Geometry of TVS's

Ryan McKinney

April 23, 2025

Contents

1 Introduction **1**

2 Prerequisite Knowledge **2**
- 2.1 Set Theory . . . 2
  - 2.1.1 Inclusion, Subset, Intersection, Union, Difference . . . 2
  - 2.1.2 Functions . . . 3
  - 2.1.3 Cardinality . . . 4
  - 2.1.4 Covers, Partitions . . . 5
  - 2.1.5 Infinite Cartesian Product . . . 5
  - 2.1.6 Relations and Orderings . . . 6
  - 2.1.7 Equivalence Relations . . . 9
  - 2.1.8 Nets . . . 10
- 2.2 Filters . . . 11
  - 2.2.1 Filter Basics . . . 11
  - 2.2.2 Filter Base . . . 14
  - 2.2.3 Ultrafilters . . . 16
  - 2.2.4 Induced Filters . . . 18
  - 2.2.5 Direct and Inverse Images of a Filter Base . . . 18
  - 2.2.6 Filter Products . . . 20
- 2.3 Topological Spaces . . . 23
  - 2.3.1 Open Sets, Closed Sets, and Neighborhoods . . . 23
  - 2.3.2 Continuous Functions . . . 28
  - 2.3.3 Subspaces And Quotient Spaces . . . 28
  - 2.3.4 Product Spaces . . . 32
  - 2.3.5 Open and Closed Maps . . . 32
  - 2.3.6 Convergence of Filters . . . 32
  - 2.3.7 Separation Axioms . . . 35
  - 2.3.8 Compactness . . . 36
  - 2.3.9 Countability Axioms . . . 36
- 2.4 Uniform Spaces . . . 36
  - 2.4.1 Uniform Topologies . . . 38
- 2.5 Algebraic Structures . . . 39
  - 2.5.1 Magma, Semigroup, Group . . . 39
  - 2.5.2 Vector Space . . . 41
- 2.6 Topological Algebra . . . 45

2.6.1 Topological Groups 45
2.6.2 Topological Vector Spaces 46
2.7 Seminormed Spaces 47
2.7.1 Pseudometrics 47
2.7.2 Quotients of Seminormed Spaces 55
2.7.3 Seminormed Hahn Banach Theorem 70
2.7.4 Seminorm Adjoints 71
2.7.5 Higher order Seminorm Duals 74
2.8 Classical Results With A Twist 77
2.8.1 Helly 77
2.8.2 Goldstine 80
2.8.3 Banach Alaoglu 81
2.8.4 Eberlein-Smulian 82
2.8.5 Bishop-Phelps 85
2.9 Reflexivity 87
2.9.1 James 88
2.9.2 Lindenstrauss On Nonseparable Reflexive Banach Spaces 96
2.10 Convexity Of Functions And Sets 97
2.11 Differentiation And SubDifferentials 100
2.12 Normalized Duality Mapping 102
2.13 Orthogonality 102
2.14 Convexity Of A Space 103
2.15 Smoothness Of A Space 105

3 Smoothness And Convexity 108
3.1 Convexity And Smoothness Of A Space 108
3.2 Convexity, Smoothness, and High Order Duals 109

4 Renorming Theory (Including Results about WCG Spaces) 110
4.1 Representations Of Reflexive Spaces 110
4.2 Local Uniform Convexafiability Of Reflexive Spaces 110

5 Convexity And Fixed Point Theory 111

List of Figures

List of Tables

Chapter 1

Introduction

Dank

Chapter 2

Prerequisite Knowledge

2.1 Set Theory

2.1.1 Inclusion, Subset, Intersection, Union, Difference

Definition 2.1.1 ($\in$)**.** *Let X and Y be sets. We use the notation $Y \in X$ to indicate that Y is an element of X. If Y is not an element of X then we write $Y \notin X$.*

Definition 2.1.2 (Subset)**.** Let X and Y be sets such that $x \in Y \implies x \in X$. Then we write $Y \subset X$ and we say that Y is a **Subset** of X and we write $X \supset Y$ and we say that X is a **Superset** of Y.

Definition 2.1.3 (Set Difference)**.** Let X and Y be sets. We define

$$X \setminus Y = \{x \in X | x \notin Y\}$$

We call $X \setminus Y$ the **Complement** of Y relative to X.

subsectionBinary Products

Definition 2.1.4 (Pair Set and Singleton)**.** Let X and Y. be sets. We then assume that the set Z containing exactly X and Y is a set which we call the **Pair Set** of X and Y. Since we did not assume $X \neq Y$, this implies that the set containing only X also exists. We refer to sets with only a single elemnt as **Singletons** .

Definition 2.1.5 (Ordered Pair)**.** Let X and Y be sets. Then we define $(X, Y) = \{\{X\}, \{X, Y\}\} \in 2^{\{X,Y\}}$. We call (X, Y) the **Ordered Pair** of X with Y.

Definition 2.1.6 (Binary Cartesian Product)**.** Let $X \neq \emptyset$ and let $Y \neq \emptyset$. We define $X \times Y = \{(x, y) \in 2^{2^{X \cup Y}} | x \in X \wedge y \in Y\}$ We call $X \times Y$ the **Binary Cartesian Product** of X with Y.

Definition 2.1.7 (Set Diagonal)**.** Let X be a set. We define $\Delta(X) = \{(x, x) \in X \times X | x \in X\}$ and we call $\Delta(X)$ the **Diagonal** of X.

2.1.2 Functions

Definition 2.1.8 (Relation)**.** Let $X \neq \emptyset$ be a set and let $Y \neq \emptyset$ be a set. We say that R is a **Relation** from X to Y if $R \subset X \times Y$. If $(a, b) \in R$, then we may write aRb.

Definition 2.1.9 (Function)**.** Let $X \neq \emptyset$ and $Y \neq \emptyset$. Let $f \subset X \times Y$ such that for each $x \in X$ there is a Unique $y \in Y$ such that $(x, y) \in f$. Then we say that f is a **Function** from X into Y. and we write $f : X \to Y$. We may also call f a **Map** or a **Mapping** from X into Y. Primarily, though we will rely on the notation $f : X \to Y$ to indicate that f is a **Function** with **Domain** X and **Codomain** Y. If $A \subset X$ and $B \subset Y$, then we denote

$$f(A) = \{f(x) \in Y | x \in A\} \quad f^{-1}(B) = \{x \in X | f(x) \in B\}$$

We call $f(A)$ the **Image** of A under f and we call $f^{-1}(B)$ the **Preimage** of B under f. we call $f(X)$ the **Range** of f. When the domain of a function is understood, we may also refer to an unnamed map f by writing,

$$x \to f(x)$$

Definition 2.1.10 (Insertion Function)**.** Let $A \subset B$ and define $f : A \to B$ by $f(x) = x$. The we call f the **Insertion Function** of A into B.

Definition 2.1.11 (Restriction)**.** Let X, Y be sets and let R be a **Relation** from X to Y. Let $A \subset X$. We define

$$(R)|_A = \{(x, y) \in R | x \in A\}$$

We call $(R)|_A$ the **Restriction** of the **Relation** R to the set A.

Definition 2.1.12 (Relation Inverse)**.** Let $X \neq \emptyset$ and $Y \neq \emptyset$. Let R be a **Relation** from X to Y. We define

$$R^{-1} = \{(y, x) \in Y \times X | (x, y) \in R\}$$

We call R^{-1} the **Inverse** of R.

Definition 2.1.13 (Extension)**.** Let X and Y be sets. Let $g : X \to Y$. Let f be a **Restriction** of g. Then we call g an **Extension** of f.

Definition 2.1.14 (Injective)**.** Let X, Y be sets and let $f : X \to Y$. We say that f is an **Injection** , or that f is **Injective** if for all $x, y \in X$, if $x \neq y$, then $f(x) \neq f(y)$.

Definition 2.1.15 (Surjective)**.** Let X, Y be sets and let $f : X \to Y$. Suppose that for each $y \in Y$, there exists an $x \in X$ such that $f(x) = y$. Then we say that f is a **Surjection** onto Y, and we call f **Surjective** onto Y. When Y is understood and the risk of misunderstanding is minimal, we may omit saying onto Y.

Definition 2.1.16 (Bijective)**.** Let X and Y be sets and let $f : X \to Y$ be **Surjective** and **Injective** . Then we say that f is **Bijective** , or we say that f is a **Bijection** .

2.1.3 Cardinality

Definition 2.1.17 (Cardinality)**.** Let $n \in \mathbb{N}$. We define

$$N_n = \{k \in \mathbb{N} | k \leq n\}$$

Let X be a set. Let $f : X \to N_n$ be a **Bijection** . Then, we say that X has **Cardinality** n and we write $\mathbf{Card}\,(\mathrm{X}) = \mathrm{n}$. More generaly, if there exists a **Bijection** between two sets Y and Z, then we write $\mathbf{Card}\,(\mathrm{Y}) = \mathbf{Card}\,(\mathrm{Z})$ and we say that they have the same **Cardinalities** . Define $X_0 = \mathbb{N}$ and for $k \in \mathbb{N}$, define $X_{k+1} = 2^{X_k}$. Then for $k \in \mathbb{N}$, we define $\aleph_k = \mathbf{Card}\,(\mathrm{X_k})$. If $\mathbf{Card}\,(\mathrm{X}) \in \mathbb{N}$, then we say that X is **Finite** . If $\mathbf{Card}\,(\mathrm{Z}) \in \mathbb{N}$ or $\mathbf{Card}\,(\mathrm{Z}) = \aleph_0$, then we say that Z is **Denumerable** . If $\mathbf{Card}\,(\mathrm{Y}) = \aleph_0$, then we say that Y is **Countable** . If $\mathbf{Card}\,(\mathrm{W}) = \alpha_{\mathrm{k}}$ for $k \geq 1$, then we say that W is **Uncountable** . If $\mathbf{Card}\,(\mathrm{V}) = \alpha_{\mathrm{j}}$ for $j \in \mathbb{N}$, then we say that V is **Infinite** .

Definition 2.1.18 (Closure Under Unions)**.** Let S be a set such that

$$\{S_\alpha | \alpha \in A\} \subset S \implies \bigcup_{\alpha \in A} S_\alpha \in S$$

for all index sets A. Then we say that S is **Closed Under Unions** or **Closed Under Aribtrary Unions** and that S posesses **Closure Under Unions** or **Closure Under Aribtrary Unions** . If this relation only holds when A is a **Countable** set then we say that S is **Closed Under Countable Unions** and that S posesses **Closure Under Countable Unions** . If this relation only holds when A is a **Finite** set then we say that S is **Closed Under Finite Unions** and that S posesses **Closure Under Finite Unions**

Definition 2.1.19 (Closure Under Intersections)**.** Let S be a set such that

$$\{S_\alpha | \alpha \in A\} \subset S \implies \bigcap_{\alpha \in A} S_\alpha \in S$$

for all index sets A. Then we say that S is **Closed Under Intersections** or **Closed Under Aribtrary Intersections** and that S posesses **Closure Under Intersections** or **Closure Under Aribtrary Intersections** . If this relation only holds when A is a **Countable** set then we say that S is **Closed Under Countable Intersections** and that S posesses **Closure Under Countable Intersections** . If this relation only holds when A is a **Finite** set then we say that S is **Closed Under Finite Intersections** and that S posesses **Closure Under Finite Intersections**

Proposition 2.1.20. *Let Let X be a set. The following are true.*

(i) *If X has the property that $\{y, z\} \subset X \implies y \cap z \in X$, then X posesses* **Closure Under Finite Intersections** .

(ii) If X has the property that $\{y, z\} \subset X \implies y \cup z \in X$, then X posesses **Closure Under Finite Unions** .

Proof of 2.1.20. i. We use induction. Let M be the set of natural numbers for which X is closed under intersections of n sets. The intersection of a single set equals that set, so $1 \in M$. $2 \in M$ by direct application of the assumption of 2.1.20. i. Let $m \in M$. Let $\{x_i\}_{i=1}^{m+1} \subset M$. Then

$$\bigcap_{i=1}^{m+1} x_i = \left(\bigcap_{i=1}^{m} x_i\right) \cap x_{m+1} \in X$$

so $m+1 \in M$. Hence $M = \mathbb{N}$ and 2.1.20. i is proven. □

Proof of 2.1.20. ii. We use induction. Let M be the set of natural numbers for which X is closed under unions of n sets. The union of a single set equals that set, so $1 \in M$. $2 \in M$ by direct application of the assumption of 2.1.20. ii. Let $m \in M$. Let $\{x_i\}_{i=1}^{m+1} \subset M$. Then

$$\bigcup_{i=1}^{m+1} x_i = \left(\bigcup_{i=1}^{m} x_i\right) \cup x_{m+1} \in X$$

so $m+1 \in M$. Hence $M = \mathbb{N}$ and 2.1.20. ii is proven. □

Definition 2.1.21 (**Nested**). *Let $F, G \neq \emptyset$. We say that* ***Nested***(F,G) *holds if, for each $g \in G$, there exists $f \in F$ such that $f \subset g$.*

2.1.4 Covers, Partitions

Definition 2.1.22 (Disjoint). Let X and Y be sets such that $X \cap Y = \emptyset$. Then we say that X and Y are **Disjoint** . Let $F = \{X_\alpha\}_{\alpha \in A}$ be a collection of sets such that for each $\alpha, \beta \in A$ with $\alpha \neq \beta$, we have X_α is **Disjoint** to X_β. Then we say that F is **Disjoint** .

Definition 2.1.23 (Cover, Subcover). Let X be a set and let $Y = \{Y_\alpha\}_{\alpha \in A}$ such that

$$X \subset \bigcup_{\alpha \in A} Y_\alpha$$

Then we say that Y is a **Cover** for X or that Y **Covers** X. In the context of talking about a **Cover** , if every member of a **Cover** posses a certain property then we may say that the **Cover** has that property. If $Z \subset Y$ **Covers** X, then we call Z a **Subcover** of Y. One exception to this is that when talking about the **Cardinality** or **Disjointedness** of a **Cover** , we are talking about **Cover** itself, not each of its constituent sets.

Definition 2.1.24 (Partition). Let X be a set and $Y \subset 2^X$ be a **Disjoint** **Cover** for X. Then we call Y a **Partition** for X.

2.1.5 Infinite Cartesian Product

Definition 2.1.25 (Infinite Cartesian Product). Let $A \neq \emptyset$. For each $\alpha \in A$, let $X_\alpha \neq \emptyset$. Define

$$\prod_{\alpha \in A} X_\alpha = \left\{ f : A \to \bigcup_{\alpha \in A} X_\alpha | (\forall \alpha \in A)(f(\alpha) \in X_\alpha) \right\}$$

We call this the **Cartesian Product** of $\{X_\alpha\}_{\alpha\in A}$. For each $\alpha \in A$, we define

$$\pi_\alpha : \prod_{\alpha\in A} X_\alpha \to X_\alpha \qquad\qquad \pi_\alpha(f) = f(\alpha)$$

We call π_α the $\alpha-$ **Projection Map** .

Definition 2.1.26 (**Diagonal**). *rm Let X be a set. Let $A \neq \emptyset$. We define*

$$\Delta_A(X) = \left\{ \{x\}_{\alpha\in A} \in \prod_{\alpha\in A} X | x \in X \right\}$$

*We call this the **Diagonal** of X with respect to A, or, when A is understood, the **Diagonal** of X.*

Definition 2.1.27 (Function Product)**.** *For $\alpha \in A$, let $f_\alpha : X_\alpha \to Y_\alpha$. Define*

$$f : \prod_{\alpha\in A} X_\alpha \to \prod_{\alpha\in A} Y_\alpha$$

by

$$f\left(\{x_\alpha\}_{\alpha\in A}\right) = \{f_\alpha(x_\alpha)\}_{\alpha\in A}$$

*Then we call f the **Function Product** of $\{f_\alpha\}_{\alpha\in A}$ and we denote $\prod_{\alpha\in A} f_\alpha := f$. or in the case where $\{X_\alpha\}_{\alpha\in A} = \{X_1, \cdots, X_n\}$, we may denote $f = f_1 \times f_2 \times \cdot \times f_n$.*

2.1.6 Relations and Orderings

Definition 2.1.28 (**Reflexive**). *Let $X \neq \emptyset$ be a set. Let R be a* **Relation** on X. We say that R is **Reflexive** with respect to X if, or equivalently we say that R posseses **Reflexivity** with respect to X if $\{(a,a)|a \in X\} \subset R$. When X is understood, we may simply say that R is **Reflexive** or that R posesses **Reflexivity** .

Definition 2.1.29 (**Transitive**). *Let $X \neq \emptyset$ be a set. Let R be a* **Relation** on X. We say that R is **Transitive** , or equivalently we say that R posseses **Transitivity** if whenever $(a,b) \in R$ and $(b,c) \in R$, we also have $(a,c) \in R$.

Definition 2.1.30 (**Preorder**). *Let $X \neq \emptyset$ be a set. Let R be a* **Relation** on X. If R is **Reflexive** and **Transitive** then we call R a **Preorder** on X, or we equivalently call R a **Preordering** of X and we call (X,R) a **Preordered Set** .

Definition 2.1.31 (**Comparable**). *Let (X,R) be a* **Preordered Set** . We say that $x, y \in X$ are **Comparable** and that they possess **Comparability** if xRy or yRx.

Definition 2.1.32 (**Symmetric**). *Let $X \neq \emptyset$ be a set. Let R be a* **Relation** on X. We say that R is **Symmetric** , or equivalently we say that R posseses **Symmetry** if whenever aRb, we also have bRa. This is equivalent to the condition $R = R^{-1}$.

Proposition 2.1.33. *Let $X \neq \emptyset$ and let R be a* **Relation** on X. The following are true.

(i) $R \cap R^{-1}$ is **Symmetric** .

(ii) $R \cup R^{-1}$ is **Symmetric** .

Proof of 2.1.33 i. If $R \cap R^{-1} = \emptyset$ then it is trivally **Symmetric** . Suppose $R \cap R^{-1} \neq \emptyset$ and let $(x, y) \in R \cap R^{-1}$. Then $(x, y) \in R$, implying by 2.1.6 that $(y, x) \in R$. Also this implies $(x, y) \in R^{-1}$ so by 2.1.6 we have $(y, x) \in (R^{-1})^{-1} = R$. Hence $(y, x) \in R \cap R^{-1}$, so **Symmetry** is verified. □

Proof of 2.1.33 ii. Let $(x, y) \in R \cup R^{-1}$. Then either $(x, y) \in R$ or $(x, y) \in R^{-1}$. In the former case, $(y, x) \in R^{-1} \subset R \cup R^{-1}$. In the latter case, $(y, x) \in R \subset R \cup R^{-1}$. Hence in either case $(y, x) \in R \cup R^{-1}$ and **Symmetry** is verified. □

Definition 2.1.34 (**Anti-Symmetric**)**.** *Let $X \neq \emptyset$ be a set. Let R be a* **Relation** on X. We say that R is **Anti-Symmetric** , or equivalently we say that R posseses **Anti-Symmetry** if whenever aRb and bRa, we must have $a = b$.

Definition 2.1.35 (**Maximal Element**)**.** *Let $X \neq \emptyset$ be a set. Let R be an* **Relation** on X. Let $Y \subset X$. Let $a \in Y$. We say that a is a **Maximal Element** of Y, or equivalently we say that a is a **Maximum** of Y if for every $b \in Y$, if aRb, then $a = b$. The Plural of **Maximum** is a **Maxima** , and we represent the set of **Maxima** of Y with respect to the relation R with $\mathbf{Maxima}_{\mathrm{R}}(\mathrm{Y})$, or if R is understood, we represent the set of **Maxima** of Y with **Maxima**(Y).

Proposition 2.1.36 (**Maximal Element** unique if R is **Anti-Symmetric**)**.** *Let $X \neq \emptyset$ be a set. Let R be an* **Anti-Symmetric Relation** on X. Let $Y \subset X$. Let a and b be each be a **Maximal Element** of Y. Then $a = b$.

Proof. Since $a \in$ **Maxima**(Y), $b \leq a$. Since $b \in$ **Maxima**(Y), $a \leq b$. By **Anti-Symmetry** , $b = a$. □

Definition 2.1.37 (**Minimal Element**)**.** *Let $X \neq \emptyset$ be a set. Let R be an* **Relation** on X. Let $Y \subset X$. Let $a \in Y$. We say that a is a **Minimal Element** of Y, or equivalently we say that a is a **Minimum** of Y if for every $b \in Y$, if bRa, then we have $a = b$. The Plural of **Minimum** is **Minima** , and we represent the set of **Minima** of Y with respect to the relation R with $\mathbf{Minima}_{\mathrm{R}}(\mathrm{Y})$, or if R is understood, we represent the set of **Minima** of Y with **Minima**(Y).

Proposition 2.1.38 (**Minimal Element** unique if R is **Anti-Symmetric**)**.** *Let $X \neq \emptyset$ be a set. Let R be an* **Anti-Symmetric Relation** on X. Let $Y \subset X$. Let a and b be each be a **Minimal Element** of Y. Then $a = b$.

Proof. Since $a \in$ **Minima**(Y), $a \leq b$. Since $b \in$ **Minima**(Y), $b \leq a$. By **Anti-Symmetry** , $b = a$. □

Definition 2.1.39 (**Upper Bound**)**.** *Let $X \neq \emptyset$ be a set. Let R be a* **Relation** on X. Let $Y \subset X$. Let $a \in X$. We say that a is an **Upper Bound** for Y if for every $x \in Y$, we have xRa. If a is an **Upper Bound** then we also say that the set Y is **Bounded From Above** by a. We denote the set of **Upper Bounds** of Y with respect to the relation R with $\mathbf{UpperBound}_{\mathrm{R}}(\mathrm{Y})$. When R is understood, we denote this set with **UpperBound**(Y).

Definition 2.1.40 (**Lower Bound**)**.** *Let $X \neq \emptyset$ be a set. Let R be a* **Relation** on X. Let $Y \subset X$. Let $a \in X$. We say that a is an **Lower Bound** for Y if for every $x \in Y$, we have aRx. If a is an **Lower Bound** then we also say that the set Y is **Bounded From Below** by a. We denote the set of **Lower Bounds** of Y with respect to the relation R with $\mathbf{LowerBound}_{\mathrm{R}}(\mathrm{Y})$. When R is understood, we denote this set with **LowerBound**(Y).

Definition 2.1.41 (**Least Upper Bound**)**.** *Let $X \neq \emptyset$ be a set. Let R be a* **Relation** on X. Let $Y \subset X$. Let $a \in X$. We say that a is a **Least Upper Bound** of Y if $a \in$ **Minima**(**UpperBound**(Y)). We denote the set of **Least Upper Bounds** for Y with **LUB**(Y). If $b \in$ **LUB**(Y), then we also call b a **Supremum** of Y. The Plural of **Supremum** is **Suprema** . If **LUB**(Y) = {c}, then we write $c =$ **Sup**(Y).

Definition 2.1.42 (**Greatest Lower Bound**)**.** *Let $X \neq \emptyset$ be a set. Let R be a* **Relation** on X. Let $Y \subset X$. Let $a \in X$. We say that a is a **Greatest Lower Bound** of Y if $a \in$ **Maxima**(**LowerBound**(Y)). We denote the set of **Greatest Lower Bounds** for Y with **GLB**(Y). If $b \in$ **GLB**(Y), then we also call b a **Infimum** of Y. The Plural of **Infimum** is **Infima** . If **GLB**(Y) = {c}, then we write $c =$ **Inf**(Y).

Definition 2.1.43 (**Equivalence Relation**)**.** *Let $X \neq \emptyset$ be a set. Let $\cong$ be a* **Preorder** on X. We say that $\cong$ is an **Equivalence Relation** on X if it is **Symmetric** .

Definition 2.1.44 (**Partial Order**)**.** *Let $X \neq \emptyset$ be a set. Let $\leq$ be a* **Preorder** on X. We say that $\leq$ is a **Partial Order** on X and we say that $\leq$ is a **Partial Ordering** of X if $\leq$ is **Anti-Symmetric** . Let $\leq$ is a **Partial Order** on X, the we refer to the pair $(X, \leq)$ as a **Partially Ordered Set** .

Definition 2.1.45 (**Total Order**)**.** *Let (X, R) be a* **Partially Ordered Set** in which every pair of elements is **Comparable** . Then we call R a **Total Order** on X and we call (X, R) a **Totally Ordered Set** .

Definition 2.1.46 (**Chain**)**.** *Let $(X, \leq)$ be a* **Partially Ordered Set** . Let $A \subset X$ such that $(A, \leq \cap (A \times A))$ is a **Totally Ordered Set** . Then we call A a **Chain** in X.

Definition 2.1.47 (**Direction**)**.** *Let $X \neq \emptyset$ be a set. Let $\leq$ be a* **Preorder** on X. If every pair of elements in X has an **Upper Bound** with respect to $\leq$, then we call $\leq$ is a **Direction** on X, , we call $\leq$ is a **Directing** of X , and we call $(X, \leq)$ is a **Directed Set** .

Definition 2.1.48 (**Section** of a **Directed Set**). *Let* $(X, \leq)$ *be a* **Directed Set** . Let $x \in X$. We define

$$S(x, \leq) = \{y \in X | x \leq y\} \tag{2.1}$$

We call $S(x, \leq)$ the **Section** of $\leq$ corresponding to $x \in X$.

Definition 2.1.49 (**Lattice** , **Join** , **Meet**). *Let* $(X, \leq)$ *be a* **Partially Ordered Set** such that, for every $x, y \in X$, the set $\{x, y\}$ has both a **Supremum** and an **Infimum** . Then we call $(X, \leq)$ **Lattice** . Furthermore, we call **Sup**{x, y} the **Join** of x and y and we call **Inf**{x, y} the **Meet** of x and y. If every nonempty subset of X has both a **Supremum** and **Infimum** then we call $(X, \leq)$ a **Complete Lattice** .

Definition 2.1.50 (**Sequence**). *Let X be a set. A* ***Sequence*** *in X is a* ***Function*** *$f : \mathbb{N} \to X$. If f is a* ***Sequence*** *in X and $f(n) = x_n$ for $n \in \mathbb{N}$, then we may refer to $\{x_n\}_{n\in\mathbb{N}}$ as the* ***Sequence*** *itself.*

2.1.7 Equivalence Relations

Definition 2.1.51 (Equivalence Class). *Let $X \neq \emptyset$. Let $\cong$ be an* **Equivalence Relation** defined on X. Let $x \in X$. We define the set $[x]_\cong$ by

$$[x]_\cong = \{y \in X | y \cong x\} \tag{2.2}$$

We call $[x]_\cong$ the **Equivalence Class** of x in $(X, \cong)$.

Proposition 2.1.52 (Equivalence Classes Partition). *Let $X \neq \emptyset$. Let $\cong$ be an* **Equivalence Relation** defined on X. Let $x, y \in X$. The following statements are equivalent.

1. $[x]_\cong \cap [y]_\cong \neq \emptyset$
2. $x \cong y$
3. $[x]_\cong = [y]_\cong$
4. $[x]_\cong \subset [y]_\cong$
5. $[y]_\cong \subset [x]_\cong$

Proof That $1 \implies 2$. Suppose $M := [x]_\cong \cap [y]_\cong \neq \emptyset$. Then there exists $z \in M$. Then $z \cong x$, so by **Symmetry** , $x \cong z$. But by **Transitivity** , pair with $z \cong y$, we conclude $x \cong y$. □

Proof That $2 \implies 4$. Let $x \cong y$ and let $z \in [x]_\cong$. Then $z \cong x \cong y$, so $z \cong y$ and $z \in [y]_\cong$. Since z was arbitrary, we're done. □

Proof That $2 \implies 5$. Let $x \cong y$. By **Symmetry** , $y \cong x$, so by $(2 \implies 4)$, we are done. □

Proof That $2 \implies 3$. Since $2 \implies 4$ and $2 \implies 5$ and 5 and 4 together imply 3, we have this. □

Proof That 5 $\implies$ 1. Let $[y]_{\cong} \subset [x]_{\cong}$. Then $y \in [y]_{\cong} = [y]_{\cong} \cap [x]_{\cong}$. Hence 1 holds. □

Definition 2.1.53 (Quotient Set)**.** *Let $X \neq \emptyset$. Let $\cong$ be an* **Equivalence Relation** defined on X. We define the set $X/\cong$ by

$$X/\cong = \{[x]_{\cong} : x \in X\} \tag{2.3}$$

We call $X/\cong$ the **Quotient Set** of X under the relation $\cong$.

Remark 2.1.54 (**Quotient Set** forms a **Partition**)**.** *2.1.52 , paired with the fact that $x \in [x]_{\cong}$, implies that $X/\cong$ is a* **Partition** of X.

Definition 2.1.55 (Quotient Map)**.** *Let $X \neq \emptyset$. Let $\cong$ be an* **Equivalence Relation** on X. Let X / $\cong$ be the **Quotient Set** of X with respect to the relation $\cong$. Define $T : X \to X/\cong$ by setting, for each $x \in X$,

$$T(x) = [x] \tag{2.4}$$

We call T the **Quotient Map** of X under $\cong$.

k

Proposition 2.1.56 (**Quotient Map** is **Surjective**)**.** *Let $X \neq \emptyset$. Let $\cong$ be an* **Equivalence Relation** on X. Let $T : X \to X/\cong$ be the **Quotient Map** of X under the **Relation** $\cong$. Then T is a **Surjection** .

Proof. Let $K \in Z/\cong$. Then for some $x \in Z$, $K = [x]$. Then $T(x) = K$. Since K was arbitrary, we are done. □

2.1.8 Nets

Definition 2.1.57 (**Net**)**.** *A **Net** is a **Function** mapping from a directed set $(A, \leq)$ into another set X. If $f : A \to X$ is a **Net** such that for $\alpha \in A$ we have $f(\alpha) = x_\alpha$, then we may use the notation $\{x_\alpha\}_{\alpha \in A} \subset X$.*

Definition 2.1.58 (**Section** of a **Net**)**.** *Let $X \neq \emptyset$, let $(A, \leq)$ be a* **Directed Set** and let $\sigma = \{x_\alpha\}_{\alpha \in A}$ be a **Net** in X. Let $\gamma \in A$. Let $S(\gamma, \leq)$ be the **Section** of $\leq$ corresponding to γ. We define

$$\{x_\alpha | \alpha \in S(\gamma, \leq)\} \tag{2.5}$$

the **Section** of x_γ in σ.

Proposition 2.1.59 (Net Section)**.** *Let $X \neq \emptyset$ and let $\{x_\alpha\}_{\alpha \in A}$ be a **Net** in X. Let $\beta \leq \gamma \in A$. For $\alpha \in A$, let $S(\alpha)$ denote the **Section** of x_α in $\{x_\alpha\}_{\alpha \in A}$. Then $S(\gamma) \subset S(\beta)$.*

Proof. Let $y \in S(\gamma)$. Then $y = x_\tau$ for some $\beta \leq \gamma \leq \tau$. Hence, $y = x_\tau \in S(\beta)$. □

Definition 2.1.60 (**Inductive Order**). *Let* $(X, \leq)$ *be a* **Partially Ordered Set** . We say that $\leq$ is an **Inductive Order** on X and we say that X is **Inductively Ordered** by $\leq$ if each **Chain** in X has an **Upper Bound** in X. we also call $(X, \leq)$ an **Inductively Ordered** set in this circumstance.

Theorem 2.1.61 (Zorns Lemma). *An* ***Inductively Ordered*** *set has a* **Maximum**

Remark 2.1.62. *2.1.61 Is equivalent to the axiom of choice*

2.2 Filters

2.2.1 Filter Basics

Definition 2.2.1 (**Filter**). *Let* $X \neq \emptyset$. *Let* $\mathcal{F} \subset 2^X$ *satisfy the following.*

(i) $\mathcal{F} \neq \emptyset$.

(ii) $\emptyset \notin \mathcal{F}$

(iii) If $G_1 \in \mathcal{F}$ *and* $G_1 \subset G_2 \subset X$, *then* $G_2 \in \mathcal{F}$.

(iv) If $\{G_1, G_2\} \subset \mathcal{F}$, *then* $G_1 \cap G_2 \in \mathcal{F}$.

Then we call $\mathcal{F}$ *a* ***Filter*** *on* X.

Proposition 2.2.2. *Let* $X \neq \emptyset$ *and let* $\mathcal{F}$ *be a* ***Filter*** *on* X. *Let* $B \subset X$. *The following are true.*

(i) $X \in \mathcal{F}$.

(ii) $\mathcal{F}$ *is* **Closed Under Finite Intersections**

(iii) The intersection of a collection of **Filters** on X is a **Filter** on X.

(iv) $\mathcal{F}_B$ defined by $\{U \cap B | U \in \mathcal{F}\}$ is a **Filter** on B if and only if $\emptyset \notin \mathcal{F}_B$.

Proof of 2.2.2 i. By 2.2.1. i, $\exists B \neq \emptyset \in \mathcal{F}$. Since $B \subset X \subset X$, by 2.2.1. iii, $X \in \mathcal{F}$, so 2.2.2 i is proven. □

Proof of 2.2.2 ii. Direct application of 2.2.1. iv paired with 2.1.20. ii. □

Proof of 2.2.2 iii. Let $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ be a collection of **Filters** on X. Define $\mathcal{F} = \bigcap\limits_{\alpha \in A} \mathcal{F}_\alpha$. By 2.2.2 i, for each $\alpha \in A$, $X \in \mathcal{F}_\alpha$, so $X \in \mathcal{F}$. Hence $\mathcal{F}$ satisfies 2.2.1. *i*. Furthermore, by 2.2.1. *ii*, for each $\alpha \in A$, $\emptyset \notin \mathcal{F}_\alpha$, so $\emptyset \notin \mathcal{F}$. Therefore $\mathcal{F}$ satisfies 2.2.1. ii. Since $G_1 \in \mathcal{F}$ and $G_1 \subset G_2 \subset X$. Then for each $\alpha \in A$, $G_1 \in \mathcal{F}_\alpha$, so by 2.2.1. iii, $G_2 \in \mathcal{F}_\alpha$. Hence $G_2 \in \mathcal{F}$, so $\mathcal{F}$ satisfies 2.2.1. iii. Finally, let $\{G_1, G_2\} \subset \mathcal{F}$. Then for each $\alpha \in A$, $\{G_1, G_2\} \subset \mathcal{F}_\alpha$, implying by 2.2.1. *iv* that $G_1 \cap G_2 \in \mathcal{F}_\alpha$, so $G_1 \cap G_2 \in \mathcal{F}$, implying $\mathcal{F}$ satisfies 2.2.1. *iv*. This concludes the proof of this result. □

Proof of 2.2.2 iv. Since $\mathcal{F}$ satisfies 2.2.1. i, no matter what $\emptyset \neq \mathcal{F}_B$, so $\mathcal{B}$ satisfies 2.2.1. *i*. If $G_1 \in \mathcal{F}_B$, then there is an $H_1 \in \mathcal{F}$ with $G_1 = H_1 \cap B$. If $G_1 \subset G_2 \subset B$, then $H_1 \cap B \subset G_2$ and $H_1 \cap (X \setminus B) \subset (X \setminus B)$, so $H_1 \subset G_2 \cup (X \setminus B)$, which implies $G_2 \cup (X \setminus B) \in \mathcal{F}$ by 2.2.1. iii. By construction, then, $G_2 = (G_2 \cup (X \setminus B)) \cap B \in \mathcal{F}_B$. Hence $\mathcal{F}_B$ satisfies 2.2.1. *iii*, no matter what. Next, if $\{G_1, G_2\} \subset \mathcal{F}_B$, then there are $H_1, H_2 \in \mathcal{F}$ with $G_i = H_i \cap B$. Since $\mathcal{F}$ satisfies 2.2.1. *iv*, $H_1 \cap H_2 \in \mathcal{F}$. This implies by construction that $B \cap (H_1 \cap H_2) \in \mathcal{F}_B$, but $B \cap (H_1 \cap H_2) = G_1 \cap G_2$, so $\mathcal{F}_B$ satisfies 2.2.1. iv. Finally, $\mathcal{F}_B$ satisfies 2.2.1. *ii* if and only if it satisfies that same assumption, concluding the proof of this result. □

Definition 2.2.3 (**Coarser** , **Finer**)**.** *Let X be a set and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be* ***Filters*** *on X such that $\mathcal{F}_1 \subset \mathcal{F}_2$. Then we say that $\mathcal{F}_1$ is* ***Coarser*** *than $\mathcal{F}_2$ and we say that $\mathcal{F}_2$ is* ***Finer*** *than $\mathcal{F}_1$. Let $A \subset X$ be a collection of filters and let $\mathcal{F}_2 \in A$ be* ***Finer*** *than every element of A. Then we say that $\mathcal{F}_2$ is the* ***Finest*** *element of A. Let $\mathcal{F}_3 \in A$ be* ***Coarser*** *than every element of A. Then we say that $\mathcal{F}_3$ is the* ***Coarsest*** *in A.* ***Filter Fineness*** *defines a* **Partial Ordering** on the collection of **Filters** on X, where $\mathcal{F}_1 \leq \mathcal{F}_2$ if $\mathcal{F}_2$ is a **Finer** than $\mathcal{F}_1$. A **Maximum** of **Filter Fineness** is called an **Ultrafilter** on X.

Proposition 2.2.4. *Let X be a set and $\emptyset \neq A \subset 2^X$. Then there is a* ***Filter*** *on X which contains A if and only if any* **Finite** intersection of elements of A is nonempty. Furthermore, if there is a **Filter** containing A, then there is a **Coarsest** containing A, and it is given by all sets containing some **Finite** intersection of elements of A.

Proof. The given condition is necessary by a combination of 2.2.2 i and 2.2.1. ii. For sufficiency, let K be the collection of finite intersections of elements of A. Define

$$\mathcal{K} = \{F \cup Y | F \in K \wedge K \subset X\} \tag{2.6}$$

Then $A \subset \mathcal{K}$. Since $A \neq \emptyset$, $\mathcal{K} \neq \emptyset$, so $\mathcal{K}$ satisfies 2.2.1. i. Since **Finite** intersections of elements of A are nonempty, $\emptyset \notin K$, implying $\emptyset \notin \mathcal{K}$, so 2.2.1. *ii*. Now, let $P \in \mathcal{K}$ and let $P \subset Q \subset X$. Then there exists $L_P \in K$ and $Y_P \subset X$ such that $P = L_P \cup Y_P$, and $Q = L_P \cup (Y_P \cup Q) \in \mathcal{K}$, so 2.2.1. iii holds for $\mathcal{K}$. Finally, let $G_1, G_2 \in \mathcal{K}$. Then $G_i = U_i \cup P_i$ for $U_i \in K$ and $P_i \subset X$. By definition of $\mathcal{K}$, there are a $\{G_1^j\}_{j=1}^{n_1} \subset A$ and $\{G_2^j\}_{j=1}^{n_2} \subset A$ with $U_i = \bigcap_{j=1}^{n_i} G_i^j$. Clearly $U_1 \cap U_2$, being a finite subset of elemnts of A, is an element of $\mathcal{K}$. Furthermore,

$$\begin{aligned} U_1 \cap U_2 &\subset (U_1 \cap U_2) \cup ((U_1 \cap P_2) \cup (U_2 \cap P_1) \cup (P_1 \cap P_2)) \\ &= (U_1 \cup P_1) \cap (U_2 \cup P_2) \\ &= G_1 \cap G_2 \end{aligned}$$

so since $\mathcal{K}$ satisfies 2.2.1. *iii*, $G_1 \cap G_2 \in \mathcal{K}$, so 2.2.1. *iv* applies for $\mathcal{K}$, so $\mathcal{K}$ is a **Filter** . By 2.2.2 ii, any **Filter** containing A would contain K. By 2.2.1. *iii*, any **Filter** containing A would contain $\mathcal{K}$, so any **Filter** on X containing A would be **Finer** than $\mathcal{K}$, and so $\mathcal{K}$ is the **Finest** **Filter** containing A. Hence, 2.2.4 is proven. □

Proposition 2.2.5 (Filter Order Facts)**.** *Let $X \neq \emptyset$. The following are true.*

(i) *Let $\mathcal{F}$ be a* ***Filter*** *on X and let $A \subset X$. Then there is a* ***Filter*** *containing A on X which is* ***Finer*** *than $\mathcal{F}$ if and only if $A \cap U \neq \emptyset$ for each $U \in \mathcal{F}$.*

(ii) *Let $K = \{\mathcal{F}_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of* ***Filters*** *on X. There exists a* ***Filter*** *on X which is* ***Finer*** *than each $\mathcal{F}_\alpha$ if and only if, for every* **Finite** subset $\{\mathcal{F}_{\alpha_i}\}_{i=1}^n \subset \mathcal{K}$, for each $\{U_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{F}_{\alpha_i}$, $\bigcap_{i=1}^n U_i \neq \emptyset$.

(iii) The union of a **Chain** of **Filters** on X is a **Filter** on X which is **Finer** than each element of the **Chain** .

(iv) The intersection of a **Chain** of **Filters** on X is a **Filter** on X which is **Coarser** than each element of the **Chain** .

(v) **Filter Fineness** is an **Inductive Order** on the collection of **Filters** on X.

(vi) **Filter Coarseness** is an **Inductive Order** on the collection of **Filters** on X.

Proof of 2.2.5. i. By 2.2.2 ii the collection of **Finite** intersections of elements of $\mathcal{K} := \{A\} \cup \mathcal{F}$ is given by

$$\mathcal{L} := \mathcal{F} \cup \{A \cap F | F \in \mathcal{F}\}$$

By 2.2.1. ii, $\emptyset \notin \mathcal{F}$. Hence, $\emptyset \notin \mathcal{L}$ if and only if $\emptyset \notin \{A \cap F | F \in \mathcal{F}\}$ By 2.2.4, $\emptyset \notin \mathcal{L}$ is equivalent to there existing a **Filter** on X containing $\mathcal{K}$, and so there existing a **Filter** on X containing $\mathcal{K}$ is equivalent to $\emptyset \notin \{A \cap F | F \in \mathcal{F}\}$. ☐

Proof of 2.2.5. ii. By 2.2.2 ii, each $\mathcal{F}_\alpha$ **Closed Under Finite Intersections** . Therefore, the collection of all **Finite** intersections of elements of

$$\bigcup_{\alpha \in A} \mathcal{F}_\alpha$$

is given by the collection intersections single element each of a **Finite** collection of $\mathcal{F}_\alpha's$. An application of 2.2.4 finishes the result. ☐

Proof of 2.2.5. iii. Let $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ be a **Chain** of **Filters** on X and let

$$\mathcal{F} = \bigcup_{\alpha \in A} \mathcal{F}_\alpha$$

Since each $\mathcal{F}_\alpha \neq \emptyset$, $\mathcal{F} \neq \emptyset$, so $\mathcal{F}$ satisfies 2.2.1. i. Furthermore, $\emptyset \notin \mathcal{F}_\alpha$ for any $\alpha \in A$, so $\mathcal{F}$ satisfies 2.2.1. ii Let $G \in \mathcal{F}$ and let $G \subset H \subset X$. Then for some $\alpha \in A$, $G \in \mathcal{F}_\alpha$. Since $\mathcal{F}_\alpha$ is a **Filter** , by 2.2.1. iii, $H \in \mathcal{F}_\alpha \subset \mathcal{F}$, so $H \in \mathcal{F}$. Hence $\mathcal{F}$ satisfies 2.2.1. *iii* Finally, let $G_1, G_2 \in \mathcal{F}$. Then there are $\alpha_1, \alpha_2 \in A$ such that $G_i \in \mathcal{F}_{\alpha_i}$ for $i \in \{1, 2\}$. Either $\mathcal{F}_{\alpha_1} \subset \mathcal{F}_{\alpha_2}$ or $\mathcal{F}_{\alpha_2} \subset \mathcal{F}_{\alpha_1}$, so without loss of generality, let $\mathcal{F}_{\alpha_1} \subset \mathcal{F}_{\alpha_2}$. Then $G_1 \in \mathcal{F}_{\alpha_2}$, implying $G_1 \cap G_2 \in \mathcal{F}_{\alpha_2} \subset \mathcal{F}$, so $G_1 \cap G_2 \in \mathcal{F}$, so $\mathcal{F}$satisfies 2.2.1. iv Hence $\mathcal{F}$ is a **Filter** on X which is **Finer** than each $\mathcal{F}_\alpha$. ☐

Proof of 2.2.5. iv. Define

$$\mathcal{F} = \bigcap_{\alpha \in A} \mathcal{F}_\alpha$$

$\emptyset \notin \mathcal{F}_\alpha$ for any $\alpha \in A$, so $\emptyset \notin \mathcal{F}$, implying $\mathcal{F}$ satisfies 2.2.1. *ii*. Furthermore, $X \in \mathcal{F}_\alpha$ for each $\alpha \in A$, so $X \in \mathcal{F}$, implying $\mathcal{F}$ satisfies 2.2.1. *i*. If $G \in \mathcal{F}$ and $G \subset H \subset X$, then for every $\alpha \in A$, $G \in \mathcal{F}_\alpha$. This implies, that by 2.2.1. iii, that for every $\alpha \in A$, $H \in \mathcal{F}_\alpha$. Hence $H \in \mathcal{F}$, so $\mathcal{F}$ satisfies 2.2.1. iii. Finally, let $G_1, G_2 \in \mathcal{F}$. Then $G_1, G_2 \in \mathcal{F}_\alpha$ for every $\alpha \in A$, so for every $\alpha \in A$, by 2.2.1. iv, $G_1 \cap G_2 \in \mathcal{F}_\alpha$ for every $\alpha \in A$, hence $G_1 \cap G_2 \in \mathcal{F}$. Hence, $\mathcal{F}$ is a **Filter** on X. Also, $\mathcal{F} \subset \mathcal{F}_\alpha$ for every $\alpha \in A$, and so is **Coarser** than every $\mathcal{F}_\alpha$. □

Proof of 2.2.5. v. Direct application of 2.2.5. iii to the collection of all **Chains** of **Filters** in X. □

Proof of 2.2.5. vi. Direct application of 2.2.5. iv to the collection of all **Chains** of **Filters** on X. □

2.2.2 Filter Base

Definition 2.2.6 (**Subbasis**). *Let X be a set and $A \subset X$ such that*

(i) $\emptyset \neq A$.

(ii)

$$\emptyset \notin K := \left\{ \bigcap_{i=1}^{n} A_i | \{A_i\}_{i=1}^{n} \subset A \wedge n \in \mathbb{N} \right\} \tag{2.7}$$

Define

$$\mathcal{K} = \{U \cup P | U \in K \wedge P \subset X\}$$

We call $\mathcal{K}$ the ***Filter*** *on X* ***Generated By*** *A. and we call A a* ***Subbasis*** *for G. By 2.2.4, $\mathcal{K}$ is in fact a* ***Filter*** *, and is the* ***Coarsest Filter*** *on X containing A.*

Definition 2.2.7 (**Filter Base**). *Let $X \neq \emptyset$. Let $\mathcal{B} \subset 2^X$ such that*

(i) $\emptyset \neq \mathcal{B}$.

(ii) $\emptyset \notin \mathcal{B}$.

