

Geometry of TVS's

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Chapter 1

Introduction

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Chapter 2

Prerequisite Knowledge

2.1 Weak Topologies And Dual Spaces Of Seminormed Spaces

2.1.1 Basics

Definition 2.1.1 (\in). Let X and Y be sets. We use the notation $Y \in X$ to indicate that Y is an element of X . If Y is not an element of X then we write $Y \notin X$.

Definition 2.1.2 (Subset). Let X and Y be sets such that $x \in Y \implies x \in X$. Then we write $Y \subset X$ and we say that Y is a **Subset** of X and we write $X \supset Y$ and we say that X is a **Superset** of Y .

Definition 2.1.3 (Set Difference). Let X and Y be sets. We define

$$X \setminus Y = \{x \in X | x \notin Y\}$$

We call $X \setminus Y$ the **Complement** of Y relative to X .

Definition 2.1.4 (Pair Set and Singleton). Let X and Y be sets. We then assume that the set Z containing exactly X and Y is a set which we call the **Pair Set** of X and Y . Since we did not assume $X \neq Y$, this implies that the set containing only X also exists. We refer to sets with only a single element as **Singletons**.

Definition 2.1.5 (Ordered Pair). Let X and Y be sets. Then we define $(X, Y) = \{\{X\}, \{X, Y\}\} \in 2^{\{X, Y\}}$. We call (X, Y) the **Ordered Pair** of X with Y .

Definition 2.1.6 (Binary Cartesian Product). Let $X \neq \emptyset$ and let $Y \neq \emptyset$. We define $X \times Y = \{(x, y) \in 2^{X \cup Y} | x \in X \wedge y \in Y\}$. We call $X \times Y$ the **Binary Cartesian Product** of X with Y .

Definition 2.1.7 (Set Diagonal). Let X be a set. We define $\Delta(X) = \{(x, x) \in X \times X | x \in X\}$ and we call $\Delta(X)$ the **Diagonal** of X .

Definition 2.1.8 (Relation). Let $X \neq \emptyset$ be a set and let $Y \neq \emptyset$ be a set. We say that R is a **Relation** from X to Y if $R \subset X \times Y$. If $(a, b) \in R$, then we may write aRb .

Definition 2.1.9 (Function). Let $X \neq \emptyset$ and $Y \neq \emptyset$. Let $f \subset X \times Y$ such that for each $x \in X$ there is a Unique $y \in Y$ such that $(x, y) \in f$. Then we say that f is a **Function** from X into Y . and we write $f : X \rightarrow Y$. We may also call f a **Map** or a **Mapping** from X into Y . Primarily, though we will rely on the notation $f : X \rightarrow Y$ to indicate that f is a **Function** with **Domain** X and **Codomain** Y . If $A \subset X$ and $B \subset Y$, then we denote

$$f(A) = \{f(x) \in Y | x \in A\} \quad f^{-1}(B) = \{x \in X | f(x) \in B\}$$

We call $f(A)$ the **Image** of A under f and we call $f^{-1}(B)$ the **Preimage** of B under f . we call $f(X)$ the **Range** of f . When the domain of a function is understood, we may also refer to an unnamed map f by writing,

$$x \rightarrow f(x)$$

Definition 2.1.10 (Insertion Function). Let $A \subset B$ and define $f : A \rightarrow B$ by $f(x) = x$. The we call f the **Insertion Function** of A into B .

Definition 2.1.11 (Restriction). Let X, Y be sets and let R be a **Relation** from X to Y . Let $A \subset X$. We define

$$(R)|_A = \{(x, y) \in R | x \in A\}$$

We call $(R)|_A$ the **Restriction** of the **Relation** R to the set A .

Definition 2.1.12 (Relation Inverse). Let $X \neq \emptyset$ and $Y \neq \emptyset$. Let R be a **Relation** from X to Y . We define

$$R^{-1} = \{(y, x) \in Y \times X | (x, y) \in R\}$$

We call R^{-1} the **Inverse** of R .

Definition 2.1.13 (Extension). Let X, Y be sets and let $f, g : X \rightarrow Y$. Let f be a **Restriction** of g . Then we call g an **Extension** of f .

Definition 2.1.14 (Injective). Let X, Y be sets and let $f : X \rightarrow Y$. We say that f is an **Injection**, or that f is **Injective** if for all $x, y \in X$, if $x \neq y$, then $f(x) \neq f(y)$.

Definition 2.1.15 (Surjective). Let X, Y be sets and let $f : X \rightarrow Y$. Suppose that for each $y \in Y$, there exists an $x \in X$ such that $f(x) = y$. Then we say that f is a **Surjection** onto Y , and we call f **Surjective** onto Y . When Y is understood and the risk of misunderstanding is minimal, we may omit saying onto Y .

Definition 2.1.16 (Bijective). Let X and Y be sets and let $f : X \rightarrow Y$ be **Surjective** and **Injective**. Then we say that f is **Bijective**, or we say that f is a **Bijection**.

Definition 2.1.17 (Cardinality). Let $n \in \mathbb{N}$. We define

$$N_n = \{k \in \mathbb{N} | k \leq n\}$$

Let X be a set. Let $f : X \rightarrow N_n$ be a **Bijection**. Then, we say that X has **Cardinality** n and we write **Card**(X) = n . More generally, if there exists a **Bijection** between two sets Y and Z , then we write **Card**(Y) = **Card**(Z) and we say that they have the same

Cardinalities . Define $X_0 = \mathbb{N}$ and for $k \in \mathbb{N}$, define $X_{k+1} = 2^{X_k}$. Then for $k \in \mathbb{N}$, we define $\aleph_k = \mathbf{Card}(X_k)$. If $\mathbf{Card}(X) \in \mathbb{N}$, then we say that X is **Finite** . If $\mathbf{Card}(Z) \in \mathbb{N}$ or $\mathbf{Card}(Z) = \aleph_0$, then we say that Z is **Denumerable** . If $\mathbf{Card}(Y) = \aleph_0$, then we say that Y is **Countable** . If $\mathbf{Card}(W) = \aleph_k$ for $k \geq 1$, then we say that W is **Uncountable** . If $\mathbf{Card}(V) = \aleph_j$ for $j \in \mathbb{N}$, then we say that V is **Infinite** .

Definition 2.1.18 (Disjoint). Let X and Y be sets such that $X \cap Y = \emptyset$. Then we say that X and Y are **Disjoint** . Let $F = \{X_\alpha\}_{\alpha \in A}$ such that for each $\alpha, \beta \in A$ with $\alpha \neq \beta$, we have X_α is **Disjoint** to X_β . Then we say that F is **Disjoint** .

Definition 2.1.19 (Cover, Subcover). Let X be a set and let $Y = \{Y_\alpha\}_{\alpha \in A}$ such that

$$X \subset \bigcup_{\alpha \in A} Y_\alpha$$

Then we say that Y is a **Cover** for X or that Y **Covers** X . In the context of talking about a **Cover** , if every member of a **Cover** posses a certain property then we may say that the **Cover** has that property. If $Z \subset Y$ **Covers** X , then we call Z a **Subcover** of Y . One exception to this is that when talking about the **Cardinality** or **Disjointedness** of a **Cover** , we are talking about **Cover** itself, not each of its constituent sets.

Definition 2.1.20 (Partition). Let X be a set and $Y \subset 2^X$ be a **Disjoint Cover** for X . Then we call Y a **Partition** for X .

Definition 2.1.21 (Infinite Cartesian Product). Let $A \neq \emptyset$. For each $\alpha \in A$, let $X_\alpha \neq \emptyset$. Define

$$\prod_{\alpha \in A} X_\alpha = \left\{ f : A \rightarrow \bigcup_{\alpha \in A} X_\alpha \mid (\forall \alpha \in A)(f(\alpha) \in X_\alpha) \right\}$$

We call this the **Cartesian Product** of $\{X_\alpha\}_{\alpha \in A}$. For each $\alpha \in A$, we define

$$\pi_\alpha : \prod_{\alpha \in A} X_\alpha \rightarrow X_\alpha \quad \pi_\alpha(f) = f(\alpha)$$

We call π_α the α - **Projection Map** .

Definition 2.1.22 (Closure Under Unions). Let S be a set such that

$$\{S_\alpha \mid \alpha \in A\} \subset S \implies \bigcup_{\alpha \in A} S_\alpha \in S$$

for all index sets A . Then we say that S is **Closed Under Unions** or **Closed Under Aribtrary Unions** and that S possesses **Closure Under Unions** or **Closure Under Aribtrary Unions** . If this relation only holds when A is a **Countable** set then we say that S is **Closed Under Countable Unions** and that S possesses **Closure Under Countable Unions** . If this relation only holds when A is a **Finite** set then we say that S is **Closed Under Finite Unions** and that S possesses **Closure Under Finite Unions** .

Definition 2.1.23 (Closure Under Intersections). Let S be a set such that

$$\{S_\alpha | \alpha \in A\} \subset S \implies \bigcap_{\alpha \in A} S_\alpha \in S$$

for all index sets A . Then we say that S is **Closed Under Intersections** or **Closed Under Arbitrary Intersections** and that S possesses **Closure Under Intersections** or **Closure Under Arbitrary Intersections**. If this relation only holds when A is a **Countable** set then we say that S is **Closed Under Countable Intersections** and that S possesses **Closure Under Countable Intersections**. If this relation only holds when A is a **Finite** set then we say that S is **Closed Under Finite Intersections** and that S possesses **Closure Under Finite Intersections**.

Proposition 2.1.24. *Let X be a set. The following are true.*

- (i) *If X has the property that $\{y, z\} \subset X \implies y \cap z \in X$, then X possesses **Closure Under Finite Intersections**.*
- (ii) *If X has the property that $\{y, z\} \subset X \implies y \cup z \in X$, then X possesses **Closure Under Finite Unions**.*

Proof of 2.1.24. i. We use induction. Let M be the set of natural numbers for which X is closed under intersections of n sets. The intersection of a single set equals that set, so $1 \in M$. $2 \in M$ by direct application of the assumption of 2.1.24. i. Let $m \in M$. Let $\{x_i\}_{i=1}^{m+1} \subset X$. Then

$$\bigcap_{i=1}^{m+1} x_i = \left(\bigcap_{i=1}^m x_i \right) \cap x_{m+1} \in X$$

so $m+1 \in M$. Hence $M = \mathbb{N}$ and 2.1.24. i is proven. \square

Proof of 2.1.24. ii. We use induction. Let M be the set of natural numbers for which X is closed under unions of n sets. The union of a single set equals that set, so $1 \in M$. $2 \in M$ by direct application of the assumption of 2.1.24. ii. Let $m \in M$. Let $\{x_i\}_{i=1}^{m+1} \subset X$. Then

$$\bigcup_{i=1}^{m+1} x_i = \left(\bigcup_{i=1}^m x_i \right) \cup x_{m+1} \in X$$

so $m+1 \in M$. Hence $M = \mathbb{N}$ and 2.1.24. ii is proven. \square

2.1.2 Relations and Orderings

Definition 2.1.25 (**Reflexive**). Let $X \neq \emptyset$ be a set. Let R be a **Relation** on X . We say that R is **Reflexive** with respect to X if, or equivalently we say that R possesses **Reflexivity** with respect to X if $\{(a, a) | a \in X\} \subset R$. When X is understood, we may simply say that R is **Reflexive** or that R possesses **Reflexivity**.

Definition 2.1.26 (Transitive). Let $X \neq \emptyset$ be a set. Let R be a **Relation** on X . We say that R is **Transitive**, or equivalently we say that R possesses **Transitivity** if whenever $(a, b) \in R$ and $(b, c) \in R$, we also have $(a, c) \in R$.

Definition 2.1.27 (Preorder). Let $X \neq \emptyset$ be a set. Let R be a **Relation** on X . If R is **Reflexive** and **Transitive** then we call R a **Preorder** on X , or we equivalently call R a **Preordering** of X and we call (X, R) a **Preordered Set**.

Definition 2.1.28 (Comparable). Let (X, R) be a **Preordered Set**. We say that $x, y \in X$ are **Comparable** and that they possess **Comparability** if xRy or yRx .

Definition 2.1.29 (Symmetric). Let $X \neq \emptyset$ be a set. Let R be a **Relation** on X . We say that R is **Symmetric**, or equivalently we say that R possesses **Symmetry** if whenever aRb , we also have bRa .

Definition 2.1.30 (Anti-Symmetric). Let $X \neq \emptyset$ be a set. Let R be a **Relation** on X . We say that R is **Anti-Symmetric**, or equivalently we say that R possesses **Anti-Symmetry** if whenever aRb and bRa , we must have $a = b$.

Definition 2.1.31 (Maximal Element). Let $X \neq \emptyset$ be a set. Let R be an **Relation** on X . Let $Y \subset X$. Let $a \in Y$. We say that a is a **Maximal Element** of Y , or equivalently we say that a is a **Maximum** of Y if for every $b \in Y$, if aRb , then $a = b$. The Plural of **Maximum** is a **Maxima**, and we represent the set of **Maxima** of Y with respect to the relation R with $\text{Maxima}_R(Y)$, or if R is understood, we represent the set of **Maxima** of Y with $\text{Maxima}(Y)$.

Proposition 2.1.32 (Maximal Element unique if R is Anti-Symmetric). Let $X \neq \emptyset$ be a set. Let R be an **Anti-Symmetric Relation** on X . Let $Y \subset X$. Let a and b be each be a **Maximal Element** of Y . Then $a = b$.

Proof. Since $a \in \text{Maxima}(Y)$, $b \leq a$. Since $b \in \text{Maxima}(Y)$, $a \leq b$. By **Anti-Symmetry**, $b = a$. □

Definition 2.1.33 (Minimal Element). Let $X \neq \emptyset$ be a set. Let R be an **Relation** on X . Let $Y \subset X$. Let $a \in Y$. We say that a is a **Minimal Element** of Y , or equivalently we say that a is a **Minimum** of Y if for every $b \in Y$, if bRa , then we have $a = b$. The Plural of **Minimum** is **Minima**, and we represent the set of **Minima** of Y with respect to the relation R with $\text{Minima}_R(Y)$, or if R is understood, we represent the set of **Minima** of Y with $\text{Minima}(Y)$.

Proposition 2.1.34 (Minimal Element unique if R is Anti-Symmetric). Let $X \neq \emptyset$ be a set. Let R be an **Anti-Symmetric Relation** on X . Let $Y \subset X$. Let a and b be each be a **Minimal Element** of Y . Then $a = b$.

Proof. Since $a \in \text{Minima}(Y)$, $a \leq b$. Since $b \in \text{Minima}(Y)$, $b \leq a$. By **Anti-Symmetry**, $b = a$. \square

Definition 2.1.35 (Upper Bound). Let $X \neq \emptyset$ be a set. Let R be a **Relation** on X . Let $Y \subset X$. Let $a \in X$. We say that a is an **Upper Bound** for Y if for every $x \in Y$, we have xRa . If a is an **Upper Bound** then we also say that the set Y is **Bounded From Above** by a . We denote the set of **Upper Bounds** of Y with respect to the relation R with $\text{UpperBound}_R(Y)$. When R is understood, we denote this set with $\text{UpperBound}(Y)$.

Definition 2.1.36 (Lower Bound). Let $X \neq \emptyset$ be a set. Let R be a **Relation** on X . Let $Y \subset X$. Let $a \in X$. We say that a is an **Lower Bound** for Y if for every $x \in Y$, we have aRx . If a is an **Lower Bound** then we also say that the set Y is **Bounded From Below** by a . We denote the set of **Lower Bounds** of Y with respect to the relation R with $\text{LowerBound}_R(Y)$. When R is understood, we denote this set with $\text{LowerBound}(Y)$.

Definition 2.1.37 (Least Upper Bound). Let $X \neq \emptyset$ be a set. Let R be a **Relation** on X . Let $Y \subset X$. Let $a \in X$. We say that a is a **Least Upper Bound** of Y if $a \in \text{Minima}(\text{UpperBound}(Y))$. We denote the set of **Least Upper Bounds** for Y with $\text{LUB}(Y)$. If $b \in \text{LUB}(Y)$, then we also call b a **Supremum** of Y . The Plural of **Supremum** is **Suprema**. If $\text{LUB}(Y) = \{c\}$, then we write $c = \text{Sup}(Y)$.

Definition 2.1.38 (Greatest Lower Bound). Let $X \neq \emptyset$ be a set. Let R be a **Relation** on X . Let $Y \subset X$. Let $a \in X$. We say that a is a **Greatest Lower Bound** of Y if $a \in \text{Maxima}(\text{LowerBound}(Y))$. We denote the set of **Greatest Lower Bounds** for Y with $\text{GLB}(Y)$. If $b \in \text{GLB}(Y)$, then we also call b a **Infimum** of Y . The Plural of **Infimum** is **Infima**. If $\text{GLB}(Y) = \{c\}$, then we write $c = \text{Inf}(Y)$.

Definition 2.1.39 (Equivalence Relation). Let $X \neq \emptyset$ be a set. Let \cong be a **Preorder** on X . We say that \cong is an **Equivalence Relation** on X if it is **Symmetric**.

Definition 2.1.40 (Partial Order). Let $X \neq \emptyset$ be a set. Let \leq be a **Preorder** on X . We say that \leq is a **Partial Order** on X and we say that \leq is a **Partial Ordering** of X if \leq is **Anti-Symmetric**. Let \leq is a **Partial Order** on X , then we refer to the pair (X, \leq) as a **Partially Ordered Set**.

Definition 2.1.41 (Total Order). Let (X, R) be a **Partially Ordered Set** in which every pair of elements is **Comparable**. Then we call R a **Total Order** on X and we call (X, R) a **Totally Ordered Set**.

Definition 2.1.42 (Chain). Let (X, \leq) be a **Partially Ordered Set**. Let $A \subset X$ such that $(A, \leq \cap (A \times A))$ is a **Totally Ordered Set**. Then we call A a **Chain** in X .

Definition 2.1.43 (**Direction**). Let $X \neq \emptyset$ be a set. Let \leq be a **Preorder** on X . If every pair of elements in X has an **Upper Bound** with respect to \leq , then we call \leq is a **Direction** on X , , we call \leq is a **Directing** of X , and we call (X, \leq) is a **Directed Set** .

Definition 2.1.44 (**Section** of a **Directed Set**). Let (X, \leq) be a **Directed Set** . Let $x \in X$. We define

$$S(x, \leq) = \{y \in X | x \leq y\} \quad (2.1)$$

We call $S(x, \leq)$ the **Section** of \leq corresponding to $x \in X$.

Definition 2.1.45 (**Lattice** , **Join** , **Meet**). Let (X, \leq) be a **Partially Ordered Set** such that, for every $x, y \in X$, the set $\{x, y\}$ has both a **Supremum** and an **Infimum** . Then we call (X, \leq) **Lattice** . Furthermore, we call $\text{Sup}\{x, y\}$ the **Join** of x and y and we call $\text{Inf}\{x, y\}$ the **Meet** of x and y . If every nonempty subset of X has both a **Supremum** and **Infimum** then we call (X, \leq) a **Complete Lattice** .

Definition 2.1.46 (**Sequence**). Let X be a set. A **Sequence** in X is a **Function** $f : \mathbb{N} \rightarrow X$. If f is a **Sequence** in X and $f(n) = x_n$ for $n \in \mathbb{N}$, then we may refer to $\{x_n\}_{n \in \mathbb{N}}$ as the **Sequence** itself.

2.1.3 Nets

Definition 2.1.47 (**Net**). A **Net** is a **Function** mapping from a directed set (A, \leq) into another set X . If $f : A \rightarrow X$ is a **Net** such that for $\alpha \in A$ we have $f(\alpha) = x_\alpha$, then we may use the notation $\{x_\alpha\}_{\alpha \in A} \subset X$.

Definition 2.1.48 (**Section** of a **Net**). Let $X \neq \emptyset$, let (A, \leq) be a **Directed Set** and let $\sigma = \{x_\alpha\}_{\alpha \in A}$ be a **Net** in X . Let $\gamma \in A$. Let $S(\gamma, \leq)$ be the **Section** of \leq corresponding to γ . We define

$$\{x_\alpha | \alpha \in S(\gamma, \leq)\} \quad (2.2)$$

the **Section** of x_γ in σ .

Proposition 2.1.49 (Net Section). Let $X \neq \emptyset$ and let $\{x_\alpha\}_{\alpha \in A}$ be a **Net** in X . Let $\beta \leq \gamma \in A$. For $\alpha \in A$, let $S(\alpha)$ denote the **Section** of x_α in $\{x_\alpha\}_{\alpha \in A}$. Then $S(\beta) \subset S(\gamma)$.

Proof. Let $y \in S(\beta)$. Then $y = x_\tau$ for some $\beta \leq \tau \leq \gamma$. Hence, $y = x_\tau \in S(\gamma)$. □

Definition 2.1.50 (**Inductive Order**). Let (X, \leq) be a **Partially Ordered Set** . We say that \leq is an **Inductive Order** on X and we say that X is **Inductively Ordered** by \leq if each **Chain** in X has an **Upper Bound** in X . we also call (X, \leq) an **Inductively Ordered** set in this circumstance.

Theorem 2.1.51 (Zorns Lemma). An **Inductively Ordered** set has a **Maximum**

Remark 2.1.52. 2.1.51 Is equivalent to the axiom of choice

2.1.4 Filters

Definition 2.1.53 (**Filter**). Let $X \neq \emptyset$. Let $\mathcal{F} \subset 2^X$ satisfy the following.

- (i) $\mathcal{F} \neq \emptyset$.
- (ii) $\emptyset \notin \mathcal{F}$
- (iii) If $G_1 \in \mathcal{F}$ and $G_1 \subset G_2 \subset X$, then $G_2 \in \mathcal{F}$.
- (iv) If $\{G_1, G_2\} \subset \mathcal{F}$, then $G_1 \cap G_2 \in \mathcal{F}$.

Then we call \mathcal{F} a **Filter** on X .

Proposition 2.1.54. Let $X \neq \emptyset$ and let \mathcal{F} be a **Filter** on X . Let $B \subset X$. The following are true.

- (i) $X \in \mathcal{F}$.
- (ii) \mathcal{F} is **Closed Under Finite Intersections**
- (iii) The intersection of a collection of **Filters** on X is a **Filter** on X .
- (iv) \mathcal{F}_B defined by $\{U \cap B \mid U \in \mathcal{F}\}$ is a **Filter** on B if and only if $\emptyset \notin \mathcal{F}_B$.

Proof of 2.1.54 i. By 2.1.53. i, $\exists B \neq \emptyset \in \mathcal{F}$. Since $B \subset X \subset X$, by 2.1.53. iii, $X \in \mathcal{F}$, so 2.1.54 i is proven. \square

Proof of 2.1.54 ii. Direct application of 2.1.53. iv paired with 2.1.24. ii. \square

Proof of 2.1.54 iii. Let $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ be a collection of **Filters** on X . Define $\mathcal{F} = \bigcap_{\alpha \in A} \mathcal{F}_\alpha$. By 2.1.54 i, for each $\alpha \in A$, $X \in \mathcal{F}_\alpha$, so $X \in \mathcal{F}$. Hence \mathcal{F} satisfies 2.1.53. i. Furthermore, by 2.1.53. ii, for each $\alpha \in A$, $\emptyset \notin \mathcal{F}_\alpha$, so $\emptyset \notin \mathcal{F}$. Therefore \mathcal{F} satisfies 2.1.53. ii. Since $G_1 \in \mathcal{F}$ and $G_1 \subset G_2 \subset X$. Then for each $\alpha \in A$, $G_1 \in \mathcal{F}_\alpha$, so by 2.1.53. iii, $G_2 \in \mathcal{F}_\alpha$. Hence $G_2 \in \mathcal{F}$, so \mathcal{F} satisfies 2.1.53. iii. Finally, let $\{G_1, G_2\} \subset \mathcal{F}$. Then for each $\alpha \in A$, $\{G_1, G_2\} \subset \mathcal{F}_\alpha$, implying by 2.1.53. iv that $G_1 \cap G_2 \in \mathcal{F}_\alpha$, so $G_1 \cap G_2 \in \mathcal{F}$, implying \mathcal{F} satisfies 2.1.53. iv. This concludes the proof of this result. \square

Proof of 2.1.54 iv. Since \mathcal{F} satisfies 2.1.53. i, no matter what $\emptyset \neq \mathcal{F}_B$, so \mathcal{B} satisfies 2.1.53. i. If $G_1 \in \mathcal{F}_B$, then there is an $H_1 \in \mathcal{F}$ with $G_1 = H_1 \cap B$. If $G_1 \subset G_2 \subset B$, then $H_1 \cap B \subset G_2$ and $H_1 \cap (X \setminus B) \subset (X \setminus B)$, so $H_1 \subset G_2 \cup (X \setminus B)$, which implies $G_2 \cup (X \setminus B) \in \mathcal{F}$ by 2.1.53. iii. By construction, then, $G_2 = (G_2 \cup (X \setminus B)) \cap B \in \mathcal{F}_B$. Hence \mathcal{F}_B satisfies 2.1.53. iii, no matter what. Next, if $\{G_1, G_2\} \subset \mathcal{F}_B$, then there are $H_1, H_2 \in \mathcal{F}$ with $G_i = H_i \cap B$. Since \mathcal{F} satisfies 2.1.53. iv, $H_1 \cap H_2 \in \mathcal{F}$. This implies by construction that $B \cap (H_1 \cap H_2) \in \mathcal{F}_B$, but $B \cap (H_1 \cap H_2) = G_1 \cap G_2$, so \mathcal{F}_B satisfies 2.1.53. iv. Finally, \mathcal{F}_B satisfies 2.1.53. ii if and only if it satisfies that same assumption, concluding the proof of this result. \square

Definition 2.1.55 (**Coarser** , **Finer**). Let X be a set and let \mathcal{F}_1 and \mathcal{F}_2 be **Filters** on X such that $\mathcal{F}_1 \subset \mathcal{F}_2$. Then we say that \mathcal{F}_1 is **Coarser** than \mathcal{F}_2 and we say that \mathcal{F}_2 is **Finer** than \mathcal{F}_1 . Let $A \subset X$ be a collection of filters and let $\mathcal{F}_2 \in A$ be **Finer** than every element of A . Then we say that \mathcal{F}_2 is the **Finest** element of A . Let $\mathcal{F}_3 \in A$ be **Coarser** than every element of A . Then we say that \mathcal{F}_3 is the **Coarsest** in A . **Filter Fineness** defines a **Partial Ordering** on the collection of **Filters** on X , where $\mathcal{F}_1 \leq \mathcal{F}_2$ if \mathcal{F}_2 is a **Finer** than \mathcal{F}_1 . A **Maximum** of **Filter Fineness** is called an **Ultrafilter** on X .

Proposition 2.1.56. Let X be a set and $\emptyset \neq A \subset 2^X$. Then there is a **Filter** on X which contains A if and only if any **Finite** intersection of elements of A is nonempty. Furthermore, if there is a **Filter** containing A , then there is a **Coarsest** containing A , and it is given by all sets containing some **Finite** intersection of elements of A .

Proof. The given condition is necessary by a combination of 2.1.54 i and 2.1.53. ii. For sufficiency, let K be the collection of finite intersections of elements of A . Define

$$\mathcal{K} = \{F \cup Y \mid F \in K \wedge K \subset X\} \quad (2.3)$$

Then $A \subset \mathcal{K}$. Since $A \neq \emptyset$, $\mathcal{K} \neq \emptyset$, so \mathcal{K} satisfies 2.1.53. i. Since **Finite** intersections of elements of A are nonempty, $\emptyset \notin K$, implying $\emptyset \notin \mathcal{K}$, so 2.1.53. ii. Now, let $P \in \mathcal{K}$ and let $P \subset Q \subset X$. Then there exists $L_P \in K$ and $Y_P \subset X$ such that $P = L_P \cup Y_P$, and $Q = L_P \cup (Y_P \cup Q) \in \mathcal{K}$, so 2.1.53. iii holds for \mathcal{K} . Finally, let $G_1, G_2 \in \mathcal{K}$. Then $G_i = U_i \cup P_i$ for $U_i \in K$ and $P_i \subset X$. By definition of \mathcal{K} , there are a $\{G_1^j\}_{j=1}^{n_1} \subset A$ and $\{G_2^j\}_{j=1}^{n_2} \subset A$ with $U_i = \bigcap_{j=1}^{n_i} G_i^j$. Clearly $U_1 \cap U_2$, being a finite subset of elements of A , is an element of \mathcal{K} . Furthermore,

$$\begin{aligned} U_1 \cap U_2 &\subset (U_1 \cap U_2) \cup ((U_1 \cap P_2) \cup (U_2 \cap P_1) \cup (P_1 \cap P_2)) \\ &= (U_1 \cup P_1) \cap (U_2 \cup P_2) \\ &= G_1 \cap G_2 \end{aligned}$$

so since \mathcal{K} satisfies 2.1.53. iii, $G_1 \cap G_2 \in \mathcal{K}$, so 2.1.53. iv applies for \mathcal{K} , so \mathcal{K} is a **Filter**. By 2.1.54 ii, any **Filter** containing A would contain \mathcal{K} . By 2.1.53. iii, any **Filter** containing A would contain \mathcal{K} , so any **Filter** on X containing A would be **Finer** than \mathcal{K} , and so \mathcal{K} is the **Finest Filter** containing A . Hence, 2.1.56 is proven. \square

Proposition 2.1.57 (Filter Order Facts). Let $X \neq \emptyset$. The following are true.

- (i) Let \mathcal{F} be a **Filter** on X and let $A \subset X$. Then there is a **Filter** containing A on X which is **Finer** than \mathcal{F} if and only if $A \cap U \neq \emptyset$ for each $U \in \mathcal{F}$.
- (ii) Let $K = \{\mathcal{F}_\alpha\}_{\alpha \in A}$ be a collection of **Filters** on X . There exists a **Filter** on X which is **Finer** than each \mathcal{F}_α if and only if, for every **Finite** subset $\{\mathcal{F}_{\alpha_i}\}_{i=1}^n \subset K$, for each $\{U_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{F}_{\alpha_i}$, $\bigcap_{i=1}^n U_i \neq \emptyset$.
- (iii) The union of a **Chain** of **Filters** on X is a **Filter** on X which is **Finer** than each element of the **Chain**.

- (iv) The intersection of a **Chain** of **Filters** on X is a **Filter** on X which is **Coarser** than each element of the **Chain**.
- (v) **Filter Fineness** is an **Inductive Order** on the collection of **Filters** on X .
- (vi) **Filter Coarseness** is an **Inductive Order** on the collection of **Filters** on X .

Proof of 2.1.57. i. By 2.1.54 ii the collection of **Finite** intersections of elements of $\mathcal{K} := \{A\} \cup \mathcal{F}$ is given by

$$\mathcal{L} := \mathcal{F} \cup \{A \cap F \mid F \in \mathcal{F}\}$$

By 2.1.53. ii, $\emptyset \notin \mathcal{F}$. Hence, $\emptyset \notin \mathcal{L}$ if and only if $\emptyset \notin \{A \cap F \mid F \in \mathcal{F}\}$. By 2.1.56, $\emptyset \notin \mathcal{L}$ is equivalent to there existing a **Filter** on X containing \mathcal{K} , and so there existing a **Filter** on X containing \mathcal{K} is equivalent to $\emptyset \notin \{A \cap F \mid F \in \mathcal{F}\}$. \square

Proof of 2.1.57. ii. By 2.1.54 ii, each \mathcal{F}_α **Closed Under Finite Intersections**. Therefore, the collection of all **Finite** intersections of elements of

$$\bigcup_{\alpha \in A} \mathcal{F}_\alpha$$

is given by the collection intersections single element each of a **Finite** collection of \mathcal{F}_α 's. An application of 2.1.56 finishes the result. \square

Proof of 2.1.57. iii. Let $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ be a **Chain** of **Filters** on X and let

$$\mathcal{F} = \bigcup_{\alpha \in A} \mathcal{F}_\alpha$$

Since each $\mathcal{F}_\alpha \neq \emptyset$, $\mathcal{F} \neq \emptyset$, so \mathcal{F} satisfies 2.1.53. i. Furthermore, $\emptyset \notin \mathcal{F}_\alpha$ for any $\alpha \in A$, so \mathcal{F} satisfies 2.1.53. ii. Let $G \in \mathcal{F}$ and let $G \subset H \subset X$. Then for some $\alpha \in A$, $G \in \mathcal{F}_\alpha$. Since \mathcal{F}_α is a **Filter**, by 2.1.53. iii, $H \in \mathcal{F}_\alpha \subset \mathcal{F}$, so $H \in \mathcal{F}$. Hence \mathcal{F} satisfies 2.1.53. iii. Finally, let $G_1, G_2 \in \mathcal{F}$. Then there are $\alpha_1, \alpha_2 \in A$ such that $G_i \in \mathcal{F}_{\alpha_i}$ for $i \in \{1, 2\}$. Either $\mathcal{F}_{\alpha_1} \subset \mathcal{F}_{\alpha_2}$ or $\mathcal{F}_{\alpha_2} \subset \mathcal{F}_{\alpha_1}$, so without loss of generality, let $\mathcal{F}_{\alpha_1} \subset \mathcal{F}_{\alpha_2}$. Then $G_1 \in \mathcal{F}_{\alpha_2}$, implying $G_1 \cap G_2 \in \mathcal{F}_{\alpha_2} \subset \mathcal{F}$, so $G_1 \cap G_2 \in \mathcal{F}$, so \mathcal{F} satisfies 2.1.53. iv. Hence \mathcal{F} is a **Filter** on X which is **Finer** than each \mathcal{F}_α . \square

Proof of 2.1.57. iv. Define

$$\mathcal{F} = \bigcap_{\alpha \in A} \mathcal{F}_\alpha$$

$\emptyset \notin \mathcal{F}_\alpha$ for any $\alpha \in A$, so $\emptyset \notin \mathcal{F}$, implying \mathcal{F} satisfies 2.1.53. ii. Furthermore, $X \in \mathcal{F}_\alpha$ for each $\alpha \in A$, so $X \in \mathcal{F}$, implying \mathcal{F} satisfies 2.1.53. i. If $G \in \mathcal{F}$ and $G \subset H \subset X$, then for every $\alpha \in A$, $G \in \mathcal{F}_\alpha$. This implies, that by 2.1.53. iii, that for every $\alpha \in A$, $H \in \mathcal{F}_\alpha$. Hence $H \in \mathcal{F}$, so \mathcal{F} satisfies 2.1.53. iii. Finally, let $G_1, G_2 \in \mathcal{F}$. Then $G_1, G_2 \in \mathcal{F}_\alpha$ for every $\alpha \in A$, so for every $\alpha \in A$, by 2.1.53. iv, $G_1 \cap G_2 \in \mathcal{F}_\alpha$ for every $\alpha \in A$, hence $G_1 \cap G_2 \in \mathcal{F}$. Hence, \mathcal{F} is a **Filter** on X . Also, $\mathcal{F} \subset \mathcal{F}_\alpha$ for every $\alpha \in A$, and so is **Coarser** than every \mathcal{F}_α . \square

Proof of 2.1.57. v. Direct application of 2.1.57. iii to the collection of all **Chains** of **Filters** in X . \square

Proof of 2.1.57. vi. Direct application of 2.1.57. iv to the collection of all **Chains** of **Filters** on X . \square

Filter Base

Definition 2.1.58 (Subbasis). Let X be a set and $A \subset X$ such that

(i) $\emptyset \neq A$.

(ii)

$$\emptyset \notin K := \left\{ \bigcap_{i=1}^n A_i \mid \{A_i\}_{i=1}^n \subset A \wedge n \in \mathbb{N} \right\} \quad (2.4)$$

Define

$$\mathcal{K} = \{U \cup P \mid U \in K \wedge P \subset X\}$$

We call \mathcal{K} the **Filter** on X **Generated By** A . and we call A a **Subbasis** for G . By 2.1.56, \mathcal{K} is in fact a **Filter**, and is the **Coarsest Filter** on X containing A .

Definition 2.1.59 (Filter Base). Let $X \neq \emptyset$. Let $\mathcal{B} \subset 2^X$ such that

(i) $\emptyset \neq \mathcal{B}$.

(ii) $\emptyset \notin \mathcal{B}$.

(iii) If $\{U, V\} \subset \mathcal{B}$, then there exists a $W \in \mathcal{B}$ with $W \subset U \cap V$.

Then we call \mathcal{B} a **Filter Base** on X . By 2.1.60, the **Filter Generated By** a **Filter Base** A is given by $\{U \subset X \mid (\exists Y \subset A)(Y \subset U)\}$.

If A, B are **Filter Bases** on X and they **Generate** the same **Filter**, then we call them **Equivalent**.

Proposition 2.1.60. Let X be a set and let $A \subset 2^X$. and define $\mathcal{U} = \{U \subset X \mid (\exists a \in A)(a \subset U)\}$. The following are equivalent.

1. A is a **Filter Base** on X .

2. \mathcal{U} is a **Filter** on X .

\implies . Suppose A is a **Filter Base** on X . By 2.1.59. ii, $\emptyset \notin A$, so $\emptyset \notin \mathcal{U}$, implying that \mathcal{U} satisfies 2.1.53. ii. Also, by 2.1.59. i $\emptyset \neq A \subset \mathcal{U}$, so \mathcal{U} satisfies 2.1.53. i. That \mathcal{U} satisfies 2.1.53. iii is obvious. Finally, if $G_1, G_2 \in \mathcal{U}$, then there exists $U_1, U_2 \in A$ such that $A_i \subset G_i$. By 2.1.59. iii, there is $B \in A$ satisfying $B \subset U_1 \cap U_2 \subset G_1 \cap G_2$, so $G_1 \cap G_2 \in \mathcal{U}$, implying \mathcal{U} is a **Filter** on X . \square

\impliedby . If $A = \emptyset$, then $\mathcal{U} = \emptyset$, so A failing 2.1.59. i implies \mathcal{U} fails 2.1.53. i. If $\emptyset \in A$, then $\emptyset \in \mathcal{U}$, so A failing 2.1.59. ii implies \mathcal{U} fails 2.1.53. ii. Finally, if \mathcal{A} fails 2.1.59. iii, then we can find $B, C \in A$ such that $B \cap C \notin \mathcal{U}$, implying \mathcal{U} fails 2.1.53. iv. Hence necessity has been proven. \square

Remark 2.1.61. If A is a **Filter Base** on X , then \mathcal{U} defined in 2.1.60 is the **Generated By** A .

Proposition 2.1.62 (FilterBaseFacts). *Let $X \neq \emptyset$. Let \mathcal{F} and \mathcal{G} be **Filters** on X . Let F be a **Filter Base** for \mathcal{F} and let G be a **Filter Base** for \mathcal{G} . The following are true.*

- (i) *The collection of **Finite** intersections of a **Subbasis** A for \mathcal{F} forms a **Filter Base** for \mathcal{F} .*
- (ii) *$B \subset \mathcal{F}$ is a **Filter Base** for \mathcal{F} if and only if for each $Y \in \mathcal{F}$, there exists $b \in B$ with $b \subset Y$.*
- (iii) *\mathcal{F} is **Finer** than \mathcal{G} if and only if for each $g \in G$, there is an $f \in F$ with $f \subset g$.*
- (iv) *F is **Equivalent** to G if and only if for each $f \in F$ there is a $g \in G$ with $g \subset f$ and for each $g \in G$ there is an $f \in F$ with $f \subset g$.*

Proof of 2.1.62. i. Let

$$\mathcal{B} = \left\{ \bigcap_{i=1}^n A_i \mid \{A_i\}_{i=1}^n \subset A \wedge n \in \mathbb{N} \right\} \quad (2.5)$$

By 2.1.58. ii, $\emptyset \notin \mathcal{B}$, so \mathcal{B} satisfies 2.1.59. ii. By 2.1.58. i, $\emptyset \neq A \subset \mathcal{B}$, so \mathcal{B} satisfies 2.1.59. i. Since $\emptyset \notin \mathcal{B}$ is **Closed Under Finite Intersections**, if $U, V \in \mathcal{B}$, then $\emptyset \neq U \cap V \in \mathcal{B}$, so \mathcal{B} can be seen to satisfy 2.1.59. iii. Hence \mathcal{B} is a **Filter Base**. Since $A \subset \mathcal{B}$, \mathcal{B} is a **Filter Base** for a **Finer** than \mathcal{F} . However, since $A \subset \mathcal{F}$, by 2.1.54 ii, $\mathcal{B} \subset \mathcal{F}$. Hence \mathcal{F} is the **Filter Generated By** \mathcal{B} . \square

Proof of 2.1.62. ii. (\Leftarrow). Let \mathcal{G} denote the **Filter Generated By** B . Then since $B \subset \mathcal{F}$, $\mathcal{G} \subset \mathcal{F}$. If for each $Y \in \mathcal{F}$ there exists $b \in B$ with $b \subset Y$, then

$$\mathcal{F} \subset \{U \subset X \mid (\exists b \in B)(b \subset U)\} \subset \mathcal{G} \subset \mathcal{F} \quad (2.6)$$

so that $\mathcal{F} = \mathcal{G}$ and $\{U \subset X \mid (\exists b \in B)(b \subset U)\} = \mathcal{F}$ is a **Filter** on X . Hence, by 2.1.60, B is a **Filter**.

