.1 i*. First observe that if $1 \leq j \leq n$ and $x_j^* = 0$, then by defining $\{\beta_i\}_{i=1}^n$ by $\beta_i = \delta_{ij}$, and by applying 2.8.1 iii, we have

$$\begin{aligned}
|\alpha_j| &= \left| \sum_{i=1}^n \beta_i \alpha_i \right| \\
&\leq M \left|\left| \sum_{i=1}^n \beta_i x_i^* \right|\right| \\
&= M \left|\left| x_j^* \right|\right| \\
&= 0
\end{aligned}$$

Hence, if every $x_j^* = 0$, then every $\alpha_j = 0$, so $x_\epsilon = 0$ satisfies 2.8.1 i for every $\epsilon > 0$.

Suppose at least some of the $\{x_i^*\}_{i=1}^n$ are nonzero. Then we can, by reordering, assume that for some integer m, $1 \leq m \leq n$, we have $\{x_i^*\}_{i=1}^m$ is **Linearly Independent**, and

$$span\left(\{x_i^*\}_{i=1}^m\right) = span\left(\{x_i^*\}_{i=1}^n\right) \tag{2.36}$$

Let $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{C}^K$. Define $S : X \to \mathbb{C}^m$ by

$$S(x) = \left(\langle x, x_1^* \rangle, \langle x, x_2^* \rangle, \cdots, \langle x, x_m^* \rangle \right)$$

By 2.4.49, S is **Surjective**. Hence there exists $\tilde{x} \in S^{-1}(\alpha_1, \cdots, \alpha_m)$. Then, for $1 \leq i \leq m$, $\langle \tilde{x}, x_i^* \rangle = \alpha_i$. Let $m < j \leq n$, $j \in \mathbb{Z}$. Then, by 2.36, There exists $\{\gamma_i\}_{i=1}^m \subset \mathbb{C}^m$ such that

$$x_j^* = \sum_{i=1}^m \gamma_i x_i^*$$

Now define $\{\beta_i\}_{i=1}^n$ by

$$\beta_i = \begin{cases} \gamma_i & 1 \leq i \leq m \\ 0 & m < i \wedge i \neq j \\ -1 & m < i \wedge i = j \end{cases} \tag{2.37}$$

Hence, by applying 2.8.1 iii, we have

$$\begin{aligned} \left| \langle \tilde{x}, x_j^* \rangle - \alpha_j \right| &= \left| \left\langle \tilde{x}, \sum_{i=1}^m \beta_i x_i^* \right\rangle - \alpha_j \right| \\ &= \left| \left(\sum_{i=1}^m \beta_i \langle \tilde{x}, x_i^* \rangle \right) + (-1)\, \alpha_j \right| \\ &= \left| \left(\sum_{i=1}^m \beta_i \alpha_i \right) + \beta_j \alpha_j \right| \\ &= \left| \sum_{i=1}^n \beta_i \alpha_i \right| \\ &\leq M \left|\left| \sum_{i=1}^n \beta_i x_i^* \right|\right| \\ &= M \left|\left| \sum_{i=1}^m \gamma_i x_i^* + (-1) x_j^* \right|\right| \\ &= 0 \end{aligned}$$

so that

$$\langle \tilde{x}, x_i \rangle = \alpha_i \qquad (1 \leq i \leq n) \tag{2.38}$$

Define $K = \bigcap_{i=1}^m Kern(x_i^*)$. Then $S^{-1}(\alpha_1, \cdots, \alpha_m) = \tilde{x} + K$. Define $T : span(\tilde{x}) + K \to \mathbb{R}$ by $Tx = d(x, K)$. Then $T \in (span(\tilde{x}) + K)^*$, $||T|| \leq 1$, and $K \subset Kernel(T)$, so we can apply 2.7.28 i to claim the existence of $x^* \in X^*$ satisfying $||x^*|| = 1$, $\langle \tilde{x}, x^* \rangle = d(\tilde{x}, K)$, and $K \subset ker(x^*)$. By definition of K, we can apply 2.4.50 to claim $x^* \in span(x_1^*, \cdots, x_m^*) \subset span(x_1^*, \cdots, x_n^*)$. Hence there is a representation $x^* = \sum_{i=1}^n \mu_i x_i^*$. Hence, using this representation and by applying 2.8.1 iii to the sequence $\{\mu_i\}_{i=1}^n \subset \mathbb{C}$,

$$\begin{aligned} d(\tilde{x}, K) &= \langle \tilde{x}, x^* \rangle \\ &= \sum_{i=1}^n \mu_i \langle \tilde{x}, x_i^* \rangle \\ &= \sum_{i=1}^n \mu_i \alpha_i \\ &\leq M \left|\left| \sum_{i=1}^n \mu_i x_i^* \right|\right| \\ &= M\, ||x^*|| \\ &= M \end{aligned}$$

Let $\epsilon > 0$. Then we can find $x_0 \in K$ such that $||\tilde{x} - x_0|| < M + \epsilon$, and since $K \subset Kern(x_i^*)$ for $1 \leq i \leq n$ $\langle \tilde{x} - x_0, x_i^* \rangle = \alpha_i$ for $1 \leq i \leq n$. Thus $x_\epsilon = \tilde{x} - x_0$ satisfies 2.8.1 i for this ϵ. Since $\epsilon > 0$ was arbitrary, the proof is complete. □

Corollary 2.8.2. Let X be a **Seminormed Space**. Let, $x^{**} \in X^{**}$. Let $\{x_i^*\}_{i=1}^n \subset X^*$. Let and $\epsilon > 0$. The following are true

(i) There exists an $x_1 \in X$ such that $||x_1|| \leq ||x^{**}|| + \epsilon$ and for $1 \leq i \leq n$, $\langle x_1, x_i^* \rangle = \langle x_i^*, x^{**} \rangle$.

(ii) There exists an $x_2 \in X$ such that $||x_2|| \leq ||x^{**}||$ and for $1 \leq i \leq n$, $|\langle x_2, x_i^* \rangle - \langle x_i^*, x^{**} \rangle| < \epsilon$.

Proof of Both. For $1 \leq i \leq n$, define $\alpha_i = \langle x_i^*, x^{**} \rangle$. Let $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$. Then,

$$\begin{aligned} \left| \sum_{i=1}^n \beta_i \alpha_i \right| &= \left| \sum_{i=1}^n \beta_i \langle x_i^*, x^{**} \rangle \right| \\ &= \left| \left\langle \sum_{i=1}^n \beta_i x_i^*, x^{**} \right\rangle \right| \\ &\leq ||x^{**}|| \left|\left| \sum_{i=1}^n \beta_i x_i^* \right|\right| \end{aligned}$$

Hence 2.8.1 *iii* is satisfied with $\alpha_i = \langle x_i^*, x^{**} \rangle$ and $M = ||x^{**}||$. From an application of 2.8.1 *iii* $\implies$ 2.8.1 *ii*, we conclude the existence of $x_1 \in X$ with $\langle x_1, x_i^* \rangle = \alpha_i = \langle x_i^*, x^{**} \rangle$ and $||x_1|| \leq ||x^* * || + \epsilon$, so 2.8.2 i holds. Also, from 2.8.1 *iii* $\implies$ 2.8.1 *i*, we conclude the existence of an $x_2 \in X$ satisfying

$$\begin{aligned} |\langle x_2, x_i^* \rangle - \langle x_i^*, x^{**} \rangle| &= |\langle x_2, x_i^* \rangle - \alpha_i| \\ &< \epsilon \end{aligned}$$

and $||x_2|| \leq ||x^{**}||$. Hence 2.8.2 ii holds. □

2.8.2 Goldstine

Theorem 2.8.3 (Goldstine)**.** Let X be a **Seminormed Space**. Let $c : X \to X^{**}$ be the **Canonical Embedding** of X into X^{**}. Let B denote the **Closed Unit Ball** about 0 in X. Let B_1 denote the **Closed Unit Ball** about 0 in X^{**}. Then $c(B)$ is $\mathfrak{weak}^*$ **Dense** in B_1.

Proof. Let $c^* : X^* \to X^{***}$ denote the **Canonical Embedding** of X^* into X^{***}. Let $\mathcal{K}$ denote the collection of **Finite** subsets of X^*. For each $x^{**} \in X^{**}$, $\epsilon > 0$ and $K \in \mathcal{K}$. define

$$\begin{aligned} U(x^{**}, \epsilon, K) &:= \bigcap_{x^* \in K} \{y \in X^{**} : |\langle x^{**} - y, c^*(x^*)\rangle| < \epsilon\} \\ &= \bigcap_{x^* \in K} \{y \in X^{**} : |\langle x^*, x^{**} - y\rangle| < \epsilon\} \end{aligned}$$

Define

$$\mathcal{B} = \{U(x^{**}, \epsilon, K) : K \in \mathcal{K} \quad x^{**} \in X^{**} \quad \epsilon \in (0, \infty)\}$$

Then by definition, the $\mathcal{B}$ is a **Basis** for the $\mathfrak{weak}^*$ **Topology** on X^{**}.

Let $x^{**} \in B_1$, $\epsilon > 0$, and $\{x_i\}_{i=1}^n = K \in \mathcal{K}$. Then by 2.8.2 ii there exists an $x \in B$ satisfying , for each $1 \le i \le n$,

$$\begin{aligned} |\langle x_i^*, x^{**} - c(x)\rangle| &= |\langle x_i^*, x^{**}\rangle - \langle x, x_i^*\rangle| \\ &< \epsilon \end{aligned}$$

Hence

$$c(x) \in \bigcap_{x_i^* \in K} \{y \in X^{**} : |\langle x_i^*, x^{**} - y\rangle| < \epsilon\} = U(x^{**}, \epsilon, K)$$

Hence $c(B)$ is $\mathfrak{weak}^*$ **Dense** in B_1. □

2.8.3 Banach Alaoglu

The following well known result concerning the $\mathfrak{weak}^*$ compactness of the unit ball of a **Banach Space** was first proven in the separable case by Banach, and then generalized in 1940 by Alaoglu [1] to **Banach Spaces**. Generalizations of this result in a general TVS satisfying sufficient conditions have also been shown but the form presented here comes from [2], who drops the assumption of completeness for one direction of the implication.

Theorem 2.8.4 (Banach-Alaoglu-Morales)**.** Let X be a **Normed Space**. Let B denote the **Closed Unit Ball** about 0 in X^*. Then B is $\mathfrak{weak}^*$ **Compact**.

Proof. Let $\mathbb{F}$ denote the underlying **Field** of X. For each $x \in X$, define

$$D_x = \{y \in \mathbb{F} |\, |y| \le ||x||\}$$

Then, for each $x \in X$, D_x is **Hausdorff** and **Compact**. □

Proof. Let $\mathbb{F}$ denote $X's$ field, and for $x \in X$, define $D_x = \{y \in \mathbb{F} : |y| \leq ||x||\}$. Then each D_x is Hausdorff and compact so by Tychonoff's theorem, $D := \prod_{x \in X} D_x$ is compact and Hausdorff when endowed with the product topology. If $T \in D$, then $T : X \to \mathbb{F}$ and $|Tx| \leq ||x||$ for each $x \in X$, so $D \cap X^* \subset B$. It is also clear that $B \subset D$, so $D \cap X^* = B$. Let $\{\gamma_\alpha\}_{\alpha \in A}$ be a net in B converging to $\gamma \in D$ in D's product topology. Then, letting π_x denote the x^{th} projection, for each $x \in X$,

$$\gamma_\alpha(x) = \pi_x(\gamma_\alpha) \to \pi_x(\gamma) = \gamma(x) \tag{2.39}$$

If $\alpha \in \mathbb{F}$ and $x, y \in X$, then

$$\langle \alpha x + y, \gamma_\alpha \rangle \to \langle \alpha x + y, \gamma \rangle \tag{2.40}$$

and also

$$\langle \alpha x + y, \gamma_\alpha \rangle = \alpha \langle x, \gamma_\alpha \rangle + \langle y, \gamma_\alpha \rangle \to \alpha \langle x, \gamma \rangle + \langle y, \gamma \rangle \tag{2.41}$$

which implies γ is linear since D is Hausdorff, and hence $\gamma \in B$. Thus B is closed in D. What remains to be shown is that the *weak*$*$ topology on B is the subspace topology on B induced by $D's$ topology, since a Closed subset of a compact Hausdorff space is compact. For notation, denote with $\mathcal{T}_D$ the subspace topology on B induced by D's topology, and denote with $\mathcal{T}_w$ the subspace topology on B induced by the *weak** topology on X^*. To see that $\mathcal{T}_w \subset \mathcal{T}_D$, let $\{\gamma_\alpha\}_{\alpha \in A} \subset B$ such that $\gamma_\alpha \overset{\mathcal{T}_D}{\to} \gamma$. For each $x \in X$, letting c be the canonical embedding,

$$\langle \gamma_\alpha, c(x) \rangle = \langle x, \gamma_\alpha \rangle = \pi_x(\gamma_\alpha) \to \pi_x(\gamma) = \langle x, \gamma \rangle = \langle \gamma, c(x) \rangle \tag{2.42}$$

Hence $\gamma_\alpha \overset{\mathcal{T}_w}{\to} \gamma$, so $\mathcal{T}_w \subset \mathcal{T}_D$. To see that $\mathcal{T}_D \subset \mathcal{T}_w$, fix $x \in X$ and let $\{\gamma_\alpha\}_{\alpha \in A} \subset B$ such that $\gamma_\alpha \overset{\mathcal{T}_w}{\to} \gamma$. Then $\pi_x(\gamma_\alpha) = \langle x, \gamma_\alpha \rangle \to \gamma(x) = \pi_x(\gamma)$, so by definition of the product topology $\gamma_\alpha \overset{\mathcal{T}_D}{\to} \gamma$, implying $\mathcal{T}_D \subset \mathcal{T}_w$. Hence B is *weak** is compact. □

Corollary 2.8.5 (Banach Alaoglu Seminorm)**.** *Let X be a seminormed space and define* $B = \{x^* \in X^* : ||x^*|| \leq 1\}$. *Then B is weak** *compact.*

Proof. This is a consequence of the fact that the *weak** topology on X^* is identical to the *weak** topology on the dual space of $X/ \left|\left|\cdot\right|\right|^{-1} \{0\}$. □

This gives us the useable result

Corollary 2.8.6 (Banach-Alaoglu-Morales)**.** *Let X be a seminormed space and* $C \subset X^*$

1. *If X is complete and C weak** *compact, then C is weak** *closed and bounded.*

2. *If C is weak** *closed and bounded, then C is weak** *compact.*

Proof. (1) Since C is *weak** compact, it is *weak** closed since the *weak** topology is Hausdorff. Since $c(x) : (X, \mathcal{T}_{w^*}) \to \mathbb{C}$ is continuous for each $x \in X$, for each $x \in X$, $c(x)(C) =$ is compact and therefore bounded. Hence, for every $x \in X$, $\{|\langle x, c \rangle| : c \in C\}$ is bounded, so by the Banach Steinhaus, C is bounded. □

Proof. (2) Since C is bounded, it is contained in some closed ball B which we know to be *weak** compact by 2.8.4. Since the *weak** topology on B is compact and Hausdorff and C is closed in this topology, it is compact in this topology. Since the subspace topology on C induced by the *weak** topology on X^* equals this topology, we are done. □

2.8.4 Eberlein-Smulian

The purpose of this section is to provide a characterization of weakly compact subsets of a complete seminormed space X, which will serve to increase the applicability of the results regarding weakly compactly generated spaces covered later in this document. The first main result of this section, 2.8.9, serves to show that even though weak topologies of Banach Spaces are not in general metrizable, an equivalence between weak compactness and sequential compactness exists. From this result ($1 \implies 2$) was first presented in the case of normed spaces in [3], and then ($2 \implies 1$) was proven in the case of normed spaces in [4]. Several different proofs have been given in the years since, and the one present here is based on that present in [5], which is also followed in [6]. We begin with a few lemmas.