(iii) Define $\mathcal{B}_{Intersection} = \{U \cap V | \{U, V\} \subset \mathcal{B}\}$. *Then* ***Nested***$(\mathcal{B}, \mathcal{B}_{Intersection})$ *holds.*

Then we call $\mathcal{B}$ a ***Filter Base*** *on X. By 2.2.8, the* ***Filter Generated By*** *a* ***Filter Base*** *A is given by $\{U \subset X | (\exists Y \subset A)(Y \subset U)\}$.*

If A, B are ***Filter Bases*** *on X and they* ***Generate*** *the same* ***Filter*** *, then we call them* ***Equivalent*** *.*

Proposition 2.2.8. *Let X be a set and let $A \subset 2^X$. and define $\mathcal{U} = \{U \subset X | (\exists a \in A)(a \subset U)\}$. The following are equivalent.*

1. *A is a* ***Filter Base*** *on X.*

2. $\mathcal{U}$ *is a* ***Filter*** *on* X*.*

$\Longrightarrow$. Supose A is a **Filter Base** on X. By 2.2.7. *ii*, $\emptyset \notin A$, so $\emptyset \notin \mathcal{U}$, implying that $\mathcal{U}$ satisfies 2.2.1. ii. Also, by 2.2.7. i $\emptyset \neq A \subset \mathcal{U}$, so $\mathcal{U}$ satisfies 2.2.1. i. That $\mathcal{U}$ satisfies 2.2.1. iii is obvious. Finally, if $G_1, G_2 \in \mathcal{U}$, then there exists $U_1, U_2 \in A$ such that $A_i \subset G_i$. By 2.2.7. iii, there is $B \in A$ satisfying $B \subset U_1 \cap U_2 \subset G_1 \cap G_2$, so $G_1 \cap G_2 \in \mathcal{U}$, implying $\mathcal{U}$ is a **Filter** on X. ☐

$\Longleftarrow$. If $A = \emptyset$, then $\mathcal{U} = \emptyset$, so A failing 2.2.7. *i* implies $\mathcal{U}$ fails 2.2.1. i. If $\emptyset \in A$, then $\emptyset \in \mathcal{U}$, so A failing 2.2.7. *ii* implies $\mathcal{U}$ fails 2.2.1. ii. Finally, if $\mathcal{A}$ fails 2.2.7. iii, then we can find $B, C \in A$ such that $B \cap C \notin \mathcal{U}$, implying $\mathcal{U}$ fails 2.2.1. *iv*. Hence necessity has been proven. ☐

Remark 2.2.9. *If* A *is a* ***Filter Base*** *on* X*, then* $\mathcal{U}$ *defined in* 2.2.8 *is the* ***Generated By*** A*.*

Proposition 2.2.10 (FilterBaseFacts)**.** *Let* $X \neq \emptyset$*. Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be* ***Filters*** *on* X*. Let* F *be a* ***Filter Base*** *for* $\mathcal{F}$ *and let* G *be a* ***Filter Base*** *for* $\mathcal{G}$*. The following are true.*

(i) The collection of **Finite** intersections of a **Subbasis** A for $\mathcal{F}$ forms a **Filter Base** for $\mathcal{F}$.

(ii) $B \subset \mathcal{F}$ is a **Filter Base** for $\mathcal{F}$ if and only if **Nested**$(B, \mathcal{F})$ holds.

(iii) $\mathcal{F}$ is **Finer** than $\mathcal{G}$ if and only if **Nested**(F, G) holds.

(iv) F is **Equivalent** to G if and only if **Nested**(F, G) and **Nested**(G, F) both hold.

(v) $\mathcal{F}$ is a **Filter Base** for $\mathcal{F}$.

Proof of 2.2.10. i. Let

$$\mathcal{B} = \left\{ \bigcap_{i=1}^{n} A_i | \{A_i\}_{i=1}^{n} \subset A \wedge n \in \mathbb{N} \right\} \tag{2.8}$$

By 2.2.6. ii, $\emptyset \notin \mathcal{B}$, so $\mathcal{B}$ satisfies 2.2.7. ii. By 2.2.6. i, $\emptyset \neq A \subset \mathcal{B}$, so $\mathcal{B}$ satisfies 2.2.7. i. Since $\emptyset \notin \mathcal{B}$ is **Closed Under Finite Intersections** , if $U, V \in \mathcal{B}$, then $\emptyset \neq U \cap V \in \mathcal{B}$, so $\mathcal{B}$ can be seen to satisfy 2.2.7. *iii*. Hence $\mathcal{B}$ is a **Filter Base** . Since $A \subset \mathcal{B}$, $\mathcal{B}$ is a **Filter Base** for a **Finer** than $\mathcal{F}$. However, since $A \subset \mathcal{F}$, by 2.2.2 ii, $\mathcal{B} \subset \mathcal{F}$. Hence $\mathcal{F}$ is the **Filter Generated By** $\mathcal{B}$. ☐

Proof of 2.2.10. ii. ($\Longleftarrow$). Let $\mathcal{G}$ denote the **Filter Generated By** B. Then since $B \subset \mathcal{F}$, $\mathcal{G} \subset \mathcal{F}$. If for each $Y \in \mathcal{F}$ there exists $b \in B$ with $b \subset Y$, then

$$\mathcal{F} \subset \{U \subset X | (\exists b \in B)(b \subset U)\} \subset \mathcal{G} \subset \mathcal{F} \tag{2.9}$$

so that $\mathcal{F} = \mathcal{G}$ and $\{U \subset X | (\exists b \in B)(b \subset U)\} = \mathcal{F}$ is a **Filter** on X. Hence, by 2.2.8, B is a **Filter** .

($\Longrightarrow$). If B is a **Filter Base** for $\mathcal{F}$, then by 2.2.8, $\mathcal{F} = \{Y \subset X | (\exists b \in B)(b \subset Y)\}$ so the desired property holds ☐

Proof of 2.2.10. iii. Let $\mathcal{F}$ be finer than $\mathcal{G}$. Then by applying 2.2.8 $\mathcal{G} \subset \mathcal{F} = \{U \subset X | (\exists f \in F)(f \subset U)\}$, which is the desired result in one direction. The other direction is equivalent again applying 2.2.8 to claim $\mathcal{G} \subset \{U \subset X | (\exists f \in F)(f \subset U)\}$. □

Proof of 2.2.10. iv. This is a result of two applications of 2.2.10. iii, one in each direction. □

Proof of 2.2.10. v. Define $\mathcal{U} = \{U \subset X | (\exists f \in \mathcal{F}) \pitchfork (f \subset U)\}$. By construction $\mathcal{F} \subset \mathcal{U}$. By 2.2.1. iii, $\mathcal{U} \subset \mathcal{F}$. Hence, by 2.2.8, $\mathcal{U} = \mathcal{F}$ is a **Filter Base** on X. Clearly $\mathcal{F} \subset \mathcal{F}$ and **Nested**$(\mathcal{F}, \mathcal{F})$ hold, so we can apply 2.2.10. ii so see that $\mathcal{F}$ is a **Filter Base** for $\mathcal{F}$. □

Proposition 2.2.11 (**Net Sections** form a **Filter Base**)**.** Let $X \neq \emptyset$ and let $\sigma = \{x_\alpha\}_{\alpha \in A}$ be a **Net** in X. For each $\alpha \in A$, denote with $S(\sigma, \alpha)$ the **Section** of x_α in σ. Define

$$\mathcal{B} = \{S(\sigma, \alpha) | \alpha \in A\}$$

For each $\alpha \in A$, let $S(\alpha, \leq)$ denote the **Section** of $\leq$ corresponding to α. Then the following are true:

(i) $\mathcal{B}$ is a **Filter Base** on X.

Proof of 2.2.11 i. Since σ is a **Net** , $(A, \leq)$ is a **Directed Set** , implying that $(A, \leq)$ is a **Preordered Set** . Hence, $\leq$ is **Reflexive** so that if $\alpha \in A$, then $x_\alpha \in S(\sigma, \alpha)$. Hence, $\emptyset \notin \mathcal{B}$, so $\mathcal{B}$ satisfies 2.2.7. ii. Furthermore, since $(A, \leq)$ is a **Preordered Set** , A is nonempty, so $\emptyset \neq \mathcal{B}$, implying $\mathcal{B}$ satisfeis 2.2.7. i. Finally, let $U, V \in \mathcal{B}$. Then we can find $u, v \in A$ such that $U = S(\sigma, U)$, $V = S(\sigma, V)$. Since A is a **Directed Set** , there exists $w \in A$ with $u \leq w$ and $v \leq w$. Hence by 2.1.59, $S(\sigma, w) \subset S(\sigma, u) \cap S(\sigma, v)$. Since $S(\sigma, w) \in \mathcal{B}$, $\mathcal{B}$ satisfies 2.2.7. iii, and we're done. □

Definition 2.2.12 (**Section Filter**)**.** Let $X \neq \emptyset$. Let $\sigma = \{x_\alpha\}_{\alpha \in A}$ be a **Net** in X. For each $\alpha \in A$, let $S(\sigma, \alpha)$ denote the **Section** of x_α in σ. Define $\mathcal{B} = \{S(\sigma, \alpha) | \alpha \in A\}$. By **??**, $\mathcal{B}$ is a **Filter Base** on X. We call the **Generated By** $\mathcal{B}$ the **Section Filter** of σ. We call the **Section Filter** of the identity **Net** in A the **Section Filter** of A. We denote the **Section Filter** of A with $\mathcal{F}_A$.

2.2.3 Ultrafilters

Definition 2.2.13 (**Ultrafilter**)**.** *Let $X \neq \emptyset$. An **Ultrafilter** on X is a* **Maximum** of the relation of **Filter Fineness** on X.

Remark 2.2.14 (**Ultrafilter** Existence)**.** *Let $\mathcal{F}$ be a **Filter** on $X \neq \emptyset$. By 2.2.5. iii Not only is **Filter Fineness** an **Inductive Order** on the set of **Filters** of X, (as stated in 2.2.5. v) , but **Filter Fineness** is also an **Inductive Order** on the set of **Filters Finer** than $\mathcal{F}$. Hence by 2.1.61, $\mathcal{F}$ is contained in an **Ultrafilter** on X. A **Filter Base** for an **Ultrafilter** is called an **Ultrafilter Base** .*

Proposition 2.2.15 (Ultrafilter Facts)**.** *Suppose the following*

(I) $X \neq \emptyset$.

(II) $\mathcal{F}$ *is an* ***Ultrafilter*** *on* X*.*

(III) $\mathcal{G}$ *is a* ***Filter*** *on* X*.*

(IV) $\mathcal{K}$ *is a* ***Subbasis*** *on* X*.*

(V) $\mathcal{M} = \mathcal{K} \cup \{K \subset X | X \setminus K \in \mathcal{K}\}$*.*

Then the following are true

(i) If $\{A, B\} \subset 2^X$ *and* $A \cup B \in \mathcal{F}$*, then* $A \in \mathcal{F}$ *or* $B \in \mathcal{F}$*.*

(ii) If $\{A_i\}_{i=1}^n \subset 2^X$ *such that* $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$*, then for some* $j \in \{1, \cdots, n\}$*,* $A_j \in \mathcal{F}$*.*

(iii) If $\mathcal{M} = 2^X$*, then* $\mathcal{K}$ *is an* ***Ultrafilter*** *on* X*.*

(iv) $\mathcal{G}$ *is the intersection of all* ***Ultrafilters*** *on* X *which contain* $\mathcal{G}$*.*

Proof of 2.2.15. i. We use contradiction. Suppose $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$. Define $\mathcal{T} = \{G \in 2^X | A \cup G \in \mathcal{F}\}$. Then $B \in \mathcal{T}$, so $\mathcal{T}$ satisfies 2.2.1. i. Furthermore, if $G_1 \in \mathcal{T}$ and $G_1 \subset G_2 \subset X$, then by 2.2.1. iii, $A \cup G_1 \subset A \cup G_2 \in \mathcal{F}$. Hence $G_2 \in \mathcal{T}$ so $\mathcal{T}$ satisfies 2.2.1. iii. Let $G_3, G_4 \in \mathcal{T}$

$$A \cup (G_3 \cap G_4) = (A \cup G_3) \cap (A \cup G_4) \in \mathcal{F}$$

so that $G_3 \cap G_4 \in \mathcal{T}$ and therefore $\mathcal{T}$ satisfies 2.2.1. iv. Finally since $A \notin \mathcal{F}$, $\emptyset \notin \mathcal{T}$, so $\mathcal{T}$ satisfeis 2.2.1. *ii*, and therefore $\mathcal{T}$ is a **Filter** on X. Trivially, $\mathcal{F} \subset \mathcal{T}$ but since $B \in \mathcal{T} \setminus \mathcal{F}$, this contradicts 2.2.15. II Hence the result holds. □

Proof of 2.2.15. ii. We use induction on n. Obviously the result holds for $n = 1$ and by 2.2.15. i, the result also holds for $n = 2$. Suppose the result holds for $n = k$ Let $\{A_i\}_{i=1}^{k+1} \subset 2^X$ such that $\bigcup_{i=1}^{k+1} A_i \in \mathcal{F}$. Then since the result holds for $n = 2$, either $A_{k+1} \in \mathcal{F}$ or $\bigcup_{i=1}^{k} A_i \in \mathcal{F}$. Since the result holds for $n = k$, either $A_{k+1} \in \mathcal{F}$ or $A_i \in \mathcal{F}$ for $i \in \{1, \cdots, k\}$. Hence the result holds for $n = k + 1$. Hence the result holds in general. □

Proof of 2.2.15. iii. I first prove that $\mathcal{M} = 2^X$, paired with 2.2.15. IV and 2.2.15. V implies $\mathcal{K}$ is a **Filter** on X. By 2.2.15. I, $\mathcal{M} \neq \emptyset$. By 2.2.15. V, then $\mathcal{K} \neq \emptyset$, so $\mathcal{K}$ satisfies 2.2.1. i. By 2.2.6. ii, $\emptyset \notin \mathcal{K}$, so $\mathcal{K}$ satisfies 2.2.1. ii. Let $G_1 \in \mathcal{K}$ and let $G_1 \subset G_2 \subset X$. Then $G_1 \cap (X \setminus G_2) = \emptyset$, which by 2.2.6. ii implies $X \setminus G_2 \notin \mathcal{K}$. Since $\mathcal{M} = 2^X$, we conclude $G_2 \in \mathcal{K}$, so $\mathcal{K}$ satisfies 2.2.1. iii. Finally, let $G_1, G_2 \in \mathcal{K}$. By assumption, either $G_1 \cap G_2 \in \mathcal{K}$ or $X \setminus (G_1 \cap G_2) \in \mathcal{K}$. If $X \setminus (G_1 \cap_G 2) \in \mathcal{K}$, then by 2.2.6. ii, $G_1 \cap G_2 \cap (X \setminus (G_1 \cap G_2)) \neq \emptyset$, a contradiction. Hence, $G_1 \cap G_2 \in \mathcal{K}$, so that 2.2.1. iv is satisfied by $\mathcal{K}$. Hence $\mathcal{K}$ is a **Filter** on X. By 2.2.14, there is an **Ultrafilter** $\mathcal{L}$ containing $\mathcal{K}$. If $\mathcal{K}$ is not an **Ultrafilter** , then $\exists B \in \mathcal{L} \setminus \mathcal{K}$. Since $\mathcal{M} = 2^X$, $X \setminus B \in \mathcal{K} \subset \mathcal{L}$, implying $\emptyset = B \cap (X \setminus B) \in \mathcal{L}$, contradicting 2.2.1. ii, thus $\mathcal{K}$ is an **Ultrafilter** . □

Proof of 2.2.15. iv. Let $\{\mathcal{P}_\alpha\}_{\alpha\in A}$ be the collection of all **Ultrafilters** on X containing $\mathcal{G}$. Define $\mathcal{P} = \bigcap_{\alpha\in A} \mathcal{P}_\alpha$. I must schow $\mathcal{P} = \mathcal{G}$. By 2.2.2 iii, $\mathcal{P}$ is a **Filter** on X, and by construction $\mathcal{G} \subset \mathcal{P}$. Let $B \subset X \setminus \mathcal{G}$. Then, by 2.2.1. iii, there is no $G \in \mathcal{G}$ with $G \subset A$. Hence, for each $G \in \mathcal{G}$, $G \cap (X \setminus A) \neq \emptyset$. Therefore, we can apply 2.2.5. i to claim that there is a **Filter** $\mathcal{G}_1$ on X which is **Finer** than $\mathcal{G}$ satisfying $X \setminus A \in \mathcal{G}_1$. By 2.2.14, there is an $\alpha \in A$ such that $X \setminus A \in \mathcal{G}_1 \subset \mathcal{P}_\alpha$, so $X \setminus A \in \mathcal{P}$. Since $\mathcal{P}$ satisfies 2.2.1. ii and 2.2.1. iv, $X \setminus A \in \mathcal{P}$ implies $A \notin \mathcal{P}$ and so $\mathcal{P} \subset \mathcal{G}$. This completes the proof. □

2.2.4 Induced Filters

Definition 2.2.16 (**Induced**). Let $X \neq \emptyset$, $\mathcal{F}$ be a **Filter** on $\mathcal{F}$ and $A \subset X$ such that

$$\mathcal{F}_A := \{U \cap A | U \in \mathcal{F}\}$$

satisfies $\emptyset \notin \mathcal{F}_A$. Then by 2.2.2 iv $\mathcal{F}_A$ is a **Filter** on A which we call the **Filter Induced** by $\mathcal{F}$.

Proposition 2.2.17 (**Induced Filter** facts). *Let $X \neq \emptyset$ Let $\mathcal{F}$ be a* ***Filter*** *on X. Let $\mathcal{B}$ be a* ***Filter Base*** *for $\mathcal{F}$. Let $\mathcal{G}$ be an* ***Ultrafilter*** *on X. Let $A \subset X$. Define $\mathcal{F}_A = \{A \cap U | U \in \mathcal{F}\}$. Define $\mathcal{B}_A = \{A \cap U | U \in \mathcal{B}\}$. Define $\mathcal{G}_A = \{A \cap U | U \in \mathcal{G}\}$. The following are true*

(i) If $\mathcal{F}_A$ is a ***Filter*** *on A, then $\mathcal{B}_A$ is a* ***Filter Base*** *for $\mathcal{F}_A$.*

(ii) $\mathcal{G}_A$ is a ***Filter*** *on A if and only if $A \in \mathcal{G}$. In this case, $\mathcal{G}_A$ is an* ***Ultrafilter*** *on A.*

Proof of 2.2.17. i. Let $U \in \mathcal{F}_\alpha$. Then there exists $V \in \mathcal{F}$ such that $U = A \cap V$. Since $\mathcal{B}$ is a **Filter Base** for $\mathcal{F}$, by 2.2.10. ii, there exists $B \in \mathcal{B}$ satisfying $B \subset V$. Then $B \cap A \subset A \cap V = U$. But $B \cap A \in \mathcal{B}_A$, so since $\mathcal{B}_A \subset \mathcal{F}_A$, we can the apply 2.2.10. *ii* to claim that $\mathcal{B}_A$ is a **Filter Base** for $\mathcal{F}_A$. □

Proof of 2.2.17. ii. Even if $\mathcal{G}$ was merely a **Filter** on X, by 2.2.1. ii and **??** $A \in \mathcal{G}$ is sufficient to guarantee that $\mathcal{G}_A$ is a **Filter** on A. Now suppose $\mathcal{G}_A$ is a **Filter** on A Then $A \cap U \neq \emptyset$ for $U \in \mathcal{G}$, Since $\mathcal{G}$ is an **Ultrafilter** on X, we can apply maximality with 2.2.5. i, to see that $A \in \mathcal{G}$. Finally, if $P \subset A$ satisfies $P \notin \mathcal{G}_A$, then $P \notin \mathcal{G}$. Since $\mathcal{G}$ is an **Ultrafilter** , by 2.2.5. i, $P \cap U = \emptyset$ for some $U \in \mathcal{G}$. This implies $P \cap (U \cap A) = \emptyset$, and since $U \cap A \in \mathcal{G}_A$, we can apply 2.2.5. i to conclude that there is no **Filter Finer** than $\mathcal{G}_A$ on A which contains P. Since $P \subset A$ was arbitrary, $\mathcal{G}_A$ is an **Ultrafilter** on A.

□

2.2.5 Direct and Inverse Images of a Filter Base

Proposition 2.2.18 (Direct Filter Image). *Suppose the following.*

1. $X, Y \neq \emptyset$

2. $f : X \to Y$ is **Surjective** .

3. For $i \in \{1, 2\}$, $\mathcal{B}_i$ is a **Filter Base** for a **Filter** $\mathcal{F}_i$ on X.
4. $\mathcal{F}_2$ is **Finer** than $\mathcal{F}_1$.
5. $\mathcal{K}$ is an **Ultrafilter Base** on X.

Then the following are true.

(i) $f(\mathcal{F}_1)$ *is a* ***Filter*** *on* Y.

(ii) $f(\mathcal{B}_1)$ *is a* ***Filter Base*** *for* $f(\mathcal{F}_1)$.

(iii) $f(\mathcal{F}_2)$ *is a* ***Finer*** *than* $f(\mathcal{F}_1)$.

(iv) $f(\mathcal{K})$ *is an* ***Ultrafilter Base*** *on* Y.

Proof of 2.2.18. i. Since $\emptyset \notin \mathcal{F}_1$, $\emptyset \notin f(\mathcal{F}_1)$, so $f(\mathcal{F}_1)$ satisfies 2.2.1. ii. Since $\emptyset \neq \mathcal{F}_1$, $f(\mathcal{F}_1) \neq \emptyset$, so $f(\mathcal{F}_1)$ satisfies 2.2.1. *i*. Let $G_1 \in f(\mathcal{F}_1)$. Let $G_1 \subset G_2 \subset Y$. Then, since f is **Surjective** , $U \subset f^{-1}(G_2)$, which by 2.2.1. iii implies $f^{-1}(G_2) \in \mathcal{F}_1$. Hence $G_2 = f\left(f^{-1}\left(G_2\right)\right) \in f(\mathcal{F}_1)$, so $f(\mathcal{F}_1)$ satisfies 2.2.1. iii. Finally, if $G_1, G_2 \in f(\mathcal{F}_1)$, then there are $K_1, K_2 \in \mathcal{F}_1$ with $f(K_i) = G_i$ for $i \in \{1, 2\}$. By 2.2.1. iv, $K_1 \cap K_2 \in \mathcal{F}_1$. Also, $f(K_1 \cap K_2) \subset f(K_1) \cap f(K_2)$, so by 2.2.1. iii, $f(K_1) \cap f(K_2) \in f(\mathcal{F}_1)$. Hence $f(\mathcal{F}_1)$ satisfies 2.2.1. *iv* and is therefore a **Filter** on Y. □

Proof of 2.2.18. ii. By 2.2.7. i, $\emptyset \neq \mathcal{B}_1$, so $\emptyset \neq f(\mathcal{B}_1)$, and thus $f(\mathcal{B}_1)$ satisfies 2.2.7. i. By 2.2.7. ii, $\emptyset \notin \mathcal{B}_1$, so $\emptyset \notin f(\mathcal{B}_1)$, implying $\mathcal{B}_1$ satisfies 2.2.7. ii. Finally, let $U_1, U_2 \in f(\mathcal{B}_1)$. Then there exists $V_i \in \mathcal{B}_1$ with $f(V_i) = U_i$. Then by 2.2.7. iii, $V_1 \cap V_2 \in \mathcal{B}_1$, and $f(V_1 \cap V_2) \subset f(V_1) \cap f(V_2)$, and $f(V_1 \cap V_2) \in f(\mathcal{B}_1)$ by construction, so $f(\mathcal{B}_1)$ satisfies 2.2.7. iii, and therefore $f(\mathcal{B}_1)$ is a **Filter Base** on Y. Now, if $V \in f(\mathcal{F}_1)$, then by definition, there exists $U \in \mathcal{F}_1$ with $f(U) = V$. By 2.2.10. ii, there exists a $b \in \mathcal{B}_1$ with $b \subset U$. This implies $f(b) \subset f(U) = V$, but $f(b) \in f(\mathcal{B}_1)$. Furthermore, since $\mathcal{B}_1 \subset \mathcal{F}_1$, $f(\mathcal{B}_1) \subset f(\mathcal{F}_1)$, so we can apply 2.2.10. ii to claim that $f(\mathcal{B}_1)$ is a **Filter Base** for $f(\mathcal{F}_1)$. □

Proof of 2.2.18. iii. If $\mathcal{F}_1 \subset \mathcal{F}_2$ then $f(\mathcal{F}_1) \subset f(\mathcal{F}_2)$. An invocation of 2.2.18. ii finishes the result. □

Proof of 2.2.18. iv. Let $\mathcal{G}$ denote the **Ultrafilter** for which $\mathcal{K}$ is an **Ultrafilter Base** . Let $U \subset Y$. Since f is **Surjective** ,

$$X = f^{-1}(U) \cup \left(X \setminus f^{-1}(U)\right) = f^{-1}(U) \cup f^{-1}\left(f(X) \setminus U\right) = f^{-1}(U) \cup f^{-1}(Y \setminus U) \quad (2.10)$$

and by 2.2.2 i, we have $f^{-1}(U) \cup f^{-1}(Y \setminus U) \in \mathcal{G}$ Since $\mathcal{G}$is an **Ultrafilter** , by **??**, either $f^{-1}(U) \in \mathcal{G}$ or $f^{-1}(Y \setminus U) \in \mathcal{G}$. This implies either $U \in f(\mathcal{G})$ or $Y \setminus U \in f(\mathcal{G})$. By **??**, $f(\mathcal{G})$ is an **Ultrafilter** on Y. An application of 2.2.18. ii completes the result. □

Proposition 2.2.19 (Inverse Filter Image)**.** *Suppose the following*

1. $X, Y \neq \emptyset$

2. $f : X \to Y$.

3. *$\mathcal{B}$ is a* ***Filter Base*** *for a* ***Filter*** *$\mathcal{F}$ on Y.*

4. $\mathcal{F}_{f(X)} := \{U \cap f(X) | U \in \mathcal{F}\}$

Then the following are true

(i) $f^{-1}(\mathcal{B})$ is a ***Filter Base*** *on X if and only if $\emptyset \notin f^{-1}(\mathcal{B})$.*

(ii) If $f^{-1}(\mathcal{B})$ is a ***Filter Base*** *on X, then $f(f^{-1}(\mathcal{B}))$ is a* ***Filter Base*** *for a* ***Filter*** *on Y finer than $\mathcal{F}$.*

(iii) If $f(f^{-1}(\mathcal{B}))$ is a ***Filter Base*** *for a* ***Filter*** *$\mathcal{G}$ on Y, then $\mathcal{G}_{f(X)} = \mathcal{F}_{f(X)}$.*

Proof of 2.2.19. i. Necessity of $\emptyset \notin f^{-1}(\mathcal{B})$ is obvious by 2.2.7. ii. For sufficiency, suppose $\emptyset \not\in f^{-1}(\mathcal{B})$. Then $f^{-1}(\mathcal{B})$ satisfies 2.2.7. ii trivially. Furthermore, by 2.2.7. *i*, $\mathcal{B} \neq \emptyset$, so $f^{-1}(\mathcal{B}) \neq \emptyset$, so $f^{-1}(\mathcal{B})$ satisfies 2.2.7. i. Finally, let $U_1, U_2 \in f^{-1}(\mathcal{B})$. Then there exist $V_1, V_2 \in \mathcal{B}$ such that $U_i = f^{-1}(V_i)$. By 2.2.7. iii, there exists $W \in \mathcal{B}$ such that $W \subset V_1 \cap V_2$, and $f^{-1}(W) \in f^{-1}(\mathcal{B})$. Clearly,

$$f^{-1}(W) \subset f^{-1}(V_1 \cap V_2) = f^{-1}(V_1) \cap f^{-1}(V_2) = U_1 \cap U_2$$

Hence $f^{-1}(\mathcal{B})$ satisfies 2.2.7. iii □

Proof of 2.2.19. ii. If $f^{-1}(\mathcal{B})$ is a **Filter Base** on X, then we can leverage 2.2.18. ii to claim that $f(f^{-1}(\mathcal{B}))$ is a **Filter Base** for a **Filter** on $f(X)$, and therefore also a **Filter Base** for a **Filter** on Y. In particular, $f(f^{-1}(\mathcal{B}))$ is a **Filter Base** . Furthermore, if $b \in \mathcal{B}$, then $f(f^{-1}(b)) \in f(f^{-1}(\mathcal{B}))$ and $f(f^{-1}(b)) = f(X) \cap b \subset b$, so by 2.2.10. iii, $f(f^{-1}(\mathcal{B}))$ is a **Filter Base** for a **Finer** **Filter** than $\mathcal{F}$. □

Proof of 2.2.19. iii. This is because given the assumptions, $f(f^{-1}(\mathcal{B})) = \{f(X) \cap \mathcal{B} | x \in \mathcal{B}\}$, which lets us apply **??** □

2.2.6 Filter Products

Definition 2.2.20 (**Product Filter**). *Suppose the following*

1. *$A \neq \emptyset$.*

2. *$\{X_\alpha\}_{\alpha \in A}$ is a collection of nonemptyset sets.*

3. *For each $\alpha \in A$, $\mathcal{F}_\alpha$ is a* ***Filter*** *on X_α.*

4. *For each $\gamma \in A$, $\pi_\gamma : \prod_{\alpha \in A} X_\alpha \to X_\gamma$ represents the* ***Projection Map*** *.*

Then we define the ***Filter*** *on $\prod_{\alpha \in A} X_\alpha$* ***Generated By***

$$\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$$

to be the ***Product Filter*** *on $\prod_{\alpha \in A} (X_\alpha, \mathcal{F}_\alpha)$*

Proposition 2.2.21. *Suppose the following*

1. $A \neq \emptyset$.

2. $\{X_\alpha\}_{\alpha \in A}$ *is a collection of nonemptyset sets.*

3. *For each* $\alpha \in A$, $\mathcal{B}_\alpha$ *is a* ***Filter Base*** *for a* ***Filter*** $\mathcal{F}_\alpha$ *on* X_α.

4. *For each* $\alpha \in A$, $\mathcal{G}_\alpha$ *is a* ***Subbasis*** *which* ***Generates*** $\mathcal{F}_\alpha$.

5. *For each* $\gamma \in A$, $\pi_\gamma : \prod_{\alpha \in A} X_\alpha \to X_\gamma$ *represents the* ***Projection Map*** .

6. $\mathcal{G} := \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{G}_\alpha)$.

7. $\mathcal{B}$ *is the collection of finite intersections of elements of* $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}_\alpha)$

8. $\mathcal{F}$ *is the* ***Product Filter*** *on* $\prod_{\alpha \in A} (X_\alpha, \mathcal{F}_\alpha)$

Then the following are true

(i) $\mathcal{F}$ *is well defined.*

(ii) $\mathcal{G}$ *is a* ***Subbasis*** *for* $\mathcal{F}$.

(iii) $\mathcal{B}$ *is a* ***Filter Base*** *for* $\mathcal{F}$.

(iv) $\mathcal{F}$ *is the* ***Coarsest Filter*** *on* $\prod_{\alpha \in A} X_\alpha$ *such that for every* $\alpha \in A$, $\pi_\alpha(\mathcal{F}) = \mathcal{F}_\alpha$.

Proof of 2.2.21. i. Since $A \neq \emptyset$, there exists a $\gamma \in A$. By 2.2.1. i, $\mathcal{F}_\gamma \neq \emptyset$. Hence

$$\emptyset \neq \pi_\gamma^{-1}(\mathcal{F}_\gamma) \subset \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$$

so that $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ satisfies 2.2.6. i. Additionally, let $\{U_i\}_{i=1}^n \subset \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$. Then for each $1 \leq i \leq n$, there exists an $\alpha_i \in A$ and a $V_i \in \mathcal{F}_{\alpha_i}$ such that $U_i = \pi_{\alpha_i}^{-1}(V_i)$. For $i, j \in \{1, \cdots, n\}$, define $i \equiv j \iff \alpha_i = \alpha_j$. It is clear that $\equiv$ is an **Equivalence Relation** on $\{1, \cdots, n\}$. Let $k \in \{1, \cdots, n\}/\equiv$. Then

$$\begin{aligned}
\bigcap_{p \in k} U_p &= \bigcap_{p \in k} \pi_{\alpha_p}^{-1}(V_{\alpha_p}) \\
&= \bigcap_{p \in k} \pi_{\alpha_{min(k)}}^{-1}(V_p) \\
&= \pi_{\alpha_{min(k)}}^{-1}\left(\bigcap_{p \in k} V_p\right)
\end{aligned}$$

Since $\{V_p\}_{p\in k} \subset \mathcal{F}_{\alpha_{min(k)}}$, 2.2.2 ii, $\bigcap_{p\in k} V_p \in \mathcal{F}_{\alpha_{min(k)}}$, so by 2.2.1. ii, $\bigcap_{p\in k} V_p \neq \emptyset$. For each $k \in \{1, \cdots, n\}/\equiv$, let $x_k \in \pi^{-1}_{\alpha_{min(k)}}\left(\bigcap_{p\in k} V_p\right)$. Partition $A = A' \cup (A \setminus A')$ where $A' = \{\alpha_{min(k)} | k \in \{1, \cdots, n\}/\equiv\}$. For each $\alpha \in A \setminus A'$, Define $F_\alpha = X_\alpha$. For each $\alpha_{min(k)} \in A'$, define $F_{\alpha_{min(k)}} = \{x_k\}$. Then by the axiom of choice,

$$\emptyset \neq \prod_{\alpha\in A} F_\alpha \subset \bigcap_{i=1}^{n} U_i$$

so that $\bigcup_{\alpha\in A} \pi^{-1}_\alpha(\mathcal{F}_\alpha)$ satisfies 2.2.6. ii. Hence $\bigcup_{\alpha\in A} \pi^{-1}_\alpha(\mathcal{F}_\alpha)$ is a **Subbasis** on X, and the **Filter** that it **Generates** is $\mathcal{F}$. □

Proof of 2.2.21. iii. Since

$$\bigcup_{\alpha\in A} \pi^{-1}_\alpha(\mathcal{B}_\alpha) \subset \bigcup_{\alpha\in A} \pi^{-1}_\alpha(\mathcal{F}_\alpha) \tag{2.11}$$

and by 2.2.21. i, $\bigcup_{\alpha\in A} \pi^{-1}_\alpha(\mathcal{B}_\alpha)$ generates a **Filter** $\mathcal{F}_0$ on $\prod_{\alpha\in A} X_\alpha$. In Particular, the above equation implies $\mathcal{F}_0 \subset \mathcal{F}$. Denote Let $\alpha \in A$. By 2.2.19. i, both $\pi^{-1}_\alpha(\mathcal{F}_\alpha)$ and $\pi^{-1}_\alpha(\mathcal{B}_\alpha)$ are **Filter Bases** for a **Filter** on the product set. Let $U \in \pi^{-1}_\alpha(\mathcal{F}_\alpha)$. Then, by 2.2.10. ii, there is a $b \in \mathcal{B}_\alpha$ such that $b \subset U$. Hence $f^{-1}(b) \subset f^{-1}(U)$. Which implies by 2.2.10. iii that $\pi^{-1}_\alpha(\mathcal{B}_\alpha)$ is a **Filter Base** for a **Finer Filter** than $\pi^{-1}_\alpha(\mathcal{F}_\alpha)$. This implies that any **Filter** containing $\pi^{-1}_\alpha(\mathcal{B}_\alpha)$ also contains $\pi^{-1}_\alpha(\mathcal{F}_\alpha)$. Since, for each $\alpha \in A$, $\pi^{-1}_\alpha(\mathcal{B}_\alpha) \subset \mathcal{F}_0$ for each $\alpha \in A$, $\pi^{-1}_\alpha(\mathcal{F}_\alpha) \subset \mathcal{F}_0$. Hence $\bigcup_{\alpha\in A} \pi^{-1}_\alpha(\mathcal{F}_\alpha) \subset \mathcal{F}_0$, so by 2.2.6, $\mathcal{F} \subset \mathcal{F}_0$. Hence $\mathcal{F} = \mathcal{F}_0$. Also, by 2.2.10. i, since $\bigcup_{\alpha\in A} \pi^{-1}_\alpha(\mathcal{B}_\alpha)$ is by construction a **Subbasis** for $\mathcal{F}_0$, Since $\mathcal{B}$ is the collection of **Finite** intersections of elements of $\bigcup_{\alpha\in A} \pi^{-1}_\alpha(\mathcal{B}_\alpha)$, $\mathcal{B}$ is a **Filter Base** for $\mathcal{F}_0 = \mathcal{F}$. □

Proof of 2.2.21. ii. For each $\alpha \in A$, let $\mathcal{K}_\alpha$ the collection of **Finite** intersections of elements of $\mathcal{G}_\alpha$. By **??**, for each $\alpha \in A$, $\mathcal{K}_\alpha$ is a **Filter Base** for $\mathcal{F}_\alpha$. An application of 2.2.21. iii implies that $\mathcal{F}$ is the **Filter** generated by $\bigcup_{\alpha\in A} \pi^{-1}_\alpha(\mathcal{K}_\alpha)$. But each

$$\pi^{-1}_\alpha(\mathcal{K}_\alpha) = \left\{\bigcap_{i=1}^{n} U_\alpha | U_\alpha \in \pi^{-1}_\alpha(\mathcal{G}_\alpha)\right\}$$

So the collection of **Finite** intersections of elements of $\bigcup_{\alpha\in A} \pi^{-1}_\alpha(\mathcal{K}_\alpha)$ equals the collection of finite intersections of elements of $\bigcup_{\alpha\in A} \pi^{-1}_\alpha(\mathcal{G}_\alpha)$, they generate the same **Filter** ! □

Proof of 2.2.21. iv. I first show that $\mathcal{F}$ has the described property, and then show that it is the **Coarsest** on the product set with the property. Fix $\gamma \in A$. Clearly, by **Surjectivity** , $\mathcal{F}_\gamma = \pi_\gamma\left(\pi^{-1}_\gamma(\mathcal{F}_\gamma)\right) \subset \pi_\gamma(\mathcal{F})$. Now, let $U \in \pi_\gamma(\mathcal{F})$. Then $U = \pi_\gamma(V)$ for some $V \in \mathcal{F}$. Since $\bigcup_{\alpha\in A} \pi^{-1}_\alpha(\mathcal{F}_\alpha)$ is a **Subbasis** for $\mathcal{F}$, by 2.2.10. i and 2.2.10. ii, there is a colleciton $\{U_i\}_{i=1}^{n} \subset$

$\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ such that $\bigcap_{i=1}^{n} U_i \subset V$. Let each $U_i \in \pi_{\alpha_i}^{-1}(\mathcal{F}_{\alpha_i})$. If, for all $i \in \{1, \cdots, n\}$, $\alpha_i \neq \gamma$ then $U = X_\gamma \in \mathcal{F}_\alpha$. Otherwise, let V' be the intersection of all $U_i's$ such that $\alpha_i = \gamma$. Then $V' \in \pi_\gamma^{-1}(\mathcal{F}_\gamma)$ and $V' \subset V$. Hence $\pi_\gamma(V') \in \mathcal{F}_\gamma$ and $\pi_\gamma(V') \subset \pi_\gamma(V) = U$, so by 2.2.1. iii, $U \in \mathcal{F}_\gamma$, implying $\pi_\gamma(\mathcal{F}) \subset \mathcal{F}_\gamma$. Since the inclusion goes both ways, $\pi_\gamma(\mathcal{F}) = \mathcal{F}_\gamma$. To see that $\mathcal{F}$ is the **Coarsest** with this filter, not that if $\mathcal{H}$ is a **Filter** on the product set such that $\pi_\gamma(\mathcal{H}) = \mathcal{F}_\gamma$, then

$$\pi_\gamma^{-1}(\mathcal{F}_\gamma) = \pi_\gamma^{-1}(\pi_\gamma(\mathcal{H})) \subset \mathcal{H}$$

Hence if this occurs for every γ then $\mathcal{F} \subset \mathcal{H}$, exactly what we are trying to show.

□

2.3 Topological Spaces

2.3.1 Open Sets, Closed Sets, and Neighborhoods

Definition 2.3.1 (Topological Space). *Let $X \neq \emptyset$ be a set and let $\{\emptyset, X\} \subset \mathcal{T} \subset 2^X$ such that*

(i) $X \in \mathcal{T}$.

(ii) $\emptyset \in \mathcal{T}$.

(iii) $\mathcal{T}$ *is* **Closed Under Aribtrary Unions** .

(iv) $\mathcal{T}$ is **Closed Under Finite Intersections** .

Then we call $\mathcal{T}$ *a* **Topology** on X and we call $(X, \mathcal{T})$ a **Topological Space** .

Definition 2.3.2 (**Discrete Topology** , **Indiscrete Topology**). *Let X be a set. We call $\{X, \emptyset\}$ the* ***Indiscrete Topology*** *on X and we call 2^X the* ***Discrete Topology*** *on X.*

Definition 2.3.3 (**Set-Open** , **Set-Closed**). *Let $(X, \mathcal{T})$ be a* **Topological Space** , and let $A \in \mathcal{T}$. We say that A is **Set-Open** in $(X, \mathcal{T})$ (or **Set-Open** in X or **Set-Open** in $\mathcal{T}$ or simply **Set-Open** in cases where confusion won't result) and that A posesses **Set-Openness** . We say that $X \setminus A$ is **Set-Closed** and that $X \setminus A$ posesses **Closedness**

Definition 2.3.4 (**Compact**). *We say that a* **Topological Space** is **Compact** if every **Set-Open** **Cover** for X has a **Finite** **Subcover** .

Definition 2.3.5 (**Compact**). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be* **Topological Spaces** . We say that $f : X \to Y$ is **Compact** if $f(K)$ is **Compact** in $(Y, \mathcal{T}_Y)$ for every **Compact** $K \in (X, \mathcal{T}_X)$.

Definition 2.3.6 (**Coarse**, **Fine**)**.** *Let X be a set. Let $\mathcal{T}_1, \mathcal{T}_2$ be* **Topology** on X such that $\mathcal{T}_1 \subset \mathcal{T}_2$. In this case, we say that $\mathcal{T}_1$ is more **Coarse**than $\mathcal{T}_2$, that $\mathcal{T}_1$ is **Coarser**than $\mathcal{T}_2$, that $\mathcal{T}_2$ is more **Fine**than $\mathcal{T}_1$, that $\mathcal{T}_2$ is **Finer**than $\mathcal{T}_1$, and we write $\mathcal{T}_1 \leq_{TopFine(X)} \mathcal{T}_2$ and we write $\mathcal{T}_2 \leq_{TopCoarse(X)} \mathcal{T}_1$.

Let K denote the collection of topologies on X. Let $A \subset K$. If one exists, a **Maximum** of A with respect to $\leq_{TopFine(X)}$ is called the **Finest**topology in A. If one exists, a **Maximum** of A with respect to $\leq_{TopCoarse(X)}$ is called the **Coarsest**topology in A.

Since $\subset$ defines a **Partial Order** on the power set of X, **Fineness**defines a **Partial Order** on the set of **Topology** of X.

The intersection of any 2 topologies on X is a topology on X, so that $\leq_{TopCoarse(X)}$ is a **Direction** on X.

Definition 2.3.7 (**Neighborhood** , **Neighborhood Filter**)**.** *Let $(X, \mathcal{T})$ be a* **Topological Space** . Let $A \subset B \subset C \subset X$ and let B be **Set-Open** in $(X, \mathcal{T})$. Then we call C a **Neighborhood** of A in $(X, \mathcal{T})$. If $x \in X$, then we call a **Neighborhood** of $\{x\}$ a **Neighborhood** of x. We denote the collection of all **Neighborhoods** of $x \in X$ with $\mathcal{U}_{\mathcal{T}}(x)$. and we call this the **Neighborhood Filter** of x. By 2.3.8, is a **Filter** on X, and se we call this the **Neighborhood Filter** of $\mathcal{T}$ at x.

Proposition 2.3.8 (**Neighborhood Filter** is a **Filter**)**.** *Let $(X, \mathcal{T})$ be a* **Topological Space** For each $x \in X$, let $\mathcal{U}_{\mathcal{T}}(x)$ denote the **Neighborhood Filter** of x. The following are true

(i) $\mathcal{U}_{\mathcal{T}}$(x) is a **Filter** on X.

(ii) For each $U \in \mathcal{U}_{\mathcal{T}}(x)$, $x \in U$.

(iii) Let $x, y \in X$. Then, if $U \in \mathcal{U}_{\mathcal{T}}(x)$, then there exists $V \in \mathcal{U}_{\mathcal{T}}(x)$ such that for each $y \in V$, $U \in \mathcal{U}_{\mathcal{T}}(y)$.

Proof of 2.3.8 i. Clearly $x \in X \subset X \subset X \subset X \in \mathcal{T}$, so $X \in \mathcal{U}_{\mathcal{T}}(x)$. Thus $\mathcal{U}_{\mathcal{T}}(x)$ satisfies 2.2.1. i. Also, since $x \notin \emptyset$, $\emptyset \notin \mathcal{U}_{\mathcal{T}}(x)$. Hence $\mathcal{U}_{\mathcal{T}}(x)$ satisfies 2.2.1. ii. If $\{G_1, G_2\} \subset \mathcal{U}_{\mathcal{T}}(x)$ with $G_1 \cap G_2 \neq \emptyset$ then there are **Set-Open** U_i with $x \in U_i \subset G_i$. For these U_i, $x \in U_1 \cap U_2 \subset U_1 \cap U_2 \subset G_1 \cap G_2$ and $U_1 \cap U_2 \in \mathcal{T}$. Hence, $\mathcal{U}_{\mathcal{T}}$ satifies 2.2.1. iv. It is obvious that $\mathcal{U}_{\mathcal{T}}(x)$ satisfies 2.2.1. iii. □

Proof of 2.3.8 ii. Painfully Obvious □

Proof of 2.3.8 iii. Let $U \in \mathcal{U}_{\mathcal{T}}(x)$. Then there exists **Set-Open** V with $x \in V \subset U$. Since V is **Set-Open** , $V \in \mathcal{U}_{\mathcal{T}}(x)$. Let $Y \in V$. Then, $y \in V \subset V \subset U$, so $U \in \mathcal{U}_{\mathcal{T}}(y)$. Hence 2.3.8 *iii* is satisfied. □

Proposition 2.3.9 (**Topology** from **Neighborhood Filters**)**.** *Let $X \neq \emptyset$. For each $x \in X$, let $\mathcal{U}(x) \subset 2^X$ such that each $\mathcal{U}(x)$ satisfies 2.2.1. iii 2.2.1. iv and 2.3.8 ii and the collection $\{\mathcal{U}(x) | x \in X\}$ satisfies 2.3.8 iii. Then there existss a Unique topology $\mathcal{T}$ on X such that for each $x \in X$, $\mathcal{U}(x)$ is the* **Neighborhood Filter** for $\mathcal{T}$ at x.

Proof. □

Definition 2.3.10 (Relation of Equal **Neighborhood Filters**). *Let* $(Z, \mathcal{T}_Z)$ *be a* **Topological Space** Define the relation $\cong \subset Z \times Z$ by setting, for $x, y \in Z$,

$$x \cong y \iff \mathcal{U}_{\mathcal{T}_Z}(x) = \mathcal{U}_{\mathcal{T}_Z}(y) \tag{2.12}$$

We call $\cong$ the **Relation Of Equal Neighborhood Filters** on $(Z, \mathcal{T}_Z)$

Proposition 2.3.11 (**Relation Of Equal Neighborhood Filters** on) **.** *The* **Relation Of Equal Neighborhood Filters** on $\cong$ on a **Topological Space** $(Z, \mathcal{T}_Z)$ forms an **Equivalence Relation** on Z.