(\Rightarrow). If B is a **Filter Base** for \mathcal{F} , then by 2.1.60, $\mathcal{F} = \{Y \subset X \mid (\exists b \in B)(b \subset Y)\}$ so the desired property holds \square

Proof of 2.1.62. iii. Let \mathcal{F} be finer than \mathcal{G} . Then by applying 2.1.60 $\mathcal{G} \subset \mathcal{F} = \{U \subset X \mid (\exists f \in F)(f \subset U)\}$, which is the desired result in one direction. The other direction is equivalent again applying 2.1.60 to claim $\mathcal{G} \subset \{U \subset X \mid (\exists f \in F)(f \subset U)\}$. \square

Proof of 2.1.62. iv. This is a result of two applications of 2.1.62. iii, one in each direction. \square

Proposition 2.1.63 (**Net Sections** form a **Filter Base**). *Let $X \neq \emptyset$ and let $\sigma = \{x_\alpha\}_{\alpha \in A}$ be a **Net** in X . For each $\alpha \in A$, denote with $S(\sigma, \alpha)$ the **Section** of x_α in σ . Define*

$$\mathcal{B} = \{S(\sigma, \alpha) \mid \alpha \in A\}$$

*Then \mathcal{B} is a **Filter Base** on X .*

Proof. Since σ is a **Net**, (A, \leq) is a **Directed Set**, implying that (A, \leq) is a **Preordered Set**. Hence, \leq is **Reflexive** so that if $\alpha \in A$, then $x_\alpha \in S(\sigma, \alpha)$. Hence, $\emptyset \notin \mathcal{B}$, so \mathcal{B} satisfies 2.1.59. ii. Furthermore, since (A, \leq) is a **Preordered Set**, A is nonempty, so $\emptyset \neq \mathcal{B}$, implying \mathcal{B} satisfies 2.1.59. i. Finally, let $U, V \in \mathcal{B}$. Then we can find $u, v \in A$ such that $U = S(\sigma, u)$, $V = S(\sigma, v)$. Since A is a **Directed Set**, there exists $w \in A$ with $u \leq w$ and $v \leq w$. Hence by 2.1.49, $S(\sigma, w) \subset S(\sigma, u) \cap S(\sigma, v)$. Since $S(\sigma, w) \in \mathcal{B}$, \mathcal{B} satisfies 2.1.59. iii, and we're done. \square

Ultrafilters

Definition 2.1.64 (**Ultrafilter**). Let $X \neq \emptyset$. An **Ultrafilter** on X is a **Maximum** of the relation of **Filter Fineness** on X .

Remark 2.1.65 (**Ultrafilter Existence**). Let \mathcal{F} be a **Filter** on $X \neq \emptyset$. By 2.1.57. iii Not only is **Filter Fineness** an **Inductive Order** on the set of **Filters** of X , (as stated in 2.1.57. v), but **Filter Fineness** is also an **Inductive Order** on the set of **Filters Finer** than \mathcal{F} . Hence by 2.1.51, \mathcal{F} is contained in an **Ultrafilter** on X . A **Filter Base** for an **Ultrafilter** is called an **Ultrafilter Base**.

Proposition 2.1.66 (Ultrafilter Facts). Suppose the following

- (I) $X \neq \emptyset$.
- (II) \mathcal{F} is an **Ultrafilter** on X .
- (III) \mathcal{G} is a **Filter** on X .
- (IV) \mathcal{K} is a **Subbasis** on X .
- (V) $\mathcal{M} = \mathcal{K} \cup \{K \subset X \mid X \setminus K \in \mathcal{K}\}$.

Then the following are true

- (i) If $\{A, B\} \subset 2^X$ and $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.
- (ii) If $\{A_i\}_{i=1}^n \subset 2^X$ such that $\bigcup_{i=1}^n A_i \in \mathcal{F}$, then for some $j \in \{1, \dots, n\}$, $A_j \in \mathcal{F}$.
- (iii) If $\mathcal{M} = 2^X$, then \mathcal{K} is an **Ultrafilter** on X .
- (iv) \mathcal{G} is the intersection of all **Ultrafilters** on X which contain \mathcal{G} .

Proof of 2.1.66. i. We use contradiction. Suppose $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$. Define $\mathcal{T} = \{G \in 2^X \mid A \cup G \in \mathcal{F}\}$. Then $B \in \mathcal{T}$, so \mathcal{T} satisfies 2.1.53. i. Furthermore, if $G_1 \in \mathcal{T}$ and $G_1 \subset G_2 \subset X$, then by 2.1.53. iii, $A \cup G_1 \subset A \cup G_2 \in \mathcal{F}$. Hence $G_2 \in \mathcal{T}$ so \mathcal{T} satisfies 2.1.53. iii. Let $G_3, G_4 \in \mathcal{T}$

$$A \cup (G_3 \cap G_4) = (A \cup G_3) \cap (A \cup G_4) \in \mathcal{F}$$

so that $G_3 \cap G_4 \in \mathcal{T}$ and therefore \mathcal{T} satisfies 2.1.53. iv. Finally since $A \notin \mathcal{F}$, $\emptyset \notin \mathcal{T}$, so \mathcal{T} satisfies 2.1.53. ii, and therefore \mathcal{T} is a **Filter** on X . Trivially, $\mathcal{F} \subset \mathcal{T}$ but since $B \in \mathcal{T} \setminus \mathcal{F}$, this contradicts 2.1.66. II Hence the result holds. \square

Proof of 2.1.66. ii. We use induction on n . Obviously the result holds for $n = 1$ and by 2.1.66. i, the result also holds for $n = 2$. Suppose the result holds for $n = k$. Let $\{A_i\}_{i=1}^{k+1} \subset 2^X$ such that $\bigcup_{i=1}^{k+1} A_i \in \mathcal{F}$. Then since the result holds for $n = 2$, either $A_{k+1} \in \mathcal{F}$ or $\bigcup_{i=1}^k A_i \in \mathcal{F}$. Since the result holds for $n = k$, either $A_{k+1} \in \mathcal{F}$ or $A_i \in \mathcal{F}$ for $i \in \{1, \dots, k\}$. Hence the result holds for $n = k + 1$. Hence the result holds in general. \square

Proof of 2.1.66. iii. I first prove that $\mathcal{M} = 2^X$, paired with 2.1.66. IV and 2.1.66. V implies \mathcal{K} is a **Filter** on X . By 2.1.66. I, $\mathcal{M} \neq \emptyset$. By 2.1.66. V, then $\mathcal{K} \neq \emptyset$, so \mathcal{K} satisfies 2.1.53. i. By 2.1.58. ii, $\emptyset \notin \mathcal{K}$, so \mathcal{K} satisfies 2.1.53. ii. Let $G_1 \in \mathcal{K}$ and let $G_1 \subset G_2 \subset X$. Then $G_1 \cap (X \setminus G_2) = \emptyset$, which by 2.1.58. ii implies $X \setminus G_2 \notin \mathcal{K}$. Since $\mathcal{M} = 2^X$, we conclude $G_2 \in \mathcal{K}$, so \mathcal{K} satisfies 2.1.53. iii. Finally, let $G_1, G_2 \in \mathcal{K}$. By assumption, either $G_1 \cap G_2 \in \mathcal{K}$ or $X \setminus (G_1 \cap G_2) \in \mathcal{K}$. If $X \setminus (G_1 \cap G_2) \in \mathcal{K}$, then by 2.1.58. ii, $G_1 \cap G_2 \cap (X \setminus (G_1 \cap G_2)) \neq \emptyset$, a contradiction. Hence, $G_1 \cap G_2 \in \mathcal{K}$, so that 2.1.53. iv is satisfied by \mathcal{K} . Hence \mathcal{K} is a **Filter** on X . By 2.1.65, there is an **Ultrafilter** \mathcal{L} containing \mathcal{K} . If \mathcal{K} is not an **Ultrafilter**, then $\exists B \in \mathcal{L} \setminus \mathcal{K}$. Since $\mathcal{M} = 2^X$, $X \setminus B \in \mathcal{K} \subset \mathcal{L}$, implying $\emptyset = B \cap (X \setminus B) \in \mathcal{L}$, contradicting 2.1.53. ii, thus \mathcal{K} is an **Ultrafilter**. \square

Proof of 2.1.66. iv. Let $\{\mathcal{P}_\alpha\}_{\alpha \in A}$ be the collection of all **Ultrafilters** on X containing \mathcal{G} . Define $\mathcal{P} = \bigcap_{\alpha \in A} \mathcal{P}_\alpha$. I must show $\mathcal{P} = \mathcal{G}$. By 2.1.54 iii, \mathcal{P} is a **Filter** on X , and by construction $\mathcal{G} \subset \mathcal{P}$. Let $B \subset X \setminus \mathcal{G}$. Then, by 2.1.53. iii, there is no $G \in \mathcal{G}$ with $G \subset A$. Hence, for each $G \in \mathcal{G}$, $G \cap (X \setminus A) \neq \emptyset$. Therefore, we can apply 2.1.57. i to claim that there is a **Filter** \mathcal{G}_1 on X which is **Finer** than \mathcal{G} satisfying $X \setminus A \in \mathcal{G}_1$. By 2.1.65, there is an $\alpha \in A$ such that $X \setminus A \in \mathcal{G}_1 \subset \mathcal{P}_\alpha$, so $X \setminus A \in \mathcal{P}$. Since \mathcal{P} satisfies 2.1.53. ii and 2.1.53. iv, $X \setminus A \in \mathcal{P}$ implies $A \notin \mathcal{P}$ and so $\mathcal{P} \subset \mathcal{G}$. This completes the proof. \square

Induced Filters

Definition 2.1.67 (**Induced**). Let $X \neq \emptyset$, \mathcal{F} be a **Filter** on \mathcal{F} and $A \subset X$ such that

$$\mathcal{F}_A := \{U \cap A \mid U \in \mathcal{F}\}$$

satisfies $\emptyset \notin \mathcal{F}_A$. Then by 2.1.54 iv \mathcal{F}_A is a **Filter** on A which we call the **Filter Induced** by \mathcal{F} .

Proposition 2.1.68 (**Induced Filter facts**). Let $X \neq \emptyset$. Let \mathcal{F} be a **Filter** on X . Let \mathcal{B} be a **Filter Base** for \mathcal{F} . Let \mathcal{G} be an **Ultrafilter** on X . Let $A \subset X$. Define $\mathcal{F}_A = \{A \cap U \mid U \in \mathcal{F}\}$. Define $\mathcal{B}_A = \{A \cap U \mid U \in \mathcal{B}\}$. Define $\mathcal{G}_A = \{A \cap U \mid U \in \mathcal{G}\}$. The following are true

- (i) If \mathcal{F}_A is a **Filter** on A , then \mathcal{B}_A is a **Filter Base** for \mathcal{F}_A .
- (ii) \mathcal{G}_A is a **Filter** on A if and only if $A \in \mathcal{G}$. In this case, \mathcal{G}_A is an **Ultrafilter** on A .

Proof of 2.1.68. i. Let $U \in \mathcal{F}_A$. Then there exists $V \in \mathcal{F}$ such that $U = A \cap V$. Since \mathcal{B} is a **Filter Base** for \mathcal{F} , by 2.1.62. ii, there exists $B \in \mathcal{B}$ satisfying $B \subset V$. Then $B \cap A \subset A \cap V = U$. But $B \cap A \in \mathcal{B}_A$, so since $\mathcal{B}_A \subset \mathcal{F}_A$, we can apply 2.1.62. ii to claim that \mathcal{B}_A is a **Filter Base** for \mathcal{F}_A . \square

Proof of 2.1.68. ii. Even if \mathcal{G} was merely a **Filter** on X , by 2.1.53. ii and ?? $A \in \mathcal{G}$ is sufficient to guarantee that \mathcal{G}_A is a **Filter** on A . Now suppose \mathcal{G}_A is a **Filter** on A . Then $A \cap U \neq \emptyset$ for $U \in \mathcal{G}$. Since \mathcal{G} is an **Ultrafilter** on X , we can apply maximality with 2.1.57. i, to see that $A \in \mathcal{G}$. Finally, if $P \subset A$ satisfies $P \notin \mathcal{G}_A$, then $P \notin \mathcal{G}$. Since \mathcal{G} is an **Ultrafilter**, by 2.1.57. i, $P \cap U = \emptyset$ for some $U \in \mathcal{G}$. This implies $P \cap (U \cap A) = \emptyset$, and since $U \cap A \in \mathcal{G}_A$, we can apply 2.1.57. i to conclude that there is no **Filter Finer** than \mathcal{G}_A on A which contains P . Since $P \subset A$ was arbitrary, \mathcal{G}_A is an **Ultrafilter** on A . \square

Direct and Inverse Images of a Filter Base

Proposition 2.1.69 (Direct Filter Image). *Suppose the following.*

1. $X, Y \neq \emptyset$
2. $f : X \rightarrow Y$ is **Surjective**.
3. For $i \in \{1, 2\}$, \mathcal{B}_i is a **Filter Base** for a **Filter** \mathcal{F}_i on X .
4. \mathcal{F}_2 is **Finer** than \mathcal{F}_1 .
5. \mathcal{K} is an **Ultrafilter Base** on X .

Then the following are true.

- (i) $f(\mathcal{F}_1)$ is a **Filter** on Y .
- (ii) $f(\mathcal{B}_1)$ is a **Filter Base** for $f(\mathcal{F}_1)$.
- (iii) $f(\mathcal{F}_2)$ is a **Finer** than $f(\mathcal{F}_1)$.
- (iv) $f(\mathcal{K})$ is an **Ultrafilter Base** on Y .

Proof of 2.1.69. i. Since $\emptyset \notin \mathcal{F}_1$, $\emptyset \notin f(\mathcal{F}_1)$, so $f(\mathcal{F}_1)$ satisfies 2.1.53. ii. Since $\emptyset \neq \mathcal{F}_1$, $f(\mathcal{F}_1) \neq \emptyset$, so $f(\mathcal{F}_1)$ satisfies 2.1.53. i. Let $G_1 \in f(\mathcal{F}_1)$. Let $G_1 \subset G_2 \subset Y$. Then, since f is **Surjective**, $U \subset f^{-1}(G_2)$, which by 2.1.53. iii implies $f^{-1}(G_2) \in \mathcal{F}_1$. Hence $G_2 = f(f^{-1}(G_2)) \in f(\mathcal{F}_1)$, so $f(\mathcal{F}_1)$ satisfies 2.1.53. iii. Finally, if $G_1, G_2 \in f(\mathcal{F}_1)$, then there are $K_1, K_2 \in \mathcal{F}_1$ with $f(K_i) = G_i$ for $i \in \{1, 2\}$. By 2.1.53. iv, $K_1 \cap K_2 \in \mathcal{F}_1$. Also, $f(K_1 \cap K_2) \subset f(K_1) \cap f(K_2)$, so by 2.1.53. iii, $f(K_1) \cap f(K_2) \in f(\mathcal{F}_1)$. Hence $f(\mathcal{F}_1)$ satisfies 2.1.53. iv and is therefore a **Filter** on Y . \square

Proof of 2.1.69. ii. By 2.1.59. i, $\emptyset \neq \mathcal{B}_1$, so $\emptyset \neq f(\mathcal{B}_1)$, and thus $f(\mathcal{B}_1)$ satisfies 2.1.59. i. By 2.1.59. ii, $\emptyset \notin \mathcal{B}_1$, so $\emptyset \notin f(\mathcal{B}_1)$, implying \mathcal{B}_1 satisfies 2.1.59. ii. Finally, let $U_1, U_2 \in f(\mathcal{B}_1)$. Then there exists $V_i \in \mathcal{B}_1$ with $f(V_i) = U_i$. Then by 2.1.59. iii, $V_1 \cap V_2 \in \mathcal{B}_1$, and $f(V_1 \cap V_2) \subset f(V_1) \cap f(V_2)$, and $f(V_1 \cap V_2) \in f(\mathcal{B}_1)$ by construction, so $f(\mathcal{B}_1)$ satisfies 2.1.59. iii, and therefore $f(\mathcal{B}_1)$ is a **Filter Base** on Y . Now, if $V \in f(\mathcal{F}_1)$, then by definition, there exists $U \in \mathcal{F}_1$ with $f(U) = V$. By 2.1.62. ii, there exists a $b \in \mathcal{B}_1$ with $b \subset U$. This implies $f(b) \subset f(U) = V$, but $f(b) \in f(\mathcal{B}_1)$. Furthermore, since $\mathcal{B}_1 \subset \mathcal{F}_1$, $f(\mathcal{B}_1) \subset f(\mathcal{F}_1)$, so we can apply 2.1.62. ii to claim that $f(\mathcal{B}_1)$ is a **Filter Base** for $f(\mathcal{F}_1)$. \square

Proof of 2.1.69. iii. If $\mathcal{F}_1 \subset \mathcal{F}_2$ then $f(\mathcal{F}_1) \subset f(\mathcal{F}_2)$. An invocation of 2.1.69. ii finishes the result. \square

Proof of 2.1.69. iv. Let \mathcal{G} denote the **Ultrafilter** for which \mathcal{K} is an **Ultrafilter Base**. Let $U \subset Y$. Since f is **Surjective**,

$$X = f^{-1}(U) \cup (X \setminus f^{-1}(U)) = f^{-1}(U) \cup f^{-1}(f(X) \setminus U) = f^{-1}(U) \cup f^{-1}(Y \setminus U) \quad (2.7)$$

and by 2.1.54 i, we have $f^{-1}(U) \cup f^{-1}(Y \setminus U) \in \mathcal{G}$. Since \mathcal{G} is an **Ultrafilter**, by ??, either $f^{-1}(U) \in \mathcal{G}$ or $f^{-1}(Y \setminus U) \in \mathcal{G}$. This implies either $U \in f(\mathcal{G})$ or $Y \setminus U \in f(\mathcal{G})$. By ??, $f(\mathcal{G})$ is an **Ultrafilter** on Y . An application of 2.1.69. ii completes the result. \square

Proposition 2.1.70 (Inverse Filter Image). *Suppose the following*

1. $X, Y \neq \emptyset$
2. $f : X \rightarrow Y$.
3. \mathcal{B} is a **Filter Base** for a **Filter** \mathcal{F} on Y .
4. $\mathcal{F}_{f(X)} := \{U \cap f(X) \mid U \in \mathcal{F}\}$

Then the following are true

- (i) $f^{-1}(\mathcal{B})$ is a **Filter Base** on X if and only if $\emptyset \notin f^{-1}(\mathcal{B})$.
- (ii) If $f^{-1}(\mathcal{B})$ is a **Filter Base** on X , then $f(f^{-1}(\mathcal{B}))$ is a **Filter Base** for a **Filter** on Y finer than \mathcal{F} .
- (iii) If $f(f^{-1}(\mathcal{B}))$ is a **Filter Base** for a **Filter** \mathcal{G} on Y , then $\mathcal{G}_{f(X)} = \mathcal{F}_{f(X)}$.

Proof of 2.1.70. i. Necessity of $\emptyset \notin f^{-1}(\mathcal{B})$ is obvious by 2.1.59. ii. For sufficiency, suppose $\emptyset \notin f^{-1}(\mathcal{B})$. Then $f^{-1}(\mathcal{B})$ satisfies 2.1.59. ii trivially. Furthermore, by 2.1.59. i, $\mathcal{B} \neq \emptyset$, so $f^{-1}(\mathcal{B}) \neq \emptyset$, so $f^{-1}(\mathcal{B})$ satisfies 2.1.59. i. Finally, let $U_1, U_2 \in f^{-1}(\mathcal{B})$. Then there exist $V_1, V_2 \in \mathcal{B}$ such that $U_i = f^{-1}(V_i)$. By 2.1.59. iii, there exists $W \in \mathcal{B}$ such that $W \subset V_1 \cap V_2$, and $f^{-1}(W) \in f^{-1}(\mathcal{B})$. Clearly,

$$f^{-1}(W) \subset f^{-1}(V_1 \cap V_2) = f^{-1}(V_1) \cap f^{-1}(V_2) = U_1 \cap U_2$$

Hence $f^{-1}(\mathcal{B})$ satisfies 2.1.59. iii \square

Proof of 2.1.70. ii. If $f^{-1}(\mathcal{B})$ is a **Filter Base** on X , then we can leverage 2.1.69. ii to claim that $f(f^{-1}(\mathcal{B}))$ is a **Filter Base** for a **Filter** on $f(X)$, and therefore also a **Filter Base** for a **Filter** on Y . In particular, $f(f^{-1}(\mathcal{B}))$ is a **Filter Base**. Furthermore, if $b \in \mathcal{B}$, then $f(f^{-1}(b)) \in f(f^{-1}(\mathcal{B}))$ and $f(f^{-1}(b)) = f(X) \cap b \subset b$, so by 2.1.62. iii, $f(f^{-1}(\mathcal{B}))$ is a **Filter Base** for a **Finer Filter** than \mathcal{F} . \square

Proof of 2.1.70. iii. This is because given the assumptions, $f(f^{-1}(\mathcal{B})) = \{f(X) \cap \mathcal{B} \mid \mathcal{B} \in \mathcal{B}\}$, which lets us apply ?? \square

Filter Products

Definition 2.1.71 (**Product Filter**). Suppose the following

1. $A \neq \emptyset$.
2. $\{X_\alpha\}_{\alpha \in A}$ is a collection of nonempty sets.
3. For each $\alpha \in A$, \mathcal{F}_α is a **Filter** on X_α .
4. For each $\gamma \in A$, $\pi_\gamma : \prod_{\alpha \in A} X_\alpha \rightarrow X_\gamma$ represents the **Projection Map**.

Then we define the **Filter** on $\prod_{\alpha \in A} X_\alpha$ **Generated By**

$$\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$$

to be the **Product Filter** on $\prod_{\alpha \in A} (X_\alpha, \mathcal{F}_\alpha)$

Proposition 2.1.72. Suppose the following

1. $A \neq \emptyset$.
2. $\{X_\alpha\}_{\alpha \in A}$ is a collection of nonempty sets.
3. For each $\alpha \in A$, \mathcal{B}_α is a **Filter Base** for a **Filter** \mathcal{F}_α on X_α .
4. For each $\alpha \in A$, \mathcal{G}_α is a **Subbasis** which **Generates** \mathcal{F}_α .
5. For each $\gamma \in A$, $\pi_\gamma : \prod_{\alpha \in A} X_\alpha \rightarrow X_\gamma$ represents the **Projection Map**.
6. $\mathcal{G} := \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{G}_\alpha)$.
7. \mathcal{B} is the collection of finite intersections of elements of $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}_\alpha)$
8. \mathcal{F} is the **Product Filter** on $\prod_{\alpha \in A} (X_\alpha, \mathcal{F}_\alpha)$

Then the following are true

- (i) \mathcal{F} is well defined.
- (ii) \mathcal{G} is a **Subbasis** for \mathcal{F} .
- (iii) \mathcal{B} is a **Filter Base** for \mathcal{F} .
- (iv) \mathcal{F} is the **Coarsest Filter** on $\prod_{\alpha \in A} X_\alpha$ such that for every $\alpha \in A$, $\pi_\alpha(\mathcal{F}) = \mathcal{F}_\alpha$.

Proof of 2.1.72. i. Since $A \neq \emptyset$, there exists a $\gamma \in A$. By 2.1.53. i, $\mathcal{F}_\gamma \neq \emptyset$. Hence

$$\emptyset \neq \pi_\gamma^{-1}(\mathcal{F}_\gamma) \subset \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$$

so that $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ satisfies 2.1.58. i. Additionally, let $\{U_i\}_{i=1}^n \subset \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$. Then for each $1 \leq i \leq n$, there exists an $\alpha_i \in A$ and a $V_i \in \mathcal{F}_{\alpha_i}$ such that $U_i = \pi_{\alpha_i}^{-1}(V_i)$. For $i, j \in \{1, \dots, n\}$, define $i \equiv j \iff \alpha_i = \alpha_j$. It is clear that \equiv is an **Equivalence Relation** on $\{1, \dots, n\}$. Let $k \in \{1, \dots, n\}/\equiv$. Then

$$\begin{aligned} \bigcap_{p \in k} U_p &= \bigcap_{p \in k} \pi_{\alpha_p}^{-1}(V_{\alpha_p}) \\ &= \bigcap_{p \in k} \pi_{\alpha_{\min(k)}}^{-1}(V_p) \\ &= \pi_{\alpha_{\min(k)}}^{-1} \left(\bigcap_{p \in k} V_p \right) \end{aligned}$$

Since $\{V_p\}_{p \in k} \subset \mathcal{F}_{\alpha_{\min(k)}}$, 2.1.54. ii, $\bigcap_{p \in k} V_p \in \mathcal{F}_{\alpha_{\min(k)}}$, so by 2.1.53. ii, $\bigcap_{p \in k} V_p \neq \emptyset$. For

each $k \in \{1, \dots, n\}/\equiv$, let $x_k \in \pi_{\alpha_{\min(k)}}^{-1} \left(\bigcap_{p \in k} V_p \right)$. Partition $A = A' \cup (A \setminus A')$ where $A' = \{\alpha_{\min(k)} | k \in \{1, \dots, n\}/\equiv\}$. For each $\alpha \in A \setminus A'$, Define $F_\alpha = X_\alpha$. For each $\alpha_{\min(k)} \in A'$, define $F_{\alpha_{\min(k)}} = \{x_k\}$. Then by the axiom of choice,

$$\emptyset \neq \prod_{\alpha \in A} F_\alpha \subset \bigcap_{i=1}^n U_i$$

so that $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ satisfies 2.1.58. ii. Hence $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ is a **Subbasis** on X , and the **Filter** that it **Generates** is \mathcal{F} . \square

Proof of 2.1.72. iii. Since

$$\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}_\alpha) \subset \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha) \quad (2.8)$$

and by 2.1.72. i, $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}_\alpha)$ generates a **Filter** \mathcal{F}_0 on $\prod_{\alpha \in A} X_\alpha$. In Particular, the above equation implies $\mathcal{F}_0 \subset \mathcal{F}$. Denote Let $\alpha \in A$. By 2.1.70. i, both $\pi_\alpha^{-1}(\mathcal{F}_\alpha)$ and $\pi_\alpha^{-1}(\mathcal{B}_\alpha)$ are **Filter Bases** for a **Filter** on the product set. Let $U \in \pi_\alpha^{-1}(\mathcal{F}_\alpha)$. Then, by 2.1.62. ii, there is a $b \in \mathcal{B}_\alpha$ such that $b \subset U$. Hence $f^{-1}(b) \subset f^{-1}(U)$. Which implies by 2.1.62. iii that $\pi_\alpha^{-1}(\mathcal{B}_\alpha)$ is a **Filter Base** for a **Finer Filter** than $\pi_\alpha^{-1}(\mathcal{F}_\alpha)$. This implies that any **Filter** containing $\pi_\alpha^{-1}(\mathcal{B}_\alpha)$ also contains $\pi_\alpha^{-1}(\mathcal{F}_\alpha)$. Since, for each $\alpha \in A$, $\pi_\alpha^{-1}(\mathcal{B}_\alpha) \subset \mathcal{F}_0$ for each $\alpha \in A$, $\pi_\alpha^{-1}(\mathcal{F}_\alpha) \subset \mathcal{F}_0$. Hence $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha) \subset \mathcal{F}_0$, so by 2.1.58, $\mathcal{F} \subset \mathcal{F}_0$. Hence $\mathcal{F} = \mathcal{F}_0$. Also, by 2.1.62. i, since $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}_\alpha)$ is by construction a **Subbasis** for \mathcal{F}_0 , Since \mathcal{B} is the collection of **Finite** intersections of elements of $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{B}_\alpha)$, \mathcal{B} is a **Filter Base** for $\mathcal{F}_0 = \mathcal{F}$. \square

Proof of 2.1.72. ii. For each $\alpha \in A$, let \mathcal{K}_α the collection of **Finite** intersections of elements of \mathcal{G}_α . By ??, for each $\alpha \in A$, \mathcal{K}_α is a **Filter Base** for \mathcal{F}_α . An application of 2.1.72. iii implies that \mathcal{F} is the **Filter** generated by $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{K}_\alpha)$. But each

$$\pi_\alpha^{-1}(\mathcal{K}_\alpha) = \left\{ \bigcap_{i=1}^n U_\alpha \mid U_\alpha \in \pi_\alpha^{-1}(\mathcal{G}_\alpha) \right\}$$

So the collection of **Finite** intersections of elements of $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{K}_\alpha)$ equals the collection of finite intersections of elements of $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{G}_\alpha)$, they generate the same **Filter** ! \square

Proof of 2.1.72. iv. I first show that \mathcal{F} has the described property, and then show that it is the **Coarsest** on the product set with the property. Fix $\gamma \in A$. Clearly, by **Surjectivity** , $\mathcal{F}_\gamma = \pi_\gamma(\pi_\gamma^{-1}(\mathcal{F}_\gamma)) \subset \pi_\gamma(\mathcal{F})$. Now, let $U \in \pi_\gamma(\mathcal{F})$. Then $U = \pi_\gamma(V)$ for some $V \in \mathcal{F}$. Since $\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ is a **Subbasis** for \mathcal{F} , by 2.1.62. i and 2.1.62. ii, there is a collection $\{U_i\}_{i=1}^n \subset \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ such that $\bigcap_{i=1}^n U_i \subset V$. Let each $U_i \in \pi_{\alpha_i}^{-1}(\mathcal{F}_{\alpha_i})$. If, for all $i \in \{1, \dots, n\}$, $\alpha_i \neq \gamma$ then $U = X_\gamma \in \mathcal{F}_\gamma$. Otherwise, let V' be the intersection of all U_i 's such that $\alpha_i = \gamma$. Then $V' \in \pi_\gamma^{-1}(\mathcal{F}_\gamma)$ and $V' \subset V$. Hence $\pi_\gamma(V') \in \mathcal{F}_\gamma$ and $\pi_\gamma(V') \subset \pi_\gamma(V) = U$, so by 2.1.53. iii, $U \in \mathcal{F}_\gamma$, implying $\pi_\gamma(\mathcal{F}) \subset \mathcal{F}_\gamma$. Since the inclusion goes both ways, $\pi_\gamma(\mathcal{F}) = \mathcal{F}_\gamma$. \square

2.1.5 Point Set Topology

Open Sets, Closed Sets, and Neighborhoods

Definition 2.1.73 (Topological Space). Let $X \neq \emptyset$ be a set and let $\{\emptyset, X\} \subset \mathcal{T} \subset 2^X$ such that

- (i) $X \in \mathcal{T}$.
- (ii) $\emptyset \in \mathcal{T}$.
- (iii) \mathcal{T} is **Closed Under Arbitrary Unions** .
- (iv) \mathcal{T} is **Closed Under Finite Intersections** .

Then we call \mathcal{T} a **Topology** on X and we call (X, \mathcal{T}) a **Topological Space** .

Definition 2.1.74 (**Set-Open** , **Set-Closed**). Let (X, \mathcal{T}) be a **Topological Space** , and let $A \in \mathcal{T}$. We say that A is **Set-Open** in (X, \mathcal{T}) (or **Set-Open** in X or **Set-Open** in \mathcal{T} or simply **Set-Open** in cases where confusion won't result) and that A possesses **Set-Openness** . We say that $X \setminus A$ is **Set-Closed** and that $X \setminus A$ possesses **Closedness** .

Definition 2.1.75 (Continuous). Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be **Topological Spaces** . We say that a function $f : X \rightarrow Y$ is **Continuous** and that it exhibits **Continuity** with respect to \mathcal{T}_1 and \mathcal{T}_2 $f^{-1}(\mathcal{T}_Y) \subset \mathcal{T}_X$. We may make the **Topologies** explicit by writing $f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$, in which case we just say that f is **Continuous** or that f possesses **Continuity** .

Definition 2.1.76 (Open). Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be **Topological Spaces** . We say that $f : X \rightarrow Y$ is **Open** if $f(U)$ is **Set-Open** in (Y, \mathcal{T}_Y) for every **Set-Open** $U \in (X, \mathcal{T}_X)$.

Definition 2.1.77 (Homeomorphism). Let X, \mathcal{T}_X and Y, \mathcal{T}_Y be **Topological Spaces** . Let $f : X \rightarrow Y$ such be a **Continuous** , **Open Bijection** . Then we say that f is a **Homeomorphism** from X to Y and we say that X and Y are **Homeomorphic** .

Definition 2.1.78 (Coarse, Fine). Let X be a set. Let $\mathcal{T}_1, \mathcal{T}_2$ be **Topology** on X such that $\mathcal{T}_1 \subset \mathcal{T}_2$. In this case, we say that \mathcal{T}_1 is more **Coarse** than \mathcal{T}_2 , that \mathcal{T}_1 is **Coarser** than \mathcal{T}_2 , that \mathcal{T}_2 is more **Fine** than \mathcal{T}_1 , that \mathcal{T}_2 is **Finer** than \mathcal{T}_1 , and we write $\mathcal{T}_1 \leq_{\text{TopFine}(X)} \mathcal{T}_2$ and we write $\mathcal{T}_2 \leq_{\text{TopCoarse}(X)} \mathcal{T}_1$.

Let K denote the collection of topologies on X . Let $A \subset K$. If one exists, a **Maximum** of A with respect to $\leq_{\text{TopFine}(X)}$ is called the **Finest** topology in A . If one exists, a **Maximum** of A with respect to $\leq_{\text{TopCoarse}(X)}$ is called the **Coarsest** topology in A .

Since \subset defines a **Partial Order** on the power set of X , **Fineness** defines a **Partial Order** on the set of **Topology** of X .

The intersection of any 2 topologies on X is a topology on X , so that $\leq_{\text{TopCoarse}(X)}$ is a **Direction** on X .

Definition 2.1.79 (Weak Topology). Let X be a set. For each $\alpha \in A$, let $(Y_\alpha, \mathcal{T}_\alpha)$ be a **Topological Space** , and let $\phi_\alpha : X \rightarrow (Y_\alpha, \mathcal{T}_\alpha)$. Let \mathcal{T} be the **Coarsest** possible **Topology** on X such that for each $\alpha \in A$, $\phi_\alpha : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}_\alpha)$ is **Continuous** . We call \mathcal{T} the **Weak Topology** on X induced by $\{\phi_\alpha\}_{\alpha \in A}$. In the literature, (Schaefer, Topological Vector Spaces 2nd ed), the weak topology is also sometime s referred to as the Projective Topology or as the Kernel Topology

Definition 2.1.80 (Inductive Topology). Let X be a set and for each $\alpha \in A$, let $(Y_\alpha, \mathcal{T}_\alpha)$ be a **Topological Space** . Furthermore, for each $\alpha \in A$, let $\phi_\alpha : (Y, \mathcal{T}_\alpha) \rightarrow X$. Let \mathcal{T} be the **Finest** topology on X for which each ϕ_α is **Continuous** . fWe call \mathcal{T} the **Inductive Topology** on X induced by $\{\phi_\alpha\}_{\alpha \in A}$.

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Definition 2.1.81 (Subspace Topology). Let (X, \mathcal{T}_X) be a **Topological Space** , Let $Y \subset X$, and let f be the **Insertion Function** of Y into X . We call the **Weak Topology** on Y generated by f which we will denote here with \mathcal{T}_Y , the **Subspace Topology** of Y relative to (X, \mathcal{T}_X) . We call (Y, \mathcal{T}_Y) the **Subspace Topological Space** . Unless otherwise specified, when referring to a subset of a **Topological Space** , we consider that subset as being a **Topological Space** which is endowed with the **Subspace Topology** , and when we say that a subset of a **Topological Space** has a particular (Topological) property which has thus far only been defined for a **Topological Space** , we mean that the **Subspace Topological Space** has that property.

Neighborhood

Definition 2.1.82 (**Neighborhood** , **Neighborhood Filter**). Let (X, \mathcal{T}) be a **Topological Space** . Let $A \subset B \subset C \subset X$ and let B be **Set-Open** in (X, \mathcal{T}) . Then we call C a **Neighborhood** of A in (X, \mathcal{T}) . If $x \in X$, then we call a **Neighborhood** of $\{x\}$ a **Neighborhood** of x . We denote the collection of all **Neighborhoods** of $x \in X$ with $\mathcal{UT}(x)$. and we call this the By 2.1.83, is a **Filter** on X , and se we call this the **Neighborhood Filter** of \mathcal{T} at x .

Proposition 2.1.83 (**Neighborhood Filter** is a **Filter**). Let (X, \mathcal{T}) be a **Topological Space** and let $x \in X$. Then $\mathcal{UT}(x)$, the **Neighborhood Filter** of x , is in fact a **Filter** on X .

Proof. For part 1 of 2.1.53. ii , since $x \notin \emptyset$, $\emptyset \notin \mathcal{UT}(x)$ If $\{G_1, G_2\} \subset \mathcal{UT}(x)$ with $G_1 \cap G_2 \neq \emptyset$ then there are **Set-Open** $x \in U_i \subset G_i$ and we have $x \in U_1 \cap U_2 \subset U_1 \cap U_2 \subset G_1 \cap G_2$ with $U_1 \cap U_2 \in \mathcal{T}$. Hence, 2.1.53. iv holds. 2.1.53. iii is obvious \square

Definition 2.1.84 (**Discrete Topology** , **Indiscrete Topology**). Let X be a set. We call $\{X, \emptyset\}$ the **Indiscrete Topology** on X and we call 2^X the **Discrete Topology** on X .

Definition 2.1.85 (**Accumulation Point** , **Closure** , **Interior** , **Boundary**). Let (X, \mathcal{T}) be a **Topological Space** . Let $A \subset X$. We define the following.

1. $A' = \{x \in X | (\forall U \in \mathcal{UT}(A))((U \setminus A) \cap \{x\} \neq \emptyset)\}$
2. $\overline{A} = A \cup A'$
3. $\partial(A) = \overline{A} \cap \overline{X \setminus A}$
4. $\overset{\circ}{A} = A \setminus \overline{X \setminus A}$

We call an element of A' an **Accumulation Point** of A . We call \overline{A} the **Closure** of A . We call $\overset{\circ}{A}$ the **Interior** of A . We call $\partial(A)$ the **Boundary** of A .

Definition 2.1.86 (**SubBasis**). Let $X \neq \emptyset$ and let $B \subset 2^X$. We denote the **Coarsest Topology** on X containing B with $\mathcal{T}_X(B)$. We say that B is a **SubBasis** for $\mathcal{T}_X(B)$ and we call $\mathcal{T}_X(B)$ the **Topology** on X **Generated By** B .

Proposition 2.1.87 (Characterization Of Generated Topology). Let $X \neq \emptyset$ and $F \subset 2^X$. Define

$$\mathcal{T}_{Prop} = \left\{ \bigcup_{\alpha \in A} \bigcap_{i=1}^{N_\alpha} U_{i,\alpha} | (\forall \alpha \in A) ((N_\alpha \in \mathbb{N}) \wedge ((\forall i \in \{1, \dots, N_\alpha\}) (U_{i,\alpha} \in F))) \right\} \cup \{X, \emptyset\}$$

Then $\mathcal{T}_X(F) = \mathcal{T}_{Prop}$.

Proof. We first show that \mathcal{T}_{Prop} is a **Topology** on X . For **Closure Under Unions**, Let $B \neq \emptyset$ and $\{B_\beta\}_{\beta \in B} \subset \mathcal{T}_{Prop}$. Then for each $\beta \in B$, we can find A_β such that for each $\alpha_\beta \in A_\beta$, there is an $N_{\alpha_\beta} \in \mathbb{N}$ such that for each $i \in \{1, \dots, N_{\alpha_\beta}\}$, $U_{i, \alpha_\beta} \in F$ and

$$B_\beta = \bigcup_{\alpha \in A_\beta} \bigcap_{i=1}^{N_{\alpha_\beta}} U_{i, \alpha_\beta} \quad (2.9)$$

Hence, we can write

$$\begin{aligned} \bigcup_{\beta \in B} B_\beta &= \bigcup_{\beta \in B} \bigcup_{\alpha_\beta \in A_\beta} \bigcap_{i=1}^{N_{\alpha_\beta}} U_{i, \alpha_\beta} \\ &= \bigcup_{\alpha_\beta \in \bigcup_{\beta \in B} A_\beta} \bigcap_{i=1}^{N_{\alpha_\beta}} U_{i, \alpha_\beta} \in \mathcal{T}_{Prop} \end{aligned}$$

For **Closure Under Finite Intersections**, let $N \in \mathbb{N}$ and $\{B_j\}_{j=1}^N \subset \mathcal{T}_{Prop}$. Then for each $j \in \{1, \dots, N\}$, there is an A_j such that for each $\alpha_j \in A_j$, there is an $N_{\alpha_j} \in \mathbb{N}$ such that for each $i \in \{1, \dots, N_{\alpha_j}\}$, $U_{i, \alpha_j} \in F$ and

$$\begin{aligned} \bigcap_{j=1}^N B_j &= \bigcap_{j=1}^N \bigcup_{\alpha_j \in A_j} \bigcap_{i=1}^{N_{\alpha_j}} U_{i, \alpha_j} \\ &= \bigcup_{\{\alpha_j\}_{j=1}^N \in \prod_{j \in \{1, \dots, N\}} A_j} \left(\bigcap_{j=1}^N \bigcap_{i=1}^{N_{\alpha_j}} U_{i, \alpha_j} \right) \in \mathcal{T}_{Prop} \end{aligned}$$

By construction, $X \in \mathcal{T}_{Prop}$ and $\emptyset \in \mathcal{T}_{Prop}$, so \mathcal{T}_{Prop} is in fact a **Topology** on X . By taking the union over the intersection of a single element, we have $F \subset \mathcal{T}_{Prop}$, so that $\mathcal{T}_X(F) \subset \mathcal{T}_{Prop}$. Furthermore, $\mathcal{T}_X(F)$ is closed under finite intersections and arbitrary unions so that it must contain \mathcal{T}_{Prop} . Hence, equality holds. □

Definition 2.1.88 (Basis). Let (X, \mathcal{T}) be a **Topological Space** and let $B \subset \mathcal{T}$ such that each element of \mathcal{T} can be written as a union of elements of B . Then we call B a **Basis** for \mathcal{T} .

