# Geometry of TVS's

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# Prerequisite Knowledge

## 2.1 Weak Topologies And Dual Spaces Of Seminormed Spaces

### 2.1.1 Quotient Maps

**Definition 2.1.1** (Topological Space)**.** $(Z, \mathcal{T}_Z)$

**Definition 2.1.2** (Neighborhood Filter)**.** *Let* $(Z, \mathcal{T}_Z)$ *be a* **Topological Space** . Let $x \in Z$. We denote with $\mathcal{U}_{\mathcal{T}_Z}(x)$ the collection of all elements of $\mathcal{T}_Z$ containing x. We call $\mathcal{U}_{\mathcal{T}_Z}(x)$ a **Neighborhood Filter** of $\mathcal{T}_Z$ at x.

**Definition 2.1.3** (relation of equal neighborhood filters)**.** *Let* $(Z, \mathcal{T}_Z)$ *be a topological space. Define the relation* $\cong$ *on* $Z$ *by setting, for* $x, y \in Z$*,*

$$x \cong y \iff \mathcal{U}_{\mathcal{T}_Z}(x) = \mathcal{U}_{\mathcal{T}_Z}(y) \tag{2.1}$$

*We call* $\cong$ *the* **Relation Of Equal Neighborhood Filters** on $(Z, \mathcal{T}_Z)$

**Proposition 2.1.4** (Equal Neighborhood Filters Equivalence Relation)**.** *The relation of equal neighborhood filters* $\cong$ *on a topological space* $(Z, \mathcal{T}_Z)$ *forms an equivalence relation on* $Z$*.*

*Proof.* Falls directly from the fact that set equality is an equivalence relation. □

**Definition 2.1.5** (Equivalence Class)**.** *Let* $X \neq \emptyset$*. Let* $\cong$ *be an equivelance relation defined on* $X$*. Let* $x \in X$*. We define the set* $[x]_{\cong}$ *by*

$$[x]_{\cong} = \{y \in X | y \cong x\} \tag{2.2}$$

*We call* $[x]_{\cong}$ *the* **Equivalence Class** of x in $(X, \cong)$.

**Proposition 2.1.6** (Equivalence Classes Partition). *Let $X \neq \emptyset$. Let $\cong$ be an equivalence relation defined on X. Let $x, y \in X$. The following are true.*

$$[x]_{\cong} \cap [y]_{\cong} \neq \emptyset \iff [x]_{\cong} = [y]_{\cong} \iff x \cong y \iff [x]_{\cong} \subset [y]_{\cong} \iff [y]_{\cong} \subset [x]_{\cong} \tag{2.3}$$

$$x \in [x]_{\cong} \tag{2.4}$$

*Proof.* OBVIOUS □

**Definition 2.1.7** (Quotient Set). *Let $X \neq \emptyset$. Let $\cong$ be an equivalence relation defined on X. We define the set $X/\cong$ by*

$$X/\cong = \{[x] : x \in X\} \tag{2.5}$$

*We call $X/\cong$ the* **Quotient Set** of X under the relation $\cong$.

**Remark 2.1.8** (Quotient Set Partition). *By 2.1.6, the quotient set forms a partition of X.*

**Definition 2.1.9** (Quotient Map). *Let $X \neq \emptyset$. Let $\cong$ be an equivalence relation on X. Let $X/\cong$ be the* **Quotient Set** of X with respect to the relation $\cong$. Define $T : X \to X/\cong$ by setting, for each $x \in X$,

$$T(x) = [x] \tag{2.6}$$

We call T the **Quotient Map** of X under the relation $\cong$.

**Proposition 2.1.10** (Quotient Map Surjective). *Let $X \neq \emptyset$. Let $\cong$ be an equivalence relation on X. Let $T : X \to X/\cong$ be the* **Quotient Map** of X under the relation $\cong$. Then T is a surjection.

*Proof.* Let $K \in Z/\cong$. Then for some $x \in Z$, $K = [x]$. Then $T(x) = K$. Since K was arbitrary, we are done. □

**Definition 2.1.11** (Quotient Space Topology). *Let $(Z, \mathcal{T}_Z)$ be a topological space. Let $\cong$ be the* **Relation Of Equal Neighborhood Filters** on $(Z, \mathcal{T}_Z)$ . Let T be the **Quotient Map** of Z under the relation $\cong$. Define $\mathcal{T}_{Z/\cong}$ by

$$\mathcal{T}_{Z/\cong} = \left\{ \bigcup_{x \in U} \{T(x)\} \in 2^{Z/\cong} | U \in \mathcal{T}_Z \right\} \tag{2.7}$$

By 2.1.12, $\mathcal{T}_{Z/\cong}$ is a topology on $Z/\cong$. We call $\mathcal{T}_{Z/\cong}$ the **Quotient Topology** and we call $\left(Z/\cong, \mathcal{T}_{Z/\cong}\right)$ the **Quotient Topological Space** of $(Z, \mathcal{T}_Z)$.

**Proposition 2.1.12** (Quotient Space Topology). *Let $(Z, \mathcal{T}_Z)$ be a topological space with* **Quotient Topological Space** $\left(Z/\cong, \mathcal{T}_{Z/\cong}\right)$ and **Quotient Map** T.

Then the following are true.

1. $\mathcal{T}_{Z/\cong}$ is a topology on $Z/\cong$.

2. $T : (Z, \mathcal{T}_Z) \to (Z/\cong, \mathcal{T}_{Z/\cong})$ is continuous.
3. If U is open (closed) in $(Z, \mathcal{T}_Z)$ then $T(U)$ and $T(Z \setminus U)$ partition $Z/\cong$.
4. If U is open in $(Z, \mathcal{T}_Z)$, then $T^{-1}(T(U)) = U$.
5. If K is closed in $(Z, \mathcal{T}_Z)$, then $T^{-1}T(K) = K$.
6. $T : (Z, \mathcal{T}_Z) \to (Z/\cong, \mathcal{T}_{Z/\cong})$ is an open mapping.
7. $T : (Z, \mathcal{T}_Z) \to (Z/\cong, \mathcal{T}_{Z/\cong})$ is a closed mapping.
8. $(Z, \mathcal{T}_Z)$ is a compact space if and only if $(Z/\cong, \mathcal{T}_{Z/\cong})$ is a compact space.
9. If $\mathcal{B}$ is a basis for $\mathcal{T}_z$, then $\{T(U) | U \in \mathcal{B}\}$ is a basis for $\mathcal{T}_{Z/\cong}$.

*Proof of 1.* Since $\emptyset \in \mathcal{T}_Z$, we have

$$\emptyset = \bigcup_{x \in \emptyset} \{Tx\} \in \mathcal{T}_{Z/\cong} \tag{2.8}$$

Since $Z \in \mathcal{T}_Z$, and by 2.1.8,

$$Z/\cong = \bigcup_{x \in Z} \{[x]\} = \bigcup_{x \in Z} \{T(x)\} \in \mathcal{T}_{Z/\cong} \tag{2.9}$$

Let $\{U_\alpha | \alpha \in A\} \subset \mathcal{T}_{Z/\cong}$. For each $\alpha \in A$, there exists $B_\alpha \in \mathcal{T}_Z$ such that we have

$$U_\alpha = \bigcup_{x \in B_\alpha} \{Tx\} \tag{2.10}$$

Since $\bigcup_{\alpha \in A} B_\alpha \in \mathcal{T}_Z$, we have

$$\bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} \bigcup_{x \in U_\alpha} \{T(x)\} = \bigcup_{x \in \bigcup\limits_{\alpha \in A} B_\alpha} \{T(x)\} \in \mathcal{T}_{Z/\cong} \tag{2.11}$$

Let $\{U_i\}_{i=1}^n \subset \mathcal{T}_{Z/\cong}$. For each $i \in \{1, ..., n\}$, there exists $B_i \in \mathcal{T}_Z$ such that

$$U_i = \bigcup_{x \in B_i} \{T(x)\} \tag{2.12}$$

Suppose

$$[x_0] \in \bigcap_{i=1}^n \bigcup_{x \in B_i} \{T(x)\} \tag{2.13}$$

Then for each $i \in \{1, ..., n\}$, there is a $y_i \in B_i$ such that $y_i \cong x_0$. Since each $B_i$ is open, the definition of $\cong$ implies that $x_0 \in B_i$ for every i. Hence,

$$x_0 \in \bigcap_{i=1}^n B_i \tag{2.14}$$

Implying

$$[x_0] \in \bigcup_{x \in \bigcap_{i=1}^{n} B_i} \{[x]\} \tag{2.15}$$

Hence,

$$\bigcap_{i=1}^{n} \bigcup_{x \in B_i} \{T(x)\} \subset \bigcup_{x \in \bigcap_{i=1}^{n} B_i} \{[x]\} \tag{2.16}$$

Furthermore, since the reverse inclusion is obvious, and since $\bigcap_{i=1}^{n} B_i \in \mathcal{T}_Z$, we have

$$\bigcap_{i=1}^{n} U_i = \bigcap_{i=1}^{n} \bigcup_{x \in B_i} \{T(x)\} = \bigcup_{x \in \bigcap_{i=1}^{n} B_i} \{T(x)\} \in \mathcal{T}_{Z/\cong} \tag{2.17}$$

□

*Proof of 2.* Let $V \in \mathcal{T}_{Z/\cong}$. Let $x_0 \in T^{-1}V$. Then $[x_0] \in V$. By definition, there is a $U \in \mathcal{T}_Z$ such that

$$T(U) \subset \bigcup_{x \in U} \{T(x)\} = V \tag{2.18}$$

Hence there is a $y_0 \in U$ such that

$$[x_0] \in T(y_0) = \{[y_0]\} \tag{2.19}$$

Therefore, $x \cong y$. Definition of the relation of equal neighborhood filters implies $\mathcal{U}(x_0) = \mathcal{U}(y_0)$. Hence, $x_0 \in U \subset T^{-1}(V)$. □

*Proof of 3.* Let $K$ be closed in $(Z, \mathcal{T}_Z)$. Then each point $x_0$ in $Z \setminus K$ has some $U_{x_0} \in \mathcal{U}_{\mathcal{T}_Z}(x_0)$ which is disjoint from K. Hence $y_0 \ncong x_0$ for any $y_0 \in K$, $x_0 \in Z \setminus K$. Hence $T(K)$ is disjoint from $T(Z \setminus K)$. This fact, paired with 2.1.1, implies $T(Z \setminus K)$ and T(K) partition $Z/\cong$. □

*Proof of 4.* Let $U \in \mathcal{T}_Z$. The nontrivial direction to prove is $T^{-1}(T(U)) \subset U$. Let $y \in T^{-1}(T(U))$. Then $[y] = Ty \in T(U)$. Hence, $[y] = T(x) = [x]$ for some $x \in U$. Since $y \cong x$ and $x \in U \in \mathcal{U}_{\mathcal{T}_Z}(x)$, we have $U \in \mathcal{U}_{\mathcal{T}_Z}(y)$. Hence $y \in U$. Since y was arbitrary, $T^{-1}(T(U)) \subset U$, and equality is obvious because the other direction of inclusion is trivial. □

*Proof of 5.* Let K be closed in $(Z, \mathcal{T}_Z)$. Part 3 Of this result implies $Z/\cong$ is partitioned by $T(K)$ and $T(Z \setminus K)$.

By part 4 of this proposition,

$$\begin{aligned} T^{-1}(T(K)) &= T^{-1}(T(Z) \setminus T(Z \setminus K)) \\ &= T^{-1}(Z/\cong \setminus T(Z \setminus K)) \\ &= T^{-1}(Z/\cong) \setminus T^{-1}(T(Z \setminus K)) \\ &= Z \setminus (Z \setminus K) \\ &= K \end{aligned}$$

□

*Proof of 6.* Let $U \in \mathcal{T}_Z$. Then by definition of the **Quotient Topology**

$$TU = \bigcup_{x \in U} \{T(x)\} \in \mathcal{T}_{Z/\cong} \tag{2.20}$$

□

*Proof of 7.* Let K be closed in $(Z, \mathcal{T}_Z)$. Then $Z \setminus K \in \mathcal{T}_Z$. By Parts 3 and five of this proposition, we know $T(K) = Z/\cong \setminus T(Z \setminus K)$ and also that $T(Z \setminus K) \in \mathcal{T}_{Z/\cong}$. Hence $T(K)$ is closed in $(Z/\cong, \mathcal{T}_{Z/\cong})$. □

*Proof of 8.* Let $(Z, \mathcal{T}_Z)$ be compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of $(Z/\cong, \mathcal{T}_{Z/\cong})$. Then $\{T^{-1}(U_\alpha) \,|\alpha \in A\}$ is an open covering of $(Z, \mathcal{T}_Z)$. Compactness of $(Z, \mathcal{T}_Z)$ guarantees the existence of a finite subcovering $\{T^{-1}(U_{\alpha_i}) \,|i \in \{1, ..., n\}\}$. Hence $\{U_{\alpha_i} | i \in \{1, ..., n\}\} = \{TT^{-1}(U_{\alpha_i}) | i \in \{1, ..., n\}\}$ is an open covering of $(Z/\cong, \mathcal{T}_{Z/\cong})$. And the compactness of $(Z/\cong, \mathcal{T}_{Z/\cong})$ is verified.

Now, suppose $(Z/\cong, \mathcal{T}_{Z/\cong})$ is compact. Let $\{V_\beta | \beta \in B\}$ be an open covering of $(Z, \mathcal{T}_Z)$. Since T is an open mapping, $\{T(V_\beta) | \beta \in B\}$ is an open covering of $(Z/\cong, \mathcal{T}_{Z/\cong})$ which by compactness has a finite subcover $\{T(V_{\beta_i}) | i \in \{1, ..., n\}\}$. By part 4 of 2.1.12, $\{V_{\beta_i} | i \in \{1, ..., n\}\} = \{T^{-1}(T(V_{\beta_i})) | i \in \{1, ..., n\}\}$ is then an open subcovering of $(Z, \mathcal{T}_Z)$. □

*Proof of 9.* Let $\mathcal{B}$ be a basis for $\mathcal{T}_z$ and let $V \in \mathcal{T}_{Z/\cong}$. Then $T^{-1}(Z) \in \mathcal{T}_Z$, and so there is a subcollection $\{U_\alpha\}_{\alpha \in A} \subset \mathcal{B}$ such that $T^{-1}(V) = \bigcup_{\alpha \in A} U_\alpha$. Hence,

$$\begin{aligned} V &= T(T^{-1}(V)) \\ &= T\left(\bigcup_{\alpha \in A} U_\alpha\right) \\ &= \bigcup_{\alpha \in A} T(U_\alpha) \end{aligned}$$

□

### 2.1.2 Pseudometrics

**Definition 2.1.13** (Triangle Inequality)**.** *Let $X$ and $Y$ be sets. We say that a map $f : X \times X \to Y$ is a* **Symmetric Map** for each $x_0, x_1 \in X$, $f(x_0, x_1) = f(x_1, x_0)$.

**Definition 2.1.14** (Symmetric Map)**.** *Let $X$ be a set and $(Y, +, \leq)$ be an ordered group. We say that a map $f : X \times X \to Y$ satisfies the* **Triangle Inequality** if for each $x_0, x_1, x_3 \in X$, we have

$$f(x_0, x_2) \leq f(x_0, x_1) + f(x_1, x_2)$$

**Definition 2.1.15** (Pseudometric)**.** *Let $X \neq \emptyset$. Let $d : X \times X \to [0, \infty)$ be a* **Symmetric Map** that satisfies the **Triangle Inequality** and further satisfies, for each $x \in X$,

$$d(x, x) = 0 \tag{2.21}$$

Under these conditions we call d a **Pseudometric** on X and we call $(X, d)$ a **Pseudometric Space** .

**Definition 2.1.16** (Pseudometric Cauchy Sequence)**.** *Let* $(X, d)$ *be a* **Pseudometric Space** . We say that a sequence $\{x_i\}_{i\in\mathbb{N}}$ is a **Pseudometric Cauchy Sequence** if, for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$, sucht that for each pair $m, n \in \mathbb{N}$ such that $m > N$ and $n > N$, we have

$$d(x_m, x_n) < \epsilon \tag{2.22}$$

**Definition 2.1.17** (Pseudometric Convergence)**.** *Let* $(X, d)$ *be a pseudometric space and let* $\{x_i\}_{i\in\mathbb{N}}$ *be a sequence in* $(X, d)$*. Let* $x_0 \in X$*. We say that* $\{x_i\}_{i\in\mathbb{N}}$ *exhibits* **Pseudometric-Convergence** to $x_0$ in d, or we say that $\{x_i\}_{i\in\mathbb{N}}$ **Pseudometric-Converges** to $x_0$ in d or we say that $\{x_i\}_{i\in\mathbb{N}}$ is **Pseudometrically-Convergent** to $x_0 \in d$ if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for every $n > N$, we have

$$d(x_0, x_n) < \epsilon \tag{2.23}$$

**Definition 2.1.18** (Pseudometric Complete)**.** *We say that a* **Pseudometric Space** $(X, d)$ is **Pseudometric-Complete** if each **Pseudometric Cauchy Sequence** sequence in $(X, d)$ **Pseudometric-Converges** to a limit in $X$.

**Definition 2.1.19** (Pseudometric Ball)**.** *Let* $(X, d)$ *be a* **Pseudometric Space** . For each $x_0 \in X$ and each $\epsilon > 0$, we define the following.

1. $B_d(x_0, \epsilon) := \{y \in X | d(x_0, y) < \epsilon\}$ denotes the **Open Ball** about $x_0$ with radius $\epsilon$.