Lemma 2.8.7 (Metrizable Weak)**.** *If X is a seminormed space and X^* contains a countable set that separates points mod $K := ||\cdot||^{-1}\{0\}$, then subspace topology induced by the weak topology on any weakly compact subset A of X is pseudometrizable.*

Proof. As a consequence of 2.5.21 and 2.7.37, it is sufficient to let X be a normed space and $\{x_i^*\}_{i\in\mathbb{N}}$ separate points in X. Let $M = 2\sup_{x\in A}||x||$, and define d to be the metric on A defined by, for $x, y \in A$,

$$d(x,y) = \sum_{k\in\mathbb{N}} \frac{|\langle x-y, x_k^*\rangle|}{||x_k^*||\,2^k} \tag{2.43}$$

Let $x \in A$, $\epsilon > 0$ be arbitrary, and define

$$n = \left\lceil 2 + log_2\left(\frac{M}{\epsilon}\right)\right\rceil \qquad U = A \cap \bigcap_{k=1}^{n} \left\{ y \in X : |\langle x-y, x_k^*\rangle| < \frac{||x_k^*||\,2^{k-1}\epsilon}{n} \right\} \tag{2.44}$$

The U is open in the subspace topology on A induced by $X's$ weak topology. Furthermore, if $y \in U$, then

$$\begin{aligned} d(x,y) &= \sum_{k\in\mathbb{N}} \frac{|\langle x-y, x_k^*\rangle|}{||x_k^*||\,2^k} \\ &\leq \sum_{k=1}^{n} \frac{|\langle x-y, x_k^*\rangle|}{||x_k^*||\,2^k} + \sum_{k=n+1}^{\infty} \frac{2M}{2^k} \\ &< \sum_{k=1}^{n} \frac{\epsilon}{2n} + \frac{M}{2^{n-1}} < \epsilon \end{aligned} \tag{2.45}$$

So that $U \subset B_d(x;\epsilon)$. This implies $Id : (A, \mathcal{T}_w) \to (A, \mathcal{T}_d)$ is continuous. Since a continuous injection from a compact space into a Hausdorff space is a homeomorphism, the subspace topology on A induced by the weak topology equals the topology on A induced by d, and so A's weak topology is metrizable. □

Lemma 2.8.8. *Let X be a seminormed space and $Y \subset X^{**}$ be a finite dimensional vector subspace. Then there exists a finite set $Z \subset \partial B_{X^*}(0;1)$ such that for each $y^{**} \in Y$,*

$$||y^{**}|| \leq 2\max_{z^*\in Z} |\langle z^*, y^{**}\rangle| \tag{2.46}$$

Proof. Let $S = \partial B_{X^{**}}(0;1) \cap Y$. Then, since Y is finite dimensional, S is compact, and therefore permits a $\frac{1}{4} - net$ $\{s_i\}_{i=1}^{n}$. Now let $\{z_k^*\}_{k=1}^{n} \subset \partial B_{X^*}(0;1)$ such that for each k, $\langle z_k^*, s_i \rangle > \frac{3}{4}$. Let $s \in S$ then there is a k such that $||s - s_k|| < \frac{1}{4}$. for this k, we have

$$\langle z_k^*, s \rangle = \langle z_k^*, s_k \rangle + \langle z_k^*, s - s_k \rangle \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2} \tag{2.47}$$

□

Theorem 2.8.9 (Eberlein-Smulian)**.** *Let X be a seminormed space and $A \subset X$. Then the following are equivalent.*

1. *A is weakly compact.*

2. *A is weakly sequentially compact.*

Proof. (1 $\implies$ 2) Let $A \subset X$ be weakly compact, and let $\{x_i\}_{i\in\mathbb{N}} \subset A$. Define $S = \overline{span\{x_i : i \in \mathbb{N}\}}$. Since S is closed and convex, it is weakly closed, and so $A \cap S$ is weakly compact as well. By construction, S is separable, and so contains a countable dense set $\{y_i\}_{i\in\mathbb{N}}$. By Hahn-Banach, for each $i \in \mathbb{N}$, there exists $y_i^* \in S^*$ such that $\langle y_i, y_i^* \rangle = 1$, and continuity of each y_i^* implies $\{y_i^*\}_{i\in\mathbb{N}}$ separates points in S mod $||\cdot||^{-1}\{0\}$. Hence we can apply 2.8.7 to claim that the subspace topology on $A \cap S$ induced by $S's$ weak topology is metrizable, and therefore $\{x_i\}_{i\in\mathbb{N}}$ has a sub-sequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ which is weakly S-convergent, and therefore weakly X-convergent since subspace topologies are no less fine than the topologies that induce them. Since $A \subset X$ is weakly closed, this sequence converges within A, and so A is weakly sequentially compact. □

Proof. (2 $\implies$ 1). Let $A \subset X$ be weakly sequentially compact, let c denote the canonical embedding of X into X^{**}, and let x^{**} in the *weak** closure of $c(A)$. Let $x_1^1 \in X^*$ have norm 1. By assumption, there exists $a_1^{**} \in c(A)$ such that $|\langle x_1^*, x^{**} - a_1^{**} \rangle| < 1$. By 2.8.8, there exists $\{x_1^2, \cdots, x_{n_2}^2\} \subset \partial B_{X^*}(0;1)$ such that for each $y^{**} \in span\{x^{**}, x^{**} - a_1^{**}\}$,

$$||y^{**}|| \leq 2 \max_{1\leq k\leq n_2} \left|\left\langle x_k^2, y^{**} \right\rangle\right| \tag{2.48}$$

Also, since x^{**} is in the *weak** closure of $c(A)$, there exists $a_2^{**} \in c(A) \cap U_2$ where

$$U_2 = \{y^{**} \in X^{**} : (\forall 1 \leq j \leq 2)(\forall 1 \leq k \leq n_j)(\left|\left\langle x_k^j, x^{**} - y^{**} \right\rangle\right| < \frac{1}{2})\} \tag{2.49}$$

Continuing inductively, , we construct a sequence $\{a_n^{**}\}_{n\in\mathbb{N}} \subset c(A)$ such that for each $j \in \mathbb{N}$, $\{x_k^j\}_{k=1}^{n_j} \subset \partial B_{X^*}(0;1)$ such that for every $y^{**} \in span\left\{x^{**}, x^{**} - a_1^{**}, \cdots, x^{**} - a_{j-1}^{**}\right\}$, we have

$$||y^{**}|| \leq 2 \max_{1\leq k\leq n_j} \left|\left\langle x_k^j, x^{**} - y^{**} \right\rangle\right| \tag{2.50}$$

and $a_j^{**} \in c(A) \cap U_j$ where U_j is the $\{x_k^m\}_{1\leq m\leq j, 1\leq k\leq n_m}$ *weak** neighborhood about x^{**} of radius $\frac{1}{j}$. For each $k \in \mathbb{N}$, let $a_k = c^{-1}(c(a_k))$. Since A is sequentially weakly compact, $\{a_k\}_{k\in\mathbb{N}}$ has a weak cluster point $x \in A$. Also, $x \in \overline{span\{a_i\}_{i\in\mathbb{N}}}$ because this is a weakly closed set, implying $c(x) \in \overline{span\{a_i^{**}\}_{i\in\mathbb{N}}}$, which then implies $c(x) \in \overline{span\{x^{**}, x^{**} - a_1^{**}, x^{**} - a_2^{**}, \cdots\}}$.

By continuity of the norm and each element of $\{x_i^k\}_{k\in\mathbb{N},1\leq i\leq n_k}$, we conclude that for each element y^{**} of $\overline{\{x^{**}, x^{**}-a_1^{**}, x^{**}-a_2^{**}, \cdots\}}$,

$$||y^{**}|| \leq 2 \sup_{k\in\mathbb{N},1\leq i\leq n_k} \left|\left\langle x_i^k, y^{**}\right\rangle\right| \tag{2.51}$$

This is useful, because for each $k \in \mathbb{N}$, $1 \leq i \leq n_k$, we have, for large enough j,

$$\begin{aligned} \left|\left\langle x_i^k, x^{**} - c(x)\right\rangle\right| &\leq \left|\left\langle x_i^k, x^{**} - a_j^{**}\right\rangle\right| + \left|\left\langle a_j^{**} - c(x), x_i^k\right\rangle\right| \\ &\leq \frac{1}{j} + \left|\left\langle x_i^k, a_j - x\right\rangle\right| \end{aligned} \tag{2.52}$$

which can be made arbitrarily small, and so $\left|\left\langle x_i^k, x^{**} - c(x)\right\rangle\right| = 0$, implying that

$$||x^{**} - c(x)|| \leq 2 \sup_{k\in\mathbb{N},1\leq i\leq n_k} \left|\left\langle x_i^k, x^{**} - x\right\rangle\right| = 0 \tag{2.53}$$

So $x^{**} = c(x)$, and therefore $c(A)$ is *weak** closed. Since A is weakly-sequentially compact, $c(A)$ is *weak** sequentially compact and therefore bounded by Banach Steinhaus. By 2.8.4, bounded *weak** closed sets are compact, and so $c(A)$ is *weak** compact. Since the weak topology on $A/\left|\left|\cdot\right|\right|^{-1}\{0\}$ is the same as the *weak** topology on $c(A)$, $A/\left|\left|\cdot\right|\right|^{-1}\{0\}$ is weakly compact. To see that A is weakly compact, apply **??**. □

2.8.5 Bishop-Phelps

In this subsection, I develop a result due to [7] which will prove useful throughout this document. I begin by presenting the concept of a convex cone and a trio of lemmas which are commonly utilized in the proof of this result.

Definition 2.8.10 (Convex Cone)**.** Let X be a seminormed space over $\mathbb{R}$. If $K \subset X$ is convex and closed under positive scalar multiples, then we call it a **convex cone**. If J is a convex cone in X, $C \subset X$, $x_0 \in C$, and $(J + x_0) \cap C = \{x_0\}$, then we say that J **supports** C at x_0. If $x^* \in \partial B_{X^*}(0; 1)$ and $\alpha > 0$ then we define

$$K(x^*, \alpha) := \{x \in X : ||x|| \leq \alpha \langle x, x^*\rangle\} \tag{2.54}$$

Remark 2.8.11. Let X be a seminormed space, $x^* \in \partial B_{X^*}(0; 1)$, and $\alpha > 0$. The following are true.

1. $K(x^*, \alpha)$ is a closed convex cone.

2. If $\alpha > 1$, $Int(K(x^*, \alpha)) \neq \emptyset$.

Proof. (1) If $\{x_n\} \subset K(x^*, \alpha)$ converges, say $x_n \to x$, then continuity of x^* implies $\langle x_n, x^*\rangle \to \langle x, x^*\rangle$. Hence, given $\epsilon > 0$, there exists $N > 0$ such that for $n > N$ we have $max\left(\left|||x|| - ||x_n||\right|, |\langle x - x_n, x^*\right.$ ϵ, so that for all $n > N$,

$$||x|| \leq ||x_n|| + \epsilon < \alpha \langle x_n, x^*\rangle + \epsilon < \alpha \langle x, x^*\rangle + (\alpha + 1)\epsilon \tag{2.55}$$

So $x \in K(x^*, \alpha)$ closedness is verified. It is obvious that $K(x^*, \alpha)$ is closed under positive scalar multiples, and for convexity, if $x, y \in K(x^*, \alpha)$ and $t \in [0,1]$, then

$$||tx + (1-t)y|| \leq t\,||x|| + (1-t)\,||y|| \leq t\alpha \langle x, x^* \rangle + (1-t)\alpha \langle y, x^* \rangle = \alpha \langle tx + (1-t)y, x^* \rangle \tag{2.56}$$

□

Proof. (2) By definition of the norm on X^*, there is an $x \in \overline{B_X(0;1)}$ such that $2/\left(\alpha\left(1+\frac{1}{\alpha}\right)\right) < \langle x, x^* \rangle$, implying by linearity that $1/\alpha < \left\langle \frac{1+(1/\alpha)}{2} x, x^* \right\rangle$. By continuity of x^* we find a neighborhood U of $\left(1+\frac{1}{\alpha}\right)/2$ contained in $B_X(0;1)$ such that for each $y \in U$, $1/\alpha < \langle y, x^* \rangle$. This implies $||y|| \leq 1 < \alpha \langle y, x^* \rangle$, so $U \subset K(x^*, \alpha)$. □

Lemma 2.8.12 (Bishop-Phelps Lemma)**.** *Let X be a complete seminormed space, $x^*, y^* \in \partial B_{X^*}(0;1)$, $C \subset X$ closed and convex, $1 > \epsilon > 0$, and $k > 1 + \frac{2}{\epsilon}$. The following are true.*

1. *If x^* is bounded on C, then for each $z \in C$, there is an $x_0 \in X$ such that $K(x^*, \epsilon)$ supports C at x_0 and $x_0 \in K(x^*, \epsilon) + z$.*
2. *If $|\langle x, y^* \rangle| \leq \frac{\epsilon}{2}$ for each $x \in Kern(x^*) \cap \overline{B_X(0;1)}$, then*

$$min\left(||x^* + y^*||\,, ||x^* - y^*||\right) \leq \epsilon \tag{2.57}$$

3. *If y^* is nonnegative on $K(x^*, k)$, then $||x^* - y^*|| \leq \epsilon$.*

Proof. (1) Let x^* be bounded on C and define, for $x, y \in X$, $y \lesssim x$ if and only if $x - y \in K(x^*, \epsilon)$. Fix $z \in C$. Define $Z = C \cap (K(x^*, \epsilon) + z)$. Since C and $K(x^*, \epsilon)$ are closed, so is Z. Let $\mathcal{C} = \{x_\alpha\}_{\alpha \in A}$ be a chain in where $(A, \leq)$ is a totally ordered set and $x_\alpha \lesssim x_\beta \iff \alpha \leq \beta$. If $x_\alpha, x_\beta \in \mathcal{C}$, where $x_\beta \lesssim x_\alpha$, then $x_\alpha - x_\beta \in K(x^*, \epsilon)$, so $0 \leq ||x_\alpha - x_\beta|| \leq \epsilon \langle x_\alpha - x_\beta, x^* \rangle$, implying $\langle x_\beta, x^* \rangle \leq \langle x_\alpha, x^* \rangle$. Thus we conclude $\{\langle x_\alpha, x^* \rangle\}_{\alpha \in A}$ is a monotone bounded net in $\mathbb{R}$ that is therefore Cauchy, which by the following inequality

$$||x_\beta - x_\alpha|| \leq \epsilon \langle x_\alpha - x_\beta, x^* \rangle = \epsilon\left(\langle x_\alpha, x^* \rangle - \langle x_\beta, x^* \rangle\right) \to 0 \tag{2.58}$$

implies $\mathcal{C}$ is a Cauchy net and therefore converges, say $x_\alpha \to y_0 \in Z$. Continuity of the norm and x^* imply together that y_0 is an upper bound for $\mathcal{C}$. Since $\mathcal{C}$ was an arbitrary chain in Z, Z has a maximal element x_0. By definition, $x_0 \in Z := K(x^*, \epsilon) + z$. Since $x_0 \in Z \subset C$, $x_0 \in C$. Further, since $0 \in K(x^*, \epsilon)$, $x_0 \in K(x^*, \epsilon) \cap C$. Let $y \in (K(x^*, \epsilon) + x_0) \cap C$. Then $y - x_0 \in K(x^*, \epsilon)$ so that $z \lesssim x_0 \lesssim y$, meaning $y \in Z$ and therefore $y = x_0$ since x_0 is maximal. Hence $(K(x^*, \epsilon) + x_0) \cap C = \{x_0\}$, so we are done. □