Proposition 2.1.89. Let (X, \mathcal{T}) be a **Topological Space** and let $\mathcal{G} \subset \mathcal{T}$ such that $\{\emptyset, X\} \subset \mathcal{G}$. The following conditions are equivalent

1. For every **Set-Open** U , for every $x \in U$, there exists an $G_x \in \mathcal{G}$ such that $x \in G_x \subset U$.
2. \mathcal{G} is a **Basis** for \mathcal{T} .

1 \implies 2. Let $U \in \mathcal{T}$. Then we can write $U = \bigcup_{x \in U} G_x$, implying that \mathcal{G} is a **Basis**. \square

2 \implies 1. Let U is **Set-Open**, then since \mathcal{G} is a **Basis**, there is a $\{G_\alpha\}_{\alpha \in A} \subset \mathcal{G}$ such that $U = \bigcup_{\alpha \in A} G_\alpha$. Hence, if $x \in U$, then $x \in G_\alpha$ for some $\alpha \in A$, and obviously $G_\alpha \subset U$, so 1 holds and we're done. \square

Proposition 2.1.90 (Basis Of Generated Topology). *Let $X \neq \emptyset$ and let $B \subset 2^X$. Then B is a **Basis** for $\mathcal{T}_X(B)$ if and only if the following hold*

1. $X \in B$

2. $\emptyset \in B$.

3. For each $U, V \in B$, For each $x \in U \cap V$, there is a $W \in B$ with $x \in W \subset U \cap V$.

Proof. I first claim that it is sufficient to show that any finite intersection of elements of B can be written as a union of elements of B . By Induction, proving for a binary intersection is sufficient. Hence, let $U, V \in B$ with $U \cap V \neq \emptyset$. Then for each $x \in U \cap V$, by assumption, there exists a $W_x \in B$ such that $x \in W_x \subset U \cap V$. Hence, we can write

$$U \cap V \subset \bigcup_{x \in U \cap V} W_x \subset U \cap V$$

showing that finite intersections of elements of B can be written as unions of elements of B . Hence, by 2.1.87, $\mathcal{T}_X(B)$ consists of exactly the unions of elements of B , finishing one direction. For the other direction, if B is a **Basis** for $\mathcal{T}_X(B)$, then by 2.1.89, since $\mathcal{T}_X(B)$ contains finite intersections of elements of B , the given properties hold. \square

Definition 2.1.91 (**Neighborhood Basis**). *Let (X, \mathcal{T}) be a **Topological Space** and let $x \in X$. Let $F \subset \mathcal{T}$ such that for each $U \in \mathcal{T}$ with $x \in U$, there exists $f \in F$ with $f \subset U$. Further, let $x \in G$ for each $G \in F$. Then we call F a **Neighborhood Basis** for \mathcal{T} at x .*

Definition 2.1.92 (**Compact**). *We say that a **Topological Space** is **Compact** if every **Set-Open Cover** for X has a **Finite Subcover**.*

Definition 2.1.93 (**Closed**). *Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be **Topological Spaces**. We say that $f : X \rightarrow Y$ is **Closed** if $f(K)$ is **Set-Closed** in (Y, \mathcal{T}_Y) for every **Set-Closed** $K \in (X, \mathcal{T}_X)$.*

Definition 2.1.94 (**Compact**). *Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be **Topological Spaces**. We say that $f : X \rightarrow Y$ is **Compact** if $f(K)$ is **Compact** in (Y, \mathcal{T}_Y) for every **Compact** $K \in (X, \mathcal{T}_X)$.*

Definition 2.1.95 (**Product Topology**). *Let $A \neq \emptyset$. For each $\alpha \in A$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a **Topological Space**. We call the **Weak Topology** on $\prod_{\alpha \in A} X_\alpha$ induced by $\{\pi_\alpha : \prod_{\alpha \in A} X_\alpha \rightarrow (X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ the **Product Topology**.*

Countability Axioms

Definition 2.1.96 (**First Countable**). Let (X, \mathcal{T}) be a **Topological Space** . We say that X is **First Countable** if for each $x \in X$, there is a **Countable Neighborhood Basis** for \mathcal{T} at X .

Definition 2.1.97 (**Second Countable**). A **Topological Space** which permits a **Countable Basis** is called **Second Countable**

Definition 2.1.98 (**Dense**). Let (X, \mathcal{T}) be a **Topological Space** and let $A \subset X$. We say that A is **Dense** in X if $\bar{A} = X$.

Definition 2.1.99 (**Separable**). We say that a **Topological Space** which permits a **Countable Dense** subset is **Separable** .

Definition 2.1.100 (**Lindelof**). A **Topological Space** in which every **Set-Open Cover** permits a **Countable Subcover** is called a **Lindelof** space.

Separation Axioms

Uniformities

Definition 2.1.101 (**Uniformity**). Let X be a st and let \mathcal{W} be a **Filter** on $X \times X$ such that

1. For each $W \in \mathcal{W}$, $\Delta(X) \subset W$.
2. $W \in \mathcal{W} \implies W^{-1} \in \mathcal{W}$.
3. For each $W \in \mathcal{W}$, there exists $V \in \mathcal{W}$ such that $V \circ V \subset W$.

Then we call \mathcal{W} a **Uniformity** on X . Furthermore, if $W \in \mathcal{W}$, then we call W an **Entourage** of W .

Definition 2.1.102 (Homeomorphism).

2.1.6 Neighborhood Filter Of A Point

Definition 2.1.103 (Relation of Equal **Neighborhood Filters**). Let (Z, \mathcal{T}_Z) be a **Topological Space** . Define the relation $\cong \subset Z \times Z$ by setting, for $x, y \in Z$,

$$x \cong y \iff \mathcal{U}_{\mathcal{T}_Z}(x) = \mathcal{U}_{\mathcal{T}_Z}(y) \quad (2.10)$$

We call \cong the **Relation Of Equal Neighborhood Filters** on (Z, \mathcal{T}_Z)

Proposition 2.1.104 (**Relation Of Equal Neighborhood Filters** on) . The **Relation Of Equal Neighborhood Filters** on \cong on a **Topological Space** (Z, \mathcal{T}_Z) forms an **Equivalence Relation** on Z .

Proof. Let $x \in (Z, \mathcal{T}_Z)$. Then $\mathcal{U}_{\mathcal{T}_Z}(x) = i \mathcal{U}_{\mathcal{T}_Z}(x)$, so $x \cong x$. Thus \cong is **Reflexive**.

Let $x, y \in (Z, \mathcal{T}_Z)$. Suppose $x \cong y$. Then $\mathcal{U}_{\mathcal{T}_Z}(x) = \mathcal{U}_{\mathcal{T}_Z}(y)$, so trivially $\mathcal{U}_{\mathcal{T}_Z}(y) = i \mathcal{U}_{\mathcal{T}_Z}(x)$, and thus $y \cong x$. Hence, \cong is **Symmetric**.

Let $x, y, z \in (Z, \mathcal{T}_Z)$. Let $x \cong y$ and $y \cong z$. Then, $\mathcal{U}_{\mathcal{T}_Z}(x) = i \mathcal{U}_{\mathcal{T}_Z}(y) = \mathcal{U}_{\mathcal{T}_Z}(z)$ so that $x \cong z$. Thus \cong is **Transitive**. Since \cong is **Reflexive**, **Symmetric**, and **Transitive**, it is an **Equivalence Relation**. □

2.1.7 Equivalence Relations, Quotient Sets, and Quotient Maps

Definition 2.1.105 (Equivalence Class). Let $X \neq \emptyset$. Let \cong be an **Equivalence Relation** defined on X . Let $x \in X$. We define the set $[x]_{\cong}$ by

$$[x]_{\cong} = \{y \in X \mid y \cong x\} \quad (2.11)$$

We call $[x]_{\cong}$ the **Equivalence Class** of x in (X, \cong) .

Proposition 2.1.106 (Equivalence Classes Partition). Let $X \neq \emptyset$. Let \cong be an **Equivalence Relation** defined on X . Let $x, y \in X$. The following statements are equivalent.

1. $[x]_{\cong} \cap [y]_{\cong} \neq \emptyset$
2. $x \cong y$
3. $[x]_{\cong} = [y]_{\cong}$
4. $[x]_{\cong} \subset [y]_{\cong}$
5. $[y]_{\cong} \subset [x]_{\cong}$

Proof That 1 \implies 2. Suppose $M := [x]_{\cong} \cap [y]_{\cong} \neq \emptyset$. Then there exists $z \in M$. Then $z \cong x$, so by **Symmetry**, $x \cong z$. But by **Transitivity**, pair with $z \cong y$, we conclude $x \cong y$. □

Proof That 2 \implies 4. Let $x \cong y$ and let $z \in [x]_{\cong}$. Then $z \cong x \cong y$, so $z \cong y$ and $z \in [y]_{\cong}$. Since z was arbitrary, we're done. □

Proof That 2 \implies 5. Let $x \cong y$. By **Symmetry**, $y \cong x$, so by (2 \implies 4), we are done. □

Proof That 2 \implies 3. Since 2 \implies 4 and 2 \implies 5 and 5 and 4 together imply 3, we have this. □

Proof That 5 \implies 1. Let $[y]_{\cong} \subset [x]_{\cong}$. Then $y \in [y]_{\cong} = [y]_{\cong} \cap [x]_{\cong}$. Hence 1 holds. □

Definition 2.1.107 (Quotient Set). Let $X \neq \emptyset$. Let \cong be an **Equivalence Relation** defined on X . We define the set X/\cong by

$$X/\cong = \{[x]_{\cong} : x \in X\} \quad (2.12)$$

We call X/\cong the **Quotient Set** of X under the relation \cong .

Remark 2.1.108 (**Quotient Set** forms a **Partition**). 2.1.106 , paired with the fact that $x \in [x]_{\cong}$, implies that X/\cong is a **Partition** of X .

Definition 2.1.109 (Quotient Map). Let $X \neq \emptyset$. Let \cong be an **Equivalence Relation** on X . Let X/\cong be the **Quotient Set** of X with respect to the relation \cong . Define $T : X \rightarrow X/\cong$ by setting, for each $x \in X$,

$$T(x) = [x] \quad (2.13)$$

We call T the **Quotient Map** of X under \cong .

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Proposition 2.1.110 (**Quotient Map** is **Surjective**). Let $X \neq \emptyset$. Let \cong be an **Equivalence Relation** on X . Let $T : X \rightarrow X/\cong$ be the **Quotient Map** of X under the **Relation** \cong . Then T is a **Surjection**.

Proof. Let $K \in Z/\cong$. Then for some $x \in Z$, $K = [x]$. Then $T(x) = K$. Since K was arbitrary, we are done. \square

2.1.8 Quotient Space Topology

Definition 2.1.111 (Quotient Space Topology). Let (Z, \mathcal{T}_Z) be a topological space. Let \cong be the **Relation Of Equal Neighborhood Filters** on (Z, \mathcal{T}_Z) . Let T be the **Quotient Map** of Z under the relation \cong . Define $\mathcal{T}_{Z/\cong}$ by

$$\mathcal{T}_{Z/\cong} = \left\{ \bigcup_{x \in U} \{T(x)\} \in 2^{Z/\cong} \mid U \in \mathcal{T}_Z \right\} \quad (2.14)$$

By 2.1.112, $\mathcal{T}_{Z/\cong}$ is a topology on Z/\cong . We call $\mathcal{T}_{Z/\cong}$ the **Quotient Topology** and we call $(Z/\cong, \mathcal{T}_{Z/\cong})$ the **Quotient Topological Space** of (Z, \mathcal{T}_Z) .

Proposition 2.1.112 (Quotient Space Topology). Let (Z, \mathcal{T}_Z) be a **Topological Space** with **Quotient Topological Space** $(Z/\cong, \mathcal{T}_{Z/\cong})$ and **Quotient Map** T .

Then the following are true.

1. $\mathcal{T}_{Z/\cong}$ is a **Topology** on Z/\cong .
2. $T : (Z, \mathcal{T}_Z) \rightarrow (Z/\cong, \mathcal{T}_{Z/\cong})$ is **Continuous**.
3. If U is **Set-Open** (**Set-Closed**) in (Z, \mathcal{T}_Z) then $T(U)$ and $T(Z \setminus U)$ **Partition** Z/\cong .
4. If U is **Set-Open** in (Z, \mathcal{T}_Z) , then $T^{-1}(T(U)) = U$.
5. If K is **Set-Closed** in (Z, \mathcal{T}_Z) , then $T^{-1}T(K) = K$.
6. $T : (Z, \mathcal{T}_Z) \rightarrow (Z/\cong, \mathcal{T}_{Z/\cong})$ is **Open**.

7. $T : (Z, \mathcal{T}_Z) \rightarrow (Z/\cong, \mathcal{T}_{Z/\cong})$ is **Closed**.
8. (Z, \mathcal{T}_Z) is a **Compact** space if and only if $(Z/\cong, \mathcal{T}_{Z/\cong})$ is a **Compact** space.
9. If \mathcal{B} is a **Basis** for \mathcal{T}_Z , then $\{T(U) | U \in \mathcal{B}\}$ is a **Basis** for $\mathcal{T}_{Z/\cong}$.
10. If T is **Injective**, then it is a **Homeomorphism**.

Proof of 1. Since $\emptyset \in \mathcal{T}_Z$, we have

$$\emptyset = \bigcup_{x \in \emptyset} \{Tx\} \in \mathcal{T}_{Z/\cong} \quad (2.15)$$

Since $Z \in \mathcal{T}_Z$, and by 2.1.108,

$$Z/\cong = \bigcup_{x \in Z} \{[x]\} = \bigcup_{x \in Z} \{T(x)\} \in \mathcal{T}_{Z/\cong} \quad (2.16)$$

Let $\{U_\alpha | \alpha \in A\} \subset \mathcal{T}_{Z/\cong}$. For each $\alpha \in A$, there exists $B_\alpha \in \mathcal{T}_Z$ such that we have

$$U_\alpha = \bigcup_{x \in B_\alpha} \{Tx\} \quad (2.17)$$

Since $\bigcup_{\alpha \in A} B_\alpha \in \mathcal{T}_Z$, we have

$$\bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} \bigcup_{x \in U_\alpha} \{T(x)\} = \bigcup_{x \in \bigcup_{\alpha \in A} B_\alpha} \{T(x)\} \in \mathcal{T}_{Z/\cong} \quad (2.18)$$

Let $\{U_i\}_{i=1}^n \subset \mathcal{T}_{Z/\cong}$. For each $i \in \{1, \dots, n\}$, there exists $B_i \in \mathcal{T}_Z$ such that

$$U_i = \bigcup_{x \in B_i} \{T(x)\} \quad (2.19)$$

Suppose

$$[x_0] \in \bigcap_{i=1}^n \bigcup_{x \in B_i} \{T(x)\} \quad (2.20)$$

Then for each $i \in \{1, \dots, n\}$, there is a $y_i \in B_i$ such that $y_i \cong x_0$. Since each B_i is **Set-Open**, the definition of \cong implies that $x_0 \in B_i$ for every i . Hence,

$$x_0 \in \bigcap_{i=1}^n B_i \quad (2.21)$$

Implying

$$[x_0] \in \bigcup_{x \in \bigcap_{i=1}^n B_i} \{[x]\} \quad (2.22)$$

Hence,

$$\bigcap_{i=1}^n \bigcup_{x \in B_i} \{T(x)\} \subset \bigcup_{x \in \bigcap_{i=1}^n B_i} \{[x]\} \quad (2.23)$$

Furthermore, since the reverse inclusion is obvious, and since $\bigcap_{i=1}^n B_i \in \mathcal{T}_Z$, we have

$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n \bigcup_{x \in B_i} \{T(x)\} = \bigcup_{x \in \bigcap_{i=1}^n B_i} \{T(x)\} \in \mathcal{T}_{Z/\cong} \quad (2.24)$$

□

Proof of 2. Let $V \in \mathcal{T}_{Z/\cong}$. Let $x_0 \in T^{-1}(V)$. Then $[x_0] \in V$. By definition, there is a $U \in \mathcal{T}_Z$ such that

$$T(U) \subset \bigcup_{x \in U} \{T(x)\} = V \quad (2.25)$$

Hence there is a $y_0 \in U$ such that

$$[x_0] \in T(y_0) = \{[y_0]\} \quad (2.26)$$

Therefore, $x \cong y$. Definition of the **Relation Of Equal Neighborhood Filters** on i implies $\mathcal{U}(x_0) = \mathcal{U}(y_0)$. Hence, $x_0 \in U \subset T^{-1}(V)$. □

Proof of 3. Let K be closed in (Z, \mathcal{T}_Z) . Then each point x_0 in $Z \setminus K$ has some Open U_{x_0} containing x_0 which is **Disjoint** from K . Hence $y_0 \not\cong x_0$ for any $y_0 \in K$, $x_0 \in Z \setminus K$. Hence $T(K)$ is **Disjoint** from $T(Z \setminus K)$. This fact, paired with 2.1.7, implies $T(Z \setminus K)$ and $T(K)$ is a **Partition** of Z/\cong . □

Proof of 4. Let $U \in \mathcal{T}_Z$. The nontrivial direction to prove is $T^{-1}(T(U)) \subset U$. Let $y \in T^{-1}(T(U))$. Then $[y] = Ty \in T(U)$. Hence, $[y] = T(x) = [x]$ for some $x \in U$. Since $y \cong x$ and $x \in U \in \mathcal{U}_{\mathcal{T}_Z}(x)$, we have $U \in \mathcal{U}_{\mathcal{T}_Z}(y)$. Hence $y \in U$. Since y was arbitrary, $T^{-1}(T(U)) \subset U$, and equality is obvious because the other direction of inclusion is trivial. □

Proof of 5. Let K be **Set-Closed** in (Z, \mathcal{T}_Z) . Part 3 Of this result implies Z/\cong is partitioned by $T(K)$ and $T(Z \setminus K)$.

By part 4 of this proposition,

$$\begin{aligned} T^{-1}(T(K)) &= T^{-1}(T(Z) \setminus T(Z \setminus K)) \\ &= T^{-1}(Z/\cong \setminus T(Z \setminus K)) \\ &= T^{-1}(Z/\cong) \setminus T^{-1}(T(Z \setminus K)) \\ &= Z \setminus (Z \setminus K) \\ &= K \end{aligned}$$

□

Proof of 6. Let $U \in \mathcal{T}_Z$. Then by definition of the **Quotient Topology**

$$TU = \bigcup_{x \in U} \{T(x)\} \in \mathcal{T}_{Z/\cong} \quad (2.27)$$

□

Proof of 7. Let K be **Set-Closed** in (Z, \mathcal{T}_Z) . Then $Z \setminus K \in \mathcal{T}_Z$. By Parts 3 and five of this proposition, we know $T(K) = Z/\cong \setminus T(Z \setminus K)$ and also that $T(Z \setminus K) \in \mathcal{T}_{Z/\cong}$. Hence $T(K)$ is closed in $(Z/\cong, \mathcal{T}_{Z/\cong})$. □

Proof of 8. Let (Z, \mathcal{T}_Z) be **Compact**. Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of $(Z/\cong, \mathcal{T}_{Z/\cong})$. Then $\{T^{-1}(U_\alpha) \mid \alpha \in A\}$ is an open covering of (Z, \mathcal{T}_Z) . **Compactness** of (Z, \mathcal{T}_Z) guarantees the existence of a finite subcovering $\{T^{-1}(U_{\alpha_i}) \mid i \in \{1, \dots, n\}\}$. Hence $\{U_{\alpha_i} \mid i \in \{1, \dots, n\}\} = \{TT^{-1}(U_{\alpha_i}) \mid i \in \{1, \dots, n\}\}$ is an **Set-Open Cover** of $(Z/\cong, \mathcal{T}_{Z/\cong})$. And the **Compactness** of $(Z/\cong, \mathcal{T}_{Z/\cong})$ is verified.

Now, suppose $(Z/\cong, \mathcal{T}_{Z/\cong})$ is **Compact**. Let $\{V_\beta \mid \beta \in B\}$ be an **Set-Open Cover** of (Z, \mathcal{T}_Z) . Since T is an **Open** mapping, $\{T(V_\beta) \mid \beta \in B\}$ is an **Set-Open Cover** of $(Z/\cong, \mathcal{T}_{Z/\cong})$ which by **Compactness** has a **Finite Subcover** $\{T(V_{\beta_i}) \mid i \in \{1, \dots, n\}\}$. By part 4 of 2.1.112, $\{V_{\beta_i} \mid i \in \{1, \dots, n\}\} = \{T^{-1}(T(V_{\beta_i})) \mid i \in \{1, \dots, n\}\}$ is then an **Set-Open Subcover** of (Z, \mathcal{T}_Z) . □

Proof of 9. Let \mathcal{B} be a basis for \mathcal{T}_Z and let $V \in \mathcal{T}_{Z/\cong}$. Then $T^{-1}(V) \in \mathcal{T}_Z$, and so there is a subcollection $\{U_\alpha\}_{\alpha \in A} \subset \mathcal{B}$ such that $T^{-1}(V) = \bigcup_{\alpha \in A} U_\alpha$. Hence,

$$\begin{aligned} V &= T(T^{-1}(V)) \\ &= T\left(\bigcup_{\alpha \in A} U_\alpha\right) \\ &= \bigcup_{\alpha \in A} T(U_\alpha) \end{aligned}$$

□

Proof of 10. If T is **Injective**, then since it is **Continuous** Part 2 of this result, open by part 6 of this result, and **Surjective** by 2.1.7, it is a **Bicontinuous Bijection**, that is, a **Homeomorphism**. □

2.1.9 Weak Topologies

2.1.10 Algebraic Structures

Definition 2.1.113 (Algebraic Declarations Placeholder).

Definition 2.1.114 (**Commutative**). Let X and Y be sets. We say that a map $f : X \times X \rightarrow Y$ is a **Symmetric Map** if for each $x_0, x_1 \in X$, $f(x_0, x_1) = f(x_1, x_0)$. In this situation, we may also refer to f as **Commutative**, or say that f possesses **Commutativity**.

Definition 2.1.115 (**Operation** , **Unary Operation** , **Binary Operation**). Let $X \neq \emptyset$ be a set. Let $A \neq \emptyset$ be a set with $\text{cardinality}(A) = n \in \mathbb{N}$. We call a mapping

$$T : \prod_{\alpha \in A} X \rightarrow X$$

an n -ary **Operation** on X . If $n = 1$ then we call T a **Unary Operation** on X . If $n = 2$, then we call T a **Binary Operation** on X . If T is a **Binary Operation** on X , we sometimes use the notation

$$xTy = T(x, y)$$

Definition 2.1.116 (**Magma**). Let X be a set and $T : X \times X \rightarrow X$ be a **Binary Operation** on X . We call the pair (X, T) a **Magma**. When it is clear what operation is being referred to, we may simply refer to X as the **Magma**. If T is **Commutative**, then we call (X, T) (or simply just X) a **Commutative Magma**. In general, this naming convention is used for any algebraic structure defined on a set via a **Binary Operation** with particular properties.

Definition 2.1.117 (**Magma Homomorphism**). Let (X, \oplus_X) and (Y, \oplus_Y) be **Magmas**. Let $T : X \rightarrow Y$ satisfy, for each $x_1, x_2 \in X$.

$$T(x_1 \oplus_X x_2) = T(x_1) \oplus_Y T(x_2)$$

Then we call T a **Magma Homomorphism**. We represent the collection of **Magma Homomorphisms** from (X, \oplus_X) to (Y, \oplus_Y) with $H_{\text{Magma}}((X, \oplus_X), (Y, \oplus_Y))$, or, when \oplus_X and \oplus_Y are clear, $H_{\text{Magma}}(X, Y)$. A **Magma Homomorphism** is called **Additive** and possesses the property **Additivity**.

Definition 2.1.118 (**Left Identity Element** , **Right Identity Element**). Let (X, L) and (X, R) be **Magmas**. Let $l, r \in X$ such that for every $x \in X$ we have

$$lLx = x$$

$$xRr = x$$

In such a scenario, we say that l is a **Left Identity Element** of (X, L) , and we say that r is a **Right Identity Element** of (X, R) .

Definition 2.1.119 (**Identity Element**). Let (X, \oplus) be a **Magma**. Let $e \in X$ be both a **Left Identity Element** and a **Right Identity Element** of \oplus . Then, we say that e is an **Identity Element** of (X, \oplus) .

Definition 2.1.120 (**Unital Magma**). Let (X, \oplus) be a **Magma**. Let e be an **Identity Element** of (X, \oplus) . Then we call (X, \oplus, e) a **Unital Magma**. If it is unambiguous what operation is being referred to, as in the case of **Magmas**, we may simply say let X be a **Unital Magma**, or potentially Let (X, e) be a **Unital Magma**.

Definition 2.1.121 (‘ **Unital Magma Homomorphism** ’). Let (X, \oplus_X, e_X) and (Y, \oplus_Y, e_Y) be **Unital Magmas** and $T : X \rightarrow Y$ be a **Magma Homomorphism** such that $T(e_X) = e_Y$. Then we call T a ‘ **Unital Magma Homomorphism** ’. We represent the set of **Unital Magma Homomorphisms** between X and Y with $H_{\text{UMagma}}(X, Y)$. Obviously, $H_{\text{UMagma}}(X, Y) \subset H_{\text{Magma}}(X, Y)$.

Definition 2.1.122 (**Left Inverse** , **Right Inverse**). Let (X, \oplus, e) be a **Unital Magma** . Let $l, r \in X$ such that

$$l \oplus r = e \quad (2.28)$$

In this scenario, we say that l is a **Left Inverse** of r in (X, \oplus, e) and we say that r is a **Right Inverse** of l in (X, \oplus, e) . Furthermore, we say that r is **Left Invertible** in (X, \oplus, e) and that l is **Right Invertible** in (X, \oplus, e) .

Definition 2.1.123 (**Inverse**). Let (X, \oplus, e) be a **Unital Magma** . Let $x, y \in X$ such that x is a **Left Inverse** of y and x is a **Right Inverse** of y . Then, we say that x is an **Inverse** of y in (X, \oplus, e) and we say y an **Invertible** element of (X, \oplus, e) .

Definition 2.1.124 (**Associative**). Let T be a **Binary Operation** on a set X . We say that T is **Associative** and we say that T possesses **Associativity** if for each $x, y, z \in X$, we have

$$T(x, T(y, z)) = T(T(x, y), z)$$

Definition 2.1.125 (**Semigroup**). Let (X, \oplus) be a **Magma** . Let \oplus be **Associative** . Then we say that (X, \oplus) is a **Semigroup** .

Definition 2.1.126 (**Monoid**). Let (X, \oplus, e) be a **Unital Magma** and let (X, \oplus) be a **Semigroup** . Then we call (X, \oplus, e) a **Monoid** .

Definition 2.1.127 (**Consistent**). Let (X, \oplus) be a **Magma** and let R be a **Relation** on X such that For each $x_0, x_1 \in X$, if $x_0 R x_1$ and $y_0 R y_1$, then $(x_0 \oplus y_0) R (x_1 \oplus y_1)$ Then we say that R is **Consistent** with (X, \oplus) , and we say that R possesses **Consistency** with respect to (X, \oplus) .

Definition 2.1.128 (**Partially Ordered Magma** , **Totally Ordered Magma** , **Directed Magma**). Let (X, \oplus) be a **Magma** . Let T be a **Total Ordering** on X which is **Consistent** with (X, \oplus) . Let P be a **Partial Ordering** on X which is **Consistent** with (X, \oplus) . Let D be a **Directing** on X which is **Consistent** with (X, \oplus) . We call (X, \oplus, T) a **Totally Ordered Magma** . (X, \oplus, P) a **Partially Ordered Magma** . (X, \oplus, D) a **Directed Magma** .

Definition 2.1.129 (**Group**). Let (X, \oplus, e) be a **Monoid** such that each $x \in X$ is an **Invertible** . Then we call (X, \oplus, e) a **Group** . Out of respect, we call a **Commutative Group** an **Abelian Group** .

Definition 2.1.130 (**Group Inverse Operator**). Let (X, \oplus, e) be a group. We denote with \mathbf{T}^{-1}_G the function defined as follows: $\mathbf{T}^{-1}_G : X \rightarrow X$,

$$\mathbf{T}^{-1}_G(x) = -x$$

We call \mathbf{T}^{-1}_G the **Group Inverse Operator** of (X, \oplus, e) .

Definition 2.1.131 (**Translation Operator**). Let (G, \oplus) be a **Magma** . Let $g \in G$. We define $\mathbf{T}^R_g : G \rightarrow G$ and $\mathbf{T}^L_g : G \rightarrow G$ by setting, for each $x \in G$,

$$\mathbf{T}^R_g(x) = x \oplus g$$

$$\mathbf{T}^L_g(x) = g \oplus x$$

We call \mathbf{T}^R_g the **Right Translation** of (G, \oplus) by g , and we call \mathbf{T}^L_g the **Left Translation** of (G, \oplus) by g . If \oplus is **Commutative** , then we define $\mathbf{T}_g = \mathbf{T}^R_g = \mathbf{T}^L_g$ which we call **Translation** of (G, \oplus) by g .

2.1.11 Topological Algebra

Definition 2.1.132 (**Topological Group** '). Let $(G, +, e)$ be a **Group** . **Topology** on G such that $+: G \times G \rightarrow G$ is **Continuous** with respect to the
and g_{-1} is **Continuous** .

In this scenario, we call (G, \mathcal{T}) a **Topological Group** ' .

Definition 2.1.133 (**Local Basis**). Let (G, \mathcal{T}) be a **Topological Group** ' with **Identity Element** e . We call a **Neighborhood Basis** of \mathcal{T} about e a **LocalBasis** for (G, \mathcal{T}) .

2.1.12 Vector Spaces

Definition 2.1.134 (Scalar Homogeneous). Let V be a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. We say that a map $p : V \rightarrow V$ is **Scalar Homogeneous** , if , for each $\alpha \in \mathbb{F}$ and each $x \in V$, we have

$$p(\alpha x) = \alpha p(x) \tag{2.29}$$

Under these circumstances, we may instead say that the operator p possesses **Scalar Homogeneity** .

Definition 2.1.135 (Scalar Homogeneous). Let V be a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. We say that a map $p : V \rightarrow V$ is **Absolutely Scalar Homogeneous** , if , for each $\alpha \in \mathbb{F}$ and each $x \in V$, we have

$$p(\alpha x) = |\alpha| p(x) \tag{2.30}$$

Under these circumstances, we may instead say that the operator p possesses **Absolute Scalar Homogeneity** .

Remark 2.1.136 (**Scalar Homogeneous** or **Absolutely Scalar Homogeneous** operator at 0 is 0). If V is a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, then for each $x \in V$, $0x = 0$. Hence, if p is a **Absolutely Scalar Homogeneous** operator on v , then for any $x \in V$

$$p(0) = p(0x) = |0|p(x) = 0p(x) = 0 \quad (2.31)$$

If instead p is **Scalar Homogeneous** operator on V , then we have

$$p(0) = p(0x) = 0p(x) = 0 \quad (2.32)$$

that is, in either case, $p(0)=0$.

Definition 2.1.137 (**Subadditive**). Let (G, \oplus_G) be a **Magma** and (H, \oplus_H, \leq) be a **Partially Ordered Magma**. We call a mapping $p : G \rightarrow H$ **Subadditive** if, for every $x, y \in G$, we have

$$p(x \oplus_G y) \leq p(x) \oplus_H p(y) \quad (2.33)$$

Under these circumstances, we may also say that p possesses **Subadditivity**.

Definition 2.1.138 (**Linear**). Let V, U be **Vector Spaces** over a **Field** \mathbb{F} . We say that $T : V \rightarrow U$ is **Linear** or that T possesses **Linearity** if T is both **Additive** and **Scalar Homogeneous**

Definition 2.1.139 (**Space of Linear Operators**). Let U, V be **Vector Spaces** over the same **Field** \mathbb{F} . We denote with $L(U, V)$ the set of **Linear** operators $T : U \rightarrow V$. We refer to $L(U, V)$ as the **Space of Linear Operators** from U to V . We endow $L(U, V)$ with the operations of pointwise addition and pointwise scalar multiplication, which the reader can verify makes $L(U, V)$ into a **Vector Space**.

Definition 2.1.140 (**Balanced**). Let V be a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $S \subset V$. We call S a **Balanced Set** and we say that S is **Balanced** if for each $\alpha \in \mathbb{F}$ with $|\alpha| \leq 1$ we have $\alpha S \subset S$.

Definition 2.1.141 (**Absorbing**). Let V be a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $A, B \subset V$. We say that A **Absorbs** B if there exists a $c > 0$ such that for every $d \in \mathbb{F}$ with $|d| > c$ we have $B \subset dA$. In such a Scenario, A is also said to **Absorb** B , and we say that B is **Absorbed** by A . If A **Absorbs** every Singleton in V , then we call A an **Absorbing Set** or we say that A is **Absorbing**.

Definition 2.1.142 (**Scaling Operator**). Let V be a **Vector Space** over a **Field** \mathbb{F} . Let $\alpha \in \mathbb{F}$. We define $M_\alpha : V \rightarrow V$ by setting, for each $x \in V$,

$$M_\alpha(x) = \alpha x \quad (2.34)$$

We call M_α the **Scaling Operator**

Proposition 2.1.143 (Scaling Operator). *Let V be a Vector Space over a Field \mathbb{F} . The following are true:*

1. If $\alpha, \beta \in \mathbb{F}$, then $M_\alpha \circ M_\beta = M_{\alpha*\beta}$.

Proof of 01. Let $v \in V$. Then

$$\begin{aligned} M_\alpha \circ M_\beta v &= M_\alpha (\beta * v) \\ &= \alpha * (\beta * v) \\ &= (\alpha * \beta) * v \\ &= M_{\alpha*\beta} v \end{aligned}$$

□

Definition 2.1.144 (Interval). *Let V be a Vector Space over a Field \mathbb{F} . Let $x, y \in V$. We define the following sets:*

$$\begin{aligned} [x, y] &= \{tx + (1 - t)y | t \in [0, 1]\} \\ [x, y) &= \{tx + (1 - t)y | t \in [0, 1)\} \\ (x, y] &= \{tx + (1 - t)y | t \in (0, 1]\} \\ (x, y) &= \{tx + (1 - t)y | t \in (0, 1)\} \end{aligned}$$

We refer to any of these sets as **Intervals** in V . Even in the absence of a topological structure, we use the following language:

1. $[x, y]$ is called a **Closed Interval**.
2. (x, y) is called an **Open Interval**.
3. $(x, y]$ and $[x, y)$ are called **Half-Open Intervals** or **Half-Closed Intervals**.

Definition 2.1.145 (Convex). *Let V be a Vector Space over a Field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $K \subset \mathbb{F}$. We say that K is **Convex** if for every pair $x, y \in K$, we have $[x, y] \subset K$.*

2.1.13 Pseudometrics

Definition 2.1.146 (Symmetric Map). *Let X be a set and $(Y, +, \leq)$ be a totally ordered magma. We say that a map $f : X \times X \rightarrow Y$ satisfies the **Triangle Inequality** if for each $x_0, x_1, x_3 \in X$, we have*

$$f(x_0, x_2) \leq f(x_0, x_1) + f(x_1, x_2)$$

Definition 2.1.147 (Pseudometric). *Let $X \neq \emptyset$. Let $d : X \times X \rightarrow [0, \infty)$ be a **Symmetric Map** that satisfies the **Triangle Inequality** and further satisfies, for each $x \in X$,*

$$d(x, x) = 0 \tag{2.35}$$

Under these conditions we call d a **Pseudometric** on X and we call (X, d) a **Pseudometric Space**.

Definition 2.1.148 (Metric). *Let (X, d) be a **Pseudometric Space**. If d has the property that for $x, y \in X$, if $x \neq y$, then*

$$d(x, y) \neq 0$$

*Then we call d a **Metric** on X and we call (X, d) a **Metric Space***

Definition 2.1.149 (Pseudometric Cauchy Sequence). *Let (X, d) be a **Pseudometric Space**. We say that a sequence $\{x_i\}_{i \in \mathbb{N}}$ is a **Pseudometric Cauchy Sequence** if, for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each pair $m, n \in \mathbb{N}$ such that $m > N$ and $n > N$, we have*

$$d(x_m, x_n) < \epsilon \quad (2.36)$$

Definition 2.1.150 (Pseudometric Convergence). *Let (X, d) be a **Pseudometric Space**. Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence in (X, d) . Let $x_0 \in X$. We say that $\{x_i\}_{i \in \mathbb{N}}$ exhibits **Pseudometric-Convergence** to x_0 in d , or we say that $\{x_i\}_{i \in \mathbb{N}}$ **Pseudometric-Converges** to x_0 in d , or we say that $\{x_i\}_{i \in \mathbb{N}}$ is **Pseudometrically-Convergent** to $x_0 \in d$ if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for every $n > N$, we have*

$$d(x_0, x_n) < \epsilon \quad (2.37)$$

Proposition 2.1.151 (Convergent Implies Cauchy). *Let (X, d) be a **Pseudometric Space**. Let $\{x_i\}_{i \in \mathbb{N}}$ be a **Pseudometrically-Convergent** sequence. Then $\{x_i\}_{i \in \mathbb{N}}$ is a **Pseudometric Cauchy Sequence**.*

Proof. Since $\{x_i\}$ converges, let $x_i \rightarrow x$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $n > N$, we have $d(x_i, x) < \frac{\epsilon}{2}$. For this N , if $m, n > N$, then we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \epsilon \quad (2.38)$$

and so the sequence is a **Pseudometric Cauchy Sequence**, as advertised. \square

Definition 2.1.152 (Uniformly Cauchy). *Let (X_α, d_α) be a **Pseudometric Space** for $\alpha \in A$ where A is some indexing set. For each $\alpha \in A$, let $\phi_\alpha := \{x_i^\alpha\}_{i \in \mathbb{N}} \subset X_\alpha$ be a sequence. We say that the collection $\{\phi_\alpha\}_{\alpha \in A}$ is **Uniformly Cauchy** if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for each pair $m, n \in \mathbb{N}$ such that $m > N$ and $n > N$, and for each $\alpha \in A$, we have*

$$d_\alpha(x_n^\alpha, x_m^\alpha) < \epsilon \quad (2.39)$$

Definition 2.1.153 (Uniform Convergence). *Let (X_α, d_α) be a **Pseudometric Space** for $\alpha \in A$ where A is some indexing set. For each $\alpha \in A$, let $\phi_\alpha := \{x_i^\alpha\}_{i \in \mathbb{N}} \subset X_\alpha$ be a sequence. We say that the collection $\{\phi_\alpha\}_{\alpha \in A}$ is **Uniformly Convergent** to $\{x_\alpha\}_{\alpha \in A} \in \prod_{\alpha \in A} X_\alpha$ if for each $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for each $n > N$, and for every $\alpha \in A$, we have*

$$d_\alpha(x_i^\alpha, x_\alpha) < \epsilon \quad (2.40)$$

In this scenario, we may equivalently say that $\{\phi_\alpha\}$ demonstrates **Uniform Convergence** to $\{x_\alpha\}_{\alpha \in A}$ or that it **Converges Uniformly**.

When we mention **Uniform Convergence** without reference to what the convergence is to, we are merely claiming the existence of such a limit.