2. $\overline{B_d}(x_0, \epsilon) := \{y \in X | d(x_0, y) \leq \epsilon\}$ denotes the **Closed Ball** about $x_0$ with radius $\epsilon$.

**Definition 2.1.20** (Pseudometric Topology)**.** *Let* $(X, d)$ *be a* **Pseudometric Space** , and let $\mathcal{B}$ be the set of **Open Ball** 's in $(X, d)$. By 2.1.2, $\mathcal{B}$ is the basis for a unique topology $\mathcal{T}_d$ on X. We call $\mathcal{T}_d$ the **Pseudometric Topology** induced by $d$ on X.

**Proposition 2.1.21** (Pseudometric Topology)**.** *Let* $(X, d)$ *by* **Pseudometric Space** and let $\mathcal{B}$ be the set of **Open Ball** 's in $(X, d)$. The following are true.

1. There exists a unique topology $\mathcal{T}_d$ on X which $\mathcal{B}$ is a basis of. That is, the **Pseudometric Topology** $\mathcal{T}_d$ is well defined.

2. The **Pseudometric Topology** is first countable. That is, each of its points permits a countable neighborhood basis.

*Proof of 1.* Uniqueness is guaranteed by closure under arbitrary unions of a topology. For existense, it is sufficient to show that the collection of arbitrary unions of elements of $\mathcal{B}$ is closed under finite intersections. Suppose that for $1 \leq i \leq n$, we have $\{U_{\alpha_i} | \alpha_i \in A_i\} \subset \mathcal{B}$ and consider the set

$$U = \bigcap_{i=1}^{n} \bigcup_{\alpha_i \in A_i} U_{\alpha_i} \tag{2.24}$$

Let $x_0 \in U$. For each $i \in \{1, ..., n\}$, there exists $\alpha_i \in A_i$ such that

$$x_0 \in U_{\alpha_i} = B_d(x_i; \epsilon_i) \tag{2.25}$$

For each $i \in \{1, ..., n\}$, define $\delta_i = d(x_0, x_i)$. Then $0 < \delta_i < \epsilon_i$. Then, for each $i \in \{1, ..., n\}$,

$$B_d(x_0; \epsilon_i - \delta_i) \subset U_{\alpha_i} \subset \bigcup_{\alpha_i \in A_i} U_{\alpha_i} \tag{2.26}$$

Define

$$\delta_{x_0} = \min_{i=1}^{n} (\epsilon_i - \delta_i) \tag{2.27}$$

Then $x_0 \in B(x_0; \delta_{x_0}) \subset U$. If $U = \{x_\alpha | \alpha \in A\}$, then the arbitrary nature of $x_0$ above means we can repeat this construction, writing

$$U \subset \bigcup_{\alpha \in A} B(x_\alpha; \delta_{x_\alpha}) \subset \bigcup_{\alpha \in A} U = U \tag{2.28}$$

Hence, $U \in B$ and the proof is complete. □

*Proof of 2.* Let $x_0 \in X$. I claim that

$$\mathcal{B}_{x_0} := \left\{ B_d \left( x_0; \frac{1}{n} \right) | n \in \mathbb{N} \right\} \tag{2.29}$$

is a neighborhood basis for $(X, \mathcal{T}_d)$ at $x_0$. Let $U \in \mathcal{U}_{\mathcal{T}_d}(x)$. Since $\mathcal{B}$ is a basis for $\mathcal{T}_d$, for some $y0 \in X$ and $\epsilon > 0$, $x_0 \in B_d(y_0; \epsilon) \subset U$. Let $\delta = d(x_0, y_0)$. Then $\epsilon - \delta > 0$. Define

$$n = \left\lceil \frac{1}{\epsilon - \delta} \right\rceil \tag{2.30}$$

Then we have

$$B_d \left( x_0; \frac{1}{n} \right) \subset B_d(x_0 : \epsilon - \delta) \subset B(y_0; \epsilon) \subset U \tag{2.31}$$

□

**Definition 2.1.22** (Relation Of Zero Distance)**.** *Let* $(X, d)$ *be a* **Pseudometric Space** . Define the relation $\cong_d$ on $X \times X$ by setting, for $x, y \in X$,

$$x \cong_d y \iff d(x, y) = 0 \tag{2.32}$$

We call $\cong_d$ the **Relation Of Zero Distance** on $(X, d)$.

**Proposition 2.1.23** (Relation Of Zero Distance is the Relation Of Equal Neighborhood Filters)**.** *Let* $(X, d)$ *be a* **Pseudometric Space** . Let $\cong_{\mathcal{T}_d}$ be the **Relation Of Equal Neighborhood Filters** on o n $(X, \mathcal{T}_d)$. Let $\cong_d$ be the **Relation Of Zero Distance** on $(X, d)$. Then $\cong_{\mathcal{T}_d} = \cong_d$.

*Proof.* Let $x, y \in X$ and suppose $x_0 \cong_d y_0$. Let $U \in \mathcal{U}_{\mathcal{T}_d}(x_0)$. Then for some $\epsilon > 0$, $x_0 \in B(x_0; \epsilon) \subset U$. Since $x_0 \cong_d y_0$, $d(x_0, y_0) = 0$, so $y_0 \in B(x_0; \epsilon) \subset U$. Hence $U \in \mathcal{U}_{\mathcal{T}_d}(y_0)$. The arbitrary nature of $U \in \mathcal{U}_{\mathcal{T}_d}(x_0)$ implies

$$\mathcal{U}_{\mathcal{T}_d}(x_0) \subset \mathcal{U}_{\mathcal{T}_d}(y_0) \tag{2.33}$$

A reverse construction would just as easily show the reverse inclusion, so we conclude that $x_0 \cong_{\mathcal{T}_d} y_0$. Now suppose $x_0 \cong_{\mathcal{T}_d} y$. Then or each $n \in \mathbb{N}$,

$$y_0 \in B_d\left(x_0; \frac{1}{n}\right) \tag{2.34}$$

Hence $d(x_0, y_0) < \frac{1}{n}$ for each natural n, therefore $d(x_0, y_0) = 0$ and $x_0 \cong_d y_0$. □

**Definition 2.1.24** (Metric Space Induced By Pseudometric)**.** *Let* $(X, d)$ *be a* **Pseudometric Space** , and let $\cong$ be the **Relation Of Zero Distance** , which by 2.1.23 is also the **Relation Of Equal Neighborhood Filters** on $(X, \mathcal{T}_d)$. Define $\tilde{d} : X/\cong \to [0, \infty)$ by

$$\tilde{d}([x], [y]) = d(x, y) \tag{2.35}$$

By 2.1.25, $\tilde{d}$ is well defined and is in fact a metric on $X/\cong$, so we call $\tilde{d}$ the **Metric Induced By The Pseudometric** d on X, or we call it the **Pseudometric Induced Metric** of $(X, d)$.

**Proposition 2.1.25** (Metric Space Induced By Pseudometric Space)**.** *Let* $(X, d)$ *be a* **Pseudometric Space** , $\cong$ the **Relation Of Zero Distance** on $(X, d)$ and $\tilde{d}$ be defined as in 2.1.24. Let $(X/\cong, \mathcal{T}_{X/\cong})$ be the **Quotient Topological Space** with **Quotient Map** T, and let $(X/\cong, \mathcal{T}_{\tilde{d}})$ be the topological space induced by the metric space $(X/\cong, \tilde{d})$. The following are true.

1. $\tilde{d}$ is in fact well defined, and is a metric on $X/\cong$, justifying calling it the **Metric Induced By The Pseudometric** d.
2. $\mathcal{T}_{X/\cong} = \mathcal{T}_{\tilde{d}}$
3. T is an isometric surjection from $(X, d)$ to $(X/\cong, \tilde{d})$
4. $(X/\cong, \tilde{d})$ is complete if and only if $(X, d)$ is **Pseudometric-Complete** .
5. T is an isometry from $(X, d)$ to $(X/\cong, \tilde{d})$.

*Proof of 01.* First we show that $\tilde{d}$ is well defined as a mapping, that is, that if $x_0, y_0 \in X$ and $x_1 \cong x_0$ and $y_1 \cong y_0$, then we should have

$$\tilde{d}([x_0], [y_0]) = \tilde{d}([x_1], [y_1]) \tag{2.36}$$

This is easy, as

$$\begin{aligned} d(x_0, y_0) &\leq d(x_0, x_1) + d(x_1, y_1) + d(y_1, y_0) \\ &= d(x_1, y_1) \\ &\leq d(x_1, x_0) + d(x_0, y_0) + d(y_0, y_1) \\ &= d(x_0, y_0) \end{aligned}$$

Nonnegativity falls directly from the nonnegativity of d. Proving that $\tilde{d}$ is a **Symmetric Map** is equally trivial

$$\tilde{d}\left([x],[y]\right)=d(x,y)=d(y,x)=\tilde{d}\left([y],[x]\right)$$

Proving that $\tilde{d}$ satisfies the **Triangle Inequality** is similarly simple, letting $x_0, y_0, z_0 \in X$, we have

$$\begin{aligned}\tilde{d}\left([x_0],[z_0]\right)&=d(x_0,z_0)\\&\leq d(x_0,y_0)+d(y_0,z_0)\\&=\tilde{d}\left([x_0],[y_0]\right)+\tilde{d}\left([y_0],[z_0]\right)\end{aligned}$$

All that remains is to show positivity on nonequal arguements. Let $x_0, y_0 \in X$ such that $[x_0] \neq [y_0]$. Then $x_0 \ncong y_0$. Hence

$$\tilde{d}\left([x_0],[y_0]\right)=d(x_0,y_0)\neq 0$$

□

*Proof of 02.* By 2.1.12, part 9, $\mathcal{B}_{\cong} := \{T(B_d(x;\epsilon))|x \in X, \epsilon > 0\}$ is a basis for $\mathcal{T}_{X/\cong}$. By definition, $\mathcal{B}_{\tilde{d}} := \{B_{\tilde{d}}([x];\epsilon)|x \in X, \epsilon > 0\}$ is a basis for $\mathcal{T}_{\tilde{x}}$.

To prove this result, I claim (and then justify) that for each $x \in X$ and $\epsilon > 0$,

$$T\left(B_d(x;\epsilon)\right)=B_{\tilde{d}}\left([x];\epsilon\right) \tag{2.37}$$

Suppose $\tilde{y} \in T\left(B_d(x;\epsilon)\right)$. Then $\tilde{y} = T(y)$ for some $y \in B_d(x;\epsilon)$. Hence

$$\begin{aligned}\tilde{d}(\tilde{y},[x])&=\tilde{d}(T(y),[x])\\&=\tilde{d}([y],[x])\\&=d(y,x)\\&<\epsilon\end{aligned}$$

Hence $\tilde{y} \in B_d([x];\epsilon)$, and so

$$T\left(B_d(x;\epsilon)\right)\subset B_{\tilde{d}}\left([x];\epsilon\right) \tag{2.38}$$

Suppose $[y] \in B_{\tilde{d}}([x];\epsilon)$. Then $d(x,y) = \tilde{d}()\,[x],[y]) < \epsilon$, so $y \in B_d(x;\epsilon)$. Hence $[y] = T(y) \in T\left(B_d(x;\epsilon)\right)$, so the reverse inclusion also holds, and we are done. □

*Proof of 03.* Falls directly from the definition $T(x) = [x]$, hence

$$d(x,y)=\tilde{d}\left([x],[y]\right)=\tilde{d}\left(T(x),T(y)\right) \tag{2.39}$$

□

*Proof of 04.* Direct consequence of part 3 of this result. Suppose □

**Definition 2.1.26** ((Pseudo)Metrizable)**.** *Let* $(X, \mathcal{T})$ *be a topological space.*

1. *We say that $(X, \mathcal{T})$ (Or $\mathcal{T}$ or $X$ which it wouldn't cause confusion) is* **Pseudometrizable** if there exists a pseudometric d on X such that $\mathcal{T}$ is the **Pseudometric Topology** on $(X, d)$.

2. We say that $(X, \mathcal{T})$ (Or $\mathcal{T}$ or X when it wouldn't cause confusion) is metrizable if there exists a metric d on X such that $\mathcal{T}$ is the metric topology on $(X, d)$.

**Proposition 2.1.27** (Pseudometrizable Prequotient)**.** *If a quotient of a topological space $X$ is pseudometrizable, then $X$ is pseudometrizable.*

*Proof.* If $\tilde{d}$ is a pseudometric on $d/\cong$ then it is easy to see that defining $d(x, y) = \tilde{d}([x], [y])$ results in a pseudometric on X which produces the topology on X. □

### 2.1.3 Seminormed Spaces

If $(X, ||\cdot||)$ is a seminormed space, that is a vector space on which $||\cdot||$ would be a norm if not for the fact that it allows some nonzero vector x to have $||x|| = 0$, then $||\cdot||$ induces a pseudometric on X. If $K = ||\cdot||^{-1}(\{0\})$, then if $x - y \in K$ and $z \in X$, then $||(z + x) - (z + y)|| = 0$ by the triangle inequality, so since K is a vector subspace, $X/||\cdot|| = \{x + K : x \in X\}$. Hence $X/K$ is clearly a normed space with norm $||[x]|| = ||x||$, and $X/K$ will preserve completeness and incompleteness of X.

**Proposition 2.1.28** (Seminorm Linear Operators)**.** *Let $(X, ||\cdot||_X)$ be a seminormed space and $(Y, ||\cdot||_Y)$ a normed space. For each continuous linear operator $T : X \to Y$, define*

$$||T|| = \sup_{||x||_X \neq 0} \frac{||Tx||_Y}{||x||_X} \tag{2.40}$$

*Then $||\cdot||$ is a norm on the space of continuous linear operators from X to Y, which we shall denote with $BL(X, Y)$. Further, if $Q : BL(X, Y) \to BL(X/||\cdot||_X^{-1}\{0\}, Y)$ is defined by $QT[x] = Tx$, then Q is a well defined linear bijective isometry.*

*Proof.* □

As a consequence of the above proposition, even if $X$ is just a seminormed space, $X^*$ is still a Banach space which is isomorphic to $(X/K)^*$, implying the possibility of extending several results known for normed spaces into the context of seminorms. Further, since $X/K$ can be embedded into X, several existence results, such as Helly's theorem, can also be generalized to the case of a seminormed space. In the context of a seminormed space, the canonical embedding $c : X \to X^{**}$ ceases to be injective but remains a linear isometry, and we shall continue to use the nomenclature that X is **reflexive** if $c$ is surjective. Since $X^{**} = (X/||\cdot||^{-1}\{0\})^{**}$, it is no surprise that $c(X) = c(X/||\cdot||^{-1}\{0\})$. Hence, a seminormed space is reflexive if and only if it's induced normed space is reflexive. The weak topology on a set X induced by set of mappings $\{\phi_\alpha : X \to Y_\alpha\}$ where each $Y_\alpha$ is a topological space is the coarsest topology on X which makes each $\phi_\alpha$ continuous. Similar to in the context of a normed space, if X is a seminormed space, we define the weak topology on X to be the topology on X generated by $X^*$, and the *weak*$^*$ topology on $X^*$ to be the topology generated by $c(X)$. Before moving on to the classical theory revamped, I present on more useful result about weak topologies of seminormed spaces.

**Proposition 2.1.29** (Weak Quotients)**.** *Let $X$ be a seminormed space and $\{Y_\alpha\}_{\alpha\in A}$ be a collection of topological spaces. For each $\alpha \in A$ let $\phi_\alpha : X \to Y_\alpha$ have the property that for every $x, y \in X$, for every $\alpha \in A$, $||x - y|| = 0 \implies \phi_\alpha(x) = \phi_\alpha(y)$. For each $\alpha \in A$, define $\tilde{\phi}_\alpha : X/\, ||\cdot||^{-1} \{0\} \to Y_\alpha$ by $\tilde{\phi}_\alpha[x] = \phi_\alpha x$. Let $\mathcal{T}_w$ denote the weak topology on $X$ induced by $\{\phi_\alpha\}_{\alpha\in A}$, and $\mathcal{T}_{\tilde{w}}$ denote the weak topology on $X/\, ||\cdot||^{-1} \{0\}$ induced by $\{\tilde{\phi}_\alpha\}_{\alpha\in A}$. Then*

$$(X, \mathcal{T}_w)/\, ||\cdot||^{-1} \{0\} = (X/\, ||\cdot||^{-1} \{0\}, \mathcal{T}_{\tilde{w}}) \tag{2.41}$$

*Proof.* □

Finally, before we move on, recall that if $X, Y$ are Topological vector spaces, we can topologize the set of continuous linear operators from X to Y, denoted $BL(X, Y)$ by saying that $\{T_\alpha\}_{\alpha\in A} \subset BL(X, Y)$ converges to $T \in BL(X, Y)$ if there is a neighborhood U of 0 in X such that $T_\alpha x \to Tx$ uniformly for $x \in U$.

## 2.2 Classical Results With A Twist

By 2.1.28 and 2.1.29, many of the classical theorems relating a normed space and its duals still hold in the context of a seminormed space without too much alteration of the proofs. Since the author has not seen these results presented in this context, they are presented with proof below.

### 2.2.1 Helly

In this subsection, we develop Helly's theorem in the context of a seminormed space, which will serve as valuable lemma throughout this document. Its location here is due to the fact that is is a generalization of a lemma commonly used to prove the Goldstine Theorem.