Proof. (2) By assumption, $\left|\left|y^*|_{Kern(x^*)}\right|\right| \leq \frac{\epsilon}{2}$, so by the Hahn-Banach theorem, we can find a $\tilde{y^*} \in X^*$ extending $y^*|_{Kern(x^*)}$ such that $\left|\left|\tilde{y^*}\right|\right| \leq \frac{\epsilon}{2}$. Since $y^* - \tilde{y^*} \neq 0$, $codim\left(kern(x^*)\right) = 1$, and $kern\left(x^*\right) \subset kern\left(y^* - \tilde{y^*}\right)$,we conclude $kern\left(y^* - \tilde{y^*}\right) = kern(x^*)$. Hence, for some $\alpha \in \mathbb{R}$, $y^* - \tilde{y^*} = \alpha x^*$. For this alpha, we have

$$|1 - |\alpha|| = \left|||y^*|| - \left|\left|\tilde{y^*} - y^*\right|\right|\right| \leq \left|\left|\tilde{y^*}\right|\right| \leq \frac{\epsilon}{2} \tag{2.59}$$

If $\alpha \geq 0$,

$$||x^* - y^*|| = \left|\left|x^* - \left(\alpha x^* + \tilde{y^*}\right)\right|\right| = \left|\left|(1-\alpha)\, x^* - \tilde{y^*}\right|\right| \leq |1-\alpha| + \left|\left|\tilde{y^*}\right|\right| \leq \epsilon \tag{2.60}$$

If $\alpha \leq 0$, then

$$||x^* + y^*|| = \left|\left|x^* + \left(\alpha x^* + \tilde{y^*}\right)\right|\right| = \left|\left|(1+\alpha)x^* + \tilde{y^*}\right|\right| \leq |1+\alpha| + \left|\left|\tilde{y^*}\right|\right| \leq \epsilon \tag{2.61}$$

□

Proof. (3) Since $||x^*|| = 1$, there exists $x \in \partial B_X(0;1)$ such that $\langle x, x^* \rangle > \frac{1}{k}\left(1 + \frac{2}{\epsilon}\right)$. If $y \in Kern(x^*) \cap \overline{B_X(0;1)}$, then

$$\left|\left|x \pm \frac{2}{\epsilon} y\right|\right| \leq 1 + \frac{2}{\epsilon} < k \langle x, x^* \rangle = k \left\langle x \pm \frac{2}{\epsilon} y, x^* \right\rangle \tag{2.62}$$

so $x \pm \frac{2}{\epsilon} y \in K(x^*, k)$, so by assumption $\left\langle x \pm \frac{2}{\epsilon} y, y^* \right\rangle \geq 0$. Since this occurs for both positive and negative, $|\langle y, y^* \rangle| = \frac{\epsilon}{2}\left|\left\langle \frac{2}{\epsilon} y, y^* \right\rangle\right| \leq \frac{\epsilon}{2} \langle y^*, x \rangle \leq \frac{\epsilon}{2}||x|| = \frac{\epsilon}{2}$. Hence by part 2, either $||x^* - y^*|| \leq \epsilon$, or $||x^* + y^*|| \leq \epsilon$. Since $||x^*|| = 1$, there exists $x \in \partial B_X(0;1)$ such that $\frac{||x||}{k} \leq max\left(\epsilon, \frac{1}{k}\right) < \langle x, x^* \rangle$, so that $x \in K(x^*, k)$, implying $\langle x, y^* \rangle \geq 0$, and therefore $\epsilon < \langle x_0, x^* + y^* \rangle \leq ||x^* + y^*||$. Hence we conclude $||x^* - y^*|| \leq \epsilon$.

□

Theorem 2.8.13 (Bishop-Phelps Theorem)**.** *Let X be a complete seminormed space, $C \subset X$ be closed, bounded, and convex, and define $M := \{f \in X^* | (\exists x_0 \in C)(\langle x_0, f \rangle = \sup_{x \in C} \langle x, f \rangle)\}$. Then $\overline{M} = X^*$*

Proof. Since M is a vector subspace independent of translations of C, we assume without loss of generality that $0 \in C$ and that it is sufficient to show that M is dense in $\partial B_{X^*}(0;1)$. Let $x^* \in \partial B_{X^*}(0;1)$. Let $\epsilon \in (0,1)$ and let $1 + \frac{2}{\epsilon} < k$. by 2.8.11, $K(x^*, k)$ is a closed convex cone with nonempty interior. Applying 2.8.12, part one, there is $x_0 \in C$ with $x_0 \in K(x^*, k)$ and $(K(x^*, k) + x_0) \cap C = \{x_0\}$. By Hahn Banach, there exists $y^* \in \partial B_{X^*}(0;1)$ satisfying

$$\sup_{x \in C} \langle x, y^* \rangle = \langle x_0, y^* \rangle = \inf_{x \in K(x^*, k) + x_0} \langle x, y^* \rangle = \inf_{\tilde{x} \in K(x^*, k)} \langle \tilde{x}, y^* \rangle + \langle x_0, y^* \rangle \tag{2.63}$$

Hence y^* is positive on $K(x^*, k)$, so by 2.8.12 part 3, $||x^* - y^*|| < \epsilon$, so we are done since $y^* \in M$ and x^* was arbitrary. □

2.9 Reflexivity

Recall that a seminormed space X is said to be **reflexive** if $c(X) = X^{**}$. Since X^{**} is always complete and c an isometry, any reflexive space is always complete. Due to the Banach-Alaoglu theorem, in reflexive space X, the closed unit ball of X is weakly compact. For this reason and others, reflexivity is a condition of interest to many mathematicians. We begin with a basic result.

Lemma 2.9.1 (Reflexive Separable)**.** *Let X be a complete seminormed space. Then the following are equivalent.*

1. *X is reflexive.*
2. *The closed unit ball of X is weakly compact.*
3. *Each closed separable subspace of X is reflexive.*
4. *All collections of closed, bounded, convex sets in X have the finite intersection property*
5. *The closed unit ball of X is weakly sequentially compact*

Proof. ($1 \implies 2$) Let X be reflexive. By 2.8.9 it is sufficient to show that any sequence $\{x_i\}_{i\in\mathbb{N}} \subset \overline{B_X(0;1)}$ has a weak cluster point $x \in \overline{B_X(0;1)}$. Since $\overline{B_{X^{**}}(0;1)}$ is $weak^*$ compact, $\{c(x_i)\}_{i\in\mathbb{N}}$ has a subsequence $\{c(x_{n_i})\}_{i\in\mathbb{N}}$ such that $c(x_{n_i}) \xrightarrow{w^*} \tilde{x} \in \overline{B_{X^{**}}(0;1)}$. Since X is reflexive, for some $x \in \overline{B_X(0;1)}$, $c(x) = \tilde{x}$. Let $x^* \in X^*$. Then,

$$|\langle x_{n_i} - x, x^* \rangle| = |\langle x^*, c(x_{n_i}) - \tilde{x} \rangle| \to 0 \tag{2.64}$$

as $i \to \infty$, and so $x_{n_i} \xrightarrow{w} x$, completing the proof. □

Proof. ($2 \implies 3$) Suppose the closed unit ball of X is weakly compact, and let $x^{**} \in \overline{B_{X^{**}}(0;1)}$. By **??**, there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ such that $c(x_n) \xrightarrow{w^*} x^{**}$. By assumption, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $x_{n_k} \xrightarrow{w} x \in \overline{B_X(0;1)}$. This implies $c(x_{n_k}) \xrightarrow{w^*} c(x)$, and so since the $weak^*$ topology is Hausdorff, $x^{**} = c(x) \in c\left(\overline{B_X(0;1)}\right)$. □

Proof. ($2 \implies 3$). Let X be reflexive and $S \subset X$ be a closed separable vector subspace. Let $\{x_i\}_{i\in\mathbb{N}} \subset \overline{B_X(0;1)}$. By assumption this sequence has an $X-weakly$ convergent subsequence, $x_{n_k} \to x \in X$. Since S is weakly closed, $x \in S$, and since the weak topology on X is finer than that on S, $x_{n_k} \xrightarrow{S-w} x$. By 2.8.9 and ($2 \implies 1$, S is reflexive. □

Proof. ($3 \implies 2$) Let $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$. Then since $S := \overline{span\{x_i\}_{i\in\mathbb{N}}}$ is a closed separable subspace, $\{x_n\}_{n\in\mathbb{N}}$ has an S-weakly convergent subsequence, $x_{n_k} \xrightarrow{S-w} x \in S$. If $x^* \in X^*$, then $x^*|_S \in S^*$, so that $|\langle x - x_{n_k}, x^* \rangle| \to 0$, implying $x_{n_k} \xrightarrow{w} x$, an application of 2.8.9 finishes the proof. □

Proof. ($4 \iff 2$) Obvious since closed convex sets are weakly closed. □

Proof. ($5 \iff 2$) Apply 2.8.9. □

2.9.1 James

As an easy application of 2.8.4 and 2.8.9, for a reflexive space X, all $x^* \in X^*$ attain their norm. The converse of this fact was, for a time, an open question of considerable interest. The converse of this result was, as is traditional in mathematics, proven in a piecemeal manner. The result was first tackled by James in [8] under the added assumption that every space Y isomorphic to X has the property that each $y^* \in Y^*$ attains its norm and that X permits a Shauder basis. This result was rapidly improved by Klee in [9] who dropped the assumption of a Schauder basis, and then by James again in [10] who proved the result in the case of a separable space X. The question of the converse in a Banach space was finally answered to the affirmative in [11], building on the arguments in [10]. This paper included generalizations all the way to quasi-complete locally convex TVS's.

Definition 2.9.2. *If X is a topological vector space and $\{x_i^*\}_{i\in\mathbb{N}} \subset X^*$, then $CoLim\{x_i^*\}_{i\in\mathbb{N}}$ is the set of all $x^* \in X^*$ such that for every $x \in X$,*

$$\liminf_{i\to\infty} \langle x, x_i^*\rangle \leq \langle x, x^*\rangle \leq \limsup_{i\to\infty} \langle x, x_i^*\rangle \tag{2.65}$$

Remark 2.9.3 (CoLim Nonempty)**.** *Let X be a complete seminormed space and let $\{x_i^*\}_{i\in\mathbb{N}} \subset X^*$ be bounded. Then $CoLim\{x_i^*\}_{i\in\mathbb{N}} \neq \emptyset$*

Proof. Since $\{x_i^*\}_{i\in\mathbb{N}}$ is bounded, it has a subsequence with a *weak** limit x^* who must live in $CoLim\{x_i^*\}_{i\in\mathbb{N}}$. □

Lemma 2.9.4. *Let X be a complete seminormed space, $\alpha \in (0,1)$, $\{x_i^*\}_{i\in\mathbb{N}} \subset X^*$, and $\{\beta_i\}_{i\in\mathbb{N}} \subset (0,1)$ such that $\sum_{i\in\mathbb{N}} \beta_i = 1$. Then if (1) or (3) hold below, there are $\{y_i^*\}_{i\in\mathbb{N}} \subset X^*$ and $\gamma \geq \alpha$ such that (2) or (4) hold respectively.*

1. *$\{x_i^*\}_{i\in\mathbb{N}} \subset \partial B_{X^*}(0;1)$ such that $d\left(0, \overline{co}\{x_i^*\}_{i\in\mathbb{N}}\right) \geq \alpha$.*

2. *$\gamma \leq 1$, and for each $i \in \mathbb{N}$,*

$$y_i^* \in \overline{co}\{x_j^*\}_{j\geq i} \qquad \left\|\sum_{j\in\mathbb{N}} \beta_j y_j^*\right\| = \gamma \qquad \left\|\sum_{j=1}^{i} \beta_j y_j^*\right\| < \gamma\left(1 - \alpha\left(\sum_{j=i+1}^{\infty} \beta_j\right)\right) \tag{2.66}$$

3. *$\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ such that $d\left(\overline{co}\{x_i^*\}_{i\in\mathbb{N}} - CoLim\{x_i^*\}_{i\in\mathbb{N}}, 0\right) \geq \alpha$.*

4. *$\gamma \leq 2$, $\{y_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$, and for each $i \in \mathbb{N}$, $y^* \in CoLim\{y_i^*\}_{i\in\mathbb{N}}$,*

$$\left\|\sum_{j\in\mathbb{N}} \beta_j \left(y_j^* - y^*\right)\right\| = \gamma \qquad \left\|\sum_{j=1}^{i} \beta_j \left(y_j^* - y^*\right)\right\| < \gamma\left(1 - \alpha\left(\sum_{j=i+1}^{\infty} \beta_j\right)\right) \tag{2.67}$$

Proof. ($1 \implies 2$) There exists a positive sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that

$$\sum_{i\in\mathbb{N}} \frac{\beta_i \delta_i}{\left(\sum_{j=i+1}^{\infty} \beta_i\right)\left(\sum_{j=i}^{\infty} \beta_i\right)} < 1 - \alpha \tag{2.68}$$

Let $\gamma_1 \in \mathbb{R}$ and choose $y_1^* \in \overline{co}\{x_i^*\}_{i\in\mathbb{N}}$ such that $\gamma_1 = d\,(0, \overline{co}\{x_i^*\}_{i\in\mathbb{N}})$ and $||y_1^*|| \leq \gamma_1\,(1+\delta_1)$. From here, for each $n \geq 1$, define $\gamma_{n+1} \in \mathbb{R}$ and choose $y_{n+1}^* \in \overline{co}\{x_i^*\}_{i\geq n+1}$ such that

$$\gamma_{n+1} = \inf\left\{\left|\left|\left(\sum_{i=1}^{n}\beta_i y_i^*\right) + \left(1-\sum_{i=1}^{n}\beta_i\right)y^*\right|\right| : y^* \in \overline{co}\{x_i^*\}_{i\geq n+1}\right\} \tag{2.69}$$

and

$$\left|\left|\sum_{i=1}^{n}\beta_i y_i^* + \left(1-\sum_{i=1}^{n}\beta_i\right)y_{n+1}^*\right|\right| < \gamma_{n+1}\,(1+\delta_{n+1}) \tag{2.70}$$