Proof. Let $x \in (Z, \mathcal{T}_Z)$. Then $\mathcal{U}_{\mathcal{T}_Z}(x) = i\, \mathcal{U}_{\mathcal{T}_Z}(x)$, so $x \cong x$. Thus $\cong$ is **Reflexive** .

Let $x, y \in (Z, \mathcal{T}_Z)$. Suppose $x \cong y$. Then $\mathcal{U}_{\mathcal{T}_Z}(x) = \mathcal{U}_{\mathcal{T}_Z}(y)$, so trivially $\mathcal{U}_{\mathcal{T}_Z}(y) = i\, \mathcal{U}_{\mathcal{T}_Z}(x)$, and thus $y \cong x$. Hence, $\cong$ is **Symmetric**

Let $x, y, z \in (Z, \mathcal{T}_Z)$. Let $x \cong y$ and $y \cong z$. Then, $\mathcal{U}_{\mathcal{T}_Z}(x) = i\, \mathcal{U}_{\mathcal{T}_Z}(y) = \mathcal{U}_{\mathcal{T}_Z}(z)$ so that $x \cong z$. Thus $\cong$ is **Transitive** Since $\cong$ is **Reflexive** , **Symmetric** , and **Transitive** , it is an **Equivalence Relation** .

□

Definition 2.3.12 (**Accumulation Point** , **Closure** , **Interior** , **Boundary**). *Let* $(X, \mathcal{T})$ *be a* **Topological Space** . Let $A \subset X$. We define the following.

1. $A' = \{x \in X | (\forall U \in \mathcal{U}_{\mathcal{T}}(A))((U \setminus A) \cap \{x\} \neq \emptyset)\}$

2. $\overline{A} = A \cup A'$

3. $\partial(A) = \overline{A} \cap \overline{X \setminus A}$

4. $\overset{\circ}{A} = A \setminus \overline{X \setminus A}$

We call an element of A' an **Accumulation Point** of A. We call $\overline{A}$ the **Closure** of A. We call $\overset{\circ}{A}$ the **Interior** of A. We call $\partial(A)$ the **Boundary** of A.

Definition 2.3.13 (**SubBasis**). *Let* $X \neq \emptyset$ *and let* $B \subset 2^X$. *We denote the* ***Coarsest*** **Topology** on X containing B with $\mathcal{T}_X(B)$. We say that B is a **SubBasis** for $\mathcal{T}_X(B)$ and we call $\mathcal{T}_X(B)$ the **Topology** on X **Generated By** B.

Proposition 2.3.14 (Characterization Of Generated Topology)**.** *Let* $X \neq \emptyset$ *and* $F \subset 2^X$. *Define*

$$\mathcal{T}_{Prop} = \left\{ \bigcup_{\alpha \in A} \bigcap_{i=1}^{N_\alpha} U_{i,\alpha} | (\forall \alpha \in A) \left((N_\alpha \in \mathbb{N}) \wedge \left((\forall i \in \{1, \cdots, N_\alpha\}) \left(U_{i,\alpha} \in F \right) \right) \right) \right\} \cup \{X, \emptyset\}$$

Then $\mathcal{T}_X(F) = \mathcal{T}_{Prop}$.

Proof. We first show that $\mathcal{T}_{Prop}$ is a **Topology** on X. For **Closure Under Unions** , Let $B \neq \emptyset$ and $\{B_\beta\}_{\beta \in B} \subset \mathcal{T}_{Prop}$ Then for each $\beta \in B$, we can find A_β such that for each $\alpha_\beta \in A_\beta$, there is an $N_{\alpha_\beta} \in \mathbb{N}$ such that for each $i \in \{1, \cdots, N_{\alpha_\beta}\}$, $U_{i,\alpha_\beta} \in F$ and

$$B_\beta = \bigcup_{\alpha \in A_\beta} \bigcap_{i=1}^{N_{\alpha_\beta}} U_{i,\alpha_\beta} \tag{2.13}$$

Hence, we can write

$$\begin{aligned}
\bigcup_{\beta \in B} B_\beta &= \bigcup_{\beta \in B} \bigcup_{\alpha_\beta \in A_\beta} \bigcap_{i=1}^{N_{\alpha_\beta}} U_{i,\alpha_\beta} \\
&= \bigcup_{\alpha_\beta \in \bigcup\limits_{\beta \in B} A_\beta} \bigcap_{i=1}^{N_{\alpha_\beta}} U_{i,\alpha_\beta} \in \mathcal{T}_{Prop}
\end{aligned}$$

For **Closure Under Finite Intersections** , let $N \in \mathbb{N}$ and $\{B_j\}_{j=1}^N \subset \mathcal{T}_{Prop}$. Then for each $j \in \{1, \cdots, N\}$, there is an A_j such that for each $\alpha_j \in A_j$, there is an $N_{\alpha_j} \in \mathbb{N}$ such that for each $i \in \{1, \cdots, N_{\alpha_j}\}$, $U_{i,\alpha_j} \in F$ and

$$\begin{aligned}
\bigcap_{j=1}^{N} B_j &= \bigcap_{j=1}^{N} \bigcup_{\alpha_j \in A_j} \bigcap_{i=1}^{N_{\alpha_j}} U_{\alpha_j,i} \\
&= \bigcup_{\{\alpha_j\}_{j=1}^N \in \prod\limits_{j \in \{1,\cdots,N\}} A_j} \left(\bigcap_{j=1}^{N} \bigcap_{i=1}^{N_{\alpha_j}} U_{\alpha_j,i} \right) \in \mathcal{T}_{Prop}
\end{aligned}$$

By construction, $X \in \mathcal{T}_{Prop}$ and $\emptyset \in \mathcal{T}_{Prop}$, so $\mathcal{T}_{Prop}$ is in fact a **Topology** on X. By taking the union over the intersection of a single element, we have $F \subset \mathcal{T}_{Prop}$, so that $\mathcal{T}_X(F) \subset \mathcal{T}_{Prop}$. Furthermore, $\mathcal{T}_X(F)$ is closed under finite intersections and arbitrary unions so that it must contain $\mathcal{T}_{Prop}$. Hence, equality holds.

□

Definition 2.3.15 (**Basis**). *Let* $(X, \mathcal{T})$ *be a* **Topological Space** and let $B \subset \mathcal{T}$ such that each element of $\mathcal{T}$ can be written as a union of elements of B. Then we call B a **Basis** for $\mathcal{T}$.

Proposition 2.3.16. *Let* $(X, \mathcal{T})$ *be a* **Topological Space** and let $\mathcal{G} \subset \mathcal{T}$ such that $\{\emptyset, X\} \subset \mathcal{G}$. The following conditions are equivalent

1. For every **Set-Open** U, for every $x \in U$, there exists an $G_x \in \mathcal{G}$ such that $x \in G_x \subset U$.

2. $\mathcal{G}$ is a **Basis** for $\mathcal{T}$.

$1 \implies 2$. Let $U \in \mathcal{T}$. Then we can write $U = \bigcup_{x \in U} G_x$, implying that $\mathcal{G}$ is a **Basis** . □

$2 \implies 1$. Let U is **Set-Open** , then since $\mathcal{G}$ is a **Basis** , there is a $\{G_\alpha\}_{\alpha \in A} \subset \mathcal{G}$ such that $U = \bigcup_{\alpha \in A} G_\alpha$. Hence, if $x \in U$, then $x \in G_\alpha$ for some $\alpha \in A$, and obviously $G_\alpha \subset U$, so 1 holds and we're done. □

Proposition 2.3.17 (Bassis Of Generated Topology). *Let $X \neq \emptyset$ and let $B \subset 2^X$ Then B is a* ***Basis*** *for* $\mathcal{T}_X(B)$ *if and only if the following hold*

1. $X \in B$
2. $\emptyset \in B$.
3. *For each $U, V \in B$, For each $x \in U \cap V$, there is a $W \in B$ with $x \in W \subset U \cap V$.*

Proof. I first claim that it is sufficient to show that any finite intersection of elements of B can be written as a union of elements of B. By Induction, proving for a binary intersection is sufficient. Hence, let $U, V \in B$ with $U \cap V \neq \emptyset$. Then for each $x \in U \cap V$, by assumption, there exists a $W_x \in B$ such that $x \in W_x \subset U \cap V$. Hence, we can write

$$U \cap V \subset \bigcup_{x \in U \cap V} W_x \subset U \cap V$$

showign that finite intersctions of elements of B can be written as unions of elements of B. Hence, by 2.3.14, $\mathcal{T}_X(B)$ consists of exactly the unions of elements of B, finishing one direction. For the other direction, if B is a **Basis** for $\mathcal{T}_X(B)$, then by 2.3.16, sicnce $\mathcal{T}_X(B)$ contains finite intersections of elements of B, the given properties hold.

□

Definition 2.3.18 (**Fundamental System Of Neighborhoods**). *Let $(X, \mathcal{T})$ be a* **Topological Space** . Let $x \in X$. Let $\mathcal{U}_{\mathcal{T}}(x)$ denote the **Neighborhood Filter** of $\mathcal{T}$ at x. We say that $\mathcal{K}$ is a **Fundamental System Of Neighborhoods** for X at x if

(i) $\mathcal{K} \subset \mathcal{U}_{\mathcal{T}}(x)$.

(ii) . For each $U \in \mathcal{U}_{\mathcal{T}}(x)$, there exists $V \in \mathcal{K}$ such that $V \subset U$.

Definition 2.3.19 (**Neighborhood Basis**). *Let $(X, \mathcal{T})$ be a* **Topological Space** and let $X \in X$. Let $F \subset \mathcal{T}$ such that for each $U \in \mathcal{T}$ with $x \in U$, there exists $f \in F$ with $f \subset U$. Further, let $x \in G$ for each $G \in F$. Then we call F a **Neighborhood Basis** for $\mathcal{T}$ at x.

Proposition 2.3.20 (**Neighborhood Basis** Facts). *Let $(X, \mathcal{T})$ be a Topological Space and let $x \in X$. The following are true*

2.3.2 Continuous Functions

Definition 2.3.21 (**Continuous**). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be* **Topological Spaces** . We say that a function $f : X \to Y$ is **Continuous** and that it exhibits **Continuity** with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ $f^{-1}(\mathcal{T}_Y) \subset \mathcal{T}_X$. We may make the **Topologies** explicit by writing $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, in which case we just say that f is **Continuous** or that f posesses **Continuity** .

Definition 2.3.22 (**Homeomorphism**). *Let $X, \mathcal{T}_X$ and $Y, \mathcal{T}_Y$ be* **Topological Spaces** . Let $f : X \to Y$ such be a **Continuous** **Bijection** such that $f^{-1} : Y \to X$ is also **Continuous** . Then we say that f is a **Homeomorphism** from X to Y and we say that X and Y are **Homeomorphic** .

Definition 2.3.23 (**Weak Topology**). *Let X be a set. For each $\alpha \in A$, let (Y_α, T_α) be a* **Topological Space** , and let $\phi_\alpha : X \to (Y_\alpha, T_\alpha)$. Let $\mathcal{T}$ be the **Coarsest**possible **Topology** on X such that for each $\alpha \in A$, $\phi_\alpha : (X, \mathcal{T}) \to (Y, \mathcal{T}_\alpha)$ is **Continuous** . We call $\mathcal{T}$ the **Weak Topology** on X induced by $\{\phi_\alpha\}_{\alpha \in A}$ In the literature, (Schaefer, Topological Vector Spaces 2nd ed), the weak topology is also sometime s referred to as the Projective Topology or as the Kernel Topology

Definition 2.3.24 (**Inductive Topology**). *Let X be a set and for each $\alpha \in A$, let $(Y_\alpha, \mathcal{T}_\alpha)$ be a* **Topological Space** . Furthermore, for each $\alpha \in A$, let $\phi_\alpha : (Y, \mathcal{T}_\alpha) \to X$. Let $\mathcal{T}$ be the **Finest**topology on X for which each ϕ_α is **Continuous** . fWe call $\mathcal{T}$ the **Inductive Topology** on X induced by $\{\phi_\alpha\}_{\alpha \in A}$.

:

Definition 2.3.25 (**Open**). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be* **Topological Spaces** . We say that $f : X \to Y$ is **Open** if $f(U)$ is **Set-Open** in $(Y, \mathcal{T}_Y)$ for every **Set-Open** $U \in (X, \mathcal{T}_X)$.

Definition 2.3.26 (**Closed**). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be* **Topological Spaces** . We say that $f : X \to Y$ is **Closed** if $f(K)$ is **Set-Closed** in $(Y, \mathcal{T}_Y)$ for every **Set-Closed** $K \in (X, \mathcal{T}_X)$.

2.3.3 Subspaces And Quotient Spaces

Definition 2.3.27 (**Subspace Topology**). *Let $(X, \mathcal{T}_X)$ be a* **Topological Space** , Let $Y \subset X$, and let f be the **Insertion Function** of Y into X. We call the **Weak Topology** on Y generated by f which we will denote here with $\mathcal{T}_Y$, the **Subspace Topology** of Y relative to $(X, \mathcal{T}_X)$. We call $(Y, \mathcal{T}_Y)$ the **Subspace Topological Space** . Unless otherwise specified, when referring to a subset of a **Topological Space** , we consider that subset as being a **Topological Space** which is endowed with the **Subspace Topology** , and when we say that a subset of a **Topological Space** has a particular (Topological) property which has thus far only been defined for a **Topological Space** , we mean that the **Subspace Topological Space** has that property.

Definition 2.3.28 (Quotient Space Topology)**.** *Let* $(Z, \mathcal{T}_Z)$ *be a topological space. Let* $\cong$ *be the* **Relation Of Equal Neighborhood Filters** on $(Z, \mathcal{T}_Z)$. Let T be the **Quotient Map** of Z under the relation $\cong$. Define $\mathcal{T}_{Z/\cong}$ by

$$\mathcal{T}_{Z/\cong} = \left\{ \bigcup_{x \in U} \{T(x)\} \in 2^{Z/\cong} | U \in \mathcal{T}_Z \right\} \tag{2.14}$$

By 2.3.29, $\mathcal{T}_{Z/\cong}$ is a topology on $Z/\cong$. We call $\mathcal{T}_{Z/\cong}$ the **Quotient Topology** and we call $\left(Z/\cong, \mathcal{T}_{Z/\cong}\right)$ the **Quotient Topological Space** of $(Z, \mathcal{T}_Z)$.

Proposition 2.3.29 (Quotient Space Topology)**.** *Let* $(Z, \mathcal{T}_Z)$ *be a* **Topological Space** with **Quotient Topological Space** $\left(Z/\cong, \mathcal{T}_{Z/\cong}\right)$ and **Quotient Map** T.

Then the following are true.

1. $\mathcal{T}_{Z/\cong}$ is a **Topology** on $Z/\cong$.
2. $T : (Z, \mathcal{T}_Z) \to (Z/\cong, \mathcal{T}_{Z/\cong})$ is **Continuous** .
3. If U is **Set-Open** (**Set-Closed**) in $(Z, \mathcal{T}_Z)$ then $T(U)$ and $T(Z \setminus U)$ **Partition** $Z/\cong$.
4. If U is **Set-Open** in $(Z, \mathcal{T}_Z)$, then $T^{-1}(T(U)) = U$.
5. If K is **Set-Closed** in $(Z, \mathcal{T}_Z)$, then $T^{-1}T(K) = K$.
6. $T : (Z, \mathcal{T}_Z) \to (Z/\cong, \mathcal{T}_{Z/\cong})$ is **Open** .
7. $T : (Z, \mathcal{T}_Z) \to (Z/\cong, \mathcal{T}_{Z/\cong})$ is **Closed** .
8. $(Z, \mathcal{T}_Z)$ is a **Compact** space if and only if $(Z/\cong, \mathcal{T}_{Z/\cong})$ is a **Compact** space.
9. If $\mathcal{B}$ is a **Basis** for $\mathcal{T}_z$, then $\{T(U)|U \in \mathcal{B}\}$ is a **Basis** for $\mathcal{T}_{Z/\cong}$.
10. If T is **Injective** , then it is a **Homeomorphism** .

Proof of 1. Since $\emptyset \in \mathcal{T}_Z$, we have

$$\emptyset = \bigcup_{x \in \emptyset} \{Tx\} \in \mathcal{T}_{Z/\cong} \tag{2.15}$$

Since $Z \in \mathcal{T}_Z$, and by 2.1.54,

$$Z/\cong = \bigcup_{x \in Z} \{[x]\} = \bigcup_{x \in Z} \{T(x)\} \in \mathcal{T}_{Z/\cong} \tag{2.16}$$

Let $\{U_\alpha | \alpha \in A\} \subset \mathcal{T}_{Z/\cong}$. For each $\alpha \in A$, there exists $B_\alpha \in \mathcal{T}_Z$ such that we have

$$U_\alpha = \bigcup_{x \in B_\alpha} \{Tx\} \tag{2.17}$$

Since $\bigcup_{\alpha\in A} B_\alpha \in \mathcal{T}_Z$, we have

$$\bigcup_{\alpha\in A} U_\alpha = \bigcup_{\alpha\in A}\bigcup_{x\in U_\alpha}\{T(x)\} = \bigcup_{x\in \bigcup\limits_{\alpha\in A} B_\alpha}\{T(x)\} \in \mathcal{T}_{Z/\cong} \tag{2.18}$$

Let $\{U_i\}_{i=1}^n \subset \mathcal{T}_{Z/\cong}$. For each $i \in \{1,...,n\}$, there exists $B_i \in \mathcal{T}_Z$ such that

$$U_i = \bigcup_{x\in B_i}\{T(x)\} \tag{2.19}$$

Suppose

$$[x_0] \in \bigcap_{i=1}^n \bigcup_{x\in B_i}\{T(x)\} \tag{2.20}$$

Then for each $i \in \{1,...,n\}$, there is a $y_i \in B_i$ such that $y_i \cong x_0$. Since each B_i is **Set-Open** , the definition of $\cong$ implies that $x_0 \in B_i$ for every i. Hence,

$$x_0 \in \bigcap_{i=1}^n B_i \tag{2.21}$$

Implying

$$[x_0] \in \bigcup_{x\in \bigcap\limits_{i=1}^n B_i}\{[x]\} \tag{2.22}$$

Hence,

$$\bigcap_{i=1}^n \bigcup_{x\in B_i}\{T(x)\} \subset \bigcup_{x\in \bigcap\limits_{i=1}^n B_i}\{[x]\} \tag{2.23}$$

Furthermore, since the reverse inclusion is obvious, and since $\bigcap_{i=1}^n B_i \in \mathcal{T}_Z$, we have

$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n \bigcup_{x\in B_i}\{T(x)\} = \bigcup_{x\in \bigcap\limits_{i=1}^n B_i}\{T(x)\} \in \mathcal{T}_{Z/\cong} \tag{2.24}$$

□

Proof of 2. Let $V \in \mathcal{T}_{Z/\cong}$. Let $x_0 \in T^{-1}(V)$. Then $[x_0] \in V$. By definition, there is a $U \in \mathcal{T}_Z$ such that

$$T(U) \subset \bigcup_{x\in U}\{T(x)\} = V \tag{2.25}$$

Hence there is a $y_0 \in U$ such that

$$[x_0] \in T(y_0) = \{[y_0]\} \tag{2.26}$$

Therefore, $x \cong y$. Definition of the **Relation Of Equal Neighborhood Filters** on i mplies $\mathcal{U}(x_0) = \mathcal{U}(y_0)$. Hence, $x_0 \in U \subset T^{-1}(V)$. □

Proof of 3. Let K be closed in $(Z, \mathcal{T}_Z)$. Then each point x_0 in $Z \setminus K$ has some Open U_{x_0} containing x_0 which is **Disjoint** from K. Hence $y_0 \ncong x_0$ for any $y_0 \in K$, $x_0 \in Z \setminus K$. Hence $T(K)$ is **Disjoint** from $T(Z \setminus K)$. This fact, paired with 2.1.7, implies $T(Z \setminus K)$ and T(K) is a **Partition** of $Z/\cong$. □

Proof of 4. Let $U \in \mathcal{T}_Z$. The nontrivial direction to prove is $T^{-1}(T(U)) \subset U$. Let $y \in T^{-1}(T(U))$. Then $[y] = Ty \in T(U)$. Hence, $[y] = T(x) = [x]$ for some $x \in U$. Since $y \cong x$ and $x \in U \in \mathcal{U}_{\mathcal{T}_Z}(x)$, we have $U \in \mathcal{U}_{\mathcal{T}_Z}(y)$. Hence $y \in U$. Since y was arbitrary, $T^{-1}(T(U)) \subset U$, and equality is obvious because the other direction of inclusion is trivial. □

Proof of 5. Let K be **Set-Closed** in $(Z, \mathcal{T}_Z)$. Part 3 Of this result implies $Z/\cong$ is partitioned by $T(K)$ and $T(Z \setminus K)$.

By part 4 of this proposition,

$$\begin{aligned} T^{-1}(T(K)) &= T^{-1}(T(Z) \setminus T(Z \setminus K)) \\ &= T^{-1}(Z/\cong \setminus T(Z \setminus K)) \\ &= T^{-1}(Z/\cong) \setminus T^{-1}(T(Z \setminus K)) \\ &= Z \setminus (Z \setminus K) \\ &= K \end{aligned}$$

□

Proof of 6. Let $U \in \mathcal{T}_Z$. Then by definition of the **Quotient Topology**

$$TU = \bigcup_{x \in U} \{T(x)\} \in \mathcal{T}_{Z/\cong} \tag{2.27}$$

□

Proof of 7. Let K be **Set-Closed** in $(Z, \mathcal{T}_Z)$. Then $Z \setminus K \in \mathcal{T}_Z$. By Parts 3 and five of this proposition, we know $T(K) = Z/\cong \setminus T(Z \setminus K)$ and also that $T(Z \setminus K) \in \mathcal{T}_{Z/\cong}$. Hence $T(K)$ is closed in $(Z/\cong, \mathcal{T}_{Z/\cong})$. □

Proof of 8. Let $(Z, \mathcal{T}_Z)$ be **Compact** . Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of $(Z/\cong, \mathcal{T}_{Z/\cong})$. Then $\{T^{-1}(U_\alpha) | \alpha \in A\}$ is an open covering of $(Z, \mathcal{T}_Z)$. **Compactness** of $(Z, \mathcal{T}_Z)$ guarantees the existence of a finite subcovering $\{T^{-1}(U_{\alpha_i}) | i \in \{1, ..., n\}\}$. Hence $\{U_{\alpha_i} | i \in \{1, ..., n\}\} = \{TT^{-1}(U_{\alpha_i}) | i \in \{1, ..., n\}\}$ is an **Set-Open Cover** of $(Z/\cong, \mathcal{T}_{Z/\cong})$. And the **Compactness** of $(Z/\cong, \mathcal{T}_{Z/\cong})$ is verified.

Now, suppose $(Z/\cong, \mathcal{T}_{Z/\cong})$ is **Compact** . Let $\{V_\beta | \beta \in B\}$ be an **Set-Open Cover** of $(Z, \mathcal{T}_Z)$. Since T is an **Open** mapping, $\{T(V_\beta) | \beta \in B\}$ is an **Set-Open Cover** of $(Z/\cong, \mathcal{T}_{Z/\cong})$ which by **Compactness** has a **Finite Subcover** $\{T(V_{\beta_i}) | i \in \{1, ..., n\}\}$. By part 4 of 2.3.29, $\{V_{\beta_i} | i \in \{1, ..., n\}\} = \{T^{-1}(T(V_{\beta_i})) | i \in \{1, ..., n\}\}$ is then an **Set-Open Subcover** of $(Z, \mathcal{T}_Z)$. □

Proof of 9. Let $\mathcal{B}$ be a basis for $\mathcal{T}_z$ and let $V \in \mathcal{T}_{Z/\cong}$. Then $T^{-1}(Z) \in \mathcal{T}_Z$, and so there is a subcollection $\{U_\alpha\}_{\alpha\in A} \subset \mathcal{B}$ such that $T^{-1}(V) = \bigcup_{\alpha\in A} U_\alpha$. Hence,

$$\begin{aligned} V &= T(T^{-1}(V)) \\ &= T\left(\bigcup_{\alpha\in A} U_\alpha\right) \\ &= \bigcup_{\alpha\in A} T(U_\alpha) \end{aligned}$$

□

Proof of 10. If T is **Injective** , then since it is **Continuous** Part 2 of this result, open by part 6 of this result, and **Surjective** by 2.1.7, it is a **Homeomorphism** . □

2.3.4 Product Spaces

Definition 2.3.30 (**Product Topology**). *Let $A \neq \emptyset$. For each $\alpha \in A$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a* **Topological Space** . We call the **Weak Topology** on $\prod_{\alpha\in A} X_\alpha$ induced by $\{\pi_\alpha : \prod_{\alpha\in A} X_\alpha \to (X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in A}$ the **Product Topology** .

2.3.5 Open and Closed Maps

2.3.6 Convergence of Filters

Definition 2.3.31 (**Convergence**). *Let $(X, \mathcal{T})$ be a* **Topological Space** . Let $\mathcal{F}$ be a **Filter** on X. Let $\mathcal{B}$ be a **Filter Base** for $\mathcal{F}$. Let $\mathcal{U}_\mathcal{T}$(x) denote the **Neighborhood Filter** of $\mathcal{T}$ at x. Let $\mathcal{F}$be **Finer** than $\mathcal{U}_\mathcal{T}$(x). Then we say the following: x is a **Limit** of $\mathcal{F}$, x is a **Limit** of $\mathcal{B}$, $\mathcal{F}$ **Converges** to x, $\mathcal{B}$ **Converges** to x, $\mathcal{F}$ is **Convergent** to x, $\mathcal{B}$ is **Convergent** to x, $\mathcal{F}$ posesses **Convergence** to x, and $\mathcal{B}$ posesses **Convergence** to x.

Remark 2.3.32. By 2.2.10. v, a **Filter** $\mathcal{F}$ is a **Filter Base** for itself. For this reason, in many of the following items, we present propositions or definitions only in terms of **Filter Bases** , understanding that the definition also then applies to **Filters**.

Proposition 2.3.33 (**Convergence** Facts). *Let $(X, \mathcal{T})$ be a* **Topological Space** . Let $x \in X$. Let $\mathcal{U}_\mathcal{T}$(x) denote the **Neighborhood Filter** for $\mathcal{T}$ at x. Let $\mathcal{F}$ be a **Filter** on X. Let $\mathcal{B}$ be a **Filter Base** for $\mathcal{F}$. The following are true.

(i) x is a **Limit** of $\mathcal{F}$ if and only if x is a **Limit** of $\mathcal{B}$.

(ii) x is a **Limit** of $\mathcal{B}$ if and only if, for some **Fundamental System Of Neighborhoods** $\mathcal{U}$ of x, **Nested**($\mathcal{B}$,$\mathcal{U}$) holds.

(iii) x is a **Limit** of $\mathcal{B}$ if and only if, for every **Fundamental System Of Neighborhoods** $\mathcal{U}$ of x, **Nested**($\mathcal{B}$,$\mathcal{U}$) holds.

(iv) If x is a **Limit** of $\mathcal{F}$ and $\mathcal{G}$ is **Finer** than $\mathcal{F}$ then x is a **Limit** of $\mathcal{G}$.

(v) If x is a **Limit** of $\mathcal{F}$ in $(X, \mathcal{T})$, and $\mathcal{T}_1$ is a **Coarser Topology** on X, then x is a **Limit** of $\mathcal{F}$ in $(X, \mathcal{T}_1)$.

(vi) Let $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ be a collection of **Filters** on X each of which have x as a **Limit** . Then $\bigcap_{\alpha \in A} \mathcal{F}_\alpha$ has x as a **Limit** .

(vii) x is a **Limit** of $\mathcal{F}$ if and only if x is a **Limit** of every **Ultrafilter** which is **Finer** than $\mathcal{F}$.

Definition 2.3.34 (**Cluster Point**)**.** *Let* $(X, \mathcal{T})$ *be a* **Topological Space** . Let $x \in X$. Let $\mathcal{B}$ be a **Filter Base** in X. We say that x is a **Cluster Point** of $\mathcal{B}$ if

$$x \in \bigcap_{U \in \mathcal{B}} \overline{U} \tag{2.28}$$

Proposition 2.3.35 (**Cluster Point** Facts)**.** *Let* $(X, \mathcal{T})$ *be a* **Topological Space** . Let $\mathcal{B}$ and $\mathcal{D}$ be **Equivalent Filter Bases** . Let $\mathcal{F}$ be the **Generated By** $\mathcal{B}$. Let $\mathcal{G}$ be an **Ultrafilter** in X. Let $x \in X$. The following are true.

(i) The set of **Cluster Points** of $\mathcal{B}$ is **Set-Closed** in $(X, \mathcal{T})$.

(ii) x is a **Cluster Point** of $\mathcal{B}$ if and only if x is a **Cluster Point** of $\mathcal{D}$.

(iii) x is a **Cluster Point** of $\mathcal{B}$ if and only if x is a **Cluster Point** of $\mathcal{F}$, viewed as a **Filter Base** .

(iv) Let $\mathcal{U}$ be a **Fundamental System Of Neighborhoods** for $\mathcal{T}$ at x. Then x is a **Cluster Point** for $\mathcal{B}$ if and only if for each $U \in \mathcal{U}$ and for each $B \in \mathcal{B}$, $U \cap B \neq \emptyset$.

(v) x is a **Cluster Point** for $\mathcal{F}$ if and only if there is a **Filter Finer** than $\mathcal{F}$ for which x is a **Limit** .

(vi) If x is a **Limit** of $\mathcal{F}$, then it is a **Cluster Point** of x.

(vii) x is a **Cluster Point** of $\mathcal{G}$ if and only if x is a **Limit** of $\mathcal{G}$.

Definition 2.3.36 (**Limit**)**.** Let X be a set and Y be a topological space. Let $\mathcal{F}$ be a **Filter** in X and let $f : X \to Y$. Let $y \in Y$. If y is a **Limit** of $f(\mathcal{F})$, then we say that y is a **Limit** of f with respect to $\mathcal{F}$ and we write $y \in \lim_{\mathcal{F}} f$ or we may write $y \in \lim\limits_{x,\mathcal{F}} f(x)$. If $\lim_{\mathcal{F}} f$ is a **Singleton** then we will, as an abuse of notation, write $y = \lim_{\mathcal{F}} f$ and $y = \lim\limits_{x,\mathcal{F}} f(x)$. If $\mathcal{B}$ is a **Filter Base** on X and If y is a **Cluster Point** $f(\mathcal{B})$ then we sayt hat y is a **Cluster Point** of f with respect to $\mathcal{B}$.

Definition 2.3.37 (**Limit**)**.** Let $(X, \mathcal{T})$ be a **Topological Space** . Let $\sigma = \{x_\alpha\}_{\alpha \in A} \subset X$ be a **Net** in X. We say that $x \in X$ is a **Limit** of σ if x is a **Limit** of σ with respect to the **Section Filter** on A. We say that $x \in X$ is a **Cluster Point** of σ if x is a **Cluster Point** of σ with respect to the **Section Filter** on A.

Proposition 2.3.38 (**Limit**). *Let $X \neq \emptyset$. Let $(Y, \mathcal{T})$ be a* **Topological Space** . Let $f : X \to Y$. Let $\mathcal{F}$ be a **Filter** in X. Let $y \in Y$. The Following are true

(i) y is a **Limit** of f with respect to $\mathcal{F}$ if and only if, for each **Neighborhood** V of y, there exists $M \in \mathcal{F}$ with $f(M) \subset V$.

(ii) y is a **Limit** of f with respect to $\mathcal{F}$ if and only if for each **Neighborhood** V of y, $f^{-1}(V) \in \mathcal{F}$.

(iii) y is a **Cluster Point** of f with respect to $\mathcal{F}$ if and only if for each **Neighborhood** V of y and each $M \in \mathcal{F}$, $f(M) \cap V \neq \emptyset$.

(iv) If y is a **Limit** of f with respect to $\mathcal{F}$ in $(Y, \mathcal{T})$ and $\mathcal{T}_1$ is a **Coarser Topology** on Y than $\mathcal{T}$, then y is a **Limit** of f with respect to $\mathcal{F}$ in $(Y, \mathcal{T}_1)$.

(v) If y is a **Cluster Point** of f with respect to $\mathcal{F}$ in $(Y, \mathcal{T})$ and $\mathcal{T}_1$ is a **Coarser Topology** on Y than $\mathcal{T}$, then y is a **Cluster Point** of f with respect to $\mathcal{F}$ in $(Y, \mathcal{T}_1)$.

(vi) If y is a **Limit** of f with respect to $\mathcal{F}$ and $\mathcal{F}_1$ is a **Finer Filter** on X than $\mathcal{F}$, then y is a **Limit** of f with respect to $\mathcal{F}_1$.

(vii) If y is a **Cluster Point** of f with respect to $\mathcal{F}$ and $\mathcal{F}_1$ is a **Finer Filter** on X than $\mathcal{F}$, then y is a **Cluster Point** of f with respect to $\mathcal{F}_1$.

(viii) y is a **Cluster Point** of f with repsect to $\mathcal{F}$ if and only if there is a **Filter** $\mathcal{G}$ on X which is **Finer** than $\mathcal{F}$ such that y is a **Limit** of f with respect to $\mathcal{G}$.

(ix) The set of **Cluster Points** of f with respect to $\mathcal{F}$ is **Set-Closed** in Y, and may be empty.

Definition 2.3.39 (**Limit** of a **Function** at a point). Let $(X, \mathcal{T}_X)$ be a **Topological Space** . Let $(Y, \mathcal{T}_Y)$ be a **Topological Space** . Let $f : X \to Y$. Let $a \in X$. Let $\mathcal{B}$ be the **Neighborhood Filter** of X at a. Let y be a **Limit** of f with respect to $\mathcal{B}$. Then instead of the standard notation

$$y \in \lim_{x,\mathcal{B}} f(x)$$

we instead write

$$y \in \lim_{x \to a} f(x)$$

and we say that y is a **Limit** of f at a. If y is a **Cluster Point** of f with respect to $\mathcal{B}$, then we say that y is a **Cluster Point** of f at a.

Proposition 2.3.40. *For $i \in \{0, 1\}$, let $(X_i, \mathcal{T}_i)$ be* **Topological Spaces** . Let $f : X_0 \to X_1$. Let $x_0 \in X_0$. The following are true.

(i) f is **Continuous** at x_0 if and only if $f(x_0) \in \lim\limits_{x \to x_0} f(x)$.

(ii) If f is **Continuous** at x_0, then for every **Filter Base** $\mathcal{B}$in X which **Converges** to a, we have $f(\mathcal{B})$ **Converges** to $f(a)$.

(iii) If, for every **Ultrafilter** $\mathcal{U}$ on X which **Converges** to a , we have $f(\mathcal{U})$ converges to $f(a)$, then f is **Continuous** at a.

(iv) Let $X_2 \neq \emptyset$ and let $\mathcal{F}$ be a **Filter** on X_2. Let $f_1 : X_2 \to X_0$. Let $x_0 \in \lim_{x,\mathcal{F}} g(x)$. If f is continuous, then $f(x_0) \in \lim_{x,\mathcal{F}} f \circ g(x)$.

2.3.7 Separation Axioms

Definition 2.3.41 (Separation Axioms)**.** *Let* $(X, \mathcal{T})$ *be a* **Topological Space** . We define the following.

(i) We say X is **Hausdorff** , or **T2** if distinct points in X have **Disjoint Neighborhoods** .

Proposition 2.3.42 (**Hausdorff** Characterizations)**.** *Let* $(X, \mathcal{T})$ *be a* $\mathcal{T}$ *Space. The following are equivalent.*

(i) X *is* ***Hausdorff*** .

(ii) For all $x \in X$*, if* $\mathcal{U}_{\mathcal{T}}(x)$ *is the* **Neighborhood Filter** of X at x, then

$$\bigcap_{U \in \mathcal{U}_{\mathcal{T}}(x)} \overline{U} = \{x\}$$

(iii) $\Delta\,(X)$ is **Set-Closed** in $X \times X$.

(iv) For any index set A, $\Delta_A\,(X)$ is **Set-Closed** in $\prod_{\alpha \in A} X$.

(v) A **Filter** $\mathcal{F}$ in X has at most one **Limit** .

(vi) If a **Filter** $\mathcal{F}$ in X **Converges** , say $\mathcal{F} \to x$, then

(vii)

(viii)

(ix)

(x)

(xi)

2.3.8 Compactness

2.3.9 Countability Axioms

Definition 2.3.43 (**First Countable**). *Let $(x, \mathcal{T})$ be a* **Topological Space** . We say that X is **First Countable** if for each $x \in X$, there is a **Countable Neighborhood Basis** for $\mathcal{T}$ at X.

Definition 2.3.44 (**Second Countable**). *A* **Topological Space** which permits a **Countable Basis** is called **Second Countable**

Definition 2.3.45 (**Dense**). *Let $(X, \mathcal{T})$ be a* **Topological Space** and let $A \subset X$. We say that A is **Dense** in X if $\overline{(A)} = X$.

Definition 2.3.46 (**Separable**). *We say that a* **Topological Space** which permits a **Countable Dense** subset is **Separable** .

Definition 2.3.47 (**Lindelof**). *A* **Topological Space** in which every **Set-Open Cover** permits a **Countable Subcover** is called a **Lindelof** space.

2.4 Uniform Spaces

Definition 2.4.1 (**Uniformity**). *Let X be a set and let $\mathcal{W} \subset 2^{X \times X}$ such that*

*(i) $\mathcal{W}$ is a **Filter** on $X \times X$.*

(ii) For each $W \in \mathcal{W}$, $\Delta(X) \subset W$.

(iii) $W \in \mathcal{W} \implies W^{-1} \in \mathcal{W}$.

*(iv) **Nested**$(\{V \circ V | V \in \mathcal{W}\}, \mathcal{W})$ holds.*

*Then we call $\mathcal{W}$ a **Uniformity** on X. Furthermore, if $W \in \mathcal{W}$, then we call W an **Entourage** of $\mathcal{W}$. If W is an **Entourage** of $\mathcal{W}$ and If $(x, y) \in W$, then we say that x and y are W- **Close** . If R is a* **Relation** on X then saying that $x_0 R y_0$ is true whenever x_0 and y_0 are **Close Enough** means that there exists an **Entourage** V such that $V \subset R$.

Proposition 2.4.2 (**Uniformity** Characterization). *The definition for a **Uniformity** $\mathcal{W}$ on a nonemptyset X is unchanged if 2.4.1 iii and 2.4.1 iv are replaced by*

*(i) **Nested**$(\{V \circ V^{-1} | V \in \mathcal{W}\}, \mathcal{W})$ holds.*

Proof. For this whole proof, let $\mathcal{W} \subset 2^{X \times X}$ satisfy 2.4.1 i and 2.4.1 ii.

($\implies$) Let $\mathcal{W}$ satisfy 2.4.1 *iii* and 2.4.1 iv Let $W \in \mathcal{W}$. Then, by 2.4.1 iv, there is a $V \in \mathcal{W}$, such that $V \circ V \subset W$. Also, by 2.4.1 iii, $V^{-1} \in \mathcal{W}$. Hence, by 2.4.1 i and 2.2.1. iv, $\tilde{V} := V \cap V^{-1} \in \mathcal{W}$. Note that $\tilde{V}^{-1} = \tilde{V}$, and

$$\tilde{V} \circ \tilde{V}^{-1} = \tilde{V} \circ \tilde{V} \subset V \circ V \subset W$$

Since W was arbitrary, 2.4.2 i is satisfied.

($\Longleftarrow$) Let $\mathcal{W}$ satisfy 2.4.2 i. Let $W \in \mathcal{W}$. Then by 2.4.2 i, there is a $V \in \mathcal{W}$ such that $V \circ V^{-1} \subset W$. By **??**, $\Delta(X) \subset V$, so

$$V^{-1} = \Delta(X) \circ V^{-1} \subset V \circ V^{-1} \subset W$$

This implies $V \subset W^{-1}$. By 2.2.1. iii and 2.4.1 i, $W^{-1} \in \mathcal{W}$, so 2.4.1 iii is satisfied. Now, letting $\tilde{V} = V \cap V^{-1}$, we have $\tilde{V} \in \mathcal{W}$ by 2.2.1. iv and 2.4.1 iii , and also $\tilde{V}^{-1} = \tilde{V}$ so

$$\tilde{V} \circ \tilde{V} = \tilde{V} \circ \tilde{V}^{-1} \subset V \circ V^{-1} \subset W$$

so that 2.4.1 iv is verified and we're done.

□

Definition 2.4.3 (**Fundamental System Of Entourages**)**.** *Let $X \neq \emptyset$ and $\mathcal{W}$be a* ***Uniformity*** *on X. Let $\mathcal{B} \subset \mathcal{W}$ such that* ***Nested***$(\mathcal{B}, \mathcal{W}, h)$ *olds. Then we call $\mathcal{B}$a* ***Fundamental System Of Entourages*** *for $\mathcal{W}$.*

Proposition 2.4.4 (Fundamental System Facts)**.** *Let $X \neq \emptyset$. Let $\mathcal{W}$ be a* ***Uniformity*** *on X. The following are true.*

(i) Let $\mathcal{B}$ be a ***Fundamental System Of Entourages*** *for $\mathcal{W}$. Let $k \in \mathbb{N} \setminus \{0\}$. Then $\mathcal{B}_n := \{U^n | U \in \mathcal{B}\}$ is a* ***Fundamental System Of Entourages*** *for $\mathcal{W}$.*

(ii) There exists a ***Fundamental System Of Entourages*** *$\mathcal{D}$ for $\mathcal{W}$ which consists only of* **Symmetric Entourages** .

Proposition 2.4.5 (Fundamental System Characterization)**.** *Let $X \neq \emptyset$ and $\mathcal{B} \subset 2^{X \times X}$. Then there exists a* ***Uniformity*** *$\mathcal{W}$on X such that $\mathcal{B}$ is a* ***Fundamental System Of Entourages*** *for $\mathcal{W}$ if and only if the following hold.*

1. *$\mathcal{B}$ satisfies 2.2.7. iii.*

2. *$\mathcal{B}$ satisfies 2.4.1 ii*

3. *$\mathcal{B}$ satisfies 2.4.1 iv*

4. *For each $U \in \mathcal{B}$, there exists $V \in B$ such that $V \subset U^{-1}$.*

Furthermore, in this case $\mathcal{B}$ is a ***Filter Base*** *for $\mathcal{W}$.*

Proposition 2.4.6 (Uniformity Facts)**.**

Definition 2.4.7 (**Isomorphism**)**.** *Let $X_1, X_2 \neq \emptyset$. Let $\mathcal{W}_i$ be a* ***Uniformity*** *on X_i for $i \in \{1, 2\}$. We say that $f : X_1 \to X_2$ is a* ***Isomorphism*** *of $(X_1, \mathcal{W}_1)$ onto $(X_2, \mathcal{W}_2)$ if $f \times f(\mathcal{W}_1) = \mathcal{W}_2$, where $f \times f$ denotes the* ***Function Product*** *.*

2.4.1 Uniform Topologies

Proposition 2.4.8. *Let $X \neq \emptyset$ and $\mathcal{W}$ be a* ***Uniformity*** *on X. For each $x \in X$, define*

$$\mathcal{B}_x := \{U(x) | U \in \mathcal{W}\}$$

Define

$$\mathcal{T}_{\mathcal{W}} = \{A \subset X | (\forall x \in A)(A \in \mathcal{B}_x)\} \tag{2.29}$$

Then $\mathcal{T}_{\mathcal{W}}$ is the Unique **Topology** on X such that for each $x \in X$, $\mathcal{B}_x$ is a **Neighborhood Filter** for $\mathcal{T}_{\mathcal{W}}$ at x.

Proof. □

Definition 2.4.9 (**Uniform Topology**)**.** *Let $X \neq \emptyset$ and let $\mathcal{W}$ be a* ***Uniformity*** *on X. Let $\mathcal{T}_{\mathcal{W}}$ be the* **Topology** defined in 2.4.8. Then we call $\mathcal{T}_{\mathcal{W}}$ the **Uniform Topology** on X **Induced** by $\mathcal{W}$. We may call $(X, \mathcal{W}, \mathcal{T}_{\mathcal{W}})$ a **Uniform Space** .

Proposition 2.4.10. *Let $(X, \mathcal{W}, \mathcal{T}_{\mathcal{W}})$ be a* ***Uniform Space*** *. Let $\mathcal{U}$ denote the collection of* **Symmetric Entourages** . Let $M \subset X \times X$. Let $N \subset X$. Let $X \times X$ be endowed with the **Product Topology** . Then the following are true

(i) If $U \in \mathcal{U}$, then UMU is a **Neighborhood** of M in $X \times X$.

(ii)
$$\overline{M} = \bigcap_{U \in \mathcal{U}} UMU$$

(iii) If $U \in \mathcal{U}$, then $U(A)$ is a **Neighborhood** of A.

(iv)

(v)
$$\overline{N} = \bigcap_{U \in \mathcal{U}} U(N) = \bigcap_{U \in \mathcal{W}} U(N)$$

Proof of 2.4.10 i. □

Proof of 2.4.10 ii. □

Proof of 2.4.10 iii. □

Proof of 2.4.10 v. □

Remark 2.4.11 (A- **Uniform-Neighborhoods**)**.** *Let $(X, \mathcal{W}, \mathcal{T}_{\mathcal{W}})$ be a* ***Uniform Space*** *. Let $A \subset X$ and let U be a* **Symmetric Entourage** . By 2.4.10 iii, $V(A)$ is a **Neighborhood** of A. We shall now refer to this set as a $V-$ **Uniform-Neighborhood** of A.

Proposition 2.4.12 (Uniform Product Fundamental System)**.** *Let $(X, \mathcal{W}, \mathcal{T}_{\mathcal{W}})$ be a* ***Uniform Space*** *. Endow $X \times X$ with the* ***Product Topology*** *. The following are true.*

(i) $\{\overset{\circ}{M} | M \in \mathcal{W}\}$ *forms a* ***Fundamental System Of Entourages*** *for* $\mathcal{W}$.

(ii) $\{\overline{M} | M \in \mathcal{W}\}$ *forms a* ***Fundamental System Of Entourages*** *for* $\mathcal{W}$.

Proof of 2.4.12 i. □

Proof of 2.4.12 ii. □

2.5 Algebraic Structures

2.5.1 Magma, Semigroup, Group

Definition 2.5.1 (Algebraic Declarations Placeholder)**.**

Definition 2.5.2 (**Commutative**)**.** *Let X and Y be sets. We say that a map $f : X \times X \to Y$ is a* **Symmetric Map** if for each $x_0, x_1 \in X$, $f(x_0, x_1) = f(x_1, x_0)$. In this situation, we may also refer to f as **Commutative** , or say that f posesses **Commutativity** .

Definition 2.5.3 (**Operation** , **Unary Operation** , **Binary Operation**)**.** *Let $X \neq \emptyset$ be a set. Let $A \neq \emptyset$ be a set with cardinality$(A) = n \in \mathbb{N}$. We call a mapping*

$$T : \prod_{\alpha \in A} X \to X$$

an n-ary **Operation** on X. If $n = 1$ then we call T a **Unary Operation** on X. If $n = 2$, then we call T a **Binary Operation** on X. If T is a **Binary Operation** on X, we sometimes use the notation

$$xTy = T(x, y)$$

Definition 2.5.4 (**Magma**)**.** *Let X be a set and $T : X \times X \to X$ be a* **Binary Operation** on X. We call the pair (X, T) a **Magma** . When it is clear what operation is being referred to, we may simply refer to X as the **Magma** . If T is **Commutative** , then we call (X, T) (or simply just X) a **Commutative Magma** . In general, this naming convention is used for any algebraic structure defined on a set via a **Binary Operation** with particular properties.