**Theorem 2.2.1** (Helly's Theorem)**.** *Let $(X, ||\cdot||)$ be a seminormed space $M > 0$, $\{\alpha_i\}_{i=1}^n \subset \mathbb{R}$, and $\{x_i^*\}_{i=1}^n \subset X^*$. Then the following are equivalent.*

1. *For every $\epsilon > 0$, there is an $x_\epsilon \in X$ such that $||x_\epsilon|| < M + \epsilon$ and $\langle x, x_i^*\rangle = \alpha_i$ for $1 \le i \le n$.*

2. *For every $\epsilon > 0$, there is an $x_\epsilon \in X$ such that $||x_\epsilon|| \le M$ and $|\langle x_\epsilon, x_i^*\rangle - \alpha_i| < \epsilon$ for $1 \le i \le n$.*

3. *For each $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$,*
$$\left|\sum_{i=1}^n \beta_i\alpha_i\right| \le M \left|\left|\sum_{i=1}^n \beta_i x_i^*\right|\right| \tag{2.42}$$

*Proof.* (1 $\implies$ 2) Obvious since $\{x_i^*\}_{i=1}^n \subset X^*$. □

*Proof.* (2 $\implies$ 3) Let $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$ and choose $\epsilon > 0$. Then

$$\begin{aligned} \left|\sum_{i=1}^n \beta_i \alpha_i\right| &= \left|\sum_{i=1}^n \beta_i \left(\alpha_i - \langle x_\epsilon, x_i^*\rangle + \langle x_\epsilon, x_i^*\rangle\right)\right| \\ &< \sum_{i=1}^n \epsilon\,|\beta_i| + \left|\left\langle x_\epsilon, \sum_{i=1}^n \beta_i x_i^*\right\rangle\right| \\ &\leq \epsilon \sum_{i=1}^n |\beta_i| + M\left|\left|\sum_{i=1}^n \beta_i x_i^*\right|\right| \end{aligned} \tag{2.43}$$

Since $\epsilon$ was arbitrary, the desired inequality holds. $\square$

*Proof.* (3 $\implies$ 1) If each $x_i^* = 0$, then by 2.42, each $\alpha_i = 0$, so $x = 0$ works for any $\epsilon$. Otherwise, we without loss of generality reorder $\{x_i^*\}_{i=1}^n$ so that $\{x_i^*\}_{i=1}^m$ is linearly independent and $span\{x_i\}_{i=1}^m = span\{x_i\}_{i=1}^n$. Define $S : X \to \mathbb{C}^m$ by $Sx = [\langle x, x_1^*\rangle, \cdots, \langle x, x_m^*\rangle]$, and recognize that linear independence of $\{x_i\}_{i=1}^n$ guarantees that S is surjective, so there exists $\tilde{x} \in S^{-1}\{[\alpha_1, \cdots, \alpha_m]\}$ and furthermore

$$S^{-1}\{[\alpha_1, \cdots, \alpha_m]\} = \tilde{x} + \bigcap_{i=1}^m kern(x_i^*) = \tilde{x} + \bigcap_{i=1}^n kern(x_i^*) := \tilde{x} + K \tag{2.44}$$

If $m < j \leq n$, then $x_j^* \in span(x_i^*, \cdots, x_n^*)$ so $x_j = \sum_{i=1}^n \gamma_i x_i^*$. Now we choose $\beta_i = \gamma_i$ for $1 \leq i \leq m$, $\beta_j = -1$, and other $\beta_k' s = 0$.

$$\left|\langle \tilde{x}, x_j^*\rangle - \alpha_j\right| = \left|\left\langle \tilde{x}, \sum_{i=1}^m \beta_i x_i^*\right\rangle - \alpha_j\right| = \left|\left(\sum_{i=1}^m \beta_i \langle \tilde{x}, x_i^*\rangle\right) + (-1)\alpha_j\right| \tag{2.45}$$

$$= \left|\sum_{i=1}^n \beta_i \alpha_i\right| \tag{2.46}$$

$$\leq M\left|\left|\sum_{i=1}^n \beta_i x_i^*\right|\right| \tag{2.47}$$

$$= M\left|\left|\sum_{i=1}^m \gamma_i x_i^* + (-1)x_j^*\right|\right| \tag{2.48}$$

Hence $\langle x, x_i^*\rangle = \alpha_i$ for $1 \leq i \leq n$ and any $x \in \tilde{x} + K$. By the Hahn Banach Theorem applied to the semi norm on X induced by the seminorm on the quotient space $X/K$, there is an $x^* \in X^*$ such that $\sup\limits_{||x|| \neq 0} \frac{|\langle x, x^*\rangle|}{||x||} = 1$, $\langle \tilde{x}, x^*\rangle = d(\tilde{x}, K)$, and $K \subset ker(x^*)$. Since $K \subset ker(x^*)$,

$x^* \in span\{x_i^*\}_{i=1}^n$, say $x^* = \sum_{i=1}^n \mu_i x_i^*$. Hence,

$$\begin{aligned} d(\tilde{x}, K) = \langle \tilde{x}, x^* \rangle &= \sum_{i=1}^n \mu_i \langle \tilde{x}, x_i^* \rangle \\ &= \sum_{i=1}^n \mu_i \alpha_i \\ &\leq M \left|\left| \sum_{i=1}^n \mu_i x_i^* \right|\right| = M \end{aligned} \tag{2.49}$$

Thus, given $\epsilon > 0$, we find an $z_\epsilon \in K$ such that $||\tilde{x} - z_\epsilon|| < M + \epsilon$ and let $x_\epsilon = \tilde{x} - z_\epsilon$. □

**Corollary 2.2.2.** *Let $X$ be a seminormed space, $x^{**} \in X^{**}$, $\{x_i^*\}_{i=1}^n \subset X^*$, and $\epsilon > 0$. Then there exists the following.*

1. *An $x_1 \in X$ such that $||x_1|| \leq ||x^{**}|| + \epsilon$ and for $1 \leq i \leq n$, $\langle x_1, x_i^* \rangle = \langle x_i^*, x^{**} \rangle$.*
2. *An $x_2 \in X$ such that $||x_2|| \leq ||x^{**}||$ and for $1 \leq i \leq n$, $|\langle x_1, x_i^* \rangle - \langle x_i^*, x^{**} \rangle| < \epsilon$.*

*Proof.* For $1 \leq i \leq n$, let $\alpha_i = \langle x_i^*, x^{**} \rangle$, and let $\{\beta_i\}_{i=1}^n \subset \mathbb{C}$. Then

$$\left| \sum_{i=1}^n \beta_i \alpha_i \right| = \left| \left\langle \sum_{i=1}^n \beta_i x_i^*, x^{**} \right\rangle \right| \leq ||x^{**}|| \left|\left| \sum_{i=1}^n \beta_i x_i^* \right|\right| \tag{2.50}$$

An application of 2.2.1 completes the proof. □

### 2.2.2 Goldstine

**Theorem 2.2.3** (Goldstine)**.** *Let $X$ be a seminormed space, $c : X \to X^{**}$ denote the canonical embedding, $B$ denote the closed unit ball of $X$, and $B_1$ denote the closed unit ball of $X^{**}$. Then $c(B)$ is weak$^*$ dense in $B_1$.*

*Proof.* Note that the *weak*$^*$ topology on $X^{**}$ has as a basis sets of the form

$$U\left(x^{**}, \epsilon, \{x_i^*\}_{i=1}^n\right) := \{y^{**} \in X^{**} : (\forall 1 \leq i \leq n)\left(|\langle x_i^*, x^{**} - y^{**} \rangle| < \epsilon\right) \tag{2.51}$$

where we range over all $x^{**} \in X^{**}$, $\epsilon > 0$, and finite subsets $\{x_i^*\}_{i=1}^n$ of $X^*$. Let $x^{**} \in B_1$, $\epsilon > 0$, and $\{x_i^*\}_{i=1}^n \subset X^*$. Let $\epsilon > 0$. Then by 2.2.2, there exists an $x \in B$ such that for $1 \leq i \leq n$, $|\langle x_i^*, x^{**} - c(x) \rangle| = |\langle x_i^*, x^{**} \rangle - \langle x, x_i^* \rangle| < \epsilon$, from which the desired result follows. □

### 2.2.3 Banach Alaoglu

The following well known result concerning the *weak*$^*$ compactness of the unit ball of a Banach space was first proven in the separable case by Banach, and then generalized in 1940 by Alaoglu [1] to Banach spaces. Generalizations of this result in a general TVS satisfying sufficient conditions have also been shown but the form presented here comes from [2], who drops the assumption of completeness for one direction of the implication.

**Theorem 2.2.4** (Banach-Alaoglu-Morales)**.** *Let $X$ be a normed space and define $B = \{x^* \in X^* : ||x^*|| \leq 1\}$. Then $B$ is weak$^*$ compact.*

*Proof.* Let $\mathbb{F}$ denote $X's$ field, and for $x \in X$, define $D_x = \{y \in \mathbb{F} : |y| \leq ||x||\}$. Then each $D_x$ is Hausdorff and compact so by Tychonoff's theorem, $D := \prod_{x \in X} D_x$ is compact and Hausdorff when endowed with the product topology. If $T \in D$, then $T : X \to \mathbb{F}$ and $|Tx| \leq ||x||$ for each $x \in X$, so $D \cap X^* \subset B$. It is also clear that $B \subset D$, so $D \cap X^* = B$. Let $\{\gamma_\alpha\}_{\alpha \in A}$ be a net in B converging to $\gamma \in D$ in D's product topology. Then, letting $\pi_x$ denote the $x^{th}$ projection, for each $x \in X$,

$$\gamma_\alpha(x) = \pi_x(\gamma_\alpha) \to \pi_x(\gamma) = \gamma(x) \tag{2.52}$$

If $\alpha \in \mathbb{F}$ and $x, y \in X$, then

$$\langle \alpha x + y, \gamma_\alpha \rangle \to \langle \alpha x + y, \gamma \rangle \tag{2.53}$$

and also

$$\langle \alpha x + y, \gamma_\alpha \rangle = \alpha \langle x, \gamma_\alpha \rangle + \langle y, \gamma_\alpha \rangle \to \alpha \langle x, \gamma \rangle + \langle y, \gamma \rangle \tag{2.54}$$

which implies $\gamma$ is linear since D is Hausdorff, and hence $\gamma \in B$. Thus $B$ is closed in D. What remains to be shown is that the *weak*$*$ topology on B is the subspace topology on B induced by $D's$ topology, since a Closed subset of a compact Hausdorff space is compact. For notation, denote with $\mathcal{T}_D$ the subspace topology on B induced by D's topology, and denote with $\mathcal{T}_w$ the subspace topology on B induced by the *weak*$^*$ topology on $X^*$. To see that $\mathcal{T}_w \subset \mathcal{T}_D$, let $\{\gamma_\alpha\}_{\alpha \in A} \subset B$ such that $\gamma_\alpha \overset{\mathcal{T}_D}{\to} \gamma$. For each $x \in X$, letting c be the canonical embedding,

$$\langle \gamma_\alpha, c(x) \rangle = \langle x, \gamma_\alpha \rangle = \pi_x(\gamma_\alpha) \to \pi_x(\gamma) = \langle x, \gamma \rangle = \langle \gamma, c(x) \rangle \tag{2.55}$$

Hence $\gamma_\alpha \overset{\mathcal{T}_w}{\to} \gamma$, so $\mathcal{T}_w \subset \mathcal{T}_D$. To see that $\mathcal{T}_D \subset \mathcal{T}_w$, fix $x \in X$ and let $\{\gamma_\alpha\}_{\alpha \in A} \subset B$ such that $\gamma_\alpha \overset{\mathcal{T}_w}{\to} \gamma$. Then $\pi_x(\gamma_\alpha) = \langle x, \gamma_\alpha \rangle \to \gamma(x) = \pi_x(\gamma)$, so by definition of the product topology $\gamma_\alpha \overset{\mathcal{T}_D}{\to} \gamma$, implying $\mathcal{T}_D \subset \mathcal{T}_w$. Hence B is *weak*$^*$ is compact. □

**Corollary 2.2.5** (Banach Alaoglu Seminorm)**.** *Let $X$ be a seminormed space and define $B = \{x^* \in X^* : ||x^*|| \leq 1\}$. Then $B$ is weak$^*$ compact.*

*Proof.* This is a consequence of the fact that the *weak*$^*$ topology on $X^*$ is identical to the *weak*$^*$ topology on the dual space of $X / ||\cdot||^{-1} \{0\}$. □

This gives us the useable result

**Corollary 2.2.6** (Banach-Alaoglu-Morales)**.** *Let $X$ be a seminormed space and $C \subset X^*$*

1. *If $X$ is complete and $C$ weak$^*$ compact, then $C$ is weak$^*$ closed and bounded.*

2. *If $C$ is weak$^*$ closed and bounded, then $C$ is weak$^*$ compact.*

*Proof.* (1) Since $C$ is *weak*$^*$ compact, it is *weak*$^*$ closed since the *weak*$^*$ topology is Hausdorff. Since $c(x) : (X, \mathcal{T}_{w^*}) \to \mathbb{C}$ is continuous for each $x \in X$, for each $x \in X$, $c(x)(C) =$ is compact and therefore bounded. Hence, for every $x \in X$, $\{|\langle x, c \rangle| : c \in C\}$ is bounded, so by the Banach Steinhaus, C is bounded. □

*Proof.* (2) Since $C$ is bounded, it is contained in some closed ball B which we know to be *weak*$^*$ compact by 2.2.4. Since the *weak*$^*$ topology on B is compact and Hausdorff and $C$ is closed in this topology, it is compact in this topology. Since the subspace topology on C induced by the *weak*$^*$ topology on $X^*$ equals this topology, we are done. □

### 2.2.4 Eberlein-Smulian

The purpose of this section is to provide a characterization of weakly compact subsets of a complete seminormed space X, which will serve to increase the applicability of the results regarding weakly compactly generated spaces covered later in this document. The first main result of this section, 2.2.9, serves to show that even though weak topologies of Banach Spaces are not in general metrizable, an equivalence between weak compactness and sequential compactness exists. From this result ($1 \implies 2$) was first presented in the case of normed spaces in [3], and then ($2 \implies 1$) was proven in the case of normed spaces in [4]. Several different proofs have been given in the years since, and the one present here is based on that present in [5], which is also followed in [6]. We begin with a few lemmas.

**Lemma 2.2.7** (Metrizable Weak)**.** *If X is a seminormed space and $X^*$ contains a countable set that separates points mod $K := ||\cdot||^{-1}\{0\}$, then subspace topology induced by the weak topology on any weakly compact subset A of X is pseudometrizable.*

*Proof.* As a consequence of 2.1.27 and 2.1.29, it is sufficient to let X be a normed space and $\{x_i^*\}_{i\in\mathbb{N}}$ separate points in X. Let $M = 2\sup_{x\in A}||x||$, and define d to be the metric on A defined by, for $x, y \in A$,

$$d(x,y) = \sum_{k\in\mathbb{N}} \frac{|\langle x-y, x_k^*\rangle|}{||x_k^*||\, 2^k} \tag{2.56}$$

Let $x \in A$, $\epsilon > 0$ be arbitrary, and define

$$n = \left\lceil 2 + log_2\left(\frac{M}{\epsilon}\right)\right\rceil \qquad U = A \cap \bigcap_{k=1}^{n} \left\{ y \in X : |\langle x-y, x_k^*\rangle| < \frac{||x_k^*||\, 2^{k-1}\epsilon}{n} \right\} \tag{2.57}$$

The U is open in the subspace topology on A induced by $X's$ weak topology. Furthermore, if $y \in U$, then

$$\begin{aligned} d(x,y) &= \sum_{k\in\mathbb{N}} \frac{|\langle x-y, x_k^*\rangle|}{||x_k^*||\, 2^k} \\ &\le \sum_{k=1}^{n} \frac{|\langle x-y, x_k^*\rangle|}{||x_k^*||\, 2^k} + \sum_{k=n+1}^{\infty} \frac{2M}{2^k} \\ &< \sum_{k=1}^{n} \frac{\epsilon}{2n} + \frac{M}{2^{n-1}} < \epsilon \end{aligned} \tag{2.58}$$

So that $U \subset B_d(x;\epsilon)$. This implies $Id : (A, \mathcal{T}_w) \to (A, \mathcal{T}_d)$ is continuous. Since a continuous injection from a compact space into a Hausdorff space is a homeomorphism, the subspace topology on A induced by the weak topology equals the topology on A induced by d, and so A's weak topology is metrizable. □

**Lemma 2.2.8.** *Let $X$ be a seminormed space and $Y \subset X^{**}$ be a finite dimensional vector subspace. Then there exists a finite set $Z \subset \partial B_{X^*}(0;1)$ such that for each $y^{**} \in Y$,*

$$||y^{**}|| \leq 2 \max_{z^* \in Z} |\langle z^*, y^{**} \rangle| \tag{2.59}$$

*Proof.* Let $S = \partial B_{X^{**}}(0;1) \cap Y$. Then, since Y is finite dimensional, S is compact, and therefore permits a $\frac{1}{4} - net$ $\{s_i\}_{i=1}^n$. Now let $\{z_k^*\}_{k=1}^n \subset \partial B_{X^*}(0;1)$ such that for each k, $\langle z_k^*, s_i \rangle > \frac{3}{4}$. Let $s \in S$ then there is a k such that $||s - s_k|| < \frac{1}{4}$. for this k, we have

$$\langle z_k^*, s \rangle = \langle z_k^*, s_k \rangle + \langle z_k^*, s - s_k \rangle \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2} \tag{2.60}$$

□

**Theorem 2.2.9** (Eberlein-Smulian)**.** *Let $X$ be a seminormed space and $A \subset X$. Then the following are equivalent.*