It is clear that since the set over which we are taking the infimum never gets larger, so $\{\gamma_i\}_{i\in\mathbb{N}}$ is a nondecreasing sequence. Further, since $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ and $\sum_{i\in\mathbb{N}}\beta_i = 1$, we have, for every i,

$$\alpha \leq \gamma_i \nearrow \gamma = \left|\left|\sum_{k\in\mathbb{N}}\beta_k y_k^*\right|\right| \leq 1 \tag{2.71}$$

So what is left to be shown is that for every $i \in \mathbb{N}$,

$$\left|\left|\sum_{j=1}^{i}\beta_j y_j^*\right|\right| < \gamma\left(1-\alpha\left(\sum_{j=i+1}^{\infty}\beta_j\right)\right) \tag{2.72}$$

Let $i \in \mathbb{N}$. Then,

$$\begin{aligned}
\left|\left|\sum_{j=1}^{i}\beta_j(y_j^*-y^*)\right|\right| &= \left|\left|\left(\left(\frac{\sum_{j=i}^{\infty}\beta_j}{\sum_{j=i}^{\infty}\beta_j}\right)\left(\sum_{j=1}^{i-1}\beta_j y_j^*\right)\right) + \left(\left(\frac{\sum_{j=i}^{\infty}\beta_j}{\sum_{j=i}^{\infty}\beta_j}\right)(\beta_i y_i^*)\right)\right|\right| \\
&= \left|\left|\left(\left(\frac{\lambda_i + \sum_{j=i+1}^{\infty}\beta_j}{\sum_{j=i}^{\infty}\beta_j}\right)\left(\sum_{j=1}^{i-1}\beta_j y_j^*\right)\right) + \left(\left(\frac{\beta_i\sum_{j=i}^{\infty}\beta_j}{\sum_{j=i}^{\infty}\beta_j}\right)(y_i^*)\right)\right|\right| \\
&\leq \frac{\beta_i}{\sum_{j=i}^{\infty}\beta_j}\left|\left|\sum_{j=1}^{i=1}\beta_j y_j^* + \left(\sum_{j=i}^{\infty}\beta_j\right)y_i^*\right|\right| + \frac{\sum_{j=i+1}^{\infty}\beta_j}{\sum_{j=i}^{\infty}\beta_j}\left|\left|\sum_{j=1}^{i-1}\beta_j y_j^*\right|\right| \\
&\leq \frac{\beta_i}{\sum_{j=i}^{\infty}\beta_j}\left|\left|\sum_{j=1}^{i=1}\beta_j y_j^* + \left(1-\sum_{j=1}^{i-1}\beta_j\right)y_i^*\right|\right| + \frac{\sum_{j=i+1}^{\infty}\beta_j}{\sum_{j=i}^{\infty}\beta_j}\left|\left|\sum_{j=1}^{i-1}\beta_j y_j^*\right|\right| \\
&< \frac{\beta_i}{\sum_{j=i}^{\infty}\beta_j}(\gamma_i)\,(1+\delta_i) + \frac{\sum_{j=i+1}^{\infty}\beta_j}{\sum_{j=i}^{\infty}\beta_j}\left|\left|\sum_{j=1}^{i-1}\beta_j y_j^*\right|\right| \\
&= \left(\sum_{j=i+1}^{\infty}\beta_j\right)\left(\left(\frac{\beta_i\gamma_i\,(1+\delta_i)}{\left(\sum_{j=i}^{\infty}\beta_j\right)\left(\sum_{j=i+1}^{\infty}\beta_j\right)}\right) + \left(\frac{1}{\sum_{j=i}^{\infty}\beta_j}\right)\left|\left|\sum_{j=1}^{i-1}\beta_j y_j^*\right|\right|\right)
\end{aligned} \tag{2.73}$$

Hence, for any $i \in \mathbb{N}$,

$$\begin{aligned}
\left\|\sum_{j=1}^{i}\beta_j y_j^*\right\| &< \left(\sum_{j=i+1}^{\infty}\beta_j\right)\left(\left(\frac{\beta_i\gamma_i\left(1+\delta_i\right)}{\left(\sum_{j=i}^{\infty}\beta_j\right)\left(\sum_{j=i+1}^{\infty}\beta_j\right)}\right)+\left(\frac{1}{\sum_{j=i}^{\infty}\beta_j}\right)\left\|\sum_{j=1}^{i-1}\beta_j y_j^*\right\|\right)\\
&< \left(\sum_{j=i+1}^{\infty}\beta_j\right)\sum_{j=1}^{i}\frac{\beta_j\gamma_j\left(1+\delta_j\right)}{\left(\sum_{k=j}^{\infty}\beta_k\right)\left(\sum_{k=j+1}^{\infty}\beta_k\right)}\\
&\le \gamma\left(\sum_{j=i+1}^{\infty}\beta_j\right)\sum_{j=1}^{i}\frac{\beta_j\left(1+\delta_j\right)}{\left(\sum_{k=j}^{\infty}\beta_k\right)\left(\sum_{k=j+1}^{\infty}\beta_k\right)}\\
&\le \gamma\left(\sum_{j=i+1}^{\infty}\beta_j\right)\left(\left(\sum_{j=1}^{i}\frac{\beta_j}{\left(\sum_{k=j}^{\infty}\beta_k\right)\left(\sum_{k=j+1}^{\infty}\beta_k\right)}\right)+(1-\alpha)\right)\\
&= \gamma\left(\sum_{j=i+1}^{\infty}\beta_j\right)\left(\left(\sum_{j=1}^{i}\left(\frac{1}{\sum_{k=j}^{\infty}\beta_k}-\frac{1}{\sum_{k=j+1}^{\infty}\beta_k}\right)\right)+(1-\alpha)\right)\\
&= \gamma\left(\sum_{j=i+1}^{\infty}\beta_j\right)\left(\frac{1}{\sum_{j=i+1}^{\infty}\beta_j}-\frac{1}{\sum_{j=1}^{\infty}\beta_j}+1-\alpha\right)\\
&= \gamma\left(\sum_{j=i+1}^{\infty}\beta_j\right)\left(\frac{1}{\sum_{j=i+1}^{\infty}\beta_j}-\alpha\right)\\
&= \gamma\left(1-\alpha\left(\sum_{j=i+1}^{\infty}\beta_j\right)\right)
\end{aligned} \tag{2.74}$$

completing the proof. $\square$

Proof. ($3 \implies 4$) There exists a positive sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that

$$\sum_{i\in\mathbb{N}}\frac{\beta_i\delta_i}{\left(\sum\limits_{j=i+1}^{\infty}\beta_i\right)\left(\sum\limits_{j=i}^{\infty}\beta_i\right)} < 1-\alpha \tag{2.75}$$

Define $\{x_i^0\}_{i\in\mathbb{N}} = \{x_i^*\}_{i\in\mathbb{N}}$,

$$\gamma_1 = \inf\left\{\sup_{y^*\in CoLim\{\phi_i\}_{i\in\mathbb{N}}}\{||x^*-y^*||\} : x^*\in\overline{co}\{x_i^*\}_{i\in\mathbb{N}}, \phi_k\in\overline{co}\{x_i^*\}_{i\geq k}, k\in\mathbb{N}\right\} \tag{2.76}$$

and pick $y_1^* \in \overline{co}\{x_i^*\}_{i\in\mathbb{N}}$, $\{\phi_i^1\}_{i\in\mathbb{N}} \subset X^*$ such that $\phi_k^1 \in \overline{co}\{x_i^*\}_{i\geq k}$ for every k and $w' \in CoLim\{\phi_i^1\}_{i\in\mathbb{N}}$ such that

$$\gamma_1(1-\delta_1) < ||y_1^* - w|| < \gamma_1(1+\delta_1) \tag{2.77}$$

So that there exists $\tilde{x} \in \overline{B_X(0;1)}$ such that $\gamma_1(1-\delta_1) < \langle \tilde{x}, y_1^* - w' \rangle$, and since $w' \in CoLim\{\phi_i^1\}_{i\in\mathbb{N}}$, we extract a subsequence $\{x_i^1\}_{i\in\mathbb{N}}$ of $\{\phi_i^1\}_{i\in\mathbb{N}}$ so that for any $w \in CoLim\{x_i^1\}_{i\in\mathbb{N}}$, we have

$$\langle \tilde{x}, w \rangle = \lim_{i\to\infty} \left\langle \tilde{x}, x_i^1 \right\rangle = \liminf_{i\to\infty} \left\langle \tilde{x}, \phi_i^1 \right\rangle \leq \langle \tilde{x}, w' \rangle \tag{2.78}$$

And so, for any $w \in CoLim\{x_i^1\}_{i\in\mathbb{N}}$, we have

$$\gamma_1(1-\delta_1) < \langle \tilde{x}, y_1^* - w \rangle \tag{2.79}$$

Continuing inductively, for $i \in \mathbb{N}$, set

$$\gamma_{i+1} = \inf\left\{\sup\left\{\left\|\left(\sum_{j=1}^{i} \beta_j y_j^*\right) + \left(\left(\sum_{j=i+1}^{\infty} \beta_j\right) y^*\right) - w\right\| : w \in CoLim\{\phi_i\}_{i\in\mathbb{N}}\right\}\right\} \tag{2.80}$$

Where the infimum is taken over all $y^* \in \overline{co}\{x_j^i\}_{j\geq i+1}$ and all $\{\phi_i\}_{i\in\mathbb{N}} \subset X^*$ such that $\phi_k \in \overline{co}\{x_j^i\}_{j\geq k}$. Next, pick $y_{i+1}^* \in \overline{co}\{x_j^i\}_{j\geq i+1}$ and $\{\phi_j^{i+1}\}_{j\in\mathbb{N}} \subset X^*$ such that for every k, $\phi_k^{i+1} \in \overline{co}\{x_j^i\}_{j\geq k}$ and pick $w' \in CoLim\{\phi_j^{i+1}\}_{j\in\mathbb{N}}$ such that

$$\gamma_{i+1}(1-\delta_{i=1}) < \left\|\sum_{j=1}^{i} \beta_j y_j^* + \left(\sum_{j=i+1}^{\infty} \beta_j\right) y_{i+1}^* - w'\right\| < \gamma_{i+1}(1+\delta_{i+1}) \tag{2.81}$$

Next, pick $\tilde{x} \in \overline{B_X(0;1)}$ satisfying

$$\gamma_{i+1}(1-\delta_{i+1}) < \left\langle \tilde{x}, \sum_{j=1}^{i} \beta_j y_j^* + \left(\left(\sum_{j=i+1}^{\infty} \beta_j\right) y_j^*\right) - w' \right\rangle \tag{2.82}$$

and apply the fact that since $\liminf_{j\to\infty} \left\langle \tilde{x}, \phi_j^{i+1} \right\rangle \leq \langle \tilde{x}, w \rangle$, we can find a subsequence $\{x_j^{i+1}\}_{j\in\mathbb{N}}$ of $\{\phi_j^{i+1}\}_{j\in\mathbb{N}}$ such that for every $w \in CoLim\{x_j^{i+1}\}_{j\in\mathbb{N}}$ we have

$$\gamma_{i+1}(1-\delta_{i+1}) < \left\langle \tilde{x}, \sum_{j=1}^{i} \beta_j y_j^* + \left(\left(\sum_{j=i+1}^{\infty} \beta_j\right) y_j^*\right) - w \right\rangle \tag{2.83}$$

completing our construction. Clearly, $CoLim\{y_j^*\}_{j\in\mathbb{N}} \subset CoLim\{\phi_j^i\}_{j\in\mathbb{N}}$ for every $i \in \mathbb{N}$. Hence, for every $w \in CoLim\{y_j^*\}_{j\in\mathbb{N}}$, for ever $i \in \mathbb{N}$, we have

$$\gamma_i(1-\delta_i) < \left\|\sum_{j=1}^{i-1} \beta_j y_j^* + \left(\left(\sum_{j=i}^{\infty} \beta_j\right) y_i^*\right) - w\right\| < \gamma_i(1-\delta_i) \tag{2.84}$$

Also, since $\{y_j^*\}_{j\in\mathbb{N}} \subset \overline{co}\{x_j\}_{j\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$, $CoLim\{y_j^*\}_{j\in\mathbb{N}} \subset \overline{co}\{y_j^*\}_{j\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$. By definition, $\gamma_1 \geq \alpha$, and $\{\gamma_i\}_{i\in\mathbb{N}}$ is a nondecreasing sequence since it is defined by taking the infimum over a set which never gains new elements as i increases. Further, $||w|| \leq 1$ for $w \in CoLim\{y_j^*\}_{j\in\mathbb{N}}$ implies that for every n, $\gamma_n \leq 2$, so by monotone convergence, $\gamma_i \nearrow \gamma = \left\|\sum_{j\in\mathbb{N}} \beta_j (y_j - w)\right\| \leq 2$. As for the final estimate, we have, for $i \in \mathbb{N}$ and $y^* \in CoLim\{y_j^*\}_{j\in\mathbb{N}}$, we have

Let $i \in \mathbb{N}$. Then, Hence, for every $i \in \mathbb{N}$, we have

$$\begin{aligned}
\left\|\sum_{j=1}^{i} \beta_j(y_j^* - y^*)\right\| &< \left(\sum_{j=i+1}^{\infty} \beta_j\right) \\
&\quad * \left(\left(\frac{\beta_i \gamma_i \left(1+\delta_i\right)}{\left(\sum_{j=i}^{\infty} \beta_j\right)\left(\sum_{j=i+1}^{\infty} \beta_j\right)}\right) + \left(\frac{1}{\sum_{j=i}^{\infty} \beta_j}\right)\left\|\sum_{j=1}^{i-1} \beta_j(y_j^* - y^*)\right\|\right) \\
&< \left(\sum_{j=i+1}^{\infty} \beta_j\right)\sum_{j=1}^{i} \frac{\beta_j \gamma_j \left(1+\delta_j\right)}{\left(\sum_{k=j}^{\infty} \beta_k\right)\left(\sum_{k=j+1}^{\infty} \beta_k\right)} \\
&\le \gamma\left(\sum_{j=i+1}^{\infty} \beta_j\right)\sum_{j=1}^{i} \frac{\beta_j \left(1+\delta_j\right)}{\left(\sum_{k=j}^{\infty} \beta_k\right)\left(\sum_{k=j+1}^{\infty} \beta_k\right)} \\
&\le \gamma\left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\left(\sum_{j=1}^{i} \frac{\beta_j}{\left(\sum_{k=j}^{\infty} \beta_k\right)\left(\sum_{k=j+1}^{\infty} \beta_k\right)}\right) + (1-\alpha)\right) \\
&= \gamma\left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\left(\sum_{j=1}^{i}\left(\frac{1}{\sum_{k=j}^{\infty} \beta_k} - \frac{1}{\sum_{k=j+1}^{\infty} \beta_k}\right)\right) + (1-\alpha)\right) \\
&= \gamma\left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\frac{1}{\sum_{j=i+1}^{\infty} \beta_j} - \frac{1}{\sum_{j=1}^{\infty} \beta_j} + 1 - \alpha\right) \\
&= \gamma\left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\frac{1}{\sum_{j=i+1}^{\infty} \beta_j} - \alpha\right) \\
&= \gamma\left(1 - \alpha\left(\sum_{j=i+1}^{\infty} \beta_j\right)\right)
\end{aligned} \tag{2.85}$$

□

It is worth noting that in the following two theorems, the assumption of completeness is necessary, as demonstrated by [12].