Proposition 2.1.154 (Uniform Cauchy and Pointwise Convergence implies Uniform Convergence). *Let (X_α, d_α) be a **Pseudometric Space** for $\alpha \in A$ where A is some indexing set. For each $\alpha \in A$, let $\phi_\alpha := \{x_i^\alpha\}_{i \in \mathbb{N}} \subset X_\alpha$ be a sequence. Suppose the collection $\{\phi_\alpha\}_{\alpha \in A}$ is **Uniformly Cauchy** and that each ϕ_α is **Pseudometrically-Convergent**, say $x_i^\alpha \rightarrow x_\alpha$. Then $\{\phi_\alpha\}_{\alpha \in A}$ is **Uniformly Convergent** to $\{x_\alpha\}_{\alpha \in A}$.*

Proof. Let $\epsilon > 0$. Then, since $\{\phi_\alpha\}_{\alpha \in A}$ is **Uniformly Cauchy**, there is an $N \in \mathbb{N}$ such that for $m, n > N$, we have $d_\alpha(x_n^\alpha, x_m^\alpha) < \frac{\epsilon}{2}$. Also, since each ϕ_α converges to x_α , there are $N_\alpha \in \mathbb{N}$ such that for any $n_\alpha > N_\alpha$, we have $d_\alpha(x_{n_\alpha}^\alpha, x_\alpha) < \frac{\epsilon}{2}$. Define $M_\alpha = \max(N + 1, N_\alpha + 1)$ for $\alpha \in A$. Let $n > N$. Then, for any $\alpha \in A$, we have.

$$\begin{aligned} d_\alpha(x_n^\alpha, x_\alpha) &\leq d_\alpha(x_n^\alpha, x_{M_\alpha}^\alpha) + d_\alpha(x_{M_\alpha}^\alpha, x_\alpha) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

completing the proof. □

Definition 2.1.155 (Pseudometric Complete). *We say that a **Pseudometric Space** (X, d) is **Pseudometric-Complete** if each **Pseudometric Cauchy Sequence** sequence in (X, d) **Pseudometric-Converges** to a limit in X .*

In the case that d is a **Metric**, then being **Pseudometric-Complete** is equivalent to being **Complete** in the classical sense, so we will commonly refer to a **Pseudometric Space** which is **Pseudometric-Complete** as simply being **Complete**.

Definition 2.1.156 (Pseudometric Ball). *Let (X, d) be a **Pseudometric Space**. For each $x_0 \in X$ and each $\epsilon > 0$, we define the following.*

1. $B_d(x_0, \epsilon) := \{y \in X \mid d(x_0, y) < \epsilon\}$ denotes the **Open Ball** about x_0 with radius ϵ .
2. $\overline{B}_d(x_0, \epsilon) := \{y \in X \mid d(x_0, y) \leq \epsilon\}$ denotes the **Closed Ball** about x_0 with radius ϵ .

Definition 2.1.157 (Pseudometric Topology). *Let (X, d) be a **Pseudometric Space**, and let \mathcal{B} be the set of **Open Ball**'s in (X, d) . By 2.1.13, \mathcal{B} is the basis for a unique topology \mathcal{T}_d on X . We call \mathcal{T}_d the **Pseudometric Topology** induced by d on X .*

Proposition 2.1.158 (Pseudometric Topology). *Let (X, d) be a **Pseudometric Space** and let \mathcal{B} be the set of **Open Ball**'s in (X, d) . The following are true.*

1. There exists a unique topology \mathcal{T}_d on X which \mathcal{B} is a basis of. That is, the **Pseudometric Topology** \mathcal{T}_d is well defined.
2. The **Pseudometric Topology** is first countable. That is, each of its points permits a countable neighborhood basis.

Proof of 1. Uniqueness is guaranteed by closure under arbitrary unions of a topology. For existence, it is sufficient to show that the collection of arbitrary unions of elements of \mathcal{B} is closed under finite intersections. Suppose that for $1 \leq i \leq n$, we have $\{U_{\alpha_i} | \alpha_i \in A_i\} \subset \mathcal{B}$ and consider the set

$$U = \bigcap_{i=1}^n \bigcup_{\alpha_i \in A_i} U_{\alpha_i} \quad (2.41)$$

Let $x_0 \in U$. For each $i \in \{1, \dots, n\}$, there exists $\alpha_i \in A_i$ such that

$$x_0 \in U_{\alpha_i} = B_d(x_i; \epsilon_i) \quad (2.42)$$

For each $i \in \{1, \dots, n\}$, define $\delta_i = d(x_0, x_i)$. Then $0 < \delta_i < \epsilon_i$. Then, for each $i \in \{1, \dots, n\}$,

$$B_d(x_0; \epsilon_i - \delta_i) \subset U_{\alpha_i} \subset \bigcup_{\alpha_i \in A_i} U_{\alpha_i} \quad (2.43)$$

Define

$$\delta_{x_0} = \min_{i=1}^n (\epsilon_i - \delta_i) \quad (2.44)$$

Then $x_0 \in B(x_0; \delta_{x_0}) \subset U$. If $U = \{x_\alpha | \alpha \in A\}$, then the arbitrary nature of x_0 above means we can repeat this construction, writing

$$U \subset \bigcup_{\alpha \in A} B(x_\alpha; \delta_{x_\alpha}) \subset \bigcup_{\alpha \in A} U = U \quad (2.45)$$

Hence, $U \in \mathcal{B}$ and the proof is complete. \square

Proof of 2. Let $x_0 \in X$. I claim that

$$\mathcal{B}_{x_0} := \left\{ B_d \left(x_0; \frac{1}{n} \right) \mid n \in \mathbb{N} \right\} \quad (2.46)$$

is a neighborhood basis for (X, \mathcal{T}_d) at x_0 . Let $U \in \mathcal{U}_{\mathcal{T}_d}(x)$ be open in \mathcal{T}_d . Since \mathcal{B} is a basis for \mathcal{T}_d , for some $y_0 \in X$ and $\epsilon > 0$, $x_0 \in B_d(y_0; \epsilon) \subset U$. Let $\delta = d(x_0, y_0)$. Then $\epsilon - \delta > 0$. Define

$$n = \left\lceil \frac{1}{\epsilon - \delta} \right\rceil \quad (2.47)$$

Then we have

$$B_d \left(x_0; \frac{1}{n} \right) \subset B_d(x_0; \epsilon - \delta) \subset B_d(y_0; \epsilon) \subset U \quad (2.48)$$

\square

Definition 2.1.159 (Relation Of Zero Distance). *Let (X, d) be a **Pseudometric Space**.* Define the relation \cong_d on $X \times X$ by setting, for $x, y \in X$,

$$x \cong_d y \iff d(x, y) = 0 \quad (2.49)$$

We call \cong_d the **Relation Of Zero Distance** on (X, d) .

Proposition 2.1.160 (Relation Of Zero Distance is the Relation Of Equal Neighborhood Filters). *Let (X, d) be a **Pseudometric Space**. Let $\cong_{\mathcal{T}_d}$ be the **Relation Of Equal Neighborhood Filters** on (X, \mathcal{T}_d) . Let \cong_d be the **Relation Of Zero Distance** on (X, d) . Then $\cong_{\mathcal{T}_d} = \cong_d$.*

Proof. Let $x, y \in X$ and suppose $x_0 \cong_d y_0$. Let $U \in \mathcal{U}_{\mathcal{T}_d}(x_0)$. Then for some $\epsilon > 0$, $x_0 \in B(x_0; \epsilon) \subset U$. Since $x_0 \cong_d y_0$, $d(x_0, y_0) = 0$, so $y_0 \in B(x_0; \epsilon) \subset U$. Hence $U \in \mathcal{U}_{\mathcal{T}_d}(y_0)$. The arbitrary nature of $U \in \mathcal{U}_{\mathcal{T}_d}(x_0)$ implies

$$\mathcal{U}_{\mathcal{T}_d}(x_0) \subset \mathcal{U}_{\mathcal{T}_d}(y_0) \quad (2.50)$$

A reverse construction would just as easily show the reverse inclusion, so we conclude that $x_0 \cong_{\mathcal{T}_d} y_0$. Now suppose $x_0 \cong_{\mathcal{T}_d} y$. Then for each $n \in \mathbb{N}$,

$$y_0 \in B_d \left(x_0; \frac{1}{n} \right) \quad (2.51)$$

Hence $d(x_0, y_0) < \frac{1}{n}$ for each natural n , therefore $d(x_0, y_0) = 0$ and $x_0 \cong_d y_0$. \square

Definition 2.1.161 (Metric Space Induced By Pseudometric). *Let (X, d) be a **Pseudometric Space**, and let \cong be the **Relation Of Zero Distance**, which by 2.1.160 is also the **Relation Of Equal Neighborhood Filters** on (X, \mathcal{T}_d) . Define $\tilde{d} : X/\cong \rightarrow [0, \infty)$ by*

$$\tilde{d}([x], [y]) = d(x, y) \quad (2.52)$$

By 2.1.162, \tilde{d} is well defined and is in fact a metric on X/\cong , so we call \tilde{d} the **Metric Induced By The Pseudometric d on X** , or we call it the **Pseudometric Induced Metric** of (X, d) .

Proposition 2.1.162 (Metric Space Induced By Pseudometric Space). *Let (X, d) be a **Pseudometric Space**, \cong the **Relation Of Zero Distance** on (X, d) and \tilde{d} be defined as in 2.1.161. Let $(X/\cong, \mathcal{T}_{X/\cong})$ be the **Quotient Topological Space** with **Quotient Map** T , and let $(X/\cong, \mathcal{T}_{\tilde{d}})$ be the topological space induced by the metric space $(X/\cong, \tilde{d})$. The following are true.*

1. \tilde{d} is in fact well defined, and is a metric on X/\cong , justifying calling it the **Metric Induced By The Pseudometric d** .
2. $\mathcal{T}_{X/\cong} = \mathcal{T}_{\tilde{d}}$
3. T is an isometric surjection (X, d) to $(X/\cong, \tilde{d})$
4. $(X/\cong, \tilde{d})$ is complete if and only if (X, d) is **Pseudometric-Complete**.

Proof of 01. First we show that \tilde{d} is well defined as a mapping, that is, that if $x_0, y_0 \in X$ and $x_1 \cong x_0$ and $y_1 \cong y_0$, then we should have

$$\tilde{d}([x_0], [y_0]) = \tilde{d}([x_1], [y_1]) \quad (2.53)$$

This is easy, as

$$\begin{aligned}
 d(x_0, y_0) &\leq d(x_0, x_1) + d(x_1, y_1) + d(y_1, y_0) \\
 &= d(x_1, y_1) \\
 &\leq d(x_1, x_0) + d(x_0, y_0) + d(y_0, y_1) \\
 &= d(x_0, y_0)
 \end{aligned}$$

Nonnegativity falls directly from the nonnegativity of d . Proving that \tilde{d} is a **Symmetric Map** is equally trivial

$$\tilde{d}([x], [y]) = d(x, y) = d(y, x) = \tilde{d}([y], [x])$$

Proving that \tilde{d} satisfies the **Triangle Inequality** is similarly simple, letting $x_0, y_0, z_0 \in X$, we have

$$\begin{aligned}
 \tilde{d}([x_0], [z_0]) &= d(x_0, z_0) \\
 &\leq d(x_0, y_0) + d(y_0, z_0) \\
 &= \tilde{d}([x_0], [y_0]) + \tilde{d}([y_0], [z_0])
 \end{aligned}$$

All that remains is to show positivity on nonequal arguments. Let $x_0, y_0 \in X$ such that $[x_0] \neq [y_0]$. Then $x_0 \not\equiv y_0$. Hence

$$\tilde{d}([x_0], [y_0]) = d(x_0, y_0) \neq 0$$

□

Proof of 02. By 2.1.112, part 9, $\mathcal{B}_\cong := \{T(B_d(x; \epsilon)) \mid x \in X, \epsilon > 0\}$ is a basis for $\mathcal{T}_{X/\cong}$. By definition, $\mathcal{B}_{\tilde{d}} := \{B_{\tilde{d}}([x]; \epsilon) \mid x \in X, \epsilon > 0\}$ is a basis for $\mathcal{T}_{\tilde{x}}$.

I claim that for each $x \in X$ and $\epsilon > 0$,

$$T(B_d(x; \epsilon)) = B_{\tilde{d}}([x]; \epsilon) \tag{2.54}$$

To see this, suppose $\tilde{y} \in T(B_d(x; \epsilon))$. Then $\tilde{y} = T(y)$ for some $y \in B_d(x; \epsilon)$. Hence

$$\begin{aligned}
 \tilde{d}(\tilde{y}, [x]) &= \tilde{d}(T(y), [x]) \\
 &= \tilde{d}([y], [x]) \\
 &= d(y, x) \\
 &< \epsilon
 \end{aligned}$$

Hence $\tilde{y} \in B_{\tilde{d}}([x]; \epsilon)$, and so

$$T(B_d(x; \epsilon)) \subset B_{\tilde{d}}([x]; \epsilon) \tag{2.55}$$

Suppose $[y] \in B_{\tilde{d}}([x]; \epsilon)$. Then $d(x, y) = \tilde{d}([x], [y]) < \epsilon$, so $y \in B_d(x; \epsilon)$. Hence $[y] = T(y) \in T(B_d(x; \epsilon))$, so the reverse inclusion also holds, and so the above claim holds. This, paired with the fact that

$$\{[x] \mid x \in X\} = X/\cong \tag{2.56}$$

finishes the result.

□

Proof of 03. Falls directly from the definition $T(x) = [x]$, hence

$$d(x, y) = \tilde{d}([x], [y]) = \tilde{d}(T(x), T(y)) \quad (2.57)$$

T is surjective by 2.1.7. □

Proof of 04. Let (X, d) be **Pseudometric-Complete**. Let $\{[x_i]\}_{i \in \mathbb{N}} \subset (X/\cong, \tilde{d})$ be a **Pseudometric Cauchy Sequence**. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $m, n > N$, we have

$$d(x_m, x_n) = \tilde{d}(Tx_m, Tx_n) = \tilde{d}([x_m], [x_n]) < \epsilon \quad (2.58)$$

So the sequence $\{x_i\}_{i \in \mathbb{N}} \subset (X, d)$ is **Pseudometric Cauchy Sequence**. Since (X, d) is **Pseudometric-Complete**, this sequence has a limit, say $x_i \rightarrow x \in (X, d)$. But, we have $[x_i] = Tx_i \rightarrow Tx = [x]$, so $\{[x_i]\}$ is convergent, and since that sequence was arbitrary, $(X/\cong, \tilde{d})$ is **Pseudometric-Complete**.

Let $(X/\cong, \tilde{d})$ be **Pseudometric-Complete**. Let $\{x_i\} \subset X$ be a **Pseudometric Cauchy Sequence**. Let $\epsilon > 0$. Then there exist $N \in \mathbb{N}$ such that for $m, n > N$, we have

$$\tilde{d}([x_m], [x_n]) = \tilde{d}(Tx_m, Tx_n) = d(x_m, x_n) < \epsilon \quad (2.59)$$

so that $\{[x_i]\}_{i \in \mathbb{N}}$ is also a **Pseudometric Cauchy Sequence**. Since $(X/\cong, \tilde{d})$ is **Pseudometric-Complete**, this sequence has a limit, say $[x_i] \rightarrow y \in X/\cong$. Since T is surjective, for some $x \in X$, $Tx \in y$, and so

$$d(x, x_i) = \tilde{d}(Tx, Tx_i) = \tilde{d}(y, [x_i]) \rightarrow 0 \quad (2.60)$$

meaning $x_i \rightarrow x$ and we are done. □

Remark 2.1.163 (Mettic Space Correspondence). *Note that in the case of a metric space, the condition of being **Pseudometric-Complete** is equivalent to the condition being Complete, and A sequence is a **Pseudometric Cauchy Sequence** if and only if it is a cauchy sequence.*

Definition 2.1.164 ((Pseudo)Metriizable). *Let (X, \mathcal{T}) be a topological space.*

1. We say that (X, \mathcal{T}) (Or \mathcal{T} or X which it wouldn't cause confusion) is **Pseudometrizable** if there exists a pseudometric d on X such that \mathcal{T} is the **Pseudometric Topology** on (X, d) .
2. We say that (X, \mathcal{T}) (Or \mathcal{T} or X when it wouldn't cause confusion) is **Metriizable** if there exists a metric d on X such that \mathcal{T} is the metric topology on (X, d) .

Proposition 2.1.165 (Pseudometrizable Prequotient). *Let (X, \mathcal{T}_X) be a topological space with **Relation Of Equal Neighborhood Filters** on \cong , and with **Quotient Topological Space** $(X/\cong, \mathcal{T}_{X/\cong})$ and **Quotient Map** T . Let $(X/\cong, \mathcal{T}_{X/\cong})$ be **Pseudometrizable** with **Pseudometric** \tilde{d} .*

The following hold.

1. (X, \mathcal{T}_X) is **Pseudometrizable** with a pseudometric $d : X \times X \rightarrow [0, \infty)$ defined by

$$d(x, y) = \tilde{d}([x], [y]).$$

2. \tilde{d} is a Metric $(X/\cong, \mathcal{T}_{X/\cong})$, and so this space is **Metrizable**.
3. If T is injective, then d as defined above is a metric on X , so that (X, \mathcal{T}_X) is **Metrizable**.

Proof Of One. First, observe that

$$d(x, y) = \tilde{d}([x], [y]) \in [0, \infty)$$

so that d is well defined.

Also,

$$d(x, y) = \tilde{d}([x], [y]) = \tilde{d}([y], [x]) = d(y, x)$$

, so d is a **Symmetric Map**.

Also,

$$\begin{aligned} d(x, z) &= \tilde{d}([x], [z]) \\ &\leq \tilde{d}([x], [y]) + \tilde{d}([y], [z]) \\ &= d(x, y) + d(y, z) \end{aligned}$$

so d satisfies the **Triangle Inequality**. Also,

$$d(x, x) = \tilde{d}([x], [x]) = 0 \tag{2.61}$$

and so d is a **Pseudometric** on X .

Let \mathcal{T}_d denote the **Pseudometric Topology** on (X, d) . What remains to show is that $\mathcal{T}_X = \mathcal{T}_d$.

Since $d(x, y) = \tilde{d}([x], [y]) = \tilde{d}(Tx, Ty)$, T is an isometry.

Let $x \in U \in \mathcal{T}_X$. Then $[x] \in T(U) \in \mathcal{T}_{X/\cong}$. Hence, there is an $\epsilon > 0$ such that $B_{\tilde{d}}([x], \epsilon) \subset T(U)$. By 2.1.112, part 4, $T^{-1}(B_{\tilde{d}}([x], \epsilon)) \subset T^{-1}(T(U)) = U$, where by part 6 of that same result, $T^{-1}(B_{\tilde{d}}([x], \epsilon)) \in \mathcal{T}_X$. Since T is an isometry $B_d(x, \epsilon) = T^{-1}(B_{\tilde{d}}([x], \epsilon)) \subset U$. Thus we have found an open ball contained in U containing an arbitrary point of U . Hence, $\mathcal{T}_X \subset \mathcal{T}_d$. As part of the preceding argument we also showed that an arbitrary d -**Open Ball** was in \mathcal{T}_X , so $\mathcal{T}_d \subset \mathcal{T}_X$, and so equality holds and we're done. \square

Proof of Two. Let $x, y \in X$ with $[x] \neq [y]$. Then $x \not\sim y$. By 2.1.160, $x \not\sim_d y$. Hence $\tilde{d}([x], [y]) = d(x, y) > 0$. \square

Proof of Three. Let T be injective, and suppose $x, y \in X$ with $x \neq y$. Then $[x] = Tx \neq Ty = [y]$, so by part 2 of this result, $d(x, y) = \tilde{d}([x], [y]) > 0$. \square

2.1.14 Topological Vector Spaces

Definition 2.1.166 (**Compatible**). Let $(V, +, \cdot, 0)$ be a **Vector Space** over \mathbb{F} and \mathcal{T} be a **Topology** on V such that $(V, +, \mathcal{T})$ is a **Topological Group** ‘ and $\cdot : \mathbb{F} \times V \rightarrow V$ is **Continuous** . Then we say that \mathcal{T} is **Compatible** with $(V, +, \cdot, 0)$, or when $+$ and \cdot are obvious, we say that \mathcal{T} is **Compatible** with \mathcal{T} .

Definition 2.1.167 (**Topological Vector Space**). Let $(V, +, \cdot, 0)$ be a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let \mathcal{T} be a **Topology** on V which is **Compatible** with $(V, +, \cdot, 0)$. Then we call (V, \mathcal{T}) a **Topological Vector Space** .

Definition 2.1.168 (**Locally Convex**). We say that a **Topological Vector Space** (X, \mathcal{T}) is **Locally Convex** if (X, \mathcal{T}) has a **Local Basis** consisting only of **Convex** sets. A **Locally Convex** space is said to possess **Local Convexity** .

Proposition 2.1.169 (Existence of **Balanced Neighborhood Basis** of 0 in a **Topological Vector Space**). Let (X, \mathcal{T}) be a **Topological Vector Space** over a **Field** \mathbb{F} . The following are True.

1. If $U \in \mathcal{U}_{\mathcal{T}}(0)$, then there is a **Balanced Set** $V \subset U$ such that $V \in \mathcal{U}_{\mathcal{T}}(0)$.
2. There exists a **Neighborhood Basis** about $0 \in X$ for \mathcal{T} consisting entirely of **Balanced Set** sets.
3. If $U \in \mathcal{U}_{\mathcal{T}}(0)$, then there is a **Balanced Set** $V \subset U$ such that $V \in \mathcal{U}_{\mathcal{T}}(0)$.
4. If (X, \mathcal{T}) is **Locally Convex** , then there exists a **Neighborhood Basis** about $0 \in X$ for \mathcal{T} consisting entirely of **Balanced Set** sets.

Part 01. TODO □

Part 02. TODO □

Part 03. TODO □

Part 04. TODO □

Definition 2.1.170 (TVS Bounded Set). Let (V, \mathcal{T}) be a **Topological Vector Space** . Let $A \subset V$. We say that A is **TVS-Bounded** with respect to \mathcal{T} , or when confusion is unlikely we simply say that A is **TVS-Bounded** if for every $U \in \mathcal{U}_{\mathcal{T}}(0)$, there exists an $\alpha \in \mathbb{F}$, $\alpha > 0$, such that $A \subset \alpha U$.

Definition 2.1.171 (**Bounded Linear Operator**). Let (V_i, \mathcal{T}_i) be a **Topological Vector Space** over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ for $i \in \{0, 1\}$. We say that a **Linear** operator $T : (V_1, \mathcal{T}_1) \rightarrow (V_2, \mathcal{T}_2)$ is a **Bounded Linear Operator** if for each $U \in \mathcal{U}_{\mathcal{T}_1}$ with U **TVS-Bounded** with respect to \mathcal{T}_0 , TU is **TVS-Bounded** with respect to \mathcal{T}_1 .

Definition 2.1.172 (**Space Of Continuous Linear Operators**). Let (U, \mathcal{T}_U) and (V, \mathcal{T}_V) each be a **Topological Vector Space** over the same **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $L(U, V)$ denote the **Space of Linear Operators** from U to V . We denote with $CL((U, \mathcal{T}_U), (V, \mathcal{T}_V))$ the subset of $L(U, V)$ consisting only of the **Continuous** operators. When \mathcal{T}_U and \mathcal{T}_V are understood, we may denote $CL((U, \mathcal{T}_U), (V, \mathcal{T}_V)) = CL(U, V)$

Remark 2.1.173 (**Space Of Continuous Linear Operators** is a **Vector Subspace**). Let (U, \mathcal{T}_U) and (V, \mathcal{T}_V) each be a **Topological Vector Space** over the same **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $L(U, V)$ denote the **Space of Linear Operators** from U to V . Let $CL(U, V)$ denote the **Space Of Continuous Linear Operators** from U to V . Then $CL(U, V)$ is a **Vector Subspace** of $L(U, V)$.

Definition 2.1.174 (**Topology of Uniform Convergence**). Let X be a set and (Y, \mathcal{T}_Y) be a **Topological Vector Space**. Let \mathcal{B} be a **Local Basis** for Y . Suppose \mathcal{F} is a **Vector Subspace** of the set of functions $T : X \rightarrow Y$. Suppose $\mathcal{G} \subset 2^X$ such that (\mathcal{G}, \subset) is a **Directed Set**. For each $x \in X$ and $y \in Y$, and define

$$M(x, y) = \{f \in \mathcal{F} | f(x) \in y\}$$

Now we define

$$\mathcal{T}(\mathcal{F}, \mathcal{T}_Y, \mathcal{G}) = \{f + M(x, y) | x \in \mathcal{G} \wedge y \in \mathcal{B} \wedge f \in \mathcal{F}\}$$

We call $\mathcal{T}(\mathcal{F}, \mathcal{T}_Y, \mathcal{G})$ the **Topology of Uniform Convergence** of \mathcal{F} on \mathcal{G} with respect to \mathcal{T}_Y . When \mathcal{F} , \mathcal{T}_Y or \mathcal{G} are understood they may be omitted from the reference. By 2.1.175, \mathcal{T} is a **Topology** on \mathcal{F} .

Proposition 2.1.175 (**Topology of Uniform Convergence**).

2.1.15 Seminormed Spaces

Definition 2.1.176 (Seminorm). Let V be a **Vector Space** over a **Field** $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. We say that a map $\|\cdot\| : V \rightarrow [0, \infty)$ is a **Seminorm** on V if it is both **Subadditive** and **Absolutely Scalar Homogeneous**. In this case, we refer to $(V, \|\cdot\|)$ as a **Seminormed Space**. We say that $\|\cdot\|$ is **Non-Degenerate** if there is at least one $v \in V$ with $\|v\| > 0$. We say that $\|\cdot\|$ is **Degenerate** if it is not **Non-Degenerate**. We may also refer to the **Seminormed Space** $(V, \|\cdot\|)$ as being **Degenerate** or **Non-Degenerate**.

Definition 2.1.177 (Norm). Let $(V, \|\cdot\|)$ be a **Seminormed Space**. If the following implication is true for $x \in V$, then we refer to $\|\cdot\|$ as a **Norm** on V , and we call $(V, \|\cdot\|)$ a **Normed Space**.

$$x \neq 0 \implies \|x\| \neq 0 \tag{2.62}$$

Proposition 2.1.178 (Subadditive Operator On a Group Induces a Metric). Let $(G, +, e)$ be a **Group** and let $(H, +, \leq)$ be a **Totally Ordered Magma**. Let $p : G \rightarrow H$ be **Subadditive**. define $d : G \times G \rightarrow H$ by setting, for each $x, y \in G$,

$$d(x, y) = p(x + (-y)) \tag{2.63}$$

Then d satisfies the triangle inequality.

Proof. let $x, y, z \in G$. Then

$$\begin{aligned}
 d(x, z) &= p(x + (-z)) \\
 &= p(x + e + (-z)) \\
 &= p(x + (-y) + y + (-z)) \\
 &\leq p(x + (-y)) + p(y + (-z)) \\
 &= d(x, y) + d(y, z)
 \end{aligned}$$

completing the proof. \square

Definition 2.1.179 (Seminorm Topology). Let $(X, \|\cdot\|)$ be a **Seminormed Space**. define $d_{\|\cdot\|} : V \times V \rightarrow [0, \infty)$ by setting, for $x, y \in X$,

$$d_{\|\cdot\|}(x, y) = \|x - y\| \quad (2.64)$$

Observe the following:

1. 2.1.12 guarantees that $d_{\|\cdot\|}(x, x) = 0$ for $x \in X$.
2. 2.1.178 guarantees that d satisfies the **Triangle Inequality**.
3. d is a **Symmetric Map**, as we have

$$d(x, y)_{\|\cdot\|} = \|x - y\| = |-1| \|x - y\| = \|y - x\| = d(y, x) \quad (2.65)$$

Hence, $d_{\|\cdot\|}$ is a **Pseudometric** on X , which we call the **Pseudometric induced by the Seminorm** on X . We refer to $(X, d_{\|\cdot\|})$ as the **Pseudometric Space induced by the Seminormed Space** $(X, \|\cdot\|)$. We refer to the **Pseudometric Topology** induced by $d_{\|\cdot\|}$ as the **Seminorm Topology** induced by $\|\cdot\|$, and unless otherwise specified, when we reference $(X, \|\cdot\|)$, we consider it to be endowed with this topology.

Definition 2.1.180 (Seminorm Kernel). Let $(V, \|\cdot\|)$ be a **Seminormed Space**. Define the set $\mathcal{K}_{(V, \|\cdot\|)}^{\text{ernel}}$ by

$$\mathcal{K}_{(V, \|\cdot\|)}^{\text{ernel}} = \{x \in V \mid \|x\| = 0\} \quad (2.66)$$

We call this set the **Seminorm Kernel** of the space $\mathcal{K}_{(V, \|\cdot\|)}^{\text{ernel}}$. When confusion is unlikely, we may denote this set with $\mathcal{K}^{\text{ernel}}$, $\mathcal{K}_V^{\text{ernel}}$, or even $\mathcal{K}_{\|\cdot\|}^{\text{ernel}}$, or we may just refer to it as the **Seminorm Kernel**, the **Seminorm Kernel** of V , or the **Seminorm Kernel** of $\|\cdot\|$.

Proposition 2.1.181 (Seminorm Kernel is a vector Subspace). Let $(X, \|\cdot\|)$ be a **Seminormed Space** over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ with corresponding **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$. Then the following are true.

1. $\mathcal{K}^{\text{ernel}}$ is a vector subspace of X .
2. $\mathcal{K}^{\text{ernel}}$ is closed in the **Seminorm Topology** on X .

3. $\mathcal{K}^{\text{ernel}} = X$ if and only if X is **Degenerate** .

Proof of One. **Subadditivity** implies that, if $x, y \in \mathcal{K}^{\text{ernel}}$, then $\|x + y\| \leq \|x\| + \|y\| = 0$. By **Scalar Homogeneity**, if $x \in \mathcal{K}^{\text{ernel}}$ and $\alpha \in \mathbb{F}$, $\|\alpha x\| = |\alpha| \|x\| = 0$ so $\mathcal{K}^{\text{ernel}}$ is in fact a vector subspace of X . \square

Proof of Two. If $x \in X \setminus \mathcal{K}^{\text{ernel}}$ then $\|x\| = \alpha > 0$ for some positive α . Hence $B(x; \alpha/2)$ is an open set containing x disjoint from $\mathcal{K}^{\text{ernel}}$. We can then write $X \setminus \mathcal{K}^{\text{ernel}}$ as the union of all such open sets to see that $\mathcal{K}^{\text{ernel}}$ is closed. \square

Proof of Three. Direct application of the definitions of the **Seminorm Kernel** and **Degenerate Seminorm** . \square

Definition 2.1.182 (Quotient Space Mod Kernel). *Let $(X, \|\cdot\|)$ be a **Seminormed Space** over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. with **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$. By 2.1.181, part 1, $\mathcal{K}^{\text{ernel}}$ is a vector subspace of X 's algebraic structure, and so if we define $\cong_{\mathcal{K}^{\text{ernel}}} \subset X \times X$ by setting, for $x, y \in X$*

$$x \cong_{\mathcal{K}^{\text{ernel}}} y \iff x - y \in \mathcal{K}^{\text{ernel}} \quad (2.67)$$

Then one recognizes $\cong_{\mathcal{K}^{\text{ernel}}}$ as **Equivalence MOD- $\mathcal{K}^{\text{ernel}}$** as would be commonly spoken of in Module or Vector Space theory. From this, alot of nice properties fall out. We list them here, without proof just to nail down notation. For proof, see any undergraduate algebra text.

1. If $x \cong_{\mathcal{K}^{\text{ernel}}} y$, then we say that x and y are **Equivalent MOD- $\mathcal{K}^{\text{ernel}}$** .
2. For $x \in X$, we denote the **Equivalence Class** $[x]_{\cong_{\mathcal{K}^{\text{ernel}}}}$ with $[x]_{\mathcal{K}^{\text{ernel}}}$ or with $x + \mathcal{K}^{\text{ernel}}$, or when confusion is unlikely, simply $[x]$.
3. We denote $X / \cong_{\mathcal{K}^{\text{ernel}}}$ with $X / \mathcal{K}^{\text{ernel}}$.
4. If we define $\oplus : X / \mathcal{K}^{\text{ernel}} \times X / \mathcal{K}^{\text{ernel}} \rightarrow X / \mathcal{K}^{\text{ernel}}$ by setting, for $x, y \in X$, $[x]_{\mathcal{K}^{\text{ernel}}} \oplus [y]_{\mathcal{K}^{\text{ernel}}} = [x + y]_{\mathcal{K}^{\text{ernel}}}$, then \oplus is well defined and endows $X / \mathcal{K}^{\text{ernel}}$ with a group structure.
5. If we further define $\odot : \mathbb{F} \times X / \mathcal{K}^{\text{ernel}} \rightarrow X / \mathcal{K}^{\text{ernel}}$ by $\alpha[x]_{\mathcal{K}^{\text{ernel}}} = [\alpha x]_{\mathcal{K}^{\text{ernel}}}$, then $(X / \mathcal{K}^{\text{ernel}}, \oplus, \odot, [0]_{\mathcal{K}^{\text{ernel}}})$ is a Vector space over \mathbb{F} .
6. Unless otherwise specified, when referring to the set $X / \mathcal{K}^{\text{ernel}}$, we endow it with the above vector space structure, and we call this space the **Seminorm Kernel Quotient Vector Space** of the seminormed space $(X, \|\cdot\|)$.

Proposition 2.1.183 (Equivalence Mod Kernel is Pseudometric Equivalence). *Let $(X, \|\cdot\|)$ be a seminormed space. with **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$. Let d denote the **Pseudometric induced by the Seminorm**. Let \cong_d denote the **Relation Of Zero Distance** with respect to d .*

Then $\cong_{\mathcal{K}^{\text{ernel}}} = \cong_d$.

Proof. Let $x, y \in X$ and let $x \cong_{\mathcal{K}^{\text{ernel}}} y$. Then, since $x - y \in \mathcal{K}^{\text{ernel}}$, Then $d(x, y) := \|x - y\| = 0$, so $x \cong_d y$. Hence $\cong_{\mathcal{K}^{\text{ernel}}} \subset \cong_d$

Now let $x, y \in X$ with $x \cong_d y$. Then $\|x - y\| = d(x, y) = 0$, so $x - y \in \mathcal{K}^{\text{ernel}}$, and therefore $x \cong_{\mathcal{K}^{\text{ernel}}} y$. Hence, $\cong_d \subset \cong_{\mathcal{K}^{\text{ernel}}}$.

Since inclusion goes both directions, $\cong_{\mathcal{K}^{\text{ernel}}} = \cong_d$.

□

Definition 2.1.184 (Quotient Norm Space). Let $(X, \|\cdot\|)$ be a **Seminormed Space** with **Pseudometric induced by the Seminorm** d , **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$, and **Seminorm Kernel Quotient Vector Space** $X/\mathcal{K}^{\text{ernel}}$. Let $\tilde{d} : X/\mathcal{K}^{\text{ernel}} \times X/\mathcal{K}^{\text{ernel}} \rightarrow [0, \infty)$ be the **Metric Induced By The Pseudometric**.

Define $\|\cdot\|_{\mathcal{K}^{\text{ernel}}} : X/\mathcal{K}^{\text{ernel}} \rightarrow [0, \infty)$ by setting, for $x \in X$,

$$\|[x]\|_{\mathcal{K}^{\text{ernel}}} = \tilde{d}([x], [0]) \quad (2.68)$$

By 2.1.185, $(X/\mathcal{K}^{\text{ernel}}, \|\cdot\|_{\mathcal{K}^{\text{ernel}}})$ is a normed space which we call the **Quotient Normed Space** of $(X, \|\cdot\|)$, and we call $\|\cdot\|_{\mathcal{K}^{\text{ernel}}}$ the **Quotient Norm**. Whenever we refer to $X/\mathcal{K}^{\text{ernel}}$, unless otherwise specified, we endow it with this norm and the topology generated by this norm. Furthermore, whenever we consider $X/\mathcal{K}^{\text{ernel}}$, unless otherwise specified, we consider it as possessing the topology generated by the norm $\|\cdot\|_{\mathcal{K}^{\text{ernel}}}$.

Proposition 2.1.185 (Quotient Normed Space). Let $(X, \|\cdot\|)$ be a **Seminormed Space** with **Pseudometric induced by the Seminorm** d , **Seminorm Kernel** $\mathcal{K}^{\text{ernel}}$, and **Seminorm Kernel Quotient Vector Space** $X/\mathcal{K}^{\text{ernel}}$. Let $\tilde{d} : X/\mathcal{K}^{\text{ernel}} \times X/\mathcal{K}^{\text{ernel}} \rightarrow [0, \infty)$ be the **Metric Induced By The Pseudometric**. Let $T : X \rightarrow X/\mathcal{K}^{\text{ernel}}$ denote the **Quotient Map** of X into $X/\mathcal{K}^{\text{ernel}}$ (Recalling that the **Relation Of Equal Neighborhood Filters** on e equals the **Relation Of Zero Distance** equals the relation of **Equivalence MOD- $\mathcal{K}^{\text{ernel}}$**), so they would all produce the same quotient map) Let $\|\cdot\|_{\mathcal{K}^{\text{ernel}}}$ denote the **Quotient Norm**.

The following are true.

1. $\|\cdot\|_{\mathcal{K}^{\text{ernel}}}$ is a norm on $X/\mathcal{K}^{\text{ernel}}$.
2. \tilde{d} is the **Pseudometric induced by the Seminorm** $\|\cdot\|_{\mathcal{K}^{\text{ernel}}}$, and thus they produce the same topology.
3. T has all of the properties described in 2.1.112.
4. T is Linear.
5. T is Surjective.
6. T is an isometry.
7. T is injective if and only if $\|\cdot\|$ is a norm.

Proof of 1. First, note that $\text{Range}(\|\cdot\|_{\mathcal{K}^{\text{ernel}}}) \subset \text{Range}(\tilde{d}) \subset [0, \infty)$, so that $\|\cdot\|_{\mathcal{K}^{\text{ernel}}}$ has the correct domain and codomain. For **Subadditivity**, let $[x], [y] \in X/\mathcal{K}^{\text{ernel}}$. Then

$$\begin{aligned} \|[x] + [y]\|_{\mathcal{K}^{\text{ernel}}} &= \|[x + y]\|_{\mathcal{K}^{\text{ernel}}} \\ &= \tilde{d}([x + y], [0]) \\ &= d(x + y, 0) \\ &= \|x + y\| \\ &\leq \|x\| + \|y\| \\ &= d(x, 0) + d(y, 0) \\ &= \tilde{d}([x], [0]) + \tilde{d}([y], [0]) \\ &= \|[x]\|_{\mathcal{K}^{\text{ernel}}} + \|[y]\|_{\mathcal{K}^{\text{ernel}}} \end{aligned}$$

For **Absolute Scalar Homogeneity**, let $\alpha \in \mathbb{F}$ and $[x] \in X/\mathcal{K}^{\text{ernel}}$. Then,

$$\begin{aligned} \|[\alpha x]\|_{\mathcal{K}^{\text{ernel}}} &= \tilde{d}([\alpha x], [0]) \\ &= d(\alpha x, 0) \\ &= \|\alpha x\| \\ &= |\alpha| \|x\| \\ &= |\alpha| \|[x]\|_{\mathcal{K}^{\text{ernel}}} \end{aligned}$$

Finally, suppose $[x] \neq 0$. Then, since the additive identity of $X/\mathcal{K}^{\text{ernel}}$ is $\mathcal{K}^{\text{ernel}}$, $x \notin \mathcal{K}^{\text{ernel}}$. Hence $\|[x]\|_{\mathcal{K}^{\text{ernel}}} = \tilde{d}([x], 0) = d(x, 0) = \|x\| > 0$. □

Proof of 2. Let D denote the **Pseudometric induced by the Seminorm** $\|\cdot\|_{\mathcal{K}^{\text{ernel}}}$. Then, for $[x], [y] \in X/\mathcal{K}^{\text{ernel}}$,

$$\begin{aligned} \tilde{d}([x], [y]) &= d(x, y) \\ &= \|x - y\| \\ &= \|x - y - 0\| \\ &= d(x - y, 0) \\ &= \tilde{d}([x - y], 0) \\ &= \|[x - y]\|_{\mathcal{K}^{\text{ernel}}} \\ &= \|[x] - [y]\|_{\mathcal{K}^{\text{ernel}}} = D([x], [y]) \end{aligned}$$

Since these two **Pseudometric**'s are equal, they produce the same topology. Furthermore, by applying 2.1.162, we see that the topology generated by $\|\cdot\|_{\mathcal{K}^{\text{ernel}}}$ is also the **Quotient Topology** on $X/\mathcal{K}^{\text{ernel}}$. □

Proof of 3. T is the topological **Quotient Map** and the norm topology is the **Quotient Topology**, so the assumptions of 2.1.112 are satisfied. □

Proof of 4. Let $x, y \in X$ and $\alpha \in \mathbb{F}$. Then

$$\begin{aligned} T(\alpha x + y) &= [\alpha x + y] \\ &= (\alpha x + y) + \mathcal{K}^{\text{ernel}} \\ &= \alpha(x + \mathcal{K}^{\text{ernel}}) + (y + \mathcal{K}^{\text{ernel}}) \\ &= \alpha[x] + [y] \\ &= \alpha T(x) + T(y) \end{aligned}$$

□

Proof of 5. Direct consequence of 2.1.7

□

Proof of 6. Consequence of part 2 of this result combined with

□

Proof of 7. If $\|\cdot\|$ is a **Norm**, then $\mathcal{K}^{\text{ernel}} = 0$, so $Tx = Ty \implies T(x - y) = 0 \implies x - y \in \mathcal{K}^{\text{ernel}} \implies x - y = 0 \implies x = y$.