1. *A is weakly compact.*
2. *A is weakly sequentially compact.*

*Proof.* $(1 \implies 2)$ Let $A \subset X$ be weakly compact, and let $\{x_i\}_{i \in \mathbb{N}} \subset A$. Define $S = \overline{span\{x_i : i \in \mathbb{N}\}}$. Since S is closed and convex, it is weakly closed, and so $A \cap S$ is weakly compact as well. By construction, S is separable, and so contains a countable dense set $\{y_i\}_{i \in \mathbb{N}}$. By Hahn-Banach, for each $i \in \mathbb{N}$, there exists $y_i^* \in S^*$ such that $\langle y_i, y_i^* \rangle = 1$, and continuity of each $y_i^*$ implies $\{y_i^*\}_{i \in \mathbb{N}}$ separates points in S mod $||\cdot||^{-1}\{0\}$. Hence we can apply 2.2.7 to claim that the subspace topology on $A \cap S$ induced by $S's$ weak topology is metrizable, and therefore $\{x_i\}_{i \in \mathbb{N}}$ has a sub-sequence $\{x_{n_i}\}_{i \in \mathbb{N}}$ which is weakly S-convergent, and therefore weakly X-convergent since subspace topologies are no less fine than the topologies that induce them. Since $A \subset X$ is weakly closed, this sequence converges within A, and so A is weakly sequentially compact. □

*Proof.* $(2 \implies 1)$. Let $A \subset X$ be weakly sequentially compact, let c denote the canonical embedding of $X$ into $X^{**}$, and let $x^{**}$ in the $weak^*$ closure of $c(A)$. Let $x_1^1 \in X^*$ have norm 1. By assumption, there exists $a_1^{**} \in c(A)$ such that $|\langle x_1^*, x^{**} - a_1^{**} \rangle| < 1$. By 2.2.8, there exists $\{x_1^2, \cdots, x_{n_2}^2\} \subset \partial B_{X^*}(0;1)$ such that for each $y^{**} \in span\{x^{**}, x^{**} - a_1^{**}\}$,

$$||y^{**}|| \leq 2 \max_{1 \leq k \leq n_2} \left|\langle x_k^2, y^{**} \rangle\right| \tag{2.61}$$

Also, since $x^{**}$ is in the $weak^*$ closure of $c(A)$, there exists $a_2^{**} \in c(A) \cap U_2$ where

$$U_2 = \{y^{**} \in X^{**} : (\forall 1 \leq j \leq 2)(\forall 1 \leq k \leq n_j)(\left|\langle x_k^j, x^{**} - y^{**} \rangle\right| < \frac{1}{2})\} \tag{2.62}$$

Continuing inductively, , we construct a sequence $\{a_n^{**}\}_{n \in \mathbb{N}} \subset c(A)$ such that for each $j \in \mathbb{N}$, $\{x_k^j\}_{k=1}^{n_j} \subset \partial B_{X^*}(0;1)$ such that for every $y^{**} \in span\left\{x^{**}, x^{**} - a_1^{**}, \cdots, x^{**} - a_{j-1}^{**}\right\}$, we have

$$||y^{**}|| \leq 2 \max_{1 \leq k \leq n_j} \left|\langle x_k^j, x^{**} - y^{**} \rangle\right| \tag{2.63}$$

and $a_j^{**} \in c(A) \cap U_j$ where $U_j$ is the $\{x_k^m\}_{1 \le m \le j, 1 \le k \le n_m}$ $weak^*$ neighborhood about $x^{**}$ of radius $\frac{1}{j}$. For each $k \in \mathbb{N}$, let $a_k = c^{-1}(c(a_k))$. Since A is sequentially weakly compact, $\{a_k\}_{k \in \mathbb{N}}$ has a weak cluster point $x \in A$. Also, $x \in \overline{span\{a_i\}_{i \in \mathbb{N}}}$ because this is a weakly closed set, implying $c(x) \in \overline{span\{a_i^{**}\}_{i \in \mathbb{N}}}$, which then implies $c(x) \in \overline{span\{x^{**}, x^{**} - a_1^{**}, x^{**} - a_2^{**}, \cdots\}}$. By continuity of the norm and each element of $\{x_i^k\}_{k \in \mathbb{N}, 1 \le i \le n_k}$, we conclude that for each element $y^{**}$ of $\overline{\{x^{**}, x^{**} - a_1^{**}, x^{**} - a_2^{**}, \cdots\}}$,

$$||y^{**}|| \le 2 \sup_{k \in \mathbb{N}, 1 \le i \le n_k} \left| \left\langle x_i^k, y^{**} \right\rangle \right| \tag{2.64}$$

This is useful, because for each $k \in \mathbb{N}$, $1 \le i \le n_k$, we have, for large enough j,

$$\begin{aligned} \left| \left\langle x_i^k, x^{**} - c(x) \right\rangle \right| &\le \left| \left\langle x_i^k, x^{**} - a_j^{**} \right\rangle \right| + \left| \left\langle a_j^{**} - c(x), x_i^k \right\rangle \right| \\ &\le \frac{1}{j} + \left| \left\langle x_i^k, a_j - x \right\rangle \right| \end{aligned} \tag{2.65}$$

which can be made arbitrarily small, and so $\left| \left\langle x_i^k, x^{**} - c(x) \right\rangle \right| = 0$, implying that

$$||x^{**} - c(x)|| \le 2 \sup_{k \in \mathbb{N}, 1 \le i \le n_k} \left| \left\langle x_i^k, x^{**} - x \right\rangle \right| = 0 \tag{2.66}$$

So $x^{**} = c(x)$, and therefore $c(A)$ is $weak^*$ closed. Since A is weakly-sequentially compact, $c(A)$ is $weak^*$ sequentially compact and therefore bounded by Banach Steinhaus. By 2.2.4, bounded $weak^*$ closed sets are compact, and so $c(A)$ is $weak^*$ compact. Since the weak topology on $A / \, ||\cdot||^{-1} \{0\}$ is the same as the $weak^*$ topology on $c(A)$, $A / \, ||\cdot||^{-1} \{0\}$ is weakly compact. To see that A is weakly compact, apply **??**. □

### 2.2.5 Bishop-Phelps

In this subsection, I develop a result due to [7] which will prove useful throughout this document. I begin by presenting the concept of a convex cone and a trio of lemmas which are commonly utilized in the proof of this result.

**Definition 2.2.10** (Convex Cone)**.** Let $X$ be a seminormed space over $\mathbb{R}$. If $K \subset X$ is convex and closed under positive scalar multiples, then we call it a **convex cone**. If J is a convex cone in X, $C \subset X$, $x_0 \in C$, and $(J + x_0) \cap C = \{x_0\}$, then we say that J **supports** $C$ at $x_0$. If $x^* \in \partial B_{X^*}(0; 1)$ and $\alpha > 0$ then we define

$$K(x^*, \alpha) := \{x \in X : ||x|| \le \alpha \langle x, x^* \rangle\} \tag{2.67}$$

**Remark 2.2.11.** Let X be a seminormed space, $x^* \in \partial B_{X^*}(0; 1)$, and $\alpha > 0$. The following are true.

1. $K(x^*, \alpha)$ is a closed convex cone.

2. If $\alpha > 1$, $Int(K(x^*, \alpha)) \neq \emptyset$.

*Proof.* (1) If $\{x_n\} \subset K(x^*, \alpha)$ converges, say $x_n \to x$, then continuity of $x^*$ implies $\langle x_n, x^* \rangle \to \langle x, x^* \rangle$. Hence, given $\epsilon > 0$, there exists $N > 0$ such that for $n > N$ we have $max\left(||\,||x|| - ||x_n||\,||, |\langle x - x_n, x^*$ $\epsilon$, so that for all $n > N$,

$$||x|| \leq ||x_n|| + \epsilon < \alpha \langle x_n, x^* \rangle + \epsilon < \alpha \langle x, x^* \rangle + (\alpha + 1)\epsilon \tag{2.68}$$

So $x \in K(x^*, \alpha)$ closedness is verified. It is obvious that $K(x^*, \alpha)$ is closed under positive scalar multiples, and for convexity, if $x, y \in K(x^*, \alpha)$ and $t \in [0, 1]$, then

$$||tx + (1-t)y|| \leq t\,||x|| + (1-t)\,||y|| \leq t\alpha \langle x, x^* \rangle + (1-t)\alpha \langle y, x^* \rangle = \alpha \langle tx + (1-t)y, x^* \rangle \tag{2.69}$$

□

*Proof.* (2) By definition of the norm on $X^*$, there is an $x \in \overline{B_X(0;1)}$ such that $2/\left(\alpha\left(1 + \frac{1}{\alpha}\right)\right) < \langle x, x^* \rangle$, implying by linearity that $1/\alpha < \left\langle \frac{1+(1/\alpha)}{2} x, x^* \right\rangle$. By continuity of $x^*$ we find a neighborhood U of $\left(1 + \frac{1}{\alpha}\right)/2$ contained in $B_X(0;1)$ such that for each $y \in U$, $1/\alpha < \langle y, x^* \rangle$. This implies $||y|| \leq 1 < \alpha \langle y, x^* \rangle$, so $U \subset K(x^*, \alpha)$. □

**Lemma 2.2.12** (Bishop-Phelps Lemma)**.** *Let $X$ be a complete seminormed space, $x^*, y^* \in \partial B_{X^*}(0;1)$, $C \subset X$ closed and convex, $1 > \epsilon > 0$, and $k > 1 + \frac{2}{\epsilon}$. The following are true.*

1. *If $x^*$ is bounded on $C$, then for each $z \in C$, there is an $x_0 \in X$ such that $K(x^*, \epsilon)$ supports $C$ at $x_0$ and $x_0 \in K(x^*, \epsilon) + z$.*
2. *If $|\langle x, y^* \rangle| \leq \frac{\epsilon}{2}$ for each $x \in Kern(x^*) \cap \overline{B_X(0;1)}$, then*
$$min\left(||x^* + y^*||, ||x^* - y^*||\right) \leq \epsilon \tag{2.70}$$
3. *If $y^*$ is nonnegative on $K(x^*, k)$, then $||x^* - y^*|| \leq \epsilon$.*

*Proof.* (1) Let $x^*$ be bounded on C and define, for $x, y \in X$, $y \lesssim x$ if and only if $x - y \in K(x^*, \epsilon)$. Fix $z \in C$. Define $Z = C \cap (K(x^*, \epsilon) + z)$. Since C and $K(x^*, \epsilon)$ are closed, so is Z. Let $\mathcal{C} = \{x_\alpha\}_{\alpha \in A}$ be a chain in where $(A, \leq)$ is a totally ordered set and $x_\alpha \lesssim x_\beta \iff \alpha \leq \beta$. If $x_\alpha, x_\beta \in \mathcal{C}$, where $x_\beta \lesssim x_\alpha$, then $x_\alpha - x_\beta \in K(x^*, \epsilon)$, so $0 \leq ||x_\alpha - x_\beta|| \leq \epsilon \langle x_\alpha - x_\beta, x^* \rangle$, implying $\langle x_\beta, x^* \rangle \leq \langle x_\alpha, x^* \rangle$. Thus we conclude $\{\langle x_\alpha, x^* \rangle\}_{\alpha \in A}$ is a monotone bounded net in $\mathbb{R}$ that is therefore Cauchy, which by the following inequality

$$||x_\beta - x_\alpha|| \leq \epsilon \langle x_\alpha - x_\beta, x^* \rangle = \epsilon \left(\langle x_\alpha, x^* \rangle - \langle x_\beta, x^* \rangle\right) \to 0 \tag{2.71}$$

implies $\mathcal{C}$ is a Cauchy net and therefore converges, say $x_\alpha \to y_0 \in Z$. Continuity of the norm and $x^*$ imply together that $y_0$ is an upper bound for $\mathcal{C}$. Since $\mathcal{C}$ was an arbitrary chain in Z, $Z$ has a maximal element $x_0$. By definition, $x_0 \in Z := K(x^*, \epsilon) + z$. Since $x_0 \in Z \subset C$, $x_0 \in C$. Further, since $0 \in K(x^*, \epsilon)$, $x_0 \in K(x^*, \epsilon) \cap C$. Let $y \in (K(x^*, \epsilon) + x_0) \cap C$. Then $y - x_0 \in K(x^*, \epsilon)$ so that $z \lesssim x_0 \lesssim y$, meaning $y \in Z$ and therefore $y = x_0$ since $x_0$ is maximal. Hence $(K(x^*, \epsilon) + x_0) \cap C = \{x_0\}$, so we are done. □

*Proof.* (2) By assumption, $\left|\left|y^*|_{Kern(x^*)}\right|\right| \leq \frac{\epsilon}{2}$, so by the Hahn-Banach theorem, we can find a $\tilde{y^*} \in X^*$ extending $y^*|_{Kern(x^*)}$ such that $\left|\left|\tilde{y^*}\right|\right| \leq \frac{\epsilon}{2}$. Since $y^* - \tilde{y^*} \neq 0$, $codim\,(kern(x^*)) = 1$, and $kern\,(x^*) \subset kern\left(y^* - \tilde{y^*}\right)$,we conclude $kern\left(y^* - \tilde{y^*}\right) = kern(x^*)$. Hence, for some $\alpha \in \mathbb{R}$, $y^* - \tilde{y^*} = \alpha x^*$. For this alpha, we have

$$|1 - |\alpha|| = \left|||y^*|| - \left|\left|\tilde{y^*} - y^*\right|\right|\right| \leq \left|\left|\tilde{y^*}\right|\right| \leq \frac{\epsilon}{2} \tag{2.72}$$

If $\alpha \geq 0$,

$$||x^* - y^*|| = \left|\left|x^* - \left(\alpha x^* + \tilde{y^*}\right)\right|\right| = \left|\left|(1-\alpha)\,x^* - \tilde{y^*}\right|\right| \leq |1-\alpha| + \left|\left|\tilde{y^*}\right|\right| \leq \epsilon \tag{2.73}$$

If $\alpha \leq 0$, then

$$||x^* + y^*|| = \left|\left|x^* + \left(\alpha x^* + \tilde{y^*}\right)\right|\right| = \left|\left|(1+\alpha)x^* + \tilde{y^*}\right|\right| \leq |1+\alpha| + \left|\left|\tilde{y^*}\right|\right| \leq \epsilon \tag{2.74}$$

□

*Proof.* (3) Since $||x^*|| = 1$, there exists $x \in \partial B_X(0;1)$ such that $\langle x, x^* \rangle > \frac{1}{k}\left(1 + \frac{2}{\epsilon}\right)$. If $y \in Kern(x^*) \cap \overline{B_X(0;1)}$, then

$$\left|\left|x \pm \frac{2}{\epsilon}y\right|\right| \leq 1 + \frac{2}{\epsilon} < k\,\langle x, x^* \rangle = k\left\langle x \pm \frac{2}{\epsilon}y, x^*\right\rangle \tag{2.75}$$

so $x \pm \frac{2}{\epsilon}y \in K(x^*, k)$, so by assumption $\left\langle x \pm \frac{2}{\epsilon}y, y^*\right\rangle \geq 0$. Since this occurs for both positive and negative, $|\langle y, y^* \rangle| = \frac{\epsilon}{2}\left|\left\langle \frac{2}{\epsilon}y, y^*\right\rangle\right| \leq \frac{\epsilon}{2}\langle y^*, x\rangle \leq \frac{\epsilon}{2}||x|| = \frac{\epsilon}{2}$. Hence by part 2, either $||x^* - y^*|| \leq \epsilon$, or $||x^* + y^*|| \leq \epsilon$. Since $||x^*|| = 1$, there exists $x \in \partial B_X(0;1)$ such that $\frac{||x||}{k} \leq max\left(\epsilon, \frac{1}{k}\right) < \langle x, x^* \rangle$, so that $x \in K(x^*, k)$, implying $\langle x, y^* \rangle \geq 0$, and therefore $\epsilon < \langle x_0, x^* + y^* \rangle \leq ||x^* + y^*||$. Hence we conclude $||x^* - y^*|| \leq \epsilon$.

□

**Theorem 2.2.13** (Bishop-Phelps Theorem)**.** *Let X be a complete seminormed space, $C \subset X$ be closed, bounded, and convex, and define $M := \{f \in X^* | (\exists x_0 \in C)(\langle x_0, f \rangle = \sup_{x \in C} \langle x, f \rangle)\}$. Then $\overline{M} = X^*$*

*Proof.* Since M is a vector subspace independent of translations of C, we assume without loss of generality that $0 \in C$ and that it is sufficient to show that M is dense in $\partial B_{X^*}(0;1)$. Let $x^* \in \partial B_{X^*}(0;1)$. Let $\epsilon \in (0,1)$ and let $1 + \frac{2}{\epsilon} < k$. by 2.2.11, $K(x^*, k)$ is a closed convex cone with nonempty interior. Applying 2.2.12, part one, there is $x_0 \in C$ with $x_0 \in K(x^*, k)$ and $(K(x^*, k) + x_0) \cap C = \{x_0\}$. By Hahn Banach, there exists $y^* \in \partial B_{X^*}(0;1)$ satisfying

$$\sup_{x \in C} \langle x, y^* \rangle = \langle x_0, y^* \rangle = \inf_{x \in K(x^*,k)+x_0} \langle x, y^* \rangle = \inf_{\tilde{x} \in K(x^*,k)} \langle \tilde{x}, y^* \rangle + \langle x_0, y^* \rangle \tag{2.76}$$

Hence $y^*$ is positive on $K(x^*, k)$, so by 2.2.12 part 3, $||x^* - y^*|| < \epsilon$, so we are done since $y^* \in M$ and $x^*$ was arbitrary. □

## 2.3 Reflexivity

Recall that a seminormed space X is said to be **reflexive** if $c(X) = X^{**}$. Since $X^{**}$ is always complete and c an isometry, any reflexive space is always complete. Due to the Banach-Alaoglu theorem, in reflexive space X, the closed unit ball of X is weakly compact. For this reason and others, reflexivity is a condition of interest to many mathematicians. We begin with a basic result.