Definition 2.5.5 (**Magma Homomorphism**)**.** *Let $(X, \oplus_X)$ and $(Y, \oplus_Y)$ be* **Magmas** . Let $T : X \to Y$ satisfy, for each $x_1, x_2 \in X$.

$$T(x_1 \oplus_X x_2) = T(x_1) \oplus_Y T(x_2)$$

Then we call T a **Magma Homomorphism** . We represent the collection of **Magma Homomorphisms** from $(X, \oplus_X)$ to $(Y, \oplus_Y)$ with $H_{\mathbf{Magma}}((X, \oplus_X), (Y, \oplus_Y))$, or, when $\oplus_X$ and $\oplus_Y$ are clear, $H_{\mathbf{Magma}}(X, Y)$. A **Magma Homomorphism** is called **Additive** and posseses the property **Additivity** .

Definition 2.5.6 (**Left Identity Element** , **Right Identity Element**). *Let* (X, L) *and* (X, R) *be* **Magmas** . Let $l, r \in X$ such that for every $x \in X$ we have

$$lLx = x$$
$$xRr = x$$

In such a scenario, we say that l is a **Left Identity Element** of (X, L), and we say that r is a **Right Identity Element** of (X, R).

Definition 2.5.7 (**Identity Element**). *Let* $(X, \oplus)$ *be a* **Magma** . Let $e \in X$ be both a **Left Identity Element** and a **Right Identity Element** of $\oplus$. Then, we say that e is an **Identity Element** of $(X, \oplus)$.

Definition 2.5.8 (**Unital Magma**). *Let* $(X, \oplus)$ *be a* **Magma** . Let e be an **Identity Element** of $(X, \oplus)$. Then we call $(X, \oplus, e)$ a **Unital Magma** . If it is unambiguous what operation is being referred to, as in the case of **Magmas** , we may simply say let X be a **Unital Magma** , or potentially Let (X, e) be a **Unital Magma** .

Definition 2.5.9 (' **Unital Magma Homomorphism**). *Let* $(X, \oplus_X, e_X)$ *and* $(Y, \oplus_Y, e_Y)$ *be* **Unital Magmas** and $T : X \to Y$ be a **Magma Homomorphism** such that $T(e_X) = e_Y$. Then we call T a ' **Unital Magma Homomorphism** . We represent the set of **Unital Magma Homomorphisms** between X and Y with $H_{\mathbf{UMagma}}(X, Y)$. Obviously, $H_{\mathbf{UMagma}}(X, Y) \subset H_{\mathbf{Magma}}(X, Y)$.

Definition 2.5.10 (**Left Inverse** , **Right Inverse**). *Let* $(X, \oplus, e)$ *be a* **Unital Magma** . Let $l, r \in X$ such that

$$l \oplus r = e \tag{2.30}$$

In this scenario, we say that l is a **Left Inverse** of r in $(X, \oplus, e)$ and we say that r is a **Right Inverse** of l in $(x, \oplus, e)$. Furthermore, we say that r is **Left Invertible** in $(X, \oplus, e)$ and that l is **Right Invertible** in $(X, \oplus, e)$

Definition 2.5.11 (**Inverse**). *Let* $(X, \oplus, e)$ *be a* **Unital Magma** . Let $x, y \in X$ such that x is a **Left Inverse** of y and x is a **Right Inverse** of y. Then, we say that x is an **Inverse** of y in $(X, \oplus, e)$ and we say y an **Invertible** element of $(X, \oplus, e)$.

Definition 2.5.12 (**Associative**). *Let* T *be a* **Binary Operation** on a set X. We say that T is **Associative** and we say that T posses **Associativity** if for each $x, y, z \in X$, we have

$$T(x, T(y, z)) = T(T(x, y), z)$$

Definition 2.5.13 (**Semigroup**). *Let* $(X, \oplus)$ *be a* **Magma** . Let $\oplus$ be **Associative** . Then we say that $(X, \oplus)$ is a **Semigroup** .

Definition 2.5.14 (**Monoid**). *Let* $(X, \oplus, e)$ *be a* **Unital Magma** and let $(X, \oplus)$ be a **Semigroup** . Then we call $(X, \oplus, e)$ a **Monoid** .

Definition 2.5.15 (**Consistent**). *Let* $(X, \oplus)$ *be a* **Magma** and let R be a **Relation** on X such that For each $x_0, x_1 \in X$, if $x_0 R x_1$ and $y_0 R y_1$, then $(x_0 \oplus y_0) \, R \, (x_1 \oplus y_1)$ Then we say that R is **Consistent** with $(X, \oplus)$, and we say that R posesses **Consistency** with respect to $(X, \oplus)$.

Definition 2.5.16 (**Partially Ordered Magma** , **Totally Ordered Magma** , **Directed Magma**). *Let* $(X, \oplus)$ *be a* **Magma** . Let T be a **Total Ordering** on X which is **Consistent** with $(X, \oplus)$. Let P be a **Partial Ordering** on X which is **Consistent** with $(X, \oplus)$. Let D be a **Directing** on X which is **Consistent** with $(X, \oplus)$. We call $(X, \oplus, T)$ a **Totally Ordered Magma** . $(X, \oplus, P)$ a **Partially Ordered Magma** . $(X, \oplus, D)$ a **Directed Magma** .

Definition 2.5.17 (**Group**). *Let* $(X, \oplus, e)$ *be a* **Monoid** such that each $x \in X$ is an **Invertible** . Then we call $(X, \oplus, e)$ a **Group** . Out of respect, we call a **Commutative Group** an **Abelian Group** .

Definition 2.5.18 (**Group Inverse Operator**). *Let* $(X, \oplus, e)$ *be a group. We denote with* $\mathbf{T^{-1}}_{\mathrm{G}}$ *the function defined as follows:* $\mathbf{T^{-1}}_{\mathrm{G}} : \mathrm{X} \to \mathrm{X}$,

$$\mathbf{T^{-1}}_{\mathrm{G}}(\mathrm{x}) = -\mathrm{x}$$

We call $\mathbf{T^{-1}}_{\mathrm{G}}$ *the* **Group Inverse Operator** of $(X, \oplus, e)$.

Definition 2.5.19 (**Translation Operator**). *Let* $(G, \oplus)$ *be a* **Magma** . Let $g \in G$. We define $\mathbf{T^{R}}_{\mathrm{g}} : \mathrm{G} \to \mathrm{G}$ and $\mathbf{T^{L}}_{\mathrm{g}} : \mathrm{G} \to \mathrm{G}$ by setting, for each $x \in G$,

$$\mathbf{T^{R}}_{\mathrm{g}}(\mathrm{x}) = \mathrm{x} \oplus \mathrm{g}$$

$$\mathbf{T^{L}}_{\mathrm{g}}(\mathrm{x}) = \mathrm{g} \oplus \mathrm{x}$$

We call $\mathbf{T^{R}}_{\mathrm{g}}$ the **Right Translation** of $(G, \oplus)$ by g, and we call $\mathbf{T^{L}}_{\mathrm{g}}$ the **Left Translation** of $(G, \oplus)$ by g. If $\oplus$ is **Commutative** , then we define $\mathbf{T}_{\mathrm{g}} = \mathbf{T^{R}}_{\mathrm{g}} = \mathbf{T^{L}}_{\mathrm{g}}$ which we call **Translation** of $(G, \oplus)$ by g.

2.5.2 Vector Space

Definition 2.5.20 (**Linearly Independent**). *Let* V *be a* ***Vector Space*** *over a* ***Field*** $\mathcal{K}$. *Let* $A \subset V$. *We say that* A *is* ***Linearly Independent*** *if, for any finite Subset* $\{x_i\}_{i=1}^n \subset A$, *the only solution to*

$$\sum_{i=1}^{n} \beta_i x_i = 0 \qquad\qquad \{\beta_i\}_{i=1}^n \subset \mathcal{K}$$

is $\beta_i = 0$ *for* $1 \leq i \leq n$. *In such a scenario, we may also say that* A *posesses* ***Linearly Independence*** . *A subset of a* ***Vector Space*** *which is not* ***Linearly Independent*** *is said to be* ***Linearly Dependent*** *and to posess* ***Linear Dependence*** .

Definition 2.5.21 (Scalar Homogeneous). *Let V be a* ***Vector Space*** *over a* ***Field*** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. *We say that a map $p : V \to V$ is* **Scalar Homogeneous** , if , for each $\alpha \in \mathbb{F}$ and each $x \in V$, we have

$$p(\alpha x) = \alpha p(x) \tag{2.31}$$

Under these circumstances, we may instead say that the operator p posesses **Scalar Homogeneity** .

Definition 2.5.22 (Scalar Homogeneous). *Let V be a* ***Vector Space*** *over a* ***Field*** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. *We say that a map $p : V \to V$ is* **Absolutely Scalar Homogeneous** , if , for each $\alpha \in \mathbb{F}$ and each $x \in V$, we have

$$p(\alpha x) = |\alpha|\, p(x) \tag{2.32}$$

Under these circumstances, we may instead say that the operator p posesses **Absolute Scalar Homogeneity** .

Remark 2.5.23 (**Scalar Homogeneous** or **Absolutely Scalar Homogeneous** operator at 0 is 0). *If V is a* ***Vector Space*** *over a* ***Field*** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, *then for each $x \in V$, $0x = 0$. Hence, if p is a* **Absolutely Scalar Homogeneous** operator on v, then for any $x \in V$

$$p(0) = p(0x) = |0|p(x) = 0p(x) = 0 \tag{2.33}$$

If instead p is **Scalar Homogeneous** operator on V, then we have

$$p(0) = p(0x) = 0p(x) = 0 \tag{2.34}$$

that is, in either case, p(0)=0.

Definition 2.5.24 (**Subadditive**). *Let $(G, \oplus_G)$ be a* **Magma** and $(H, \oplus_H, \leq)$ be a **Partially Ordered Magma** . We call a mapping $p : G \to H$ **Subadditive** if, for every $x, y \in G$, we have

$$p(x \oplus_G y) \leq p(x) \oplus_H p(y) \tag{2.35}$$

Under these circumstances, we may also say that p posesses **Subadditivity**.

Definition 2.5.25 (**Linear**). *Let V, U be* ***Vector Spaces****over a* ***Field*** $\mathbb{F}$. *We say that $T : V \to U$ is* **Linear** or that T possesses **Linearity** if T is both **Additive** and **Scalar Homogeneous**

Definition 2.5.26 (**Space of Linear Operators**). *Let U, V be* ***Vector Spaces****over the same* ***Field*** $\mathbb{F}$. *We denote with $L(U, V)$ the set of* **Linear** operators $T : U \to V$. We refer to $L(U, V)$ as the **Space of Linear Operators** from U to V. We endow $L(U, V)$ with the operations of pointwise addition and pointwise scalar multiplication, which the reader can verify makes $L(U, V)$ into a **Vector Space** .

Definition 2.5.27 (**Algebraic Dual**). *Let V be a* ***Vector Space*** *over a* ***Field*** *$\mathbb{F}$. We define $V' = L(V, \mathbb{F})$. We call V' the* ***Algebraic Dual*** *of V. If $x^* \in V'$, then we call x^* a* ***Linear Functional*** *.*

Proposition 2.5.28 (**Linearly Independent Linear Functionals**). *Let V be a* ***Vector Space*** *over a* ***Field*** *$\mathbb{F}$. Let $\{x_i^*\}_{i=1}^n \subset V'$ be* ***Linearly Independent*** *. Let $\{e_i\}_{i=1}^n$ be the* ***Standard Basis*** *for $\mathbb{F}^n$. Define $S : V \to \mathbb{F}^n$ by*

$$S(v) = \sum_{i=1}^{n} \langle v, x_i^* \rangle \, e_i \tag{2.36}$$

Then S is **Surjective** *.*

Proof. Supose otherwise. Then there exists $0 \neq c \in \mathbb{F}^n$, represented

$$c = \sum_{i=1}^{n} c_i e_i$$

such that $c \perp Range(S)$. Hence, for every $x \in X$,

$$\begin{aligned}
0 &= \left\langle \sum_{i=1}^{n} \langle x, x_i^* \rangle \, e_i, c \right\rangle \\
&= \left\langle \sum_{i=1}^{n} \langle x, x_i^* \rangle \, e_i, \sum_{i=1}^{n} c_i e_i \right\rangle \\
&= \sum_{i=1}^{n} \langle x, x_i^* \rangle \, c_i \\
&= \left\langle x, \sum_{i=1}^{n} c_i x_i^* \right\rangle
\end{aligned}$$

This implies that $\sum_{i=1}^{n} c_i x_i^* = 0$, a contradiction. □

Proposition 2.5.29. *Let V be a* ***Vector Space*** *over a* ***Field*** *$\mathbb{F}$. Let $\{v_i\}_{i=1}^n \subset V'$. Let $v \in V'$. Suppose*

$$\bigcap_{i=1}^{n} Kernel(v_i) \subset Kernel(v)$$

Then $v \in span(v_1, \cdots, v_n)$.

Proof. Without loss of generality we let $\{v_i\}_{i=1}^n$ be **Linearly Independent** . Let $\{e_i\}_{i=1}^n$ be the **Standard Basis** of $\mathbb{F}^n$. Define $S : V \to \mathbb{F}^n$ by $S(x) = \sum_{i=1}^{n} \langle x, v_i \rangle \, e_i$. By 2.5.28, S is **Surjective** . Hence, if we let $Q : V \to V/Kernel(S)$ be the quotient map. Then S has a invertible quotient $\tilde{S} : V/Kernel(S) \to \mathbb{F}^n$ which satisfies $S = \tilde{S} \circ Q$ and $Q = \tilde{S}^{-1} \circ S$.

Since $Kernel(S) = \bigcap_{i=1}^{n} Kernel(v_i) \subset Kernel(v)$, there exists $\tilde{v} : X/Kernel(S) \to \mathbb{F}$ such that $v = \tilde{v} \circ Q$. Also, $\tilde{v} \circ \tilde{S}^{-1} : \mathbb{F}^n \to \mathbb{F}$ is linear, so there are $\{\sigma_i\}_{i=1}^n \subset \mathbb{F}$ such that if $\sum_{i=1}^n x_i e_i \in \mathbb{F}^n$, we have

$$\tilde{v} \circ \tilde{S}^{-1} \left(\sum_{i=1}^{n} x_i e_i \right) = \sum_{i=1}^{n} x_i \left(\tilde{v} \circ \tilde{S}^{-1} \right) (e_i) = \sum_{i=1}^{n} x_i \sigma_i$$

Hence, for $x \in V$, we have

$$\begin{aligned} \langle x, v \rangle &= \tilde{v} \circ Q(x) \\ &= \tilde{v} \circ \tilde{S}^{-1} \circ S(x) \\ &= \left(\tilde{v} \circ \tilde{S}^{-1} \right) \left(\sum_{i=1}^{n} \langle x, v_i \rangle e_i \right) \\ &= \sum_{i=1}^{n} \langle x, v_i \rangle \sigma_i \\ &= \left\langle x, \sum_{i=1}^{n} \sigma_i v_i \right\rangle \end{aligned}$$

Hence $v \in span(v_1, \cdots, v_n)$. □

Definition 2.5.30 (**Balanced**). *Let V be a* ***Vector Space*** *over a* ***Field*** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. *Let $S \subset V$. We call S a* **Balanced Set** and we say that S is **Balanced** if for each $\alpha \in \mathbb{F}$ with $|\alpha| \leq 1$ we have $\alpha S \subset S$.

Definition 2.5.31 (**Absorbing**). *Let V be a* ***Vector Space*** *over a* ***Field*** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. *Let $A, B \subset V$. We say that A* **Absorbs** B if there exists a $c > 0$ such that for every $d \in \mathbb{F}$ with $|d| > c$ we have $B \subset dA$. In such a Scenario, A is also said to **Absorb** B, and we say that B is **Absorbed** by A. If A **Absorbs** every Singleton in V, then we call A an **Absorbing Set** or we say that A is **Absorbing** .

Definition 2.5.32 (**Scaling Operator**). *Let V be a* ***Vector Space*** *over a* ***Field*** $\mathbb{F}$. *Let $\alpha \in \mathbb{F}$. We define $M_\alpha : V \to V$ by setting, for each $x \in V$,*

$$M_\alpha(x) = \alpha x \tag{2.37}$$

We call M_α the **Scaling Operator**

Proposition 2.5.33 (**Scaling Operator**). *Let V be a* ***Vector Space*** *over a* ***Field*** $\mathbb{F}$. *The following are true:*

1. *If $\alpha, \beta \in \mathbb{F}$, then $M_\alpha \circ M_\beta = M_{\alpha * \beta}$.*

Proof of 01. Let $v \in V$. Then

$$\begin{aligned} M_\alpha \circ M_\beta v &= M_\alpha(\beta * v) \\ &= \alpha * (\beta * v) \\ &= (\alpha * \beta) * v \\ &= M_{\alpha * \beta} v \end{aligned}$$

□

Definition 2.5.34 (**Interval**). *Let V be a* ***Vector Space*** *over a* ***Field*** $\mathbb{F}$. *Let $x, y \in V$. We define the following sets:*

$$\begin{aligned} [x, y] &= \{tx + (1-t)y | t \in [0,1]\} \\ [x, y) &= \{tx + (1-t)y | t \in [0,1)\} \\ (x, y] &= \{tx + (1-t)y | t \in (0,1]\} \\ (x, y) &= \{tx + (1-t)y | t \in (0,1)\} \end{aligned}$$

We refer to any of these sets as **Intervals** in V. Even in the absence of a topological structure, we use the following language:

1. $[x, y]$ is called a **Closed Interval** .
2. (x, y) is called an **Open Interval** .
3. $(x, y]$ and $[x, y)$ are called **Half-Open Intervals** or **Half-Closed Intervals** .

Definition 2.5.35 (**Convex**). *Let V be a* ***Vector Space*** *over a* ***Field*** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. *Let $K \subset \mathbb{F}$. We say that K is* **Convex** if for every pair $x, y \in K$, we have $[x, y] \subset K$.

2.6 Topological Algebra

2.6.1 Topological Groups

Definition 2.6.1 (**Topological Group** ‘). *Let $(G, +, e)$ be a* **Group** . **Topology** on G such that $+ : G \times G \to G$ is **Continuous** with respect to the

and g_{-1} is **Continuous** .

In this scenario, we call $(G, \mathcal{T})$ a **Topological Group** ‘ .

Definition 2.6.2 (**Local Basis**). *Let $(G, \mathcal{T})$ be a* **Topological Group** ‘ with **Identity Element** e. We call a **Neighborhood Basis** of $\mathcal{T}$ about e a **LocalBasis** for $(G, \mathcal{T})$.

2.6.2 Topological Vector Spaces

Definition 2.6.3 (**Compatible**). *Let* $(V, +, \cdot, 0)$ *be a* ***Vector Space*** *over* $\mathbb{F}$ *and* $\mathcal{T}$ *be a* **Topology** on V such that $(V, +, \mathcal{T})$ is a **Topological Group** ' and $\cdot : \mathbb{F} \times V \to V$ is **Continuous** . Then we say that $\mathcal{T}$ is **Compatible** with $(V, +, \cdot, 0)$, or when $+$ and $\cdot$ are obvious, we say that $\mathcal{T}$ is **Compatible** with $\mathcal{T}$.

Definition 2.6.4 (**Topological Vector Space**). *Let* $(V, +, \cdot, 0)$ *be a* ***Vector Space*** *over a* ***Field*** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. *Let* $\mathcal{T}$ *be a* **Topology** on V which is **Compatible** with $(V, +, \cdot, 0)$. Then we call $(V, \mathcal{T})$ a **Topological Vector Space** .

Definition 2.6.5 (**Locally Convex**). *We say that a* **Topological Vector Space** $(X, \mathcal{T})$ is **Locally Convex** if $(X, \mathcal{T})$ has a **Local Basis** consisting only of **Convex** sets. A **Locally Convex** space is said to posess **Local Convexity** .

Proposition 2.6.6 (Existence of **Balanced Neighborhood Basis** of 0 in a **Topological Vector Space**). *Let* $(X, \mathcal{T})$ *be a* **Topological Vector Space** over a **Field** $\mathbb{F}$. The following are True.

1. If $U \in \mathcal{U}_{\mathcal{T}}(0)$, then there is a **Balanced Set** $V \subset U$ such that $V \in \mathcal{U}_{\mathcal{T}}(0)$.
2. There exists a **Neighborhood Basis** about $0 \in X$ for $\mathcal{T}$ consisting entirely of **Balanced Set** sets.
3. If $U \in \mathcal{U}_{\mathcal{T}}(0)$, then there is a **Balanced Set** $V \subset U$ such that $V \in \mathcal{U}_{\mathcal{T}}(0)$.
4. If $(X, \mathcal{T})$ is **Locally Convex** , then there exists a **Neighborhood Basis** about $0 \in X$ for $\mathcal{T}$ consisting entirely of **Balanced Set** sets.

Part 01. TODO □

Part 02. TODO □

Part 03. TODO □

Part 04. TODO □

Definition 2.6.7 (TVS Bounded Set). *Let* $(V, \mathcal{T})$ *be a* **Topological Vector Space** . Let $A \subset V$. We say that A is **TVS-Bounded** with respect to $\mathcal{T}$, or when confusion is unlikely we simply say that A is **TVS-Bounded** if for every $U \in \mathcal{U}_{\mathcal{T}}(0)$, there exists an $\alpha \in \mathbb{F}$, $\alpha > 0$, such that $A \subset \alpha U$.

Definition 2.6.8 (**Bounded Linear Operator**). *Let* $(V_i, \mathcal{T}_i)$ *be a* **Topological Vector Space** over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ for $i \in \{0, 1\}$. We say that a **Linear** operator $T : (V_1, \mathcal{T}_1) \to (V_2, \mathcal{T}_2)$ is a **Bounded Linear Operator** if for each $U \in V_1$ with U **TVS-Bounded** with respect to $\mathcal{T}_0$, TU is **TVS-Bounded** with respect to $\mathcal{T}_1$.

Definition 2.6.9 (**Space Of Continuous Linear Operators**). *Let* $(U, \mathcal{T}_U)$ *and* $(V, \mathcal{T}_V)$ *each be a* **Topological Vector Space** over the same **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $L(U, V)$ denote the **Space of Linear Operators** from U to V. We denote with $CL((U, \mathcal{T}_U), (V, \mathcal{T}_V))$ the subset of $L(U, V)$ consisting only of the **Continuous** operators. When $\mathcal{T}_U$ and $\mathcal{T}_V$ are understood, we may denote $CL((U, \mathcal{T}_U), (V, \mathcal{T}_V)) = CL(U, V)$

Remark 2.6.10 (**Space Of Continuous Linear Operators** is a **Vector Subspace**). *Let* $(U, \mathcal{T}_U)$ *and* $(V, \mathcal{T}_V)$ *each be a* **Topological Vector Space** over the same **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $L(U, V)$ denote the **Space of Linear Operators** from U to V. Let $CL(U, V)$ denote the **Space Of Continuous Linear Operators** from U to V. Then $CL(U, V)$ is a **Vector Subspace** of $L(U, V)$.

Definition 2.6.11 (**Topology of Uniform Convergence**). *Let* X *be a set and* $(Y, \mathcal{T}_Y)$ *be a* **Topological Vector Space** . Let $\mathcal{B}$ be a **Local Basis** for Y. Suppose $\mathcal{F}$ is a **Vector Subspace** of the set of functions $T : X \to Y$. Suppose $\mathcal{G} \subset 2^X$ such that $(\mathcal{G}, \subset)$ is a **Directed Set** . For each $x \subset X$ and $y \subset Y$, and define

$$M(x, y) = \{f \in \mathcal{F} | f(x) \subset y\}$$

Now we define

$$\mathcal{T}(\mathcal{F}, \mathcal{T}_Y, \mathcal{G}) = \{f + M(x, y) | x \in \mathcal{G} \wedge y \in \mathcal{B} \wedge f \in \mathcal{F}\}$$

We call $\mathcal{T}(\mathcal{F}, \mathcal{T}_Y, \mathcal{G})$ the **Topology of Uniform Convergence** of $\mathcal{F}$ on $\mathcal{G}$ with respect to $\mathcal{T}_Y$. When $\mathcal{F}$, $\mathcal{T}_Y$ or $\mathcal{G}$ are understood they may be omitted from the reference. By 2.6.12, $\mathcal{T}$ is a **Topology** on $\mathcal{F}$.

Proposition 2.6.12 (**Topology of Uniform Convergence**).

2.7 Seminormed Spaces

2.7.1 Pseudometrics

Definition 2.7.1 (Symmetric Map). *Let* X *be a set and* $(Y, +, \leq)$ *be a totally ordered magma. We say that a map* $f : X \times X \to Y$ *satisfies the* **Triangle Inequality** if for each $x_0, x_1, x_3 \in X$, we have

$$f(x_0, x_2) \leq f(x_0, x_1) + f(x_1, x_2)$$

Definition 2.7.2 (Pseudometric). *Let* $X \neq \emptyset$. *Let* $d : X \times X \to [0, \infty)$ *be a* **Symmetric Map** that satisfies the **Triangle Inequality** and further satisfies, for each $x \in X$,

$$d(x, x) = 0 \tag{2.38}$$

Under these conditions we call d a **Pseudometric** on X and we call (X, d) a **Pseudometric Space** .

Definition 2.7.3 (Metric)**.** *Let* (X, d) *be a* **Pseudometric Space** . If d has the property that for $x, y \in X$, if $x \neq y$, then

$$d(x, y) \neq 0$$

Then we call d a **Metric** on X and we call (X, d) a **Metric Space**

Definition 2.7.4 (Pseudometric Cauchy Sequence)**.** *Let* (X, d) *be a* **Pseudometric Space** . We say that a sequence $\{x_i\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence** if, for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each pair $m, n \in \mathbb{N}$ such that $m > N$ and $n > N$, we have

$$d(x_m, x_n) < \epsilon \tag{2.39}$$

Definition 2.7.5 (Pseudometric Convergence)**.** *Let* (X, d) *be a* **Pseudometric Space** . Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in (X, d). Let $x_0 \in X$. We say that $\{x_i\}_{i\in\mathbb{N}}$ exhibits **Pseudometric-Convergence** to x_0 in d, or we say that $\{x_i\}_{i\in\mathbb{N}}$ **Pseudometric-Converges** to x_0 in d, or we say that $\{x_i\}_{i\in\mathbb{N}}$ is **Pseudometrically-Convergent** to $x_0 \in d$ if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for every $n > N$, we have

$$d(x_0, x_n) < \epsilon \tag{2.40}$$

Proposition 2.7.6 (Convergent Implies Cauchy)**.** *Let* (X, d) *be a* **Pseudometric Space** . Let $\{x_i\}_{i\in\mathbb{N}}$ be a **Pseudometrically-Convergent** sequence. Then $\{x_i\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence** .

Proof. Since $\{x_i\}$ converges, let $x_i \to x$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $n > N$, we have $d(x_i, x) < \frac{\epsilon}{2}$. For this N, if $m, n > N$, then we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \epsilon \tag{2.41}$$

and so the sequence is a **Pseudometric Cauchy Sequence** , as advertised. □

Definition 2.7.7 (Uniformly Cauchy)**.** *Let* (X_α, d_α) *be a* **Pseudometric Space** for $\alpha \in A$ where A is some indexing set. For each $\alpha \in A$, let $\phi_\alpha := \{x_i^\alpha\}_{i\in\mathbb{N}} \subset X_\alpha$ be a sequence. We say that the collection $\{\phi_\alpha\}_{\alpha\in A}$ is **Uniformly Cauchy** if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for each pair $m, n \in N$ such that $m > N$ and $n > N$, and for each $\alpha \in A$, we have

$$d_\alpha\left(x_n^\alpha, x_m^\alpha\right) < \epsilon \tag{2.42}$$

Definition 2.7.8 (Uniform Convergence)**.** *Let* (X_α, d_α) *be a* **Pseudometric Space** for $\alpha \in A$ where A is some indexing set. For each $\alpha \in A$, let $\phi_\alpha := \{x_i^\alpha\}_{i\in\mathbb{N}} \subset X_\alpha$ be a sequence. We say that the collection $\{\phi_\alpha\}_{\alpha\in A}$ is **Uniformly Convergent** to $\{x_\alpha\}_{\alpha\in A} \in \prod\limits_{\alpha\in A} X_\alpha$ if for each $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for each $n > N$, and for every $\alpha \in A$, we have

$$d_\alpha(x_i^\alpha, x_\alpha) < \epsilon \tag{2.43}$$

In this scenario, we may equivalently say that $\{\phi_\alpha\}$ demonstrates **Uniform Convergence** to $\{x_\alpha\}_{\alpha\in A}$ or that it **Converges Uniformly** .

When we mention **Uniform Convergence** without reference to what the convergence is to, we are merely claiming the existence of such a limit.

Proposition 2.7.9 (Uniform Cauchy and Pointwise Convergence implies Uniform Convergence)**.** *Let* (X_α, d_α) *be a* **Pseudometric Space** for $\alpha \in A$ where A is some indexing set. For each $\alpha \in A$, let $\phi_\alpha := \{x_i^\alpha\}_{i\in\mathbb{N}} \subset X_\alpha$ be a sequence. Suppose the collection $\{\phi_\alpha\}_{\alpha\in A}$ is **Uniformly Cauchy** and that each ϕ_α is **Pseudometrically-Convergent** , say $x_i^\alpha \to x_\alpha$. Then $\{\phi_\alpha\}_{\alpha\in A}$ is **Uniformly Convergent** to $\{x_\alpha\}_{\alpha\in A}$.

Proof. Let $\epsilon > 0$. Then, since $\{\phi_\alpha\}_{\alpha\in A}$ is **Uniformly Cauchy** , there is an $N \in \mathbb{N}$ such that for $m, n > N$, we have $d_\alpha(x_n^\alpha, x_m^\alpha) < \frac{\epsilon}{2}$. Also, since each ϕ_α converges to x_α, there are $N_\alpha \in \mathbb{N}$. such that for any $n_\alpha > N_\alpha$, we have $d_\alpha(x_{n_\alpha}^\alpha, x_\alpha) < \frac{\epsilon}{2}$. Define $M_\alpha = max(N+1, N_\alpha+1)$ for $\alpha \in A$. Let $n > N$. Then, for any $\alpha \in A$, we have.

$$\begin{aligned}
d_\alpha(x_n^\alpha, x_\alpha) &\le d_\alpha(x_n^\alpha, x_{M_\alpha}^\alpha) + d_\alpha(x_{M_\alpha}, x_\alpha) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

completing the proof.

□

Definition 2.7.10 (Pseudometric Complete)**.** *We say that a* **Pseudometric Space** (X, d) is **Pseudometric-Complete** if each **Pseudometric Cauchy Sequence** sequence in (X, d) **Pseudometric-Converges** to a limit in X.

In the case that d is a **Metric** , then being **Pseudometric-Complete** is equivalent to being **Complete** in the classical sense, so we will commonly refer to a **Pseudometric Space** which is **Pseudometric-Complete** as simply being **Complete** .

Definition 2.7.11 (Pseudometric Ball)**.** *Let* (X, d) *be a* **Pseudometric Space** . For each $x_0 \in X$ and each $\epsilon > 0$, we define the following.

1. $B_d(x_0, \epsilon) := \{y \in X | d(x_0, y) < \epsilon\}$ denotes the **Open Ball** about x_0 with radius ϵ.

2. $\overline{B_d}(x_0, \epsilon) := \{y \in X | d(x_0, y) \le \epsilon\}$ denotes the **Closed Ball**about x_0 with radius ϵ.

3. We call the **Open Ball** about x_0 with radius 1 the **Open Unit Ball** about x_0.

4. We call the **Closed Ball**about x_0 with radius 1 the **Closed Unit Ball** about x_0.

Definition 2.7.12 (Pseudometric Topology)**.** *Let* (X, d) *be a* **Pseudometric Space** , and let $\mathcal{B}$ be the set of **Open Ball** 's in (X, d). By 2.7.1, $\mathcal{B}$ is the basis for a unique topology $\mathcal{T}_d$ on X. We call $\mathcal{T}_d$ the **Pseudometric Topology** induced by d on X.

Proposition 2.7.13 (Pseudometric Topology)**.** *Let* (X, d) *by* **Pseudometric Space** and let $\mathcal{B}$ be the set of **Open Ball** 's in (X, d). The following are true.

1. There exists a unique topology $\mathcal{T}_d$ on X which $\mathcal{B}$ is a basis of. That is, the **Pseudometric Topology** $\mathcal{T}_d$ is well defined.

2. The **Pseudometric Topology** is first countable. That is, each of its points permits a countable neighborhood basis.

Proof of 1. Uniqueness is guaranteed by closure under arbitrary unions of a topology. For existense, it is sufficient to show that the collection of arbitrary unions of elements of $\mathcal{B}$ is closed under finite intersections. Suppose that for $1 \le i \le n$, we have $\{U_{\alpha_i} | \alpha_i \in A_i\} \subset \mathcal{B}$ and consider the set

$$U = \bigcap_{i=1}^{n} \bigcup_{\alpha_i \in A_i} U_{\alpha_i} \tag{2.44}$$

Let $x_0 \in U$. For each $i \in \{1, ..., n\}$, there exists $\alpha_i \in A_i$ such that

$$x_0 \in U_{\alpha_i} = B_d(x_i; \epsilon_i) \tag{2.45}$$

For each $i \in \{1, ..., n\}$, define $\delta_i = d(x_0, x_i)$. Then $0 < \delta_i < \epsilon_i$. Then, for each $i \in \{1, ..., n\}$,

$$B_d(x_0; \epsilon_i - \delta_i) \subset U_{\alpha_i} \subset \bigcup_{\alpha_i \in A_i} U_{\alpha_i} \tag{2.46}$$

Define

$$\delta_{x_0} = \min_{i=1}^{n} (\epsilon_i - \delta_i) \tag{2.47}$$

Then $x_0 \in B(x_0; \delta_{x_0}) \subset U$. If $U = \{x_\alpha | \alpha \in A\}$, then the arbitrary nature of x_0 above means we can repeat this construction, writing

$$U \subset \bigcup_{\alpha \in A} B(x_\alpha; \delta_{x_\alpha}) \subset \bigcup_{\alpha \in A} U = U \tag{2.48}$$

Hence, $U \in B$ and the proof is complete. □

Proof of 2. Let $x_0 \in X$. I claim that

$$\mathcal{B}_{x_0} := \left\{ B_d\left(x_0; \frac{1}{n}\right) | n \in \mathbb{N} \right\} \tag{2.49}$$

is a neighborhood basis for $(X, \mathcal{T}_d)$ at x_0. Let $U \in \mathcal{U}_{\mathcal{T}_d}(x)$ be open in $\mathcal{T}_d$. Since $\mathcal{B}$ is a basis for $\mathcal{T}_d$, for some $y0 \in X$ and $\epsilon > 0$, $x_0 \in B_d(y_0; \epsilon) \subset U$. Let $\delta = d(x_0, y_0)$. Then $\epsilon - \delta > 0$. Define

$$n = \left\lceil \frac{1}{\epsilon - \delta} \right\rceil \tag{2.50}$$

Then we have

$$B_d\left(x_0; \frac{1}{n}\right) \subset B_d(x_0 : \epsilon - \delta) \subset B(y_0; \epsilon) \subset U \tag{2.51}$$

□

Definition 2.7.14 (Relation Of Zero Distance)**.** *Let* (X, d) *be a* **Pseudometric Space** . Define the relation $\cong_d$ on $X \times X$ by setting, for $x, y \in X$,

$$x \cong_d y \iff d(x, y) = 0 \tag{2.52}$$

We call $\cong_d$ the **Relation Of Zero Distance** on (X, d).

Proposition 2.7.15 (Relation Of Zero Distance is the Relation Of Equal Neighborhood Filters)**.** *Let* (X, d) *be a* **Pseudometric Space** . Let $\cong_{\mathcal{T}_d}$ be the **Relation Of Equal Neighborhood Filters** on $(X, \mathcal{T}_d)$. Let $\cong_d$ be the **Relation Of Zero Distance** on (X, d). Then $\cong_{\mathcal{T}_d} = \cong_d$.

Proof. Let $x, y \in X$ and suppose $x_0 \cong_d y_0$. Let $U \in \mathcal{U}_{\mathcal{T}_d}(x_0)$. Then for some $\epsilon > 0$, $x_0 \in B(x_0; \epsilon) \subset U$. Since $x_0 \cong_d y_0$, $d(x_0, y_0) = 0$, so $y_0 \in B(x_0; \epsilon) \subset U$. Hence $U \in \mathcal{U}_{\mathcal{T}_d}(y_0)$. The arbitrary nature of $U \in \mathcal{U}_{\mathcal{T}_d}(x_0)$ implies

$$\mathcal{U}_{\mathcal{T}_d}(x_0) \subset \mathcal{U}_{\mathcal{T}_d}(y_0) \tag{2.53}$$

A reverse construction would just as easily show the reverse inclusion, so we conclude that $x_0 \cong_{\mathcal{T}_d} y_0$. Now suppose $x_0 \cong_{\mathcal{T}_d} y$. Then or each $n \in \mathbb{N}$,

$$y_0 \in B_d\left(x_0; \frac{1}{n}\right) \tag{2.54}$$

Hence $d(x_0, y_0) < \frac{1}{n}$ for each natural n, therefore $d(x_0, y_0) = 0$ and $x_0 \cong_d y_0$. □

Definition 2.7.16 (Metric Space Induced By Pseudometric)**.** *Let* (X, d) *be a* **Pseudometric Space** , and let $\cong$ be the **Relation Of Zero Distance** , which by 2.7.15 is also the **Relation Of Equal Neighborhood Filters** on $(X, \mathcal{T}_d)$. Define $\tilde{d} : X/\cong \to [0, \infty)$ by

$$\tilde{d}\left([x], [y]\right) = d(x, y) \tag{2.55}$$

By 2.7.17, $\tilde{d}$ is well defined and is in fact a metric on $X/\cong$, so we call $\tilde{d}$ the **Metric Induced By The Pseudometric** d on X, or we call it the **Pseudometric Induced Metric** of (X, d).

Proposition 2.7.17 (Metric Space Induced By Pseudometric Space)**.** *Let* (X, d) *be a* **Pseudometric Space** , $\cong$ the **Relation Of Zero Distance** on (X, d) and $\tilde{d}$ be defined as in 2.7.16. Let $(X/\cong, \mathcal{T}_{X/\cong})$ be the **Quotient Topological Space** with **Quotient Map** T, and let $(X/\cong, \mathcal{T}_{\tilde{d}})$ be the topological space induced by the metric space $(X/\cong, \tilde{d})$. The following are true.

1. $\tilde{d}$ is in fact well defined, and is a metric on $X/\cong$, justifying calling it the **Metric Induced By The Pseudometric** d.

2. $\mathcal{T}_{X/\cong} = \mathcal{T}_{\tilde{d}}$

3. T is an isometric surjection (X, d) to $(X/\cong, \tilde{d})$

4. $(X/\cong, \tilde{d})$ is complete if and only if (X, d) is **Pseudometric-Complete** .

Proof of 01. First we show that $\tilde{d}$ is well defined as a mapping, that is, that if $x_0, y_0 \in X$ and $x_1 \cong x_0$ and $y_1 \cong y_0$, then we should have

$$\tilde{d}\left([x_0], [y_0]\right) = \tilde{d}\left([x_1], [y_1]\right) \tag{2.56}$$

This is easy, as

$$\begin{aligned} d(x_0, y_0) &\leq d(x_0, x_1) + d(x_1, y_1) + d(y_1, y_0) \\ &= d(x_1, y_1) \\ &\leq d(x_1, x_0) + d(x_0, y_0) + d(y_0, y_1) \\ &= d(x_0, y_0) \end{aligned}$$

Nonnegativity falls directly from the nonnegativity of d. Proving that $\tilde{d}$ is a **Symmetric Map** is equally trivial

$$\tilde{d}\left([x], [y]\right) = d(x, y) = d(y, x) = \tilde{d}\left([y], [x]\right)$$

Proving that $\tilde{d}$ satisfies the **Triangle Inequality** is similarly simple, letting $x_0, y_0, z_0 \in X$, we have

$$\begin{aligned} \tilde{d}\left([x_0], [z_0]\right) &= d(x_0, z_0) \\ &\leq d(x_0, y_0) + d(y_0, z_0) \\ &= \tilde{d}\left([x_0], [y_0]\right) + \tilde{d}\left([y_0], [z_0]\right) \end{aligned}$$

All that remains is to show positivity on nonequal arguements. Let $x_0, y_0 \in X$ such that $[x_0] \neq [y_0]$. Then $x_0 \ncong y_0$. Hence

$$\tilde{d}\left([x_0], [y_0]\right) = d(x_0, y_0) \neq 0$$

□

Proof of 02. By 2.3.29, part 9, $\mathcal{B}_{\cong} := \{T(B_d(x;\epsilon)) | x \in X, \epsilon > 0\}$ is a basis for $\mathcal{T}_{X/\cong}$. By definition, $\mathcal{B}_{\tilde{d}} := \{B_{\tilde{d}}([x]\,;\epsilon) | x \in X, \epsilon > 0\}$ is a basis for $\mathcal{T}_{\tilde{x}}$.

I claim that for each $x \in X$ and $\epsilon > 0$,

$$T\left(B_d(x;\epsilon)\right) = B_{\tilde{d}}\left([x]\,;\epsilon\right) \tag{2.57}$$

To see this, suppose $\tilde{y} \in T\left(B_d(x;\epsilon)\right)$. Then $\tilde{y} = T(y)$ for some $y \in B_d(x;\epsilon)$. Hence

$$\begin{aligned} \tilde{d}(\tilde{y}, [x]) &= \tilde{d}(T(y), [x]) \\ &= \tilde{d}([y], [x]) \\ &= d(y, x) \\ &< \epsilon \end{aligned}$$

Hence $\tilde{y} \in B_d([x]\,;\epsilon)$, and so

$$T\left(B_d(x;\epsilon)\right) \subset B_{\tilde{d}}\left([x]\,;\epsilon\right) \tag{2.58}$$

Suppose $[y] \in B_{\tilde{d}}\left([x]\,;\epsilon\right)$. Then $d(x, y) = \tilde{d}()\,[x], [y]) < \epsilon$, so $y \in B_d(x;\epsilon)$. Hence $[y] = T(y) \in T\left(B_d(x;\epsilon)\right)$, so the reverse inclusion also holds, and so the above claim holds. This, paired witht he fact that

$$\{[x] | x \in X\} = X/\cong \tag{2.59}$$

finishes the result.

□

Proof of 03. Falls directly from the definition $T(x) = [x]$, hence

$$d(x, y) = \tilde{d}([x], [y]) = \tilde{d}(T(x), T(y)) \tag{2.60}$$

T is surjective by 2.1.7. □

Proof of 04. Let (X, d) be **Pseudometric-Complete** . Let $\{[x_i]\}_{i\in\mathbb{N}} \subset (X/\cong, \tilde{d})$ be a **Pseudometric Cauchy Sequence** . Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $m, n > N$, we have

$$d(x_m, x_n) = \tilde{d}(Tx_m, Ty_m) = \tilde{d}([x_m], [x_n]) < \epsilon \tag{2.61}$$

So the sequence $\{x_i\}_{i\in\mathbb{N}} \subset (X, d)$ is **Pseudometric Cauchy Sequence** . Since (X, d) is **Pseudometric-Complete** , this sequence has a limit, say $x_i \to x \in (X, d)$. But, we have $[x_i] = Tx_i \to Tx = [x]$, so $\{[x_i]\}$ is convergent, and since that sequence was arbitrary, $(X/\cong, \tilde{d})$ is **Pseudometric-Complete** .

Let $(X/\cong, \tilde{d})$ be **Pseudometric-Complete** . Let $\{x_i\} \subset X$ be a **Pseudometric Cauchy Sequence** . Let $\epsilon > 0$. Then there exist $N \in \mathbb{N}$ such that for $m, n > N$, we have

$$\tilde{d}([x_m], [x_n]) = \tilde{d}(Tx_m, Tx_n) = d(x_m, x_n) < \epsilon \tag{2.62}$$

so that $\{[x_i]\}_{i\in\mathbb{N}}$ is also a **Pseudometric Cauchy Sequence** . Since $(X/\cong, \tilde{d})$ is **Pseudometric-Complete** , this sequence has a limit, say $[x_i] \to y \in X/\cong$. Since T is surjective, for some $x \in X$, $Tx \in y$, and so

$$d(x, x_i) = \tilde{d}(Tx, Tx_i) = \tilde{d}(y, [x_i]) \to 0 \tag{2.63}$$

meaning $x_i \to x$ and we are done.

□

Remark 2.7.18 (Mettic Space Correspondence)**.** *Note that in the case of a metric space, the condition of being* **Pseudometric-Complete** is equivalent to the condition being Complete, and A sequence is a **Pseudometric Cauchy Sequence** if and only if it is a cauchy sequence.

Definition 2.7.19 ((Pseudo)Metrizable)**.** *Let $(X, \mathcal{T})$ be a topological space.*

1. *We say that $(X, \mathcal{T})$ (Or $\mathcal{T}$ or X which it wouldn't cause confusion) is* **Pseudometrizable** if there exists a pseudometric d on X such that $\mathcal{T}$ is the **Pseudometric Topology** on (X, d).

2. We say that $(X, \mathcal{T})$ (Or $\mathcal{T}$ or X when it wouldn't cause confusion) is **Metrizable** if there exists a metric d on X such that $\mathcal{T}$ is the metric topology on (X, d).

Proposition 2.7.20 (Pseudometrizable Prequotient)**.** *Let $(X, \mathcal{T}_X)$ be a topological space with* **Relation Of Equal Neighborhood Filters** on $\cong$, and with **Quotient Topological Space** $(X/\cong, \mathcal{T}_{X/\cong})$ and **Quotient Map** T. Let $(X/\cong, \mathcal{T}_{X/\cong})$ be **Pseudometrizable** with **Pseudometric** $\tilde{d}$.

The following hold.

1. $(X, \mathcal{T}_X)$ is **Pseudometrizable** with a pseudometric $d : X \times X \to [0, \infty)$ defined by
$$d(x, y) = \tilde{d}\left([x], [y]\right).$$
2. $\tilde{d}$ is a Metric $(X/\cong, \mathcal{T}_{X/\cong})$, and so this space is **Metrizable** .
3. If T is injective, then d as defined above is a metric on X, so that $(X, \mathcal{T}_X)$ is **Metrizable** .

Proof Of One. First, observe that
$$d(x, y) = \tilde{d}([x], [y]) \in [0, \infty)$$
so that d is well defined.

Also,
$$d(x, y) = \tilde{d}([x], [y]) = \tilde{d}([y], [x]) = d(y, x)$$
, so d is a **Symmetric Map** .

Also,
$$\begin{aligned} d(x, z) &= \tilde{d}([x], [z]) \\ &\leq \tilde{d}([x], [y]) + \tilde{d}([y], [z]) \\ &= d(x, y) + d(y, z) \end{aligned}$$
so d satisfies the **Triangle Inequality** . Also,
$$d(x, x) = \tilde{d}([x], [x]) = 0 \tag{2.64}$$
and so d is a **Pseudometric** on X.

Let $\mathcal{T}_d$ denote the **Pseudometric Topology** on (X, d). What remains to show is that $\mathcal{T}_X = \mathcal{T}_d$.