□

Remark 2.1.186 (Quotient Normed Space). If $(X, \|\cdot\|_X)$ is a **Normed Space** then by parts 4, 5, 6, and 7 of 2.1.185, $T : X \rightarrow \mathcal{K}_X^{\text{ernel}}$ is an isomorphism of **Normed Spaces** whose definition is literally

$$Tx = \{x\} \tag{2.69}$$

For this reason, as an admitted abuse of notation, later in this document, I may not distinguish between the quotient $X/\mathcal{K}_X^{\text{ernel}}$ and the space X if X is a **Normed Space**, and similarly, I may not distinguish between $x \in X$ and $\{x\} \in X/\mathcal{K}_X^{\text{ernel}}$.

Proposition 2.1.187. Let $(X, \|\cdot\|)$ be a **Seminormed Space** with **Quotient Normed Space** $(X/\mathcal{K}^{\text{ernel}}, \|\cdot\|_{\mathcal{K}^{\text{ernel}}})$.

Then X is **Pseudometric-Complete** if and only if $X/\mathcal{K}^{\text{ernel}}$ is complete.

Proof. Let X be **Pseudometric-Complete**. Let $\{[x_i]\}_{i \in \mathbb{N}} \subset X/\mathcal{K}^{\text{ernel}}$ be a **Pseudometric Cauchy Sequence**. Let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for $m, n > N$ we have

$$\|[x_m - x_n]\|_{\mathcal{K}^{\text{ernel}}} < \epsilon \tag{2.70}$$

For this N , we have

$$\|x_m - x_n\| = \|[x_m - x_n]\|_{\mathcal{K}^{\text{ernel}}} < \epsilon \tag{2.71}$$

so that $\{x_i\}_{i \in \mathbb{N}}$ is a **Pseudometric Cauchy Sequence**. Since X is **Pseudometric-Complete**, there is a $x \in X$ such that $\|x_i - x\| \rightarrow 0$, but since T is an isometry,

$$\|[x] - [x_i]\| = \|[x_i - x]\|_{\mathcal{K}^{\text{ernel}}} \rightarrow 0 \tag{2.72}$$

and so $[x_i] \rightarrow [x]$. so that $X/\mathcal{K}^{\text{ernel}}$ is complete.

Now suppose instead that $X/\mathcal{K}^{\text{ernel}}$ is complete and suppose $\{x_i\}_{i \in \mathbb{N}}$ is a **Pseudometric Cauchy Sequence** in X . Since $\|[x_i - x_j]\|_{\mathcal{K}^{\text{ernel}}} = \|x_i - x_j\|$, $\{[x_i]\}_{i \in \mathbb{N}}$ is a **Pseudometric Cauchy Sequence** in $X/\mathcal{K}^{\text{ernel}}$, which therefore has a limit $y \in X/\mathcal{K}^{\text{ernel}}$. Since T is surjective, $y = [x]$ for some $x \in X$, and it is easy to see that $x_i \rightarrow x$ so that X is **Pseudometric-Complete**.

□

Definition 2.1.188 (Space of Continuous Linear Operators From a Seminormed Space into a Normed Space). Let $(X, \|\cdot\|_X)$ be a **Non-Degenerate Seminormed Space**. Let $(Y, \|\cdot\|_Y)$ be a **Seminormed Space**. We denote with $BL((X, \|\cdot\|_X), (Y, \|\cdot\|_Y))$ the collection of **Continuous Linear** operators $T : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$. When the topologies on X and Y are understood, we denote this set with $BL(X, Y)$. We refer to $BL(X, Y)$ as the **Space of Bounded Linear Operators** from $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$, or when $\|\cdot\|_X$ and $\|\cdot\|_Y$ are understood, from X to Y .

We endow $BL(X, Y)$ with the algebraic operations of pointwise scalar multiplication and pointwise addition, making $BL(X, Y)$ a vector space.

We define $\|\cdot\| : BL(X, Y) \rightarrow [0, \infty)$ by defining, for $T \in BL(X, Y)$

$$\|T\| = \sup_{\|x\|_X \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} \quad (2.73)$$

As will be proven in 2.1.189, $\|\cdot\|$ is a **Seminorm** on $BL(X, Y)$, which we refer to as the **Operator Seminorm** on $BL(X, Y)$. induced by the **Seminorm** $\|\cdot\|_X$ on X and the **Seminorm** $\|\cdot\|_Y$ on Y .

In the case that $\|\cdot\|_Y$ is a **Norm**, rather than just a **Seminorm**, by 2.1.189, $\|\cdot\|$ is a **Norm** on $BL(X, Y)$, which we instead call the **Operator Norm**.

Proposition 2.1.189 (Space of Bounded Linear Operators On Seminormed Spaces). Let $(X, \|\cdot\|_X)$ be a **Seminormed Space**. Let $(Y, \|\cdot\|_Y)$ be a **Seminormed Space**. Let $BL(X, Y)$ denote the **Space of Bounded Linear Operators** from X to Y . Let $\|\cdot\|$ denote the **Operator Seminorm**.

The following are true.

1. $\|\cdot\|$ is in fact a well-defined **Seminorm** on $BL(X, Y)$.
2. If $\|\cdot\|_Y$ is a **Norm**, then so is $\|\cdot\|$.
3. If $T \in BL(X, Y)$ and $\alpha \in (0, \infty)$, then $\|T\| = \sup_{\|x\|_X = \alpha} \frac{\|Tx\|_Y}{\|x\|_X}$.
4. If $T \in BL(X, Y)$ and $\alpha \in (0, \infty)$, then $\|T\| = \sup_{0 < \|x\|_X \leq \alpha} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{0 < \|x\|_X < \alpha} \frac{\|Tx\|_Y}{\|x\|_X}$.
5. If $T \in BL(X, Y)$ and $x \in X$, then $\|Tx\|_Y \leq \|T\| \|x\|_X$.
6. $S : X \rightarrow Y$ is linear, $S(\mathcal{K}_X^{\text{ernel}}) \subset \mathcal{K}_Y^{\text{ernel}}$, and $\sup_{\|x\|_X \neq 0} \frac{\|Sx\|_Y}{\|x\|_X} < \infty$, if and only if $S \in BL(X, Y)$.
7. A sequence $\{T_i\}_{i \in \mathbb{N}}$ is a **Pseudometric Cauchy Sequence** if and only if there exists an $\alpha > 0$ such that the collection of sequences $\{\{T_i x\}_{i \in \mathbb{N}} | x \in B_X(0; \alpha)\}$ is **Uniformly Cauchy** if and only if for every $\beta > 0$, the collection of sequences $\{\{T_i x\}_{i \in \mathbb{N}} | x \in B_X(0; \beta)\}$ is **Uniformly Cauchy**.
8. If $T_i \rightarrow T$ with respect to $\|\cdot\|$, then $T_i x \rightarrow Tx$ with respect to $\|\cdot\|_Y$ for each $x \in X$.

9. A sequence $\{T_i\}_{i \in \mathbb{N}} \subset BL(X, Y)$ converges with respect to $\|\cdot\|$ if and only if it is a **Pseudometric Cauchy Sequence** and for each x_α in some Hamel basis $\{x_\alpha\}_{\alpha \in A} \subset X$, the sequence $\{T_i x_\alpha\}_{i \in \mathbb{N}}$ converges with respect to $\|\cdot\|_Y$.
10. Let X be **Non-Degenerate**. Then $BL(X, Y)$ is complete if Y is complete.
11. If $BL(X, Y)$ is **Non-Degenerate** then Y is **Non-Degenerate**.
12. If $S : X \rightarrow Y$ is linear, then $S \in BL(X, Y)$ if and only if there exists a constant $c \in (0, \infty)$ such that for every $x \in X$, $\|Tx\| \leq c\|x\|$.
13. If $T \in BL(X, Y)$, and $A = \{c \in (0, \infty) \mid \|Tx\| \leq c\|x\| \ (\forall x \in X)\}$, then $\|T\| = \inf(A)$.

Proof of 1. Since X is **Non-Degenerate**, there exists an $x \in X$ with $\|x\|_X \neq 0$, so for each $T \in BL(X, Y)$, the set that the supremum is being taken over is nonempty. Also, it is clear that $\text{Range}(\|\cdot\|) \subset [0, \infty)$,

For **Subadditivity**, let $T_i \in BL(X, Y)$ for $i \in \{0, 1\}$. and $x \in X$ with $\|x\| > 0$. Then, since $\|\cdot\|_Y$ is **Subadditive**,

$$\frac{\|(T_0 + T_1)x\|_Y}{\|x\|_X} \leq \frac{\|T_0 x\|_Y}{\|x\|_X} + \frac{\|T_1 x\|_Y}{\|x\|_X}$$

Since this is true for each x with $\|x\|_X \neq 0$, taking the supremum of each side yields

$$\begin{aligned} \sup_{\|x\|_X \neq 0} \left(\frac{\|(T_0 + T_1)x\|_Y}{\|x\|_X} \right) &\leq \sup_{\|x\|_X \neq 0} \left(\frac{\|T_0 x\|_Y}{\|x\|_X} + \frac{\|T_1 x\|_Y}{\|x\|_X} \right) \\ &\leq \sup_{\|x\|_X \neq 0} \left(\frac{\|T_0 x\|_Y}{\|x\|_X} \right) + \sup_{\|x\|_X \neq 0} \left(\frac{\|T_1 x\|_Y}{\|x\|_X} \right) \end{aligned}$$

Hence, $\|T_0 + T_1\| \leq \|T_0\| + \|T_1\|$ so that $\|\cdot\|$ is **Subadditive**. For **Absolute Scalar Homogeneity**, let $T \in BL(X, Y)$, $\alpha \in \mathbb{F}$, and $x \in X$ with $\|x\|_X \neq 0$. Then

$$\frac{\|(\alpha T)x\|_Y}{\|x\|_X} = \frac{\|\alpha(Tx)\|_Y}{\|x\|_X} = |\alpha| \frac{\|Tx\|_Y}{\|x\|_X}$$

Hence taking the supremum finishes the proof. \square

Proof of 2. Let $T \neq 0 \in BL(X, Y)$. Then for some $x \in X$, $Tx \neq 0$. Then Tx has a neighborhood U disjoint from 0_Y , Hence $x \in T^{-1}(U)$ but not $0_X \in T^{-1}(U)$, since $T0_X = 0_Y$. Since U is a neighborhood of x disjoint from 0 , there is an $\epsilon > 0$ such that $0_X \subset \mathcal{C}\overline{B_X}(x; \epsilon)$, and therefore $\|x\|_X > \epsilon$. Since $\|x\|_X > 0$, it is ranged over in the supremum defining $\|T\|$, and so

$$0 < \frac{\|Tx\|_Y}{\|x\|_X} \leq \sup_{\|x\|_X \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \|T\| \quad (2.74)$$

\square

Proof of 3. Let $\alpha \in (0, \infty)$ Let $T \in BL(X, Y)$. Then, there is a sequence $\{x_i\} \subset X$ with each $\|x_i\|_X \neq 0$ such that

$$\frac{\|Tx_i\|_Y}{\|x_i\|_X} \rightarrow \|T\| \quad (2.75)$$

For each $i \in \mathbb{N}$, define $y_i = \alpha x_i / \|x_i\|_X$. then each $\|y_i\| = \alpha$, and by **Absolute Scalar Homogeneity** of T , we have

$$\frac{\|Ty_i\|_Y}{\|y_i\|_X} = \frac{\|Tx_i\|_Y}{\|x_i\|_X} \rightarrow \|T\| \quad (2.76)$$

, completing the proof. \square

Proof of 4. If we define, for $T \in BL(X, Y)$, $f(T) = \sup_{0 < \|x\|_X \leq \alpha} \frac{\|Tx\|_Y}{\|x\|_X}$, then since $\|\cdot\|^{-1}((0, \alpha)) \subset \|\cdot\|^{-1}((0, \infty))$, we have $f(T) \leq \|T\|$ and since $\|\cdot\|^{-1}(\{\alpha\}) \subset \|\cdot\|^{-1}((0, \alpha))$, we have $\|T\| \leq f(T)$. proving the first equality. The second is found by applying the same argument to $\alpha/2$ and realizing that $(0, \alpha/2] \subset (0, \alpha)$. \square

Proof of 5. Let $T \in BL(X, Y)$ and $x \in X$. If $\|Tx\|_Y \neq 0$, then $B_Y(Tx, \frac{\|Tx\|_Y}{2})$ is a neighborhood of Tx disjoint from 0. Continuity of T implies x then has a neighborhood disjoint from $0 \in T^{-1}(0)$, implying that $\|x\|_X \neq 0$.

Hence if $\|x\|_X = 0$, then we know $\|Tx\|_Y = 0$, so that the relation

$$\|Tx\|_Y \leq \|T\| \|x\|_X \quad (2.77)$$

If $\|x\|_X \neq 0$, then by definition of supremum,

$$\frac{\|Tx\|_Y}{\|x\|_X} \leq \|T\|$$

so that $\|Tx\|_Y \leq \|T\| \|x\|_X$. \square

Proof of 6. I assume the first 3 conditions and show that $S \in BL(X, Y)$. It is necessary and sufficient to show that S is continuous. Let $F = \sup_{\|x\|_X \neq 0} \frac{\|Sx\|_Y}{\|x\|_X}$. If $F = 0$, then $S(X) \subset \mathcal{K}_Y^{\text{ernel}}$.

Every neighborhood of every point in $\mathcal{K}_Y^{\text{ernel}}$ contains $\mathcal{K}_Y^{\text{ernel}}$, so in that case continuity holds. Suppose $F \neq 0$. By translation invariance of the topology, it is sufficient to consider neighborhoods of $0_Y \in Y$. Let $\epsilon > 0$. Define $V = B_X(0; \frac{\epsilon}{F})$. Let $x_0 \in V$. If $\|x_0\|_X = 0$, then $S(x_0) \in S(\mathcal{K}_X^{\text{ernel}}) \subset \mathcal{K}_Y^{\text{ernel}} \subset B_Y(0; \epsilon)$. If $\|x_0\|_X \neq 0$, then $\|Sx_0\|_Y \leq F \|x_0\|_X < \epsilon$, so $s(x_0) \in B_Y(0; \epsilon)$. Hence $S(B_X(0; \frac{\epsilon}{F})) \subset B_Y(0; \epsilon)$. so S is continuous, and this direction for the proof is complete.

Suppose conversely that $S \in BL(X, Y)$. Then S is **Linear** by definition, and the supremum expression is finite by part 1 of this result. Since S is **Linear**, $S0_X = 0_Y$. Since S is **Continuous**,

$$\begin{aligned} S(\mathcal{K}_X^{\text{ernel}}) &= S(\overline{\{0_X\}}) \\ &\subset \overline{S(\{0_X\})} \\ &= \overline{\{0_Y\}} \\ &= \mathcal{K}_Y^{\text{ernel}} \end{aligned}$$

□

Proof of 7. (3 \implies 2) is trivial, as is (2 \implies 3).

I now prove (1 \implies 3). Let $\{T_i\}_{i \in \mathbb{N}}$ be a **Pseudometric Cauchy Sequence**. Let $\beta > 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $m, n > N$,

$$\|T_n - T_m\| < \frac{\epsilon}{\beta}$$

Let $x \in B_X(0; \beta)$. Then

$$\begin{aligned} \|T_m x - T_n x\|_Y &= \|(T_m - T_n)x\|_Y \\ &\leq \|T_m - T_n\| \|x\|_X \\ &< \epsilon \end{aligned}$$

Since $x \in B_X(0; \beta)$ was arbitrary, $\{\{T_i x\}_{i \in \mathbb{N}} \mid x \in B_X(0; \beta)\}$ is **Uniformly Cauchy**.

I now prove (3 \implies 1). Let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for $m, n > N$, for each $x \in B_X(0; 2)$,

$$\|T_m x - T_n x\| < \epsilon$$

In particular, if $\|x\| = 1$, then

$$\frac{\|(T_m - T_n)x\|_Y}{\|x\|_X} = \|(T_m - T_n)x\|_Y < \epsilon \quad (2.78)$$

Hence, by taking the supremum over such x and applying part 3 of this result, $\|T_m - T_n\| < \epsilon$. □

Proof of 8. Let $T_i \rightarrow T$. Let $x \in X$. If $x \in \mathcal{K}_X^{\text{ernel}}$, then $T_i(x) \in \mathcal{K}_Y^{\text{ernel}}$ for $i \in \mathbb{N}$ and $T_x \in \mathcal{K}_Y^{\text{ernel}}$, so convergence is obvious. Suppose $\|x\|_X > 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $n > N$, $\|T_n - T\| < \frac{\epsilon}{\|x\|_X}$. For such n ,

$$\|T_i x - T x\|_Y \leq \|T_i - T\| \|x\|_X < \epsilon$$

□

Proof of 9. (\implies) Suppose $T_i \rightarrow T$. Then, by 2.1.151, $\{T_i\}_{i \in \mathbb{N}}$ is a **Pseudometric Cauchy Sequence**. An application of part 8 of this result implies the pointwise convergence on a hamel basis.

(\Leftarrow) Let $\{x_\alpha\}_{\alpha \in A}$ be a Hamel basis for X . Let $T_i x_\alpha \rightarrow y_\alpha$ for $\alpha \in A$. Define $T : X \rightarrow Y$ by setting, for $x \in X$, for any $\{\alpha_i\}_{i=1}^n \subset A$ $\{\beta_i\}_{i=1}^n \subset \mathbb{F}$,

$$T \left(\sum_{i=1}^n \beta_i x_{\alpha_i} \right) = \sum_{i=1}^n \beta_i y_{\alpha_i} \quad (2.79)$$

The uniqueness of a hamel basis representation implies that T is well defined. It is clear also that T is linear, and that $T(\mathcal{K}_X^{\text{ernel}}) \subset \mathcal{K}_Y^{\text{ernel}}$.

Let $x \in X$. Then we can find a unique representation, $x = \sum_{j=1}^n \beta_j x_{\alpha_j}$ where $x_{\alpha_j} \in A$ and $\beta_j \in \mathbb{F}$ for every j . For each $j \in \{1, \dots, n\}$, there is an N_j such that if $n_j > N_j$, the

$$\|T_{n_j} x_{\alpha_j} - y_{\alpha_j}\| < \frac{\epsilon}{n(|\beta_j| + 1)} \quad (2.80)$$

Let $N = \max\{N_j\}_{j=1}^n$. Let $m > N$. Then, we have

$$\begin{aligned} \|T_m x - Tx\| &= \left\| T_m \left(\sum_{j=1}^n \beta_j x_{\alpha_j} \right) - T \left(\sum_{j=1}^n \beta_j x_{\alpha_j} \right) \right\| \\ &= \left\| \sum_{j=1}^n \beta_j (T_m x_{\alpha_j} - T x_{\alpha_j}) \right\| \\ &= \sum_{j=1}^n |\beta_j| \|T_m x_{\alpha_j} - T x_{\alpha_j}\| \\ &< \epsilon \end{aligned}$$

Since $m > N$ was arbitrary, $T_i x \rightarrow Tx$ for $x \in X$.

Since $T_i x \rightarrow Tx$ for $x \in B_X(0; 2)$, and since part 7 of this result, paired with the assumption that $\{T_i\}_{i \in \mathbb{N}}$ is a **Pseudometric Cauchy Sequence**, $\{\{T_i x\}_{i \in \mathbb{N}}\}_{x \in B_X(0; 2)}$ is **Uniformly Convergent** to $\{Tx\}_{x \in B_X(0; 2)}$.

Let $\epsilon > 0$. By **Uniform Convergence**, there is an $N \in \mathbb{N}$ such that for $n > N$, $x \in B_X(0; 2)$, we have

$$\|T_n x - Tx\| < \epsilon \quad (2.81)$$

In particular, if $\|x\| = 1$,

$$\frac{\|(T_n - T)x\|_Y}{\|x\|_X} < \epsilon \quad (2.82)$$

Implying first through part 6 of this result that $T \in BL(X, Y)$, and second that Hence $T_i \rightarrow T$ with respect to $\|\cdot\|$. \square

Proof of 10. Let Y be Pseudometric Complete. and let X be **Non-Degenerate**. Let $\{T_\alpha\}_{\alpha \in A} \subset BL(X, Y)$ be a **Pseudometric Cauchy Sequence**. Let $\{x_\alpha\}_{\alpha \in A}$ be a Hamel basis for X . Let $\alpha \in A$. If $x_\alpha \in \mathcal{K}_X^{\text{ernel}}$, then $T_i x_\alpha \in \mathcal{K}_Y^{\text{ernel}}$ for $i \in \mathbb{N}$, and so $T_i x_\alpha \rightarrow 0$. Otherwise, $\|x_\alpha\|_X > 0$, so for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n > N$, we have $\|T_m - T_n\| < \frac{\epsilon}{\|x_\alpha\|}$. For such m and n ,

$$\begin{aligned} \frac{\|T_m x_\alpha - T_n x_\alpha\|_Y}{\|x_\alpha\|_X} &= \frac{\|(T_m - T_n)x_\alpha\|_Y}{\|x_\alpha\|_X} \\ &\leq \|T_m - T_n\| \\ &< \frac{\epsilon}{\|x_\alpha\|} \end{aligned}$$

By multiplying by $\|x_\alpha\|_X$, we see that $\{T_i x_\alpha\}_{i \in \mathbb{N}}$ is a **Pseudometric Cauchy Sequence**. Since Y is **Complete**, these sequences converge, say $T_i x_\alpha \rightarrow y_\alpha$. This allows us apply part 9 of this result to claim that $\{T_i\}_{i \in \mathbb{N}}$ converges in $\|\cdot\|$. \square

Proof of 11. If $BL(X, Y)$ is nondegenerate, then for some $T \in BL(X, Y)$, for some $x \in X$, $\|Tx\|_Y \neq 0$. \square

Proof of 12. Let $S \in BL(X, Y)$. Let $c = \|S\| + 1$. Then, if $\|x\| \neq 0$, we have

$$\frac{\|Sx\|}{\|x\|} \leq \sup_{\|x\| \neq 0} \frac{\|Sx\|}{\|x\|} = \|S\| = c \quad (2.83)$$

so multiplying by $\|x\|$ gives $\|Sx\| \leq c\|x\|$. If $\|x\| = 0$, then $x \in \mathcal{K}_X^{\text{ernel}}$, so by part 6 of this result, $Sx \in \mathcal{K}_Y^{\text{ernel}}$, so

$$\|Sx\| = 0 = c0 = c\|Sx\| \quad (2.84)$$

finishing this direction of the proof.

Suppose instead, now, that there was a constant $c \in (0, 1)$ such that for each $x \in X$, we had $\|Sx\| \leq c\|x\|$. Then, if $\|x\| = 0$, we must have $\|Sx\| \leq c\|x\| = 0$, and so $S(\mathcal{K}_X^{\text{ernel}}) \subset \mathcal{K}_Y^{\text{ernel}}$. Furthermore, if $\|x\| \neq 0$, then we can divide by $\|x\|$ to get

$$\frac{\|Sx\|}{\|x\|} \leq c \quad (2.85)$$

Taking the supremum over all x with $\|x\| \neq 0$ gives us the desired result. \square

Proof of 13. By part 5 of this result, $\|T\|$ is one possible value for c , so $\|T\| \in A$. That is, $\inf(A) \leq \|T\|$. Furthermore, if $\epsilon > 0$, then there is an $x \in X$ with

$$\frac{\|Tx\|}{\|x\|} > \|T\| - \epsilon \quad (2.86)$$

implying $\|Tx\| > (\|T\| - \epsilon)\|x\|$ so that $\|T\| - \epsilon \notin A$. Hence $\|T\| - \epsilon < \inf(A)$. Since this is true for every positive epsilon, $\|T\| < \inf(A)$. \square

Remark 2.1.190 (Converses). *The converses to parts 10 and 11 of 2.1.189 are true, but their proof relies upon the Hahn Banach theorem for Seminormed spaces, and so their official claim is delayed until later in the narrative.*

Definition 2.1.191 (Codomain Quotient Operator). *Let X and Y be Seminormed Spaces . Define $\mathcal{Q}_Y : BL(X, Y) \rightarrow BL(X, Y/\mathcal{K}_Y^{\text{ernel}})$ by setting, for each $x \in X$,*

$$\mathcal{Q}_Y Tx = [Tx]$$

Let $T \in BL(X, Y)$. We call \mathcal{Q}_Y the **Codomain Quotient Map** of X and Y and we call $\mathcal{Q}_Y T$ the **Codomain Quotient Operator** of T .

Proposition 2.1.192 (Codomain Quotient Operator). *Let X and Y be Seminormed Spaces with Codomain Quotient Map \mathcal{Q}_Y . The following are true.*

1. \mathcal{Q}_Y is a well defined continuous linear surjective isometry.
2. If Y is a Normed Space , then \mathcal{Q}_Y is invertible with a continuous inverse.

Proof Of 1. Since $Tx \in Y$ for any $x \in X$, $[Tx]_Y$ is defined for any $x \in X$. Furthermore, if $q_y : Y \rightarrow Y/\mathcal{K}^{\text{ernel}}$ is the **Quotient Map** of Y under **Equivalence MOD- $\mathcal{K}^{\text{ernel}}$** , then $\mathcal{Q}_Y T = q_y \circ T$. By 2.1.185, $q_y \in BL(Y, Y/\mathcal{K}^{\text{ernel}})$. Hence, $\mathcal{Q}_Y T = q_y \circ T \in BL(X, Y/\mathcal{K}^{\text{ernel}})$. Hence \mathcal{Q}_Y is well defined.

For linearity, let $\alpha \in \mathbb{F}$ and $S, T \in BL(X, Y)$. Let $x \in X$. Then,

$$\begin{aligned} \mathcal{Q}_Y (\alpha T + S) x &= [(\alpha T + S) x]_Y \\ &= [\alpha T x + S x]_Y \\ &= [\alpha T x]_Y + [S x]_Y \\ &= \alpha [T x]_Y + [S x]_Y \\ &= \alpha \mathcal{Q}_Y T x + \mathcal{Q}_Y S x \\ &= (\alpha \mathcal{Q}_Y T + \mathcal{Q}_Y S) x \end{aligned}$$

For being an isometry, let $T \in BL(X, Y)$ and let $x \in X$. Then, since $\|[Tx]_Y\|_{Y/\mathcal{K}^{\text{ernel}}} = \|Tx\|_Y$,

$$\begin{aligned} \frac{\|\mathcal{Q}_Y T x\|_{Y/\mathcal{K}^{\text{ernel}}}}{\|x\|_X} &= \frac{\|[Tx]_Y\|_{Y/\mathcal{K}^{\text{ernel}}}}{\|x\|_X} \\ &= \frac{\|Tx\|_Y}{\|x\|_X} \end{aligned}$$

and thus taking the norm over x with $\|x\|_X \neq 0$ will yield the same result. Hence $\|T\| = \|\mathcal{Q}_Y T\|$.

For surjectivity, let $\tilde{T} \in BL(X, Y/\mathcal{K}_Y^{\text{ernel}})$. Let $\{x_\alpha\}_{\alpha \in A}$ be a hamel basis for X . For each $\alpha \in A$, let $y_\alpha \in \tilde{T}x_\alpha$. Define $T : X \rightarrow Y$ by

$$T \left(\sum_{i=1}^n \beta_{\alpha_i} x_{\alpha_i} \right) = \sum_{i=1}^n \beta_{\alpha_i} y_{\alpha_i} \quad (2.87)$$

T is obviously linear and has the property $[Tx] = \tilde{T}x$. and since $\tilde{T} \in BL(X, Y/\mathcal{K}_Y^{\text{ernel}})$, $\tilde{T}\mathcal{K}_X^{\text{ernel}} \subset \mathcal{K}_{(Y/\mathcal{K}_Y^{\text{ernel}})}^{\text{ernel}} = 0$. Hence $T\mathcal{K}_X^{\text{ernel}} \subset \mathcal{K}_Y^{\text{ernel}}$. Furthermore, if $x \in X$ with $\|x\|_X \neq 0$, then

$$\begin{aligned} \frac{\|Tx\|_Y}{\|x\|_X} &= \frac{\|[Tx]_Y\|_{Y/\mathcal{K}^{\text{ernel}}}}{\|x\|_X} \\ &= \frac{\|\tilde{T}x\|_{Y/\mathcal{K}^{\text{ernel}}}}{\|x\|_X} \end{aligned}$$

Therefore T is bounded. Hence $T \in BL(X, Y)$, and $\mathcal{Q}_Y T = \tilde{T}$. Thus we have surjectivity, and are done. \square

Proof Of 2. If Y is a **Normed Space**, a linear isometric homeomorphism by 2.1.185. In particular, in this case, q_y is injective, meaning that if $T, S \in BL(X, Y)$ where $T \neq S$, then $Tx_0 \neq Sx_0$ for some $x_0 \in X$. For this x_0 , $q_y Tx_0 \neq q_y Sx_0$, so $\mathcal{Q}_Y T \neq \mathcal{Q}_Y S$. Therefore \mathcal{Q}_Y is injective, and therefore a bijection. The inverse of an isometry is also an isometry and therefore continuous, finishing this proof. \square

Definition 2.1.193 (Quotient Operator). Let X, Y be **Seminormed Spaces** with **Seminorm Kernels** $\mathcal{K}_X^{\text{ernel}}, \mathcal{K}_Y^{\text{ernel}}$. Define $Q : BL(X, Y) \rightarrow BL(X/\mathcal{K}_X^{\text{ernel}}, Y/\mathcal{K}_Y^{\text{ernel}})$ by setting, for $T \in BL(X, Y)$, for $x \in X$,

$$QT[x]_X = [Tx]_Y \quad (2.88)$$

We call Q the **Operator Quotient Map** of X and Y and we call QT the **Quotient Operator** of T .

Proposition 2.1.194 (Quotient Operator). Let X, Y be **Seminormed Spaces** with **Seminorm Kernels** $\mathcal{K}_X^{\text{ernel}}, \mathcal{K}_Y^{\text{ernel}}$ and **Operator Quotient Map** Q . Then Q is a well-defined linear surjective isometry.

Proof. We first show that Q is well defined. Let $T \in BL(X, Y)$ and let $x_0, x_1 \in X$ such that $[x_0] = [x_1]$. Then $\|x_0 - x_1\|_X = 0$, so since T is continuous, $\|Tx_0 - Tx_1\|_Y = 0$. Hence $Tx_0 \cong Tx_1$, so $[Tx_0] = [Tx_1]$.

For linearity, let $\alpha \in \mathbb{F}$, and let $T, S \in BL(X, Y)$. Let $x \in X$. Then

$$\begin{aligned} Q(\alpha T + S)[x]_X &= [(\alpha T + S)x]_Y \\ &= \alpha [Tx]_Y + [Sx]_Y \\ &= \alpha QT[x]_X + QS[x]_X \\ &= (\alpha QT + QS)[x]_X \end{aligned}$$

Since $x \in X$ was arbitrary, Q is linear.

As for being an isometry, let $T \in BL(X, Y)$ and let $x \in X$. Since $\|[x]\| = \|x\|$ and $\|Tx\| = \|[Tx]\|$, we have

$$\begin{aligned} \frac{\|QT[x]_X\|_{Y/\mathcal{K}_Y^{\text{ernel}}}}{\|[x]\|_{X/\mathcal{K}_X^{\text{ernel}}}} &= \frac{\|[Tx]\|_{Y/\mathcal{K}_Y^{\text{ernel}}}}{\|[x]\|_{X/\mathcal{K}_X^{\text{ernel}}}} \\ &= \frac{\|Tx\|_Y}{\|x\|_X} \end{aligned}$$

and so taking the supremum over $\|x\| \neq 0$ gives us that this is an isometry.

For surjectivity, let $\tilde{T} \in BL(X/\mathcal{K}_X^{\text{ernel}}, Y/\mathcal{K}_Y^{\text{ernel}})$. Let $\{x_\alpha\}_{\alpha \in A}$ be a Hamel basis for X . For each $\alpha \in A$, let $y_\alpha \in \tilde{T}[x_\alpha]_X$. Now define

$$T \sum_{i=1}^n \beta_i x_{\alpha_i} = \sum_{i=1}^n \beta_i y_{\alpha_i} \quad (2.89)$$

Then $T : X \rightarrow Y$ is obviously linear, and $Tx \in \tilde{T}[x]_X$ for $x \in X$. Hence,

$$\frac{\|Tx\|_Y}{\|x\|_X} = \frac{\|\tilde{T}[x]_X\|_{Y/\mathcal{K}_Y^{\text{ernel}}}}{\|[x]_X\|_{X/\mathcal{K}_X^{\text{ernel}}}} \quad (2.90)$$

so T is bounded, and hence $T \in BL(X, Y)$, but that also implies that by definition, $QT = \tilde{T}$, so we have proven surjectivity. \square

Definition 2.1.195 (Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators). *Let X, Y be Seminormed Spaces with Seminorm Kernels $\mathcal{K}_X^{\text{ernel}}, \mathcal{K}_Y^{\text{ernel}}$. Let $\mathcal{K}^{\text{ernel}}$ denote the Seminorm Kernel of $BL(X, Y)$. Let Q denote the Operator Quotient Map of X and Y . Define $\Theta_{(X, Y)} : BL(X, Y)/\mathcal{K}^{\text{ernel}} \rightarrow BL(X/\mathcal{K}_X^{\text{ernel}}, Y/\mathcal{K}_Y^{\text{ernel}})$ by setting, for each $T \in BL(X, Y)$.*

$$\Theta_{(X, Y)}([T]) = QT \quad (2.91)$$

We call $\Theta_{(X, Y)}$ the Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators from X to Y . When X and Y are understood, we may denote the Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators simply with Θ . By 2.1.196, $\Theta_{(X, Y)}$ is an isomorphism of Normed Spaces. That is, Θ is Linear, Bijective, Bicontinuous, and an isometry.

Proposition 2.1.196 (Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators). *Let X, Y be Seminormed Spaces. Let Θ denote the Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators from X to Y . Then Θ is a bijective, bicontinuous, linear, isometry.*

Proof. By 2.1.185, part 1, $Y/\mathcal{K}_Y^{\text{ernel}}$ is a Normed Space, Hence by 2.1.189, part 2, $BL(X/\mathcal{K}_X^{\text{ernel}}, Y/\mathcal{K}_Y^{\text{ernel}})$ is a Normed Space. Similarly, by 2.1.185, part 1, $BL(X, Y)/\mathcal{K}^{\text{ernel}}$ is a normed space. Hence, it is sufficient to show that Θ is a well-defined surjective linear isometry.

For well definedness, let $T, S \in BL(X, Y)$ with $[T] = [S]$. Then, $\|T - S\| = 0$, so if $x \in X$, $\|Tx - Sx\| = 0$. Hence $Tx \cong Sx$ and since x was arbitrary, $QT = QS$.

Let q denote the Quotient Map $q : BL(X, Y) \rightarrow BL(X, Y)/\mathcal{K}^{\text{ernel}}$. By parts 4, 5, and 6 of 2.1.185, q is a linear surjective isometry. Also, by definition, $\Theta \circ q = Q$. Since Q is surjective, Θ is surjective. Since Q is an isometry, and q is a surjective isometry, Θ is an isometry. Since Q is linear, and since q is surjective and linear, Θ is linear. \square

Definition 2.1.197 (Seminorm Topological Dual Space). *Let $(X, \|\cdot\|)$ be a Seminormed Space over a field \mathbb{F} . We denote with X^* Normed Space $BL(X, \mathbb{F})$, and we call X^* the Topological Dual Space of X . If $x^* \in X^*$, then we may denote, for $x \in X$, $x^*(x)$ with $\langle x, x^* \rangle$.*

Since \mathbb{F} is a Normed Space, by 2.1.189, part 02, X^* is as well.

Since X^* is a normed space

Also, $q : \mathbb{F} \rightarrow \mathbb{F}/\mathcal{K}_{\mathbb{F}}^{\text{ernel}}$ is a linear bijective isometry by 2.1.185, so if $Q : X^* \rightarrow BL(X/\mathcal{K}_X^{\text{ernel}}, \mathbb{F}/\mathcal{K}_{\mathbb{F}}^{\text{ernel}})$ is the Operator Quotient Map and if $\Theta : BL(X, \mathbb{F})/\mathcal{K}^{\text{ernel}} \rightarrow BL(X/\mathcal{K}_X^{\text{ernel}}, \mathbb{F}/\mathcal{K}_{\mathbb{F}}^{\text{ernel}})$ is the Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators, then we have

$$\Theta = Q \circ q^{-1} \quad (2.92)$$

Definition 2.1.198 (Dual Space). *Let $(X, \|\cdot\|)$ be a Seminormed Space. We call $BL(X, \mathbb{F})$ the Topological Dual Space of $(X, \|\cdot\|)$, and we denote $BL(X, \mathbb{F})$ with the symbol X^* . If $x^* \in X^*$, then we use the notational convention of writing, for $x \in X$.*

$$\langle x, x^* \rangle := x^*(x) \quad (2.93)$$

It would also be correct to refer to the **Topological Dual Space** of $(X, \|\cdot\|)$ as the **1st Topological Dual Space** of X

Remark 2.1.199 (**Topological Dual Space** is a **Normed Space**). *Let X be a **Seminormed Space** . Then, using 2.1.189, since \mathbb{F} is a **Normed Space** , so is X^* .*

Theorem 2.1.200 (**Topological Dual Space** Isomorphism). *Let X be a **Seminormed Space** . Define $\Omega : X^* \rightarrow (X/\mathcal{K}_X^{\text{ernel}})^*$ Spa by setting, for $x^* \in X^*$, and for $x \in X$,*

$$\langle x, x^* \rangle = \langle [x], \Omega x^* \rangle \quad (2.94)$$

Then Ω is a Linear, Bijective, Isometric, Bicontinuous operator. That is, X^* and $(X/\mathcal{K}_X^{\text{ernel}})^*$ are isomorphic, and that isomorphism is explicitly given by Ω .

Proof. Consider the following

$$BL(X, \mathbb{F}) \xrightarrow{q} BL(X, \mathbb{F})/\mathcal{K}_{BL(X/\mathbb{F})}^{\text{ernel}} \xrightarrow{\Theta} BL(X/\mathcal{K}_X^{\text{ernel}}, \mathbb{F}/\mathcal{K}_{\mathbb{F}}^{\text{ernel}}) \xrightarrow{\mathcal{Q}_{\mathbb{F}}^{-1}} BL(X/\mathcal{K}_X^{\text{ernel}}, \mathbb{F}) \quad (2.95)$$

where q is the **Quotient Map** , which is an linear bijective bicontinuous isometry in this case by parts 4, 5, 6, and 7 of of 2.1.185, Θ is the **Canonical Isomorphism Of The Quotient Space Of Continuous Linear Operators** , which is a linear bijective bicontinuous isometry by 2.1.196 and $\mathcal{Q}_{\mathbb{F}}$ is the **Codomain Quotient Map** . which is in this case a linear, bijective, bicontinuous isometry by 2.1.192

Since $\Omega = \mathcal{Q}_{\mathbb{F}}^{-1} \circ \Theta \circ q$, and since each of the described properties are preserved under composition, Ω is also a linear bijective bicontinuous isometry. \square

Remark 2.1.201 (**Topological Dual Space** is a **Normed Space**). *Let X be a **Seminormed Space** . Since $X/\mathcal{K}_X^{\text{ernel}}$ is a **Normed Space** , so is $(X/\mathcal{K}_X^{\text{ernel}})^*$. By 2.1.200, we have a linear, bijective isometry between X^* and $(X/\mathcal{K}_X^{\text{ernel}})^*$. Hence X^* is a **Normed Space** .*

Theorem 2.1.202 (Hahn Banach Theorem For Seminormed Spaces). *Let $(X, \|\cdot\|)$ be a **Seminormed Space** , let $x_i \in X$ for $i \in \{0, 1\}$ such that $\|x_0 - x_1\|_X \neq 0$, and let X^* denote X 's **Topological Dual Space** . The following are true.*

1. If $Z \subset X$ is a subspace and $z^* \in Z^*$, then there is an extension x^* of z^* , $x^* \in X^*$ such that

$$\|z^*\|_{Z^*} = \|x^*\|_{X^*} \quad (2.96)$$

2. If $x \in X$, with $\|x\| \neq 0$, then there exists an $x^* \in X^*$ with $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|_X$.

3. If $x \in X$, then

$$\|x\|_X = \sup_{0 \neq x^* \in X^*} \frac{\langle x, x^* \rangle}{\|x^*\|} \quad (2.97)$$

4. If Y is a **Non-Degenerate Seminormed Space** , and if $x_0 \in X$, with $\|x_0\| \neq 0$, then there exists an $S \in BL(X, Y)$ with $\|S\| = 1$ and

$$\|Sx_0\| = \|x_0\| \quad (2.98)$$

Proof of 01. For $\alpha \in \{Z, X\}$, let $\Omega_\alpha : \alpha^* \rightarrow (\alpha/\mathcal{K}_\alpha^{\text{ernel}})^*$ denote the isomorphism defined in 2.1.200. Let q denote the quotient operator $q : X \rightarrow X/\mathcal{K}^{\text{ernel}}$. Define $T : Z/\mathcal{K}_Z^{\text{ernel}} \rightarrow q(Z)$ by $T([z]_{\cong_Z}) = [z]_{\cong_X}$. Since Z is endowed with the subspace Topology, T is obviously a Linear Bijective Bicontinuous Isometry.