**Lemma 2.3.1** (Reflexive Separable)**.** *Let $X$ be a complete seminormed space. Then the following are equivalent.*

1. *$X$ is reflexive.*
2. *The closed unit ball of $X$ is weakly compact.*
3. *Each closed separable subspace of $X$ is reflexive.*
4. *All collections of closed, bounded, convex sets in $X$ have the finite intersection property*
5. *The closed unit ball of $X$ is weakly sequentially compact*

*Proof.* (1 $\implies$ 2) Let $X$ be reflexive. By 2.2.9 it is sufficient to show that any sequence $\{x_i\}_{i\in\mathbb{N}} \subset \overline{B_X(0;1)}$ has a weak cluster point $x \in \overline{B_X(0;1)}$. Since $\overline{B_{X^{**}}(0;1)}$ is *weak*$^*$ compact, $\{c(x_i)\}_{i\in\mathbb{N}}$ has a subsequence $\{c(x_{n_i})\}_{i\in\mathbb{N}}$ such that $c(x_{n_i}) \overset{w^*}{\to} \tilde{x} \in \overline{B_{X^{**}}(0;1)}$. Since X is reflexive, for some $x \in \overline{B_X(0;1)}$, $c(x) = \tilde{x}$. Let $x^* \in X^*$. Then,

$$|\langle x_{n_i} - x, x^* \rangle| = |\langle x^*, c(x_{n_i}) - \tilde{x} \rangle| \to 0 \tag{2.77}$$

as $i \to \infty$, and so $x_{n_i} \overset{w}{\to} x$, completing the proof. □

*Proof.* (2 $\implies$ 3) Suppose the closed unit ball of X is weakly compact, and let $x^{**} \in \overline{B_{X^{**}}(0;1)}$. By 2.2.3, there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ such that $c(x_n) \overset{w^*}{\to} x^{**}$. By assumption, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $x_{n_k} \overset{w}{\to} x \in \overline{B_X(0;1)}$. This implies $c(x_{n_k}) \overset{w^*}{\to} c(x)$, and so since the *weak*$^*$ topology is Hausdorff, $x^{**} = c(x) \in c\left(\overline{B_X(0;1)}\right)$. □

*Proof.* (2 $\implies$ 3). Let X be reflexive and $S \subset X$ be a closed separable vector subspace. Let $\{x_i\}_{i\in\mathbb{N}} \subset \overline{B_X(0;1)}$. By assumption this sequence has an $X-$*weakly* convergent subsequence, $x_{n_k} \to x \in X$. Since S is weakly closed, $x \in S$, and since the weak topology on X is finer than that on S, $x_{n_k} \overset{S-w}{\to} x$. By 2.2.9 and (2 $\implies$ 1, S is reflexive. □

*Proof.* (3 $\implies$ 2) Let $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$. Then since $S := \overline{span\{x_i\}_{i\in\mathbb{N}}}$ is a closed separable subspace, $\{x_n\}_{n\in\mathbb{N}}$ has an S-weakly convergent subsequence, $x_{n_k} \overset{S-w}{\to} x \in S$. If $x^* \in X^*$, then $x^*|_S \in S^*$, so that $|\langle x - x_{n_k}, x^* \rangle| \to 0$, implying $x_{n_k} \overset{w}{\to} x$, an application of 2.2.9 finishes the proof. □

*Proof.* (4 $\iff$ 2) Obvious since closed convex sets are weakly closed. □

*Proof.* (5 $\iff$ 2) Apply 2.2.9. □

### 2.3.1 James

As an easy application of 2.2.4 and 2.2.9, for a reflexive space X, all $x^* \in X^*$ attain their norm. The converse of this fact was, for a time, an open question of considerable interest. The converse of this result was, as is traditional in mathematics, proven in a piecemeal manner. The result was first tackled by James in [8] under the added assumption that every space Y isomorphic to X has the property that each $y^* \in Y^*$ attains its norm and that X permits a Shauder basis. This result was rapidly improved by Klee in [9] who dropped the assumption of a Schauder basis, and then by James again in [10] who proved the result in the case of a separable space X. The question of the converse in a Banach space was finally answered to the affirmative in [11], building on the arguments in [10]. This paper included generalizations all the way to quasi-complete locally convex TVS's.

**Definition 2.3.2.** *If X is a topological vector space and $\{x_i^*\}_{i\in\mathbb{N}} \subset X^*$, then $CoLim\{x_i^*\}_{i\in\mathbb{N}}$ is the set of all $x^* \in X^*$ such that for every $x \in X$,*

$$\liminf_{i\to\infty} \langle x, x_i^* \rangle \leq \langle x, x^* \rangle \leq \limsup_{i\to\infty} \langle x, x_i^* \rangle \tag{2.78}$$

**Remark 2.3.3** (CoLim Nonempty)**.** *Let X be a complete seminormed space and let $\{x_i^*\}_{i\in\mathbb{N}} \subset X^*$ be bounded. Then $CoLim\{x_i^*\}_{i\in\mathbb{N}} \neq \emptyset$*

*Proof.* Since $\{x_i^*\}_{i\in\mathbb{N}}$ is bounded, it has a subsequence with a *weak*$^*$ limit $x^*$ who must live in $CoLim\{x_i^*\}_{i\in\mathbb{N}}$. □

**Lemma 2.3.4.** *Let X be a complete seminormed space, $\alpha \in (0,1)$, $\{x_i^*\}_{i\in\mathbb{N}} \subset X^*$, and $\{\beta_i\}_{i\in\mathbb{N}} \subset (0,1)$ such that $\sum_{i\in\mathbb{N}} \beta_i = 1$. Then if (1) or (3) hold below, there are $\{y_i^*\}_{i\in\mathbb{N}} \subset X^*$ and $\gamma \geq \alpha$ such that (2) or (4) hold respectively.*

1. *$\{x_i^*\}_{i\in\mathbb{N}} \subset \partial B_{X^*}(0;1)$ such that $d\left(0, \overline{co}\{x_i^*\}_{i\in\mathbb{N}}\right) \geq \alpha$.*
2. *$\gamma \leq 1$, and for each $i \in \mathbb{N}$,*

$$y_i^* \in \overline{co}\{x_j^*\}_{j\geq i} \qquad \left\|\sum_{j\in\mathbb{N}} \beta_j y_j^*\right\| = \gamma \qquad \left\|\sum_{j=1}^{i} \beta_j y_j^*\right\| < \gamma\left(1 - \alpha\left(\sum_{j=i+1}^{\infty} \beta_j\right)\right) \tag{2.79}$$

3. *$\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ such that $d\left(\overline{co}\{x_i^*\}_{i\in\mathbb{N}} - CoLim\{x_i^*\}_{i\in\mathbb{N}}, 0\right) \geq \alpha$.*
4. *$\gamma \leq 2$, $\{y_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$, and for each $i \in \mathbb{N}$, $y^* \in CoLim\{y_i^*\}_{i\in\mathbb{N}}$,*

$$\left\|\sum_{j\in\mathbb{N}} \beta_j \left(y_j^* - y^*\right)\right\| = \gamma \qquad \left\|\sum_{j=1}^{i} \beta_j \left(y_j^* - y^*\right)\right\| < \gamma\left(1 - \alpha\left(\sum_{j=i+1}^{\infty} \beta_j\right)\right) \tag{2.80}$$

*Proof.* ($1 \implies 2$) There exists a positive sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that

$$\sum_{i\in\mathbb{N}} \frac{\beta_i \delta_i}{\left(\sum_{j=i+1}^{\infty} \beta_i\right)\left(\sum_{j=i}^{\infty} \beta_i\right)} < 1 - \alpha \tag{2.81}$$

Let $\gamma_1 \in \mathbb{R}$ and choose $y_1^* \in \overline{co}\{x_i^*\}_{i\in\mathbb{N}}$ such that $\gamma_1 = d\left(0, \overline{co}\{x_i^*\}_{i\in\mathbb{N}}\right)$ and $||y_1^*|| \leq \gamma_1\left(1+\delta_1\right)$. From here, for each $n \geq 1$, define $\gamma_{n+1} \in \mathbb{R}$ and choose $y_{n+1}^* \in \overline{co}\{x_i^*\}_{i\geq n+1}$ such that

$$\gamma_{n+1} = \inf\left\{\left|\left|\left(\sum_{i=1}^{n} \beta_i y_i^*\right) + \left(1 - \sum_{i=1}^{n} \beta_i\right) y^*\right|\right| : y^* \in \overline{co}\{x_i^*\}_{i\geq n+1}\right\} \tag{2.82}$$

and

$$\left|\left|\sum_{i=1}^{n} \beta_i y_i^* + \left(1 - \sum_{i=1}^{n} \beta_i\right) y_{n+1}^*\right|\right| < \gamma_{n+1}\left(1+\delta_{n+1}\right) \tag{2.83}$$

It is clear that since the set over which we are taking the infimum never gets larger, so $\{\gamma_i\}_{i\in\mathbb{N}}$ is a nondecreasing sequence. Further, since $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ and $\sum_{i\in\mathbb{N}} \beta_i = 1$, we have, for every i,

$$\alpha \leq \gamma_i \nearrow \gamma = \left|\left|\sum_{k\in\mathbb{N}} \beta_k y_k^*\right|\right| \leq 1 \tag{2.84}$$

So what is left to be shown is that for every $i \in \mathbb{N}$,

$$\left|\left|\sum_{j=1}^{i} \beta_j y_j^*\right|\right| < \gamma\left(1 - \alpha\left(\sum_{j=i+1}^{\infty} \beta_j\right)\right) \tag{2.85}$$

Let $i \in \mathbb{N}$. Then,

$$\begin{aligned}
\left|\left|\sum_{j=1}^{i} \beta_j (y_j^* - y^*)\right|\right| &= \left|\left|\left(\left(\frac{\sum_{j=i}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j}\right)\left(\sum_{j=1}^{i-1} \beta_j y_j^*\right)\right) + \left(\left(\frac{\sum_{j=i}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j}\right)(\beta_i y_i^*)\right)\right|\right| \\
&= \left|\left|\left(\left(\frac{\lambda_i + \sum_{j=i+1}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j}\right)\left(\sum_{j=1}^{i-1} \beta_j y_j^*\right)\right) + \left(\left(\frac{\beta_i \sum_{j=i}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j}\right)(y_i^*)\right)\right|\right| \\
&\leq \frac{\beta_i}{\sum_{j=i}^{\infty} \beta_j}\left|\left|\sum_{j=1}^{i=1} \beta_j y_j^* + \left(\sum_{j=i}^{\infty} \beta_j\right) y_i^*\right|\right| + \frac{\sum_{j=i+1}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j}\left|\left|\sum_{j=1}^{i-1} \beta_j y_j^*\right|\right| \\
&\leq \frac{\beta_i}{\sum_{j=i}^{\infty} \beta_j}\left|\left|\sum_{j=1}^{i=1} \beta_j y_j^* + \left(1 - \sum_{j=1}^{i-1} \beta_j\right) y_i^*\right|\right| + \frac{\sum_{j=i+1}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j}\left|\left|\sum_{j=1}^{i-1} \beta_j y_j^*\right|\right| \\
&< \frac{\beta_i}{\sum_{j=i}^{\infty} \beta_j}\left(\gamma_i\right)\left(1+\delta_i\right) + \frac{\sum_{j=i+1}^{\infty} \beta_j}{\sum_{j=i}^{\infty} \beta_j}\left|\left|\sum_{j=1}^{i-1} \beta_j y_j^*\right|\right| \\
&= \left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\left(\frac{\beta_i \gamma_i \left(1+\delta_i\right)}{\left(\sum_{j=i}^{\infty} \beta_j\right)\left(\sum_{j=i+1}^{\infty} \beta_j\right)}\right) + \left(\frac{1}{\sum_{j=i}^{\infty} \beta_j}\right)\left|\left|\sum_{j=1}^{i-1} \beta_j y_j^*\right|\right|\right)
\end{aligned} \tag{2.86}$$

Hence, for any $i \in \mathbb{N}$,

$$\begin{aligned}
\left|\left|\sum_{j=1}^{i} \beta_j y_j^*\right|\right| &< \left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\left(\frac{\beta_i \gamma_i (1+\delta_i)}{\left(\sum_{j=i}^{\infty} \beta_j\right)\left(\sum_{j=i+1}^{\infty} \beta_j\right)}\right) + \left(\frac{1}{\sum_{j=i}^{\infty} \beta_j}\right)\left|\left|\sum_{j=1}^{i-1} \beta_j y_j^*\right|\right|\right) \\
&< \left(\sum_{j=i+1}^{\infty} \beta_j\right)\sum_{j=1}^{i} \frac{\beta_j \gamma_j (1+\delta_j)}{\left(\sum_{k=j}^{\infty} \beta_k\right)\left(\sum_{k=j+1}^{\infty} \beta_k\right)} \\
&\leq \gamma\left(\sum_{j=i+1}^{\infty} \beta_j\right)\sum_{j=1}^{i} \frac{\beta_j (1+\delta_j)}{\left(\sum_{k=j}^{\infty} \beta_k\right)\left(\sum_{k=j+1}^{\infty} \beta_k\right)} \\
&\leq \gamma\left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\left(\sum_{j=1}^{i} \frac{\beta_j}{\left(\sum_{k=j}^{\infty} \beta_k\right)\left(\sum_{k=j+1}^{\infty} \beta_k\right)}\right) + (1-\alpha)\right) \\
&= \gamma\left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\left(\sum_{j=1}^{i}\left(\frac{1}{\sum_{k=j}^{\infty} \beta_k} - \frac{1}{\sum_{k=j+1}^{\infty} \beta_k}\right)\right) + (1-\alpha)\right) \\
&= \gamma\left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\frac{1}{\sum_{j=i+1}^{\infty} \beta_j} - \frac{1}{\sum_{j=1}^{\infty} \beta_j} + 1 - \alpha\right) \\
&= \gamma\left(\sum_{j=i+1}^{\infty} \beta_j\right)\left(\frac{1}{\sum_{j=i+1}^{\infty} \beta_j} - \alpha\right) \\
&= \gamma\left(1 - \alpha\left(\sum_{j=i+1}^{\infty} \beta_j\right)\right)
\end{aligned} \tag{2.87}$$

completing the proof. □

*Proof.* (3 $\implies$ 4) There exists a positive sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that

$$\sum_{i\in\mathbb{N}} \frac{\beta_i \delta_i}{\left(\sum\limits_{j=i+1}^{\infty} \beta_i\right)\left(\sum\limits_{j=i}^{\infty} \beta_i\right)} < 1 - \alpha \tag{2.88}$$

Define $\{x_i^0\}_{i\in\mathbb{N}} = \{x_i^*\}_{i\in\mathbb{N}}$,

$$\gamma_1 = \inf\left\{\sup_{y^* \in CoLim\{\phi_i\}_{i\in\mathbb{N}}} \{||x^* - y^*||\} : x^* \in \overline{co}\{x_i^*\}_{i\in\mathbb{N}}, \phi_k \in \overline{co}\{x_i^*\}_{i\geq k}, k \in \mathbb{N}\right\} \tag{2.89}$$

and pick $y_1^* \in \overline{co}\{x_i^*\}_{i\in\mathbb{N}}$, $\{\phi_i^1\}_{i\in\mathbb{N}} \subset X^*$ such that $\phi_k^1 \in \overline{co}\{x_i^*\}_{i\geq k}$ for every k and $w' \in CoLim\{\phi_i^1\}_{i\in\mathbb{N}}$ such that

$$\gamma_1(1-\delta_1) < ||y_1^* - w|| < \gamma_1(1+\delta_1) \tag{2.90}$$

So that there exists $\tilde{x} \in \overline{B_X(0;1)}$ such that $\gamma_1(1-\delta_1) < \langle \tilde{x}, y_1^* - w' \rangle$, and since $w' \in CoLim\{\phi_i^1\}_{i\in\mathbb{N}}$, we extract a subsequence $\{x_i^1\}_{i\in\mathbb{N}}$ of $\{\phi_i^1\}_{i\in\mathbb{N}}$ so that for any $w \in CoLim\{x_i^1\}_{i\in\mathbb{N}}$, we have

$$\langle \tilde{x}, w \rangle = \lim_{i\to\infty} \left\langle \tilde{x}, x_i^1 \right\rangle = \liminf_{i\to\infty} \left\langle \tilde{x}, \phi_i^1 \right\rangle \le \langle \tilde{x}, w' \rangle \tag{2.91}$$

And so, for any $w \in CoLim\{x_i^1\}_{i\in\mathbb{N}}$, we have

$$\gamma_1(1-\delta_1) < \langle \tilde{x}, y_1^* - w \rangle \tag{2.92}$$

Continuing inductively, for $i \in \mathbb{N}$, set

$$\gamma_{i+1} = \inf\left\{ \sup\left\{ \left\| \left( \sum_{j=1}^{i} \beta_j y_j^* \right) + \left( \left( \sum_{j=i+1}^{\infty} \beta_j \right) y^* \right) - w \right\| : w \in CoLim\{\phi_i\}_{i\in\mathbb{N}} \right\} \right\} \tag{2.93}$$