Since $d(x, y) = \tilde{d}([x], [y]) = \tilde{d}(Tx, Ty)$, T is an isometry.

Let $x \in U \in \mathcal{T}_X$. Then $[x] \in T(U) \in \mathcal{T}_{X/\cong}$. Hence, there is an $\epsilon > 0$ such that $B_{\tilde{d}}([x], \epsilon) \subset T(U)$. By 2.3.29, part 4, $T^{-1}(B_{\tilde{d}}([x], \epsilon) \subset T^{-1}(T(U)) = U$, where by part 6 of that same result, $T^{-1}(B_{\tilde{d}}([x], \epsilon) \in \mathcal{T}_X$. Since T is an isometry $B_d(x, \epsilon) = T^{-1}(B_{\tilde{d}}([x], \epsilon) \subset U$. Thus we have found an open ball contained in U containing ana arbitrary point of U. Hence, $\mathcal{T}_X \subset \mathcal{T}_d$. As part of the preceeding arguement we also showed that an arbitrary d- **Open Ball** was in $\mathcal{T}_X$, so $\mathcal{T}_d \subset \mathcal{T}_X$, and so equality holds and we're done. ☐

Proof of Two. Let $x, y \in X$ wtih $[x] \neq [y]$. Then $x \ncong y$. By 2.7.15, $x \ncong_d y$. Hence $\tilde{d}([x], [y]) = d(x, y) > 0$. ☐

Proof of Three. Let T be injective, and suppose $x, y \in X$ with $x \neq y$. Then $[x] = Tx \neq Ty = [y]$, so by part 2 of this result, $d(x, y) = \tilde{d}([x], [y]) > 0$.

☐

2.7.2 Quotients of Seminormed Spaces

Definition 2.7.21 (Seminorm)**.** *Let V be a* ***Vector Space*** *over a* ***Field*** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. *We say that a map* $||\cdot|| : V \to [0, \infty)$ *is a* **Seminorm** on V if it is both **Subadditive** and **Absolutely Scalar Homogeneous** . In this case, we refer to $(V, ||\cdot||)$ as a **Seminormed Space** . We say that $||\cdot||$ is **Non-Degenerate** if there is at least one $v \in V$ with $||v|| > 0$. We say that $||\cdot||$ is **Degenerate** if it is not **Non-Degenerate** . We may also refer to the **Seminormed Space** $(V, ||\cdot||)$ as being **Degenerate** or **Non-Degenerate** .

Definition 2.7.22 (Norm)**.** *Let* $(V, ||\cdot||)$ *be a* **Seminormed Space** . If the following implication is true for $x \in V$, then we refer to $||\cdot||$ as a **Norm** on V, and we call $(V, ||\cdot||)$ a **Normed Space** .

$$x \neq 0 \implies ||x|| \neq 0 \tag{2.65}$$

Proposition 2.7.23 (Subadditive Operator On a Group Induces a Metric)**.** *Let* $(G, +, e)$ *be a* **Group** and let $(H, +, \leq)$ be a **Totally Ordered Magma** . Let $p : G \to H$ be **Subadditive** . define $d : G \times G \to H$ by setting, for each $x, y \in G$,

$$d(x, y) = p(x + (-y)) \tag{2.66}$$

Then d satisfies the triangle inequality.

Proof. let $x, y, z \in G$. Then

$$\begin{aligned} d(x, z) &= p(x + (-z)) \\ &= p(x + e + (-z)) \\ &= p(x + (-y) + y + (-z)) \\ &\leq p(x + (-y)) + p(y + (-z)) \\ &= d(x, y) + d(y, z) \end{aligned}$$

completing the proof. □

Definition 2.7.24 (Seminorm Topology)**.** *Let* $(X, ||\cdot||)$ *be a* **Seminormed Space** . define $d_{||\cdot||} : V \times V \to [0, \infty)$ by setting, for $x, y \in X$,

$$d_{||\cdot||}(x, y) = ||x - y|| \tag{2.67}$$

Observe the following:

1. 2.5.2 guarantees that $d_{||\cdot||}(x, x) = 0$ for $x \in X$.

2. 2.7.23 guarantees that d satisfies the **Triangle Inequality** .

3. d is a **Symmetric Map** , as we have

$$d(x, y)_{||\cdot||} = ||x - y|| = |-1|\,||x - y|| = ||y - x|| = d(y, x) \tag{2.68}$$

Hence, $d_{||\cdot||}$ is a **Pseudometric** on X, which we call the **Pseudometric induced by the Seminorm** on X. We refer to $(X, d_{||\cdot||})$ as the **Pseudometric Space induced by the Seminormed Space** $(X, ||\cdot||$. We refer to the **Pseudometric Topology** induced by $d_{||\cdot||}$ as the **Seminorm Topology** induced by $||\cdot||$, and unless otherwise specified, when we reference $(X, ||\cdot||)$, we consider it to be endowed with this topology.

Definition 2.7.25 (**Complete Seminormed Space**, **Banach Space**)**.** *Let* $(X, ||\cdot||)$ *be a* **Seminormed Space** . Let d be the **Pseudometric** induced on X by $||\cdot||$. If (X, d) is **Pseudometric-Complete** , then we call $(X, ||\cdot||)$ a **Complete Seminormed Space**. $(X, ||\cdot||)$ is a **Normed Space** then under these same circumstances we call $(X, ||\cdot||)$ a **Complete Normed Space**. A **Complete Normed Space**is called a **Banach Space**.

Definition 2.7.26 (Seminorm Kernel)**.** *Let* $(V, ||\cdot||)$ *be a* **Seminormed Space** . Define the set $\mathcal{K}^{\text{ernel}}_{(\text{V},||\cdot||)}$ by

$$\mathcal{K}^{\text{ernel}}_{(\text{B},||\cdot||)} = \{\text{x} \in \text{V}|\ ||\text{x}|| = 0\} \tag{2.69}$$

We call this set the **Seminorm Kernel** of the space $\mathcal{K}^{\text{ernel}}_{(\text{V},||\cdot||)}$. When confusion is unlikely, we may denote this set with $\mathcal{K}^{\text{ernel}}$, $\mathcal{K}^{\text{ernel}}_{\text{V}}$, or even $\mathcal{K}^{\text{ernel}}_{||\cdot||}$, or we may just refer to it as the **Seminorm Kernel** , the **Seminorm Kernel** of V, or the **Seminorm Kernel** of $||\cdot||$.

Proposition 2.7.27 (Seminorm Kernel is a vector Subspace)**.** *Let* $(X, ||\cdot||)$ *be a* **Seminormed Space** over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ with corresponding **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$. Then the following are true.

1. $\mathcal{K}^{\text{ernel}}$ is a vector subspace of X.
2. $\mathcal{K}^{\text{ernel}}$ is closed in the **Seminorm Topology** on X.
3. $\mathcal{K}^{\text{ernel}} =$ X if and only if X is **Degenerate** .

Proof of One. **Subadditivity** implies that, if $x, y \in \mathcal{K}^{\text{ernel}}$, then $||x + y|| \leq ||x|| + ||y|| = 0$. By **Scalar Homogeneity** , if $x \in \mathcal{K}^{\text{ernel}}$ and $\alpha \in \mathbb{F}$, $||\alpha x|| = |\alpha|\, ||x|| = 0$ so $\mathcal{K}^{\text{ernel}}$ is in fact a vector subspace of X. □

Proof of Two. If $x \in X \setminus \mathcal{K}^{\text{ernel}}$ then $||x|| = \alpha > 0$ for some positive α. Hence $B(x; \alpha/2)$ is an open set containing x disjoint from $\mathcal{K}^{\text{ernel}}$. We can then write $X \setminus \mathcal{K}^{\text{ernel}}$ as the union of all such open sets to see that $\mathcal{K}^{\text{ernel}}$ is closed. □

Proof of Three. Direct application of the definitions of the **Seminorm Kernel** and **Degenerate Seminorm** . □

Definition 2.7.28 (Quotient Space Mod Kernel)**.** *Let* $(X, ||\cdot||)$ *be a* **Seminormed Space** over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. with **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$. By 2.7.27, part 1, $\mathcal{K}^{\text{ernel}}$ is a vector subspace of $X's$ algebraic structure, and so if we define $\cong_{\mathcal{K}^{\text{ernel}}} \subset X \times X$ by setting, for $x, y \in X$

$$x \cong_{\mathcal{K}^{\text{ernel}}} y \iff x - y \in \mathcal{K}^{\text{ernel}} \tag{2.70}$$

Then one recognizes $\cong_{\mathcal{K}^{\text{ernel}}}$ as **Equivalence MOD-$\mathcal{K}^{\text{ernel}}$** as would be commonly spoken of in Module or Vector Space theory. From this, alot of nice properties fall out. We list them here, without proof just to nail down notation. For proof, see any undergraduate algebra text.

1. If $x \cong_{\mathcal{K}^{\text{ernel}}} y$, then we say that x and y are **Equivalent MOD-$\mathcal{K}^{\text{ernel}}$** .

2. For $x \in X$, we denote the **Equivalence Class** $[x]_{\cong_{\mathcal{K}^{\text{ernel}}}}$ with $[x]_{\mathcal{K}^{\text{ernel}}}$ or with $x + \mathcal{K}^{\text{ernel}}$, or when confusion is unlikely, simply $[x]$.

3. We denote $X/\cong_{\mathcal{K}^{\text{ernel}}}$ with $X/\mathcal{K}^{\text{ernel}}$.

4. If we define $\oplus : X/\mathcal{K}^{\text{ernel}} \times X/\mathcal{K}^{\text{ernel}} \to X/\mathcal{K}^{\text{ernel}}$ by setting , for $x, y \in X$, $[x]_{\mathcal{K}^{\text{ernel}}} \oplus [y]_{\mathcal{K}^{\text{ernel}}} = [x + y]_{\mathcal{K}^{\text{ernel}}}$, then $\oplus$ is well defined and endows $X/\mathcal{K}^{\text{ernel}}$ with a group structure.

5. If we further define $\odot : \mathbb{F} \times X/\mathcal{K}^{\text{ernel}} \to X/\mathcal{K}^{\text{ernel}}$ by $\alpha[x]_{\mathcal{K}^{\text{ernel}}} = [\alpha x]_{\mathcal{K}^{\text{ernel}}}$, then $\left(X/\mathcal{K}^{\text{ernel}}, \oplus, \odot, [0]_{\mathcal{K}^{\text{ernel}}}\right)$ is a Vector space over $\mathbb{F}$.

6. Unless otherwise specified , when referring to the set $X/\mathcal{K}^{\text{ernel}}$, we endow it with the above vector space structure , and we call this space the **Seminorm Kernel Quotient Vector Space** of the seminormed space $(X, ||\cdot||)$.

Proposition 2.7.29 (Equivalence Mod Kernel is Pseudometric Equivalence)**.** *Let* $(X, ||\cdot||)$ *be a seminromed space. with* **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$. Let d denote the **Pseudometric induced by the Seminorm** . Let $\cong_d$ denote the **Relation Of Zero Distance** with respect to d.

Then $\cong_{\mathcal{K}^{\text{ernel}}} = \cong_d$.

Proof. Let $x, y \in X$ and let $x \cong_{\mathcal{K}^{\text{ernel}}} y$. Then, since $x - y \in \mathcal{K}^{\text{ernel}}$, Then $d(x, y) := ||x - y|| = 0$, so $x \cong_d y$. Hence $\cong_{\mathcal{K}^{\text{ernel}}} \subset \cong_d$

Now let $x, y \in X$ with $x \cong_d y$. Then $||x - y|| = d(x, y) = 0$, so $x - y \in \mathcal{K}^{\text{ernel}}$, and therefore $x \cong_{\mathcal{K}^{\text{ernel}}} y$. Hence, $\cong_d \subset \cong_{\mathcal{K}^{\text{ernel}}}$.

Since inclusion goes both directions, $\cong_{\mathcal{K}^{\text{ernel}}} = \cong_d$.

□

Definition 2.7.30 (Quotient Norm Space)**.** *Let* $(X, ||\cdot||)$ *be a* **Seminormed Space** with **Pseudometric induced by the Seminorm** d, **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$, and **Seminorm Kernel Quotient Vector Space** $X/\mathcal{K}^{\text{ernel}}$. Let $\tilde{d} : X/\mathcal{K}^{\text{ernel}} \times X/\mathcal{K}^{\text{ernel}} \to [0, \infty)$ be the **Metric Induced By The Pseudometric** .

Define $||\cdot||_{\mathcal{K}^{\text{ernel}}} : X/\mathcal{K}^{\text{ernel}} \to [0, \infty)$ by setting, for $x \in X$,

$$||[x]||_{\mathcal{K}^{\text{ernel}}} = \tilde{d}([x], [0]) \tag{2.71}$$

By 2.7.31, $(X/\mathcal{K}^{\text{ernel}}, ||\cdot||_{\mathcal{K}^{\text{ernel}}})$ is a normed space which we call the **Quotient Normed Space** of $(X, ||\cdot||)$, and we call $||\cdot||_{\mathcal{K}^{\text{ernel}}}$ the **Quotient Norm** . Whenever we refer to $X/\mathcal{K}^{\text{ernel}}$, unless otherwise specified, we endow it with this norm and the topology generated by this norm. Furthermore, whenever we consider $X/\mathcal{K}^{\text{ernel}}$, unless otherwise specified, we consider it as possesing the topology generated by the norm $||\cdot||_{\mathcal{K}^{\text{ernel}}}$.

Proposition 2.7.31 (Quotient Normed Space). *Let* $(X, ||\cdot||)$ *be a* **Seminormed Space** with **Pseudometric induced by the Seminorm** d, **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$, and **Seminorm Kernel Quotient Vector Space** $X/\mathcal{K}^{\text{ernel}}$. Let $\tilde{d} : X/\mathcal{K}^{\text{ernel}} \times \mathrm{X}/\mathcal{K}^{\text{ernel}} \to [0, \infty)$ be the **Metric Induced By The Pseudometric** . Let $T : X \to X/\mathcal{K}^{\text{ernel}}$ denote the **Quotient Map** of X into $X/\mathcal{K}^{\text{ernel}}$ (Recalling that the **Relation Of Equal Neighborhood Filters** on e quals the **Relation Of Zero Distance** equals the relation of **Equivalence MOD-$\mathcal{K}^{\text{ernel}}$**), so they would all produce the same quotient map) Let $||\cdot||_{\mathcal{K}^{\text{ernel}}}$ denote the **Quotient Norm** .

The following are true.

1. $||\cdot||_{\mathcal{K}^{\text{ernel}}}$ is a norm on $X/\mathcal{K}^{\text{ernel}}$.
2. $\tilde{d}$ is the **Pseudometric induced by the Seminorm** $||\cdot||_{\mathcal{K}^{\text{ernel}}}$, and thus they produce the same topology.
3. T has all of the properties described in 2.3.29.
4. T is Linear.
5. T is Surjective.
6. T is an isometry.
7. T is injective if and only if $||\cdot||$ is a norm.

Proof of 1. First, note that $Range(||\cdot||_{\mathcal{K}^{\text{ernel}}}) \subset Range(\tilde{d}) \subset [0, \infty)$, so that $||\cdot||_{\mathcal{K}^{\text{ernel}}}$ has the correct domain and codomain. For **Subadditivity** , let $[x], [y] \in X/\mathcal{K}^{\text{ernel}}$. Then

$$\begin{aligned}
||[x] + [y]||_{\mathcal{K}^{\text{ernel}}} &= ||[x + y]||_{\mathcal{K}^{\text{ernel}}} \\
&= \tilde{d}\,([x + y], [0]) \\
&= d(x + y, 0) \\
&= ||x + y|| \\
&\leq ||x|| + ||y|| \\
&= d(x, 0) + d(y, 0) \\
&= \tilde{d}\,([x], [0]) + \tilde{d}\,([y], [0]) \\
&= ||[x]||_{\mathcal{K}^{\text{ernel}}} + ||[y]||_{\mathcal{K}^{\text{ernel}}}
\end{aligned}$$

For **Absolute Scalar Homogeneity** , let $\alpha \in \mathbb{F}$ and $[x] \in X/\mathcal{K}^{\text{ernel}}$. Then,

$$\begin{aligned}
||[\alpha x]||_{\mathcal{K}^{\text{ernel}}} &= \tilde{d}\,([\alpha x], [0]) \\
&= d(\alpha x, 0) \\
&= ||\alpha x|| \\
&= |\alpha|\,||x|| \\
&= |\alpha|\,||[x]||_{\mathcal{K}^{\text{ernel}}}
\end{aligned}$$

Finally, suppose $[x] \neq 0$. Then, since the additive identity of $X/\mathcal{K}^{\text{ernel}}$ is $\mathcal{K}^{\text{ernel}}$, $x \notin \mathcal{K}^{\text{ernel}}$. Hence $||[x]||_{\mathcal{K}^{\text{ernel}}} = \tilde{d}([x], 0) = d(x, 0) = ||x|| > 0$.

□

Proof of 2. Let D denote the **Pseudometric induced by the Seminorm** $||\cdot||_{\mathcal{K}^{\text{ernel}}}$. Then, for $[x], [y] \in X/\mathcal{K}^{\text{ernel}}$,

$$\begin{aligned}
\tilde{d}([x], [y]) &= d(x, y) \\
&= ||x - y|| \\
&= ||x - y - 0|| \\
&= d(x - y, 0) \\
&= \tilde{d}([x - y], 0) \\
&= ||[x - y]||_{\mathcal{K}^{\text{ernel}}} \\
&= ||[x] - [y]||_{\mathcal{K}^{\text{ernel}}} \qquad\qquad = D([x], [y])
\end{aligned}$$

Since these two **Pseudometric** 's are equal, they produce the same topology. Furthermore, by applying 2.7.17, we see that the topology generated by $||\cdot||_{\mathcal{K}^{\text{ernel}}}$ is also the **Quotient Topology** on $X/\mathcal{K}^{\text{ernel}}$. □

Proof of 3. T is the topological **Quotient Map** and the norm topology is the **Quotient Topology** , so the assumptions of 2.3.29 are satisfied. □

Proof of 4. Let $x, y \in X$ and $\alpha \in \mathbb{F}$. Then

$$\begin{aligned}
T(\alpha x + y) &= [\alpha x + y] \\
&= (\alpha x + y) + \mathcal{K}^{\text{ernel}} \\
&= \alpha\left(x + \mathcal{K}^{\text{ernel}}\right) + \left(y + \mathcal{K}^{\text{ernel}}\right) \\
&= \alpha[x] + [y] \\
&= \alpha T(x) + T(y)
\end{aligned}$$

□

Proof of 5. Direct consequence of 2.1.7 □

Proof of 6. Consequence of part 2 of this result combined with □

Proof of 7. If $||\cdot||$ is a **Norm** , then $\mathcal{K}^{\text{ernel}} = 0$, so $Tx = Ty \implies T(x - y) = 0 \implies x - y \in \mathcal{K}^{\text{ernel}} \implies \mathrm{x - y = 0} \implies \mathrm{x = y}$. □

Remark 2.7.32 (Quotient Normed Space)**.** *If* $(X, ||\cdot||_X)$ *is a* **Normed Space** then by parts 4, 5, 6, and 7 of 2.7.31, $T : X \to \mathcal{K}_{\mathrm{X}}^{\text{ernel}}$ is an isomorphism of **Normed Spaces** whose definition is literally

$$Tx = \{x\} \tag{2.72}$$

For this reason, as an admitted abuse of notation, later in this document, I may not distinguish between the quotient $X/\mathcal{K}_{\mathrm{X}}^{\text{ernel}}$ and the space X if X is a **Normed Space** , and similarly, I may not distinguish between $x \in X$ and $\{x\} \in X/\mathcal{K}_{\mathrm{X}}^{\text{ernel}}$.

Proposition 2.7.33. *Let* $(X, ||\cdot||)$ *be a* **Seminormed Space** with **Quotient Normed Space** $(X/\mathcal{K}^{\text{ernel}}, ||\cdot||_{\mathcal{K}^{\text{ernel}}})$.

Then X is **Pseudometric-Complete** if and only if $X/\mathcal{K}^{\text{ernel}}$ is complete.

Proof. Let X be **Pseudometric-Complete** . Let $\{[x_i]\}_{i\in\mathbb{N}} \subset X/\mathcal{K}^{\text{ernel}}$ be a **Pseudometric Cauchy Sequence** . Let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for $m, n > N$ we have

$$||[x_m - x_n]||_{\mathcal{K}^{\text{ernel}}} < \epsilon \tag{2.73}$$

For this N, we have

$$||x_m - x_n|| = ||[x_m - x_n]||_{\mathcal{K}^{\text{ernel}}} < \epsilon \tag{2.74}$$

so that $\{x_i\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence** . Since X is **Pseudometric-Complete** , there is a $x \in X$ such that $||x_i - x|| \to 0$, but since T is an isometry,

$$||[x] - [x_i]|| = ||[x_i - x]||_{\mathcal{K}^{\text{ernel}}} \to 0 \tag{2.75}$$

and so $[x_i] \to [x]$. so that $X/\mathcal{K}^{\text{ernel}}$ is complete.

Now suppose instead that $X/\mathcal{K}^{\text{ernel}}$ is complete and suppose $\{x_i\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence** in X. Since $||[x_i - x_j]||_{\mathcal{K}^{\text{ernel}}} = ||x_i - x_j||$, $\{[x_i]\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence** in $X/\mathcal{K}^{\text{ernel}}$, which therefore has a limit $y \in X/\mathcal{K}^{\text{ernel}}$. Since T is surjective, $y = [x]$ for some $x \in X$, and it is easy to see that $x_i \to x$ so that X is **Pseudometric-Complete** .

□

Definition 2.7.34 (Space of Continuous Linear Operators From a Seminormed Space into a Normed Space)**.** *Let* $(X, ||\cdot||_X)$ *be a* **Non-Degenerate Seminormed Space** . Let $(Y, ||\cdot||_Y)$ be a **Seminormed Space** . We denote with $BL\left((X, ||\cdot||_X), (Y, ||\cdot||_Y)\right)$ the collection of **Continuous Linear** operators $T : (X, ||\cdot||_X) \to (Y, ||\cdot||_Y)$. When the topologies on X and Y are understood, we denote this set with $BL\left(X, Y\right)$. We refer to $BL\left(X, Y\right)$ as the **Space of Bounded Linear Operators** from $(X, ||\cdot||_X)$ to $(Y, ||\cdot||_Y)$, or when $||\cdot||_X$ and $||\cdot||_Y$ are understood, from X to Y.

We endow $BL\left(X, Y\right)$ with the algebraic operations of pointwise scalar multiplication and pointwise addition, making $BL\left(X, Y\right)$ a vector space.

We define $||\cdot|| : BL(X, Y) \to [0, \infty)$ by defining, for $T \in BL(X, Y)$

$$||T|| = \sup_{||x||_X \neq 0} \frac{||Tx||_Y}{||x||_X} \tag{2.76}$$

As will be proven in 2.7.35, $||\cdot||$ is a **Seminorm** on $BL(X, Y)$, which we refer to as the **Operator Seminorm** on $BL(X, Y)$. induced by the **Seminorm** $||\cdot||_X$ on X and the **Seminorm** $||\cdot||_Y$ on Y.

In the case that $||\cdot||_Y$ is a **Norm** , rather than just a **Seminorm** , by 2.7.35 , $||\cdot||$ is a **Norm** on $BL(X, Y)$, which we instead call the **Operator Norm** .

Proposition 2.7.35 (Space of Bounded Linear Operators On Seminormed Spaces)**.** *Let* $(X, ||\cdot||_X)$ *be a* **Seminormed Space** . Let $(Y, ||\cdot||_Y)$ be a **Seminormed Space** . Let $BL(X, Y)$ denote the **Space of Bounded Linear Operators** from X to Y. Let $||\cdot||$ denote the **Operator Seminorm** .

The following are true.

1. $||\cdot||$ is in fact a well-defined **Seminorm** on $BL(X, Y)$.
2. If $||\cdot||_Y$ is a **Norm** , then so is $||\cdot||$.
3. If $T \in BL(X, Y)$ and $\alpha \in (0, \infty)$, then $||T|| = \sup\limits_{||x||_X = \alpha} \frac{||Tx||_Y}{||x||_X}$.
4. If $T \in BL(X, Y)$ and $\alpha \in (0, \infty)$, , then $||T|| = \sup\limits_{0 < ||x||_X \leq \alpha} \frac{||Tx||_Y}{||x||_X} = \sup\limits_{0 < ||x||_X < \alpha} \frac{||Tx||_Y}{||x||_X}$.
5. If $T \in BL(X, Y)$ and $x \in X$, then $||Tx||_Y \leq ||T|| \, ||x||_X$.
6. $S : X \to Y$ is linear , $S(\mathcal{K}^{\text{ernel}}_{\text{X}}) \subset \mathcal{K}^{\text{ernel}}_{\text{Y}}$, and $\sup\limits_{||x||_X \neq 0} \frac{||Sx||_Y}{||x||_X} < \infty$, if and only if $S \in BL(X, Y)$.
7. A sequence $\{T_i\}_{i \in \mathbb{N}}$ is a **Pseudometric Cauchy Sequence** if and only if there exists an $\alpha > 0$ such that the collection of sequences $\{\{T_i x\}_{i \in \mathbb{N}} | x \in B_X(0; \alpha)\}$ is **Uniformly Cauchy** if and only if for every $\beta > 0$, the collection of sequences $\{\{T_i x\}_{i \in \mathbb{N}} | x \in B_X(0; \beta)\}$ is is **Uniformly Cauchy**
8. If $T_i \to T$ with respect to $||\cdot||$, then $T_i x \to Tx$ with respect to $||\cdot||_Y$ for each $x \in X$
9. A sequence $\{T_i\}_{i \in \mathbb{N}} \subset BL(X, Y)$ converges with respect to $||\cdot||$ if and only if it is a **Pseudometric Cauchy Sequence** and for each x_α in some Hamel basis $\{x_\alpha\}_{\alpha \in A} \subset X$, the sequence $\{T_i x_\alpha\}_{\alpha \in A}$ converges with respect to $||\cdot||_Y$.
10. Let X be **Non-Degenerate** . Then $BL(X, Y)$ is complete if Y is complete.
11. If $BL(X, Y)$ is **Non-Degenerate** then Y is **Non-Degenerate** .
12. If $S : X \to Y$ is linear , then $S \in BL(X, Y)$ if and only if there exists a constant $c \in (0, \infty)$ such that for every $x \in X$. $||Tx|| \leq c \, ||x||$.
13. If $T \in BL(X, Y)$, and $A = \{c \in (0, \infty) | \, ||Tx|| \leq c \, ||x|| \, (\forall x \in X)\}$, then $||T|| = \inf(A)$.

Proof of 1. Since X is **Non-Degenerate** , there exists an $x \in X$ with $||x||_X \neq 0$, so for each $T \in BL(X, Y)$, the set that the supremum is being taken over is nonempty. Also, it is clear that $Range(||\cdot||) \subset [0, \infty)$,

For **Subadditivity** , let $T_i \in BL(X, Y)$ for $i \in \{0, 1\}$. and $x \in X$ with $||x|| > 0$. Then, since $||\cdot||_Y$ is **Subadditive** ,

$$\frac{||(T_0 + T_1)x||_Y}{||x||_X} \leq \frac{||T_0 x||_Y}{||x||_X} + \frac{||T_1 x||_Y}{||x||_X}$$

Since this is true for each x with $||x||_X \neq 0$, taking the supremum of each side yields

$$\sup_{||x||_X \neq 0} \left(\frac{||(T_0 + T_1)x||_Y}{||x||_X} \right) \leq \sup_{||x||_X \neq 0} \left(\frac{||T_0 x||_Y}{||x||_X} + \frac{||T_1 x||_Y}{||x||_X} \right)$$
$$\leq \sup_{||x||_X \neq 0} \left(\frac{||T_0 x||_Y}{||x||_X} \right) + \sup_{||x||_X \neq 0} \left(\frac{||T_1 x||_Y}{||x||_X} \right)$$

Hence, $||T_0 + T_1|| \leq ||T_0|| + ||T_1||$ so that $||\cdot||$ is **Subadditive** . For **Absolute Scalar Homogeneity** , let $T \in BL(X, Y)$, $\alpha \in \mathbb{F}$, and $x \in X$ with $||x||_X \neq 0$. Then

$$\frac{||(\alpha T)x||_Y}{||x||_X} = \frac{||\alpha (Tx)||_Y}{||x||_X} = |\alpha| \frac{||Tx||_Y}{||x||_X}$$

Hence taking the supremum finishes the proof. □

Proof of 2. Let $T \neq 0 \in BL(X, Y)$. Then for some $x \in X$, $Tx \neq 0$. Then Tx has a neighborhood U disjoint from 0_Y, Hence $x \in T^{-1}(U)$ but not $0_X \in T^{-1}(U)$, since $T0_X = 0_Y$. Since U is a neighborhood of x disjoint from 0, there is an $\epsilon > 0$ such that $0_X \subset \complement U \subset \complement \overline{B_X}(x; \epsilon)$, and therefore $||x||_X > \epsilon$. Since $||x||_X > 0$, it is ranged over in the supremum defining $||T||$, and so

$$0 < \frac{||Tx||_Y}{||x||_X} \leq \sup_{||x||_X \neq 0} \frac{||Tx||_X}{||x||_X} = ||T|| \tag{2.77}$$

□

Proof of 3. Let $\alpha \in (0, \infty)$ Let $T \in BL(X, Y)$. Then, there is a sequence $\{x_i\} \subset X$ with each $||x_i||_X \neq 0$ such that

$$\frac{||Tx_i||_Y}{||x_i||_X} \to ||T|| \tag{2.78}$$

For each $i \in \mathbb{N}$, define $y_i = \alpha x_i / ||x_i||_X$. then each $||y_i|| = \alpha$, and by **Absolute Scalar Homogeneity** of T, we have

$$\frac{||Ty_i||_Y}{||y_i||_X} = \frac{||Tx_i||_Y}{||x_i||_X} \to ||T|| \tag{2.79}$$

, completing the proof. □

Proof of 4. If we define, for $T \in BL(X, Y)$, $f(T) = \sup_{0 < norm x_X \leq \alpha} \frac{||Tx||_Y}{||x||_X}$, then since $||\cdot||^{-1}((0, \alpha)) \subset ||\cdot||^{-1}((0, \infty))$, we have $f(T) \leq ||T||$ and since $||\cdot||^{-1}(\{\alpha\}) \subset ||\cdot||^{-1}((0, \alpha))$, we have $||T|| \leq f(T)$. proving the first equality. The second is found by applying the same arguement to $\alpha/2$ and realizing that $(0, \alpha/2] \subset (0, \alpha)$. □

Proof of 5. Let $T \in BL(X, Y)$ and $x \in X$. If $||Tx||_Y \neq 0$, then $B_Y(Tx, \frac{||Tx||_Y}{2})$ is a neighborhood of Tx disjoint from 0. Continuity of T impliese x then has a neighborhood disjoint from $0 \in T^{-1}(0)$, implying that $||x||_X \neq 0$.

Hence if $||x||_X = 0$, then we know $||Tx||_Y = 0$, so that the relation

$$||Tx||_Y \leq ||T||\,||x||_X \tag{2.80}$$

If $||x||_X \neq 0$, then by definition of supremum,

$$\frac{||Tx||_Y}{||x||_X} \leq ||T||$$

so that $||Tx||_Y \leq ||T||\,||x||_X$. □

Proof of 6. I assume the first 3 conditions and show that $S \in BL(X, Y)$. It is necessary and sufficient to show that S is continuous. Let $F = \sup_{||x||_X \neq 0} \frac{||Sx||_Y}{||x||_X}$. If $F = 0$, then $S(X) \subset \mathcal{K}^{\text{ernel}}_{\text{Y}}$. Every neighborhood of every point in $\mathcal{K}^{\text{ernel}}_{\text{Y}}$ contains $\mathcal{K}^{\text{ernel}}_{\text{Y}}$, so in that case continuity holds. Suppose $F \neq 0$. By translation invariance of the topology, it is sufficient to consider neighborhoods of $0_Y \in Y$. Let $\epsilon > 0$. Define $V = B_X\left(0; \frac{\epsilon}{F}\right)$. Let $x_0 \in V$. If $||x_0||_X = 0$, then $S(x_0) \in S(\mathcal{K}^{\text{ernel}}_{\text{X}}) \subset \mathcal{K}^{\text{ernel}}_{\text{Y}} \subset \text{B}_{\text{Y}}(0; \epsilon)$. If $||x_0||_X \neq 0$, then $||Sx||_Y \leq F\,||x||_X < \epsilon$, so $s(x_0) \in B_Y(0; \epsilon)$. Hence $S\left(B_X\left(0; \frac{\epsilon}{F}\right)\right) \subset B_Y(0; \epsilon)$. so S is continuous,and this direction fo the proof is complete.

Suppose conversely that $S \in BL(X, Y)$. Then S is **Linear** by definition , and the supremum expression is finite by part 1 of this result. Since S is **Linear** , $S0_X = 0_Y$. Since S is **Continuous** ,

$$\begin{aligned} S(\mathcal{K}^{\text{ernel}}_{\text{X}}) &= S\left(\overline{\{0_X\}}\right) \\ &\subset \overline{S\left(\{0_X\}\right)} \\ &= \overline{\{0_Y\}} \\ &= \mathcal{K}^{\text{ernel}}_{\text{Y}} \end{aligned}$$

□

Proof of 7. ($3 \implies 2$) is trivial, as is ($2 \implies 3$).

I now prove ($1 \implies 3$). Let $\{T_i\}_{i\in\mathbb{N}}$ be a **Pseudometric Cauchy Sequence** . Let $\beta > 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $m, n > N$,

$$||T_n - T_m|| < \frac{\epsilon}{\beta}$$

Let $x \in B_X(0; \beta)$. Then

$$\begin{aligned} ||T_m x - T_n x||_Y &= ||(T_m - T_n)x||_Y \\ &\leq ||T_m - T_n||\,||x||_X \\ &< \epsilon \end{aligned}$$

Since $x \in B_X(0; \beta)$ was arbitrary, $\{\{T_i x\}_{i\in\mathbb{N}} | x \in B_X(0; \beta)\}$ is **Uniformly Cauchy** .

I now prove ($3 \implies 1$). Let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for $m, n > N$, for each $x \in B_X(0; 2)$,

$$||T_m x - T_n x|| < \epsilon$$

In particular, if $||x|| = 1$, then

$$\frac{||(T_m - T_n)x||_Y}{||x||_X} = ||(T_m - T_n)x||_Y < \epsilon \tag{2.81}$$

Hence, by taking the supremum over such x and applying part 3 of this result, $||T_m - T_n|| < \epsilon$. □

Proof of 8. Let $T_i \to T$. Let $x \in X$. If $x \in \mathcal{K}_{\text{X}}^{\text{ernel}}$, then $T_i(x) \in \mathcal{K}_{\text{Y}}^{\text{ernel}}$ for $i \in \mathbb{N}$ and $T_x \in \mathcal{K}_{\text{Y}}^{\text{ernel}}$, so convergence is obvious. Suppose $||x||_X > 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $n > N$, $||T_n - T|| < \frac{\epsilon}{||x||_X}$. For such n,

$$||T_i x - Tx||_Y \leq ||T_i - T|| \, ||x||_X < \epsilon$$

□

Proof of 9. ($\implies$) Suppose $T_i \to T$. Then, by 2.7.6, $\{T_i\}_{i \in \mathbb{N}}$ is a **Pseudometric Cauchy Sequence** . An application of part 8 of this result implies the pointwise convergence on a hamel basis.

($\impliedby$) Let $\{x_\alpha\}_{\alpha \in A}$ be a Hamel basis for X. Let $T_i x_\alpha \to y_\alpha$ for $\alpha \in A$. Define $T : X \to Y$ by setting, for $x \in X$, for any $\{\alpha_i\}_{i=1}^n \subset A$ $\{\beta_i\}_{i=1}^n \subset \mathbb{F}$,

$$T\left(\sum_{i=1}^n \beta_i x_{\alpha_i}\right) = \sum_{i=1}^n \beta_i y_{\alpha_i} \tag{2.82}$$

The uniqueness of a hamel basis representation implies that T is well defined. It is clear also that T is linear, and that $T\left(\mathcal{K}_{\text{X}}^{\text{ernel}}\right) \subset \mathcal{K}_{\text{Y}}^{\text{ernel}}$.

Let $x \in X$. Then we can find a unique representation, $x = \sum_{j=1}^n \beta_j x_{\alpha_j}$ where $x_{\alpha_j} \in A$ and $\beta_j \in \mathbb{F}$ for every j. For each $j \in \{1, ..., n\}$, there is an N_j such that if $n_j > N_j$, the

$$\left|\left|T_{n_j} x_{\alpha_j} - y_{\alpha_j}\right|\right| < \frac{\epsilon}{n(|\beta_j| + 1)} \tag{2.83}$$

Let $N = max\{N_j\}_{j=1}^n$. Let $m > N$. Then, we have

$$\begin{aligned}
||T_m x - Tx|| &= \left|\left|T_m\left(\sum_{j=1}^n \beta_j x_{\alpha_j}\right) - T\left(\sum_{j=1}^n \beta_j x_{\alpha_j}\right)\right|\right| \\
&= \left|\left|\sum_{j=1}^n \beta_j \left(T_m x_{\alpha_j} - T x_{\alpha_j}\right)\right|\right| \\
&= \sum_{j=1}^n |\beta_j| \left|\left|T_m x_{\alpha_j} - y_{\alpha_j}\right|\right| \\
&< \epsilon
\end{aligned}$$

Since $m > N$ was arbitrtary , $T_i x \to Tx$ for $x \in X$.

Since $T_i x \to Tx$ for $x \in B_X(0;2)$, and since part 7 of this result, paried with the assumption that $\{T_i\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence** , $\{\{T_i x\}_{i\in\mathbb{N}}\}_{x\in B_X(0;2)}$ is **Uniformly Convergent** to $\{Tx\}_{x\in B_X(0;2)}$.

Let $\epsilon > 0$. By **Uniform Convergence** , there is an $N \in \mathbb{N}$ such that for $n > N$, $x \in B_X(0;2)$, we have

$$||T_n x - Tx|| < \epsilon \tag{2.84}$$

In particular, if $||x|| = 1$,

$$\frac{||(T_n - T)x||_Y}{||x||_X} < \epsilon \tag{2.85}$$

Implying first through part 6 of this result that $T \in BL(X,Y)$, and second that Hence $T_i \to T$ with respect to $||\cdot||$. □

Proof of 10. Let Y be Pseudometric Complete. and let X be **Non-Degenerate** . Let $\{T_\alpha\}_{\alpha\in A} \subset BL(X,Y)$ be a **Pseudometric Cauchy Sequence** . Let $\{x_\alpha\}_{\alpha\in A}$ be a Hamel basis for X. Let $\alpha \in A$. If $x_\alpha \in \mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}$, then $T_i x_\alpha \in \mathcal{K}_{\mathrm{Y}}^{\mathrm{ernel}}$ for $i \in \mathbb{N}$, and so $T_i x_\alpha \to 0$. Otherwise, $||x_\alpha||_X > 0$, so for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m,n > N$, we have $||T_m - T_n|| < \frac{\epsilon}{||x_\alpha||}$. For such m and n,

$$\begin{aligned}
\frac{||T_m x_\alpha - T_n x_\alpha||_Y}{||x_\alpha||_X} &= \frac{||(T_m - T_n)\, x_\alpha||_Y}{||x_\alpha||_X} \\
&\leq ||T_n - T_m|| \\
&< \frac{\epsilon}{||x_\alpha||}
\end{aligned}$$

By multiplyging by $||x_\alpha||_X$, we see that $\{T_i x_\alpha\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence** . Since Y is **Complete** , these sequences converge, say $T_i x_\alpha \to y_\alpha$. This allows us apply part 9 of this result to claim that $\{T_i\}_{i\in\mathbb{N}}$ converges in $||\cdot||$. □

Proof of 11. If $BL(X,Y)$ is nondegenerate, then for some $T \in BL(X,Y)$, for some $x \in X$, $||Tx||_Y \neq 0$. □

Proof of 12. Let $S \in BL(X,Y)$. Let $c = ||S|| + 1$. Then, if $||x|| \neq 0$, we have

$$\frac{||Sx||}{||x||} \leq \sup_{||x||\neq 0} \frac{||Sx||}{||x||} = ||S|| = c \tag{2.86}$$

so multiplying by $||x||$ gives $||Sx|| \leq c\,||x||$. If $||x|| = 0$, then $x \in \mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}$, so by part 6 of this result, $Sx \in \mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}$, so

$$||Sx|| = 0 = c0 = c\,||Sx|| \tag{2.87}$$

finishing this direction of the proof.

Suppose instead, now, that there was a constant $c \in (0,1)$ such that for each $x \in X$, we had $||Sx|| \leq c\,||x||$. Then, if $||x|| = 0$, we must have $||Sx|| \leq c\,||x|| = 0$, and so $S(\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}) \subset \mathcal{K}_{\mathrm{Y}}^{\mathrm{ernel}}$. Furthermore, if $||x|| \neq 0$, then we can divide by $||x||$ to get

$$\frac{||Sx||}{||x||} \leq c \tag{2.88}$$

Taking the supremum over all x with $||x|| \neq 0$ gives us the desired result. $\square$

Proof of 13. By part 5 of this result, $||T||$ is one possible value for c, so $||T|| \in A$. That is, $inf(A) \leq ||T||$. Furthermore, if $\epsilon > 0$, then there is an $x \in X$ with

$$\frac{||Tx||}{||x||} > ||T|| - \epsilon \tag{2.89}$$

implying $||Tx|| > (||T|| - \epsilon)\,||x||$ so that $||T|| - \epsilon \notin A$. Hence $||T|| - \epsilon < inf(A)$. Since this is true for every positive epsilon, $||T|| < inf(A)$. $\square$

Remark 2.7.36 (Converses)**.** *The converses to parts 10 and 11 of 2.7.35 are true, but their proof relies upon the Hahn Banach theorem for Seminormed spaces, and so their official claim is delayed until later in the narrative.*

Definition 2.7.37 (Codomain Quotient Operator)**.** *Let X and Y be* **Seminormed Spaces** . Define $\mathcal{Q}_Y : BL(X,Y) \to BL(X, Y/\mathcal{K}^{\text{ernel}}_{\text{Y}})$ by setting, for each $x \in X$,

$$\mathcal{Q}_Y Tx = [Tx]$$

Let $T \in BL(X,Y)$. We call $\mathcal{Q}_Y$ the **Codomain Quotient Map** of X and Y and we call $\mathcal{Q}_Y T$ the **Codomain Quotient Operator** of T.

Proposition 2.7.38 (Codomain Quotient Operator)**.** *Let X and Y be* **Seminormed Spaces** with **Codomain Quotient Map** $\mathcal{Q}_Y$. The following are true.

1. $\mathcal{Q}_Y$ is a well defined continuous linear surjective isometry.

2. If Y is a **Normed Space** , then $\mathcal{Q}_Y$ is invertible with a continuous inverse.

Proof Of 1. Since $Tx \in Y$ for any $x \in X$, $[Tx]_Y$ is defined for any $x \in X$. Furthermore, if $q_y : Y \to Y/\mathcal{K}^{\text{ernel}}$ is the **Quotient Map** of Y under **Equivalence MOD-$\mathcal{K}^{\text{ernel}}$** , then $\mathcal{Q}_Y T = q_y \circ T$. By 2.7.31, $q_y \in BL(Y, Y/\mathcal{K}^{\text{ernel}})$. Hence, $\mathcal{Q}_Y T = q_y \circ T \in BL(X, Y/\mathcal{K}^{\text{ernel}})$. Hence $\mathcal{Q}_Y$ is well defined.

For linearity, let $\alpha \in \mathbb{F}$ and $S, T \in BL(X,Y)$. Let $x \in X$. Then,

$$\begin{aligned}
\mathcal{Q}_Y\left(\alpha T + S\right)x &= \left[\left(\alpha T + S\right)x\right]_Y \\
&= \left[\alpha Tx + Sx\right]_Y \\
&= \left[\alpha Tx\right]_Y + \left[Sx\right]_Y \\
&= \alpha\left[Tx\right]_Y + \left[Sx_Y\right] \\
&= \alpha \mathcal{Q}_Y Tx + \mathcal{Q}_Y Sx \\
&= \left(\alpha \mathcal{Q}_Y T + \mathcal{Q}_Y S\right)x
\end{aligned}$$

For being an isometry, let $T \in BL(X,Y)$ and let $x \in X$. Then, since $||[Tx]_Y||_{Y/\mathcal{K}^{\text{ernel}}} = ||Tx||_Y$,

$$\begin{aligned}
\frac{||\mathcal{Q}Tx||_{Y/\mathcal{K}^{\text{ernel}}}}{||x||_X} &= \frac{||[Tx]_Y||_{Y/\mathcal{K}^{\text{ernel}}}}{||x||_X} \\
&= \frac{||Tx||_Y}{||x||_X}
\end{aligned}$$

and thus taking the norm over x with $||x||_X \neq 0$ will yield the same result. Hence $||T|| = ||\mathcal{Q}_Y T||$.

For surjectivity, let $\tilde{T} \in BL(X, Y/\mathcal{K}^{\text{ernel}}_{\text{Y}})$. Let $\{x_\alpha\}_{\alpha \in A}$ be a hamel basis for X. For each $\alpha \in A$, let $y_\alpha \in \tilde{T}x_\alpha$. Define $T : X \to Y$ by

$$T\left(\sum_{i=1}^{n} \beta_{\alpha_i} x_{\alpha_i}\right) = \sum_{i=1}^{n} \beta_{\alpha_i} y_{\alpha_i} \tag{2.90}$$

T is obviously linear and has the property $[Tx] = \tilde{T}x$. and since $\tilde{T} \in BL(X, Y/\mathcal{K}^{\text{ernel}})$, $\tilde{T}\mathcal{K}^{\text{ernel}}_{\text{X}} \subset \mathcal{K}^{\text{ernel}}_{(\text{Y}/\mathcal{K}^{\text{ernel}}_{\text{Y}})} = 0$. Hence $T\mathcal{K}^{\text{ernel}}_{\text{X}} \subset \mathcal{K}^{\text{ernel}}_{\text{Y}}$. Furthermore, if $x \in X$ with $||x||_X \neq 0$, then

$$\begin{aligned}
\frac{||Tx||_Y}{||x||_X} &= \frac{||[Tx]_Y||_{Y/\mathcal{K}^{\text{ernel}}}}{||x||_X} \\
&= \frac{\left|\left|\tilde{T}x\right|\right|_{Y/\mathcal{K}^{\text{ernel}}}}{||x||_X}
\end{aligned}$$

Therefore T is bounded. Hence $T \in BL(X, Y)$, and $\mathcal{Q}_Y T = \tilde{T}$. Thus we have surjectivity, and are done. □

Proof Of 2. If Y is a **Normed Space** , a linear isometric homeomorphism by 2.7.31. In particular, in this case, q_y is injective, meaning that if $T, S \in BL(X, Y)$ where $T \neq S$, then $Tx_0 \neq Sx_0$ for some $x_0 \in X$. For this x_0, $q_y T x_0 \neq q_y S x_0$, so $\mathcal{Q}_Y T \neq \mathcal{Q}_Y S$. Therefore $\mathcal{Q}_Y$ is injective, and therefore a bijection. The inverse of an isometry is also an isometry and therefore continuous, finishing this proof. □

Definition 2.7.39 (Quotient Operator)**.** *Let X, Y be* **Seminormed Spaces** with **Seminorm Kernels** $\mathcal{K}^{\text{ernel}}_{\text{X}}$, $\mathcal{K}^{\text{ernel}}_{\text{Y}}$. Define $Q : BL(X, Y) \to BL(X/\mathcal{K}^{\text{ernel}}_{\text{X}}, \text{Y}/\mathcal{K}^{\text{ernel}}_{\text{Y}})$ by setting, for $T \in BL(X, Y)$, for $x \in X$,

$$QT\,[x]_X = [Tx]_Y \tag{2.91}$$

We call Q the **Operator Quotient Map** of X and Y and we call QT the **Quotient Operator** of T.