Define $\Gamma_Z : (Z/\mathcal{K}_Z^{\text{ernel}})^* \rightarrow q(Z)^*$ by setting, for $\phi^* \in (Z/\mathcal{K}_Z^{\text{ernel}})^*$, for $[z]_Z \in Z/\mathcal{K}_Z^{\text{ernel}}$,

$$\langle T[z]_Z, \Gamma_Z \phi^* \rangle = \langle [z]_Z, \phi^* \rangle \quad (2.99)$$

Then Γ_Z is a Linear Bijective Isometry. Hence $\Gamma_Z \circ \Omega_Z z^* \in q(Z)^*$ with $\|\Gamma_Z \circ \Omega_Z z^*\|_{q(Z)^*} = \|z^*\|_{Z^*}$.

Thus we can apply the Hahn Banach theorem for **Normed Spaces** to claim the existence of $x_q^* \in (X/\mathcal{K}_X^{\text{ernel}})^*$ where x_q^* is an extension of $\Gamma_Z \circ \Omega_Z z^*$ and

$$\|x_q^*\|_{(X/\mathcal{K}_X^{\text{ernel}})^*} = \|\Gamma_Z \circ \Omega_Z z^*\|_{(q(Z))^*} = \|z^*\|_{Z^*} \quad (2.100)$$

Finally, letting $x^* = \Omega_X^{-1} x_q^*$, we have $x^* \in X^*$, $\|x^*\|_{X^*} = \|x_q^*\|_{(X/\mathcal{K}_X^{\text{ernel}})^*} = \|z^*\|_{Z^*}$, and if $z \in Z$, then

$$\begin{aligned} \langle z, x^* \rangle &= \langle [z]_X, x_q^* \rangle \\ &= \langle [z]_X, \Gamma_Z \circ \Omega_Z z^* \rangle \\ &= \langle [z]_Z, \Omega_Z z^* \rangle \\ &= \langle z, z^* \rangle \end{aligned}$$

□

Proof of 2. Let $Z = \text{span}(x)$. Define $z^* \in Z^*$ by $\langle \alpha x, z^* \rangle = \alpha \|x\|$. Then $\|z^*\| = 1$. Also, by part 1 of this result, it has an extension $x^* \in X^*$ with $\|x^*\| = \|z^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$. □

Proof of 3. If $\|x\| = 0$, then for every $x^* \in X^*$, $\langle x, x^* \rangle = 0$. Hence

$$\|x\|_X = \sup_{0 \neq x^* \in X^*} \frac{\langle x, x^* \rangle}{\|x^*\|} = \sup_{x^* \in \partial B_{X^*}(0;1)} \frac{\langle x, x^* \rangle}{\|x^*\|} = 0 \quad (2.101)$$

Otherwise, let $x^* \in X^*$ guaranteed to exist by part 2 which satisfies $\|x^*\| = 1$, $\langle x, x^* \rangle = \|x\|$. Then

$$\begin{aligned} \|x\| &= \frac{\langle x, x^* \rangle}{\|x^*\|} \\ &\leq \sup_{x^* \in \partial B_{X^*}(0;1)} \frac{\langle x, x^* \rangle}{\|x^*\|} \\ &\leq \sup_{0 \neq x^* \in X^*} \frac{\langle x, x^* \rangle}{\|x^*\|} \end{aligned}$$

The other direction of the inequality falls directly from the definition of the norm on X^* , and is trivial, so we are done. □

Proof of 4. By part 2 of this result, there exists $x_0^* \in X^*$ with $\|x_0^*\| = 1$ and $\langle x_0, x_0^* \rangle = \|x_0\|$. Since Y is **Non-Degenerate**, there exists $y_0 \in Y$ with $\|y_0\| = 1$. Define $T : \mathbb{F} \rightarrow Y$ by $T\alpha = \alpha y_0$. Then $\|T\| = \|y_0\| = 1$. Define $S : X \rightarrow Y$ by $S = T \circ x_0^*$. Then $\|S\| \leq \|T\| \|x_0^*\| = 1$, and $\|Sx_0\| = \|\langle x_0, x_0^* \rangle y_0\| = \langle x_0, x_0^* \rangle = \|x_0\|$. Hence $\|S\| \geq 1$ and therefore $\|S\| = 1$. \square

Proposition 2.1.203 (Linear Operator Notation). . *When dealing with mappings of spaces of linear operators into spaces of other linear operators, or even functions in general, notation can get confusing, and presenting such things using ordinary notation without ambiguity can often require a plethora of parenthesis, which hamper readability of an argument.*

For this reason, at points in this document, I sometimes express the image $\beta(\alpha)$ using

$$\langle \alpha, \beta \rangle$$

Where $\beta : X \rightarrow Y$ and $\alpha \in X$.

I combine this notation with usual function notation, particularly in cases similar to the following. For $i \in \{0, 1\}$, let X_i, Y_i, Z_i be sets. For $\alpha \in \{X, Y, Z\}$, let F_α be the set of maps $f : \alpha_0 \rightarrow \alpha_1$. If $T : F_X \rightarrow F_Y$, $y \in Y_0$, and $f \in F_X$, then I would notate

$$\langle y, Tf \rangle$$

rather than $Tf(y)$ or $(T(f))(y)$

Definition 2.1.204 (**Adjoint Operator**). Let X, Y , and Z be **Seminormed Spaces** over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. . Let $T \in BL(X, Y)$. We define the operator $T_Z^\times : BL(Y, Z) \rightarrow BL(X, Z)$ by setting, for $S \in BL(Y, Z)$ and $x \in X$,

$$\langle x, \mathcal{T}_Z^\times S \rangle = \langle Tx, S \rangle \quad (2.102)$$

or, equivalently,

$$\mathcal{T}_Z^\times S = S \circ T \quad (2.103)$$

We call T_Z^\times the **Adjoint Operator** of T relative to the space Z , we denote $T_\mathbb{F}^\times = T^\times$, and we refer to $T^\times : Y^* \rightarrow X^*$ as simply the **Adjoint Operator** of T .

Proposition 2.1.205 (**Adjoint Operator**). Let X, Y , and Z be **Seminormed Spaces** over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. . Let $T \in BL(X, Y)$. Let $\mathcal{T} = T_Z^\times$ denote the **Adjoint Operator** of T relative to the space Z . Let Q_y denote the **Quotient Map**

The following are true.

1. \mathcal{T} is Linear.
2. If $S \in BL(Y, Z)$, then $\mathcal{T}S \in BL(X, Z)$. (That is, the **Adjoint Operator** is well defined as a concept).
3. $\mathcal{T} \in BL(BL(Y, Z), BL(X, Z))$.
4. $\|\mathcal{T}\| = \|T\|$

5. If T is surjective, then $\inf_{\|x\|=1} \|Tx\| \leq \inf_{\|S\|_{BL(Y,Z)}=1} \|\mathcal{T}S\|$. Also TODO: Weaken T surjectivity condition To $\text{Range}(T)$ dense in Y .
6. If $\text{Range}(T)$ is not dense in Y , then $\inf_{\|S\|_{BL(Y,Z)}=1} \|\mathcal{T}S\| = 0$
7. \mathcal{T} is surjective iff T is injective and has closed range in Y .

Proof of 01. Let $S, R \in BL(Y, Z)$, $\alpha \in \mathbb{F}$, and $x \in X$. Then,

$$\begin{aligned}
 \langle x, \mathcal{T}(\alpha S + R) \rangle &= \langle Tx, \alpha S + R \rangle \\
 &= \alpha \langle Tx, S \rangle + \langle Tx, R \rangle \\
 &= \alpha \langle x, \mathcal{T}S \rangle + \langle x, \mathcal{T}R \rangle &= \langle x, \alpha \mathcal{T}S \rangle + \langle x, \mathcal{T}R \rangle \\
 &= \langle x, \alpha \mathcal{T}S + \mathcal{T}R \rangle
 \end{aligned}$$

Since $x \in X$ was arbitrary, linearity is verified. \square

Proof of 02. Let $S \in BL(Y, Z)$. Then, $\mathcal{T}S = S \circ T$. The composition of continuous operators is continuous, so $\mathcal{T}S$ is continuous. The composition of linear operators is linear, so $\mathcal{T}S$ is linear. This, paired with linearity, implies $\mathcal{T}S \in BL(X, Z)$. \square

Proof of 3. Let $S \in BL(Y, Z)$. Then, if $x \in X$

$$\|\langle x, \mathcal{T}S \rangle\| = \|\langle Tx, S \rangle\| \leq \|S\| \|Tx\| \leq \|S\| \|T\| \|x\| \quad (2.104)$$

Hence $\|\mathcal{T}S\| \leq \|S\| \|T\|$. Since T is linear, and since S was arbitrary, by part 12 of 2.1.189, $\mathcal{T} \in BL(BL(Y, Z), BL(X, Z))$. \square

Proof of 4. For any $S \in BL(Y, Z)$, $\mathcal{T}S = S \circ T$, so $\|\mathcal{T}S\| \leq \|S\| \|T\|$. Hence $\|\mathcal{T}\| \leq \|T\|$. Now let $x_0 \in X$. Then, by part 4 of 2.1.15, there exists $S \in BL(Y, Z)$ with $\|S\| = 1$ and $\|STx_0\| = \|Tx_0\|$. Hence,

$$\begin{aligned}
 \|Tx_0\| &= \|STx_0\| \\
 &= \|(S \circ T)x_0\| \\
 &= \|(\mathcal{T}S)x_0\| \\
 &\leq \|\mathcal{T}\| \|S\| \|x_0\| \\
 &= \|\mathcal{T}\| \|x_0\|
 \end{aligned}$$

Since $x_0 \in X$ is arbitrary, $\|T\| \leq \|\mathcal{T}\|$. Since the inequality goes both ways, $\|T\| = \|\mathcal{T}\|$. \square

Proof of 05. Let $\Gamma = \inf_{\|x\|=1} \|Tx\|$, and let $S \in BL(Y, Z)$ with $\|S\| = 1$. Then,

$$\{x \mid \|Tx\| \leq \Gamma\} \subset B_X(0; 1)$$

so

$$\sup_{\|x\| \leq 1} |\langle Tx, S \rangle| \geq \sup_{\|Tx\| \leq \Gamma} |\langle Tx, S \rangle|$$

Also, since T is surjective by assumption,

$$\sup_{\|Tx\| \leq \Gamma} |\langle Tx, S \rangle| = \sup_{\|y\| \leq \Gamma} |\langle y, S \rangle|$$

From these two we arrive at the inequality

$$\begin{aligned} \|\mathcal{T}S\| &= \sup_{\|x\| \leq 1} |\langle x, \mathcal{T}S \rangle| \\ &= \sup_{\|x\| \leq 1} |\langle Tx, S \rangle| \\ &\geq \sup_{\|Tx\| \leq \Gamma} |\langle Tx, S \rangle| \\ &= \sup_{\|y\| \leq \Gamma} |\langle y, S \rangle| \\ &= \Gamma \\ &= \inf_{\|x\|=1} \|Tx\| \end{aligned}$$

Since $S \in \partial B_{BL(Y,Z)}(0; 1)$ was arbitrary, we conclude $\inf_{\|S\|=1} \|\mathcal{T}S\| \geq \inf_{\|x\|=1} \|Tx\|$ \square

Definition 2.1.206 (Higher Order Dual Spaces). *Let X be a **Seminormed Space**. From 2.1.15 we know that the **Topological Dual Space** of X , X^* , is also called the **1st Topological Dual Space** of X . Building on this, for $n \in \{2, 3, 4, \dots\}$ we call the **1st Topological Dual Space** of X^* the **2nd Topological Dual Space** of X , we call the **1st Topological Dual Space** of the **2nd Topological Dual Space** of X the **3rd Topological Dual Space** of X , and in general the **1st Topological Dual Space** of the $(n)^{th}$ **Topological Dual Space** of X the $(n+1)^{th}$ **Topological Dual Space** of X .*

In general, we denote the $(n)^{th}$ **Topological Dual Space** of X with X^{n*} , though when n is small, we may denote $X^{**} = X^{2*}$, $X^{***} = X^{3*}$, et cetera.

Definition 2.1.207 (Higher Order Dual Space Isomorphism). *Let X be a **Seminormed Space** over a field \mathbb{F} . Let $\Omega : X^* \rightarrow (X/\mathcal{K}_X^{\text{ernel}})^*$ be the Linear Bijective Isometry defined in 2.1.200. Define*

$$\Omega_1 = \Omega$$

and also define, for $2 \leq n \in \mathbb{N}$, $\Omega_n : X^{n*} \rightarrow (X/\mathcal{K}_X^{\text{ernel}})^{n*}$ by

$$\Omega_n = (\Omega_{n-1}^\times)^{-1}$$

By it is clear that the adjoint of a Linear Bijective isometry of normed spaces is also a Linear Bijective isometry of normed spaces, and so each Ω_n is as well.

Definition 2.1.208 (**Canonical Embedding** of X into X^{**}). *Let X be a **Seminormed Space**. Define $c_X : X \rightarrow X^{**}$ by setting, for each $x^* \in X^*$, for each $x \in X$*

$$\langle x^*, c(x) \rangle = \langle x, x^* \rangle \quad (2.105)$$

We call c_X the **Canonical Embedding** of X into X^* . As normal, if X is understood, we may denote $c_X = c$. If c is Surjective, then we say that X is **Reflexive**.

Proposition 2.1.209 (Canonical Embedding). *Let X be a **Seminormed Space** and let c denote its **Canonical Embedding**. The following are true.*

1. c is well defined
2. c is Linear.
3. c is an isometry.
4. c is an injection if and only if X is a **Normed Space**.
5. If $q : X \rightarrow X/\mathcal{K}^{\text{ernel}}$ is the **Quotient Map**, $c_{X/\mathcal{K}^{\text{ernel}}}$ is the **Canonical Embedding** of $(X/\mathcal{K}^{\text{ernel}})$ into $(X/\mathcal{K}^{\text{ernel}})^{**}$ and $\Omega_2 : X^{**} \rightarrow (X/\mathcal{K}^{\text{ernel}})^{**}$ is the linear bijective isometry defined in 2.1.15, then $c = \Omega_2^{-1} \circ c_{X/\mathcal{K}^{\text{ernel}}} \circ q$. //TODO: COME BACK TO THIS AND PROVE IT ONCE THE ISOS ARE CLEARED UP
6. c_X is surjective if and only if $c_{X/\mathcal{K}^{\text{ernel}}}$ is surjective.
7. X is **Reflexive** if and only if $X/\mathcal{K}^{\text{ernel}}$ is **Reflexive**.

Proof of 1. For any $x \in X$, $c(x)$ as a function is obviously well defined. Hence, I just need to show that, for any $x \in X$, $c(x) \in X^{**}$. That is, I must show that $c(x)$ is continuous and linear.

For linearity, if $x^*, y^* \in X^*$ and $\alpha \in \mathbb{F}$, we have

$$\begin{aligned} \langle \alpha x^* + y^*, c(x) \rangle &= \langle x, \alpha x^* + \alpha y^* \rangle \\ &= \alpha \langle x, x^* \rangle + \langle y, y^* \rangle \\ &= \alpha \langle x^*, c(x) \rangle + \langle y^*, c(x) \rangle \end{aligned}$$

Thus linearity holds.

For continuity, let $x \in X$ and let $x^* \in X^*$.

$$\begin{aligned} |\langle x^*, c(x) \rangle| &= |\langle x, x^* \rangle| \\ &\leq \|x\| \|x^*\| \end{aligned}$$

so that $c(x)$ is bounded with $\|c(x)\| \leq \|x\|$. □

Proof of 2. Let $\alpha \in \mathbb{F}$ and $x, y \in X$. Let $x^* \in X^*$. Then,

$$\begin{aligned} \langle x^*, c(\alpha x + y) \rangle &= \langle \alpha x + y, x^* \rangle \\ &= \alpha \langle x, x^* \rangle + \langle y, x^* \rangle \\ &= \alpha \langle x^*, c(x) \rangle + \langle x^*, c(y) \rangle \end{aligned}$$

, finishing the proof. □

Proof of 3. Let $x_0 \in X$ and $x^* \in X^*$. Then,

$$\begin{aligned} |\langle x^*, c(x_0) \rangle| &= |\langle x_0, x^* \rangle| \\ &\leq \|x_0\| \|x^*\| \end{aligned}$$

so that $\|c(x_0)\| \leq \|x_0\|$. For the other direction, by 2.1.15 part 2, there exists an $x_0^* \in X^*$ satisfying $\|x_0^*\| = 1$ and $\langle x_0, x_0^* \rangle = \|x_0\|$ □

We see that $\langle x_0^*, c(x_0) \rangle = \langle x_0, x_0^* \rangle = \|x\| = \|x_0\| \|x_0^*\|$ so that $\|c(x_0)\| \geq \|x_0\|$. Since the inequality goes both ways, $\|x_0\| = \|c(x_0)\|$, and c is therefore an isometry.

Proof of 4. Let X be a **Normed Space**. Then X^* separates points in X . Let $x \in X$ and $y \in X$ with $x \neq y$. Since X^* separates points in X , there exists $x^* \in X^*$ with $\langle x^*, c(x) \rangle = \langle x, x^* \rangle \neq \langle y, x^* \rangle = \langle x^*, c(y) \rangle$ so that $c(x) \neq c(y)$. Hence c is injective.

Now suppose instead that c is injective and let $x, y \in X$ with $\|x - y\| = 0$. We find that for any $x^* \in X^*$,

$$\begin{aligned} |\langle x^*, c(x) - c(y) \rangle| &= |\langle x^*, c(x - y) \rangle| \\ &= |\langle x - y, x^* \rangle| \\ &\leq \|x^*\| \|x - y\| \\ &= 0 \end{aligned}$$

so that $\|c(x) - c(y)\| = 0$. Since X^{**} is a normed space, this implies $c(x) = c(y)$, which through injectivity implies $x = y$. Hence we have the implication $\|x - y\| = 0 \implies x = y$, so that X is a **NormedSpace**. \square

Proof of 5. Proceeding directly from the definition, we have

$$\begin{aligned} \langle x^*, \Omega_2^{-1} \circ c_{X/\mathcal{K}^{\text{ernel}}} \circ q(x) \rangle &= \langle x^*, \Omega^\times \circ c_{X/\mathcal{K}^{\text{ernel}}} \circ q(x) \rangle \\ &= \langle \Omega x^*, c_{X/\mathcal{K}^{\text{ernel}}} \circ q(x) \rangle \\ &= \langle q(x), \Omega x^* \rangle \\ &= \langle x, x^* \rangle \\ &= \langle x^*, c(x) \rangle \end{aligned}$$

So we are done. \square

Proof of 6. Since Ω_2 is a Bijection, by the prior part of this result, c is a surjection if and only if $c_{X/\mathcal{K}^{\text{ernel}}} \circ q$ is a surjection where $q : X \rightarrow X/\mathcal{K}^{\text{ernel}}$ is the **Quotient Map**. Since q is a surjection, c is a surjection if and only if $c_{X/\mathcal{K}^{\text{ernel}}}$ is a surjection. \square

Proof of 7. This is a direct restatement of Part 06 of this result. \square

Definition 2.1.210 (Weak Topologies Relating To Seminormed and Normed Spaces). *latex*

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Similar to in the context of a normed space, if X is a seminormed space, we define the weak topology on X to be the topology on X generated by X^* , and the *weak** topology on X^* to be the topology generated by $c(X)$. Before moving on to the classical theory revamped, I present on more useful result about weak topologies of seminormed spaces.

Proposition 2.1.211 (Weak Quotients). *Let X be a seminormed space and $\{Y_\alpha\}_{\alpha \in A}$ be a collection of topological spaces. For each $\alpha \in A$ let $\phi_\alpha : X \rightarrow Y_\alpha$ have the property that for every $x, y \in X$, for every $\alpha \in A$, $\|x - y\| = 0 \implies \phi_\alpha(x) = \phi_\alpha(y)$. For each $\alpha \in A$, define*

$\tilde{\phi}_\alpha : X/\|\cdot\|^{-1}\{0\} \rightarrow Y_\alpha$ by $\tilde{\phi}_\alpha[x] = \phi_\alpha x$. Let \mathcal{T}_w denote the weak topology on X induced by $\{\phi_\alpha\}_{\alpha \in A}$, and $\mathcal{T}_{\tilde{w}}$ denote the weak topology on $X/\|\cdot\|^{-1}\{0\}$ induced by $\{\tilde{\phi}_\alpha\}_{\alpha \in A}$. Then

$$(X, \mathcal{T}_w)/\|\cdot\|^{-1}\{0\} = (X/\|\cdot\|^{-1}\{0\}, \mathcal{T}_{\tilde{w}}) \quad (2.106)$$

Proof. □

Finally, before we move on, recall that if X, Y are Topological vector spaces, we can topologize the set of continuous linear operators from X to Y , denoted $BL(X, Y)$ by saying that $\{T_\alpha\}_{\alpha \in A} \subset BL(X, Y)$ converges to $T \in BL(X, Y)$ if there is a neighborhood U of 0 in X such that $T_\alpha x \rightarrow Tx$ uniformly for $x \in U$.

2.2 Classical Results With A Twist

By ?? and 2.1.211, many of the classical theorems relating a normed space and its duals still hold in the context of a seminormed space without too much alteration of the proofs. Since the author has not seen these results presented in this context, they are presented with proof below.

2.2.1 Helly

In this subsection, we develop Helly's theorem in the context of a seminormed space, which will serve as valuable lemma throughout this document. Its location here is due to the fact that it is a generalization of a lemma commonly used to prove the Goldstine Theorem.

Theorem 2.2.1 (Helly's Theorem). *Let $(X, \|\cdot\|)$ be a seminormed space $M > 0$, $\{\alpha_i\}_{i=1}^n \subset \mathbb{R}$, and $\{x_i^*\}_{i=1}^n \subset X^*$. Then the following are equivalent.*

1. *For every $\epsilon > 0$, there is an $x_\epsilon \in X$ such that $\|x_\epsilon\| < M + \epsilon$ and $\langle x, x_i^* \rangle = \alpha_i$ for $1 \leq i \leq n$.*
2. *For every $\epsilon > 0$, there is an $x_\epsilon \in X$ such that $\|x_\epsilon\| \leq M$ and $|\langle x_\epsilon, x_i^* \rangle - \alpha_i| < \epsilon$ for $1 \leq i \leq n$.*
3. *For each $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$,*

$$\left| \sum_{i=1}^n \beta_i \alpha_i \right| \leq M \left\| \sum_{i=1}^n \beta_i x_i^* \right\| \quad (2.107)$$

Proof. (1 \implies 2) Obvious since $\{x_i^*\}_{i=1}^n \subset X^*$. □

Proof. (2 \implies 3) Let $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$ and choose $\epsilon > 0$. Then

$$\begin{aligned} \left| \sum_{i=1}^n \beta_i \alpha_i \right| &= \left| \sum_{i=1}^n \beta_i (\alpha_i - \langle x_\epsilon, x_i^* \rangle + \langle x_\epsilon, x_i^* \rangle) \right| \\ &< \sum_{i=1}^n \epsilon |\beta_i| + \left| \left\langle x_\epsilon, \sum_{i=1}^n \beta_i x_i^* \right\rangle \right| \\ &\leq \epsilon \sum_{i=1}^n |\beta_i| + M \left\| \sum_{i=1}^n \beta_i x_i^* \right\| \end{aligned} \quad (2.108)$$

Since ϵ was arbitrary, the desired inequality holds. \square

Proof. (3 \implies 1) If each $x_i^* = 0$, then by 2.107, each $\alpha_i = 0$, so $x = 0$ works for any ϵ . Otherwise, we without loss of generality reorder $\{x_i^*\}_{i=1}^n$ so that $\{x_i^*\}_{i=1}^m$ is linearly independent and $\text{span}\{x_i^*\}_{i=1}^m = \text{span}\{x_i^*\}_{i=1}^n$. Define $S : X \rightarrow \mathbb{C}^m$ by $Sx = [\langle x, x_1^* \rangle, \dots, \langle x, x_m^* \rangle]$, and recognize that linear independence of $\{x_i^*\}_{i=1}^m$ guarantees that S is surjective, so there exists $\tilde{x} \in S^{-1}\{[\alpha_1, \dots, \alpha_m]\}$ and furthermore

$$S^{-1}\{[\alpha_1, \dots, \alpha_m]\} = \tilde{x} + \bigcap_{i=1}^m \text{kern}(x_i^*) = \tilde{x} + \bigcap_{i=1}^n \text{kern}(x_i^*) := \tilde{x} + K \quad (2.109)$$

If $m < j \leq n$, then $x_j^* \in \text{span}(x_1^*, \dots, x_m^*)$ so $x_j = \sum_{i=1}^m \gamma_i x_i^*$. Now we choose $\beta_i = \gamma_i$ for $1 \leq i \leq m$, $\beta_j = -1$, and other $\beta_k = 0$.

$$|\langle \tilde{x}, x_j^* \rangle - \alpha_j| = \left| \left\langle \tilde{x}, \sum_{i=1}^m \beta_i x_i^* \right\rangle - \alpha_j \right| = \left| \left(\sum_{i=1}^m \beta_i \langle \tilde{x}, x_i^* \rangle \right) + (-1)\alpha_j \right| \quad (2.110)$$

$$= \left| \sum_{i=1}^n \beta_i \alpha_i \right| \quad (2.111)$$

$$\leq M \left\| \sum_{i=1}^n \beta_i x_i^* \right\| \quad (2.112)$$

$$= M \left\| \sum_{i=1}^m \gamma_i x_i^* + (-1)x_j^* \right\| \quad (2.113)$$

Hence $\langle x, x_i^* \rangle = \alpha_i$ for $1 \leq i \leq n$ and any $x \in \tilde{x} + K$. By the Hahn Banach Theorem applied to the semi norm on X induced by the seminorm on the quotient space X/K , there is an $x^* \in X^*$ such that $\sup_{\|x\| \neq 0} \frac{|\langle x, x^* \rangle|}{\|x\|} = 1$, $\langle \tilde{x}, x^* \rangle = d(\tilde{x}, K)$, and $K \subset \text{kern}(x^*)$. Since $K \subset \text{kern}(x^*)$, $x^* \in \text{span}\{x_i^*\}_{i=1}^n$, say $x^* = \sum_{i=1}^n \mu_i x_i^*$. Hence,

$$\begin{aligned} d(\tilde{x}, K) &= \langle \tilde{x}, x^* \rangle = \sum_{i=1}^n \mu_i \langle \tilde{x}, x_i^* \rangle \\ &= \sum_{i=1}^n \mu_i \alpha_i \\ &\leq M \left\| \sum_{i=1}^n \mu_i x_i^* \right\| = M \end{aligned} \quad (2.114)$$

Thus, given $\epsilon > 0$, we find an $z_\epsilon \in K$ such that $\|\tilde{x} - z_\epsilon\| < M + \epsilon$ and let $x_\epsilon = \tilde{x} - z_\epsilon$. \square

Corollary 2.2.2. *Let X be a seminormed space, $x^{**} \in X^{**}$, $\{x_i^*\}_{i=1}^n \subset X^*$, and $\epsilon > 0$. Then there exists the following.*

1. An $x_1 \in X$ such that $\|x_1\| \leq \|x^{**}\| + \epsilon$ and for $1 \leq i \leq n$, $\langle x_1, x_i^* \rangle = \langle x_i^*, x^{**} \rangle$.

2. An $x_2 \in X$ such that $\|x_2\| \leq \|x^{**}\|$ and for $1 \leq i \leq n$, $|\langle x_1, x_i^* \rangle - \langle x_i^*, x^{**} \rangle| < \epsilon$.

Proof. For $1 \leq i \leq n$, let $\alpha_i = \langle x_i^*, x^{**} \rangle$, and let $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$. Then

$$\left| \sum_{i=1}^n \beta_i \alpha_i \right| = \left| \left\langle \sum_{i=1}^n \beta_i x_i^*, x^{**} \right\rangle \right| \leq \|x^{**}\| \left\| \sum_{i=1}^n \beta_i x_i^* \right\| \quad (2.115)$$

An application of 2.2.1 completes the proof. \square

2.2.2 Goldstine

Theorem 2.2.3 (Goldstine). *Let X be a seminormed space, $c : X \rightarrow X^{**}$ denote the canonical embedding, B denote the closed unit ball of X , and B_1 denote the closed unit ball of X^{**} . Then $c(B)$ is weak* dense in B_1 .*

Proof. Note that the weak* topology on X^{**} has as a basis sets of the form

$$U(x^{**}, \epsilon, \{x_i^*\}_{i=1}^n) := \{y^{**} \in X^{**} : (\forall 1 \leq i \leq n) (|\langle x_i^*, x^{**} - y^{**} \rangle| < \epsilon)\} \quad (2.116)$$

where we range over all $x^{**} \in X^{**}$, $\epsilon > 0$, and finite subsets $\{x_i^*\}_{i=1}^n$ of X^* . Let $x^{**} \in B_1$, $\epsilon > 0$, and $\{x_i^*\}_{i=1}^n \subset X^*$. Let $\epsilon > 0$. Then by 2.2.2, there exists an $x \in B$ such that for $1 \leq i \leq n$, $|\langle x_i^*, x^{**} - c(x) \rangle| = |\langle x_i^*, x^{**} \rangle - \langle x, x_i^* \rangle| < \epsilon$, from which the desired result follows. \square

2.2.3 Banach Alaoglu

The following well known result concerning the weak* compactness of the unit ball of a Banach space was first proven in the separable case by Banach, and then generalized in 1940 by Alaoglu [1] to Banach spaces. Generalizations of this result in a general TVS satisfying sufficient conditions have also been shown but the form presented here comes from [2], who drops the assumption of completeness for one direction of the implication.

Theorem 2.2.4 (Banach-Alaoglu-Morales). *Let X be a normed space and define $B = \{x^* \in X^* : \|x^*\| \leq 1\}$. Then B is weak* compact.*

Proof. Let \mathbb{F} denote X 's field, and for $x \in X$, define $D_x = \{y \in \mathbb{F} : |y| \leq \|x\|\}$. Then each D_x is Hausdorff and compact so by Tychonoff's theorem, $D := \prod_{x \in X} D_x$ is compact and Hausdorff when endowed with the product topology. If $T \in D$, then $T : X \rightarrow \mathbb{F}$ and $|Tx| \leq \|x\|$ for each $x \in X$, so $D \cap X^* \subset B$. It is also clear that $B \subset D$, so $D \cap X^* = B$. Let $\{\gamma_\alpha\}_{\alpha \in A}$ be a net in B converging to $\gamma \in D$ in D 's product topology. Then, letting π_x denote the x^{th} projection, for each $x \in X$,

$$\gamma_\alpha(x) = \pi_x(\gamma_\alpha) \rightarrow \pi_x(\gamma) = \gamma(x) \quad (2.117)$$

If $\alpha \in \mathbb{F}$ and $x, y \in X$, then

$$\langle \alpha x + y, \gamma_\alpha \rangle \rightarrow \langle \alpha x + y, \gamma \rangle \quad (2.118)$$

and also

$$\langle \alpha x + y, \gamma_\alpha \rangle = \alpha \langle x, \gamma_\alpha \rangle + \langle y, \gamma_\alpha \rangle \rightarrow \alpha \langle x, \gamma \rangle + \langle y, \gamma \rangle \quad (2.119)$$

which implies γ is linear since D is Hausdorff, and hence $\gamma \in B$. Thus B is closed in D . What remains to be shown is that the *weak** topology on B is the subspace topology on B induced by D 's topology, since a Closed subset of a compact Hausdorff space is compact. For notation, denote with \mathcal{T}_D the subspace topology on B induced by D 's topology, and denote with \mathcal{T}_w the subspace topology on B induced by the *weak** topology on X^* . To see that $\mathcal{T}_w \subset \mathcal{T}_D$, let $\{\gamma_\alpha\}_{\alpha \in A} \subset B$ such that $\gamma_\alpha \xrightarrow{\mathcal{T}_D} \gamma$. For each $x \in X$, letting c be the canonical embedding,

$$\langle \gamma_\alpha, c(x) \rangle = \langle x, \gamma_\alpha \rangle = \pi_x(\gamma_\alpha) \rightarrow \pi_x(\gamma) = \langle x, \gamma \rangle = \langle \gamma, c(x) \rangle \quad (2.120)$$

Hence $\gamma_\alpha \xrightarrow{\mathcal{T}_w} \gamma$, so $\mathcal{T}_w \subset \mathcal{T}_D$. To see that $\mathcal{T}_D \subset \mathcal{T}_w$, fix $x \in X$ and let $\{\gamma_\alpha\}_{\alpha \in A} \subset B$ such that $\gamma_\alpha \xrightarrow{\mathcal{T}_D} \gamma$. Then $\pi_x(\gamma_\alpha) = \langle x, \gamma_\alpha \rangle \rightarrow \gamma(x) = \pi_x(\gamma)$, so by definition of the product topology $\gamma_\alpha \xrightarrow{\mathcal{T}_D} \gamma$, implying $\mathcal{T}_D \subset \mathcal{T}_w$. Hence B is *weak** compact. \square

Corollary 2.2.5 (Banach Alaoglu Seminorm). *Let X be a seminormed space and define $B = \{x^* \in X^* : \|x^*\| \leq 1\}$. Then B is *weak** compact.*

Proof. This is a consequence of the fact that the *weak** topology on X^* is identical to the *weak** topology on the dual space of $X/\|\cdot\|^{-1}\{0\}$. \square

This gives us the useable result

Corollary 2.2.6 (Banach-Alaoglu-Morales). *Let X be a seminormed space and $C \subset X^*$*

1. *If X is complete and C *weak** compact, then C is *weak** closed and bounded.*
2. *If C is *weak** closed and bounded, then C is *weak** compact.*

Proof. (1) Since C is *weak** compact, it is *weak** closed since the *weak** topology is Hausdorff. Since $c(x) : (X, \mathcal{T}_{w^*}) \rightarrow \mathbb{C}$ is continuous for each $x \in X$, for each $x \in X$, $c(x)(C)$ is compact and therefore bounded. Hence, for every $x \in X$, $\{|\langle x, c \rangle| : c \in C\}$ is bounded, so by the Banach Steinhaus, C is bounded. \square

Proof. (2) Since C is bounded, it is contained in some closed ball B which we know to be *weak** compact by 2.2.4. Since the *weak** topology on B is compact and Hausdorff and C is closed in this topology, it is compact in this topology. Since the subspace topology on C induced by the *weak** topology on X^* equals this topology, we are done. \square

2.2.4 Eberlein-Smulian

The purpose of this section is to provide a characterization of weakly compact subsets of a complete seminormed space X , which will serve to increase the applicability of the results regarding weakly compactly generated spaces covered later in this document. The first main result of this section, 2.2.9, serves to show that even though weak topologies of Banach Spaces are not in general metrizable, an equivalence between weak compactness and sequential compactness exists. From this result (1 \implies 2) was first presented in the case of normed

spaces in [3], and then $(2 \implies 1)$ was proven in the case of normed spaces in [4]. Several different proofs have been given in the years since, and the one present here is based on that present in [5], which is also followed in [6]. We begin with a few lemmas.

Lemma 2.2.7 (Metrizible Weak). *If X is a seminormed space and X^* contains a countable set that separates points mod $K := \|\cdot\|^{-1}\{0\}$, then subspace topology induced by the weak topology on any weakly compact subset A of X is pseudometrizable.*

Proof. As a consequence of 2.1.165 and 2.1.211, it is sufficient to let X be a normed space and $\{x_i^*\}_{i \in \mathbb{N}}$ separate points in X . Let $M = 2 \sup_{x \in A} \|x\|$, and define d to be the metric on A defined by, for $x, y \in A$,

$$d(x, y) = \sum_{k \in \mathbb{N}} \frac{|\langle x - y, x_k^* \rangle|}{\|x_k^*\| 2^k} \quad (2.121)$$

Let $x \in A$, $\epsilon > 0$ be arbitrary, and define

$$n = \left\lceil 2 + \log_2 \left(\frac{M}{\epsilon} \right) \right\rceil \quad U = A \cap \bigcap_{k=1}^n \left\{ y \in X : |\langle x - y, x_k^* \rangle| < \frac{\|x_k^*\| 2^{k-1} \epsilon}{n} \right\} \quad (2.122)$$

The U is open in the subspace topology on A induced by X 's weak topology. Furthermore, if $y \in U$, then

$$\begin{aligned} d(x, y) &= \sum_{k \in \mathbb{N}} \frac{|\langle x - y, x_k^* \rangle|}{\|x_k^*\| 2^k} \\ &\leq \sum_{k=1}^n \frac{|\langle x - y, x_k^* \rangle|}{\|x_k^*\| 2^k} + \sum_{k=n+1}^{\infty} \frac{2M}{2^k} \\ &< \sum_{k=1}^n \frac{\epsilon}{2n} + \frac{M}{2^{n-1}} < \epsilon \end{aligned} \quad (2.123)$$

So that $U \subset B_d(x; \epsilon)$. This implies $Id : (A, \mathcal{T}_w) \rightarrow (A, \mathcal{T}_d)$ is continuous. Since a continuous injection from a compact space into a Hausdorff space is a homeomorphism, the subspace topology on A induced by the weak topology equals the topology on A induced by d , and so A 's weak topology is metrizable. \square

Lemma 2.2.8. *Let X be a seminormed space and $Y \subset X^{**}$ be a finite dimensional vector subspace. Then there exists a finite set $Z \subset \partial B_{X^*}(0; 1)$ such that for each $y^{**} \in Y$,*

$$\|y^{**}\| \leq 2 \max_{z^* \in Z} |\langle z^*, y^{**} \rangle| \quad (2.124)$$

Proof. Let $S = \partial B_{X^{**}}(0; 1) \cap Y$. Then, since Y is finite dimensional, S is compact, and therefore permits a $\frac{1}{4}$ -net $\{s_i\}_{i=1}^n$. Now let $\{z_k^*\}_{k=1}^n \subset \partial B_{X^*}(0; 1)$ such that for each k , $\langle z_k^*, s_i \rangle > \frac{3}{4}$. Let $s \in S$ then there is a k such that $\|s - s_k\| < \frac{1}{4}$. for this k , we have

$$\langle z_k^*, s \rangle = \langle z_k^*, s_k \rangle + \langle z_k^*, s - s_k \rangle \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2} \quad (2.125)$$

\square

Theorem 2.2.9 (Eberlein-Smulian). *Let X be a seminormed space and $A \subset X$. Then the following are equivalent.*

1. A is weakly compact.
2. A is weakly sequentially compact.

Proof. (1 \implies 2) Let $A \subset X$ be weakly compact, and let $\{x_i\}_{i \in \mathbb{N}} \subset A$. Define $S = \overline{\text{span}\{x_i : i \in \mathbb{N}\}}$. Since S is closed and convex, it is weakly closed, and so $A \cap S$ is weakly compact as well. By construction, S is separable, and so contains a countable dense set $\{y_i\}_{i \in \mathbb{N}}$. By Hahn-Banach, for each $i \in \mathbb{N}$, there exists $y_i^* \in S^*$ such that $\langle y_i, y_i^* \rangle = 1$, and continuity of each y_i^* implies $\{y_i^*\}_{i \in \mathbb{N}}$ separates points in $S \text{ mod } \|\cdot\|^{-1}\{0\}$. Hence we can apply 2.2.7 to claim that the subspace topology on $A \cap S$ induced by S' 's weak topology is metrizable, and therefore $\{x_i\}_{i \in \mathbb{N}}$ has a sub-sequence $\{x_{n_i}\}_{i \in \mathbb{N}}$ which is weakly S -convergent, and therefore weakly X -convergent since subspace topologies are no less fine than the topologies that induce them. Since $A \subset X$ is weakly closed, this sequence converges within A , and so A is weakly sequentially compact. \square

Proof. (2 \implies 1). Let $A \subset X$ be weakly sequentially compact, let c denote the canonical embedding of X into X^{**} , and let x^{**} in the *weak** closure of $c(A)$. Let $x_1^* \in X^*$ have norm 1. By assumption, there exists $a_1^{**} \in c(A)$ such that $|\langle x_1^*, x^{**} - a_1^{**} \rangle| < 1$. By 2.2.8, there exists $\{x_1^2, \dots, x_{n_2}^2\} \subset \partial B_{X^*}(0; 1)$ such that for each $y^{**} \in \text{span}\{x^{**}, x^{**} - a_1^{**}\}$,

$$\|y^{**}\| \leq 2 \max_{1 \leq k \leq n_2} |\langle x_k^2, y^{**} \rangle| \quad (2.126)$$

Also, since x^{**} is in the *weak** closure of $c(A)$, there exists $a_2^{**} \in c(A) \cap U_2$ where

$$U_2 = \{y^{**} \in X^{**} : (\forall 1 \leq j \leq 2)(\forall 1 \leq k \leq n_j)(|\langle x_k^j, x^{**} - y^{**} \rangle| < \frac{1}{2})\} \quad (2.127)$$

Continuing inductively, we construct a sequence $\{a_n^{**}\}_{n \in \mathbb{N}} \subset c(A)$ such that for each $j \in \mathbb{N}$, $\{x_k^j\}_{k=1}^{n_j} \subset \partial B_{X^*}(0; 1)$ such that for every $y^{**} \in \text{span}\{x^{**}, x^{**} - a_1^{**}, \dots, x^{**} - a_{j-1}^{**}\}$, we have

$$\|y^{**}\| \leq 2 \max_{1 \leq k \leq n_j} |\langle x_k^j, x^{**} - y^{**} \rangle| \quad (2.128)$$

and $a_j^{**} \in c(A) \cap U_j$ where U_j is the $\{x_k^m\}_{1 \leq m \leq j, 1 \leq k \leq n_m}$ *weak** neighborhood about x^{**} of radius $\frac{1}{j}$. For each $k \in \mathbb{N}$, let $a_k = c^{-1}(c(a_k))$. Since A is sequentially weakly compact, $\{a_k\}_{k \in \mathbb{N}}$ has a weak cluster point $x \in A$. Also, $x \in \overline{\text{span}\{a_i\}_{i \in \mathbb{N}}}$ because this is a weakly closed set, implying $c(x) \in \overline{\text{span}\{a_i^{**}\}_{i \in \mathbb{N}}}$, which then implies $c(x) \in \overline{\text{span}\{x^{**}, x^{**} - a_1^{**}, x^{**} - a_2^{**}, \dots\}}$. By continuity of the norm and each element of $\{x_i^k\}_{k \in \mathbb{N}, 1 \leq i \leq n_k}$, we conclude that for each element y^{**} of $\overline{\text{span}\{x^{**}, x^{**} - a_1^{**}, x^{**} - a_2^{**}, \dots\}}$,

$$\|y^{**}\| \leq 2 \sup_{k \in \mathbb{N}, 1 \leq i \leq n_k} |\langle x_i^k, y^{**} \rangle| \quad (2.129)$$

This is useful, because for each $k \in \mathbb{N}$, $1 \leq i \leq n_k$, we have, for large enough j ,

$$\begin{aligned} |\langle x_i^k, x^{**} - c(x) \rangle| &\leq |\langle x_i^k, x^{**} - a_j^{**} \rangle| + |\langle a_j^{**} - c(x), x_i^k \rangle| \\ &\leq \frac{1}{j} + |\langle x_i^k, a_j - x \rangle| \end{aligned} \quad (2.130)$$

which can be made arbitrarily small, and so $|\langle x_i^k, x^{**} - c(x) \rangle| = 0$, implying that

$$\|x^{**} - c(x)\| \leq 2 \sup_{k \in \mathbb{N}, 1 \leq i \leq n_k} |\langle x_i^k, x^{**} - x \rangle| = 0 \quad (2.131)$$

So $x^{**} = c(x)$, and therefore $c(A)$ is *weak** closed. Since A is weakly-sequentially compact, $c(A)$ is *weak** sequentially compact and therefore bounded by Banach Steinhaus. By 2.2.4, bounded *weak** closed sets are compact, and so $c(A)$ is *weak** compact. Since the weak topology on $A/\|\cdot\|^{-1}\{0\}$ is the same as the *weak** topology on $c(A)$, $A/\|\cdot\|^{-1}\{0\}$ is weakly compact. To see that A is weakly compact, apply ?? \square

2.2.5 Bishop-Phelps

In this subsection, I develop a result due to [7] which will prove useful throughout this document. I begin by presenting the concept of a convex cone and a trio of lemmas which are commonly utilized in the proof of this result.