Where the infimum is taken over all $y^* \in \overline{co}\{x_j^i\}_{j\ge i+1}$ and all $\{\phi_i\}_{i\in\mathbb{N}} \subset X^*$ such that $\phi_k \in \overline{co}\{x_j^i\}_{j\ge k}$. Next, pick $y_{i+1}^* \in \overline{co}\{x_j^i\}_{j\ge i+1}$ and $\{\phi_j^{i+1}\}_{j\in\mathbb{N}} \subset X^*$ such that for every k, $\phi_k^{i+1} \in \overline{co}\{x_j^i\}_{j\ge k}$ and pick $w' \in CoLim\{\phi_j^{i+1}\}_{j\in\mathbb{N}}$ such that

$$\gamma_{i+1}(1-\delta_{i=1}) < \left\| \sum_{j=1}^{i} \beta_j y_j^* + \left( \sum_{j=i+1}^{\infty} \beta_j \right) y_{i+1}^* - w' \right\| < \gamma_{i+1}(1+\delta_{i+1}) \tag{2.94}$$

Next, pick $\tilde{x} \in \overline{B_X(0;1)}$ satisfying

$$\gamma_{i+1}(1-\delta_{i+1}) < \left\langle \tilde{x}, \sum_{j=1}^{i} \beta_j y_j^* + \left( \left( \sum_{j=i+1}^{\infty} \beta_j \right) y_j^* \right) - w' \right\rangle \tag{2.95}$$

and apply the fact that since $\liminf_{j\to\infty} \left\langle \tilde{x}, \phi_j^{i+1} \right\rangle \le \langle \tilde{x}, w \rangle$, we can find a subsequence $\{x_j^{i+1}\}_{j\in\mathbb{N}}$ of $\{\phi_j^{i+1}\}_{j\in\mathbb{N}}$ such that for every $w \in CoLim\{x_j^{i+1}\}_{j\in\mathbb{N}}$ we have

$$\gamma_{i+1}(1-\delta_{i+1}) < \left\langle \tilde{x}, \sum_{j=1}^{i} \beta_j y_j^* + \left( \left( \sum_{j=i+1}^{\infty} \beta_j \right) y_j^* \right) - w \right\rangle \tag{2.96}$$

completing our construction. Clearly, $CoLim\{y_j^*\}_{j\in\mathbb{N}} \subset CoLim\{\phi_j^i\}_{j\in\mathbb{N}}$ for every $i \in \mathbb{N}$. Hence, for every $w \in CoLim\{y_j^*\}_{j\in\mathbb{N}}$, for ever $i \in \mathbb{N}$, we have

$$\gamma_i(1-\delta_i) < \left\| \sum_{j=1}^{i-1} \beta_j y_j^* + \left( \left( \sum_{j=i}^{\infty} \beta_j \right) y_i^* \right) - w \right\| < \gamma_i(1-\delta_i) \tag{2.97}$$

Also, since $\{y_j^*\}_{j\in\mathbb{N}} \subset \overline{co}\{x_j\}_{j\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$, $CoLim\{y_j^*\}_{j\in\mathbb{N}} \subset \overline{co}\{y_j^*\}_{j\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$. By definition, $\gamma_1 \ge \alpha$, and $\{\gamma_i\}_{i\in\mathbb{N}}$ is a nondecreasing sequence since it is defined by taking the infimum over a set which never gains new elements as $i$ increases. Further, $||w|| \le 1$ for $w \in CoLim\{y_j^*\}_{j\in\mathbb{N}}$ implies that for every $n$, $\gamma_n \le 2$, so by monotone convergence, $\gamma_i \nearrow \gamma = \left\| \sum_{j\in\mathbb{N}} \beta_j (y_j - w) \right\| \le 2$. As for the final estimate, we have, for $i \in \mathbb{N}$ and $y^* \in CoLim\{y_j^*\}_{j\in\mathbb{N}}$, we have

Let $i \in \mathbb{N}$. Then, Hence, for every $i \in \mathbb{N}$, we have

$$\begin{aligned}
\left\|\sum_{j=1}^{i} \beta_j (y_j^* - y^*)\right\| &< \left(\sum_{j=i+1}^{\infty} \beta_j\right) \\
&\quad * \left(\left(\frac{\beta_i \gamma_i (1+\delta_i)}{\left(\sum_{j=i}^{\infty} \beta_j\right)\left(\sum_{j=i+1}^{\infty} \beta_j\right)}\right) + \left(\frac{1}{\sum_{j=i}^{\infty} \beta_j}\right)\left\|\sum_{j=1}^{i-1} \beta_j (y_j^* - y^*)\right\|\right) \\
&< \left(\sum_{j=i+1}^{\infty} \beta_j\right) \sum_{j=1}^{i} \frac{\beta_j \gamma_j (1+\delta_j)}{\left(\sum_{k=j}^{\infty} \beta_k\right)\left(\sum_{k=j+1}^{\infty} \beta_k\right)} \\
&\leq \gamma \left(\sum_{j=i+1}^{\infty} \beta_j\right) \sum_{j=1}^{i} \frac{\beta_j (1+\delta_j)}{\left(\sum_{k=j}^{\infty} \beta_k\right)\left(\sum_{k=j+1}^{\infty} \beta_k\right)} \\
&\leq \gamma \left(\sum_{j=i+1}^{\infty} \beta_j\right) \left(\left(\sum_{j=1}^{i} \frac{\beta_j}{\left(\sum_{k=j}^{\infty} \beta_k\right)\left(\sum_{k=j+1}^{\infty} \beta_k\right)}\right) + (1-\alpha)\right) \\
&= \gamma \left(\sum_{j=i+1}^{\infty} \beta_j\right) \left(\left(\sum_{j=1}^{i} \left(\frac{1}{\sum_{k=j}^{\infty} \beta_k} - \frac{1}{\sum_{k=j+1}^{\infty} \beta_k}\right)\right) + (1-\alpha)\right) \\
&= \gamma \left(\sum_{j=i+1}^{\infty} \beta_j\right) \left(\frac{1}{\sum_{j=i+1}^{\infty} \beta_j} - \frac{1}{\sum_{j=1}^{\infty} \beta_j} + 1 - \alpha\right) \\
&= \gamma \left(\sum_{j=i+1}^{\infty} \beta_j\right) \left(\frac{1}{\sum_{j=i+1}^{\infty} \beta_j} - \alpha\right) \\
&= \gamma \left(1 - \alpha \left(\sum_{j=i+1}^{\infty} \beta_j\right)\right)
\end{aligned} \tag{2.98}$$

□

It is worth noting that in the following two theorems, the assumption of completeness is necessary, as demonstrated by [12].

**Theorem 2.3.5** (James Separable). *If $X$ is a separable complete seminormed space then the following are equivalent.*

1. *$X$ is not reflexive.*
2. *For every $\alpha \in (0,1)$ there is some sequence $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ satisfying $d\left(0, \overline{co}\{x_i^*\}_{i\in\mathbb{N}}\right) \geq \alpha$ and $x_i^* \overset{w^*}{\to} 0$.*
3. *For every $\alpha \in (0,1)$ and $\{\beta_i\}_{i\in\mathbb{N}} \subset (0,1)$ satisfying $\sum_{i\in\mathbb{N}} \beta_i = 1$, there is a $\gamma \in [0,1]$*

*and* $\{y_i^*\}_{i\in\mathbb{N}} \subset X^*$ *such that* $y_i^* \xrightarrow{w^*} 0$ *and for each* $i \in \mathbb{N}$,

$$\left|\left|\sum_{j\in\mathbb{N}} \beta_j y_j^*\right|\right| = \gamma \qquad \left|\left|\sum_{j=1}^{i} \beta_j y_j^*\right|\right| \leq \gamma\left(1-\alpha\left(\sum_{j=i+1}^{\infty}\beta_j\right)\right) \tag{2.99}$$

4. *There exists* $x^* \in X^*$ *not achieving its norm.*

*Proof.* ($1 \implies 2$) Let $\alpha \in (0,1)$. Since X is nonreflexive and $c(X)$ is complete, by Riesz's lemma [13], there exists an $x^{**} \in B_{X^{**}}(0;1)$ such that $d(x^{**}, c(X)) > \alpha$. Since X is separable it has a countable dense set $\{x_i\}_{i\in\mathbb{N}}$. Fix $i \in \mathbb{N}$, let $\alpha_1 = \alpha_2 = \cdots = \alpha_{i-1} = 0$, $\alpha_i = \alpha$, and let $\{\beta_j\}_{j=1}^{i} \subset \mathbb{C}$ where $\beta_i \neq 0$ without loss of generality. Then, since c(X) is a subspace,

$$\begin{aligned}\left|\sum_{j=1}^{i} \beta_j \alpha_j\right| &= |\beta_i \alpha_i| = |\beta_i|\,\alpha \\ &\leq \frac{|\beta_i|\,\alpha}{d\left(x^{**}, c(X)\right)}\left|\left|x^{**} + \sum_{j=1}^{i-1}\frac{\beta_j}{\beta_i}c(x_j)\right|\right| \\ &= \frac{\alpha}{d\left(x^{**}, c(X)\right)}\left|\left|\beta_i x^{**} + \sum_{j=1}^{i-1}\beta_j c(x_j)\right|\right|\end{aligned} \tag{2.100}$$

Since $\alpha < d\left(x^{**}, c(X)\right)$, for some $\epsilon > 0$, $\epsilon + \frac{\alpha}{d(x^{**},c(X))} < 1$, so by 2.2.1, since $X^{**} = (X^*)^*$, there exists an $x_i^* \in \overline{B_{X^*}(0;1)}$ such that for $1 \leq j \leq i-1$ we have $\langle x_j, x_i^*\rangle = \langle x_i^*, c(x_j)\rangle = 0$ and $\langle x_i^*, c(x_i)\rangle \geq \alpha$. Using this method we construct a sequence $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ such that for each $1 \leq j \leq i-1$, $\langle x_j, x_i^*\rangle = 0$ and $\langle x_i, x^{**}\rangle \geq \alpha$. Without loss of generality, we let $||x_i^*|| = 1$. Density of $\{x_j\}_{j\in\mathbb{N}}$ and the boundedness of $\{x_j^*\}_{j\in\mathbb{N}}$ implies $x_j^* \xrightarrow{w^*} 0$. Furthermore, any convex combination of the $(x_i^*)'s$ satisfies

$$\alpha \leq \left\langle \sum_{j=1}^{n} \lambda_j x_{k_j}^*, x^{**}\right\rangle \leq ||x^{**}||\left|\left|\sum_{j=1}^{n}\lambda_j x_{k_j}^*\right|\right| \leq \left|\left|\sum_{j=1}^{n}\lambda_j x_{k_j}^*\right|\right| \tag{2.101}$$

so that $d\left(0, \overline{co}\{x_i^*\}_{i\in\mathbb{N}}\right) \geq \alpha$, completing the proof. □

*Proof.* ($2 \implies 3$). This is a direct application of 2.3.4 part ($1 \implies 2$), paired with the fact that if for every i, $y_i^* \in \overline{co}\{x_j^*\}_{j\geq i}$ and $x_j^* \xrightarrow{w^*} x$, then $y_j^* \xrightarrow{w^*} x$. □

*Proof.* ($3 \implies 4$). Let $x^* = \sum_{j\in\mathbb{N}} \beta_j y_j^*$ and let $x \in \overline{B_X(0;1)}$. Since $y_j^* \xrightarrow{w^*} 0$, for some $N \in \mathbb{N}$,

$\left\langle x, y_j^*\right\rangle < \gamma\alpha$ for every $j > \mathbb{N}$. Then

$$\begin{aligned} |\langle x, x^*\rangle| &\leq \left|\left\langle x, \sum_{j=1}^{N} \beta_j y_j^*\right\rangle\right| + \left|\left\langle x, \sum_{j=N+1}^{\infty} \beta_j y_j^*\right\rangle\right| \\ &< \left|\left\langle x, \sum_{j=1}^{N} \beta_j y_j^*\right\rangle\right| + \alpha\gamma \sum_{j=N+1}^{\infty} \beta_j \\ &\leq \left|\left|\sum_{j=1}^{N} \beta_j y_j^*\right|\right| + \alpha\gamma \sum_{j=N+1}^{\infty} \beta_j \leq \gamma\left(1 - \alpha \sum_{j=N+1}^{\infty} \beta_j\right) + \gamma\alpha \sum_{j=N+1}^{\infty} \beta_j \\ &= \gamma = \left|\left|\sum_{j\in\mathbb{N}} \beta_j y_j^*\right|\right| = ||x^*|| \end{aligned} \tag{2.102}$$

Since the inequality is strict and $x \in \overline{B_X(0;1)}$ was arbitrary, we are done. □

*Proof.* ($4 \implies 1$). If X is reflexive and $x^* \in X$, then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ such that $\langle x_n, x^*\rangle \to ||x^*||$. By 2.3.1, $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence $x_{k_n} \overset{w}{\to} x \in \overline{B_X(0;1)}$. This x satisfies $\langle x, x^*\rangle = ||x^*||$. □

**Theorem 2.3.6** (James)**.** *If $X$ is a complete seminormed space, then the following are equivalent.*

1. *$X$ is non-reflexive.*
2. *For each $\alpha \in (0,1)$, there exists an $\{x_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ and a subspace $Y \subset X$ such that $d\left(\overline{co}\{x_i^*\}_{i\in\mathbb{N}} - Y^\perp, 0\right) \geq \alpha$ and that $\langle y, x_i^*\rangle \to 0$ for each $y \in Y$.*
3. *For every $\alpha \in (0,1)$ and $\{\beta_i\}_{i\in\mathbb{N}} \subset [0,\infty)$ such that $\sum_{i\in\mathbb{N}} \beta_i = 1$, there is a $\gamma \in [0,2]$ and $\{y_i^*\}_{i\in\mathbb{N}} \subset \overline{B_{X^*}(0;1)}$ such that for each $y^* \in CoLim\{y_i^*\}_{i\in\mathbb{N}}$, each $i \in \mathbb{N}$,*
$$\left|\left|\sum_{j\in\mathbb{N}} \beta_j\left(y_j^* - y^*\right)\right|\right| = \gamma \qquad \left|\left|\sum_{j=1}^{i} \beta_j\left(y_j^* - y_j\right)\right|\right| < \gamma\left(1 - \alpha\left(\sum_{j=i+1}^{\infty} \beta_j\right)\right) \tag{2.103}$$
4. *There exists $x^* \in X^*$ which doesn't achieve its norm.*

*Proof.* ($1 \implies 2$) If X is non-reflexive, then X contains a non-reflexive closed separable subspace S. An application of 2.3.5 implies the existence of a sequence $\{x_i\}_{i\in\mathbb{N}} \subset \overline{B_{S^*}(0;1)}$ such that

$$d_s\left(0, \overline{co}\{x_i\}_{i\in\mathbb{N}}\right) \geq \alpha \qquad x_i \overset{S-w^*}{\to} 0 \tag{2.104}$$

If $y^\perp \in X^*$ such that $Y \subset kern(y^*)$, then letting for each $i \in \mathbb{N}$ $x_i^*$ be a Hahn-Banach extension living in $\overline{B_X^*(0;1)}$, and let $x^* \in \overline{co}\{x_i^*\}_{i\in\mathbb{N}}$. Then $x := x^*|_S \in \overline{co}\{x_i\}_{i\in\mathbb{N}}$. If $y \in Y$, then $\langle y, x_i^*\rangle = \langle y, x\rangle \to 0$. If $y^\perp \in Y^\perp$, then

$$\left|\left|x^* - y^\perp\right|\right| \geq \left|\left|x - \left(y^\perp|_S\right)\right|\right|_S = ||x|| \geq \alpha \tag{2.105}$$

so $d\left(\overline{co}\{x_i^*\}_{i\in\mathbb{N}} - Y^\perp, 0\right) \geq \alpha$ □

*Proof.* (2 $\implies$ 3) Since $CoLim\{x_i^*\}_{i\in\mathbb{N}} \subset Y^\perp$ from the previous part, this is an easy application of 2.3.4 (3 $\implies$ 4). $\square$

*Proof.* (3 $\implies$ 4) Define $\eta = \frac{\alpha^2}{4}$, and then let $\lambda_1 \in [0, \infty)$ such that for every natural n, $\lambda_{n+1} < \eta\lambda_n$ and $\sum_{k\in\mathbb{N}} \lambda_k = 1$. Let $y^* \in CoLim\{y_i^*\}_{i\in\mathbb{N}}$ where $\{y_i^*\}_{i\in\mathbb{N}}$ are as in part (3) of this theorem. Let $x \in \overline{B_X(0;1)}$. Since $y^* \in Colim\{y_i^*\}_{i\in\mathbb{N}}$, and since $\alpha \leq \gamma$, there exists $i \in \mathbb{N}$ such that

$$\langle x, y_{i+1}^* - y^* \rangle < \alpha^2 - 2\eta \leq \alpha\gamma - 2\eta \tag{2.106}$$

For this x, we have

$$\begin{aligned}
\left\langle x, \sum_{j\in\mathbb{N}} \beta_j (y_j^* - y^*) \right\rangle &< \sum_{j=1}^{i} \beta_j \langle x, y_j^* - y^* \rangle + \beta_{i+1}(\alpha\gamma - 2\eta) + \sum_{j=i+2}^{\infty} \beta_j \langle x, y_j^* - y^* \rangle \\
&\leq \left\| \sum_{j=1}^{i} \beta_j (y_j^* - y^*) \right\| + \beta_{i+1}(\alpha\gamma - 2\eta) + 2\sum_{j=i+2}^{\infty} \beta_j \\
&\leq \gamma \left( 1 - \alpha \left( \sum_{j=i+1}^{\infty} \beta_j \right) \right) + \beta_{i+1}(\alpha\gamma - 2\eta) + 2\sum_{j=i+2}^{\infty} \beta_j \qquad (2.107) \\
&= \gamma - \gamma\alpha \sum_{j=i+2}^{\infty} \beta_j - 2\eta\beta_{i+1} + 2\sum_{j=i+1}^{\infty} \eta\beta_j \\
&\leq \gamma - (\gamma\alpha - 2\eta) \sum_{j=i+1}^{\infty} \beta_j < \gamma = \left\| \sum_{j\in\mathbb{N}} \beta_j \left( y_j^* - y^* \right) \right\|
\end{aligned}$$

Since $x \in \overline{B_X(0;1)}$ was arbitrary, we are done. $\square$

*Proof.* (4 $\implies$ 1) If X is reflexive and $x^* \in X$, then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ such that $\langle x_n, x^* \rangle \to ||x^*||$. By 2.3.1, $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence $x_{k_n} \xrightarrow{w} x \in \overline{B_X(0;1)}$. This x satisfies $\langle x, x^* \rangle = ||x^*||$. $\square$

**Corollary 2.3.7.** *Let X be a complete seminormed space. Then the following are equivalent.*

1. *X is reflexive.*
2. *Each element of $X^*$ attains its norm.*

*Proof.* Direct consequence of 2.3.6. $\square$

### 2.3.2 Lindenstrauss On Nonseparable Reflexive Banach Spaces

If $\Gamma \neq \emptyset$, then $c_0(\Gamma)$ denotes the space of mappings $f : \Gamma \to \mathbb{C}$ such that for every $\epsilon > 0$, $card\{x \in \Gamma : |f(x)| > \epsilon\} \in \mathbb{N}$.