Theorem 2.9.5 (James Separable). *If X is a separable complete seminormed space then the following are equivalent.*

1. *X is not reflexive.*
2. *For every $\alpha \in (0,1)$ there is some sequence $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ satisfying $d\left(0, \overline{co}\{x_i^*\}_{i\in\mathbb{N}}\right) \geq \alpha$ and $x_i^* \xrightarrow{w^*} 0$.*
3. *For every $\alpha \in (0,1)$ and $\{\beta_i\}_{i\in\mathbb{N}} \subset (0,1)$ satisfying $\sum_{i\in\mathbb{N}} \beta_i = 1$, there is a $\gamma \in [0,1]$*

and $\{y_i^*\}_{i\in\mathbb{N}} \subset X^*$ *such that* $y_i^* \xrightarrow{w^*} 0$ *and for each* $i \in \mathbb{N}$,

$$\left\|\sum_{j\in\mathbb{N}} \beta_j y_j^*\right\| = \gamma \qquad \left\|\sum_{j=1}^{i} \beta_j y_j^*\right\| \leq \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j\right)\right) \tag{2.86}$$

4. *There exists* $x^* \in X^*$ *not achieving its norm.*

Proof. ($1 \implies 2$) Let $\alpha \in (0,1)$. Since X is nonreflexive and $c(X)$ is complete, by Riesz's lemma [13], there exists an $x^{**} \in B_{X^{**}}(0;1)$ such that $d(x^{**}, c(X)) > \alpha$. Since X is separable it has a countable dense set $\{x_i\}_{i\in\mathbb{N}}$. Fix $i \in \mathbb{N}$, let $\alpha_1 = \alpha_2 = \cdots = \alpha_{i-1} = 0$, $\alpha_i = \alpha$, and let $\{\beta_j\}_{j=1}^{i} \subset \mathbb{C}$ where $\beta_i \neq 0$ without loss of generality. Then, since c(X) is a subspace,

$$\begin{aligned}\left|\sum_{j=1}^{i} \beta_j \alpha_j\right| &= |\beta_i \alpha_i| = |\beta_i|\,\alpha \\ &\leq \frac{|\beta_i|\,\alpha}{d\left(x^{**}, c(X)\right)} \left\|x^{**} + \sum_{j=1}^{i-1} \frac{\beta_j}{\beta_i} c(x_j)\right\| \\ &= \frac{\alpha}{d\left(x^{**}, c(X)\right)} \left\|\beta_i x^{**} + \sum_{j=1}^{i-1} \beta_j c(x_j)\right\|\end{aligned} \tag{2.87}$$

Since $\alpha < d\left(x^{**}, c(X)\right)$, for some $\epsilon > 0$, $\epsilon + \frac{\alpha}{d(x^{**},c(X))} < 1$, so by **??**, since $X^{**} = (X^*)^*$, there exists an $x_i^* \in \overline{B_{X^*}(0;1)}$ such that for $1 \leq j \leq i-1$ we have $\langle x_j, x_i^*\rangle = \langle x_i^*, c(x_j)\rangle = 0$ and $\langle x_i^*, c(x_i)\rangle \geq \alpha$. Using this method we construct a sequence $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ such that for each $1 \leq j \leq i-1$, $\langle x_j, x_i^*\rangle = 0$ and $\langle x_i, x^{**}\rangle \geq \alpha$. Without loss of generality, we let $||x_i^*|| = 1$. Density of $\{x_j\}_{j\in\mathbb{N}}$ and the boundedness of $\{x_j^*\}_{j\in\mathbb{N}}$ implies $x_j^* \xrightarrow{w^*} 0$. Furthermore, any convex combination of the $(x_i^*)'s$ satisfies

$$\alpha \leq \left\langle \sum_{j=1}^{n} \lambda_j x_{k_j}^*, x^{**} \right\rangle \leq ||x^{**}|| \left\|\sum_{j=1}^{n} \lambda_j x_{k_j}^*\right\| \leq \left\|\sum_{j=1}^{n} \lambda_j x_{k_j}^*\right\| \tag{2.88}$$

so that $d\left(0, \overline{co}\{x_i^*\}_{i\in\mathbb{N}}\right) \geq \alpha$, completing the proof. ☐

Proof. ($2 \implies 3$). This is a direct application of 2.9.4 part ($1 \implies 2$), paired with the fact that if for every i, $y_i^* \in \overline{co}\{x_j^*\}_{j\geq i}$ and $x_j^* \xrightarrow{w^*} x$, then $y_j^* \xrightarrow{w^*} x$. ☐

Proof. ($3 \implies 4$). Let $x^* = \sum_{j\in\mathbb{N}} \beta_j y_j^*$ and let $x \in \overline{B_X(0;1)}$. Since $y_j^* \xrightarrow{w^*} 0$, for some $N \in \mathbb{N}$,

$\langle x, y_j^* \rangle < \gamma\alpha$ for every $j > \mathbb{N}$. Then

$$\begin{aligned}
|\langle x, x^* \rangle| &\leq \left| \left\langle x, \sum_{j=1}^{N} \beta_j y_j^* \right\rangle \right| + \left| \left\langle x, \sum_{j=N+1}^{\infty} \beta_j y_j^* \right\rangle \right| \\
&< \left| \left\langle x, \sum_{j=1}^{N} \beta_j y_j^* \right\rangle \right| + \alpha\gamma \sum_{j=N+1}^{\infty} \beta_j \\
&\leq \left|\left| \sum_{j=1}^{N} \beta_j y_j^* \right|\right| + \alpha\gamma \sum_{j=N+1}^{\infty} \beta_j \leq \gamma \left(1 - \alpha \sum_{j=N+1}^{\infty} \beta_j \right) + \gamma\alpha \sum_{j=N+1}^{\infty} \beta_j \\
&= \gamma = \left|\left| \sum_{j \in \mathbb{N}} \beta_j y_j^* \right|\right| = ||x^*||
\end{aligned} \tag{2.89}$$

Since the inequality is strict and $x \in \overline{B_X(0;1)}$ was arbitrary, we are done. □

Proof. ($4 \implies 1$). If X is reflexive and $x^* \in X$, then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ such that $\langle x_n, x^* \rangle \to ||x^*||$. By 2.9.1, $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence $x_{k_n} \overset{w}{\to} x \in \overline{B_X(0;1)}$. This x satisfies $\langle x, x^* \rangle = ||x^*||$. □

Theorem 2.9.6 (James)**.** *If X is a complete seminormed space, then the following are equivalent.*

1. *X is non-reflexive.*
2. *For each $\alpha \in (0,1)$, there exists an $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ and a subspace $Y \subset X$ such that $d\left(\overline{co}\{x_i^*\}_{i\in\mathbb{N}} - Y^\perp, 0\right) \geq \alpha$ and that $\langle y, x_i^* \rangle \to 0$ for each $y \in Y$.*
3. *For every $\alpha \in (0,1)$ and $\{\beta_i\}_{i\in\mathbb{N}} \subset [0,\infty)$ such that $\sum_{i\in\mathbb{N}} \beta_i = 1$, there is a $\gamma \in [0,2]$ and $\{y_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ such that for each $y^* \in CoLim\{y_i^*\}_{i\in\mathbb{N}}$, each $i \in \mathbb{N}$,*
$$\left|\left| \sum_{j\in\mathbb{N}} \beta_j \left(y_j^* - y^*\right) \right|\right| = \gamma \qquad \left|\left| \sum_{j=1}^{i} \beta_j \left(y_j^* - y_j\right) \right|\right| < \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j \right) \right) \tag{2.90}$$
4. *There exists $x^* \in X^*$ which doesn't achieve its norm.*

Proof. ($1 \implies 2$) If X is non-reflexive, then X contains a non-reflexive closed separable subspace S. An application of 2.9.5 implies the existence of a sequence $\{x_i\}_{i\in\mathbb{N}} \subset \overline{B_{S^*}(0;1)}$ such that

$$d_s\left(0, \overline{co}\{x_i\}_{i\in\mathbb{N}}\right) \geq \alpha \qquad x_i \overset{S-w^*}{\to} 0 \tag{2.91}$$

If $y^\perp \in X^*$ such that $Y \subset kern(y^*)$, then letting for each $i \in \mathbb{N}$ x_i^* be a Hahn-Banach extension living in $\overline{B_X^*(0;1)}$, and let $x^* \in \overline{co}\{x_i^*\}_{i\in\mathbb{N}}$. Then $x := x^*|_S \in \overline{co}\{x_i\}_{i\in\mathbb{N}}$. If $y \in Y$, then $\langle y, x_i^* \rangle = \langle y, x \rangle \to 0$. If $y^\perp \in Y^\perp$, then

$$\left|\left|x^* - y^\perp\right|\right| \geq \left|\left|x - \left(y^\perp|_S\right)\right|\right|_S = ||x|| \geq \alpha \tag{2.92}$$

so $d\left(\overline{co}\{x_i^*\}_{i\in\mathbb{N}} - Y^\perp, 0\right) \geq \alpha$ □

Proof. ($2 \implies 3$) Since $CoLim\{x_i^*\}_{i\in\mathbb{N}} \subset Y^\perp$ from the previous part, this is an easy application of 2.9.4 ($3 \implies 4$). □

Proof. ($3 \implies 4$) Define $\eta = \frac{\alpha^2}{4}$, and then let $\lambda_1 \in [0,\infty)$ such that for every natural n, $\lambda_{n+1} < \eta\lambda_n$ and $\sum_{k\in\mathbb{N}} \lambda_k = 1$. Let $y^* \in CoLim\{y_i^*\}_{i\in\mathbb{N}}$ where $\{y_i^*\}_{i\in\mathbb{N}}$ are as in part (3) of this theorem. Let $x \in \overline{B_X(0;1)}$. Since $y^* \in Colim\{y_i^*\}_{i\in\mathbb{N}}$, and since $\alpha \leq \gamma$, there exists $i \in \mathbb{N}$ such that

$$\left\langle x, y_{i+1}^* - y^* \right\rangle < \alpha^2 - 2\eta \leq \alpha\gamma - 2\eta \tag{2.93}$$

For this x, we have

$$\begin{aligned}
\left\langle x, \sum_{j\in\mathbb{N}} \beta_j (y_j^* - y^*) \right\rangle &< \sum_{j=1}^{i} \beta_j \left\langle x, y_j^* - y^* \right\rangle + \beta_{i+1}(\alpha\gamma - 2\eta) + \sum_{j=i+2}^{\infty} \beta_j \left\langle x, y_j^* - y^* \right\rangle \\
&\leq \left\| \sum_{j=1}^{i} \beta_j (y_j^* - y^*) \right\| + \beta_{i+1}(\alpha\gamma - 2\eta) + 2\sum_{j=i+2}^{\infty} \beta_j \\
&\leq \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j \right) \right) + \beta_{i+1}(\alpha\gamma - 2\eta) + 2\sum_{j=i+2}^{\infty} \beta_j \qquad (2.94) \\
&= \gamma - \gamma\alpha \sum_{j=i+2}^{\infty} \beta_j - 2\eta\beta_{i+1} + 2\sum_{j=i+1}^{\infty} \eta\beta_j \\
&\leq \gamma - (\gamma\alpha - 2\eta) \sum_{j=i+1}^{\infty} \beta_j < \gamma = \left\| \sum_{j\in\mathbb{N}} \beta_j \left(y_j^* - y^* \right) \right\|
\end{aligned}$$

Since $x \in \overline{B_X(0;1)}$ was arbitrary, we are done. □

Proof. ($4 \implies 1$) If X is reflexive and $x^* \in X$, then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ such that $\langle x_n, x^* \rangle \to ||x^*||$. By 2.9.1, $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence $x_{k_n} \xrightarrow{w} x \in \overline{B_X(0;1)}$. This x satisfies $\langle x, x^* \rangle = ||x^*||$. □

Corollary 2.9.7. *Let X be a complete seminormed space. Then the following are equivalent.*

1. *X is reflexive.*
2. *Each element of X^* attains its norm.*

Proof. Direct consequence of 2.9.6. □

2.9.2 Lindenstrauss On Nonseparable Reflexive Banach Spaces

If $\Gamma \neq \emptyset$, then $c_0(\Gamma)$ denotes the space of mappings $f : \Gamma \to \mathbb{C}$ such that for every $\epsilon > 0$, $card\{x \in \Gamma : |f(x)| > \epsilon\} \in \mathbb{N}$.

Definition 2.9.8 (**Convex**)**.** Let $\mathcal{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let V be a **Vector Space** over $\mathcal{F}$. Let $T : V \to (-\infty, \infty]$ We say that T is **Convex** if for each $x, y \in V$,

$$T\left(\frac{x+y}{2}\right) \leq \frac{T(x) + T(y)}{2}$$

We say that T is **Strictly Convex** if for each $x, y \in V$,

$$T\left(\frac{x+y}{2}\right) < \frac{T(x)+T(y)}{2}$$

Proposition 2.9.9 (**Convex** function)**.** Let $\mathcal{F} \in \{\mathbb{C}, \mathbb{R}\}$. Let X be a **Vector Space** over $\mathcal{F}$. The following are equivalent.

(i) T is **Convex** and for each $x, y \in X$, $T([x, y])$ is bounded from above.