Definition 2.2.10 (Convex Cone). Let X be a seminormed space over \mathbb{R} . If $K \subset X$ is convex and closed under positive scalar multiples, then we call it a **convex cone**. If J is a convex cone in X , $C \subset X$, $x_0 \in C$, and $(J + x_0) \cap C = \{x_0\}$, then we say that J **supports** C at x_0 . If $x^* \in \partial B_{X^*}(0; 1)$ and $\alpha > 0$ then we define

$$K(x^*, \alpha) := \{x \in X : \|x\| \leq \alpha \langle x, x^* \rangle\} \quad (2.132)$$

Remark 2.2.11. Let X be a seminormed space, $x^* \in \partial B_{X^*}(0; 1)$, and $\alpha > 0$. The following are true.

1. $K(x^*, \alpha)$ is a closed convex cone.
2. If $\alpha > 1$, $\text{Int}(K(x^*, \alpha)) \neq \emptyset$.

Proof. (1) If $\{x_n\} \subset K(x^*, \alpha)$ converges, say $x_n \rightarrow x$, then continuity of x^* implies $\langle x_n, x^* \rangle \rightarrow \langle x, x^* \rangle$. Hence, given $\epsilon > 0$, there exists $N > 0$ such that for $n > N$ we have $\max(\|x\| - \|x_n\|, |\langle x - x_n, x^* \rangle|) < \epsilon$, so that for all $n > N$,

$$\|x\| \leq \|x_n\| + \epsilon < \alpha \langle x_n, x^* \rangle + \epsilon < \alpha \langle x, x^* \rangle + (\alpha + 1)\epsilon \quad (2.133)$$

So $x \in K(x^*, \alpha)$ closedness is verified. It is obvious that $K(x^*, \alpha)$ is closed under positive scalar multiples, and for convexity, if $x, y \in K(x^*, \alpha)$ and $t \in [0, 1]$, then

$$\|tx + (1 - t)y\| \leq t\|x\| + (1 - t)\|y\| \leq t\alpha \langle x, x^* \rangle + (1 - t)\alpha \langle y, x^* \rangle = \alpha \langle tx + (1 - t)y, x^* \rangle \quad (2.134)$$

\square

Proof. (2) By definition of the norm on X^* , there is an $x \in \overline{B_X(0; 1)}$ such that $2/(\alpha(1 + \frac{1}{\alpha})) < \langle x, x^* \rangle$, implying by linearity that $1/\alpha < \langle \frac{1 + (1/\alpha)}{2}x, x^* \rangle$. By continuity of x^* we find a neighborhood U of $(1 + \frac{1}{\alpha})/2$ contained in $B_X(0; 1)$ such that for each $y \in U$, $1/\alpha < \langle y, x^* \rangle$. This implies $\|y\| \leq 1 < \alpha \langle y, x^* \rangle$, so $U \subset K(x^*, \alpha)$. \square

Lemma 2.2.12 (Bishop-Phelps Lemma). *Let X be a complete seminormed space, $x^*, y^* \in \partial B_{X^*}(0; 1)$, $C \subset X$ closed and convex, $1 > \epsilon > 0$, and $k > 1 + \frac{2}{\epsilon}$. The following are true.*

1. *If x^* is bounded on C , then for each $z \in C$, there is an $x_0 \in X$ such that $K(x^*, \epsilon)$ supports C at x_0 and $x_0 \in K(x^*, \epsilon) + z$.*
2. *If $|\langle x, y^* \rangle| \leq \frac{\epsilon}{2}$ for each $x \in \text{Kern}(x^*) \cap \overline{B_X(0; 1)}$, then*

$$\min(\|x^* + y^*\|, \|x^* - y^*\|) \leq \epsilon \quad (2.135)$$

3. *If y^* is nonnegative on $K(x^*, k)$, then $\|x^* - y^*\| \leq \epsilon$.*

Proof. (1) Let x^* be bounded on C and define, for $x, y \in X$, $y \lesssim x$ if and only if $x - y \in K(x^*, \epsilon)$. Fix $z \in C$. Define $Z = C \cap (K(x^*, \epsilon) + z)$. Since C and $K(x^*, \epsilon)$ are closed, so is Z . Let $\mathcal{C} = \{x_\alpha\}_{\alpha \in A}$ be a chain in where (A, \leq) is a totally ordered set and $x_\alpha \lesssim x_\beta \iff \alpha \leq \beta$. If $x_\alpha, x_\beta \in \mathcal{C}$, where $x_\beta \lesssim x_\alpha$, then $x_\alpha - x_\beta \in K(x^*, \epsilon)$, so $0 \leq \|x_\alpha - x_\beta\| \leq \epsilon \langle x_\alpha - x_\beta, x^* \rangle$, implying $\langle x_\beta, x^* \rangle \leq \langle x_\alpha, x^* \rangle$. Thus we conclude $\{\langle x_\alpha, x^* \rangle\}_{\alpha \in A}$ is a monotone bounded net in \mathbb{R} that is therefore Cauchy, which by the following inequality

$$\|x_\beta - x_\alpha\| \leq \epsilon \langle x_\alpha - x_\beta, x^* \rangle = \epsilon (\langle x_\alpha, x^* \rangle - \langle x_\beta, x^* \rangle) \rightarrow 0 \quad (2.136)$$

implies \mathcal{C} is a Cauchy net and therefore converges, say $x_\alpha \rightarrow y_0 \in Z$. Continuity of the norm and x^* imply together that y_0 is an upper bound for \mathcal{C} . Since \mathcal{C} was an arbitrary chain in Z , Z has a maximal element x_0 . By definition, $x_0 \in Z := K(x^*, \epsilon) + z$. Since $x_0 \in Z \subset C$, $x_0 \in C$. Further, since $0 \in K(x^*, \epsilon)$, $x_0 \in K(x^*, \epsilon) \cap C$. Let $y \in (K(x^*, \epsilon) + x_0) \cap C$. Then $y - x_0 \in K(x^*, \epsilon)$ so that $z \lesssim x_0 \lesssim y$, meaning $y \in Z$ and therefore $y = x_0$ since x_0 is maximal. Hence $(K(x^*, \epsilon) + x_0) \cap C = \{x_0\}$, so we are done. \square

Proof. (2) By assumption, $\|y^*|_{\text{Kern}(x^*)}\| \leq \frac{\epsilon}{2}$, so by the Hahn-Banach theorem, we can find a $\tilde{y}^* \in X^*$ extending $y^*|_{\text{Kern}(x^*)}$ such that $\|\tilde{y}^*\| \leq \frac{\epsilon}{2}$. Since $y^* - \tilde{y}^* \neq 0$, $\text{codim}(\text{kern}(x^*)) = 1$, and $\text{kern}(x^*) \subset \text{kern}(y^* - \tilde{y}^*)$, we conclude $\text{kern}(y^* - \tilde{y}^*) = \text{kern}(x^*)$. Hence, for some $\alpha \in \mathbb{R}$, $y^* - \tilde{y}^* = \alpha x^*$. For this alpha, we have

$$|1 - |\alpha|| = \|y^* - \tilde{y}^*\| = \|y^* - (y^* - \alpha x^*)\| = \|\alpha x^*\| \leq \|\tilde{y}^*\| \leq \frac{\epsilon}{2} \quad (2.137)$$

If $\alpha \geq 0$,

$$\|x^* - y^*\| = \|x^* - (\alpha x^* + \tilde{y}^*)\| = \|(1 - \alpha)x^* - \tilde{y}^*\| \leq |1 - \alpha| + \|\tilde{y}^*\| \leq \epsilon \quad (2.138)$$

If $\alpha \leq 0$, then

$$\|x^* + y^*\| = \|x^* + (\alpha x^* + \tilde{y}^*)\| = \|(1 + \alpha)x^* + \tilde{y}^*\| \leq |1 + \alpha| + \|\tilde{y}^*\| \leq \epsilon \quad (2.139)$$

\square

Proof. (3) Since $\|x^*\| = 1$, there exists $x \in \partial B_X(0; 1)$ such that $\langle x, x^* \rangle > \frac{1}{k} (1 + \frac{2}{\epsilon})$. If $y \in \text{Kern}(x^*) \cap \overline{B_X(0; 1)}$, then

$$\left\|x \pm \frac{2}{\epsilon} y\right\| \leq 1 + \frac{2}{\epsilon} < k \langle x, x^* \rangle = k \left\langle x \pm \frac{2}{\epsilon} y, x^* \right\rangle \quad (2.140)$$

so $x \pm \frac{2}{\epsilon}y \in K(x^*, k)$, so by assumption $\langle x \pm \frac{2}{\epsilon}y, y^* \rangle \geq 0$. Since this occurs for both positive and negative, $|\langle y, y^* \rangle| = \frac{\epsilon}{2} |\langle \frac{2}{\epsilon}y, y^* \rangle| \leq \frac{\epsilon}{2} \langle y^*, x \rangle \leq \frac{\epsilon}{2} \|x\| = \frac{\epsilon}{2}$. Hence by part 2, either $\|x^* - y^*\| \leq \epsilon$, or $\|x^* + y^*\| \leq \epsilon$. Since $\|x^*\| = 1$, there exists $x \in \partial B_X(0; 1)$ such that $\frac{\|x\|}{k} \leq \max(\epsilon, \frac{1}{k}) < \langle x, x^* \rangle$, so that $x \in K(x^*, k)$, implying $\langle x, y^* \rangle \geq 0$, and therefore $\epsilon < \langle x_0, x^* + y^* \rangle \leq \|x^* + y^*\|$. Hence we conclude $\|x^* - y^*\| \leq \epsilon$. \square

Theorem 2.2.13 (Bishop-Phelps Theorem). *Let X be a complete seminormed space, $C \subset X$ be closed, bounded, and convex, and define $M := \{f \in X^* | (\exists x_0 \in C)(\langle x_0, f \rangle = \sup_{x \in C} \langle x, f \rangle)\}$.*

Then $\overline{M} = X^$*

Proof. Since M is a vector subspace independent of translations of C , we assume without loss of generality that $0 \in C$ and that it is sufficient to show that M is dense in $\partial B_{X^*}(0; 1)$. Let $x^* \in \partial B_{X^*}(0; 1)$. Let $\epsilon \in (0, 1)$ and let $1 + \frac{2}{\epsilon} < k$. by 2.2.11, $K(x^*, k)$ is a closed convex cone with nonempty interior. Applying 2.2.12, part one, there is $x_0 \in C$ with $x_0 \in K(x^*, k)$ and $(K(x^*, k) + x_0) \cap C = \{x_0\}$. By Hahn Banach, there exists $y^* \in \partial B_{X^*}(0; 1)$ satisfying

$$\sup_{x \in C} \langle x, y^* \rangle = \langle x_0, y^* \rangle = \inf_{x \in K(x^*, k) + x_0} \langle x, y^* \rangle = \inf_{\tilde{x} \in K(x^*, k)} \langle \tilde{x}, y^* \rangle + \langle x_0, y^* \rangle \quad (2.141)$$

Hence y^* is positive on $K(x^*, k)$, so by 2.2.12 part 3, $\|x^* - y^*\| < \epsilon$, so we are done since $y^* \in M$ and x^* was arbitrary. \square

2.3 Reflexivity

Recall that a seminormed space X is said to be **reflexive** if $c(X) = X^{**}$. Since X^{**} is always complete and c an isometry, any reflexive space is always complete. Due to the Banach-Alaoglu theorem, in reflexive space X , the closed unit ball of X is weakly compact. For this reason and others, reflexivity is a condition of interest to many mathematicians. We begin with a basic result.

Lemma 2.3.1 (Reflexive Separable). *Let X be a complete seminormed space. Then the following are equivalent.*

1. X is reflexive.
2. The closed unit ball of X is weakly compact.
3. Each closed separable subspace of X is reflexive.
4. All collections of closed, bounded, convex sets in X have the finite intersection property
5. The closed unit ball of X is weakly sequentially compact

Proof. (1 \implies 2) Let X be reflexive. By 2.2.9 it is sufficient to show that any sequence $\{x_i\}_{i \in \mathbb{N}} \subset \overline{B_X(0; 1)}$ has a weak cluster point $x \in \overline{B_X(0; 1)}$. Since $\overline{B_{X^{**}}(0; 1)}$ is weak* compact,

$\{c(x_i)\}_{i \in \mathbb{N}}$ has a subsequence $\{c(x_{n_i})\}_{i \in \mathbb{N}}$ such that $c(x_{n_i}) \xrightarrow{w^*} \tilde{x} \in \overline{B_{X^{**}}(0;1)}$. Since X is reflexive, for some $x \in \overline{B_X(0;1)}$, $c(x) = \tilde{x}$. Let $x^* \in X^*$. Then,

$$|\langle x_{n_i} - x, x^* \rangle| = |\langle x^*, c(x_{n_i}) - \tilde{x} \rangle| \rightarrow 0 \quad (2.142)$$

as $i \rightarrow \infty$, and so $x_{n_i} \xrightarrow{w} x$, completing the proof. \square

Proof. (2 \implies 3) Suppose the closed unit ball of X is weakly compact, and let $x^{**} \in \overline{B_{X^{**}}(0;1)}$. By 2.2.3, there is a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \overline{B_X(0;1)}$ such that $c(x_n) \xrightarrow{w^*} x^{**}$. By assumption, there is a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ such that $x_{n_k} \xrightarrow{w} x \in \overline{B_X(0;1)}$. This implies $c(x_{n_k}) \xrightarrow{w^*} c(x)$, and so since the *weak** topology is Hausdorff, $x^{**} = c(x) \in c(\overline{B_X(0;1)})$. \square

Proof. (2 \implies 3). Let X be reflexive and $S \subset X$ be a closed separable vector subspace. Let $\{x_i\}_{i \in \mathbb{N}} \subset \overline{B_X(0;1)}$. By assumption this sequence has an X -weakly convergent subsequence, $x_{n_k} \rightarrow x \in X$. Since S is weakly closed, $x \in S$, and since the weak topology on X is finer than that on S , $x_{n_k} \xrightarrow{S-w} x$. By 2.2.9 and (2 \implies 1, S is reflexive. \square

Proof. (3 \implies 2) Let $\{x_n\}_{n \in \mathbb{N}} \subset \overline{B_X(0;1)}$. Then since $S := \overline{\text{span}\{x_i\}_{i \in \mathbb{N}}}$ is a closed separable subspace, $\{x_n\}_{n \in \mathbb{N}}$ has an S -weakly convergent subsequence, $x_{n_k} \xrightarrow{S-w} x \in S$. If $x^* \in X^*$, then $x^*|_S \in S^*$, so that $|\langle x - x_{n_k}, x^* \rangle| \rightarrow 0$, implying $x_{n_k} \xrightarrow{w} x$, an application of 2.2.9 finishes the proof. \square

Proof. (4 \iff 2) Obvious since closed convex sets are weakly closed. \square

Proof. (5 \iff 2) Apply 2.2.9. \square

2.3.1 James

As an easy application of 2.2.4 and 2.2.9, for a reflexive space X , all $x^* \in X^*$ attain their norm. The converse of this fact was, for a time, an open question of considerable interest. The converse of this result was, as is traditional in mathematics, proven in a piecemeal manner. The result was first tackled by James in [8] under the added assumption that every space Y isomorphic to X has the property that each $y^* \in Y^*$ attains its norm and that X permits a Schauder basis. This result was rapidly improved by Klee in [9] who dropped the assumption of a Schauder basis, and then by James again in [10] who proved the result in the case of a separable space X . The question of the converse in a Banach space was finally answered to the affirmative in [11], building on the arguments in [10]. This paper included generalizations all the way to quasi-complete locally convex TVS's.

Definition 2.3.2. *If X is a topological vector space and $\{x_i^*\}_{i \in \mathbb{N}} \subset X^*$, then $\text{CoLim}\{x_i^*\}_{i \in \mathbb{N}}$ is the set of all $x^* \in X^*$ such that for every $x \in X$,*

$$\liminf_{i \rightarrow \infty} \langle x, x_i^* \rangle \leq \langle x, x^* \rangle \leq \limsup_{i \rightarrow \infty} \langle x, x_i^* \rangle \quad (2.143)$$

Remark 2.3.3 (CoLim Nonempty). *Let X be a complete seminormed space and let $\{x_i^*\}_{i \in \mathbb{N}} \subset X^*$ be bounded. Then $\text{CoLim}\{x_i^*\}_{i \in \mathbb{N}} \neq \emptyset$*

Proof. Since $\{x_i^*\}_{i \in \mathbb{N}}$ is bounded, it has a subsequence with a *weak** limit x^* who must live in $\text{CoLim}\{x_i^*\}_{i \in \mathbb{N}}$. \square

Lemma 2.3.4. *Let X be a complete seminormed space, $\alpha \in (0, 1)$, $\{x_i^*\}_{i \in \mathbb{N}} \subset X^*$, and $\{\beta_i\}_{i \in \mathbb{N}} \subset (0, 1)$ such that $\sum_{i \in \mathbb{N}} \beta_i = 1$. Then if (1) or (3) hold below, there are $\{y_i^*\}_{i \in \mathbb{N}} \subset X^*$ and $\gamma \geq \alpha$ such that (2) or (4) hold respectively.*

1. $\{x_i^*\}_{i \in \mathbb{N}} \subset \partial B_{X^*}(0; 1)$ such that $d(0, \overline{\text{co}}\{x_i^*\}_{i \in \mathbb{N}}) \geq \alpha$.

2. $\gamma \leq 1$, and for each $i \in \mathbb{N}$,

$$y_i^* \in \overline{\text{co}}\{x_j^*\}_{j \geq i} \quad \left\| \sum_{j \in \mathbb{N}} \beta_j y_j^* \right\| = \gamma \quad \left\| \sum_{j=1}^i \beta_j y_j^* \right\| < \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j \right) \right) \quad (2.144)$$

3. $\{x_i^*\}_{i \in \mathbb{N}} \subset \overline{B_{X^*}(0; 1)}$ such that $d(\overline{\text{co}}\{x_i^*\}_{i \in \mathbb{N}} - \text{CoLim}\{x_i^*\}_{i \in \mathbb{N}}, 0) \geq \alpha$.

4. $\gamma \leq 2$, $\{y_i^*\}_{i \in \mathbb{N}} \subset \overline{B_{X^*}(0; 1)}$, and for each $i \in \mathbb{N}$, $y^* \in \text{CoLim}\{y_i^*\}_{i \in \mathbb{N}}$,

$$\left\| \sum_{j \in \mathbb{N}} \beta_j (y_j^* - y^*) \right\| = \gamma \quad \left\| \sum_{j=1}^i \beta_j (y_j^* - y^*) \right\| < \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j \right) \right) \quad (2.145)$$

Proof. (1 \implies 2) There exists a positive sequence $\{\delta_i\}_{i \in \mathbb{N}}$ such that

$$\sum_{i \in \mathbb{N}} \frac{\beta_i \delta_i}{\left(\sum_{j=i+1}^{\infty} \beta_j \right) \left(\sum_{j=i}^{\infty} \beta_j \right)} < 1 - \alpha \quad (2.146)$$

Let $\gamma_1 \in \mathbb{R}$ and choose $y_1^* \in \overline{\text{co}}\{x_i^*\}_{i \in \mathbb{N}}$ such that $\gamma_1 = d(0, \overline{\text{co}}\{x_i^*\}_{i \in \mathbb{N}})$ and $\|y_1^*\| \leq \gamma_1 (1 + \delta_1)$. From here, for each $n \geq 1$, define $\gamma_{n+1} \in \mathbb{R}$ and choose $y_{n+1}^* \in \overline{\text{co}}\{x_i^*\}_{i \geq n+1}$ such that

$$\gamma_{n+1} = \inf \left\{ \left\| \left(\sum_{i=1}^n \beta_i y_i^* \right) + \left(1 - \sum_{i=1}^n \beta_i \right) y^* \right\| : y^* \in \overline{\text{co}}\{x_i^*\}_{i \geq n+1} \right\} \quad (2.147)$$

and

$$\left\| \sum_{i=1}^n \beta_i y_i^* + \left(1 - \sum_{i=1}^n \beta_i \right) y_{n+1}^* \right\| < \gamma_{n+1} (1 + \delta_{n+1}) \quad (2.148)$$

It is clear that since the set over which we are taking the infimum never gets larger, so $\{\gamma_i\}_{i \in \mathbb{N}}$ is a nondecreasing sequence. Further, since $\{x_i^*\}_{i \in \mathbb{N}} \subset \overline{B_{X^*}(0; 1)}$ and $\sum_{i \in \mathbb{N}} \beta_i = 1$, we have, for every i ,

$$\alpha \leq \gamma_i \nearrow \gamma = \left\| \sum_{k \in \mathbb{N}} \beta_k y_k^* \right\| \leq 1 \quad (2.149)$$

So what is left to be shown is that for every $i \in \mathbb{N}$,

$$\left\| \sum_{j=1}^i \beta_j y_j^* \right\| < \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j \right) \right) \quad (2.150)$$

Let $i \in \mathbb{N}$. Then,

$$\begin{aligned}
\left\| \sum_{j=1}^i \beta_j (y_j^* - y^*) \right\| &= \left\| \left(\left(\frac{\sum_{j=i}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j} \right) \left(\sum_{j=1}^{i-1} \beta_j y_j^* \right) \right) + \left(\left(\frac{\sum_{j=i}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j} \right) (\beta_i y_i^*) \right) \right\| \\
&= \left\| \left(\left(\frac{\lambda_i + \sum_{j=i+1}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j} \right) \left(\sum_{j=1}^{i-1} \beta_j y_j^* \right) \right) + \left(\left(\frac{\beta_i \sum_{j=i}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j} \right) (y_i^*) \right) \right\| \\
&\leq \frac{\beta_i}{\sum_{j=i}^{\infty} \beta_j} \left\| \sum_{j=1}^{i-1} \beta_j y_j^* + \left(\sum_{j=i}^{\infty} \beta_j \right) y_i^* \right\| + \frac{\sum_{j=i+1}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j} \left\| \sum_{j=1}^{i-1} \beta_j y_j^* \right\| \\
&\leq \frac{\beta_i}{\sum_{j=i}^{\infty} \beta_j} \left\| \sum_{j=1}^{i-1} \beta_j y_j^* + \left(1 - \sum_{j=1}^{i-1} \beta_j \right) y_i^* \right\| + \frac{\sum_{j=i+1}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j} \left\| \sum_{j=1}^{i-1} \beta_j y_j^* \right\| \\
&< \frac{\beta_i}{\sum_{j=i}^{\infty} \beta_j} (\gamma_i) (1 + \delta_i) + \frac{\sum_{j=i+1}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j} \left\| \sum_{j=1}^{i-1} \beta_j y_j^* \right\| \\
&= \left(\sum_{j=i+1}^{\infty} \beta_j \right) \left(\left(\frac{\beta_i \gamma_i (1 + \delta_i)}{\left(\sum_{j=i}^{\infty} \beta_j \right) \left(\sum_{j=i+1}^{\infty} \beta_j \right)} \right) + \left(\frac{1}{\sum_{j=i}^{\infty} \beta_j} \right) \left\| \sum_{j=1}^{i-1} \beta_j y_j^* \right\| \right)
\end{aligned} \tag{2.151}$$

Hence, for any $i \in \mathbb{N}$,

$$\begin{aligned}
\left\| \sum_{j=1}^i \beta_j y_j^* \right\| &< \left(\sum_{j=i+1}^{\infty} \beta_j \right) \left(\left(\frac{\beta_i \gamma_i (1 + \delta_i)}{\left(\sum_{j=i}^{\infty} \beta_j \right) \left(\sum_{j=i+1}^{\infty} \beta_j \right)} \right) + \left(\frac{1}{\sum_{j=i}^{\infty} \beta_j} \right) \left\| \sum_{j=1}^{i-1} \beta_j y_j^* \right\| \right) \\
&< \left(\sum_{j=i+1}^{\infty} \beta_j \right) \sum_{j=1}^i \frac{\beta_j \gamma_j (1 + \delta_j)}{\left(\sum_{k=j}^{\infty} \beta_k \right) \left(\sum_{k=j+1}^{\infty} \beta_k \right)} \\
&\leq \gamma \left(\sum_{j=i+1}^{\infty} \beta_j \right) \sum_{j=1}^i \frac{\beta_j (1 + \delta_j)}{\left(\sum_{k=j}^{\infty} \beta_k \right) \left(\sum_{k=j+1}^{\infty} \beta_k \right)} \\
&\leq \gamma \left(\sum_{j=i+1}^{\infty} \beta_j \right) \left(\left(\sum_{j=1}^i \frac{\beta_j}{\left(\sum_{k=j}^{\infty} \beta_k \right) \left(\sum_{k=j+1}^{\infty} \beta_k \right)} \right) + (1 - \alpha) \right) \\
&= \gamma \left(\sum_{j=i+1}^{\infty} \beta_j \right) \left(\left(\sum_{j=1}^i \left(\frac{1}{\sum_{k=j}^{\infty} \beta_k} - \frac{1}{\sum_{k=j+1}^{\infty} \beta_k} \right) \right) + (1 - \alpha) \right) \\
&= \gamma \left(\sum_{j=i+1}^{\infty} \beta_j \right) \left(\frac{1}{\sum_{j=i+1}^{\infty} \beta_j} - \frac{1}{\sum_{j=1}^{\infty} \beta_j} + 1 - \alpha \right) \\
&= \gamma \left(\sum_{j=i+1}^{\infty} \beta_j \right) \left(\frac{1}{\sum_{j=i+1}^{\infty} \beta_j} - \alpha \right) \\
&= \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j \right) \right)
\end{aligned} \tag{2.152}$$

completing the proof. \square

Proof. (3 \implies 4) There exists a positive sequence $\{\delta_i\}_{i \in \mathbb{N}}$ such that

$$\sum_{i \in \mathbb{N}} \frac{\beta_i \delta_i}{\left(\sum_{j=i+1}^{\infty} \beta_j \right) \left(\sum_{j=i}^{\infty} \beta_j \right)} < 1 - \alpha \tag{2.153}$$

Define $\{x_i^0\}_{i \in \mathbb{N}} = \{x_i^*\}_{i \in \mathbb{N}}$,

$$\gamma_1 = \inf \left\{ \sup_{y^* \in \text{CoLim}\{\phi_i\}_{i \in \mathbb{N}}} \{ \|x^* - y^*\| : x^* \in \overline{\text{co}}\{x_i^*\}_{i \in \mathbb{N}}, \phi_k \in \overline{\text{co}}\{x_i^*\}_{i \geq k}, k \in \mathbb{N} \right\} \tag{2.154}$$

and pick $y_1^* \in \overline{\text{co}}\{x_i^*\}_{i \in \mathbb{N}}$, $\{\phi_i^1\}_{i \in \mathbb{N}} \subset X^*$ such that $\phi_k^1 \in \overline{\text{co}}\{x_i^*\}_{i \geq k}$ for every k and $w' \in \text{CoLim}\{\phi_i^1\}_{i \in \mathbb{N}}$ such that

$$\gamma_1(1 - \delta_1) < \|y_1^* - w'\| < \gamma_1(1 + \delta_1) \tag{2.155}$$

So that there exists $\tilde{x} \in \overline{B_X(0;1)}$ such that $\gamma_1(1 - \delta_1) < \langle \tilde{x}, y_1^* - w' \rangle$, and since $w' \in \text{CoLim}\{\phi_i^1\}_{i \in \mathbb{N}}$, we extract a subsequence $\{x_i^1\}_{i \in \mathbb{N}}$ of $\{\phi_i^1\}_{i \in \mathbb{N}}$ so that for any $w \in \text{CoLim}\{x_i^1\}_{i \in \mathbb{N}}$, we have

$$\langle \tilde{x}, w \rangle = \lim_{i \rightarrow \infty} \langle \tilde{x}, x_i^1 \rangle = \liminf_{i \rightarrow \infty} \langle \tilde{x}, \phi_i^1 \rangle \leq \langle \tilde{x}, w' \rangle \quad (2.156)$$

And so, for any $w \in \text{CoLim}\{x_i^1\}_{i \in \mathbb{N}}$, we have

$$\gamma_1(1 - \delta_1) < \langle \tilde{x}, y_1^* - w \rangle \quad (2.157)$$

Continuing inductively, for $i \in \mathbb{N}$, set

$$\gamma_{i+1} = \inf \left\{ \sup \left\{ \left\| \left(\sum_{j=1}^i \beta_j y_j^* \right) + \left(\left(\sum_{j=i+1}^{\infty} \beta_j \right) y^* \right) - w \right\| : w \in \text{CoLim}\{\phi_i\}_{i \in \mathbb{N}} \right\} \right\} \quad (2.158)$$

Where the infimum is taken over all $y^* \in \overline{\text{co}}\{x_j^i\}_{j \geq i+1}$ and all $\{\phi_i\}_{i \in \mathbb{N}} \subset X^*$ such that $\phi_k \in \overline{\text{co}}\{x_j^i\}_{j \geq k}$. Next, pick $y_{i+1}^* \in \overline{\text{co}}\{x_j^i\}_{j \geq i+1}$ and $\{\phi_j^{i+1}\}_{j \in \mathbb{N}} \subset X^*$ such that for every k , $\phi_k^{i+1} \in \overline{\text{co}}\{x_j^i\}_{j \geq k}$ and pick $w' \in \text{CoLim}\{\phi_j^{i+1}\}_{j \in \mathbb{N}}$ such that

$$\gamma_{i+1}(1 - \delta_{i+1}) < \left\| \sum_{j=1}^i \beta_j y_j^* + \left(\sum_{j=i+1}^{\infty} \beta_j \right) y_{i+1}^* - w' \right\| < \gamma_{i+1}(1 + \delta_{i+1}) \quad (2.159)$$

Next, pick $\tilde{x} \in \overline{B_X(0;1)}$ satisfying

$$\gamma_{i+1}(1 - \delta_{i+1}) < \left\langle \tilde{x}, \sum_{j=1}^i \beta_j y_j^* + \left(\left(\sum_{j=i+1}^{\infty} \beta_j \right) y_j^* \right) - w' \right\rangle \quad (2.160)$$

and apply the fact that since $\liminf_{j \rightarrow \infty} \langle \tilde{x}, \phi_j^{i+1} \rangle \leq \langle \tilde{x}, w \rangle$, we can find a subsequence $\{x_j^{i+1}\}_{j \in \mathbb{N}}$ of $\{\phi_j^{i+1}\}_{j \in \mathbb{N}}$ such that for every $w \in \text{CoLim}\{x_j^{i+1}\}_{j \in \mathbb{N}}$ we have

$$\gamma_{i+1}(1 - \delta_{i+1}) < \left\langle \tilde{x}, \sum_{j=1}^i \beta_j y_j^* + \left(\left(\sum_{j=i+1}^{\infty} \beta_j \right) y_j^* \right) - w \right\rangle \quad (2.161)$$

completing our construction. Clearly, $\text{CoLim}\{y_j^*\}_{j \in \mathbb{N}} \subset \text{CoLim}\{\phi_j^i\}_{j \in \mathbb{N}}$ for every $i \in \mathbb{N}$. Hence, for every $w \in \text{CoLim}\{y_j^*\}_{j \in \mathbb{N}}$, for every $i \in \mathbb{N}$, we have

$$\gamma_i(1 - \delta_i) < \left\| \sum_{j=1}^{i-1} \beta_j y_j^* + \left(\left(\sum_{j=i}^{\infty} \beta_j \right) y_i^* \right) - w \right\| < \gamma_i(1 + \delta_i) \quad (2.162)$$

Also, since $\{y_j^*\}_{j \in \mathbb{N}} \subset \overline{\text{co}}\{x_j\}_{j \in \mathbb{N}} \subset \overline{B_{X^*}(0;1)}$, $\text{CoLim}\{y_j^*\}_{j \in \mathbb{N}} \subset \overline{\text{co}}\{y_j^*\}_{j \in \mathbb{N}} \subset \overline{B_{X^*}(0;1)}$. By definition, $\gamma_1 \geq \alpha$, and $\{\gamma_i\}_{i \in \mathbb{N}}$ is a nondecreasing sequence since it is defined by taking the infimum over a set which never gains new elements as i increases. Further, $\|w\| \leq 1$ for $w \in \text{CoLim}\{y_j^*\}_{j \in \mathbb{N}}$ implies that for every n , $\gamma_n \leq 2$, so by monotone convergence, $\gamma_i \nearrow \gamma = \left\| \sum_{j \in \mathbb{N}} \beta_j (y_j - w) \right\| \leq 2$. As for the final estimate, we have, for $i \in \mathbb{N}$ and $y^* \in \text{CoLim}\{y_j^*\}_{j \in \mathbb{N}}$, we have

Let $i \in \mathbb{N}$. Then, Hence, for every $i \in \mathbb{N}$, we have

$$\begin{aligned}
\left\| \sum_{j=1}^i \beta_j (y_j^* - y^*) \right\| &< \left(\sum_{j=i+1}^{\infty} \beta_j \right) \\
&\quad * \left(\left(\frac{\beta_i \gamma_i (1 + \delta_i)}{\left(\sum_{j=i}^{\infty} \beta_j \right) \left(\sum_{j=i+1}^{\infty} \beta_j \right)} \right) + \left(\frac{1}{\sum_{j=i}^{\infty} \beta_j} \right) \left\| \sum_{j=1}^{i-1} \beta_j (y_j^* - y^*) \right\| \right) \\
&< \left(\sum_{j=i+1}^{\infty} \beta_j \right) \sum_{j=1}^i \frac{\beta_j \gamma_j (1 + \delta_j)}{\left(\sum_{k=j}^{\infty} \beta_k \right) \left(\sum_{k=j+1}^{\infty} \beta_k \right)} \\
&\leq \gamma \left(\sum_{j=i+1}^{\infty} \beta_j \right) \sum_{j=1}^i \frac{\beta_j (1 + \delta_j)}{\left(\sum_{k=j}^{\infty} \beta_k \right) \left(\sum_{k=j+1}^{\infty} \beta_k \right)} \\
&\leq \gamma \left(\sum_{j=i+1}^{\infty} \beta_j \right) \left(\left(\sum_{j=1}^i \frac{\beta_j}{\left(\sum_{k=j}^{\infty} \beta_k \right) \left(\sum_{k=j+1}^{\infty} \beta_k \right)} \right) + (1 - \alpha) \right) \\
&= \gamma \left(\sum_{j=i+1}^{\infty} \beta_j \right) \left(\left(\sum_{j=1}^i \left(\frac{1}{\sum_{k=j}^{\infty} \beta_k} - \frac{1}{\sum_{k=j+1}^{\infty} \beta_k} \right) \right) + (1 - \alpha) \right) \\
&= \gamma \left(\sum_{j=i+1}^{\infty} \beta_j \right) \left(\frac{1}{\sum_{j=i+1}^{\infty} \beta_j} - \frac{1}{\sum_{j=1}^{\infty} \beta_j} + 1 - \alpha \right) \\
&= \gamma \left(\sum_{j=i+1}^{\infty} \beta_j \right) \left(\frac{1}{\sum_{j=i+1}^{\infty} \beta_j} - \alpha \right) \\
&= \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j \right) \right)
\end{aligned} \tag{2.163}$$

□

It is worth noting that in the following two theorems, the assumption of completeness is necessary, as demonstrated by [12].