## 2.4 Convexity Of Functions And Sets

**Definition 2.4.1** (Convex Functions)**.** *Let $X$ be a vector space, $Y$ a topological vector space, $\mathcal{U}$ the set of neighborhoods of 0 in $Y$ except $Y$ itself, $f : X \to (-\infty, \infty]$, and $g : Y \to (-\infty, \infty]$. Let $x, y \in X$.*

1. *We call $D(f) := f^{-1}(\mathbb{R})$ the* **effective domain** of $f$.
2. If $D(f) \neq \emptyset$, then we call $f$ **proper.**
3. We call $Epi\,(f) := \{(x,t) \in X \times \mathbb{R} : f(x) \leq t\}$ the **Epigraph** of $f$.
4. We denote $[x,y] = \{tx + (1-t)y : t \in [0,1]\}$.
5. We denote $(x,y) = \{tx + (1-t)y : t \in (0,1)\}$.
6. We say that $C \subset X$ is **convex** if $[x,y] \subset C$ for each $x, y \in C$.
7. We say that $C \subset X$ is **strictly convex** if for each $x, y \in C$, for each $z_0 \in (x,y)$, and for each $z_1 \in X$, there is a $t > 0$ such that $[z_0, z_0 + tz_1] \subset C$.
8. We say that $f$ is **convex** if for each $x, y \in X$, $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$.
9. We say that $f$ is **strictly convex** if for each $x, y \in X$, $f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$.
10. If g is a convex function, then then we define the **modulus of local uniform convexity** of g, $\tilde{\Delta} : \mathcal{U} \times Y \to \mathbb{R}$ by
$$\tilde{\Delta}\,(U,x) = \frac{1}{2} \inf_{y \in Y \setminus (x+U)} \left\{ f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \right\} \tag{2.108}$$
11. If g is a convex function, then we define the **modulus of uniform convexity** of g, $\Delta : \mathcal{U} \to \mathbb{R}$ by $\Delta(U) = \inf\limits_{g(x)=1} \tilde{\Delta}(U,x)$.
12. If g is a convex function, then we say that g is **locally uniformly convex at** $x \in Y$ if for each $U \in \mathcal{U}$, $\tilde{\Delta}(U,x) > 0$.
13. We say that g is **locally uniformly convex** if it is **locally uniformly convex** at each of its points.
14. We say that g is **uniformly convex** if for each $U \in \mathcal{U}$, $\Delta(U) > 0$.
15. We say that g is **lower semi-continuous**, or LSC if it is continuous with respect to the topology on $(-\infty, \infty]$ generated by sets of the form $(-\infty, \alpha)$ where $\alpha \in \mathbb{R}$, along with $(-\infty, \infty]$ itself.

**Remark 2.4.2** (Basis Independence)**.** *It is easy to see that a mapping is locally uniform convex at a point (locally uniformly convex) [uniformly convex] if we define $\Delta$, $\tilde{\Delta}$ in terms of a single neighborhood basis of $Y$ at 0 instead of all neighborhoods of 0 in $Y$.*

**Remark 2.4.3** (Strictly Convex Real Valued)**.** *If $X$ is a vector space and $f : X \to (-\infty, \infty]$ is strictly convex, and $f$ is finite everywhere.*

*Proof.* If $f(x) = \infty$ where $x \in X$ and f is strictly convex, then we must have $\infty = f(x) < \frac{f(0)+f(2x)}{2}$, a contradiction. □

**Proposition 2.4.4.** *Let $X$ be a vector space and $T : X \to (-\infty, \infty]$. Then the following are equivalent.*

1. *$T$ is (strictly) convex.*
2. *For each $x_1 \neq x_2 \in X$ and $\lambda \in (0, 1)$.*
$$T\left(\lambda x_1 + (1-\lambda)x_2\right)(<) \leq \lambda T x_1 + (1-\lambda)T x_2 \tag{2.109}$$
3. *For each $\{\lambda_i\}_{i=1}^n \subset (0, 1)$ which sums to 1, for each $\{x_i\}_{i=1}^n \subset X$,*
$$T\left(\sum_{j=1}^n \lambda_j x_j\right)(<) \leq \sum_{j=1}^n \lambda_j T x_j \tag{2.110}$$
4. *If $x_1 \neq x_3 \in X$ and $x_2 \in (x_1, x_3)$, say $x_2 = \lambda x_1 + (1-\lambda)x_3$ then*
$$\frac{Tx_2 - Tx_1}{\lambda}(<) \leq Tx_3 - Tx_1 (<) \leq \frac{Tx_3 - Tx_2}{1-\lambda} \tag{2.111}$$
5. *Epi$(T)$ is (strictly) convex*

*Proof.* $(1 \implies 2)$

□

*Proof.* $(2 \implies 3)$ I utilize induction on n. Since f is assumed to be (strictly) convex, the proposition holds for $n = 1$ and $n = 2$. Let $k \in \mathbb{N}$. Suppose that for any $(\psi_1, \cdots, \psi_k) \in [0, 1]^k$ such that $\sum_{j=1}^k \psi_j = 1$, and for each $(x_1, \cdots, x_k) \in X^k$, we have

$$f\left(\sum_{j=1}^k \lambda_j x_j\right)(<) \leq \sum_{j=1}^k \lambda_j f(x_j) \tag{2.112}$$

Let $(\lambda_1, \cdots, \lambda_{k+1}) \in [0, 1]^{k+1}$. Let $(x_1, \cdots, x_{k+1}) \in X^{k+1}$. Without loss of generality, we assume $\lambda_{k+1} \neq 0$. Then

$$\sum_{j=1}^k \frac{\lambda_j}{1-\lambda_{k+1}} = 1 \qquad \left(\frac{\lambda_1}{1-\lambda_{k+1}}, \cdots, \frac{\lambda_k}{1-\lambda_{k+1}}\right) \in [0, 1]^k \tag{2.113}$$

Hence, because f is (strictly) convex,

$$\begin{aligned} f\left(\sum_{j=1}^{k+1}\lambda_j x_j\right) &= f\left(\lambda_{k+1}x_{k+1} + (1-\lambda_{k+1})\sum_{j=1}^{k}\frac{\lambda_j}{1-\lambda_{k+1}}x_j\right) \\ &(<) \leq \lambda_{k+1}f(x_{k+1}) + (1-\lambda_{k+1})f\left(\sum_{j=1}^{k}\frac{\lambda_j}{1-\lambda_{k+1}}x_j\right) \\ &(<) \leq \sum_{j=1}^{k+1}\lambda_j f(x_j) \end{aligned} \tag{2.114}$$

□

*Proof.* ($3 \implies 4$) □

*Proof.* ($4 \implies 1$). □

*Proof.* ($2 \iff 5$) □

The epigraph of a function provides us with a nice characterization of lower semi-continuity.

**Proposition 2.4.5** (Convex Continuity)**.** *Let $X$ be a locally convex space and $f : X \to (-\infty, \infty]$. The following conditions are equivalent.*

1. *$f$ is LSC on $X$.*
2. *$Epi(f)$ is closed in $X \times \mathbb{R}$.*

*Proof.* Define $F : X \times \mathbb{R} \to \tilde{\mathbb{R}}$ by $F(x, \alpha) = f(x) - \alpha$. Then $f$ is (weakly) LSC on $X$ if and only if $F$ is (weakly) LSC on $X \times \mathbb{R}$. Suppose $f$ is (weakly) LSC. Then $F$ is (weakly) LSC, so $F^{-1}((-\infty, 0]) = Epi(f)$ is (weakly) closed, so we're done. Suppose $Epi(f)$ is (weakly) closed. Then $F^{-1}((-\infty, 0])$ is (weakly) closed. Further, for any $\beta \in \mathbb{R}$, $F^{-1}((-\infty, \beta]) = F^{-1}((-\infty, 0]) - (0, \beta)$, and so is also closed. Hence $F$ is (weakly) LSC, and so $f$ is too. □

In the case of a convex function, the above proposition allows us to equate weak and strong lower semicontinuity.

**Corollary 2.4.6** (Weak To Strong Convex)**.** *Let $X$ be locally convex Hausdorff space and $f : X \to (-\infty, \infty]$ be convex. Then $f$ is LSC if and only if it is weakly LSC.*

*Proof.* Since $Epi(f)$ is convex, it is closed if and only if it is weakly closed, allowing us to apply 2.4.5. □

**Theorem 2.4.7** (Point Continuous)**.** *Let $X$ be a locally convex space and $f : X \to (-\infty, \infty]$ be convex and proper. Then $f$ is bounded on some open set if and only if $f$ is continuous on the interior of its domain.*

*Proof.* (*implies*) Without loss of generality, we assume that f is bounded from above by M on a (weakly) open set $\mathcal{U}$ which is symmetric and contains 0. Further, we can also suppose $f(0) = 0$. For each $\epsilon \in (0, 1)$ and each $x \in \epsilon\mathcal{U}$, we have

$$f(x) = f\left(\epsilon\frac{x}{\epsilon} + (1-\epsilon)0\right) \leq \epsilon f\left(\frac{x}{\epsilon}\right) \leq \epsilon M \tag{2.115}$$

and since $0 = \frac{x}{1+\epsilon} + \left(1 - \frac{1}{1+\epsilon}\left(\frac{-x}{\epsilon}\right)\right)$,

$$0 = f(0) = f\left(\frac{x}{1+\epsilon} + \left(1 - \frac{1}{1+\epsilon}\right)\left(\frac{-x}{\epsilon}\right)\right) \leq \frac{f(x)}{1+\epsilon} + \frac{\epsilon f\left(-\frac{x}{\epsilon}\right)}{1+\epsilon} \tag{2.116}$$

so, since $\frac{-x}{\epsilon} \in \mathcal{U}$,

$$-\epsilon M \leq -\epsilon f\left(\frac{x}{\epsilon}\right) \leq f(x) \tag{2.117}$$

Hence, $|f(x)| \leq \epsilon M$ for $x \in \epsilon\mathcal{U}$, and f is (weakly) continuous at 0. Hence, it is sufficient to show that for any $y$ in the (weak) interior of $D(f)$, there is a (weak) neighborhood of y on which f is bounded from above. To see this, let $y$ in the (weak) interior of $D(f)$. Since scalar multiplication is continuous, there is a $\rho > 1$ such that $\rho y \in D(f)$. If $\mathcal{U}_y = y + \left(1 - \frac{1}{\rho}\right)\mathcal{U}$, then $x \in \mathcal{U}_y$ can be written, for some $z \in \mathcal{U}$, as

$$x = y + \left(1 - \frac{1}{\rho}\right) z = \frac{1}{\rho}(\rho y) + \left(1 - \frac{1}{\rho}\right) z \tag{2.118}$$

Since f is convex, $D(f)$ is convex, and so $x \in D(f)$, implying $\mathcal{U}_y \subset D(f)$. Since f is a convex function, we also have that

$$f(x) \leq \frac{1}{\rho} f(\rho y) + \left(1 - \frac{1}{\rho}\right) f(x) \leq \frac{1}{\rho} f(\rho y) + \left(1 - \frac{1}{\rho}\right) M \tag{2.119}$$

so that f is bounded on $\mathcal{U}_y$ and is therefore continuous at y. □

*Proof.* ( $\Longleftarrow$ ) This is obvious. □

**Corollary 2.4.8.** *Let $X$ be a locally convex, space and $f : X \to (-\infty, \infty]$ be LSC at some point in its effective domain.*

1. *if $f$ is convex, then it is continuous on the interior of $D(f)$.*
2. *If $f$ is strictly convex, then it is continuous on $X$.*

## 2.5 Differentiation And SubDifferentials

**Definition 2.5.1** (Types Of Differentiability)**.** *Let $X$ be a topological vector space, $Y$ a Hausdorff topological vector space, $C \subset X$, $f : C \to X$, and $x_0 \in C$.*

1. *If $y \in X$ such that $x_0$ is an accumulation point of $[x_0, x_0 + y] \cap C$, then we say that the* **directional derivative in the direction of y** exists at $x_0$ and we write

$$\lim_{t \to 0^+} \frac{f(x_0 + ty) - f(x_0)}{t} = f'_+(x_0, y) \tag{2.120}$$

if the limit above actually exists.

2. If $f$ is differentiable at $x_0$ in the direction of $-y$, then we denote $f'_-(x_0, y) := -f'_+(x, -y)$.

3. If there exists $f'(x_0) \in BL(X, Y)$ such that for every $y \in X$, $f'_+(x_0, y) = f'(x_0)y$ then we call $f'(x_0)$ the **Gateaux derivative** of f at $x_0$ and we call $f$ **Gateaux differentiable** .

4. If $f$ is Gateaux differentiable at $x_0$ and $\frac{f(x_0+ty)-f(x_0)}{t} \to f'(x_0)y$ uniformly for y in some neighborhood of 0 in X, then we say that $f$ is **Frechet differentiable** at $x_0$ and we call $f'(x_0)$ the **Frechet derivative** of $f$ at $x_0$.

5. If $h : X \to (-\infty, \infty]$ and there is an $x^* \in X^*$ such that

$$\langle y - x, x^* \rangle \leq h(y) - h(x) \qquad (\forall y \in X) \tag{2.121}$$

then we say that h is **subdifferentiable** at $x_0$, call $x^*$ a **subgradient** of h at $x_0$, and denote the set of all subgradients of h at $x_0$ with $\partial h\,(x_0)$. if $D$ is the set on which h is subdifferentiable, then we call $\partial h : D \to 2^{X^*}$ the **subdifferential** of $h$.

One useful feature of convex functions is how they interact with the various forms of differentiability.