(ii) For each $x, y \in X$ and for each $\lambda \in (0,1)$,

$$T(\lambda x + (1-\lambda)y) \leq \lambda T(x) + (1-\lambda)T(y)$$

(iii) For every $\{\lambda_i\}_{i=1}^n \subset (0,1)$ such that $\sum\limits_{i=1}^n \lambda_i = 1$ and for every $\{x_i\}_{i=1}^n \subset X$ where each x_i is distinct, ,

$$T\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i T(x_i)$$

(iv) For each $x, y, z \in X$ with $x \neq z$ and $y = \lambda z + (1-\lambda)x$ where $\lambda \in (0,1)$, we have

$$\frac{T(y)-T(x)}{\lambda} \leq T(z) - T(x) \leq \frac{T(z)-T(y)}{1-\lambda}$$

(v) $Epi(T)$ is **Convex**

Proof of 2.9.9 i implies 2.9.9 ii. Let $x, y \in X$. Let M be the collection of all $n \in \mathbb{Z}^+$ such that for all $\lambda \in (0,1)$ there exists nonnegative integers j_n and k_n with $j + k = 2^n - 1$, $\frac{j_n}{2^n} \leq \lambda$ and $\frac{k_n}{2^n} \leq 1 - \lambda$ and there exists $\lambda_n \in (0,1)$ such that

$$T(\lambda x + (1-\lambda)y) \leq \frac{j_n}{2^n}T(x) + \frac{k_n}{2^n}T(y) + \frac{1}{2^n}T(\lambda_n x + (1-\lambda_n)y)$$

I first prove $1 \in M$. Without loss of generality, suppose $\lambda \geq \frac{1}{2}$. Define $\lambda_1 = 2\left(\lambda - \frac{1}{2}\right)$. Then $\lambda_1 \in (0,1)$, $\lambda = \frac{\lambda_1}{2} + \frac{1}{2}$, and $2(1-\lambda) = 1 - \lambda_1$

$$\begin{aligned}
T(\lambda x + (1-\lambda)y) &= T\left(\frac{1}{2}x + \frac{1}{2}\lambda_1 x + \frac{2}{2}(1-\lambda)y\right) \\
&= T\left(\frac{1}{2}x + \frac{1}{2}(\lambda_1 x + (1-\lambda_1)y)\right) \\
&\leq \frac{1}{2}T(x) + \frac{1}{2}T(\lambda_1 x + (1-\lambda_1)y)
\end{aligned}$$

Since $\frac{1}{2} \leq \lambda$ and $0 \leq 1-\lambda$, $1 \in M$. Now suppose $k \in M$. Then since $1 \in M$. There are integers a and b with either $a = 0$ and $b = 1$ or with $a = 1$ and $b = 0$ and there is a $\lambda_{k+1} \in [0,1]$ such that

$$T(\lambda_k x + (1-\lambda_k)y) \leq \frac{a}{2}T(x) + \frac{b}{2}T(y) + \frac{1}{2}T(\lambda_{k+1}x + (1-\lambda_{k+1})y)$$

Hence

$$\begin{aligned}
T\left(\lambda x+(1-\lambda)y\right) &\leq \frac{j_k}{2^k}T(x)+\frac{m_k}{2^k}T(y)+\frac{1}{2^k}T\left(\lambda_k x+(1-\lambda_k)y\right)\\
&\leq \frac{j_k}{2^k}T(x)+\frac{m_k}{2^k}T(y)+\frac{1}{2^k}\left(\frac{a}{2}T(x)+\frac{b}{2}T(y)+\frac{1}{2}T\left(\lambda_{k+1}x+(1-\lambda_{k+1}T(y))\right)\right)\\
&= \frac{2j_k+a}{2^{k+1}}T(x)+\frac{2m_k+b}{2^{k+1}}+\frac{1}{2^{k+1}}T\left(\lambda_{k+1}x+(1-\lambda_{k+1})y\right)
\end{aligned}$$

where we have

$$2j_k+a+2m_k+b=2\left(j_k+m_k\right)+a+b=2\left(2^k-1\right)+1=2^{k+1}-1$$

so that $k+1\in M$. Hence, $M=\mathbb{Z}^+$. Thus, for each $n\in\mathbb{N}$, if Γ is an **Upper Bound** for $T([x,y])$, we have

$$T\left(\lambda x+(1-\lambda)y\right)\leq\frac{j_n}{2^n}T(x)+\frac{m_n}{2^n}T(y)+\frac{\Gamma}{2^n}$$

Now since $\frac{j_n}{2^n}\leq\lambda$, $\frac{m_n}{2^n}\leq 1-\lambda$, and $\frac{j_n}{2^n}+\frac{m_n}{2^n}\to 1$, it is clear that $\frac{j_n}{2^n}\to\lambda$ and $\frac{m_n}{2^n}\to 1-\lambda$, so clearly the result holds. □

Proof of 2.9.9 ii implies 2.9.9 iii. I utilize induction on n. Since f is **Convex**, the proposition holds for $n=1$ and $n=2$. Let $k\in\mathbb{N}$. Suppose that for any $(\psi_1,\cdots,\psi_k)\in(0,1)^k$ such that $\sum_{j=1}^k\psi_j=1$, and for each $(x_1,\cdots,x_k)\in X^k$, we have

$$f\left(\sum_{j=1}^k\lambda_j x_j\right)\leq\sum_{j=1}^k\lambda_j f(x_j)$$

Let $(\lambda_1,\cdots,\lambda_{k+1})\in[0,1]^{k+1}$. Let $(x_1,\cdots,x_{k+1})\in X^{k+1}$. Then

$$\sum_{j=1}^k\frac{\lambda_j}{1-\lambda_{k+1}}=1\qquad\left(\frac{\lambda_1}{1-\lambda_{k+1}},\cdots,\frac{\lambda_k}{1-\lambda_{k+1}}\right)\in(0,1)^k$$

Since f is **Convex**, ,

$$\begin{aligned}
f\left(\sum_{j=1}^{k+1}\lambda_j x_j\right) &= f\left(\lambda_{k+1}x_{k+1}+(1-\lambda_{k+1})\sum_{j=1}^k\frac{\lambda_j}{1-\lambda_{k+1}}x_j\right)\\
&\leq \lambda_{k+1}f(x_{k+1})+(1-\lambda_{k+1})f\left(\sum_{j=1}^k\frac{\lambda_j}{1-\lambda_{k+1}}x_j\right)\\
&\leq \sum_{j=1}^{k+1}\lambda_j f(x_j)
\end{aligned}$$

□

Proof of 2.9.9 iii if and only if 2.9.9 iv. For the first inequality, we have

$$T(y) = T(\lambda_z + (1-\lambda)x) \leq \lambda T(z) + (1-\lambda)T(x)$$

which is equivalent to

$$T(y) - T(x) \leq \lambda (T(z) - T(x))$$

which is equivalent to

$$\frac{T(y) - T(x)}{\lambda} \leq T(z) - T(x)$$

For the second inequality,

$$T(y) \leq \lambda T(z) + (1-\lambda)T(x)$$

which is equivalent to

$$-\lambda T(z) + T(z) - (1-\lambda) T(x) \leq T(z) - T(y)$$

which is equivalent to

$$(1-\lambda)(T(z) - T(x)) \leq T(z) - T(y)$$

which is equivalent to

$$T(z) - T(x) \leq \frac{T(z) - T(y)}{1-\lambda}$$

□

Proof of 2.9.9 iv implies 2.9.9 i. □

Proof of 2.9.9 ii implies 2.9.9 v. □

Proof of 2.9.9 v implies 2.9.9 ii. □

2.10 Convexity Of Functions And Sets

Definition 2.10.1 (Convex Functions). *Let X be a vector space, Y a topological vector space, $\mathcal{U}$ the set of neighborhoods of 0 in Y except Y itself, $f : X \to (-\infty, \infty]$, and $g : Y \to (-\infty, \infty]$. Let $x, y \in X$.*

1. *We call $D(f) := f^{-1}(\mathbb{R})$ the* **effective domain** of f.
2. If $D(f) \neq \emptyset$, then we call f **proper.**
3. We call $Epi\,(f) := \{(x,t) \in X \times \mathbb{R} : f(x) \leq t\}$ the **Epigraph** of f.
4. We say that $C \subset X$ is **strictly convex** if for each $x, y \in C$, for each $z_0 \in (x, y)$, and for each $z_1 \in X$, there is a $t > 0$ such that $[z_0, z_0 + tz_1] \subset C$.

5. If g is a convex function, then then we define the **modulus of local uniform convexity** of g, $\tilde{\Delta} : \mathcal{U} \times Y \to \mathbb{R}$ by

$$\tilde{\Delta}\,(U,x) = \frac{1}{2} \inf_{y \in Y \setminus (x+U)} \left\{ f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \right\} \tag{2.95}$$

6. If g is a convex function, then we define the **modulus of uniform convexity** of g, $\Delta : \mathcal{U} \to \mathbb{R}$ by $\Delta(U) = \inf_{g(x)=1} \tilde{\Delta}(U,x)$.

7. If g is a convex function, then we say that g is **locally uniformly convex at** $x \in Y$ if for each $U \in \mathcal{U}$, $\tilde{\Delta}(U,x) > 0$.

8. We say that g is **locally uniformly convex** if it is **locally uniformly convex** at each of its points.

9. We say that g is **uniformly convex** if for each $U \in \mathcal{U}$, $\Delta(U) > 0$.

10. We say that g is **lower semi-continuous**, or LSC if it is continuous with respect to the topology on $(-\infty, \infty]$ generated by sets of the form $(-\infty, \alpha)$ where $\alpha \in \mathbb{R}$, along with $(-\infty, \infty]$ itself.

Remark 2.10.2 (Basis Independence)**.** *It is easy to see that a mapping is locally uniform convex at a point (locally uniformly convex) [uniformly convex] if we define Δ, $\tilde{\Delta}$ in terms of a single neighborhood basis of Y at 0 instead of all neighborhoods of 0 in Y.*

Remark 2.10.3 (Strictly Convex Real Valued)**.** *If X is a vector space and $f : X \to (-\infty, \infty]$ is strictly convex, and f is finite everywhere.*

Proof. If $f(x) = \infty$ where $x \in X$ and f is strictly convex, then we must have $\infty = f(x) < \frac{f(0)+f(2x)}{2}$, a contradiction. □

Proposition 2.10.4. *Let X be a vector space and $T : X \to (-\infty, \infty]$. Then the following are equivalent.*

1. *T is (strictly) convex.*

2. *For each $x_1 \neq x_2 \in X$ and $\lambda \in (0,1)$.*

$$T\left(\lambda x_1 + (1-\lambda)x_2\right)(<) \leq \lambda T x_1 + (1-\lambda) T x_2 \tag{2.96}$$

3. *For each $\{\lambda_i\}_{i=1}^n \subset (0,1)$ which sums to 1, for each $\{x_i\}_{i=1}^n \subset X$,*

$$T\left(\sum_{j=1}^n \lambda_j x_j\right)(<) \leq \sum_{j=1}^n \lambda_j T x_j \tag{2.97}$$

4. *If $x_1 \neq x_3 \in X$ and $x_2 \in (x_1, x_3)$, say $x_2 = \lambda x_1 + (1-\lambda)x_3$ then*

$$\frac{Tx_2 - Tx_1}{\lambda}\,(<) \leq Tx_3 - Tx_1\,(<) \leq \frac{Tx_3 - Tx_2}{1-\lambda} \tag{2.98}$$

5. *Epi(T) is (strictly) convex*

Proof. (1 $\implies$ 2)

$\square$

Proof. (2 $\implies$ 3) I utilize induction on n. Since f is assumed to be (strictly) convex, the proposition holds for $n = 1$ and $n = 2$. Let $k \in \mathbb{N}$. Suppose that for any $(\psi_1, \cdots, \psi_k) \in [0,1]^k$ such that $\sum_{j=1}^k \psi_j = 1$, and for each $(x_1, \cdots, x_k) \in X^k$, we have

$$f\left(\sum_{j=1}^k \lambda_j x_j\right) (<) \leq \sum_{j=1}^k \lambda_j f(x_j) \tag{2.99}$$

Let $(\lambda_1, \cdots, \lambda_{k+1}) \in [0,1]^{k+1}$. Let $(x_1, \cdots, x_{k+1}) \in X^{k+1}$. Without loss of generality, we assume $\lambda_{k+1} \neq 0$. Then

$$\sum_{j=1}^k \frac{\lambda_j}{1-\lambda_{k+1}} = 1 \qquad \left(\frac{\lambda_1}{1-\lambda_{k+1}}, \cdots, \frac{\lambda_k}{1-\lambda_{k+1}}\right) \in [0,1]^k \tag{2.100}$$

Hence, because f is (strictly) convex,

$$\begin{aligned} f\left(\sum_{j=1}^{k+1} \lambda_j x_j\right) &= f\left(\lambda_{k+1}x_{k+1} + (1-\lambda_{k+1})\sum_{j=1}^k \frac{\lambda_j}{1-\lambda_{k+1}} x_j\right) \\ &(<) \leq \lambda_{k+1} f(x_{k+1}) + (1-\lambda_{k+1}) f\left(\sum_{j=1}^k \frac{\lambda_j}{1-\lambda_{k+1}} x_j\right) \\ &(<) \leq \sum_{j=1}^{k+1} \lambda_j f(x_j) \end{aligned} \tag{2.101}$$

$\square$

Proof. (3 $\implies$ 4) $\square$

Proof. (4 $\implies$ 1). $\square$

Proof. (2 $\iff$ 5) $\square$

The epigraph of a function provides us with a nice characterization of lower semi-continuity.

Proposition 2.10.5 (Convex Continuity). *Let X be a locally convex space and $f : X \to (-\infty, \infty]$. The following conditions are equivalent.*

1. *f is LSC on X.*

2. *$Epi(f)$ is closed in $X \times \mathbb{R}$.*

Proof. Define $F : X \times \mathbb{R} \to \tilde{\mathbb{R}}$ by $F(x, \alpha) = f(x) - \alpha$. Then f is (weakly) LSC on X if and only if F is (weakly) LSC on $X \times \mathbb{R}$. Suppose f is (weakly) LSC. Then F is (weakly) LSC, so $F^{-1}((-\infty, 0]) = Epi(f)$ is (weakly) closed, so we're done. Suppose $Epi(f)$ is (weakly) closed. Then $F^{-1}((-\infty, 0])$ is (weakly) closed. Further, for any $\beta \in \mathbb{R}$, $F^{-1}((-\infty, \beta]) = F^{-1}((-\infty, 0]) - (0, \beta)$, and so is also closed. Hence F is (weakly) LSC, and so f is too. □

In the case of a convex function, the above proposition allows us to equate weak and strong lower semicontinuity.