Proposition 2.7.40 (Quotient Operator)**.** *Let X, Y be* **Seminormed Spaces** with **Seminorm Kernels** $\mathcal{K}^{\text{ernel}}_{\text{X}}$, $\mathcal{K}^{\text{ernel}}_{\text{Y}}$ and **Operator Quotient Map** Q. Then Q is a well-defined linear surjective isometry.

Proof. We first show that Q is well defined. Let $T \in BL(X, Y)$ and let $x_0, x_1 \in X$ such that $[x_0] = [x_1]$. Then $||x_0 - x_1||_X = 0$, so since T is continuous, $||Tx_0 - Tx_1||_Y = 0$. Hence $Tx_0 \cong Tx_1$, so $[Tx_0] = [Tx_1]$.

For linearity, let $\alpha \in \mathbb{F}$, and let $T, S \in BL(X, Y)$. Let $x \in X$. Then

$$\begin{aligned}
Q(\alpha T + S)\,[x]_X &= [(\alpha T + S)\,x]_Y \\
&= \alpha\,[Tx]_Y + [Sx]_Y \\
&= \alpha QT[x]_X + QS[x]_X \\
&= (\alpha QT + QS)\,[x]_X
\end{aligned}$$

Since $x \in X$ was arbitrary, Q is linear.

As for being an isometry, let $T \in BL(X,Y)$ and let $x \in X$. Since $||[x]|| = ||x||$ and $||Tx|| = ||[Tx]||$, we have

$$\frac{\left|\left|QT\,[x]_{X/\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}}\right|\right|_{Y/\mathcal{K}_{\mathrm{Y}}^{\mathrm{ernel}}}}{||[x]||_{X/\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}}} = \frac{||[Tx]||_{Y/\mathcal{K}_{\mathrm{Y}}^{\mathrm{ernel}}}}{||[X]||_{X/\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}}} = \frac{||Tx||_{Y}}{||x||_{X}}$$

and so taking the supremum over $||x|| \neq 0$ gives us that this is an isometry.

For surjectivity, let $\tilde{T} \in BL(X/\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}, \mathrm{Y}/\mathcal{K}_{\mathrm{Y}}^{\mathrm{ernel}})$. Let $\{x_\alpha\}_{\alpha\in A}$ be a Hamel basis for X. For each $\alpha \in A$, let $y_\alpha \in \tilde{T}[x_\alpha]_X$. Now define

$$T\sum_{i=1}^{n}\beta_i x_{\alpha_i} = \sum_{i=1}^{n}\beta_i y_{\alpha_i} \tag{2.92}$$

Then $T : X \to Y$ is obviously linear, and $Tx \in \tilde{T}[x]_X$ for $x \in X$. Hence,

$$`\frac{||Tx||_Y}{||x||_X} = \frac{\left|\left|\tilde{T}[x]_X\right|\right|_{Y/\mathcal{K}_{\mathrm{Y}}^{\mathrm{ernel}}}}{||[x]_X||_{X/\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}}} \tag{2.93}$$

so T is bounded, and hence $T \in BL(X,Y)$, but that also implies that by definition, $QT = \tilde{T}$, so we have proven surjectivity. □

Definition 2.7.41 (Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators)**.** *Let* X, Y *be* **Seminormed Spaces** with **Seminorm Kernels** $\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}$, $\mathcal{K}_{\mathrm{Y}}^{\mathrm{ernel}}$. Let $\mathcal{K}^{\mathrm{ernel}}$ denote the **Seminorm Kernel** of $BL(X,Y)$. Let Q denote the **Operator Quotient Map** of X and Y. Define $\Theta_{(X,Y)} : BL(X,Y)/\mathcal{K}^{\mathrm{ernel}} \to$ BL(X/$\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}$, Y/$\mathcal{K}_{\mathrm{Y}}^{\mathrm{ernel}}$) by setting, for each $T \in BL(X,Y)$.

$$\Theta_{(X,Y)}([T]) = QT \tag{2.94}$$

We call $\Theta_{(X,Y)}$ the **Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators** from X to Y. When X and Y are understood, we may denote the **Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators** simply with Θ. By 2.7.42, $\Theta_{(X,Y)}$ is an isomorphism of **Normed Spaces** . That is, Θ is Linear, Bijective, Bicontinuous, and an isometry.

Proposition 2.7.42 (Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators)**.** *Let* X, Y *be* **Seminormed Spaces** . Let Θ denote the **Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators** from X to Y. Then Θ is a bijective, bicontinuous, linear, isometry.

Proof. By 2.7.31, part 1, $Y/\mathcal{K}_{\mathrm{Y}}^{\mathrm{ernel}}$ is a **Normed Space** , Hence by 2.7.35, part 2, $BL(X/\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}, \mathrm{Y}/\mathcal{K}_{\mathrm{Y}}^{\mathrm{ernel}})$ is a **Normed Space** . Similarly, by 2.7.31, part 1, $BL(X,Y)/\mathcal{K}^{\mathrm{ernel}}$

is a normed space. Hence, it is sufficient to show that Θ is a well-defined surjective linear isometry.

For well definedness, let $T, S \in BL(X, Y)$ with $[T] = [S]$. Then, $||T - S|| = 0$, so if $x \in X$, $||Tx - Sx|| = 0$. Hence $Tx \cong Sx$ and since x was arbitrary, $QT = QS$.

Let q denote the **Quotient Map** $q : BL(X, Y) \to BL(X, Y)/\mathcal{K}^{\text{ernel}}$. By parts 4, 5, and 6 of 2.7.31, q is a linear surjective isometry. Also, by definition, $\Theta \circ q = Q$. Since Q is surjective, Θ is surjective. Since Q is an isometry, and q is a surjecive isometry, $Theta$ is an isometry. Since Q is linear, and since q is surjective and linear, Θ is linear. □

Definition 2.7.43 (Seminorm Topological Dual Space). *Let* $(X, ||\cdot||)$ *be a* **Seminormed Space** over a field $\mathbb{F}$. We denote with X^* **Normed Space** $BL(X, \mathbb{F})$, and we call X^* the **Topological Dual Space** of X. If $x^* \in X^*$, then we may denote, for $x \in X$, $x^*(x)$ with $\langle x, x^* \rangle$.

Since $\mathbb{F}$ is a **Normed Space** , by 2.7.35, part 02, X^* is as well.

Since X^* is a normed space

Also, $q : \mathbb{F} \to \mathbb{F}/\mathcal{K}^{\text{ernel}}_{\mathbb{F}}$ is a linear bijective isometry by 2.7.31, so if $Q : X^* \to BL(X/\mathcal{K}^{\text{ernel}}_{\text{X}}, \mathbb{F}/\mathcal{K}^{\text{ernel}}_{\mathbb{F}})$ is the **Operator Quotient Map** and if $\Theta : BL(X, \mathbb{F})/\mathcal{K}^{\text{ernel}} \to \text{BL}(\text{X}/\mathcal{K}^{\text{ernel}}_{\text{X}}, \mathbb{F}/\mathcal{K}^{\text{ernel}}_{\mathbb{F}})$ is the **Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators** , then we have

$$\Theta = Q \circ q^{-1} \tag{2.95}$$

Definition 2.7.44 (Dual Space). *Let* $(X, ||\cdot||)($ *be a* **Seminormed Space** . We call $BL(X, \mathbb{F})$ the **Topological Dual Space** of $(X, ||\cdot||)$, and we denote $BL(X, \mathbb{F})$ with the symbol X^*. If $x^* \in X^*$, then we use the notational convention of writing, for $x \in X$.

$$\langle x, x^* \rangle := x^*(x) \tag{2.96}$$

It would also be correct to refer to the **Topological Dual Space** of $(X, ||\cdot||)$ as the 1^{st} **Topological Dual Space** of X

Remark 2.7.45 (**Topological Dual Space** is a **Normed Space**). *Let* X *be a* **Seminormed Space** . Then, using 2.7.35, since $\mathbb{F}$ is a **Normed Space** , so is X^*.

Theorem 2.7.46 (**Topological Dual Space** Isomorphism). *Let* X *be a* **Seminormed Space** . Define $\Omega : X^* \to \left(X/\mathcal{K}^{\text{ernel}}_{\text{X}}\right)^*$ Spa by setting, for $x^* \in X$, and for $x \in X$,

$$\langle x, x^* \rangle = \langle [x], \Omega x^* \rangle \tag{2.97}$$

Then Ω is a Linear, Bijective, Isometric, Bicontinuous operator. That is, X^* and $(X/\mathcal{K}^{\text{ernel}}_{\text{X}})^*$ are isomorphic, and that isomorphism is explicitly given by Ω.

Proof. Consider the following

$$BL(X, \mathbb{F}) \xrightarrow{q} BL(X, \mathbb{F})/\mathcal{K}^{\text{ernel}}_{\text{BL(X/}\mathbb{F})} \xrightarrow{\Theta} \text{BL}(\text{X}/\mathcal{K}^{\text{ernel}}_{\text{X}}, \mathbb{F}/\mathcal{K}^{\text{ernel}}_{\mathbb{F}}) \xrightarrow{Q_{\mathbb{F}}^{-1}} \text{BL}(\text{X}/\mathcal{K}^{\text{ernel}}_{\text{X}}, \mathbb{F}) \tag{2.98}$$

where q is the **Quotient Map** , which is an linear bijective bicontinuous isometry in this case by parts 4, 5, 6, and 7 of of 2.7.31, Θ is the **Canonical Isomorphism Of**

The Quotient Space Of Continuous Linear Operators , which is a linear bijective bicontinuous isometry by 2.7.42 and $\mathcal{Q}_{\mathbb{F}}$ is the **Codomain Quotient Map** . which is in this case a linear, bijective, bicontinuous isometry by 2.7.38

Since $\Omega = \mathcal{Q}_{\mathbb{F}}^{-1} \circ \Theta \circ q$, and since each of the described properties are preserved under composition, Ω is also a linear bijective bicontinuous isometry. □

Remark 2.7.47 (**Topological Dual Space** is a **Normed Space**). *Let X be a* **Seminormed Space** . Since $X/\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}}$ is a **Normed Space** , so is $(X/\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}})^*$. By 2.7.46, we have a linear, bijective isometry between X^* and $(X/\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}})^*$. Hence X^* is a **Normed Space** .

2.7.3 Seminormed Hahn Banach Theorem

Theorem 2.7.48 (Hahn Banach Theorem For Seminormed Spaces)**.** *Let $(X, ||\cdot||)$ be a* **Seminormed Space** , let $x_i \in X$ for $i \in \{0, 1\}$ such that $||x_0 - x_1||_X \neq 0$, and let X^* denote $X's$ **Topological Dual Space** . The following are true.

(i) If $Z \subset X$ is a subspace and $z^* \in Z^*$, then there is an extension x^* of z^*, $x^* \in X^*$ such that

$$||z^*||_{Z^*} = ||x^*||_{X^*} \tag{2.99}$$

(ii) If $x \in X$, with $||x|| \neq 0$, then there exists an $x^* \in X$ with $||x^*|| = 1$ and $\langle x, x^* \rangle = ||x||_X$.

(iii) If $x \in X$, then

$$||x||_X = \sup_{0 \neq x^* \in X^*} \frac{\langle x, x^* \rangle}{||x^*||} \tag{2.100}$$

(iv) If Y is a **Non-Degenerate Seminormed Space** , and if $x_0 \in X$, with $||x_0|| \neq 0$, then there exists an $S \in BL(X, Y)$ with $||S|| = 1$ and

$$||Sx_0|| = ||x_0|| \tag{2.101}$$

Proof of 01. For $\alpha \in \{Z, X\}$, let $\Omega_\alpha : \alpha^* \to (\alpha/\mathcal{K}_{\alpha}^{\mathrm{ernel}})^*$ denote the isomorphism defined in 2.7.46. Let q denote the quotient operator $q : X \to X/\mathcal{K}^{\mathrm{ernel}}$. Define $T : Z/\mathcal{K}_{\mathrm{Z}}^{\mathrm{ernel}} \to$ q(Z) bv $T([z]_{\cong_Z}) = [z]_{\cong_X}$. Since Z is endowed with the subspace Topology, T is obviously a Linear Bijective Bicontinuous Isometry.

Define $\Gamma_Z : (Z/\mathcal{K}_{\mathrm{Z}}^{\mathrm{ernel}})^* \to$ q(Z)* by setting, for $\phi^* \in (Z/\mathcal{K}_{\mathrm{Z}}^{\mathrm{ernel}})^*$, for $[z]_Z \in Z/\mathcal{K}_{\mathrm{Z}}^{\mathrm{ernel}}$,

$$\langle T[z]_Z, \Gamma_Z \phi^* \rangle = \langle [z]_Z, \phi^* \rangle \tag{2.102}$$

Then Γ_Z is a Linear Bijective Isometry. Hence $\Gamma_Z \circ \Omega_Z z^* \in q(Z)^*$ with $||\Gamma_Z \circ \Omega_Z z^*||_{q(Z)^*} = ||z^*||_{Z^*}$.

Thus we can apply the Hahn Banach theorem for **Normed Spaces** to claim the existence of $x_q^* \in (X/\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}})^*$ where x_q^* is an extension of $\Gamma_Z \circ \Omega_Z z^*$ and

$$||x_q^*||_{(X/\mathcal{K}_{\mathrm{X}}^{\mathrm{ernel}})^*} = ||\Gamma_Z \circ \Omega_Z z^*||_{(q(Z))^*} = ||z^*||_{Z^*} \tag{2.103}$$

Finally, letting $x^* = \Omega_X^{-1} x_q^*$, we have $x^* \in X^*$, $||x^*||_{X^*} = \left|\left|x_q^*\right|\right|_{(X/\mathcal{K}_X^{\text{ernel}})^*} = ||z^*||_{Z^*}$, and if $z \in Z$, then

$$\begin{aligned}\langle z, x^* \rangle &= \left\langle [z]_X, x_q^* \right\rangle \\ &= \langle [z]_X, \Gamma_Z \circ \Omega_Z z^* \rangle \\ &= \langle [z]_Z, \Omega_Z z^* \rangle \\ &= \langle z, z^* \rangle\end{aligned}$$

□

Proof of 2. Let $Z = span(x)$. Define $z^* \in Z^*$ by $\langle \alpha x, z^* \rangle = \alpha \, ||x||$. Then $||z^*|| = 1$. Also, by part 1 of this result, it has an extension $x^* \in X^*$ with $||x^*|| = ||z^*|| = 1$ and $\langle x, x^* \rangle = ||x||$. □

Proof of 3. If $||x|| = 0$, then for every $x^* \in X$, $\langle x, x^* \rangle = 0$. Hence

$$||x||_X = \sup_{0 \neq x^* \in X^*} \frac{\langle x, x^* \rangle}{||x^*||} = \sup_{x^* \in \partial B_{X^*}(0;1)} \frac{\langle x, x^* \rangle}{||x^*||} = 0 \tag{2.104}$$

Otherwise, let $x^* \in X^*$ guaranteed to exist by part 2 which satisfies $||x^*|| = 1$, $\langle x, x^* \rangle = ||x||$. Then

$$\begin{aligned}||x|| &= \frac{\langle x, x^* \rangle}{||x^*||} \\ &\leq \sup_{x^* \in \partial B_{X^*}(0;1)} \frac{\langle x, x^* \rangle}{||x^*||} \\ &\leq \sup_{0 \neq x^* \in X^*} \frac{\langle x, x^* \rangle}{||x^*||}\end{aligned}$$

The other direction of the inequality falls directly from the definition of the norm on X^*, and is trivial, so we are done. □

Proof of 4. By part 2 of this result, there exists $x_0^* \in X^*$ with $||x_0^*|| = 1$ and $\langle x_0, x_0^* \rangle = ||x_0||$. Since Y is **Non-Degenerate** , there exists $y_0 \in Y$ with $||y_0|| = 1$. Define $T : \mathbb{F} \to Y$ by $T\alpha = \alpha y$. Then $||T|| = ||y|| = 1$. Define $S : X \to Y$ by $S = T \circ x_0^*$. Then $||S|| \leq ||T|| \, ||x_0^*|| = 1$, and $||Sx_0|| = ||\langle x_0, x_0^* \rangle \, y|| = \langle x_0, x_0^* \rangle = ||x_0||$. Hence $||S|| \geq 1$ and therefore $||S|| = 1$. □

2.7.4 Seminorm Adjoints

Proposition 2.7.49 (Linear Operator Notation)**.** . *When dealing with mappings of spaces of linear operators into spaces of other linear operators, or even functions in general, notation can get confusing, and presenting such things using ordinary notation without ambiguity can often require a plethora of parenthesis, which hamper readability of an arguement.*

For this reason, at points in this document, I sometimes express the image $\beta(\alpha)$ using

$$\langle \alpha, \beta \rangle$$

Where $\beta : X \to Y$ and $\alpha \in X$.

I combine this notation with usual function notation, particularly in cases similar to the following. For $i \in \{0, 1\}$, let X_i, Y_i, Z_i be sets. For $\alpha \in \{X, Y, Z\}$, let F_α be the set of maps $f : \alpha_0 \to \alpha_1$. If $T : F_X \to F_Y$, $y \in Y_0$, and $f \in F_X$, then I would notate

$$\langle y, Tf \rangle$$

rather than $Tf(y)$ or $(T(f)(y))$

Definition 2.7.50 (**Adjoint Operator**). *Let X, Y, and Z be* **Seminormed Spaces** over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. . Let $T \in BL(X, Y)$. We define the operator $T_Z^\times : BL(Y, Z) \to BL(X, Z)$ by setting, for $S \in BL(Y, Z)$ and $x \in X$,

$$\left\langle x, \mathcal{T}_Z^\times S \right\rangle = \langle Tx, S \rangle \tag{2.105}$$

or, equivalently,

$$\mathcal{T}_Z^\times S = S \circ T \tag{2.106}$$

We call $T_Z^\times$ the **Adjoint Operator** of T relative to the space Z, we denote $T_{\mathbb{F}}^\times = T^\times$, and we refer to $T^\times : Y^* \to X^*$ as simply the **Adjoint Operator** of T.

Proposition 2.7.51 (**Adjoint Operator**). *Let X, Y, and Z be* **Seminormed Spaces** over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. . Let $T \in BL(X, Y)$. Let $\mathcal{T} = T_Z^\times$ denote the **Adjoint Operator** of T relative to the space Z. Let Q_y denote the **Quotient Map**

The following are true.

1. $\mathcal{T}$ is Linear.

2. If $S \in BL(Y, Z)$, then $\mathcal{T}S \in BL(X, Z)$. (That is, the **Adjoint Operator** is well defined as a concept).

3. $\mathcal{T} \in BL\left(BL(Y, Z), BL(X, Z)\right)$.

4. $||\mathcal{T}|| = ||T||$

5. If T is surjective, then $\inf\limits_{||x||=1} ||Tx|| \leq \inf\limits_{||S||_{BL(Y,Z)}=1} ||\mathcal{T}S||$. Also TODO: Weaken T surjectivity condition To Range(T) dense in Y.

6. If Range(T) is not dense in Y, then $\inf\limits_{||S||_{BL(Y,Z)}=1} ||\mathcal{T}S|| = 0$

7. $\mathcal{T}$ is surjective ifff T is injective and has closed range in Y.

Proof of 01. Let $S, R \in BL(Y, Z)$, $\alpha \in \mathbb{F}$, and $x \in X$. Then,

$$\begin{aligned}
\langle x, \mathcal{T}(\alpha S + R) \rangle &= \langle Tx, \alpha S + R \rangle \\
&= \alpha \langle Tx, S \rangle + \langle Tx, R \rangle \\
&= \alpha \langle x, \mathcal{T}S \rangle + \langle x, \mathcal{T}R \rangle & &= \langle x, \alpha\mathcal{T}S \rangle + \langle x, \mathcal{T}R \rangle \\
&= \langle x, \alpha\mathcal{T}S + \mathcal{T}R \rangle
\end{aligned}$$

Since $x \in X$ was arbitrary, linearity is verified. □

Proof of 02. Let $S \in BL(Y,Z)$. Then, $\mathcal{T}S = S \circ T$. The composition of continuous operators is continuous, so $\mathcal{T}S$ is continuous. The composition of linear operators is linear, so $\mathcal{T}S$ is linear. This, paired with linearity, implies $\mathcal{T}S \in BL(X,Z)$. □

Proof of 3. Let $S \in BL(Y,Z)$. Then, if $x \in X$

$$||\langle x, \mathcal{T}S\rangle|| = ||\langle Tx, S\rangle|| \leq ||S||\,||Tx|| \leq ||S||\,||T||\,||x|| \tag{2.107}$$

Hence $||\mathcal{T}S|| \leq ||S||\,||T||$ Since T is linear, and since S was arbitrary, by part 12 of 2.7.35, $\mathcal{T} \in BL\left(BL(Y,Z), BL(X,Z)\right)$. □

Proof of 4. For any $S \in BL(Y,Z)$, $\mathcal{T}S = S \circ T$, so $||\mathcal{T}S|| \leq ||S||\,||T||$. Hence $||\mathcal{T}|| \leq ||T||$. Now let $x_0 \in X$. Then, by part 4 of 2.7.3, there exists $S \in BL(Y,Z)$ with $||S|| = 1$ and $||STx_0|| = ||Tx_0||$. Hence,

$$\begin{aligned}
||Tx_0|| &= ||STx_0|| \\
&= ||(S \circ T)x_0|| \\
&= ||(\mathcal{T}S)x_0|| \\
&\leq ||\mathcal{T}||\,||S||\,||x_0|| \\
&= ||\mathcal{T}||\,||x_0||
\end{aligned}$$

Since $x_0 \in X$ is arbitrary, $||T|| \leq ||\mathcal{T}||$. Since the inequality goes both ways, $||T|| = ||\mathcal{T}||$. □

Proof of 05. Let $\Gamma = \inf\limits_{||x||=1} ||Tx||$, and let $S \in BL(Y,Z)$ with $||S|| = 1$. Then,

$$\{x|\,||Tx|| \leq \Gamma\} \subset B_X(0;1)$$

so

$$\sup_{||x||\leq 1} |\langle Tx, S\rangle| \geq \sup_{||Tx||\leq \Gamma} |\langle Tx, S\rangle|$$

Also, since T is surjective by assumption,

$$\sup_{||Tx||\leq \Gamma} |\langle Tx, S\rangle| = \sup_{||y||\leq \Gamma} |\langle y, S\rangle|$$

From these two we arrive at the inequality

$$\begin{aligned}
||\mathcal{T}S|| &= \sup_{||x||\leq 1} |\langle x, \mathcal{T}S\rangle| \\
&= \sup_{||x||\leq 1} |\langle Tx, S\rangle| \\
&\geq \sup_{||Tx||\leq \Gamma} |\langle Tx, S\rangle| \\
&= \sup_{||y||\leq \Gamma} |\langle y, S\rangle| \\
&= \Gamma \\
&= \inf_{||x||=1} ||Tx||
\end{aligned}$$

Since $S \in \partial B_{BL(Y,Z)}(0;1)$ was arbitrary, we conclude $\inf\limits_{||S||=1} ||\mathcal{T}S|| \geq \inf\limits_{||x||=1} ||Tx||$ □

2.7.5 Higher order Seminorm Duals

Definition 2.7.52 (Higher Order Dual Spaces). *Let X be a* **Seminormed Space** . From 2.7.2 we know that the **Topological Dual Space** of X, X^*, is also called the 1^{st} **Topological Dual Space** of X. Building on this, for $n \in \{2,3,4,...,\}$ we call the 1^{st} **Topological Dual Space** of X^* the 2^{nd} **Topological Dual Space** of X, we call the 1^{st} **Topological Dual Space** of the 2^{nd} **Topological Dual Space** of X the 3^{rd} **Topological Dual Space** of X, and in general the 1^{st} **Topological Dual Space** of the $(n)^{th}$ **Topological Dual Space** of X the $(n+1)^{th}$ **Topological Dual Space** of X.

In general, we denote the $(n)^{th}$ **Topological Dual Space** of X with X^{n*}, though when n is small, we may denote $X^{**} = X^{2*}$, $X^{***} = X^{3*}$, et cetera.

Definition 2.7.53 (Higher Order Dual Space Isomorphism). *Let X be a* **Seminormed Space** over a field $\mathbb{F}$. Let $\Omega : X^* \to (X/\mathcal{K}^{\text{ernel}}_{\text{X}})^*$ be the Linear Bijective Isometry defined in 2.7.46. Define

$$\Omega_1 = \Omega$$

and also define, for $2 \leq n \in \mathbb{N}$, $\Omega_n : X^{n*} \to \left(X/\mathcal{K}^{\text{ernel}}_{\text{X}}\right)^{n*}$ by

$$\Omega_n = \left(\Omega^{\times}_{n-1}\right)^{-1}$$

By it is clear that the adjoint of a Linear Bijective isometry of normed spaces is also a Linear Bijective isometry of normed spaces, and so each Ω_n is as well.

Definition 2.7.54 (**Canonical Embedding** of X into X^{**}). *Let X be a* **Seminormed Space** . Define $c_X : X \to X^{**}$ by setting, for each $x^* \in X^*$, for each $x \in X$

$$\langle x^*, c(x)\rangle = \langle x, x^*\rangle \tag{2.108}$$

We call c_X the **Canonical Embedding** of X into X^*. As normal, if X is understood, we may denote $c_X = c$. If c is Surjective, then we say that X is **Reflexive** .

Proposition 2.7.55 (Canonical Embedding). *Let X be a* **Seminormed Space** and let c denote its **Canonical Embedding** . The following are true.

1. c is well defined
2. c is Linear.
3. c is an isometry.
4. c is an injection if and only if X is a **Normed Space** .
5. If $q : X \to X/\mathcal{K}^{\text{ernel}}$ is the **Quotient Map** , $c_{X/\mathcal{K}^{\text{ernel}}}$ is the **Canonical Embedding** of $(X/\mathcal{K}^{\text{ernel}})$ into $(X/\mathcal{K}^{\text{ernel}})^{**}$ and $\Omega_2 : X^{**} \to (X/\mathcal{K}^{\text{ernel}})^{**}$ is the linear bijective isometry defined in 2.7.5, then $c = \Omega_2^{-1} \circ c_{X/\mathcal{K}^{\text{ernel}}} \circ q$. //TODO: COME BACK TO THIS AND PROVE IT ONCE THE ISOS ARE CLEARED UP

6. c_X is surjective if and only if $c_{X/\mathcal{K}^{\text{ernel}}}$ is surjective.
7. X is **Reflexive** if and only if $X/\mathcal{K}^{\text{ernel}}$ is **Reflexive** .

Proof of 1. For any $x \in X$, $c(x)$ as a function is obviously well defined. Hence, I just need to show that, for any $x \in X$, $c(x) \in X^{**}$. That is , I must show that $c(x)$ is continuous and linear.

For linearity, if $x^*, y^* \in X^*$ and $\alpha \in \mathbb{F}$, we have

$$\begin{aligned} \langle \alpha x^* + y^*, c(x)\rangle &= \langle x, \alpha x^* + \alpha y^*\rangle \\ &= \alpha \langle x, x^*\rangle , \langle y, y^*\rangle \\ &= \alpha \langle x^*, c(x)\rangle + \langle y^*, c(x)\rangle \end{aligned}$$

Thus linearity holds.

For continuity, let $x \in X$ and let $x^* \in X^*$.

$$\begin{aligned} |\langle x^*, c(x)\rangle| &= |\langle x, x^*\rangle| \\ &\leq ||x||\,||x^*|| \end{aligned}$$

so that $c(x)$ is bounded with $||c(x)|| \leq ||x||$. □

Proof of 2. Let $\alpha \in \mathbb{F}$ and $x, y \in X$. Let $x^* \in X$. Then,

$$\begin{aligned} \langle x^*, c(\alpha x + y)\rangle &= \langle \alpha x + y, x^*\rangle \\ &= \alpha \langle x, x^*\rangle + \langle y, x^*\rangle \\ &= \alpha \langle x^*, c(x)\rangle + \langle x^*, c(y)\rangle \end{aligned}$$

, finishing the proof. □

Proof of 3. Let $x_0 \in X$ and $x^* \in X^*$. Then,

$$\begin{aligned} |\langle x^*, c(x_0)\rangle| &= |\langle x_0, x^*\rangle| \\ &\leq ||x_0||\,||x^*|| \end{aligned}$$

so that $||c(x_0)|| \leq ||(||\,x_0)$. For the other direciton, by 2.7.3 part 2, there exists an $x_0^* \in X^*$ satisfying $||x_0^*|| = 1$ and $\langle x_0, x_0^*\rangle = ||x||$ □

We see that $\langle x_0^*, c(x_0)\rangle = \langle x_0, x_0^*\rangle = ||x|| = ||x_0||\,||x_0^*||$ so that $||c(x_0)|| \geq ||x_0||$. Since the inequality goes both ways, $||x_0|| = ||c(x_0)||$, and c is therefore an isometry.

Proof of 4. Let X be a **Normed Space** . Then X^* separates points in X. Let $x \in X$ and $y \in X$ with $x \neq y$. Since X^* separates points in X, there exists $x^* \in X^*$ with $\langle x^*, c(x)\rangle = \langle x, x^*\rangle \neq \langle y, x^*\rangle , \langle x^*, c(y)\rangle$ so that $c(x) \neq c(y)$. Hence c is injective.

Now supose instead that c is injective and let $x, y \in X$ with $||x - y|| = 0$. We find that for any $x^* \in X^*$,

$$\begin{aligned} |\langle x^*, c(x) - c(y)\rangle| &= |\langle x^*, c(x - y)\rangle| \\ &= |\langle x - y, x^*\rangle| \\ &=\leq ||x^*||\,||x - y|| \\ &= 0 \end{aligned}$$

so that $||c(x) - c(y)|| = 0$. Since X^{**} is a normed space, this implies $c(x) = c(y)$, which through injectivity implies $x = y$. Hence we have the implication $||x - y|| = 0 \implies x = y$, so that X is a **NormedSpace**. □

Proof of 5. Proceeding directly from the definition, we have

$$\begin{aligned}
\left\langle x^*, \Omega_2^{-1} \circ c_{X/\mathcal{K}^{\text{ernel}}} \circ q(x) \right\rangle &= \left\langle x^*, \Omega^{\times} \circ c_{X/\mathcal{K}^{\text{ernel}}} \circ q(x) \right\rangle \\
&= \left\langle \Omega x^*, c_{X/\mathcal{K}^{\text{ernel}}} \circ q(x) \right\rangle \\
&= \langle q(x), \Omega x^* \rangle \\
&= \langle x, x^* \rangle \\
&= \langle x^*, c(x) \rangle
\end{aligned}$$

So we are done. □

Proof of 6. Since Ω_2 is a Bijection, by the prior part of this result, c is a surjection if and only if $c_{X/\mathcal{K}^{\text{ernel}}} \circ q$ is a surjection where $q : X \to X/\mathcal{K}^{\text{ernel}}$ is the **Quotient Map** . Since q is a surjection, c is a surjection if and only if$c_{X/\mathcal{K}^{\text{ernel}}}$ is a surjection. □

Proof of 7. This is a direct restatement of Part 06 of this result. □

Definition 2.7.56 (Weak Topologies Relating To Seminormed and Normed Spaces)**.** *latex*
$\mathfrak{weak}$
$\mathfrak{weak}^*$

Similar to in the context of a normed space, if X is a seminormed space, we define the weak topology on X to be the topology on X generated by X^*, and the *weak** topology on X^* to be the topology generated by $c(X)$. Before moving on to the classical theory revamped, I present on more useful result about weak topologies of seminormed spaces.

Proposition 2.7.57 (Weak Quotients)**.** *Let X be a seminormed space and $\{Y_\alpha\}_{\alpha \in A}$ be a collection of topological spaces. For each $\alpha \in A$ let $\phi_\alpha : X \to Y_\alpha$ have the property that for every $x, y \in X$, for every $\alpha \in A$, $||x - y|| = 0 \implies \phi_\alpha(x) = \phi_\alpha(y)$. For each $\alpha \in A$, define $\tilde{\phi}_\alpha : X/\left|\left|\cdot\right|\right|^{-1}\{0\} \to Y_\alpha$ by $\tilde{\phi}_\alpha[x] = \phi_\alpha x$. Let $\mathcal{T}_w$ denote the weak topology on X induced by $\{\phi_\alpha\}_{\alpha \in A}$, and $\mathcal{T}_{\tilde{w}}$ denote the weak topology on $X/\left|\left|\cdot\right|\right|^{-1}\{0\}$ induced by $\{\tilde{\phi}_\alpha\}_{\alpha \in A}$. Then*

$$(X, \mathcal{T}_w)/\left|\left|\cdot\right|\right|^{-1}\{0\} = (X/\left|\left|\cdot\right|\right|^{-1}\{0\}, \mathcal{T}_{\tilde{w}}) \tag{2.109}$$

Proof. □

Finally, before we move on, recall that if X, Y are Topological vector spaces, we can topologize the set of continuous linear operators from X to Y, denoted $BL(X, Y)$ by saying that $\{T_\alpha\}_{\alpha \in A} \subset BL(X, Y)$ converges to $T \in BL(X, Y)$ if there is a neighborhood U of 0 in X such that $T_\alpha x \to Tx$ uniformly for $x \in U$.

2.8 Classical Results With A Twist

By **??** and 2.7.57, many of the classical theorems relating a normed space and its duals still hold in the context of a seminormed space without too much alteration of the proofs. Since the author has not seen these results presented in this context, they are presented with proof below.

2.8.1 Helly

In this subsection, we develop Helly's theorem in the context of a seminormed space, which will serve as valuable lemma throughout this document. Its location here is due to the fact that is is a generalization of a lemma commonly used to prove the Goldstine Theorem.

Theorem 2.8.1 (Helly's Theorem)**.** *Let* $(X, ||\cdot||)$ *be a* **Seminormed Space** . Let $M > 0$. Let $\{\alpha_i\}_{i=1}^n \subset \mathbb{R}$, Let $\{x_i^*\}_{i=1}^n \subset X^*$. Then the following are equivalent.

(i) For each $\epsilon > 0$, there is an $x_\epsilon \in X$ such that $||x_\epsilon|| < M + \epsilon$ and $\langle x_\epsilon, x_i^* \rangle = \alpha_i$ for $1 \leq i \leq n$.

(ii) For every $\epsilon > 0$, there is an $x_\epsilon \in X$ such that $||x_\epsilon|| \leq M$ and $|\langle x_\epsilon, x_i^* \rangle - \alpha_i| < \epsilon$ for $1 \leq i \leq n$.

(iii) For each $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$,

$$\left| \sum_{i=1}^n \beta_i \alpha_i \right| \leq M \left|\left| \sum_{i=1}^n \beta_i x_i^* \right|\right| \tag{2.110}$$

2.8.1 i $\implies$ *2.8.1 ii*. Let $\epsilon > 0$. Let $K = \sup\limits_{i=1}^{n} ||x_i^*||$. Define $\epsilon_2 = \frac{\epsilon}{2K}$. Then by 2.8.1 i, there exists an $\tilde{x} \in X$ with $||\tilde{x}|| < M + \epsilon_2$ and $\langle \tilde{x}, x_i^* \rangle = \alpha_i$ for $1 \leq i \leq n$. Define $x_0 = \frac{M}{M+\epsilon_2}\tilde{x}$. Then $||x_0|| < M$. Also, $||\tilde{x} - x_0|| \leq \epsilon_2$. Furthermore, if $1 \leq i \leq n$, then we have

$$\begin{aligned} |\langle x_0, x_i^* \rangle - \alpha_i| &\leq |\langle x_0 - \tilde{x}, x_i^* \rangle| + |\langle \tilde{x}, x_i^* \rangle - \alpha_i| \\ &= |\langle x_0 - \tilde{x}, x_i^* \rangle| && \leq K\epsilon_2 \\ &\leq \frac{\epsilon}{2} < \epsilon \end{aligned}$$

and so we are done □

2.8.1 ii $\implies$ *2.8.1 iii*. Let $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$. Let $\epsilon > 0$. Then by 2.8.1 ii, there exists $x_0 \in X$

with $||x_0|| \leq M$ and $|\langle x_0, x_i^* \rangle - \alpha_i| < \epsilon$ for $1 \leq i \leq n$. Then

$$\begin{aligned}\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| &= \left|\sum_{i=1}^{n} \beta_i \left(\alpha_i - \langle x_0, x_i^* \rangle + \langle x_0, x_i^* \rangle\right)\right| \\ &\leq \left|\sum_{i=1}^{n} \beta_i \left(\alpha_i - \langle x_0, x_i^* \rangle\right)\right| + \left|\sum_{i=1}^{n} \beta_i \langle x_0, x_i^* \rangle\right| \\ &< \epsilon \sum_{i=1}^{n} |\beta_i| + \left|\left\langle x_0, \sum_{i=1}^{n} \beta_i x_i^* \right\rangle\right| \\ &\leq \epsilon \sum_{i=1}^{n} |\beta_i| + M \left|\left|\sum_{i=1}^{n} \beta_i x_i^*\right|\right|\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we are done. □

2.8.1 iii $\implies$ *2.8.1 i*. If any $x_j^* = 0$, then by selecting $\beta_i = \delta_{ij}$, by 2.8.1 iii, we see that $|\alpha_i| \leq M \, ||x_i^*|| = 0$. Hence, if all $x_i^* = 0$, then all $\alpha_i = 0$, so $x_\epsilon = 0$ works for any ϵ. Suppose at least some of the $\{x_i^*\}_{i=1}^n$ are nonzero. Then we can, by reordering, assume that for some integer m, $1 \leq m \leq n$, we have $\{x_i^*\}_{i=1}^m$ is **Linearly Independent** , and also

$$span\left(\{x_i^*\}_{i=1}^m\right) = span\left(\{x_i^*\}_{i=1}^n\right) \tag{2.111}$$

Let $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{C}^K$. Define $S : X \to \mathbb{C}^m$ by

$$S(x) = \left(\langle x, x_1^* \rangle, \langle x, x_2^* \rangle, \cdots, \langle x, x_m^* \rangle\right)$$

By 2.5.28, S is **Surjective** . Hence there exists $\tilde{x} \in S^{-1}\left(\alpha_1, \cdots, \alpha_m\right)$. Then, for $1 \leq i \leq m$, $\langle \tilde{x}, x_i^* \rangle = \alpha_i$. Define $K = \bigcap_{i=1}^{m} Kern(x_i^*)$. Then $S^{-1}\left(\alpha_1, \cdots, \alpha_m\right) = \tilde{X} + K$. Let $m < j \leq n$, $j \in \mathbb{Z}$. Then, by 2.111, There exists $\{\gamma_i\}_{i=1}^m \subset \mathbb{C}^m$ such that

$$x_j^* = \sum_{i=1}^{m} \gamma_i x_i^*$$

Now define $\{\beta_i\}_{i=1}^n$ by

$$\beta_i = \begin{cases} \gamma_i & 1 \leq i \leq m \\ 0 & m < i \wedge i \neq j \\ -1 & m < i \wedge i = j \end{cases} \tag{2.112}$$

Hence, by applying 2.8.1 iii, we have

$$
\begin{aligned}
\left|\left\langle \tilde{x}, x_j^* \right\rangle - \alpha_j\right| &= \left|\left\langle \tilde{x}, \sum_{i=1}^{m} \beta_i x_i^* \right\rangle - \alpha_j\right| \\
&= \left|\left(\sum_{i=1}^{n} \beta_i \left\langle \tilde{x}, x_i^* \right\rangle\right) + (-1)\,\alpha_j\right| \\
&= \left|\sum_{i=1}^{n} \beta_i \alpha_i\right| \\
&\leq M \left|\left|\sum_{i=1}^{n} \beta_i x_i^*\right|\right| \\
&= M \left|\left|\sum_{i=1}^{m} \gamma_i x_i^* + (-1) x_j^*\right|\right| \\
&= 0
\end{aligned}
$$

Hence $\left\langle \tilde{x}, x_j^* \right\rangle = \alpha_j$. Since $m < j$ was arbitrary, $\langle \tilde{x}, x_i^* \rangle = \alpha_i$ for $m < i \leq n$, $i \in \mathbb{Z}$.

$$\langle x_0, x_i^* \rangle = \alpha_i$$

for $1 \leq i \leq n$ and for any $x_0 \in \tilde{x} + K$. Define $T : x + K \to \mathbb{R}$ by $Tx = d(x, K)$. Then $T \in (x + K)^*$, so by 2.7.3 i there exists $x^* \in X^*$ satisfying $||x^*|| = 1$, $\langle \tilde{x}, x^* \rangle = d(\tilde{x}, K)$, and $Kernel(S) = K \subset ker(x^*)$. Since $\bigcap_{i=1}^{n} Kernel(x_i^*) = Kernel(S) \subset Kernl(x^*)$, by 2.5.29, $x^* \in span(x_1^*, \cdots, x_n^*)$. Hence we can find a representation $x^* = \sum_{i=1}^{n} \mu_i x_i^*$. (The representation is not unique) Hence,

$$
\begin{aligned}
d\left(\tilde{x}, K\right) &= \langle \tilde{x}, x^* \rangle \\
&= \sum_{i=1}^{n} \mu_i \left\langle \tilde{x}, x_i^* \right\rangle \\
&= \sum_{i=1}^{n} \mu_i \alpha_i \\
&\leq M \left|\left|\sum_{i=1}^{n} \mu_i x_i^*\right|\right| \\
&= M \left|\left|x^*\right|\right| \\
&= M
\end{aligned}
$$

Let $\epsilon > 0$. Then we can find $x_0 \in K$ such that $||\tilde{x} - x_0|| < M + \epsilon$, and by 2.8.1, $\langle \tilde{x} - x_0, x_i^* \rangle = \alpha_i$ for $1 \leq i \leq n$. $\square$

Corollary 2.8.2. *Let X be a* **Seminormed Space** . Let, $x^{**} \in X^{**}$. Let $\{x_i^*\}_{i=1}^n \subset X^*$. Let and $\epsilon > 0$. The following are true

(i) There exists an $x_1 \in X$ such that $||x_1|| \leq ||x^{**}|| + \epsilon$ and for $1 \leq i \leq n$, $\langle x_1, x_i^* \rangle = \langle x_i^*, x^{**} \rangle$.

(ii) There exists an $x_2 \in X$ such that $||x_2|| \leq ||x^{**}||$ and for $1 \leq i \leq n$, $|\langle x_2, x_i^* \rangle - \langle x_i^*, x^{**} \rangle| < \epsilon$.

Proof of Both. For $1 \leq i \leq n$, define $\alpha_i = \langle x_i^*, x^{**} \rangle$. Let $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$. Then,

$$\begin{aligned} \left| \sum_{i=1}^n \beta_i \alpha_i \right| &= \left| \sum_{i=1}^n \beta_i \left\langle x_i^*, x^* * \right\rangle \right| \\ &= \left| \left\langle \sum_{i=1}^n \beta_i x_i^*, x^{**} \right\rangle \right| \\ &\leq ||x^* *|| \left|\left| \sum_{i=1}^n \beta_i x_i^* \right|\right| \end{aligned}$$

Hence 2.8.1 *iii* is satisfied with $\alpha_i = \langle x_i^*, x^* * \rangle$ and $M = ||x^{**}||$. From an application of 2.8.1 *iii* $\implies$ 2.8.1 *ii*, we conclude the existence of $x_1 \in X$ with $\langle x_1, x_i^* \rangle = \alpha_i = \langle x_i^*, x^{**} \rangle$ and $||x_1|| \leq ||x^* *|| + \epsilon$, so 2.8.2 i holds. Also, from 2.8.1 *iii* $\implies$ 2.8.1 *i*, we conclude the existence of an $x_2 \in X$ satisfying

$$\begin{aligned} |\langle x_2, x_i^* \rangle - \langle x_i^*, x^{**} \rangle| &= |\langle x_2, x_i^* \rangle - \alpha_i| \\ &< \epsilon \end{aligned}$$

and $||x_2|| \leq ||x^{**}||$. Hence 2.8.2 ii holds. □

2.8.2 Goldstine

Theorem 2.8.3 (Goldstine)**.** *Let X be a* **Seminormed Space** . Let $c : X \to X^{**}$ be the **Canonical Embedding** of X into X^*. Let B denote the **Closed Unit Ball** about 0 in X. Let B_1 denote the **Closed Unit Ball** about 0 in X^{**}. Then $c(B)$ is **weak* Dense** in B_1.

Proof. Let $c^* : X^* \to X^{***}$ denote the **Canonical Embedding** of X^* into X^{***}. Let $\mathcal{K}$ denote the collection of **Finite** subsets of X^*. For each $x^{**} \in X^{**}$, $\epsilon > 0$ and $K \in \mathcal{K}$. define

$$\begin{aligned} U(x^{**}, \epsilon, K) &:= \bigcap_{x^* \in K} \{y \in X^{**} | \, |\langle x^{**} - y, c^*(x^*) \rangle| < \epsilon\} \\ &= \bigcap_{x^* \in K} \{y \in X^{**} | \, |\langle x^*, x^{**} - y \rangle| < \epsilon\} \end{aligned}$$

Define

$$\mathcal{B} = \{U(x^{**}, \epsilon, K) | K \in \mathcal{K} \wedge x^{**} \in X^{**} \wedge \epsilon \in (0, \infty)\}$$

Then by definition, the $\mathcal{B}$ is a **Basis** for the **weak* Topology** on X^{**}.

Let $x^{**} \in B_1$, $\epsilon > 0$, and $\{x_i\}_{i=1}^n = K \in \mathcal{K}$. Then by 2.8.2 ii there exists an $x \in B$ satisfying , for each $1 \leq i \leq n$,

$$\begin{aligned} |\langle x_i^*, x^{**} - c(x)\rangle| &= |\langle x_i^*, x^{**}\rangle - \langle x, x_i^*\rangle| \\ &< \epsilon \end{aligned}$$

Hence

$$c(x) \in \bigcap_{x_i^* \in K} \{y \in X^{**} | \, |\langle x_i^*, x^{**} - y\rangle| < \epsilon\} = U(x^{**}, \epsilon, K)$$

Hence $c(B)$ is **$\mathfrak{weak}^*$ Dense** in B_1. □

2.8.3 Banach Alaoglu

The following well known result concerning the $\mathfrak{weak}^*$compactness of the unit ball of a **Banach Space**was first proven in the separable case by Banach, and then generalized in 1940 by Alaoglu [1] to **Banach Spaces**. Generalizations of this result in a general TVS satisfying sufficient conditions have also been shown but the form presented here comes from [2], who drops the assumption of completeness for one direction of the implication.

Theorem 2.8.4 (Banach-Alaoglu-Morales)**.** Let X be a **Normed Space** . Let B denote the **Closed Unit Ball** about 0 in X^*. Then B is **$\mathfrak{weak}^*$ Compact** .