Theorem 2.3.5 (James Separable). *If X is a separable complete seminormed space then the following are equivalent.*

1. X is not reflexive.
2. For every $\alpha \in (0, 1)$ there is some sequence $\{x_i^*\}_{i \in \mathbb{N}} \subset \overline{B_{X^*}(0; 1)}$ satisfying $d(0, \overline{\text{co}}\{x_i^*\}_{i \in \mathbb{N}}) \geq \alpha$ and $x_i^* \xrightarrow{w^*} 0$.
3. For every $\alpha \in (0, 1)$ and $\{\beta_i\}_{i \in \mathbb{N}} \subset (0, 1)$ satisfying $\sum_{i \in \mathbb{N}} \beta_i = 1$, there is a $\gamma \in [0, 1]$

and $\{y_i^*\}_{i \in \mathbb{N}} \subset X^*$ such that $y_i^* \xrightarrow{w^*} 0$ and for each $i \in \mathbb{N}$,

$$\left\| \sum_{j \in \mathbb{N}} \beta_j y_j^* \right\| = \gamma \quad \left\| \sum_{j=1}^i \beta_j y_j^* \right\| \leq \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j \right) \right) \quad (2.164)$$

4. There exists $x^* \in X^*$ not achieving its norm.

Proof. (1 \implies 2) Let $\alpha \in (0, 1)$. Since X is nonreflexive and $c(X)$ is complete, by Riesz's lemma [13], there exists an $x^{**} \in B_{X^{**}}(0; 1)$ such that $d(x^{**}, c(X)) > \alpha$. Since X is separable it has a countable dense set $\{x_i\}_{i \in \mathbb{N}}$. Fix $i \in \mathbb{N}$, let $\alpha_1 = \alpha_2 = \dots = \alpha_{i-1} = 0$, $\alpha_i = \alpha$, and let $\{\beta_j\}_{j=1}^i \subset \mathbb{C}$ where $\beta_i \neq 0$ without loss of generality. Then, since $c(X)$ is a subspace,

$$\begin{aligned} \left| \sum_{j=1}^i \beta_j \alpha_j \right| &= |\beta_i \alpha_i| = |\beta_i| \alpha \\ &\leq \frac{|\beta_i| \alpha}{d(x^{**}, c(X))} \left\| x^{**} + \sum_{j=1}^{i-1} \frac{\beta_j}{\beta_i} c(x_j) \right\| \\ &= \frac{\alpha}{d(x^{**}, c(X))} \left\| \beta_i x^{**} + \sum_{j=1}^{i-1} \beta_j c(x_j) \right\| \end{aligned} \quad (2.165)$$

Since $\alpha < d(x^{**}, c(X))$, for some $\epsilon > 0$, $\epsilon + \frac{\alpha}{d(x^{**}, c(X))} < 1$, so by 2.2.1, since $X^{**} = (X^*)^*$, there exists an $x_i^* \in \overline{B_{X^*}(0; 1)}$ such that for $1 \leq j \leq i-1$ we have $\langle x_j, x_i^* \rangle = \langle x_i^*, c(x_j) \rangle = 0$ and $\langle x_i^*, c(x_i) \rangle \geq \alpha$. Using this method we construct a sequence $\{x_i^*\}_{i \in \mathbb{N}} \subset \overline{B_{X^*}(0; 1)}$ such that for each $1 \leq j \leq i-1$, $\langle x_j, x_i^* \rangle = 0$ and $\langle x_i, x^{**} \rangle \geq \alpha$. Without loss of generality, we let $\|x_i^*\| = 1$. Density of $\{x_j\}_{j \in \mathbb{N}}$ and the boundedness of $\{x_j^*\}_{j \in \mathbb{N}}$ implies $x_j^* \xrightarrow{w^*} 0$. Furthermore, any convex combination of the $(x_i^*)'$ s satisfies

$$\alpha \leq \left\langle \sum_{j=1}^n \lambda_j x_{k_j}^*, x^{**} \right\rangle \leq \|x^{**}\| \left\| \sum_{j=1}^n \lambda_j x_{k_j}^* \right\| \leq \left\| \sum_{j=1}^n \lambda_j x_{k_j}^* \right\| \quad (2.166)$$

so that $d(0, \overline{co}\{x_i^*\}_{i \in \mathbb{N}}) \geq \alpha$, completing the proof. \square

Proof. (2 \implies 3). This is a direct application of 2.3.4 part (1 \implies 2), paired with the fact that if for every i , $y_i^* \in \overline{co}\{x_j^*\}_{j \geq i}$ and $x_j^* \xrightarrow{w^*} x$, then $y_j^* \xrightarrow{w^*} x$. \square

Proof. (3 \implies 4). Let $x^* = \sum_{j \in \mathbb{N}} \beta_j y_j^*$ and let $x \in \overline{B_X(0; 1)}$. Since $y_j^* \xrightarrow{w^*} 0$, for some $N \in \mathbb{N}$,

$\langle x, y_j^* \rangle < \gamma\alpha$ for every $j > \mathbb{N}$. Then

$$\begin{aligned}
 |\langle x, x^* \rangle| &\leq \left| \left\langle x, \sum_{j=1}^N \beta_j y_j^* \right\rangle \right| + \left| \left\langle x, \sum_{j=N+1}^{\infty} \beta_j y_j^* \right\rangle \right| \\
 &< \left| \left\langle x, \sum_{j=1}^N \beta_j y_j^* \right\rangle \right| + \alpha\gamma \sum_{j=N+1}^{\infty} \beta_j \\
 &\leq \left\| \sum_{j=1}^N \beta_j y_j^* \right\| + \alpha\gamma \sum_{j=N+1}^{\infty} \beta_j \leq \gamma \left(1 - \alpha \sum_{j=N+1}^{\infty} \beta_j \right) + \gamma\alpha \sum_{j=N+1}^{\infty} \beta_j \\
 &= \gamma = \left\| \sum_{j \in \mathbb{N}} \beta_j y_j^* \right\| = \|x^*\|
 \end{aligned} \tag{2.167}$$

Since the inequality is strict and $x \in \overline{B_X(0;1)}$ was arbitrary, we are done. \square

Proof. (4 \implies 1). If X is reflexive and $x^* \in X$, then there is a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \overline{B_X(0;1)}$ such that $\langle x_n, x^* \rangle \rightarrow \|x^*\|$. By 2.3.1, $\{x_n\}_{n \in \mathbb{N}}$ has a subsequence $x_{k_n} \xrightarrow{w} x \in \overline{B_X(0;1)}$. This x satisfies $\langle x, x^* \rangle = \|x^*\|$. \square

Theorem 2.3.6 (James). *If X is a complete seminormed space, then the following are equivalent.*

1. X is non-reflexive.
2. For each $\alpha \in (0, 1)$, there exists an $\{x_i^*\}_{i \in \mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ and a subspace $Y \subset X$ such that $d(\overline{\text{co}}\{x_i^*\}_{i \in \mathbb{N}} - Y^\perp, 0) \geq \alpha$ and that $\langle y, x_i^* \rangle \rightarrow 0$ for each $y \in Y$.
3. For every $\alpha \in (0, 1)$ and $\{\beta_i\}_{i \in \mathbb{N}} \subset [0, \infty)$ such that $\sum_{i \in \mathbb{N}} \beta_i = 1$, there is a $\gamma \in [0, 2]$ and $\{y_i^*\}_{i \in \mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ such that for each $y^* \in \text{CoLim}\{y_i^*\}_{i \in \mathbb{N}}$, each $i \in \mathbb{N}$,

$$\left\| \sum_{j \in \mathbb{N}} \beta_j (y_j^* - y^*) \right\| = \gamma \quad \left\| \sum_{j=1}^i \beta_j (y_j^* - y_j) \right\| < \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j \right) \right) \tag{2.168}$$

4. There exists $x^* \in X^*$ which doesn't achieve its norm.

Proof. (1 \implies 2) If X is non-reflexive, then X contains a non-reflexive closed separable subspace S . An application of 2.3.5 implies the existence of a sequence $\{x_i\}_{i \in \mathbb{N}} \subset \overline{B_{S^*}(0;1)}$ such that

$$d_s(0, \overline{\text{co}}\{x_i\}_{i \in \mathbb{N}}) \geq \alpha \quad x_i \xrightarrow{S-w^*} 0 \tag{2.169}$$

If $y^\perp \in X^*$ such that $Y \subset \text{kern}(y^*)$, then letting for each $i \in \mathbb{N}$ x_i^* be a Hahn-Banach extension living in $\overline{B_{X^*}(0;1)}$, and let $x^* \in \overline{\text{co}}\{x_i^*\}_{i \in \mathbb{N}}$. Then $x := x^*|_S \in \overline{\text{co}}\{x_i\}_{i \in \mathbb{N}}$. If $y \in Y$, then $\langle y, x_i^* \rangle = \langle y, x \rangle \rightarrow 0$. If $y^\perp \in Y^\perp$, then

$$\|x^* - y^\perp\| \geq \|x - (y^\perp|_S)\|_S = \|x\| \geq \alpha \tag{2.170}$$

so $d(\overline{\text{co}}\{x_i^*\}_{i \in \mathbb{N}} - Y^\perp, 0) \geq \alpha$ \square

Proof. (2 \implies 3) Since $CoLim\{x_i^*\}_{i \in \mathbb{N}} \subset Y^\perp$ from the previous part, this is an easy application of 2.3.4 (3 \implies 4). \square

Proof. (3 \implies 4) Define $\eta = \frac{\alpha^2}{4}$, and then let $\lambda_1 \in [0, \infty)$ such that for every natural n , $\lambda_{n+1} < \eta\lambda_n$ and $\sum_{k \in \mathbb{N}} \lambda_k = 1$. Let $y^* \in CoLim\{y_i^*\}_{i \in \mathbb{N}}$ where $\{y_i^*\}_{i \in \mathbb{N}}$ are as in part (3) of this theorem. Let $x \in \overline{B_X(0; 1)}$. Since $y^* \in Colim\{y_i^*\}_{i \in \mathbb{N}}$, and since $\alpha \leq \gamma$, there exists $i \in \mathbb{N}$ such that

$$\langle x, y_{i+1}^* - y^* \rangle < \alpha^2 - 2\eta \leq \alpha\gamma - 2\eta \quad (2.171)$$

For this x , we have

$$\begin{aligned} \left\langle x, \sum_{j \in \mathbb{N}} \beta_j (y_j^* - y^*) \right\rangle &< \sum_{j=1}^i \beta_j \langle x, y_j^* - y^* \rangle + \beta_{i+1} (\alpha\gamma - 2\eta) + \sum_{j=i+2}^{\infty} \beta_j \langle x, y_j^* - y^* \rangle \\ &\leq \left\| \sum_{j=1}^i \beta_j (y_j^* - y^*) \right\| + \beta_{i+1} (\alpha\gamma - 2\eta) + 2 \sum_{j=i+2}^{\infty} \beta_j \\ &\leq \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j \right) \right) + \beta_{i+1} (\alpha\gamma - 2\eta) + 2 \sum_{j=i+2}^{\infty} \beta_j \quad (2.172) \\ &= \gamma - \gamma\alpha \sum_{j=i+2}^{\infty} \beta_j - 2\eta\beta_{i+1} + 2 \sum_{j=i+1}^{\infty} \eta\beta_j \\ &\leq \gamma - (\gamma\alpha - 2\eta) \sum_{j=i+1}^{\infty} \beta_j < \gamma = \left\| \sum_{j \in \mathbb{N}} \beta_j (y_j^* - y^*) \right\| \end{aligned}$$

Since $x \in \overline{B_X(0; 1)}$ was arbitrary, we are done. \square

Proof. (4 \implies 1) If X is reflexive and $x^* \in X$, then there is a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \overline{B_X(0; 1)}$ such that $\langle x_n, x^* \rangle \rightarrow \|x^*\|$. By 2.3.1, $\{x_n\}_{n \in \mathbb{N}}$ has a subsequence $x_{k_n} \xrightarrow{w} x \in \overline{B_X(0; 1)}$. This x satisfies $\langle x, x^* \rangle = \|x^*\|$. \square

Corollary 2.3.7. *Let X be a complete seminormed space. Then the following are equivalent.*

1. X is reflexive.
2. Each element of X^* attains its norm.

Proof. Direct consequence of 2.3.6. \square

2.3.2 Lindenstrauss On Nonseparable Reflexive Banach Spaces

If $\Gamma \neq \emptyset$, then $c_0(\Gamma)$ denotes the space of mappings $f : \Gamma \rightarrow \mathbb{C}$ such that for every $\epsilon > 0$, $\text{card}\{x \in \Gamma : |f(x)| > \epsilon\} \in \mathbb{N}$.

2.4 Convexity Of Functions And Sets

Definition 2.4.1 (Convex Functions). *Let X be a vector space, Y a topological vector space, \mathcal{U} the set of neighborhoods of 0 in Y except Y itself, $f : X \rightarrow (-\infty, \infty]$, and $g : Y \rightarrow (-\infty, \infty]$. Let $x, y \in X$.*

1. We call $D(f) := f^{-1}(\mathbb{R})$ the **effective domain** of f .
2. If $D(f) \neq \emptyset$, then we call f **proper**.
3. We call $\text{Epi}(f) := \{(x, t) \in X \times \mathbb{R} : f(x) \leq t\}$ the **Epigraph** of f .
4. We denote $[x, y] = \{tx + (1 - t)y : t \in [0, 1]\}$.
5. We denote $(x, y) = \{tx + (1 - t)y : t \in (0, 1)\}$.
6. We say that $C \subset X$ is **convex** if $[x, y] \subset C$ for each $x, y \in C$.
7. We say that $C \subset X$ is **strictly convex** if for each $x, y \in C$, for each $z_0 \in (x, y)$, and for each $z_1 \in X$, there is a $t > 0$ such that $[z_0, z_0 + tz_1] \subset C$.
8. We say that f is **convex** if for each $x, y \in X$, $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$.
9. We say that f is **strictly convex** if for each $x, y \in X$, $f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$.
10. If g is a convex function, then then we define the **modulus of local uniform convexity** of g , $\tilde{\Delta} : \mathcal{U} \times Y \rightarrow \mathbb{R}$ by

$$\tilde{\Delta}(U, x) = \frac{1}{2} \inf_{y \in Y \setminus (x+U)} \left\{ f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \right\} \quad (2.173)$$

11. If g is a convex function, then we define the **modulus of uniform convexity** of g , $\Delta : \mathcal{U} \rightarrow \mathbb{R}$ by $\Delta(U) = \inf_{g(x)=1} \tilde{\Delta}(U, x)$.
12. If g is a convex function, then we say that g is **locally uniformly convex at $x \in Y$** if for each $U \in \mathcal{U}$, $\tilde{\Delta}(U, x) > 0$.
13. We say that g is **locally uniformly convex** if it is **locally uniformly convex** at each of its points.
14. We say that g is **uniformly convex** if for each $U \in \mathcal{U}$, $\Delta(U) > 0$.
15. We say that g is **lower semi-continuous**, or LSC if it is continuous with respect to the topology on $(-\infty, \infty]$ generated by sets of the form $(-\infty, \alpha)$ where $\alpha \in \mathbb{R}$, along with $(-\infty, \infty]$ itself.

Remark 2.4.2 (Basis Independence). *It is easy to see that a mapping is locally uniform convex at a point (locally uniformly convex) [uniformly convex] if we define Δ , $\tilde{\Delta}$ in terms of a single neighborhood basis of Y at 0 instead of all neighborhoods of 0 in Y .*

Remark 2.4.3 (Strictly Convex Real Valued). *If X is a vector space and $f : X \rightarrow (-\infty, \infty]$ is strictly convex, and f is finite everywhere.*

Proof. If $f(x) = \infty$ where $x \in X$ and f is strictly convex, then we must have $\infty = f(x) < \frac{f(0)+f(2x)}{2}$, a contradiction. \square

Proposition 2.4.4. *Let X be a vector space and $T : X \rightarrow (-\infty, \infty]$. Then the following are equivalent.*

1. T is (strictly) convex.
2. For each $x_1 \neq x_2 \in X$ and $\lambda \in (0, 1)$.

$$T(\lambda x_1 + (1 - \lambda)x_2)(<) \leq \lambda T x_1 + (1 - \lambda)T x_2 \quad (2.174)$$

3. For each $\{\lambda_i\}_{i=1}^n \subset (0, 1)$ which sums to 1, for each $\{x_i\}_{i=1}^n \subset X$,

$$T\left(\sum_{j=1}^n \lambda_j x_j\right)(<) \leq \sum_{j=1}^n \lambda_j T x_j \quad (2.175)$$

4. If $x_1 \neq x_3 \in X$ and $x_2 \in (x_1, x_3)$, say $x_2 = \lambda x_1 + (1 - \lambda)x_3$ then

$$\frac{T x_2 - T x_1}{\lambda}(<) \leq T x_3 - T x_1(<) \leq \frac{T x_3 - T x_2}{1 - \lambda} \quad (2.176)$$

5. $\text{Epi}(T)$ is (strictly) convex

Proof. (1 \implies 2)

\square

Proof. (2 \implies 3) I utilize induction on n . Since f is assumed to be (strictly) convex, the proposition holds for $n = 1$ and $n = 2$. Let $k \in \mathbb{N}$. Suppose that for any $(\psi_1, \dots, \psi_k) \in [0, 1]^k$ such that $\sum_{j=1}^k \psi_j = 1$, and for each $(x_1, \dots, x_k) \in X^k$, we have

$$f\left(\sum_{j=1}^k \lambda_j x_j\right)(<) \leq \sum_{j=1}^k \lambda_j f(x_j) \quad (2.177)$$

Let $(\lambda_1, \dots, \lambda_{k+1}) \in [0, 1]^{k+1}$. Let $(x_1, \dots, x_{k+1}) \in X^{k+1}$. Without loss of generality, we assume $\lambda_{k+1} \neq 0$. Then

$$\sum_{j=1}^k \frac{\lambda_j}{1 - \lambda_{k+1}} = 1 \quad \left(\frac{\lambda_1}{1 - \lambda_{k+1}}, \dots, \frac{\lambda_k}{1 - \lambda_{k+1}}\right) \in [0, 1]^k \quad (2.178)$$

Hence, because f is (strictly) convex,

$$\begin{aligned}
 f\left(\sum_{j=1}^{k+1} \lambda_j x_j\right) &= f\left(\lambda_{k+1} x_{k+1} + (1 - \lambda_{k+1}) \sum_{j=1}^k \frac{\lambda_j}{1 - \lambda_{k+1}} x_j\right) \\
 &\leq \lambda_{k+1} f(x_{k+1}) + (1 - \lambda_{k+1}) f\left(\sum_{j=1}^k \frac{\lambda_j}{1 - \lambda_{k+1}} x_j\right) \\
 &\leq \sum_{j=1}^{k+1} \lambda_j f(x_j)
 \end{aligned} \tag{2.179}$$

□

Proof. (3 \implies 4) □

Proof. (4 \implies 1). □

Proof. (2 \iff 5) □

The epigraph of a function provides us with a nice characterization of lower semi-continuity.

Proposition 2.4.5 (Convex Continuity). *Let X be a locally convex space and $f : X \rightarrow (-\infty, \infty]$. The following conditions are equivalent.*

1. f is LSC on X .
2. $\text{Epi}(f)$ is closed in $X \times \mathbb{R}$.

Proof. Define $F : X \times \mathbb{R} \rightarrow \tilde{\mathbb{R}}$ by $F(x, \alpha) = f(x) - \alpha$. Then f is (weakly) LSC on X if and only if F is (weakly) LSC on $X \times \mathbb{R}$. Suppose f is (weakly) LSC. Then F is (weakly) LSC, so $F^{-1}((-\infty, 0]) = \text{Epi}(f)$ is (weakly) closed, so we're done. Suppose $\text{Epi}(f)$ is (weakly) closed. Then $F^{-1}((-\infty, 0])$ is (weakly) closed. Further, for any $\beta \in \mathbb{R}$, $F^{-1}((-\infty, \beta]) = F^{-1}((-\infty, 0]) - (0, \beta)$, and so is also closed. Hence F is (weakly) LSC, and so f is too. □

In the case of a convex function, the above proposition allows us to equate weak and strong lower semicontinuity.

Corollary 2.4.6 (Weak To Strong Convex). *Let X be locally convex Hausdorff space and $f : X \rightarrow (-\infty, \infty]$ be convex. Then f is LSC if and only if it is weakly LSC.*

Proof. Since $\text{Epi}(f)$ is convex, it is closed if and only if it is weakly closed, allowing us to apply 2.4.5. □

Theorem 2.4.7 (Point Continuous). *Let X be a locally convex space and $f : X \rightarrow (-\infty, \infty]$ be convex and proper. Then f is bounded on some open set if and only if f is continuous on the interior of its domain.*

Proof. (implies) Without loss of generality, we assume that f is bounded from above by M on a (weakly) open set \mathcal{U} which is symmetric and contains 0. Further, we can also suppose $f(0) = 0$. For each $\epsilon \in (0, 1)$ and each $x \in \epsilon\mathcal{U}$, we have

$$f(x) = f\left(\epsilon\frac{x}{\epsilon} + (1-\epsilon)0\right) \leq \epsilon f\left(\frac{x}{\epsilon}\right) \leq \epsilon M \quad (2.180)$$

and since $0 = \frac{x}{1+\epsilon} + \left(1 - \frac{1}{1+\epsilon}\right)\left(\frac{-x}{\epsilon}\right)$,

$$0 = f(0) = f\left(\frac{x}{1+\epsilon} + \left(1 - \frac{1}{1+\epsilon}\right)\left(\frac{-x}{\epsilon}\right)\right) \leq \frac{f(x)}{1+\epsilon} + \frac{\epsilon f\left(\frac{-x}{\epsilon}\right)}{1+\epsilon} \quad (2.181)$$

so, since $\frac{-x}{\epsilon} \in \mathcal{U}$,

$$-\epsilon M \leq -\epsilon f\left(\frac{x}{\epsilon}\right) \leq f(x) \quad (2.182)$$

Hence, $|f(x)| \leq \epsilon M$ for $x \in \epsilon\mathcal{U}$, and f is (weakly) continuous at 0. Hence, it is sufficient to show that for any y in the (weak) interior of $D(f)$, there is a (weak) neighborhood of y on which f is bounded from above. To see this, let y in the (weak) interior of $D(f)$. Since scalar multiplication is continuous, there is a $\rho > 1$ such that $\rho y \in D(f)$. If $\mathcal{U}_y = y + \left(1 - \frac{1}{\rho}\right)\mathcal{U}$, then $x \in \mathcal{U}_y$ can be written, for some $z \in \mathcal{U}$, as

$$x = y + \left(1 - \frac{1}{\rho}\right)z = \frac{1}{\rho}(\rho y) + \left(1 - \frac{1}{\rho}\right)z \quad (2.183)$$

Since f is convex, $D(f)$ is convex, and so $x \in D(f)$, implying $\mathcal{U}_y \subset D(f)$. Since f is a convex function, we also have that

$$f(x) \leq \frac{1}{\rho}f(\rho y) + \left(1 - \frac{1}{\rho}\right)f(x) \leq \frac{1}{\rho}f(\rho y) + \left(1 - \frac{1}{\rho}\right)M \quad (2.184)$$

so that f is bounded on \mathcal{U}_y and is therefore continuous at y . □

Proof. (\Leftarrow) This is obvious. □

Corollary 2.4.8. *Let X be a locally convex, space and $f : X \rightarrow (-\infty, \infty]$ be LSC at some point in its effective domain.*

1. *if f is convex, then it is continuous on the interior of $D(f)$.*
2. *If f is strictly convex, then it is continuous on X .*

2.5 Differentiation And SubDifferentials

Definition 2.5.1 (Types Of Differentiability). *Let X be a topological vector space, Y a Hausdorff topological vector space, $C \subset X$, $f : C \rightarrow Y$, and $x_0 \in C$.*

1. If $y \in X$ such that x_0 is an accumulation point of $[x_0, x_0 + y] \cap C$, then we say that the **directional derivative in the direction of y** exists at x_0 and we write

$$\lim_{t \rightarrow 0^+} \frac{f(x_0 + ty) - f(x_0)}{t} = f'_+(x_0, y) \quad (2.185)$$

if the limit above actually exists.

2. If f is differentiable at x_0 in the direction of $-y$, then we denote $f'_-(x_0, y) := -f'_+(x_0, -y)$.
3. If there exists $f'(x_0) \in BL(X, Y)$ such that for every $y \in X$, $f'_+(x_0, y) = f'(x_0)y$ then we call $f'(x_0)$ the **Gateaux derivative** of f at x_0 and we call f **Gateaux differentiable**.
4. If f is Gateaux differentiable at x_0 and $\frac{f(x_0 + ty) - f(x_0)}{t} \rightarrow f'(x_0)y$ uniformly for y in some neighborhood of 0 in X , then we say that f is **Frechet differentiable** at x_0 and we call $f'(x_0)$ the **Frechet derivative** of f at x_0 .
5. If $h : X \rightarrow (-\infty, \infty]$ and there is an $x^* \in X^*$ such that

$$\langle y - x, x^* \rangle \leq h(y) - h(x) \quad (\forall y \in X) \quad (2.186)$$

then we say that h is **subdifferentiable** at x_0 , call x^* a **subgradient** of h at x_0 , and denote the set of all subgradients of h at x_0 with $\partial h(x_0)$. if D is the set on which h is subdifferentiable, then we call $\partial h : D \rightarrow 2^{X^*}$ the **subdifferential** of h .

One useful feature of convex functions is how they interact with the various forms of differentiability.

Proposition 2.5.2 (Gateaux). *Let X be a topological vector space, Y a Hausdorff topological vector space, $f : X \rightarrow Y$ Gateaux differentiable at $x_0 \in X$, and $y \in X$. The following are true.*

1. $f'_+(x_0, y) = f'_+(x_0, -y) = f'_-(x_0, y)$.
2. $\langle y, f'(x_0) \rangle = \frac{d}{dt} f(x_0 + ty) |_{t=0}$
3. If f is Gateaux differentiable on a neighborhood of x_0 and $f' : X \rightarrow BL(X, Y)$ is Gateaux Differentiable at x_0 , then $f'' : X \rightarrow BL(X, BL(X, Y))$ satisfies

$$\langle y, f''(x)y \rangle = \frac{d^2}{dt^2} f(x_0 + ty) |_{t=0} \quad (2.187)$$

Proof. (1) □

Proof. (2) □

Proof. (3) □

Proposition 2.5.3 (Convex Differentiable). *Let X be a seminormed space and $f : X \rightarrow (-\infty, \infty]$ be convex. Then for each $y \in X$ and each x_0 in the interior of $D(f)$, the directional derivative in the direction of y at x_0 exists and $f'_-(x, y) \leq f'_+(x, y)$, and for any $t > 0$,*

$$f'_+(x, y) \leq \frac{f(x + ty) - f(x)}{t} \leq f'_-(x + ty, y) \quad (2.188)$$

2.6 Normalized Duality Mapping

Definition 2.6.1 (Normalized Duality Mapping). *If X is a seminormed space, then we call $J : X \rightarrow 2^{X^*}$ defined by*

$$Jx = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad (2.189)$$

the normalized duality mapping of X .

Proposition 2.6.2 (Normalized Duality Inequality). *Let X be a seminormed space and J be X 's normalized duality mapping. Then, if $x, y \in X$ and $\|x + \lambda y\| \neq 0$ where $\lambda > 0$, and $j_x \in Jx$, $j_{x+\lambda y} \in J(x + \lambda y)$, then*

$$\frac{\langle y, j_x \rangle}{\|x\|} \leq \frac{\|x + \lambda y\| - \|x\|}{\lambda} \leq \frac{\langle y, j_{x+\lambda y} \rangle}{\|x + \lambda y\|} \quad (2.190)$$

Proof.

$$\begin{aligned} \frac{\langle y, j_x \rangle}{\|x\|} &= \frac{\langle x + \lambda y, j_x \rangle - \|x\|^2}{\lambda \|x\|} \leq \frac{|\langle x + \lambda y, j_x \rangle| - \|x\|^2}{\lambda \|x\|} \leq \frac{\|j_x\| \|x + \lambda y\| - \|x\|^2}{\lambda \|x\|} \\ &= \frac{\|x + \lambda y\| - \|x\|}{\lambda} \\ &= \frac{\|x + \lambda y\|}{\|x + \lambda y\|} \frac{\|x + \lambda y\| - \|x\|}{\lambda} \leq \frac{\|x + \lambda y\|^2 - |\langle x, j_{x+\lambda y} \rangle|}{\lambda \|x + \lambda y\|} \\ &= \frac{\lambda \langle y, j_{x+\lambda y} \rangle + \langle x, j_{x+\lambda y} \rangle - |\langle x, j_{x+\lambda y} \rangle|}{\lambda \|x + \lambda y\|} \\ &\leq \frac{\lambda \langle y, j_{x+\lambda y} \rangle}{\lambda \|x + \lambda y\|} = \frac{\langle y, j_{x+\lambda y} \rangle}{\|x + \lambda y\|} \end{aligned} \quad (2.191)$$

□

Theorem 2.6.3 (Asplund). *[14] Let X be a seminormed space and J be its normalized duality mapping. Then for any $x \in X$, $Jx = \partial(\|x\|^2)$.*

Proof.

□

Proposition 2.6.4. *Let X be a (Possibly Complete, maybe not) seminormed space and let J be its normalized duality mapping. Then J is norm to weak* upper semicontinuous on X .*

Proof.

□

2.7 Orthogonality

Definition 2.7.1 (Orthogonality). *Let X be a seminormed space and $x, y \in X$. We say that x is **orthogonal** to y and we write $x \perp y$ if for each scalar λ , we have*

$$\|x\| \leq \|x + \lambda y\| \quad (2.192)$$

Proposition 2.7.2 (Orthogonality). *Let X be a seminormed space, $x, z \in X$, and $x^* \in X^*$.*

1. $\langle x, x^* \rangle = \|x\| \|x^*\|$ if and only if for each $y \in \ker(x^*)$, $x \perp y$.
2. x is orthogonal to each element of some hyperplane in X .
3. For some $\alpha \neq 0$ $x \perp (\alpha x + z)$.
4. The mapping $T : \mathbb{F} \rightarrow \mathbb{R}$ defined by $T\alpha = \|\alpha x + z\|$ achieves its minimum, and if λ_0 is a point at which it achieves this minimum, then $(\lambda_0 x + y) \perp x$ for any λ_0 which minimizes T . Furthermore, since T as defined earlier is a convex function, the set of λ for which $(\lambda x + y) \perp x$ is a convex set.

2.8 Convexity Of A Space

Definition 2.8.1 (Uniform Convexity, Weak Uniform Convexity, Local Uniform Convexity, Weak Local Uniform Convexity, Strict Convexity). *Let $(X, \|\cdot\|)$ be a seminormed space.*

1. We call the local modulus of uniform convexity of $\|\cdot\|$, denoted by the symbol $\tilde{\Delta}$, the **local modulus of uniform convexity** of X .
2. If Δ is the modulus of uniform convexity of $\|\cdot\|$, then we call Δ the **modulus of uniform convexity** of X .
3. We define $\tilde{\Delta}_w : (0, \infty) \times \partial B_X(0; 1) \rightarrow (0, \infty)$ by

$$\tilde{\Delta}_w(\epsilon, x) = \inf \{2 - \langle x + y, x^* \rangle : x^* \in \partial B_{X^*}(0; 1), y \in \partial B_X(0; 1), \|x - y\| \geq \epsilon\} \quad (2.193)$$

We call $\tilde{\Delta}_w$ the **modulus of weak local uniform convexity** of X .

4. We define $\Delta_w : (0, \infty) \rightarrow (0, \infty)$ by

$$\Delta_w(\epsilon) = \inf_{x \in \partial B_X(0; 1)} \tilde{\Delta}_w(\epsilon, x) \quad (2.194)$$

We call this the **modulus of weak uniform convexity** of X .

1. We say that X is **strictly convex** if for each $x, y \in X$ such that $\|x - y\| \neq 0$, $\|x + y\| < \|x\| + \|y\|$.
2. We say that X is **uniformly convex at x** if $\|\cdot\|$ is uniformly convex at x .
3. We say that X is **locally uniformly convex** if $\|\cdot\|$ is locally uniformly convex.
4. We say that X is **uniformly convex** if $\|\cdot\|$ is uniformly convex.
5. We say that X is **weakly uniformly convex at $x \in \partial B_X(0; 1)$** if for each $\epsilon > 0$, $\tilde{\Delta}_w(\epsilon, x) > 0$.

6. We say that X is **locally weakly uniformly convex** if it is weakly uniformly convex at each point on the boundary of X 's unit sphere.
7. We say that X is **weakly uniformly convex** if for each $\epsilon > 0$, $\Delta_w(\epsilon) > 0$.

Proposition 2.8.2 (Strictly Convex Spaces). *Let X be a normed space. and let J be a duality mapping on X of weight ϕ . Then the following are equivalent.*

1. X is strictly convex.
2. If $x, y \in X$ and $\|x + y\| = \|x\| + \|y\|$, then for some $\alpha \geq 0$, $\|x - \alpha y\| = 0$.
3. If $\|x\| = \|y\| = 1$ where $0 \neq \|x - y\|$, then $\|x + y\| < 2$.
4. If $x, y, z \in X$ and $\|x - y\| = \|x - z\| + \|z - y\|$, $\|z - z_0\| = 0$ for some $z_0 \in [x, y]$.
5. If $x^* \in X^*$ and $\|x\| = \|y\| = 1$ such that $\langle x, x^* \rangle = \langle y, y^* \rangle = \sup_{\|z\|=1} \langle z, x^* \rangle$, then $\|x - y\| = 0$.
6. $\|\cdot\|^2$ is strictly convex.
7. J is strictly monotone. That is, if $x, y \in X$, $\|x - y\| \neq 0$, $x^* \in Jx$, and $y^* \in Jy$, then

$$\langle x - y, x^* - y^* \rangle > 0 \quad (2.195)$$

8. Orthogonality in X is left-unique. That is, for $x, y \in X$, there is a unique $\alpha \in \mathbb{F}$ such that $(\alpha x + y) \perp x$.

Proposition 2.8.3 (Locally Uniformly Convex Spaces). *Let X be a Banach Space. Then the following are equivalent.*

1. X is locally uniformly convex
2. For each $\epsilon > 0$ and $x \in X$ with $\|x\| = 1$, there is a $\delta > 0$ such that if $y \in X$ satisfies $\|y\| = 1$ and $\|x - y\| \geq \epsilon$, then $\|x + y\| \leq 2(1 - \delta)$.
3. If $x \in \partial B_X(0; 1)$, $\{x_n\}_{n \in \mathbb{N}} \subset \partial B_X(0; 1)$, and $\|x + x_n\| \rightarrow 2$, then $x_n \rightarrow x$.
4. $\frac{1}{2} \|\cdot\|^2$ is locally uniformly convex.

Proposition 2.8.4 (Uniformly Convex Spaces). *Let X be a Banach space. The following are equivalent.*

1. X is uniformly Convex.
2. For each $\epsilon > 0$, there is a $\delta > 0$ such that if $x, y \in \overline{B_X(0; 1)}$ and $\|x - y\| \geq \epsilon$, then $\|x + y\| \leq 2(1 - \delta)$.
3. If $\{x_i\}_{i \in \mathbb{N}}, \{y_i\}_{i \in \mathbb{N}} \subset \overline{B_X(0; 1)}$ and $\|x_n + y_n\| \rightarrow 2$, then $x_n - y_n \rightarrow 0$.
4. $\frac{1}{2} \|\cdot\|^2$ is uniformly convex.

Proposition 2.8.5 (Weakly Locally Uniformly Convex Spaces). *Let X be a Banach space. The following conditions are equivalent.*

1. X is weakly locally uniformly convex.
2. For each $\epsilon > 0$, $x^* \in \partial B_{X^*}(0; 1)$, and $x \in \overline{B_X(0; 1)}$ there is a $\delta > 0$ such that if $y \in \overline{B_X(0; 1)}$ and $\langle x - y, x^* \rangle \geq \epsilon$, then $\|x + y\| \leq 2(1 - \delta)$.
3. If $x \in \overline{B_X(0; 1)}$ and $\{x_n\}_{n \in \mathbb{N}} \subset \overline{B_X(0; 1)}$ such that $\|x + x_n\| \rightarrow 2$, then $x_n \xrightarrow{w} x$.

Proposition 2.8.6 (Weakly Uniformly Convex Spaces). *Let X be a Banach space. The following conditions are equivalent.*

1. X is weakly uniformly convex.
2. For each $\epsilon > 0$ and $x^* \in \partial B_{X^*}(0; 1)$, there is a $\delta > 0$ such that if $x, y \in \overline{B_X(0; 1)}$ such that $\langle x - y, x^* \rangle \geq \epsilon$, then $\|x + y\| \leq 2(1 - \delta)$.
3. If $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \subset \overline{B_X(0; 1)}$ and $\|x_n + y_n\| \rightarrow 2$, then $x_n - y_n \xrightarrow{w} 0$.

Proposition 2.8.7 (Degrees Of Convexity). *Let X be a seminormed space.*

1. If X is uniformly convex, then X is weakly uniformly convex.
2. If X is uniformly convex, then X is locally uniformly convex.
3. If X is locally uniformly convex, then X is weakly locally uniformly convex.
4. If X is weakly locally uniformly convex, then X is strictly convex.

Proposition 2.8.8 (Local Weak To Strong). *If a seminormed space X is locally uniformly convex and $\{x_n\} \subset X$ satisfies $x_n \xrightarrow{w} x$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \rightarrow x$.*

Proof.

□

Theorem 2.8.9. (Milman Pettis) *A complete uniformly convex seminormed space X is reflexive.*

2.9 Smoothness Of A Space

Define Moduli Of Smoothness, remark on positiveness

Definition 2.9.1. *Let X be a seminormed space and $x_0 \in X$.*

1. We define the **modulus of smoothness** of X at x_0 , $\tilde{\rho} : [0, \infty) \times X \rightarrow \mathbb{R}$ by

$$\tilde{\rho}(\epsilon, x) = \frac{1}{2} \sup_{\|y\| < 1} (\|x + \epsilon y\| + \|x - \epsilon y\| - 2\|x\|) \quad (2.196)$$

2. We define the **modulus of smoothness** of X , $\rho : [0, \infty) \rightarrow \mathbb{R}$ by

$$\rho(\epsilon) = \sup_{\|x\| \geq 1} \rho(\epsilon, x) \quad (2.197)$$

3. We say that X is **locally uniformly smooth** at x_0 if $\lim_{\epsilon \rightarrow 0} \frac{\tilde{\rho}(\epsilon, x_0)}{\epsilon} = 0$.
4. We say that X is **locally uniformly smooth** if it is locally uniformly smooth at each of its points.
5. We say that X is **uniformly smooth** if $\lim_{\epsilon \rightarrow 0} \frac{\rho(\epsilon)}{\epsilon} = 0$.
6. We say that X is **smooth** at x_0 if Jx_0 is a singleton.
7. We say that X is **smooth** if it is smooth at each of its points.
8. 2.9.2 motivates defining X to be called **very smooth** at x_0 if it is smooth and J is seminorm to weak continuous at x_0 .
9. We say that X is **very smooth** if it is very smooth at each of its points.

Proposition 2.9.2 (Smooth Characterization). *Let X be a seminormed space, $x_0 \in X$, and J it's normalized duality mapping. The following are equivalent.*

1. X is a smooth at x_0 .
2. Every selection of J is norm to weak* continuous at x_0 .
3. There exists a selection of J which is norm to weak* continuous at x_0 .
4. $\|\cdot\|$ is Gateaux Differentiable at x_0 .
5. For every $y \in X$, there is a unique $\alpha \in \mathbb{C}$ such that $x \perp (\alpha x + y)$.
6. For every $y, z \in X$, if $x \perp y$ and $x \perp z$, then $x \perp y + z$.
7. (NEED TO DEFINE HYPERPLANE) There is a supporting hyperplane for $\overline{B_X(0; \|x\|)}$ at x .

Proposition 2.9.3 (Degrees Of Smoothness). *Let X be a seminormed space.*

1. If X is uniformly smooth, then it is locally uniformly smooth.
2. If X is locally uniformly smooth, then it is very smooth.
3. If X is very smooth, then it is smooth.

Proposition 2.9.4 (Very Smooth).

Proposition 2.9.5 (Local Uniformly Smooth At A Point). *Let X be a seminormed space, $x_0 \in X$, and J be X' 's normalized duality mapping. The following conditions are equivalent.*

1. X is locally uniformly smooth at x_0 .
2. J is continuous at x_0 .
3. $\|\cdot\|$ is Frechet differentiable at x_0 .

Corollary 2.9.6 (Local Uniformly Smooth). *Let X be a seminormed space and J be X 's normalized duality mapping. The following conditions are equivalent.*

1. X is locally uniformly smooth.
2. J is continuous.
3. $\|\cdot\|$ is Frechet differentiable.

Proposition 2.9.7 (Uniformly Smooth). *Let X be a seminormed space and J be X 's normalized duality mapping. The following are equivalent.*

1. X is uniformly smooth.
2. J is uniformly continuous on bounded subsets of X .
3. $\|\cdot\|$ is uniformly Frechet differentiable on bounded subsets of X .

Chapter 3

Smoothness And Convexity

3.1 Convexity And Smoothness Of A Space

Proposition 3.1.1 (Smoothness and Strict Convexity). *Let X be a seminormed space and J be X 's normalized duality mapping.*

1. *If X^* is smooth, then X is strictly convex.*
2. *If X^* is strictly convex, then X is smooth.*
3. *If X is reflexive, then X is strictly convex if and only if X^* is smooth.*
4. *If X is reflexive, then X is smooth if and only if X^* is strictly convex.*
5. *If X^* is strictly convex, then J is single valued and norm to weak $*$ continuous.*
6. *X is smooth and strictly convex if and only if J is single valued and strictly monotone.*

Proposition 3.1.2 (Lindenstrauss Duality Formula).

Proposition 3.1.3 (Very Smooth and Weak Local convexity).

Proposition 3.1.4 (Local Uniform Smoothness and Local Uniform Convexity). *Let X be a seminormed space and J be X 's normalized duality mapping.*

Proposition 3.1.5 (Uniform Smoothness and Convexity). *Let X be a (possibly complete) seminormed space and J be X 's normalized duality mapping.*

1. *X is uniformly convex if and only if X^* is uniformly smooth.*
2. *X is uniformly smooth if and only if X^* is uniformly convex.*
3. *X^* is uniformly convex if and only if J is single valued and uniformly continuous on bounded subsets of X .*

Corollary 3.1.6. *Uniformly Smooth Banach Spaces are Reflexive*

Proposition 3.1.7 (Normalized Convergence). *Let X be a smooth locally uniformly convex seminormed space with normalized duality mapping J . If $\{x_n\}_{n \in \mathbb{N}} \subset X$, $x \in X$, $j_x \in Jx$, and $j_n \in Jx_n$ for each $n \in \mathbb{N}$, then*

$$\langle x_n - x, j_n - j \rangle \rightarrow 0 \implies x_n \rightarrow x \quad (3.1)$$

3.2 Convexity, Smoothness, and High Order Duals

If you start with a poorly behaved Space, then things can only get worse.

Theorem 3.2.1. *If X is a seminormed space and X^* is very smooth, then X is reflexive.*

Corollary 3.2.2. *Let X be a seminormed space.*

1. *If X^* 's norm is Frechet differentiable, then X is reflexive.*
2. *If X^{**} is weakly locally uniformly convex, then X is reflexive.*
3. *If X^{***} is smooth, then X is reflexive.*
4. *If X^{****} is strictly convex, then X is reflexive.*

Chapter 4

Renorming Theory (Including Results about WCG Spaces)

4.1 Representations Of Reflexive Spaces

(Lindenstrauss' Theorem Goes Here)

4.2 Local Uniform Convexifiability Of Reflexive Spaces

-Trojanski's Theorem Goes Here

Chapter 5

Convexity And Fixed Point Theory

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