Corollary 2.10.6 (Weak To Strong Convex)**.** *Let X be locally convex Hausdorff space and $f : X \to (-\infty, \infty]$ be convex. Then f is LSC if and only if it is weakly LSC.*

Proof. Since $Epi(f)$ is convex, it is closed if and only if it is weakly closed, allowing us to apply 2.10.5. □

Theorem 2.10.7 (Point Continuous)**.** *Let X be a locally convex space and $f : X \to (-\infty, \infty]$ be convex and proper. Then f is bounded on some open set if and only if f is continuous on the interior of its domain.*

Proof. (*implies*) Without loss of generality, we assume that f is bounded from above by M on a (weakly) open set $\mathcal{U}$ which is symmetric and contains 0. Further, we can also suppose $f(0) = 0$. For each $\epsilon \in (0, 1)$ and each $x \in \epsilon\mathcal{U}$, we have

$$f(x) = f\left(\epsilon\frac{x}{\epsilon} + (1-\epsilon)0\right) \leq \epsilon f\left(\frac{x}{\epsilon}\right) \leq \epsilon M \tag{2.102}$$

and since $0 = \frac{x}{1+\epsilon} + \left(1 - \frac{1}{1+\epsilon}\left(\frac{-x}{\epsilon}\right)\right)$,

$$0 = f(0) = f\left(\frac{x}{1+\epsilon} + \left(1 - \frac{1}{1+\epsilon}\right)\left(\frac{-x}{\epsilon}\right)\right) \leq \frac{f(x)}{1+\epsilon} + \frac{\epsilon f\left(-\frac{x}{\epsilon}\right)}{1+\epsilon} \tag{2.103}$$

so, since $\frac{-x}{\epsilon} \in \mathcal{U}$,

$$-\epsilon M \leq -\epsilon f\left(\frac{x}{\epsilon}\right) \leq f(x) \tag{2.104}$$

Hence, $|f(x)| \leq \epsilon M$ for $x \in \epsilon\mathcal{U}$, and f is (weakly) continuous at 0. Hence, it is sufficient to show that for any y in the (weak) interior of $D(f)$, there is a (weak) neighborhood of y on which f is bounded from above. To see this, let y in the (weak) interior of $D(f)$. Since scalar multiplication is continuous, there is a $\rho > 1$ such that $\rho y \in D(f)$. If $\mathcal{U}_y = y + \left(1 - \frac{1}{\rho}\right)\mathcal{U}$, then $x \in \mathcal{U}_y$ can be written, for some $z \in \mathcal{U}$, as

$$x = y + \left(1 - \frac{1}{\rho}\right)z = \frac{1}{\rho}(\rho y) + \left(1 - \frac{1}{\rho}\right)z \tag{2.105}$$

Since f is convex, $D(f)$ is convex, and so $x \in D(f)$, implying $\mathcal{U}_y \subset D(f)$. Since f is a convex function, we also have that

$$f(x) \leq \frac{1}{\rho}f(\rho y) + \left(1 - \frac{1}{\rho}\right)f(x) \leq \frac{1}{\rho}f(\rho y) + \left(1 - \frac{1}{\rho}\right)M \tag{2.106}$$

so that f is bounded on $\mathcal{U}_y$ and is therefore continuous at y. □

Proof. ($\Longleftarrow$) This is obvious. □

Corollary 2.10.8. *Let X be a locally convex, space and $f : X \to (-\infty, \infty]$ be LSC at some point in its effective domain.*

1. *if f is convex, then it is continuous on the interior of $D(f)$.*
2. *If f is strictly convex, then it is continuous on X.*

2.11 Differentiation And SubDifferentials

Definition 2.11.1 (**Directional Derivative**)**.** Let X be a **Topological Vector Space**. Let Y be a **Hausdorff Topological Vector Space**. Let $C \subset X$. Let $y_0 \in X$ and let $x_0 \in C$. Let $f : C \to X$. Suppose x_0 is an **Accumulation Point** of $[x_0, x_0 + y_0]$. Suppose

$$\lim_{t \searrow 0} \frac{f\left(x_0 + ty_0\right) - f(x_0)}{t}$$

exists. Then we say that the **Directional Derivative** of f in the direction of y_0 exists at x_0. Let $D \subset C \times X$ denote the collection of pairs (x, y) such that the **Directional Derivative** of f in the direction of y exists at x_0. Then we define, for each $x, y \in D$.

$$f'_+(x, y) = \lim_{t \searrow 0} \frac{f(x + ty) - f(x)}{t}$$

We call $f'_+(x, y)$ the **Directional Derivative** of f in the direction of y at x, and we call f_+ the **Directional Derivative** of f. Let $F \subset C \times X$ denote the collection of pairs (x, y) such that the **Directional Derivative** of f in the direction of $-y$ exists at x. Then we define, for each $(x, y) \in F$, $f'_-(x, y) = f'_+(x, -y)$.

Definition 2.11.2 (**Gateaux Derivative**)**.** Let X be a **Topological Vector Space**. Let Y be a **Hausdorff Topological Vector Space**. Let $C \subset X$. Let $x_0 \in C$. Let $f : C \to X$. Suppose $\{x_0\} \times X$ is a subset of the **Domain** of f'_+. Furthermore, suppose there exists $T \in BL(X, Y)$ such that for each $y \in X$,

$$Ty = f'_+(x_0, y)$$

The we say that the **Gateaux Derivative** of f exists at x_0, we write $T = f'(x_0)$, and we call $f'(x_0)$ the **Gateaux Derivative** of f at x_0. Furthermore, we say that f is **Gateaux Differentiable** at x_0.

Definition 2.11.3 (**Frechet Derivative**)**.** Let X be a **Topological Vector Space**. Let Y be a **Hausdorff Topological Vector Space**. Let $C \subset X$. Let $x_0 \in C$. Let $f : C \to X$. Let f be **Gateaux Differentiable** at x_0. Let U be a **Neighborhood** of 0 in X. Futhermore, suppose that for each **Neighborhood** V of 0 in Y, there exists a $\delta > 0$ such that whenever $|t| < \delta$ and $y \in U$,

$$\frac{f(x_0 + ty) - f(x_0)}{t} \in U + f'(x_0)y$$

Then we say that f is **Frechet Differentiable** at x_0 and that $f'(x_0)$ is the **Frechet Derivative** of f at x_0.

Definition 2.11.4 (**Subgradient**)**.** Let X be a **Topological Vector Space**.

Let $h : X \to (-\infty, \infty]$. Let $x_0 \in X$. Suppose there exists $x^* \in X^*$ such that for every $y \in X$, we have

$$\langle y - x, x^* \rangle \leq h(y) - h(x)$$

Then we say that h is **Subdifferentiable** at x_0 and we call x^* a **Subgradient** of h at x_0. We denote the collection of all **Subgradients** of h at x_0 with $\partial h(x_0)$. If $D \subset X$ is the set on which h is **Subdifferentiable**, then we call $\partial h : D \to 2^{X^*}$ the **Subdifferential** of h.

Proposition 2.11.5 (Gateaux)**.** Let X be a **Topological Vector Space**. Let Y be a **Hausdorff Topological Vector Space**. Let $C \subset X$. Let $f : C \to Y$. Let $x_0 \in C$. Let f be **Gateaux Differentiable** at x_0. Let $y_0 \in X$. Let $g(t) = f(x_0 + ty_0)$.

(i) $f'_+(x_0, y_0) = -f'_+(x_0, -y) = -f'_-(x_0, y_0)$.

(ii) $\langle y, f'(x_0) \rangle = g'(0)$

(iii) If f is **Gateaux Differentiable** on $[x_0 - \epsilon y_0, x_0 + \epsilon y_0]$ for some $\epsilon > 0$ and if $f' : Domain(f') \to BL\left(X, BL(X, Y)\right)$ is **Gateaux Differentiable** at x_0, then

$$\langle y, f''(x) y \rangle = g''(0)$$

Proof of 2.11.5 i. Since f is **Gateaux Differentiable** at x_0 there exists $f'(x_0) \in BL(X, Y)$ such that

$$F'_+(x_0, y_0) = f'(x_0) y_0 = -f'(x_0)\left(-y_0\right) = -f'_+(x_0, -y_0) = -f'_-(x_0, y_0)$$

□

Proof of 2.11.5 ii. We have

$$\langle y, f'(x_0) \rangle = f'_+(x_0, y_0) = \lim_{t \searrow 0} \frac{f(x_0 + ty_0) - f(x_0)}{t} = \lim_{t \searrow 0} \frac{g(t) - g(0)}{t}$$

Also, letting $s = -t$, we have

$$\begin{aligned} \langle -y, f'(x_0) \rangle = f'_+(x_0, -y_0) &= \lim_{t \searrow 0} \frac{f(x_0 + t(-y_0)) - f(x_0)}{t} \\ &= \lim_{s \nearrow 0} -\frac{f(x_0 + sy_0) - f(x_0)}{s} \\ &= -\lim_{s \nearrow 0} \frac{f(x_0 + sy_0) - f(x_0)}{s} \end{aligned}$$

Hence, by applying 2.11.5 i, we have

$$\lim_{t \nearrow 0} \frac{f(x_0 + ty_0) - f(x_0)}{t} = -f'_+(x_0, -y_0) = f'_+(x_0, y_0) = \lim_{t \searrow 0} \frac{f(x_0 + ty_0) - f(x_0)}{t}$$

so the two sided limit exists, and

$$\langle y_0, f'(x_0) \rangle = f'_+(x_0, y_0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = g'(0)$$

□

Proof of 2.11.5 iii. Since f' is **Gateaux Differentiable** at x_0, there is a $T \in BL(X, BL(X,Y))$ such that for each $y \in X$,

$$Ty = \lim_{t\to 0} \frac{f'(x_0+ty) - f(x_0)}{t}$$

We denote $T = f''(x_0)$. The convergence of this limit is at least pointwise. That is, for $z \in X$, we have

$$\lim_{t\to 0}\left\langle z, \frac{f'(x_0+ty) - f'(x_0)}{t}\right\rangle = \langle z, f'(x_0)y\rangle$$

Furthermore, if f is **Gateaux Differentiable** on $[x_0 - \epsilon y_0, x_0 + \epsilon y_0]$, then we, for $t_0 \in [-\epsilon, \epsilon]$, applying 2.11.5 ii,

$$\langle y, f'(x_0+t_0y_0)\rangle = \frac{d}{dt}\left(f(x_0+t_0y_0+ty_0\right)|_{t=0} = \frac{d}{dt}\left(f(x_0+ty)\right)|_{t=t_0} = g'(t_0)$$

Hence,

$$\begin{aligned}
g''(0) &= \lim_{t\to 0}\frac{g'(t)-g'(0)}{t}\\
&= \lim_{t\to 0}\frac{\langle y_0, f'(x_0+t_0y_0)\rangle - \langle y_0, f'(x_0)\rangle}{t}\\
&= \lim_{t\to 0}\left\langle y_0, \frac{f'(x_0+ty_0)-f'(x_0)}{t}\right\rangle\\
&= \left\langle y_0, \lim_{t\to 0}\frac{f'(x_0+ty_0)-f'(x_0)}{t}\right\rangle\\
&= \langle y_0, f''(x_0)y_0\rangle
\end{aligned}$$

□

Proposition 2.11.6 (**Convex Directional Derivative**)**.** Let X be a **Topological Vector Space**. Let $f : X \to (-\infty, \infty]$ be **Convex**. Let $x_0 \in Interior\,(D(f))$. Let $y_0 \in X$. Then $f'_+(x_0, y_0)$ exists. Furthermore, $-f'_-(x_0,y_0) \le f'_+(x_0,y_0)$, and for any $t > 0$ for which $x_0 + ty_0 \in Interior\,(D(f))$, we have

$$f'_+(x_0,y_0) \le \frac{f(x_0+ty_0)-f(x_0)}{t_0} \le f'_-(x'_0+ty_0, y_0)$$

Proof. This result mostly falls from several applications of 2.9.9 iv. Let $0 < t_1 < t_0$. Then if $x = x_0 - t_0y_0$, $y = x_0 - t_1y_0$, and $z = x_0$, we have

$$\frac{f(x_0)-f(x_0-t_0y_0)}{t_0} \le \frac{f(x_0)-f(x_0-t_1y_0)}{t_1}$$

Alternatively, if $x = x_0 - t_1y_0$, $y = x_0$, and $z = x_0 + t_1y_0$, we have

$$\frac{f(x_0)-f(x_0-t_1y_0)}{t_1} \le \frac{f(x_0+t_1y_0)-f(x_0)}{t_1}$$

Finally, if $x = x_0$, $y = x_+ t_1 y_0$, and $z = x_0 + t_0 y_0$, we derive

$$\frac{f(x_0 + t_1 y_0) - f(x_0)}{t_1} \leq \frac{f(x_0 + t_0 y_0) - f(x_0)}{t_0}$$

Putting the above inequalities together, we have

$$\frac{f(x_0) - f(x_0 - t_0 y_0)}{t_0} \leq \frac{f(x_0) - f(x_0 - t_1 y_0)}{t_1} \leq \frac{f(xl_0 + t_1 y_0) - f(x_0)}{t_1} \leq \frac{f(x_0 + t_0 y_0) - f(x_0)}{t_0}$$

whenever $0 < t_1 < t_0$. From this, we derive that g_- defined by

$$g_-(t) = \frac{f(x_0) - f(x_0 - t y_0)}{t}$$

is Monotone decreasing on $(0, \infty)$ and g_+ defined by

$$g_+(t) = \frac{f(x_0 + t y_0) - f(x_0)}{t}$$

is MONOTONE INCREASING on $(0, \infty)$. Let $t_0 > 0$. Furthermore, for every $t > 0$, we have

$$g_-(t) \leq g_+(t_0)$$

Hence, since t was arbitrary,

$$-f'_-(x_0, y_0) = \lim_{t \searrow 0} g_-(t) \leq g_+(t_0)$$

Now, since $t_0 > 0$ was arbitrary, we have

$$-f'_-(x_0, y_0) \leq \lim_{t \searrow 0} g_+(t) = f'_+(x_0, y_0)$$

Let $t_0 > 0$. Now, finally leveraging the monotonicity of g_+, we have, for any $t > 0$,

$$f'_+(x_0, y_0) \leq \frac{f(x_0 + t y_0) - f(x_0)}{t}$$

g_-, except centering the inequalities at $x_0 + t_0 y_0$, For any $t > 0$, we have

$$-f'_-(x_0 + t_0 y_0, y_0) \geq \frac{f(x_0 + t_0 y_0) - f(x_0 + t_0 y_0 - t y_0)}{t}$$

Thus, if $t = t_0$, we find

$$f'_+(x_0, y_0) \leq \frac{f(x_0 + t_0 y_0) - f(x_0)}{t_0} \leq -f'_-(x_0 + t_0 y_0, y_0)$$

$\square$

2.12 Normalized Duality Mapping

Definition 2.12.1 (**Normalized Duality Map**)**.** Let X be a **Seminormed Space**. Define $J : X \to 2^{X^*}$ by

$$Jx = \left\{x^* \in X^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\right\}$$

We call J the **Normalized Duality Map** of X.