Proof. Let $\mathbb{F}$ denote the underlying **Field** of X. For each $x \in X$, define

$$D_x = \{y \in \mathbb{F} | \, |y| \leq ||x||\}$$

Then, for each $x \in X$, D_x is **Hausdorff** and **Compact** . □

Proof. Let $\mathbb{F}$ denote $X's$ field, and for $x \in X$, define $D_x = \{y \in \mathbb{F} : |y| \leq ||x||\}$. Then each D_x is Hausdorff and compact so by Tychonoff's theorem, $D := \prod_{x \in X} D_x$ is compact and Hausdorff when endowed with the product topology. If $T \in D$, then $T : X \to \mathbb{F}$ and $|Tx| \leq ||x||$ for each $x \in X$, so $D \cap X^* \subset B$. It is also clear that $B \subset D$, so $D \cap X^* = B$. Let $\{\gamma_\alpha\}_{\alpha \in A}$ be a net in B converging to $\gamma \in D$ in D's product topology. Then, letting π_x denote the x^{th} projection, for each $x \in X$,

$$\gamma_\alpha(x) = \pi_x(\gamma_\alpha) \to \pi_x(\gamma) = \gamma(x) \tag{2.113}$$

If $\alpha \in \mathbb{F}$ and $x, y \in X$, then

$$\langle \alpha x + y, \gamma_\alpha\rangle \to \langle \alpha x + y, \gamma\rangle \tag{2.114}$$

and also

$$\langle \alpha x + y, \gamma_\alpha\rangle = \alpha \langle x, \gamma_\alpha\rangle + \langle y, \gamma_\alpha\rangle \to \alpha \langle x, \gamma\rangle + \langle y, \gamma\rangle \tag{2.115}$$

which implies γ is linear since D is Hausdorff, and hence $\gamma \in B$. Thus B is closed in D. What remains to be shown is that the *weak∗* topology on B is the subspace topology on B induced by $D's$ topology, since a Closed subset of a compact Hausdorff space is compact. For notation, denote with $\mathcal{T}_D$ the subspace topology on B induced by D's topology, and denote with $\mathcal{T}_w$ the subspace topology on B induced by the $weak^*$ topology on X^*. To see that

$\mathcal{T}_w \subset \mathcal{T}_D$, let $\{\gamma_\alpha\}_{\alpha\in A} \subset B$ such that $\gamma_\alpha \overset{\mathcal{T}_D}{\to} \gamma$. For each $x \in X$, letting c be the canonical embedding,

$$\langle \gamma_\alpha, c(x)\rangle = \langle x, \gamma_\alpha\rangle = \pi_x(\gamma_\alpha) \to \pi_x(\gamma) = \langle x, \gamma\rangle = \langle \gamma, c(x)\rangle \tag{2.116}$$

Hence $\gamma_\alpha \overset{\mathcal{T}_w}{\to} \gamma$, so $\mathcal{T}_w \subset \mathcal{T}_D$. To see that $\mathcal{T}_D \subset \mathcal{T}_w$, fix $x \in X$ and let $\{\gamma_\alpha\}_{\alpha\in A} \subset B$ such that $\gamma_\alpha \overset{\mathcal{T}_w}{\to} \gamma$. Then $\pi_x(\gamma_\alpha) = \langle x, \gamma_\alpha\rangle \to \gamma(x) = \pi_x(\gamma)$, so by definition of the product topology $\gamma_\alpha \overset{\mathcal{T}_D}{\to} \gamma$, implying $\mathcal{T}_D \subset \mathcal{T}_w$. Hence B is *weak** is compact. □

Corollary 2.8.5 (Banach Alaoglu Seminorm). *Let X be a seminormed space and define $B = \{x^* \in X^* : ||x^*|| \leq 1\}$. Then B is weak* compact.*

Proof. This is a consequence of the fact that the *weak** topology on X^* is identical to the *weak** topology on the dual space of $X/\,||\cdot||^{-1}\{0\}$. □

This gives us the useable result

Corollary 2.8.6 (Banach-Alaoglu-Morales). *Let X be a seminormed space and $C \subset X^*$*

1. *If X is complete and C weak* compact, then C is weak* closed and bounded.*

2. *If C is weak* closed and bounded, then C is weak* compact.*

Proof. (1) Since C is *weak** compact, it is *weak** closed since the *weak** topology is Hausdorff. Since $c(x) : (X, \mathcal{T}_{w^*}) \to \mathbb{C}$ is continuous for each $x \in X$, for each $x \in X$, $c(x)(C) =$ is compact and therefore bounded. Hence, for every $x \in X$, $\{|\langle x, c\rangle| : c \in C\}$ is bounded, so by the Banach Steinhaus, C is bounded. □

Proof. (2) Since C is bounded, it is contained in some closed ball B which we know to be *weak** compact by 2.8.4. Since the *weak** topology on B is compact and Hausdorff and C is closed in this topology, it is compact in this topology. Since the subspace topology on C induced by the *weak** topology on X^* equals this topology, we are done. □

2.8.4 Eberlein-Smulian

The purpose of this section is to provide a characterization of weakly compact subsets of a complete seminormed space X, which will serve to increase the applicability of the results regarding weakly compactly generated spaces covered later in this document. The first main result of this section, 2.8.9, serves to show that even though weak topologies of Banach Spaces are not in general metrizable, an equivalence between weak compactness and sequential compactness exists. From this result ($1 \implies 2$) was first presented in the case of normed spaces in [3], and then ($2 \implies 1$) was proven in the case of normed spaces in [4]. Several different proofs have been given in the years since, and the one present here is based on that present in [5], which is also followed in [6]. We begin with a few lemmas.

Lemma 2.8.7 (Metrizable Weak). *If X is a seminormed space and X^* contains a countable set that separates points mod $K := ||\cdot||^{-1}\{0\}$, then subspace topology induced by the weak topology on any weakly compact subset A of X is pseudometrizable.*

Proof. As a consequence of 2.7.20 and 2.7.57, it is sufficient to let X be a normed space and $\{x_i^*\}_{i\in\mathbb{N}}$ separate points in X. Let $M = 2\sup_{x\in A}||x||$, and define d to be the metric on A defined by, for $x, y \in A$,

$$d(x,y) = \sum_{k\in\mathbb{N}} \frac{|\langle x-y, x_k^*\rangle|}{||x_k^*||\, 2^k} \tag{2.117}$$

Let $x \in A$, $\epsilon > 0$ be arbitrary, and define

$$n = \left\lceil 2 + log_2\left(\frac{M}{\epsilon}\right)\right\rceil \qquad U = A \cap \bigcap_{k=1}^{n} \left\{ y \in X : |\langle x-y, x_k^*\rangle| < \frac{||x_k^*||\, 2^{k-1}\epsilon}{n} \right\} \tag{2.118}$$

The U is open in the subspace topology on A induced by $X's$ weak topology. Furthermore, if $y \in U$, then

$$\begin{aligned} d(x,y) &= \sum_{k\in\mathbb{N}} \frac{|\langle x-y, x_k^*\rangle|}{||x_k^*||\, 2^k} \\ &\le \sum_{k=1}^{n} \frac{|\langle x-y, x_k^*\rangle|}{||x_k^*||\, 2^k} + \sum_{k=n+1}^{\infty} \frac{2M}{2^k} \\ &< \sum_{k=1}^{n} \frac{\epsilon}{2n} + \frac{M}{2^{n-1}} < \epsilon \end{aligned} \tag{2.119}$$

So that $U \subset B_d(x;\epsilon)$. This implies $Id : (A, \mathcal{T}_w) \to (A, \mathcal{T}_d)$ is continuous. Since a continuous injection from a compact space into a Hausdorff space is a homeomorphism, the subspace topology on A induced by the weak topology equals the topology on A induced by d, and so A's weak topology is metrizable. □

Lemma 2.8.8. *Let X be a seminormed space and $Y \subset X^{**}$ be a finite dimensional vector subspace. Then there exists a finite set $Z \subset \partial B_{X^*}(0;1)$ such that for each $y^{**} \in Y$,*

$$||y^{**}|| \le 2\max_{z^*\in Z} |\langle z^*, y^{**}\rangle| \tag{2.120}$$

Proof. Let $S = \partial B_{X^{**}}(0;1) \cap Y$. Then, since Y is finite dimensional, S is compact, and therefore permits a $\frac{1}{4} - net$ $\{s_i\}_{i=1}^n$. Now let $\{z_k^*\}_{k=1}^n \subset \partial B_{X^*}(0;1)$ such that for each k, $\langle z_k^*, s_i\rangle > \frac{3}{4}$. Let $s \in S$ then there is a k such that $||s - s_k|| < \frac{1}{4}$. for this k, we have

$$\langle z_k^*, s\rangle = \langle z_k^*, s_k\rangle + \langle z_k^*, s - s_k\rangle \ge \frac{3}{4} - \frac{1}{4} = \frac{1}{2} \tag{2.121}$$

□

Theorem 2.8.9 (Eberlein-Smulian)**.** *Let X be a seminormed space and $A \subset X$. Then the following are equivalent.*

1. *A is weakly compact.*

2. *A is weakly sequentially compact.*

Proof. ($1 \implies 2$) Let $A \subset X$ be weakly compact, and let $\{x_i\}_{i\in\mathbb{N}} \subset A$. Define $S = \overline{span\{x_i : i \in \mathbb{N}\}}$. Since S is closed and convex, it is weakly closed, and so $A \cap S$ is weakly compact as well. By construction, S is separable, and so contains a countable dense set $\{y_i\}_{i\in\mathbb{N}}$. By Hahn-Banach, for each $i \in \mathbb{N}$, there exists $y_i^* \in S^*$ such that $\langle y_i, y_i^*\rangle = 1$, and continuity of each y_i^* implies $\{y_i^*\}_{i\in\mathbb{N}}$ separates points in S mod $||\cdot||^{-1}\{0\}$. Hence we can apply 2.8.7 to claim that the subspace topology on $A \cap S$ induced by $S's$ weak topology is metrizable, and therefore $\{x_i\}_{i\in\mathbb{N}}$ has a sub-sequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ which is weakly S-convergent, and therefore weakly X-convergent since subspace topologies are no less fine than the topologies that induce them. Since $A \subset X$ is weakly closed, this sequence converges within A, and so A is weakly sequentially compact. □

Proof. ($2 \implies 1$). Let $A \subset X$ be weakly sequentially compact, let c denote the canonical embedding of X into X^{**}, and let x^{**} in the *weak** closure of $c(A)$. Let $x_1^1 \in X^*$ have norm 1. By assumption, there exists $a_1^{**} \in c(A)$ such that $|\langle x_1^*, x^{**} - a_1^{**}\rangle| < 1$. By 2.8.8, there exists $\{x_1^2, \cdots, x_{n_2}^2\} \subset \partial B_{X^*}(0;1)$ such that for each $y^{**} \in span\{x^{**}, x^{**} - a_1^{**}\}$,

$$||y^{**}|| \leq 2 \max_{1\leq k\leq n_2} \left|\left\langle x_k^2, y^{**}\right\rangle\right| \tag{2.122}$$

Also, since x^{**} is in the *weak** closure of $c(A)$, there exists $a_2^{**} \in c(A) \cap U_2$ where

$$U_2 = \{y^{**} \in X^{**} : (\forall 1 \leq j \leq 2)(\forall 1 \leq k \leq n_j)(\left|\left\langle x_k^j, x^{**} - y^{**}\right\rangle\right| < \frac{1}{2})\} \tag{2.123}$$

Continuing inductively, , we construct a sequence $\{a_n^{**}\}_{n\in\mathbb{N}} \subset c(A)$ such that for each $j \in \mathbb{N}$, $\{x_k^j\}_{k=1}^{n_j} \subset \partial B_{X^*}(0;1)$ such that for every $y^{**} \in span\left\{x^{**}, x^{**} - a_1^{**}, \cdots, x^{**} - a_{j-1}^{**}\right\}$, we have

$$||y^{**}|| \leq 2 \max_{1\leq k\leq n_j} \left|\left\langle x_k^j, x^{**} - y^{**}\right\rangle\right| \tag{2.124}$$

and $a_j^{**} \in c(A) \cap U_j$ where U_j is the $\{x_k^m\}_{1\leq m\leq j, 1\leq k\leq n_m}$ *weak** neighborhood about x^{**} of radius $\frac{1}{j}$. For each $k \in \mathbb{N}$, let $a_k = c^{-1}(c(a_k))$. Since A is sequentially weakly compact, $\{a_k\}_{k\in\mathbb{N}}$ has a weak cluster point $x \in A$. Also, $x \in \overline{span\{a_i\}_{i\in\mathbb{N}}}$ because this is a weakly closed set, implying $c(x) \in \overline{span\{a_i^{**}\}_{i\in\mathbb{N}}}$, which then implies $c(x) \in \overline{span\{x^{**}, x^{**} - a_1^{**}, x^{**} - a_2^{**}, \cdots\}}$. By continuity of the norm and each element of $\{x_i^k\}_{k\in\mathbb{N}, 1\leq i\leq n_k}$, we conclude that for each element y^{**} of $\overline{\{x^{**}, x^{**} - a_1^{**}, x^{**} - a_2^{**}, \cdots\}}$,

$$||y^{**}|| \leq 2 \sup_{k\in\mathbb{N}, 1\leq i\leq n_k} \left|\left\langle x_i^k, y^{**}\right\rangle\right| \tag{2.125}$$

This is useful, because for each $k \in \mathbb{N}$, $1 \leq i \leq n_k$, we have, for large enough j,

$$\begin{aligned} \left|\left\langle x_i^k, x^{**} - c(x)\right\rangle\right| &\leq \left|\left\langle x_i^k, x^{**} - a_j^{**}\right\rangle\right| + \left|\left\langle a_j^{**} - c(x), x_i^k\right\rangle\right| \\ &\leq \frac{1}{j} + \left|\left\langle x_i^k, a_j - x\right\rangle\right| \end{aligned} \tag{2.126}$$

which can be made arbitrarily small, and so $\left|\left\langle x_i^k, x^{**} - c(x)\right\rangle\right| = 0$, implying that

$$||x^{**} - c(x)|| \leq 2 \sup_{k\in\mathbb{N}, 1\leq i\leq n_k} \left|\left\langle x_i^k, x^{**} - x\right\rangle\right| = 0 \tag{2.127}$$

So $x^{**} = c(x)$, and therefore $c(A)$ is *weak** closed. Since A is weakly-sequentially compact, $c(A)$ is *weak** sequentially compact and therefore bounded by Banach Steinhaus. By 2.8.4, bounded *weak** closed sets are compact, and so $c(A)$ is *weak** compact. Since the weak topology on $A/ \|\cdot\|^{-1}\{0\}$ is the same as the *weak** topology on $c(A)$, $A/ \|\cdot\|^{-1}\{0\}$ is weakly compact. To see that A is weakly compact, apply **??**. □

2.8.5 Bishop-Phelps

In this subsection, I develop a result due to [7] which will prove useful throughout this document. I begin by presenting the concept of a convex cone and a trio of lemmas which are commonly utilized in the proof of this result.

Definition 2.8.10 (Convex Cone)**.** Let X be a seminormed space over $\mathbb{R}$. If $K \subset X$ is convex and closed under positive scalar multiples, then we call it a **convex cone**. If J is a convex cone in X, $C \subset X$, $x_0 \in C$, and $(J + x_0) \cap C = \{x_0\}$, then we say that J **supports** C at x_0. If $x^* \in \partial B_{X^*}(0; 1)$ and $\alpha > 0$ then we define

$$K(x^*, \alpha) := \{x \in X : \|x\| \leq \alpha \langle x, x^* \rangle\} \tag{2.128}$$

Remark 2.8.11. Let X be a seminormed space, $x^* \in \partial B_{X^*}(0; 1)$, and $\alpha > 0$. The following are true.

1. $K(x^*, \alpha)$ is a closed convex cone.
2. If $\alpha > 1$, $Int(K(x^*, \alpha)) \neq \emptyset$.

Proof. (1) If $\{x_n\} \subset K(x^*, \alpha)$ converges, say $x_n \to x$, then continuity of x^* implies $\langle x_n, x^* \rangle \to \langle x, x^* \rangle$. Hence, given $\epsilon > 0$, there exists $N > 0$ such that for $n > N$ we have $max\left(\left|\|x\| - \|x_n\|\right|, |\langle x - x_n, x^*\right.$ ϵ, so that for all $n > N$,

$$\|x\| \leq \|x_n\| + \epsilon < \alpha \langle x_n, x^* \rangle + \epsilon < \alpha \langle x, x^* \rangle + (\alpha + 1)\epsilon \tag{2.129}$$

So $x \in K(x^*, \alpha)$ closedness is verified. It is obvious that $K(x^*, \alpha)$ is closed under positive scalar multiples, and for convexity, if $x, y \in K(x^*, \alpha)$ and $t \in [0, 1]$, then

$$\|tx + (1-t)y\| \leq t\|x\| + (1-t)\|y\| \leq t\alpha \langle x, x^* \rangle + (1-t)\alpha \langle y, x^* \rangle = \alpha \langle tx + (1-t)y, x^* \rangle \tag{2.130}$$

□

Proof. (2) By definition of the norm on X^*, there is an $x \in \overline{B_X(0;1)}$ such that $2/\left(\alpha\left(1 + \frac{1}{\alpha}\right)\right) < \langle x, x^* \rangle$, implying by linearity that $1/\alpha < \left\langle \frac{1+(1/\alpha)}{2} x, x^* \right\rangle$. By continuity of x^* we find a neighborhood U of $\left(1 + \frac{1}{\alpha}\right)/2$ contained in $B_X(0; 1)$ such that for each $y \in U$, $1/\alpha < \langle y, x^* \rangle$. This implies $\|y\| \leq 1 < \alpha \langle y, x^* \rangle$, so $U \subset K(x^*, \alpha)$. □

Lemma 2.8.12 (Bishop-Phelps Lemma)**.** *Let X be a complete seminormed space, $x^*, y^* \in \partial B_{X^*}(0; 1)$, $C \subset X$ closed and convex, $1 > \epsilon > 0$, and $k > 1 + \frac{2}{\epsilon}$. The following are true.*

1. *If x^* is bounded on C, then for each $z \in C$, there is an $x_0 \in X$ such that $K(x^*, \epsilon)$ supports C at x_0 and $x_0 \in K(x^*, \epsilon) + z$.*

2. *If* $|\langle x, y^* \rangle| \leq \frac{\epsilon}{2}$ *for each* $x \in Kern(x^*) \cap \overline{B_X(0;1)}$*, then*

$$min\left(||x^* + y^*||\,, ||x^* - y^*||\right) \leq \epsilon \tag{2.131}$$

3. *If* y^* *is nonnegative on* $K(x^*, k)$*, then* $||x^* - y^*|| \leq \epsilon$*.*

Proof. (1) Let x^* be bounded on C and define, for $x, y \in X$, $y \lesssim x$ if and only if $x - y \in K(x^*, \epsilon)$. Fix $z \in C$. Define $Z = C \cap (K(x^*, \epsilon) + z)$. Since C and $K(x^*, \epsilon)$ are closed, so is Z. Let $\mathcal{C} = \{x_\alpha\}_{\alpha \in A}$ be a chain in where $(A, \leq)$ is a totally ordered set and $x_\alpha \lesssim x_\beta \iff \alpha \leq \beta$. If $x_\alpha, x_\beta \in \mathcal{C}$, where $x_\beta \lesssim x_\alpha$, then $x_\alpha - x_\beta \in K(x^*, \epsilon)$, so $0 \leq ||x_\alpha - x_\beta|| \leq \epsilon \langle x_\alpha - x_\beta, x^* \rangle$, implying $\langle x_\beta, x^* \rangle \leq \langle x_\alpha, x^* \rangle$. Thus we conclude $\{\langle x_\alpha, x^* \rangle\}_{\alpha \in A}$ is a monotone bounded net in $\mathbb{R}$ that is therefore Cauchy, which by the following inequality

$$||x_\beta - x_\alpha|| \leq \epsilon \langle x_\alpha - x_\beta, x^* \rangle = \epsilon \left(\langle x_\alpha, x^* \rangle - \langle x_\beta, x^* \rangle\right) \to 0 \tag{2.132}$$

implies $\mathcal{C}$ is a Cauchy net and therefore converges, say $x_\alpha \to y_0 \in Z$. Continuity of the norm and x^* imply together that y_0 is an upper bound for $\mathcal{C}$. Since $\mathcal{C}$ was an arbitrary chain in Z, Z has a maximal element x_0. By definition, $x_0 \in Z := K(x^*, \epsilon) + z$. Since $x_0 \in Z \subset C$, $x_0 \in C$. Further, since $0 \in K(x^*, \epsilon)$, $x_0 \in K(x^*, \epsilon) \cap C$. Let $y \in (K(x^*, \epsilon) + x_0) \cap C$. Then $y - x_0 \in K(x^*, \epsilon)$ so that $z \lesssim x_0 \lesssim y$, meaning $y \in Z$ and therefore $y = x_0$ since x_0 is maximal. Hence $(K(x^*, \epsilon) + x_0) \cap C = \{x_0\}$, so we are done. □

Proof. (2) By assumption, $\left|\left|y^*|_{Kern(x^*)}\right|\right| \leq \frac{\epsilon}{2}$, so by the Hahn-Banach theorem, we can find a $\tilde{y^*} \in X^*$ extending $y^*|_{Kern(x^*)}$ such that $\left|\left|\tilde{y^*}\right|\right| \leq \frac{\epsilon}{2}$. Since $y^* - \tilde{y^*} \neq 0$, $codim\left(kern(x^*)\right) = 1$, and $kern\left(x^*\right) \subset kern\left(y^* - \tilde{y^*}\right)$,we conclude $kern\left(y^* - \tilde{y^*}\right) = kern(x^*)$. Hence, for some $\alpha \in \mathbb{R}$, $y^* - \tilde{y^*} = \alpha x^*$. For this alpha, we have

$$|1 - |\alpha|| = \left|||y^*|| - \left|\left|\tilde{y^*} - y^*\right|\right|\right| \leq \left|\left|\tilde{y^*}\right|\right| \leq \frac{\epsilon}{2} \tag{2.133}$$

If $\alpha \geq 0$,

$$||x^* - y^*|| = \left|\left|x^* - \left(\alpha x^* + \tilde{y^*}\right)\right|\right| = \left|\left|\left(1 - \alpha\right) x^* - \tilde{y^*}\right|\right| \leq |1 - \alpha| + \left|\left|\tilde{y^*}\right|\right| \leq \epsilon \tag{2.134}$$

If $\alpha \leq 0$, then

$$||x^* + y^*|| = \left|\left|x^* + \left(\alpha x^* + \tilde{y^*}\right)\right|\right| = \left|\left|(1 + \alpha)x^* + \tilde{y^*}\right|\right| \leq |1 + \alpha| + \left|\left|\tilde{y^*}\right|\right| \leq \epsilon \tag{2.135}$$

□

Proof. (3) Since $||x^*|| = 1$, there exists $x \in \partial B_X(0;1)$ such that $\langle x, x^* \rangle > \frac{1}{k}\left(1 + \frac{2}{\epsilon}\right)$. If $y \in Kern(x^*) \cap \overline{B_X(0;1)}$, then

$$\left|\left|x \pm \frac{2}{\epsilon} y\right|\right| \leq 1 + \frac{2}{\epsilon} < k \langle x, x^* \rangle = k \left\langle x \pm \frac{2}{\epsilon} y, x^* \right\rangle \tag{2.136}$$

so $x \pm \frac{2}{\epsilon} y \in K(x^*, k)$, so by assumption $\left\langle x \pm \frac{2}{\epsilon} y, y^* \right\rangle \geq 0$. Since this occurs for both positive and negative, $|\langle y, y^* \rangle| = \frac{\epsilon}{2} \left|\left\langle \frac{2}{\epsilon} y, y^* \right\rangle\right| \leq \frac{\epsilon}{2} \langle y^*, x \rangle \leq \frac{\epsilon}{2} ||x|| = \frac{\epsilon}{2}$. Hence by part 2, either $||x^* - y^*|| \leq \epsilon$, or $||x^* + y^*|| \leq \epsilon$. Since $||x^*|| = 1$, there exists $x \in \partial B_X(0;1)$ such that $\frac{||x||}{k} \leq max\left(\epsilon, \frac{1}{k}\right) < \langle x, x^* \rangle$, so that $x \in K(x^*, k)$, implying $\langle x, y^* \rangle \geq 0$, and therefore $\epsilon < \langle x_0, x^* + y^* \rangle \leq ||x^* + y^*||$. Hence we conclude $||x^* - y^*|| \leq \epsilon$.

□

Theorem 2.8.13 (Bishop-Phelps Theorem)**.** *Let X be a complete seminormed space, $C \subset X$ be closed, bounded, and convex, and define $M := \{f \in X^* | (\exists x_0 \in C)(\langle x_0, f\rangle = \sup_{x \in C} \langle x, f\rangle)\}$. Then $\overline{M} = X^*$*

Proof. Since M is a vector subspace independent of translations of C, we assume without loss of generality that $0 \in C$ and that it is sufficient to show that M is dense in $\partial B_{X^*}(0;1)$. Let $x^* \in \partial B_{X^*}(0;1)$. Let $\epsilon \in (0,1)$ and let $1 + \frac{2}{\epsilon} < k$. by 2.8.11, $K(x^*, k)$ is a closed convex cone with nonempty interior. Applying 2.8.12, part one, there is $x_0 \in C$ with $x_0 \in K(x^*, k)$ and $(K(x^*, k) + x_0) \cap C = \{x_0\}$. By Hahn Banach, there exists $y^* \in \partial B_{X^*}(0;1)$ satisfying

$$\sup_{x \in C} \langle x, y^* \rangle = \langle x_0, y^* \rangle = \inf_{x \in K(x^*,k)+x_0} \langle x, y^* \rangle = \inf_{\tilde{x} \in K(x^*,k)} \langle \tilde{x}, y^* \rangle + \langle x_0, y^* \rangle \tag{2.137}$$

Hence y^* is positive on $K(x^*, k)$, so by 2.8.12 part 3, $||x^* - y^*|| < \epsilon$, so we are done since $y^* \in M$ and x^* was arbitrary. □

2.9 Reflexivity

Recall that a seminormed space X is said to be **reflexive** if $c(X) = X^{**}$. Since X^{**} is always complete and c an isometry, any reflexive space is always complete. Due to the Banach-Alaoglu theorem, in reflexive space X, the closed unit ball of X is weakly compact. For this reason and others, reflexivity is a condition of interest to many mathematicians. We begin with a basic result.

Lemma 2.9.1 (Reflexive Separable)**.** *Let X be a complete seminormed space. Then the following are equivalent.*

1. *X is reflexive.*
2. *The closed unit ball of X is weakly compact.*
3. *Each closed separable subspace of X is reflexive.*
4. *All collections of closed, bounded, convex sets in X have the finite intersection property*
5. *The closed unit ball of X is weakly sequentially compact*

Proof. ($1 \implies 2$) Let X be reflexive. By 2.8.9 it is sufficient to show that any sequence $\{x_i\}_{i \in \mathbb{N}} \subset \overline{B_X(0;1)}$ has a weak cluster point $x \in \overline{B_X(0;1)}$. Since $\overline{B_{X^{**}}(0;1)}$ is *weak** compact, $\{c(x_i)\}_{i \in \mathbb{N}}$ has a subsequence $\{c(x_{n_i})\}_{i \in \mathbb{N}}$ such that $c(x_{n_i}) \xrightarrow{w^*} \tilde{x} \in \overline{B_{X^{**}}(0;1)}$. Since X is reflexive, for some $x \in \overline{B_X(0;1)}$, $c(x) = \tilde{x}$. Let $x^* \in X^*$. Then,

$$|\langle x_{n_i} - x, x^* \rangle| = |\langle x^*, c(x_{n_i}) - \tilde{x} \rangle| \to 0 \tag{2.138}$$

as $i \to \infty$, and so $x_{n_i} \xrightarrow{w} x$, completing the proof. □

Proof. (2 $\implies$ 3) Suppose the closed unit ball of X is weakly compact, and let $x^{**} \in \overline{B_{X^{**}}(0;1)}$. By **??**, there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ such that $c(x_n) \stackrel{w^*}{\to} x^{**}$. By assumption, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $x_{n_k} \stackrel{w}{\to} x \in \overline{B_X(0;1)}$. This implies $c(x_{n_k}) \stackrel{w^*}{\to} c(x)$, and so since the *weak** topology is Hausdorff, $x^{**} = c(x) \in c\left(\overline{B_X(0;1)}\right)$. $\square$

Proof. (2 $\implies$ 3). Let X be reflexive and $S \subset X$ be a closed separable vector subspace. Let $\{x_i\}_{i\in\mathbb{N}} \subset \overline{B_X(0;1)}$. By assumption this sequence has an $X-$*weakly* convergent subsequence, $x_{n_k} \to x \in X$. Since S is weakly closed, $x \in S$, and since the weak topology on X is finer than that on S, $x_{n_k} \stackrel{S-w}{\to} x$. By 2.8.9 and (2 $\implies$ 1, S is reflexive. $\square$

Proof. (3 $\implies$ 2) Let $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$. Then since $S := \overline{span\{x_i\}_{i\in\mathbb{N}}}$ is a closed separable subspace, $\{x_n\}_{n\in\mathbb{N}}$ has an S-weakly convergent subsequence, $x_{n_k} \stackrel{S-w}{\to} x \in S$. If $x^* \in X^*$, then $x^*|_S \in S^*$, so that $|\langle x - x_{n_k}, x^*\rangle| \to 0$, implying $x_{n_k} \stackrel{w}{\to} x$, an application of 2.8.9 finishes the proof. $\square$

Proof. (4 $\iff$ 2) Obvious since closed convex sets are weakly closed. $\square$

Proof. (5 $\iff$ 2) Apply 2.8.9. $\square$

2.9.1 James

As an easy application of 2.8.4 and 2.8.9, for a reflexive space X, all $x^* \in X^*$ attain their norm. The converse of this fact was, for a time, an open question of considerable interest. The converse of this result was, as is traditional in mathematics, proven in a piecemeal manner. The result was first tackled by James in [8] under the added assumption that every space Y isomorphic to X has the property that each $y^* \in Y^*$ attains its norm and that X permits a Shauder basis. This result was rapidly improved by Klee in [9] who dropped the assumption of a Schauder basis, and then by James again in [10] who proved the result in the case of a separable space X. The question of the converse in a Banach space was finally answered to the affirmative in [11], building on the arguments in [10]. This paper included generalizations all the way to quasi-complete locally convex TVS's.

Definition 2.9.2. *If X is a topological vector space and $\{x_i^*\}_{i\in\mathbb{N}} \subset X^*$, then $CoLim\{x_i^*\}_{i\in\mathbb{N}}$ is the set of all $x^* \in X^*$ such that for every $x \in X$,*

$$\liminf_{i\to\infty} \langle x, x_i^*\rangle \leq \langle x, x^*\rangle \leq \limsup_{i\to\infty} \langle x, x_i^*\rangle \tag{2.139}$$

Remark 2.9.3 (CoLim Nonempty)**.** *Let X be a complete seminormed space and let $\{x_i^*\}_{i\in\mathbb{N}} \subset X^*$ be bounded. Then $CoLim\{x_i^*\}_{i\in\mathbb{N}} \neq \emptyset$*

Proof. Since $\{x_i^*\}_{i\in\mathbb{N}}$ is bounded, it has a subsequence with a *weak** limit x^* who must live in $CoLim\{x_i^*\}_{i\in\mathbb{N}}$. $\square$

Lemma 2.9.4. *Let X be a complete seminormed space, $\alpha \in (0,1)$, $\{x_i^*\}_{i\in\mathbb{N}} \subset X^*$, and $\{\beta_i\}_{i\in\mathbb{N}} \subset (0,1)$ such that $\sum_{i\in\mathbb{N}} \beta_i = 1$. Then if (1) or (3) hold below, there are $\{y_i^*\}_{i\in\mathbb{N}} \subset X^*$ and $\gamma \geq \alpha$ such that (2) or (4) hold respectively.*

1. $\{x_i^*\}_{i\in\mathbb{N}} \subset \partial B_{X^*}(0;1)$ *such that* $d\left(0, \overline{co}\{x_i^*\}_{i\in\mathbb{N}}\right) \geq \alpha$.
2. $\gamma \leq 1$, *and for each* $i \in \mathbb{N}$,

$$y_i^* \in \overline{co}\{x_j^*\}_{j\geq i} \qquad \left\|\sum_{j\in\mathbb{N}} \beta_j y_j^*\right\| = \gamma \qquad \left\|\sum_{j=1}^{i} \beta_j y_j^*\right\| < \gamma\left(1-\alpha\left(\sum_{j=i+1}^{\infty}\beta_j\right)\right) \tag{2.140}$$

3. $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ *such that* $d\left(\overline{co}\{x_i^*\}_{i\in\mathbb{N}} - CoLim\{x_i^*\}_{i\in\mathbb{N}}, 0\right) \geq \alpha$.
4. $\gamma \leq 2$, $\{y_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$, *and for each* $i \in \mathbb{N}$, $y^* \in CoLim\{y_i^*\}_{i\in\mathbb{N}}$,

$$\left\|\sum_{j\in\mathbb{N}} \beta_j\left(y_j^* - y^*\right)\right\| = \gamma \qquad \left\|\sum_{j=1}^{i} \beta_j\left(y_j^* - y^*\right)\right\| < \gamma\left(1-\alpha\left(\sum_{j=i+1}^{\infty}\beta_j\right)\right) \tag{2.141}$$

Proof. ($1 \implies 2$) There exists a positive sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that

$$\sum_{i\in\mathbb{N}} \frac{\beta_i\delta_i}{\left(\sum\limits_{j=i+1}^{\infty}\beta_i\right)\left(\sum\limits_{j=i}^{\infty}\beta_i\right)} < 1-\alpha \tag{2.142}$$

Let $\gamma_1 \in \mathbb{R}$ and choose $y_1^* \in \overline{co}\{x_i^*\}_{i\in\mathbb{N}}$ such that $\gamma_1 = d\left(0, \overline{co}\{x_i^*\}_{i\in\mathbb{N}}\right)$ and $\|y_1^*\| \leq \gamma_1\left(1+\delta_1\right)$. From here, for each $n \geq 1$, define $\gamma_{n+1} \in \mathbb{R}$ and choose $y_{n+1}^* \in \overline{co}\{x_i^*\}_{i\geq n+1}$ such that

$$\gamma_{n+1} = \inf\left\{\left\|\left(\sum_{i=1}^{n}\beta_i y_i^*\right) + \left(1-\sum_{i=1}^{n}\beta_i\right)y^*\right\| : y^* \in \overline{co}\{x_i^*\}_{i\geq n+1}\right\} \tag{2.143}$$

and

$$\left\|\sum_{i=1}^{n}\beta_i y_i^* + \left(1-\sum_{i=1}^{n}\beta_i\right)y_{n+1}^*\right\| < \gamma_{n+1}\left(1+\delta_{n+1}\right) \tag{2.144}$$

It is clear that since the set over which we are taking the infimum never gets larger, so $\{\gamma_i\}_{i\in\mathbb{N}}$ is a nondecreasing sequence. Further, since $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ and $\sum_{i\in\mathbb{N}}\beta_i = 1$, we have, for every i,

$$\alpha \leq \gamma_i \nearrow \gamma = \left\|\sum_{k\in\mathbb{N}}\beta_k y_k^*\right\| \leq 1 \tag{2.145}$$

So what is left to be shown is that for every $i \in \mathbb{N}$,

$$\left\|\sum_{j=1}^{i}\beta_j y_j^*\right\| < \gamma\left(1-\alpha\left(\sum_{j=i+1}^{\infty}\beta_j\right)\right) \tag{2.146}$$

Let $i \in \mathbb{N}$. Then,

$$\begin{aligned}
\left\|\sum_{j=1}^{i} \beta_j (y_j^* - y^*)\right\| &= \left\|\left(\left(\frac{\sum_{j=i}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j}\right)\left(\sum_{j=1}^{i-1} \beta_j y_j^*\right)\right) + \left(\left(\frac{\sum_{j=i}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j}\right)(\beta_i y_i^*)\right)\right\| \\
&= \left\|\left(\left(\frac{\lambda_i + \sum_{j=i+1}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j}\right)\left(\sum_{j=1}^{i-1} \beta_j y_j^*\right)\right) + \left(\left(\frac{\beta_i \sum_{j=i}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j}\right)(y_i^*)\right)\right\| \\
&\leq \frac{\beta_i}{\sum_{j=i}^{\infty} \beta_j} \left\|\sum_{j=1}^{i=1} \beta_j y_j^* + \left(\sum_{j=i}^{\infty} \beta_j\right) y_i^*\right\| + \frac{\sum_{j=i+1}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j} \left\|\sum_{j=1}^{i-1} \beta_j y_j^*\right\| \\
&\leq \frac{\beta_i}{\sum_{j=i}^{\infty} \beta_j} \left\|\sum_{j=1}^{i=1} \beta_j y_j^* + \left(1 - \sum_{j=1}^{i-1} \beta_j\right) y_i^*\right\| + \frac{\sum_{j=i+1}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j} \left\|\sum_{j=1}^{i-1} \beta_j y_j^*\right\| \\
&< \frac{\beta_i}{\sum_{j=i}^{\infty} \beta_j} (\gamma_i)(1 + \delta_i) + \frac{\sum_{j=i+1}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j} \left\|\sum_{j=1}^{i-1} \beta_j y_j^*\right\| \\
&= \left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\left(\frac{\beta_i \gamma_i (1 + \delta_i)}{\left(\sum_{j=i}^{\infty} \beta_j\right)\left(\sum_{j=i+1}^{\infty} \beta_j\right)}\right) + \left(\frac{1}{\sum_{j=i}^{\infty} \beta_j}\right)\left\|\sum_{j=1}^{i-1} \beta_j y_j^*\right\|\right)
\end{aligned} \tag{2.147}$$

Hence, for any $i \in \mathbb{N}$,

$$\begin{aligned}
\left|\left|\sum_{j=1}^{i} \beta_j y_j^*\right|\right| &< \left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\left(\frac{\beta_i \gamma_i (1+\delta_i)}{\left(\sum_{j=i}^{\infty} \beta_j\right)\left(\sum_{j=i+1}^{\infty} \beta_j\right)}\right) + \left(\frac{1}{\sum_{j=i}^{\infty} \beta_j}\right)\left|\left|\sum_{j=1}^{i-1} \beta_j y_j^*\right|\right|\right) \\
&< \left(\sum_{j=i+1}^{\infty} \beta_j\right)\sum_{j=1}^{i} \frac{\beta_j \gamma_j (1+\delta_j)}{\left(\sum_{k=j}^{\infty} \beta_k\right)\left(\sum_{k=j+1}^{\infty} \beta_k\right)} \\
&\leq \gamma \left(\sum_{j=i+1}^{\infty} \beta_j\right)\sum_{j=1}^{i} \frac{\beta_j (1+\delta_j)}{\left(\sum_{k=j}^{\infty} \beta_k\right)\left(\sum_{k=j+1}^{\infty} \beta_k\right)} \\
&\leq \gamma \left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\left(\sum_{j=1}^{i} \frac{\beta_j}{\left(\sum_{k=j}^{\infty} \beta_k\right)\left(\sum_{k=j+1}^{\infty} \beta_k\right)}\right) + (1-\alpha)\right) \\
&= \gamma \left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\left(\sum_{j=1}^{i}\left(\frac{1}{\sum_{k=j}^{\infty} \beta_k} - \frac{1}{\sum_{k=j+1}^{\infty} \beta_k}\right)\right) + (1-\alpha)\right) \\
&= \gamma \left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\frac{1}{\sum_{j=i+1}^{\infty} \beta_j} - \frac{1}{\sum_{j=1}^{\infty} \beta_j} + 1 - \alpha\right) \\
&= \gamma \left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\frac{1}{\sum_{j=i+1}^{\infty} \beta_j} - \alpha\right) \\
&= \gamma \left(1 - \alpha\left(\sum_{j=i+1}^{\infty} \beta_j\right)\right)
\end{aligned} \tag{2.148}$$

completing the proof. □

Proof. (3 $\implies$ 4) There exists a positive sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that

$$\sum_{i\in\mathbb{N}} \frac{\beta_i \delta_i}{\left(\sum\limits_{j=i+1}^{\infty} \beta_i\right)\left(\sum\limits_{j=i}^{\infty} \beta_i\right)} < 1 - \alpha \tag{2.149}$$

Define $\{x_i^0\}_{i\in\mathbb{N}} = \{x_i^*\}_{i\in\mathbb{N}}$,

$$\gamma_1 = \inf\left\{\sup_{y^*\in CoLim\{\phi_i\}_{i\in\mathbb{N}}}\{||x^* - y^*||\} : x^* \in \overline{co}\{x_i^*\}_{i\in\mathbb{N}}, \phi_k \in \overline{co}\{x_i^*\}_{i\geq k}, k \in \mathbb{N}\right\} \tag{2.150}$$

and pick $y_1^* \in \overline{co}\{x_i^*\}_{i\in\mathbb{N}}$, $\{\phi_i^1\}_{i\in\mathbb{N}} \subset X^*$ such that $\phi_k^1 \in \overline{co}\{x_i^*\}_{i\geq k}$ for every k and $w' \in CoLim\{\phi_i^1\}_{i\in\mathbb{N}}$ such that

$$\gamma_1(1-\delta_1) < ||y_1^* - w|| < \gamma_1(1+\delta_1) \tag{2.151}$$

So that there exists $\tilde{x} \in \overline{B_X(0;1)}$ such that $\gamma_1(1-\delta_1) < \langle \tilde{x}, y_1^* - w' \rangle$, and since $w' \in CoLim\{\phi_i^1\}_{i\in\mathbb{N}}$, we extract a subsequence $\{x_i^1\}_{i\in\mathbb{N}}$ of $\{\phi_i^1\}_{i\in\mathbb{N}}$ so that for any $w \in CoLim\{x_i^1\}_{i\in\mathbb{N}}$, we have

$$\langle \tilde{x}, w \rangle = \lim_{i\to\infty} \left\langle \tilde{x}, x_i^1 \right\rangle = \liminf_{i\to\infty} \left\langle \tilde{x}, \phi_i^1 \right\rangle \leq \langle \tilde{x}, w' \rangle \tag{2.152}$$

And so, for any $w \in CoLim\{x_i^1\}_{i\in\mathbb{N}}$, we have

$$\gamma_1(1-\delta_1) < \langle \tilde{x}, y_1^* - w \rangle \tag{2.153}$$

Continuing inductively, for $i \in \mathbb{N}$, set

$$\gamma_{i+1} = \inf\left\{ \sup\left\{ \left|\left| \left(\sum_{j=1}^{i} \beta_j y_j^*\right) + \left(\left(\sum_{j=i+1}^{\infty} \beta_j\right) y^*\right) - w \right|\right| : w \in CoLim\{\phi_i\}_{i\in\mathbb{N}} \right\}\right\} \tag{2.154}$$

Where the infimum is taken over all $y^* \in \overline{co}\{x_j^i\}_{j\geq i+1}$ and all $\{\phi_i\}_{i\in\mathbb{N}} \subset X^*$ such that $\phi_k \in \overline{co}\{x_j^i\}_{j\geq k}$. Next, pick $y_{i+1}^* \in \overline{co}\{x_j^i\}_{j\geq i+1}$ and $\{\phi_j^{i+1}\}_{j\in\mathbb{N}} \subset X^*$ such that for every k, $\phi_k^{i+1} \in \overline{co}\{x_j^i\}_{j\geq k}$ and pick $w' \in CoLim\{\phi_j^{i+1}\}_{j\in\mathbb{N}}$ such that

$$\gamma_{i+1}(1-\delta_{i=1}) < \left|\left| \sum_{j=1}^{i} \beta_j y_j^* + \left(\sum_{j=i+1}^{\infty} \beta_j\right) y_{i+1}^* - w' \right|\right| < \gamma_{i+1}(1+\delta_{i+1}) \tag{2.155}$$

Next, pick $\tilde{x} \in \overline{B_X(0;1)}$ satisfying

$$\gamma_{i+1}(1-\delta_{i+1}) < \left\langle \tilde{x}, \sum_{j=1}^{i} \beta_j y_j^* + \left(\left(\sum_{j=i+1}^{\infty} \beta_j\right) y_j^*\right) - w' \right\rangle \tag{2.156}$$

and apply the fact that since $\liminf_{j\to\infty} \left\langle \tilde{x}, \phi_j^{i+1} \right\rangle \leq \langle \tilde{x}, w \rangle$, we can find a subsequence $\{x_j^{i+1}\}_{j\in\mathbb{N}}$ of $\{\phi_j^{i+1}\}_{j\in\mathbb{N}}$ such that for every $w \in CoLim\{x_j^{i+1}\}_{j\in\mathbb{N}}$ we have

$$\gamma_{i+1}(1-\delta_{i+1}) < \left\langle \tilde{x}, \sum_{j=1}^{i} \beta_j y_j^* + \left(\left(\sum_{j=i+1}^{\infty} \beta_j\right) y_j^*\right) - w \right\rangle \tag{2.157}$$

completing our construction. Clearly, $CoLim\{y_j^*\}_{j\in\mathbb{N}} \subset CoLim\{\phi_j^i\}_{j\in\mathbb{N}}$ for every $i \in \mathbb{N}$. Hence, for every $w \in CoLim\{y_j^*\}_{j\in\mathbb{N}}$, for ever $i \in \mathbb{N}$, we have

$$\gamma_i(1-\delta_i) < \left|\left| \sum_{j=1}^{i-1} \beta_j y_j^* + \left(\left(\sum_{j=i}^{\infty} \beta_j\right) y_i^*\right) - w \right|\right| < \gamma_i(1-\delta_i) \tag{2.158}$$

Also, since $\{y_j^*\}_{j\in\mathbb{N}} \subset \overline{co}\{x_j\}_{j\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$, $CoLim\{y_j^*\}_{j\in\mathbb{N}} \subset \overline{co}\{y_j^*\}_{j\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$. By definition, $\gamma_1 \geq \alpha$, and $\{\gamma_i\}_{i\in\mathbb{N}}$ is a nondecreasing sequence since it is defined by taking the infimum over a set which never gains new elements as i increases. Further, $||w|| \leq 1$ for $w \in CoLim\{y_j^*\}_{j\in\mathbb{N}}$ implies that for every n, $\gamma_n \leq 2$, so by monotone convergence, $\gamma_i \nearrow \gamma = \left|\left| \sum_{j\in\mathbb{N}} \beta_j (y_j - w) \right|\right| \leq 2$. As for the final estimate, we have, for $i \in \mathbb{N}$ and $y^* \in CoLim\{y_j^*\}_{j\in\mathbb{N}}$, we have

Let $i \in \mathbb{N}$. Then, Hence, for every $i \in \mathbb{N}$, we have

$$\begin{aligned}
\left\|\sum_{j=1}^{i}\beta_j(y_j^* - y^*)\right\| &< \left(\sum_{j=i+1}^{\infty}\beta_j\right) \\
&\quad * \left(\left(\frac{\beta_i\gamma_i\left(1+\delta_i\right)}{\left(\sum_{j=i}^{\infty}\beta_j\right)\left(\sum_{j=i+1}^{\infty}\beta_j\right)}\right) + \left(\frac{1}{\sum_{j=i}^{\infty}\beta_j}\right)\left\|\sum_{j=1}^{i-1}\beta_j(y_j^* - y^*)\right\|\right) \\
&< \left(\sum_{j=i+1}^{\infty}\beta_j\right)\sum_{j=1}^{i}\frac{\beta_j\gamma_j\left(1+\delta_j\right)}{\left(\sum_{k=j}^{\infty}\beta_k\right)\left(\sum_{k=j+1}^{\infty}\beta_k\right)} \\
&\leq \gamma\left(\sum_{j=i+1}^{\infty}\beta_j\right)\sum_{j=1}^{i}\frac{\beta_j\left(1+\delta_j\right)}{\left(\sum_{k=j}^{\infty}\beta_k\right)\left(\sum_{k=j+1}^{\infty}\beta_k\right)} \\
&\leq \gamma\left(\sum_{j=i+1}^{\infty}\beta_j\right)\left(\left(\sum_{j=1}^{i}\frac{\beta_j}{\left(\sum_{k=j}^{\infty}\beta_k\right)\left(\sum_{k=j+1}^{\infty}\beta_k\right)}\right) + (1-\alpha)\right) \\
&= \gamma\left(\sum_{j=i+1}^{\infty}\beta_j\right)\left(\left(\sum_{j=1}^{i}\left(\frac{1}{\sum_{k=j}^{\infty}\beta_k} - \frac{1}{\sum_{k=j+1}^{\infty}\beta_k}\right)\right) + (1-\alpha)\right) \\
&= \gamma\left(\sum_{j=i+1}^{\infty}\beta_j\right)\left(\frac{1}{\sum_{j=i+1}^{\infty}\beta_j} - \frac{1}{\sum_{j=1}^{\infty}\beta_j} + 1 - \alpha\right) \\
&= \gamma\left(\sum_{j=i+1}^{\infty}\beta_j\right)\left(\frac{1}{\sum_{j=i+1}^{\infty}\beta_j} - \alpha\right) \\
&= \gamma\left(1 - \alpha\left(\sum_{j=i+1}^{\infty}\beta_j\right)\right)
\end{aligned} \tag{2.159}$$

□

It is worth noting that in the following two theorems, the assumption of completeness is necessary, as demonstrated by [12].