**Proposition 2.5.2** (Gateaux)**.** *Let X be a topological vector space, Y a Hausdorff topological vector space, $f : X \to Y$ Gateaux differentiable at $x_0 \in X$, and $y \in X$. The following are true.*

1. $f'_+(x_0, y) = f'_+(x_0, -y) = f'_-(x_0, y)$.
2. $\langle y, f'(x_0) \rangle = \frac{d}{dt} f\,(x_0 + ty)\,|_{t=0}$
3. *If $f$ is Gateaux differentiable on a neighborhood of $x_0$ and $f' : X \to BL(X, Y)$ is Gateaux Differentiable at $x_0$, then $f'' : X \to BL(X, BL(X, Y))$ satisfies*

$$\langle y, f''(x)y \rangle = \frac{d^2}{dt^2} f\,(x_0 + ty)\,|_{t=0} \tag{2.122}$$

*Proof.* (1) □

*Proof.* (2) □

*Proof.* (3) □

**Proposition 2.5.3** (Convex Differentiable)**.** *Let X be a seminormed space and $f : X \to (-\infty, \infty]$ be convex. Then for each $y \in X$ and each $x_0$ in the interior of $D(f)$, the directional derivative in the direction of y at $x_0$ exists and $f'_-(x, y) \leq f'_+(x, y)$, and for any $t > 0$,*

$$f'_+(x, y) \leq \frac{f(x + ty) - f(x + ty)}{t} \leq f'_-(x + ty, y) \tag{2.123}$$

## 2.6 Normalized Duality Mapping

**Definition 2.6.1** (Normalized Duality Mapping). *If $X$ is a seminormed space, then we call $J : X \to 2^{X^*}$ defined by*

$$Jx = \left\{x^* \in X^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\right\} \tag{2.124}$$

*the normalized duality mapping of $X$.*

**Proposition 2.6.2** (Normalized Duality Inequality). *Let $X$ be a seminormed space and $J$ be $X'$s normalized duality mapping. Then, if $x, y \in X$ and $||x + \lambda y|| \neq 0$ where $\lambda > 0$, and $j_x \in Jx$, $j_{x+\lambda y} \in J(x + \lambda y)$, then*

$$\frac{\langle y, j_x \rangle}{||x||} \leq \frac{||x + \lambda y|| - ||x||}{\lambda} \leq \frac{\langle y, j_{x+\lambda y} \rangle}{||x + \lambda y||} \tag{2.125}$$

*Proof.*

$$\begin{aligned}
\frac{\langle y, j_x \rangle}{||x||} &= \frac{\langle x + \lambda y, j_x \rangle - ||x||^2}{\lambda\, ||x||} \leq \frac{|\langle x + \lambda y, j_x \rangle| - ||x||^2}{\lambda\, ||x||} \leq \frac{||j_x||\, ||x + \lambda y|| - ||x||^2}{\lambda\, ||x||} \\
&= \frac{||x + \lambda y|| - ||x||}{\lambda} \\
&= \frac{||x + \lambda y||}{||x + \lambda y||} \frac{||x + \lambda y|| - ||x||}{\lambda} \leq \frac{||x + \lambda y||^2 - |\langle x, j_{x+\lambda y} \rangle|}{\lambda\, ||x + \lambda y||} \\
&= \frac{\lambda \langle y, j_{x+\lambda y} \rangle + \langle x, j_{x+\lambda y} \rangle - |\langle x, j_{x+\lambda y} \rangle|}{\lambda\, ||x + \lambda y||} \\
&\leq \frac{\lambda \langle y, j_{x+\lambda y} \rangle}{\lambda\, ||x + \lambda y||} = \frac{\langle y, j_{x+\lambda y} \rangle}{||x + \lambda y||}
\end{aligned} \tag{2.126}$$

□

**Theorem 2.6.3** (Asplund). *[14] Let $X$ be a seminormed space and $J$ be its normalized duality mapping. Then for any $x \in X$, $Jx = \partial\left(||x||^2\right)$.*

*Proof.* □

**Proposition 2.6.4.** *Let $X$ be a (Possibly Complete, maybe not) seminormed space and let $J$ be it's normalized duality mapping. Then $J$ is norm to weak$^*$ upper semicontinuous on $X$.*

*Proof.* □

## 2.7 Orthogonality

**Definition 2.7.1** (Orthogonality). *Let $X$ be a seminormed space and $x, y \in X$. We say that $x$ is* **orthogonal** to y and we write $x \perp y$ if for each scalar $\lambda$, we have

$$||x|| \leq ||x + \lambda y|| \tag{2.127}$$

**Proposition 2.7.2** (Orthogonality)**.** *Let $X$ be a seminormed space, $x, z \in X$, and $x^* \in X^*$.*

1. *$\langle x, x^* \rangle = ||x|| \, ||x^*||$ if and only if for each $y \in kern\,(x^*)$, $x \perp y$.*

2. *$x$ is orthogonal to each element of some hyperplane in X.*

3. *For some $\alpha \neq 0$ $x \perp (\alpha x + z)$.*

4. *The mapping $T : \mathbb{F} \to \mathbb{R}$ defined by $T\alpha = ||\alpha x + z||$ achieves its minimum, and if $\lambda_0$ is a point at which it achieves this minimum, then $(\lambda_0 x + y) \perp x$ for any $\lambda_0$ which minimizes T. Furthermore, since T as defined earlier is a convex function, the set of $\lambda$ for which $(\lambda x + y) \perp x$ is a convex set.*

## 2.8 Convexity Of A Space

**Definition 2.8.1** (Uniform Convexity, Weak Uniform Convexity, Local Uniform Convexity, Weak Local Uniform Convexity, Strict Convexity)**.** *Let $(X, ||\cdot||)$ be a seminormed space.*

1. *We call the local modulus of uniform covexity of $||\cdot||$, denoted by the symbol $\tilde{\Delta}$, the* **local modulus of uniform convexity** of X.

2. If $\Delta$ is the modulus of uniform convexity of $||\cdot||$, then we call $\Delta$ the **modulus of uniform convexity** of X.

3. We define $\tilde{\Delta}_w : (0, \infty) \times \partial B_X(0; 1) \to (0, \infty)$ by
$$\tilde{\Delta}_w(\epsilon, x) = \inf \{2 - \langle x + y, x^* \rangle : x^* \in \partial B_{X^*}(0; 1), y \in \partial B_X(0; 1), ||x - y|| \geq \epsilon\} \tag{2.128}$$
We call $\tilde{\Delta_w}$ the **modulus of weak local uniform convexity** of X.

4. We define $\Delta_w : (0, \infty) \to (0, \infty)$ by
$$\Delta_w(\epsilon) = \inf_{x \in \partial B_X(0;1)} \tilde{\Delta}_w(\epsilon, x) \tag{2.129}$$
We call this the **modulus of weak uniform convexity** of X.

1. *We say that X is* **strictly convex** if for each $x, y \in X$ such that $||x - y|| \neq 0$, $||x + y|| < ||x|| + ||y||$.

2. We say that X is **uniformly convex at** x if $||\cdot||$ is uniformly convex at x.

3. We say that X is **locally uniformly convex** if $||\cdot||$ is locally uniformly convex.

4. We say that X is **uniformly convex** if $||\cdot||$ is uniformly convex.

5. We say that X is **weakly uniformly convex at** $x \in \partial B_X(0; 1)$ if for each $\epsilon > 0$, $\tilde{\Delta}_w(\epsilon, x) > 0$.

6. We say that X is **locally weakly uniformly convex** if it is weakly uniformly convex at each point on the boundary of $X's$ unit sphere.

7. We say that X is **weakly uniformly convex** if for each $\epsilon > 0$, $\Delta_w(\epsilon) > 0$.

**Proposition 2.8.2** (Strictly Convex Spaces)**.** *Let X be a normed space. and let J be a duality mapping on X of weight $\phi$. Then the following are equivalent.*

1. *X is strictly convex.*

2. *If $x, y \in X$ and $||x + y|| = ||x|| + ||y||$, then for some $\alpha \geq 0$, $||x - \alpha y|| = 0$.*

3. *If $||x|| = ||y|| = 1$ where $0 \neq ||x - y||$, then $||x + y|| < 2$.*

4. *If $x, y, z \in X$ and $||x - y|| = ||x - z|| + ||z - y||$, $||z - z_0|| = 0$ for some $z_0 \in [x, y]$.*

5. *If $x^* \in X^*$ and $||x|| = ||y|| = 1$ such that $\langle x, x^* \rangle = \langle y, y^* \rangle = \sup\limits_{||z||=1} \langle z, x^* \rangle$, then $||x - y|| = 0$.*

6. *$||\cdot||^2$ is strictly convex.*

7. *J is strictly monotone. That is, if $x, y \in X$, $||x - y|| \neq 0$, $x^* \in Jx$, and $y^* \in Jy$, then*
$$\langle x - y, x^* - y^* \rangle > 0 \tag{2.130}$$

8. *Orthogonality in X is left-unique. That is, for $x, y \in X$, there is a unique $\alpha \in \mathbb{F}$ such that $(\alpha x + y) \perp x$.*

**Proposition 2.8.3** (Locally Uniformly Convex Spaces)**.** *Let X be a Banach Space. Then the following are equivalent.*

1. *X is locally uniformly convex*

2. *For each $\epsilon > 0$ and $x \in X$ with $||x|| = 1$, there is a $\delta > 0$ such that if $y \in X$ satisfies $||y|| = 1$ and $||x - y|| \geq \epsilon$, then $||x + y|| \leq 2(1 - \delta)$.*

3. *If $x \in \partial B_X(0; 1)$, $\{x_n\}_{n \in \mathbb{N}} \subset \partial B_X(0; 1)$, and $||x + x_n|| \to 2$, then $x_n \to x$.*

4. *$\frac{1}{2} ||\cdot||^2$ is locally uniformly convex.*

**Proposition 2.8.4** (Uniformly Convex Spaces)**.** *Let X be a Banach space. The following are equivalent.*

1. *X is uniformly Convex.*

2. *For each $\epsilon > 0$, there is a $\delta > 0$ such that if $x, y \in \overline{B_X(0; 1)}$ and $||x - y|| \geq \epsilon$, then $||x + y|| \leq 2(1 - \delta)$.*

3. *If $\{x_i\}_{i \in \mathbb{N}}, \{y_i\}_{i \in \mathbb{N}} \subset \overline{B_X(0; 1)}$ and $||x_n + y_n|| \to 2$, then $x_n - y_n \to 0$.*

4. *$\frac{1}{2} ||\cdot||^2$ is uniformly convex.*

**Proposition 2.8.5** (Weakly Locally Uniformly Convex Spaces)**.** *Let $X$ be a Banach space. The following conditions are equivalent.*

1. *$X$ is weakly locally uniformly convex.*
2. *For each $\epsilon > 0$, $^* \in \partial B_{X^*}(0;1)$, and $x \in \overline{B_X(0;1)}$ there is a $\delta > 0$ such that if $y \in \overline{B_X(0;1)}$ and $\langle x - y, x^* \rangle \geq \epsilon$, then $||x + y|| \leq 2(1 - \delta)$.*
3. *If $x \in \overline{B_X(0;1)}$ and $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ such that $||x + x_n|| \to 2$, then $x_n \xrightarrow{w} x$.*

**Proposition 2.8.6** (Weakly Uniformly Convex Spaces)**.** *Let $X$ be a Banach space. The following conditions are equivalent.*

1. *$X$ is weakly uniformly convex.*
2. *For each $\epsilon > 0$ and $x^* \in \overline{B_{X^*}(0;1)}$, there is a $\delta > 0$ such that if $x, y \in \overline{B_X(0;1)}$ such that $\langle x - y, x^* \rangle \geq \epsilon$, then $||x + y|| \leq 2(1 - \delta)$.*
3. *If $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset \overline{B_X(0;1)}$ and $||x_n + y_n|| \to 2$, then $x_n - y_n \xrightarrow{w} 0$.*

**Proposition 2.8.7** (Degrees Of Convexity)**.** *Let $X$ be a seminormed space.*

1. *If $X$ is uniformly convex, then $X$ is weakly uniformly convex.*
2. *If $X$ is uniformly convex, then $X$ is locally uniformly convex.*
3. *if $X$ is locally uniformly convex, then $X$ is weakly locally uniformly convex.*
4. *if $X$ is weakly locally uniformly convex, then $X$ is strictly convex.*

**Proposition 2.8.8** (Local Weak To Strong)**.** *If a seminormed space $X$ is locally uniformly convex and $\{x_n\} \subset X$ satisfies $x_n \xrightarrow{w} x$ and $||x_n|| \to ||x||$, then $x_n \to x$.*

*Proof.* □

**Theorem 2.8.9.** *(Milman Pettis) A complete uniformly convex seminormed space $X$ is reflexive.*

## 2.9 Smoothness Of A Space

Define Moduli Of Smoothness, remark on positiveness

**Definition 2.9.1.** *Let $X$ be a seminormed space and $x_0 \in X$.*

1. *We define the* **modulus of smoothness** of $X$ at $x_0$, $\tilde{\rho} : [0, \infty) \times X \to \mathbb{R}$ by

$$\tilde{\rho}(\epsilon, x) = \frac{1}{2} \sup_{||y||<1} (||x + \epsilon y|| + ||x - \epsilon y|| - 2\,||x||) \tag{2.131}$$

2. We define the **modulus of smoothness** of $X$, $\rho : [0, \infty) \to \mathbb{R}$ by

$$\rho(\epsilon) = \sup_{||x|| \geq 1} \rho(\epsilon, x) \tag{2.132}$$

3. We say that X is **locally uniformly smooth** at $x_0$ if $\lim\limits_{\epsilon \to 0} \frac{\tilde{\rho}(\epsilon, x_0)}{\epsilon} = 0$.

4. We say that X is **locally uniformly smooth** if it is locally uniformly smooth at each of its points.

5. We say that X is **uniformly smooth** if $\lim\limits_{\epsilon \to 0} \frac{\rho(\epsilon)}{\epsilon} = 0$.

6. We say that X is **smooth** at $x_0$ if $Jx_0$ is a singleton.

7. We say that X is **smooth** if it is smooth at each of its points.

8. 2.9.2 motivates defining $X$ to be called **very smooth** at $x_0$ if it is smooth and $J$ is seminorm to weak continuous at $x_0$.

9. We say that X is **very smooth** if it is very smooth at each of its points.

**Proposition 2.9.2** (Smooth Characterization)**.** *Let $X$ be a seminormed space, $x_0 \in X$, and $J$ it's normalized duality mapping. The following are equivalent.*

1. *$X$ is a smooth at $x_0$.*

2. *Every selection of $J$ is norm to weak$^*$ continuous at $x_0$.*

3. *There exists a selection of $J$ which is norm to weak$^*$ continuous at $x_0$.*

4. *$||\cdot||$ is Gateaux Differentiable at $x_0$.*

5. *For every $y \in X$, there is a unique $\alpha \in \mathbb{C}$ such that $x \perp (\alpha x + y)$.*

6. *For every $y, z \in X$, if $x \perp y$ and $x \perp z$, then $x \perp y + z$.*

7. *(NEED TO DEFINE HYPERPLANE)There is a supporting hyperplane for $\overline{B_X(0; ||x||)}$ at $x$.*

**Proposition 2.9.3** (Degrees Of Smoothness)**.** *Let $X$ be a seminormed space.*

1. *If $X$ is uniformly smooth, then it is locally uniformly smooth.*

2. *If $X$ is locally uniformly smooth, then it is very smooth.*

3. *If $X$ is very smooth, then it is smooth.*

**Proposition 2.9.4** (Very Smooth)**.**

**Proposition 2.9.5** (Local Uniformly Smooth At A Point)**.** *Let $X$ be a seminormed space, $x_0 \in X$, and $J$ be $X'$s normalized duality mapping. The following conditions are equivalent.*

1. *$X$ is locally uniformly smooth at $x_0$.*

2. *$J$ is continuous at $x_0$.*

3. *$||\cdot||$ is Frechet differentiable at $x_0$.*

**Corollary 2.9.6** (Local Uniformly Smooth)**.** *Let $X$ be a seminormed space and $J$ be $X'$s normalized duality mapping. The following conditions are equivalent.*

1. *$X$ is locally uniformly smooth.*

2. *$J$ is conitnuous.*

3. *$||\cdot||$ is Frechet differentiable.*

**Proposition 2.9.7** (Uniformly Smooth)**.** *Let $X$ be a seminormed space and $J$ be $X'$s normalized duality mapping. The following are equivalent.*

1. *$X$ is uniformly smooth.*

2. *$J$ is uniformly continuous on bounded subsets of $X$.*

3. *$||\cdot||$ is uniformly Frechet differentiable on bounded subsets of $X$.*

# Chapter 3

# Smoothness And Convexity

## 3.1 Convexity And Smoothness Of A Space

**Proposition 3.1.1** (Smoothness and Strict Convexity). *Let $X$ be a seminormed space and $J$ be $X'$s normalized duality mapping.*

1. *If $X^*$ is smooth, then $X$ is strictly convex.*
2. *If $X^*$ is strictly convex, then $X$ is smooth.*
3. *If $X$ is reflexive, then $X$ is strictly convex if and only if $X^*$ is smooth.*
4. *If $X$ is reflexive, then $X$ is smooth if and only if $X^*$ is strictly convex.*
5. *If $X^*$ is strictly convex, then $J$ is single valued and norm to weak $- *$ continuous.*
6. *$X$ is smooth and strictly convex if and only if $J$ is single valued and strictly monotone.*

**Proposition 3.1.2** (Lindenstraus Duality Formula).

**Proposition 3.1.3** (Very Smooth and Weak Local convexity).

**Proposition 3.1.4** (Local Uniform Smoothness and Local Uniform Convexity). *Let $X$ be a seminormed space and $J$ be $X'$s normalized duality mapping.*

**Proposition 3.1.5** (Uniform Smoothness and Convexity). *Let $X$ be a (possibly complete) seminormed space and $J$ be X's normalized duality mapping.*

1. *$X$ is uniformly convex if and only if $X^*$ is uniformly smooth.*
2. *$X$ is uniformly smooth if and only if $X^*$ is uniformly convex.*
3. *$X^*$ is uniformly convex if and only if $J$ is single valued and uniformly continuous on bounded subsets of $X$.*

**Corollary 3.1.6.** *Uniformly Smooth Banach Spaces are Reflexive*

**Proposition 3.1.7** (Normalized Convergence). *Let $X$ be a smooth locally uniformly convex seminormed space with normalized duality mapping $J$. If $\{x_n\}_{n\in\mathbb{N}} \subset X$, $x \in X$, $j_x \in Jx$, and $j_n \in Jx_n$ for each $n \in \mathbb{N}$, then*

$$\langle x_n - x, j_n - j\rangle \to 0 \implies x_n \to x \tag{3.1}$$

## 3.2 Convexity, Smoothness, and High Order Duals

If you start with a poorly behaved Space, then things can only get worse.

**Theorem 3.2.1.** *If $X$ is a seminormed space and $X^*$ is very smooth, then $X$ is reflexive.*

**Corollary 3.2.2.** *Let $X$ be a seminormed space.*

1. *If $X^*$'s norm is Frechet differentiable, then $X$ is reflexive.*
2. *If $X^{**}$ is weakly locally uniformly convex, then $X$ is reflexive.*
3. *If $X^{***}$ is smooth, then $X$ is reflexive.*
4. *If $X^{****}$ is strictly convex, then $X$ is reflexive.*

# Chapter 4

# Renorming Theory (Including Results about WCG Spaces)

## 4.1 Representations Of Reflexive Spaces

(Lindenstrauss' Theorem Goes Here

## 4.2 Local Uniform Convexafiability Of Reflexive Spaces

-Trojanski's Theorem Goes Here

# Chapter 5

# Convexity And Fixed Point Theory

# Bibliography

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