Proposition 2.12.2 (Normalized Duality Inequality)**.** *Let X be a seminormed space and J be X's normalized duality mapping. Then, if $x, y \in X$ and $||x + \lambda y|| \neq 0$ where $\lambda > 0$, and $j_x \in Jx$, $j_{x+\lambda y} \in J(x + \lambda y)$, then*

$$\frac{\langle y, j_x \rangle}{||x||} \leq \frac{||x + \lambda y|| - ||x||}{\lambda} \leq \frac{\langle y, j_{x+\lambda y} \rangle}{||x + \lambda y||} \tag{2.107}$$

Proof.

$$\begin{aligned}
\frac{\langle y, j_x \rangle}{||x||} &= \frac{\langle x + \lambda y, j_x \rangle - ||x||^2}{\lambda\, ||x||} \leq \frac{|\langle x + \lambda y, j_x \rangle| - ||x||^2}{\lambda\, ||x||} \leq \frac{||j_x||\, ||x + \lambda y|| - ||x||^2}{\lambda\, ||x||} \\
&= \frac{||x + \lambda y|| - ||x||}{\lambda} \\
&= \frac{||x + \lambda y||}{||x + \lambda y||} \frac{||x + \lambda y|| - ||x||}{\lambda} \leq \frac{||x + \lambda y||^2 - |\langle x, j_{x+\lambda y} \rangle|}{\lambda\, ||x + \lambda y||} \\
&= \frac{\lambda \langle y, j_{x+\lambda y} \rangle + \langle x, j_{x+\lambda y} \rangle - |\langle x, j_{x+\lambda y} \rangle|}{\lambda\, ||x + \lambda y||} \\
&\leq \frac{\lambda \langle y, j_{x+\lambda y} \rangle}{\lambda\, ||x + \lambda y||} = \frac{\langle y, j_{x+\lambda y} \rangle}{||x + \lambda y||}
\end{aligned} \tag{2.108}$$

□

Theorem 2.12.3 (Asplund)**.** *[14] Let X be a seminormed space and J be its normalized duality mapping. Then for any $x \in X$, $Jx = \partial\left(||x||^2\right)$.*

Proof. □

Proposition 2.12.4. *Let X be a (Possibly Complete, maybe not) seminormed space and let J be it's normalized duality mapping. Then J is norm to weak* upper semicontinuous on X.*

Proof. □

2.13 Orthogonality

Definition 2.13.1 (Orthogonality)**.** *Let X be a seminormed space and $x, y \in X$. We say that x is* **orthogonal** to y and we write $x \perp y$ if for each scalar λ, we have

$$||x|| \leq ||x + \lambda y|| \tag{2.109}$$

Proposition 2.13.2 (Orthogonality)**.** *Let X be a seminormed space, $x, z \in X$, and $x^* \in X^*$.*

1. *$\langle x, x^* \rangle = ||x|| \, ||x^*||$ if and only if for each $y \in kern\,(x^*)$, $x \perp y$.*

2. *x is orthogonal to each element of some hyperplane in X.*

3. *For some $\alpha \neq 0$ $x \perp (\alpha x + z)$.*

4. *The mapping $T : \mathbb{F} \to \mathbb{R}$ defined by $T\alpha = ||\alpha x + z||$ achieves its minimum, and if λ_0 is a point at which it achieves this minimum, then $(\lambda_0 x + y) \perp x$ for any λ_0 which minimizes T. Furthermore, since T as defined earlier is a convex function, the set of λ for which $(\lambda x + y) \perp x$ is a convex set.*

2.14 Convexity Of A Space

Definition 2.14.1 (Uniform Convexity, Weak Uniform Convexity, Local Uniform Convexity, Weak Local Uniform Convexity, Strict Convexity)**.** *Let $(X, ||\cdot||)$ be a seminormed space.*

1. *We call the local modulus of uniform covexity of $||\cdot||$, denoted by the symbol $\tilde{\Delta}$, the* **local modulus of uniform convexity** of X.

2. If Δ is the modulus of uniform convexity of $||\cdot||$, then we call Δ the **modulus of uniform convexity** of X.

3. We define $\tilde{\Delta}_w : (0, \infty) \times \partial B_X(0; 1) \to (0, \infty)$ by

$$\tilde{\Delta}_w(\epsilon, x) = \inf \{2 - \langle x + y, x^* \rangle : x^* \in \partial B_{X^*}(0; 1), y \in \partial B_X(0; 1), ||x - y|| \geq \epsilon\} \tag{2.110}$$

   We call $\tilde{\Delta_w}$ the **modulus of weak local uniform convexity** of X.

4. We define $\Delta_w : (0, \infty) \to (0, \infty)$ by

$$\Delta_w(\epsilon) = \inf_{x \in \partial B_X(0;1)} \tilde{\Delta}_w(\epsilon, x) \tag{2.111}$$

   We call this the **modulus of weak uniform convexity** of X.

1. *We say that X is* **strictly convex** if for each $x, y \in X$ such that $||x - y|| \neq 0$, $||x + y|| < ||x|| + ||y||$.

2. We say that X is **uniformly convex at** x if $||\cdot||$ is uniformly convex at x.

3. We say that X is **locally uniformly convex** if $||\cdot||$ is locally uniformly convex.

4. We say that X is **uniformly convex** if $||\cdot||$ is uniformly convex.

5. We say that X is **weakly uniformly convex at** $x \in \partial B_X(0; 1)$ if for each $\epsilon > 0$, $\tilde{\Delta}_w(\epsilon, x) > 0$.

6. We say that X is **locally weakly uniformly convex** if it is weakly uniformly convex at each point on the boundary of $X's$ unit sphere.

7. We say that X is **weakly uniformly convex** if for each $\epsilon > 0$, $\Delta_w(\epsilon) > 0$.

Proposition 2.14.2 (Strictly Convex Spaces)**.** *Let X be a normed space. and let J be a duality mapping on X of weight ϕ. Then the following are equivalent.*

1. *X is strictly convex.*
2. *If $x, y \in X$ and $||x + y|| = ||x|| + ||y||$, then for some $\alpha \geq 0$, $||x - \alpha y|| = 0$.*
3. *If $||x|| = ||y|| = 1$ where $0 \neq ||x - y||$, then $||x + y|| < 2$.*
4. *If $x, y, z \in X$ and $||x - y|| = ||x - z|| + ||z - y||$, $||z - z_0|| = 0$ for some $z_0 \in [x, y]$.*
5. *If $x^* \in X^*$ and $||x|| = ||y|| = 1$ such that $\langle x, x^* \rangle = \langle y, y^* \rangle = \sup\limits_{||z||=1} \langle z, x^* \rangle$, then $||x - y|| = 0$.*
6. *$||\cdot||^2$ is strictly convex.*
7. *J is strictly monotone. That is, if $x, y \in X$, $||x - y|| \neq 0$, $x^* \in Jx$, and $y^* \in Jy$, then*
$$\langle x - y, x^* - y^* \rangle > 0 \tag{2.112}$$
8. *Orthogonality in X is left-unique. That is, for $x, y \in X$, there is a unique $\alpha \in \mathbb{F}$ such that $(\alpha x + y) \perp x$.*

Proposition 2.14.3 (Locally Uniformly Convex Spaces)**.** *Let X be a Banach Space. Then the following are equivalent.*

1. *X is locally uniformly convex*
2. *For each $\epsilon > 0$ and $x \in X$ with $||x|| = 1$, there is a $\delta > 0$ such that if $y \in X$ satisfies $||y|| = 1$ and $||x - y|| \geq \epsilon$, then $||x + y|| \leq 2(1 - \delta)$.*
3. *If $x \in \partial B_X(0; 1)$, $\{x_n\}_{n \in \mathbb{N}} \subset \partial B_X(0; 1)$, and $||x + x_n|| \to 2$, then $x_n \to x$.*
4. *$\frac{1}{2} ||\cdot||^2$ is locally uniformly convex.*

Proposition 2.14.4 (Uniformly Convex Spaces)**.** *Let X be a Banach space. The following are equivalent.*

1. *X is uniformly Convex.*
2. *For each $\epsilon > 0$, there is a $\delta > 0$ such that if $x, y \in \overline{B_X(0; 1)}$ and $||x - y|| \geq \epsilon$, then $||x + y|| \leq 2(1 - \delta)$.*
3. *If $\{x_i\}_{i \in \mathbb{N}}, \{y_i\}_{i \in \mathbb{N}} \subset \overline{B_X(0; 1)}$ and $||x_n + y_n|| \to 2$, then $x_n - y_n \to 0$.*
4. *$\frac{1}{2} ||\cdot||^2$ is uniformly convex.*

Proposition 2.14.5 (Weakly Locally Uniformly Convex Spaces)**.** *Let X be a Banach space. The following conditions are equivalent.*

1. *X is weakly locally uniformly convex.*

2. *For each $\epsilon > 0$, $^* \in \partial B_{X^*}(0;1)$, and $x \in \overline{B_X(0;1)}$ there is a $\delta > 0$ such that if $y \in \overline{B_X(0;1)}$ and $\langle x - y, x^* \rangle \geq \epsilon$, then $||x + y|| \leq 2(1 - \delta)$.*

3. *If $x \in \overline{B_X(0;1)}$ and $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ such that $||x + x_n|| \to 2$, then $x_n \xrightarrow{w} x$.*

Proposition 2.14.6 (Weakly Uniformly Convex Spaces)**.** *Let X be a Banach space. The following conditions are equivalent.*

1. *X is weakly uniformly convex.*

2. *For each $\epsilon > 0$ and $x^* \in \overline{B_{X^*}(0;1)}$, there is a $\delta > 0$ such that if $x, y \in \overline{B_X(0;1)}$ such that $\langle x - y, x^* \rangle \geq \epsilon$, then $||x + y|| \leq 2(1 - \delta)$.*

3. *If $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ and $||x_n + y_n|| \to 2$, then $x_n - y_n \xrightarrow{w} 0$.*

Proposition 2.14.7 (Degrees Of Convexity)**.** *Let X be a seminormed space.*

1. *If X is uniformly convex, then X is weakly uniformly convex.*

2. *If X is uniformly convex, then X is locally uniformly convex.*

3. *if X is locally uniformly convex, then X is weakly locally uniformly convex.*

4. *if X is weakly locally uniformly convex, then X is strictly convex.*

Proposition 2.14.8 (Local Weak To Strong)**.** *If a seminormed space X is locally uniformly convex and $\{x_n\} \subset X$ satisfies $x_n \xrightarrow{w} x$ and $||x_n|| \to ||x||$, then $x_n \to x$.*

Proof. □

Theorem 2.14.9. *(Milman Pettis) A complete uniformly convex seminormed space X is reflexive.*

2.15 Smoothness Of A Space

Define Moduli Of Smoothness, remark on positiveness

Definition 2.15.1. *Let X be a seminormed space and $x_0 \in X$.*

1. *We define the* **modulus of smoothness** of X at x_0, $\tilde{\rho} : [0, \infty) \times X \to \mathbb{R}$ by

$$\tilde{\rho}(\epsilon, x) = \frac{1}{2} \sup_{||y||<1} \left(||x + \epsilon y|| + ||x - \epsilon y|| - 2\,||x|| \right) \tag{2.113}$$

2. We define the **modulus of smoothness** of X, $\rho : [0, \infty) \to \mathbb{R}$ by

$$\rho(\epsilon) = \sup_{||x||\geq 1} \rho(\epsilon, x) \tag{2.114}$$

3. We say that X is **locally uniformly smooth** at x_0 if $\lim\limits_{\epsilon\to 0}\frac{\tilde{\rho}(\epsilon,x_0)}{\epsilon}=0$.

4. We say that X is **locally uniformly smooth** if it is locally uniformly smooth at each of its points.

5. We say that X is **uniformly smooth** if $\lim\limits_{\epsilon\to 0}\frac{\rho(\epsilon)}{\epsilon}=0$.

6. We say that X is **smooth** at x_0 if Jx_0 is a singleton.

7. We say that X is **smooth** if it is smooth at each of its points.

8. 2.15.2 motivates defining X to be called **very smooth** at x_0 if it is smooth and J is seminorm to weak continuous at x_0.

9. We say that X is **very smooth** if it is very smooth at each of its points.

Proposition 2.15.2 (Smooth Characterization)**.** *Let X be a seminormed space, $x_0 \in X$, and J it's normalized duality mapping. The following are equivalent.*

1. *X is a smooth at x_0.*

2. *Every selection of J is norm to weak* continuous at x_0.*

3. *There exists a selection of J which is norm to weak* continuous at x_0.*

4. *$||\cdot||$ is Gateaux Differentiable at x_0.*

5. *For every $y \in X$, there is a unique $\alpha \in \mathbb{C}$ such that $x \perp (\alpha x + y)$.*

6. *For every $y, z \in X$, if $x \perp y$ and $x \perp z$, then $x \perp y + z$.*

7. *(NEED TO DEFINE HYPERPLANE)There is a supporting hyperplane for $\overline{B_X(0;||x||)}$ at x.*

Proposition 2.15.3 (Degrees Of Smoothness)**.** *Let X be a seminormed space.*

1. *If X is uniformly smooth, then it is locally uniformly smooth.*

2. *If X is locally uniformly smooth, then it is very smooth.*

3. *If X is very smooth, then it is smooth.*

Proposition 2.15.4 (Very Smooth)**.**

Proposition 2.15.5 (Local Uniformly Smooth At A Point)**.** *Let X be a seminormed space, $x_0 \in X$, and J be X's normalized duality mapping. The following conditions are equivalent.*

1. *X is locally uniformly smooth at x_0.*

2. *J is continuous at x_0.*

3. *$||\cdot||$ is Frechet differentiable at x_0.*

Corollary 2.15.6 (Local Uniformly Smooth)**.** *Let X be a seminormed space and J be X's normalized duality mapping. The following conditions are equivalent.*

1. *X is locally uniformly smooth.*

2. *J is conitnuous.*

3. *$||\cdot||$ is Frechet differentiable.*

Proposition 2.15.7 (Uniformly Smooth)**.** *Let X be a seminormed space and J be X's normalized duality mapping. The following are equivalent.*

1. *X is uniformly smooth.*

2. *J is uniformly continuous on bounded subsets of X.*

3. *$||\cdot||$ is uniformly Frechet differentiable on bounded subsets of X.*

Chapter 3

Smoothness And Convexity

3.1 Convexity And Smoothness Of A Space

Proposition 3.1.1 (Smoothness and Strict Convexity). *Let X be a seminormed space and J be X's normalized duality mapping.*

1. *If X^* is smooth, then X is strictly convex.*
2. *If X^* is strictly convex, then X is smooth.*
3. *If X is reflexive, then X is strictly convex if and only if X^* is smooth.*
4. *If X is reflexive, then X is smooth if and only if X^* is strictly convex.*
5. *If X^* is strictly convex, then J is single valued and norm to weak $- *$ continuous.*
6. *X is smooth and strictly convex if and only if J is single valued and strictly monotone.*

Proposition 3.1.2 (Lindenstraus Duality Formula).

Proposition 3.1.3 (Very Smooth and Weak Local convexity).

Proposition 3.1.4 (Local Uniform Smoothness and Local Uniform Convexity). *Let X be a seminormed space and J be X's normalized duality mapping.*

Proposition 3.1.5 (Uniform Smoothness and Convexity). *Let X be a (possibly complete) seminormed space and J be X's normalized duality mapping.*

1. *X is uniformly convex if and only if X^* is uniformly smooth.*
2. *X is uniformly smooth if and only if X^* is uniformly convex.*
3. *X^* is uniformly convex if and only if J is single valued and uniformly continuous on bounded subsets of X.*

Corollary 3.1.6. *Uniformly Smooth Banach Spaces are Reflexive*

Proposition 3.1.7 (Normalized Convergence). *Let X be a smooth locally uniformly convex seminormed space with normalized duality mapping J. If $\{x_n\}_{n\in\mathbb{N}} \subset X$, $x \in X$, $j_x \in Jx$, and $j_n \in Jx_n$ for each $n \in \mathbb{N}$, then*

$$\langle x_n - x, j_n - j\rangle \to 0 \implies x_n \to x \tag{3.1}$$

3.2 Convexity, Smoothness, and High Order Duals

If you start with a poorly behaved Space, then things can only get worse.

Theorem 3.2.1. *If X is a seminormed space and X^* is very smooth, then X is reflexive.*

Corollary 3.2.2. *Let X be a seminormed space.*

1. *If X^*'s norm is Frechet differentiable, then X is reflexive.*
2. *If X^{**} is weakly locally uniformly convex, then X is reflexive.*
3. *If X^{***} is smooth, then X is reflexive.*
4. *If X^{****} is strictly convex, then X is reflexive.*

Chapter 4

Renorming Theory (Including Results about WCG Spaces)

4.1 Representations Of Reflexive Spaces

(Lindenstrauss' Theorem Goes Here

4.2 Local Uniform Convexafiability Of Reflexive Spaces

-Trojanski's Theorem Goes Here

Chapter 5

Convexity And Fixed Point Theory

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