Theorem 2.9.5 (James Separable)**.** *If X is a separable complete seminormed space then the following are equivalent.*

1. *X is not reflexive.*
2. *For every $\alpha \in (0,1)$ there is some sequence $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ satisfying $d\left(0, \overline{co}\{x_i^*\}_{i\in\mathbb{N}}\right) \geq \alpha$ and $x_i^* \overset{w^*}{\to} 0$.*
3. *For every $\alpha \in (0,1)$ and $\{\beta_i\}_{i\in\mathbb{N}} \subset (0,1)$ satisfying $\sum_{i\in\mathbb{N}}\beta_i = 1$, there is a $\gamma \in [0,1]$*

and $\{y_i^*\}_{i\in\mathbb{N}} \subset X^*$ *such that* $y_i^* \xrightarrow{w^*} 0$ *and for each* $i \in \mathbb{N}$,

$$\left|\left|\sum_{j\in\mathbb{N}} \beta_j y_j^*\right|\right| = \gamma \qquad \left|\left|\sum_{j=1}^{i} \beta_j y_j^*\right|\right| \leq \gamma\left(1 - \alpha\left(\sum_{j=i+1}^{\infty} \beta_j\right)\right) \tag{2.160}$$

4. *There exists* $x^* \in X^*$ *not achieving its norm.*

Proof. $(1 \implies 2)$ Let $\alpha \in (0,1)$. Since X is nonreflexive and $c(X)$ is complete, by Riesz's lemma [13], there exists an $x^{**} \in B_{X^{**}}(0;1)$ such that $d(x^{**}, c(X)) > \alpha$. Since X is separable it has a countable dense set $\{x_i\}_{i\in\mathbb{N}}$. Fix $i \in \mathbb{N}$, let $\alpha_1 = \alpha_2 = \cdots = \alpha_{i-1} = 0$, $\alpha_i = \alpha$, and let $\{\beta_j\}_{j=1}^{i} \subset \mathbb{C}$ where $\beta_i \neq 0$ without loss of generality. Then, since c(X) is a subspace,

$$\begin{aligned}
\left|\sum_{j=1}^{i} \beta_j \alpha_j\right| &= |\beta_i \alpha_i| = |\beta_i|\,\alpha \\
&\leq \frac{|\beta_i|\,\alpha}{d\left(x^{**}, c(X)\right)} \left|\left|x^{**} + \sum_{j=1}^{i-1} \frac{\beta_j}{\beta_i} c(x_j)\right|\right| \\
&= \frac{\alpha}{d\left(x^{**}, c(X)\right)} \left|\left|\beta_i x^{**} + \sum_{j=1}^{i-1} \beta_j c(x_j)\right|\right|
\end{aligned} \tag{2.161}$$

Since $\alpha < d\left(x^{**}, c(X)\right)$, for some $\epsilon > 0$, $\epsilon + \frac{\alpha}{d(x^{**}, c(X))} < 1$, so by **??**, since $X^{**} = (X^*)^*$, there exists an $x_i^* \in \overline{B_{X^*}(0;1)}$ such that for $1 \leq j \leq i-1$ we have $\langle x_j, x_i^* \rangle = \langle x_i^*, c(x_j) \rangle = 0$ and $\langle x_i^*, c(x_i) \rangle \geq \alpha$. Using this method we construct a sequence $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ such that for each $1 \leq j \leq i-1$, $\langle x_j, x_i^* \rangle = 0$ and $\langle x_i, x^{**} \rangle \geq \alpha$. Without loss of generality, we let $||x_i^*|| = 1$. Density of $\{x_j\}_{j\in\mathbb{N}}$ and the boundedness of $\{x_j^*\}_{j\in\mathbb{N}}$ implies $x_j^* \xrightarrow{w^*} 0$. Furthermore, any convex combination of the $(x_i^*)'s$ satisfies

$$\alpha \leq \left\langle \sum_{j=1}^{n} \lambda_j x_{k_j}^*, x^{**} \right\rangle \leq ||x^{**}|| \left|\left|\sum_{j=1}^{n} \lambda_j x_{k_j}^*\right|\right| \leq \left|\left|\sum_{j=1}^{n} \lambda_j x_{k_j}^*\right|\right| \tag{2.162}$$

so that $d\left(0, \overline{co}\{x_i^*\}_{i\in\mathbb{N}}\right) \geq \alpha$, completing the proof. □

Proof. $(2 \implies 3)$. This is a direct application of 2.9.4 part $(1 \implies 2)$, paired with the fact that if for every i, $y_i^* \in \overline{co}\{x_j^*\}_{j\geq i}$ and $x_j^* \xrightarrow{w^*} x$, then $y_j^* \xrightarrow{w^*} x$. □

Proof. $(3 \implies 4)$. Let $x^* = \sum_{j\in\mathbb{N}} \beta_j y_j^*$ and let $x \in \overline{B_X(0;1)}$. Since $y_j^* \xrightarrow{w^*} 0$, for some $N \in \mathbb{N}$,

$\left\langle x, y_j^* \right\rangle < \gamma\alpha$ for every $j > \mathbb{N}$. Then

$$\begin{aligned} |\langle x, x^* \rangle| &\leq \left| \left\langle x, \sum_{j=1}^{N} \beta_j y_j^* \right\rangle \right| + \left| \left\langle x, \sum_{j=N+1}^{\infty} \beta_j y_j^* \right\rangle \right| \\ &< \left| \left\langle x, \sum_{j=1}^{N} \beta_j y_j^* \right\rangle \right| + \alpha\gamma \sum_{j=N+1}^{\infty} \beta_j \\ &\leq \left|\left| \sum_{j=1}^{N} \beta_j y_j^* \right|\right| + \alpha\gamma \sum_{j=N+1}^{\infty} \beta_j \leq \gamma \left(1 - \alpha \sum_{j=N+1}^{\infty} \beta_j \right) + \gamma\alpha \sum_{j=N+1}^{\infty} \beta_j \\ &= \gamma = \left|\left| \sum_{j\in\mathbb{N}} \beta_j y_j^* \right|\right| = ||x^*|| \end{aligned} \tag{2.163}$$

Since the inequality is strict and $x \in \overline{B_X(0;1)}$ was arbitrary, we are done. ☐

Proof. ($4 \implies 1$). If X is reflexive and $x^* \in X$, then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ such that $\langle x_n, x^* \rangle \to ||x^*||$. By 2.9.1, $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence $x_{k_n} \overset{w}{\to} x \in \overline{B_X(0;1)}$. This x satisfies $\langle x, x^* \rangle = ||x^*||$. ☐

Theorem 2.9.6 (James). *If X is a complete seminormed space, then the following are equivalent.*

1. *X is non-reflexive.*

2. *For each $\alpha \in (0,1)$, there exists an $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ and a subspace $Y \subset X$ such that $d\left(\overline{co}\{x_i^*\}_{i\in\mathbb{N}} - Y^\perp, 0\right) \geq \alpha$ and that $\langle y, x_i^* \rangle \to 0$ for each $y \in Y$.*

3. *For every $\alpha \in (0,1)$ and $\{\beta_i\}_{i\in\mathbb{N}} \subset [0,\infty)$ such that $\sum_{i\in\mathbb{N}} \beta_i = 1$, there is a $\gamma \in [0,2]$ and $\{y_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ such that for each $y^* \in CoLim\{y_i^*\}_{i\in\mathbb{N}}$, each $i \in \mathbb{N}$,*

$$\left|\left| \sum_{j\in\mathbb{N}} \beta_j \left(y_j^* - y^*\right) \right|\right| = \gamma \qquad \left|\left| \sum_{j=1}^{i} \beta_j \left(y_j^* - y_j\right) \right|\right| < \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j \right) \right) \tag{2.164}$$

4. *There exists $x^* \in X^*$ which doesn't achieve its norm.*

Proof. ($1 \implies 2$) If X is non-reflexive, then X contains a non-reflexive closed separable subspace S. An application of 2.9.5 implies the existence of a sequence $\{x_i\}_{i\in\mathbb{N}} \subset \overline{B_{S^*}(0;1)}$ such that

$$d_s\left(0, \overline{co}\{x_i\}_{i\in\mathbb{N}}\right) \geq \alpha \qquad x_i \overset{S-w^*}{\to} 0 \tag{2.165}$$

If $y^\perp \in X^*$ such that $Y \subset kern(y^*)$, then letting for each $i \in \mathbb{N}$ x_i^* be a Hahn-Banach extension living in $\overline{B_X^*(0;1)}$, and let $x^* \in \overline{co}\{x_i^*\}_{i\in\mathbb{N}}$. Then $x := x^*|_S \in \overline{co}\{x_i\}_{i\in\mathbb{N}}$. If $y \in Y$, then $\langle y, x_i^* \rangle = \langle y, x \rangle \to 0$. If $y^\perp \in Y^\perp$, then

$$\left|\left|x^* - y^\perp\right|\right| \geq \left|\left|x - \left(y^\perp|_S\right)\right|\right|_S = ||x|| \geq \alpha \tag{2.166}$$

so $d\left(\overline{co}\{x_i^*\}_{i\in\mathbb{N}} - Y^\perp, 0\right) \geq \alpha$ ☐

Proof. (2 $\implies$ 3) Since $CoLim\{x_i^*\}_{i\in\mathbb{N}} \subset Y^{\perp}$ from the previous part, this is an easy application of 2.9.4 (3 $\implies$ 4). $\square$

Proof. (3 $\implies$ 4) Define $\eta = \frac{\alpha^2}{4}$, and then let $\lambda_1 \in [0,\infty)$ such that for every natural n, $\lambda_{n+1} < \eta\lambda_n$ and $\sum_{k\in\mathbb{N}} \lambda_k = 1$. Let $y^* \in CoLim\{y_i^*\}_{i\in\mathbb{N}}$ where $\{y_i^*\}_{i\in\mathbb{N}}$ are as in part (3) of this theorem. Let $x \in \overline{B_X(0;1)}$. Since $y^* \in Colim\{y_i^*\}_{i\in\mathbb{N}}$, and since $\alpha \leq \gamma$, there exists $i \in \mathbb{N}$ such that

$$\left\langle x, y_{i+1}^* - y^* \right\rangle < \alpha^2 - 2\eta \leq \alpha\gamma - 2\eta \tag{2.167}$$

For this x, we have

$$\begin{aligned}
\left\langle x, \sum_{j\in\mathbb{N}} \beta_j (y_j^* - y^*) \right\rangle &< \sum_{j=1}^{i} \beta_j \left\langle x, y_j^* - y^* \right\rangle + \beta_{i+1}(\alpha\gamma - 2\eta) + \sum_{j=i+2}^{\infty} \beta_j \left\langle x, y_j^* - y^* \right\rangle \\
&\leq \left\| \sum_{j=1}^{i} \beta_j (y_j^* - y^*) \right\| + \beta_{i+1}(\alpha\gamma - 2\eta) + 2\sum_{j=i+2}^{\infty} \beta_j \\
&\leq \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j\right)\right) + \beta_{i+1}(\alpha\gamma - 2\eta) + 2\sum_{j=i+2}^{\infty} \beta_j \qquad (2.168)\\
&= \gamma - \gamma\alpha \sum_{j=i+2}^{\infty} \beta_j - 2\eta\beta_{i+1} + 2\sum_{j=i+1}^{\infty} \eta\beta_j \\
&\leq \gamma - (\gamma\alpha - 2\eta) \sum_{j=i+1}^{\infty} \beta_j < \gamma = \left\| \sum_{j\in\mathbb{N}} \beta_j \left(y_j^* - y^*\right) \right\|
\end{aligned}$$

Since $x \in \overline{B_X(0;1)}$ was arbitrary, we are done. $\square$

Proof. (4 $\implies$ 1) If X is reflexive and $x^* \in X$, then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ such that $\langle x_n, x^* \rangle \to ||x^*||$. By 2.9.1, $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence $x_{k_n} \xrightarrow{w} x \in \overline{B_X(0;1)}$. This x satisfies $\langle x, x^* \rangle = ||x^*||$. $\square$

Corollary 2.9.7. *Let X be a complete seminormed space. Then the following are equivalent.*

1. *X is reflexive.*
2. *Each element of X^* attains its norm.*

Proof. Direct consequence of 2.9.6. $\square$

2.9.2 Lindenstrauss On Nonseparable Reflexive Banach Spaces

If $\Gamma \neq \emptyset$, then $c_0(\Gamma)$ denotes the space of mappings $f : \Gamma \to \mathbb{C}$ such that for every $\epsilon > 0$, $card\,\{x \in \Gamma : |f(x)| > \epsilon\} \in \mathbb{N}$.

2.10 Convexity Of Functions And Sets

Definition 2.10.1 (Convex Functions)**.** *Let X be a vector space, Y a topological vector space, $\mathcal{U}$ the set of neighborhoods of 0 in Y except Y itself, $f : X \to (-\infty, \infty]$, and $g : Y \to (-\infty, \infty]$. Let $x, y \in X$.*

1. *We call $D(f) := f^{-1}(\mathbb{R})$ the* **effective domain** of f.
2. If $D(f) \neq \emptyset$, then we call f **proper.**
3. We call $Epi\,(f) := \{(x,t) \in X \times \mathbb{R} : f(x) \leq t\}$ the **Epigraph** of f.
4. We denote $[x, y] = \{tx + (1-t)y : t \in [0,1]\}$.
5. We denote $(x, y) = \{tx + (1-t)y : t \in (0,1)\}$.
6. We say that $C \subset X$ is **convex** if $[x, y] \subset C$ for each $x, y \in C$.
7. We say that $C \subset X$ is **strictly convex** if for each $x, y \in C$, for each $z_0 \in (x, y)$, and for each $z_1 \in X$, there is a $t > 0$ such that $[z_0, z_0 + tz_1] \subset C$.
8. We say that f is **convex** if for each $x, y \in X$, $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$.
9. We say that f is **strictly convex** if for each $x, y \in X$, $f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$.
10. If g is a convex function, then then we define the **modulus of local uniform convexity** of g, $\tilde{\Delta} : \mathcal{U} \times Y \to \mathbb{R}$ by
$$\tilde{\Delta}\,(U, x) = \frac{1}{2} \inf_{y \in Y \setminus (x+U)} \left\{ f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \right\} \tag{2.169}$$
11. If g is a convex function, then we define the **modulus of uniform convexity** of g, $\Delta : \mathcal{U} \to \mathbb{R}$ by $\Delta(U) = \inf\limits_{g(x)=1} \tilde{\Delta}(U, x)$.
12. If g is a convex function, then we say that g is **locally uniformly convex at** $x \in Y$ if for each $U \in \mathcal{U}$, $\tilde{\Delta}(U, x) > 0$.
13. We say that g is **locally uniformly convex** if it is **locally uniformly convex** at each of its points.
14. We say that g is **uniformly convex** if for each $U \in \mathcal{U}$, $\Delta(U) > 0$.
15. We say that g is **lower semi-continuous**, or LSC if it is continuous with respect to the topology on $(-\infty, \infty]$ generated by sets of the form $(-\infty, \alpha)$ where $\alpha \in \mathbb{R}$, along with $(-\infty, \infty]$ itself.

Remark 2.10.2 (Basis Independence)**.** *It is easy to see that a mapping is locally uniform convex at a point (locally uniformly convex) [uniformly convex] if we define Δ, $\tilde{\Delta}$ in terms of a single neighborhood basis of Y at 0 instead of all neighborhoods of 0 in Y.*

Remark 2.10.3 (Strictly Convex Real Valued)**.** *If X is a vector space and $f : X \to (-\infty, \infty]$ is strictly convex, and f is finite everywhere.*

Proof. If $f(x) = \infty$ where $x \in X$ and f is strictly convex, then we must have $\infty = f(x) < \frac{f(0)+f(2x)}{2}$, a contradiction. □

Proposition 2.10.4. *Let X be a vector space and $T : X \to (-\infty, \infty]$. Then the following are equivalent.*

1. *T is (strictly) convex.*
2. *For each $x_1 \neq x_2 \in X$ and $\lambda \in (0, 1)$.*
$$T(\lambda x_1 + (1-\lambda)x_2)\,(<) \leq \lambda T x_1 + (1-\lambda)T x_2 \tag{2.170}$$
3. *For each $\{\lambda_i\}_{i=1}^n \subset (0, 1)$ which sums to 1, for each $\{x_i\}_{i=1}^n \subset X$,*
$$T\left(\sum_{j=1}^n \lambda_j x_j\right)(<) \leq \sum_{j=1}^n \lambda_j T x_j \tag{2.171}$$
4. *If $x_1 \neq x_3 \in X$ and $x_2 \in (x_1, x_3)$, say $x_2 = \lambda x_1 + (1-\lambda)x_3$ then*
$$\frac{Tx_2 - Tx_1}{\lambda}\,(<) \leq Tx_3 - Tx_1\,(<) \leq \frac{Tx_3 - Tx_2}{1-\lambda} \tag{2.172}$$
5. *Epi(T) is (strictly) convex*

Proof. ($1 \implies 2$) □

Proof. ($2 \implies 3$) I utilize induction on n. Since f is assumed to be (strictly) convex, the proposition holds for $n = 1$ and $n = 2$. Let $k \in \mathbb{N}$. Suppose that for any $(\psi_1, \cdots, \psi_k) \in [0, 1]^k$ such that $\sum_{j=1}^k \psi_j = 1$, and for each $(x_1, \cdots, x_k) \in X^k$, we have
$$f\left(\sum_{j=1}^k \lambda_j x_j\right)(<) \leq \sum_{j=1}^k \lambda_j f(x_j) \tag{2.173}$$

Let $(\lambda_1, \cdots, \lambda_{k+1}) \in [0, 1]^{k+1}$. Let $(x_1, \cdots, x_{k+1}) \in X^{k+1}$. Without loss of generality, we assume $\lambda_{k+1} \neq 0$. Then
$$\sum_{j=1}^k \frac{\lambda_j}{1-\lambda_{k+1}} = 1 \qquad \left(\frac{\lambda_1}{1-\lambda_{k+1}}, \cdots, \frac{\lambda_k}{1-\lambda_{k+1}}\right) \in [0, 1]^k \tag{2.174}$$

Hence, because f is (strictly) convex,

$$\begin{aligned} f\left(\sum_{j=1}^{k+1}\lambda_j x_j\right) &= f\left(\lambda_{k+1}x_{k+1} + (1-\lambda_{k+1})\sum_{j=1}^{k}\frac{\lambda_j}{1-\lambda_{k+1}}x_j\right) \\ &(<) \leq \lambda_{k+1}f(x_{k+1}) + (1-\lambda_{k+1})f\left(\sum_{j=1}^{k}\frac{\lambda_j}{1-\lambda_{k+1}}x_j\right) \\ &(<) \leq \sum_{j=1}^{k+1}\lambda_j f(x_j) \end{aligned} \tag{2.175}$$

□

Proof. (3 $\implies$ 4) □

Proof. (4 $\implies$ 1). □

Proof. (2 $\iff$ 5) □

The epigraph of a function provides us with a nice characterization of lower semi-continuity.

Proposition 2.10.5 (Convex Continuity). *Let X be a locally convex space and $f : X \to (-\infty, \infty]$. The following conditions are equivalent.*

1. *f is LSC on X.*
2. *$Epi(f)$ is closed in $X \times \mathbb{R}$.*

Proof. Define $F : X \times \mathbb{R} \to \tilde{\mathbb{R}}$ by $F(x, \alpha) = f(x) - \alpha$. Then f is (weakly) LSC on X if and only if F is (weakly) LSC on $X \times \mathbb{R}$. Suppose f is (weakly) LSC. Then F is (weakly) LSC, so $F^{-1}((-\infty, 0]) = Epi(f)$ is (weakly) closed, so we're done. Suppose $Epi(f)$ is (weakly) closed. Then $F^{-1}((-\infty, 0])$ is (weakly) closed. Further, for any $\beta \in \mathbb{R}$, $F^{-1}((-\infty, \beta]) = F^{-1}((-\infty, 0]) - (0, \beta)$, and so is also closed. Hence F is (weakly) LSC, and so f is too. □

In the case of a convex function, the above proposition allows us to equate weak and strong lower semicontinuity.

Corollary 2.10.6 (Weak To Strong Convex). *Let X be locally convex Hausdorff space and $f : X \to (-\infty, \infty]$ be convex. Then f is LSC if and only if it is weakly LSC.*

Proof. Since $Epi(f)$ is convex, it is closed if and only if it is weakly closed, allowing us to apply 2.10.5. □

Theorem 2.10.7 (Point Continuous). *Let X be a locally convex space and $f : X \to (-\infty, \infty]$ be convex and proper. Then f is bounded on some open set if and only if f is continuous on the interior of its domain.*

Proof. (*implies*) Without loss of generality, we assume that f is bounded from above by M on a (weakly) open set $\mathcal{U}$ which is symmetric and contains 0. Further, we can also suppose $f(0) = 0$. For each $\epsilon \in (0,1)$ and each $x \in \epsilon\mathcal{U}$, we have

$$f(x) = f\left(\epsilon\frac{x}{\epsilon} + (1-\epsilon)0\right) \leq \epsilon f\left(\frac{x}{\epsilon}\right) \leq \epsilon M \tag{2.176}$$

and since $0 = \frac{x}{1+\epsilon} + \left(1 - \frac{1}{1+\epsilon}\left(\frac{-x}{\epsilon}\right)\right)$,

$$0 = f(0) = f\left(\frac{x}{1+\epsilon} + \left(1 - \frac{1}{1+\epsilon}\right)\left(\frac{-x}{\epsilon}\right)\right) \leq \frac{f(x)}{1+\epsilon} + \frac{\epsilon f\left(-\frac{x}{\epsilon}\right)}{1+\epsilon} \tag{2.177}$$

so, since $\frac{-x}{\epsilon} \in \mathcal{U}$,

$$-\epsilon M \leq -\epsilon f\left(\frac{x}{\epsilon}\right) \leq f(x) \tag{2.178}$$

Hence, $|f(x)| \leq \epsilon M$ for $x \in \epsilon\mathcal{U}$, and f is (weakly) continuous at 0. Hence, it is sufficient to show that for any y in the (weak) interior of $D(f)$, there is a (weak) neighborhood of y on which f is bounded from above. To see this, let y in the (weak) interior of $D(f)$. Since scalar multiplication is continuous, there is a $\rho > 1$ such that $\rho y \in D(f)$. If $\mathcal{U}_y = y + \left(1 - \frac{1}{\rho}\right)\mathcal{U}$, then $x \in \mathcal{U}_y$ can be written, for some $z \in \mathcal{U}$, as

$$x = y + \left(1 - \frac{1}{\rho}\right) z = \frac{1}{\rho}(\rho y) + \left(1 - \frac{1}{\rho}\right) z \tag{2.179}$$

Since f is convex, $D(f)$ is convex, and so $x \in D(f)$, implying $\mathcal{U}_y \subset D(f)$. Since f is a convex function, we also have that

$$f(x) \leq \frac{1}{\rho} f(\rho y) + \left(1 - \frac{1}{\rho}\right) f(x) \leq \frac{1}{\rho} f(\rho y) + \left(1 - \frac{1}{\rho}\right) M \tag{2.180}$$

so that f is bounded on $\mathcal{U}_y$ and is therefore continuous at y. □

Proof. ($\Longleftarrow$) This is obvious. □

Corollary 2.10.8. *Let X be a locally convex, space and $f : X \to (-\infty, \infty]$ be LSC at some point in its effective domain.*

1. *if f is convex, then it is continuous on the interior of $D(f)$.*

2. *If f is strictly convex, then it is continuous on X.*

2.11 Differentiation And SubDifferentials

Definition 2.11.1 (Types Of Differentiability)**.** *Let X be a topological vector space, Y a Hausdorff topological vector space, $C \subset X$, $f : C \to X$, and $x_0 \in C$.*

1. *If $y \in X$ such that x_0 is an accumulation point of $[x_0, x_0 + y] \cap C$, then we say that the* **directional derivative in the direction of y** exists at x_0 and we write

$$\lim_{t \to 0^+} \frac{f(x_0 + ty) - f(x_0)}{t} = f'_+(x_0, y) \tag{2.181}$$

if the limit above actually exists.

2. If f is differentiable at x_0 in the direction of $-y$, then we denote $f'_-(x_0, y) := -f'_+(x, -y)$.

3. If there exists $f'(x_0) \in BL(X, Y)$ such that for every $y \in X$, $f'_+(x_0, y) = f'(x_0)y$ then we call $f'(x_0)$ the **Gateaux derivative** of f at x_0 and we call f **Gateaux differentiable** .

4. If f is Gateaux differentiable at x_0 and $\frac{f(x_0+ty)-f(x_0)}{t} \to f'(x_0)y$ uniformly for y in some neighborhood of 0 in X, then we say that f is **Frechet differentiable** at x_0 and we call $f'(x_0)$ the **Frechet derivative** of f at x_0.

5. If $h : X \to (-\infty, \infty]$ and there is an $x^* \in X^*$ such that

$$\langle y - x, x^* \rangle \leq h(y) - h(x) \qquad (\forall y \in X) \tag{2.182}$$

then we say that h is **subdifferentiable** at x_0, call x^* a **subgradient** of h at x_0, and denote the set of all subgradients of h at x_0 with $\partial h\,(x_0)$. if D is the set on which h is subdifferentiable, then we call $\partial h : D \to 2^{X^*}$ the **subdifferential** of h.

One useful feature of convex functions is how they interact with the various forms of differentiability.

Proposition 2.11.2 (Gateaux)**.** *Let X be a topological vector space, Y a Hausdorff topological vector space, $f : X \to Y$ Gateaux differentiable at $x_0 \in X$, and $y \in X$. The following are true.*

1. $f'_+(x_0, y) = f'_+(x_0, -y) = f'_-(x_0, y)$.
2. $\langle y, f'(x_0) \rangle = \frac{d}{dt} f\,(x_0 + ty)\,|_{t=0}$
3. *If f is Gateaux differentiable on a neighborhood of x_0 and $f' : X \to BL(X, Y)$ is Gateaux Differentiable at x_0, then $f'' : X \to BL(X, BL(X, Y))$ satisfies*

$$\langle y, f''(x)y \rangle = \frac{d^2}{dt^2} f\,(x_0 + ty)\,|_{t=0} \tag{2.183}$$

Proof. (1) □

Proof. (2) □

Proof. (3) □

Proposition 2.11.3 (Convex Differentiable)**.** *Let X be a seminormed space and $f : X \to (-\infty, \infty]$ be convex. Then for each $y \in X$ and each x_0 in the interior of $D(f)$, the directional derivative in the direction of y at x_0 exists and $f'_-(x, y) \leq f'_+(x, y)$, and for any $t > 0$,*

$$f'_+(x, y) \leq \frac{f(x + ty) - f(x + ty)}{t} \leq f'_-(x + ty, y) \tag{2.184}$$

2.12 Normalized Duality Mapping

Definition 2.12.1 (Normalized Duality Mapping). *If X is a seminormed space, then we call $J : X \to 2^{X^*}$ defined by*

$$Jx = \left\{x^* \in X^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\right\} \tag{2.185}$$

the normalized duality mapping of X.

Proposition 2.12.2 (Normalized Duality Inequality). *Let X be a seminormed space and J be X's normalized duality mapping. Then, if $x, y \in X$ and $||x + \lambda y|| \neq 0$ where $\lambda > 0$, and $j_x \in Jx$, $j_{x+\lambda y} \in J(x + \lambda y)$, then*

$$\frac{\langle y, j_x \rangle}{||x||} \leq \frac{||x + \lambda y|| - ||x||}{\lambda} \leq \frac{\langle y, j_{x+\lambda y} \rangle}{||x + \lambda y||} \tag{2.186}$$

Proof.

$$\begin{aligned}
\frac{\langle y, j_x \rangle}{||x||} &= \frac{\langle x + \lambda y, j_x \rangle - ||x||^2}{\lambda\, ||x||} \leq \frac{|\langle x + \lambda y, j_x \rangle| - ||x||^2}{\lambda\, ||x||} \leq \frac{||j_x||\, ||x + \lambda y|| - ||x||^2}{\lambda\, ||x||} \\
&= \frac{||x + \lambda y|| - ||x||}{\lambda} \\
&= \frac{||x + \lambda y||}{||x + \lambda y||} \frac{||x + \lambda y|| - ||x||}{\lambda} \leq \frac{||x + \lambda y||^2 - |\langle x, j_{x+\lambda y} \rangle|}{\lambda\, ||x + \lambda y||} \\
&= \frac{\lambda \langle y, j_{x+\lambda y} \rangle + \langle x, j_{x+\lambda y} \rangle - |\langle x, j_{x+\lambda y} \rangle|}{\lambda\, ||x + \lambda y||} \\
&\leq \frac{\lambda \langle y, j_{x+\lambda y} \rangle}{\lambda\, ||x + \lambda y||} = \frac{\langle y, j_{x+\lambda y} \rangle}{||x + \lambda y||}
\end{aligned} \tag{2.187}$$

□

Theorem 2.12.3 (Asplund). *[14] Let X be a seminormed space and J be its normalized duality mapping. Then for any $x \in X$, $Jx = \partial\left(||x||^2\right)$.*

Proof. □

Proposition 2.12.4. *Let X be a (Possibly Complete, maybe not) seminormed space and let J be it's normalized duality mapping. Then J is norm to weak* upper semicontinuous on X.*

Proof. □

2.13 Orthogonality

Definition 2.13.1 (Orthogonality). *Let X be a seminormed space and $x, y \in X$. We say that x is* **orthogonal** to y and we write $x \perp y$ if for each scalar λ, we have

$$||x|| \leq ||x + \lambda y|| \tag{2.188}$$

Proposition 2.13.2 (Orthogonality)**.** *Let X be a seminormed space, $x, z \in X$, and $x^* \in X^*$.*

1. *$\langle x, x^* \rangle = ||x||\,||x^*||$ if and only if for each $y \in kern\,(x^*)$, $x \perp y$.*
2. *x is orthogonal to each element of some hyperplane in X.*
3. *For some $\alpha \neq 0$ $x \perp (\alpha x + z)$.*
4. *The mapping $T : \mathbb{F} \to \mathbb{R}$ defined by $T\alpha = ||\alpha x + z||$ achieves its minimum, and if λ_0 is a point at which it achieves this minimum, then $(\lambda_0 x + y) \perp x$ for any λ_0 which minimizes T. Furthermore, since T as defined earlier is a convex function, the set of λ for which $(\lambda x + y) \perp x$ is a convex set.*

2.14 Convexity Of A Space

Definition 2.14.1 (Uniform Convexity, Weak Uniform Convexity, Local Uniform Convexity, Weak Local Uniform Convexity, Strict Convexity)**.** *Let $(X, ||\cdot||)$ be a seminormed space.*

1. *We call the local modulus of uniform covexity of $||\cdot||$, denoted by the symbol $\tilde{\Delta}$, the* **local modulus of uniform convexity** of X.
2. If Δ is the modulus of uniform convexity of $||\cdot||$, then we call Δ the **modulus of uniform convexity** of X.
3. We define $\tilde{\Delta}_w : (0, \infty) \times \partial B_X(0; 1) \to (0, \infty)$ by
$$\tilde{\Delta}_w(\epsilon, x) = \inf \{2 - \langle x + y, x^* \rangle : x^* \in \partial B_{X^*}(0; 1), y \in \partial B_X(0; 1), ||x - y|| \geq \epsilon\} \tag{2.189}$$
We call $\tilde{\Delta_w}$ the **modulus of weak local uniform convexity** of X.
4. We define $\Delta_w : (0, \infty) \to (0, \infty)$ by
$$\Delta_w(\epsilon) = \inf_{x \in \partial B_X(0;1)} \tilde{\Delta}_w(\epsilon, x) \tag{2.190}$$
We call this the **modulus of weak uniform convexity** of X.

1. *We say that X is* **strictly convex** if for each $x, y \in X$ such that $||x - y|| \neq 0$, $||x + y|| < ||x|| + ||y||$.
2. We say that X is **uniformly convex at** x if $||\cdot||$ is uniformly convex at x.
3. We say that X is **locally uniformly convex** if $||\cdot||$ is locally uniformly convex.
4. We say that X is **uniformly convex** if $||\cdot||$ is uniformly convex.
5. We say that X is **weakly uniformly convex at** $x \in \partial B_X(0; 1)$ if for each $\epsilon > 0$, $\tilde{\Delta}_w(\epsilon, x) > 0$.

6. We say that X is **locally weakly uniformly convex** if it is weakly uniformly convex at each point on the boundary of $X's$ unit sphere.

7. We say that X is **weakly uniformly convex** if for each $\epsilon > 0$, $\Delta_w(\epsilon) > 0$.

Proposition 2.14.2 (Strictly Convex Spaces). *Let X be a normed space. and let J be a duality mapping on X of weight ϕ. Then the following are equivalent.*

1. *X is strictly convex.*

2. *If $x, y \in X$ and $||x + y|| = ||x|| + ||y||$, then for some $\alpha \geq 0$, $||x - \alpha y|| = 0$.*

3. *If $||x|| = ||y|| = 1$ where $0 \neq ||x - y||$, then $||x + y|| < 2$.*

4. *If $x, y, z \in X$ and $||x - y|| = ||x - z|| + ||z - y||$, $||z - z_0|| = 0$ for some $z_0 \in [x, y]$.*

5. *If $x^* \in X^*$ and $||x|| = ||y|| = 1$ such that $\langle x, x^* \rangle = \langle y, y^* \rangle = \sup_{||z||=1} \langle z, x^* \rangle$, then $||x - y|| = 0$.*

6. *$||\cdot||^2$ is strictly convex.*

7. *J is strictly monotone. That is, if $x, y \in X$, $||x - y|| \neq 0$, $x^* \in Jx$, and $y^* \in Jy$, then*

$$\langle x - y, x^* - y^* \rangle > 0 \tag{2.191}$$

8. *Orthogonality in X is left-unique. That is, for $x, y \in X$, there is a unique $\alpha \in \mathbb{F}$ such that $(\alpha x + y) \perp x$.*

Proposition 2.14.3 (Locally Uniformly Convex Spaces). *Let X be a Banach Space. Then the following are equivalent.*

1. *X is locally uniformly convex*

2. *For each $\epsilon > 0$ and $x \in X$ with $||x|| = 1$, there is a $\delta > 0$ such that if $y \in X$ satisfies $||y|| = 1$ and $||x - y|| \geq \epsilon$, then $||x + y|| \leq 2(1 - \delta)$.*

3. *If $x \in \partial B_X(0; 1)$, $\{x_n\}_{n \in \mathbb{N}} \subset \partial B_X(0; 1)$, and $||x + x_n|| \to 2$, then $x_n \to x$.*

4. *$\frac{1}{2} ||\cdot||^2$ is locally uniformly convex.*

Proposition 2.14.4 (Uniformly Convex Spaces). *Let X be a Banach space. The following are equivalent.*

1. *X is uniformly Convex.*

2. *For each $\epsilon > 0$, there is a $\delta > 0$ such that if $x, y \in \overline{B_X(0; 1)}$ and $||x - y|| \geq \epsilon$, then $||x + y|| \leq 2(1 - \delta)$.*

3. *If $\{x_i\}_{i \in \mathbb{N}}, \{y_i\}_{i \in \mathbb{N}} \subset \overline{B_X(0; 1)}$ and $||x_n + y_n|| \to 2$, then $x_n - y_n \to 0$.*

4. *$\frac{1}{2} ||\cdot||^2$ is uniformly convex.*

Proposition 2.14.5 (Weakly Locally Uniformly Convex Spaces). *Let X be a Banach space. The following conditions are equivalent.*

1. *X is weakly locally uniformly convex.*
2. *For each $\epsilon > 0$, $^* \in \partial B_{X^*}(0;1)$, and $x \in \overline{B_X(0;1)}$ there is a $\delta > 0$ such that if $y \in \overline{B_X(0;1)}$ and $\langle x-y, x^* \rangle \geq \epsilon$, then $||x+y|| \leq 2(1-\delta)$.*
3. *If $x \in \overline{B_X(0;1)}$ and $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ such that $||x+x_n|| \to 2$, then $x_n \xrightarrow{w} x$.*

Proposition 2.14.6 (Weakly Uniformly Convex Spaces). *Let X be a Banach space. The following conditions are equivalent.*

1. *X is weakly uniformly convex.*
2. *For each $\epsilon > 0$ and $x^* \in \overline{B_{X^*}(0;1)}$, there is a $\delta > 0$ such that if $x, y \in \overline{B_X(0;1)}$ such that $\langle x-y, x^* \rangle \geq \epsilon$, then $||x+y|| \leq 2(1-\delta)$.*
3. *If $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ and $||x_n+y_n|| \to 2$, then $x_n - y_n \xrightarrow{w} 0$.*

Proposition 2.14.7 (Degrees Of Convexity). *Let X be a seminormed space.*

1. *If X is uniformly convex, then X is weakly uniformly convex.*
2. *If X is uniformly convex, then X is locally uniformly convex.*
3. *if X is locally uniformly convex, then X is weakly locally uniformly convex.*
4. *if X is weakly locally uniformly convex, then X is strictly convex.*

Proposition 2.14.8 (Local Weak To Strong). *If a seminormed space X is locally uniformly convex and $\{x_n\} \subset X$ satisfies $x_n \xrightarrow{w} x$ and $||x_n|| \to ||x||$, then $x_n \to x$.*

Proof. □

Theorem 2.14.9. *(Milman Pettis) A complete uniformly convex seminormed space X is reflexive.*

2.15 Smoothness Of A Space

Define Moduli Of Smoothness, remark on positiveness

Definition 2.15.1. *Let X be a seminormed space and $x_0 \in X$.*

1. *We define the* **modulus of smoothness** of X at x_0, $\tilde{\rho} : [0,\infty) \times X \to \mathbb{R}$ by

$$\tilde{\rho}(\epsilon, x) = \frac{1}{2} \sup_{||y||<1} \left(||x+\epsilon y|| + ||x-\epsilon y|| - 2\,||x||\right) \tag{2.192}$$

2. We define the **modulus of smoothness** of X, $\rho : [0, \infty) \to \mathbb{R}$ by

$$\rho(\epsilon) = \sup_{||x|| \geq 1} \rho(\epsilon, x) \tag{2.193}$$

3. We say that X is **locally uniformly smooth** at x_0 if $\lim_{\epsilon \to 0} \frac{\tilde{\rho}(\epsilon, x_0)}{\epsilon} = 0$.

4. We say that X is **locally uniformly smooth** if it is locally uniformly smooth at each of its points.

5. We say that X is **uniformly smooth** if $\lim_{\epsilon \to 0} \frac{\rho(\epsilon)}{\epsilon} = 0$.

6. We say that X is **smooth** at x_0 if Jx_0 is a singleton.

7. We say that X is **smooth** if it is smooth at each of its points.

8. 2.15.2 motivates defining X to be called **very smooth** at x_0 if it is smooth and J is seminorm to weak continuous at x_0.

9. We say that X is **very smooth** if it is very smooth at each of its points.

Proposition 2.15.2 (Smooth Characterization)**.** *Let X be a seminormed space, $x_0 \in X$, and J it's normalized duality mapping. The following are equivalent.*

1. *X is a smooth at x_0.*

2. *Every selection of J is norm to weak* continuous at x_0.*

3. *There exists a selection of J which is norm to weak* continuous at x_0.*

4. *$||\cdot||$ is Gateaux Differentiable at x_0.*

5. *For every $y \in X$, there is a unique $\alpha \in \mathbb{C}$ such that $x \perp (\alpha x + y)$.*

6. *For every $y, z \in X$, if $x \perp y$ and $x \perp z$, then $x \perp y + z$.*

7. *(NEED TO DEFINE HYPERPLANE)There is a supporting hyperplane for $\overline{B_X(0; ||x||)}$ at x.*

Proposition 2.15.3 (Degrees Of Smoothness)**.** *Let X be a seminormed space.*

1. *If X is uniformly smooth, then it is locally uniformly smooth.*

2. *If X is locally uniformly smooth, then it is very smooth.*

3. *If X is very smooth, then it is smooth.*

Proposition 2.15.4 (Very Smooth)**.**

Proposition 2.15.5 (Local Uniformly Smooth At A Point)**.** *Let X be a seminormed space, $x_0 \in X$, and J be X's normalized duality mapping. The following conditions are equivalent.*

1. *X is locally uniformly smooth at x_0.*

2. *J is continuous at x_0.*

3. *$||\cdot||$ is Frechet differentiable at x_0.*

Corollary 2.15.6 (Local Uniformly Smooth)**.** *Let X be a seminormed space and J be X's normalized duality mapping. The following conditions are equivalent.*

1. *X is locally uniformly smooth.*

2. *J is conitnuous.*

3. *$||\cdot||$ is Frechet differentiable.*

Proposition 2.15.7 (Uniformly Smooth)**.** *Let X be a seminormed space and J be X's normalized duality mapping. The following are equivalent.*

1. *X is uniformly smooth.*

2. *J is uniformly continuous on bounded subsets of X.*

3. *$||\cdot||$ is uniformly Frechet differentiable on bounded subsets of X.*

Chapter 3

Smoothness And Convexity

3.1 Convexity And Smoothness Of A Space

Proposition 3.1.1 (Smoothness and Strict Convexity). *Let X be a seminormed space and J be X's normalized duality mapping.*

1. *If X^* is smooth, then X is strictly convex.*
2. *If X^* is strictly convex, then X is smooth.*
3. *If X is reflexive, then X is strictly convex if and only if X^* is smooth.*
4. *If X is reflexive, then X is smooth if and only if X^* is strictly convex.*
5. *If X^* is strictly convex, then J is single valued and norm to weak $- *$ continuous.*
6. *X is smooth and strictly convex if and only if J is single valued and strictly monotone.*

Proposition 3.1.2 (Lindenstraus Duality Formula).

Proposition 3.1.3 (Very Smooth and Weak Local convexity).

Proposition 3.1.4 (Local Uniform Smoothness and Local Uniform Convexity). *Let X be a seminormed space and J be X's normalized duality mapping.*

Proposition 3.1.5 (Uniform Smoothness and Convexity). *Let X be a (possibly complete) seminormed space and J be X's normalized duality mapping.*

1. *X is uniformly convex if and only if X^* is uniformly smooth.*
2. *X is uniformly smooth if and only if X^* is uniformly convex.*
3. *X^* is uniformly convex if and only if J is single valued and uniformly continuous on bounded subsets of X.*

Corollary 3.1.6. *Uniformly Smooth Banach Spaces are Reflexive*

Proposition 3.1.7 (Normalized Convergence). *Let X be a smooth locally uniformly convex seminormed space with normalized duality mapping J. If $\{x_n\}_{n\in\mathbb{N}} \subset X$, $x \in X$, $j_x \in Jx$, and $j_n \in Jx_n$ for each $n \in \mathbb{N}$, then*

$$\langle x_n - x, j_n - j\rangle \to 0 \implies x_n \to x \tag{3.1}$$

3.2 Convexity, Smoothness, and High Order Duals

If you start with a poorly behaved Space, then things can only get worse.

Theorem 3.2.1. *If X is a seminormed space and X^* is very smooth, then X is reflexive.*

Corollary 3.2.2. *Let X be a seminormed space.*

1. *If X^*'s norm is Frechet differentiable, then X is reflexive.*
2. *If X^{**} is weakly locally uniformly convex, then X is reflexive.*
3. *If X^{***} is smooth, then X is reflexive.*
4. *If X^{****} is strictly convex, then X is reflexive.*

Chapter 4

Renorming Theory (Including Results about WCG Spaces)

4.1 Representations Of Reflexive Spaces

(Lindenstrauss' Theorem Goes Here

4.2 Local Uniform Convexafiability Of Reflexive Spaces

-Trojanski's Theorem Goes Here

Chapter 5

Convexity And Fixed Point Theory

Bibliography

[1] Leonidas Alaoglu. Weak topologies of normed linear spaces. *Annals Of Mathematics*, 41(1):252–267, January 1940.

[2] Anthony Hester. Semigroups generated by pseudo-contractive mappings under the nagumo condition. *Disseratation*, pages 1–122, 2007.

[3] V.L. Smulian. Uber lineare topologische raume. *Raume. Mat. Sbornik N.S.*, 7(49):425–448, 1940.

[4] W.F. Eberlein. Weak compactness in banach spaces i. *Proceedings Of The National Academy Of Science, USA*, 33:51–53, 1947.

[5] R.J. Whitley. An elementary proof of the eberlein-smulian theorem. *Math. Ann.*, 172:116–118, 1967.

[6] Joseph Diestel. *Sequences And Series In Banach Spaces.* Springer-Verlag, New York, 1984.

[7] E. Bishop and R. R. Phelps. Support functionals of convex sets. *Convexity. Proceedings of the Seventh Symposium in Pure Mathematics of the American Mathematical Society*, 7:27–35, 1963.

[8] R.C. James. Bases and reflexivity of banach spaces. *Bull. Amer. Math.*, 56:58, 1950.

[9] V.L. Klee. Some characterizations of reflexivity. *Reviasta de ciencias*, 52:15–23, 1950.

[10] R.C. James. Reflexivity and the supremum of linear functionals. *Ann. of Math.*, 66:159–169, 1957.

[11] R.C. James. Characterizations of reflexivity. *Studia Mathematica*, 23:205–216, 1964.

[12] R.C. James. A counterexample for a sup theorem in normed spaces. *Israel Journal Of Mathematics*, 9(4):511–512, 1971.

[13] Erwin Kreyszig. *Introductory Functional Analysis With Applications.* Wiley, wiely student edition edition, 1989.

[14] Edgar Asplund. Positivity of duality mappings. *Bulletin of the American Mathematical Society*, 73:200–203, 1967.