2 Modules

2.6 Localization

1. (1) For any $a, b \in R$ and any $s_1, s_2 \in S$ such that $\frac{a}{s_1} \cdot \frac{b}{s_2} = \frac{ab}{s_1 s_2} = \frac{0}{t}$ where $t \in S$, then there is $t' \in S$ such that $t'(abt - s_1 s_2 \cdot 0) = 0$, thus $ab = 0$, therefore $a = 0$ or $b = 0$, hence $\frac{a}{s_1} = 0$ or $\frac{a}{s_1}$.

   (2) If $\frac{s}{s} = \frac{0}{t}$ where $s, t \in S$, then there is $u \in S$ such that $u(st - 0 \cdot s) = 0$, thus $ust = 0$, hence $s = 0$. It is a contradiction that $s \in S$. Therefore $\frac{s}{s} \in R[S^{-1}]$ is nonzero element.

   Hence $R[S^{-1}]$ is an integral domain.

2. Let $I = \{a \in \mathbb{Z}_6 | a \cdot s = 0 \ for \ s \in S\} = \{0, 3\}$, then I is an ideal of $\mathbb{Z}_6$. Let $\pi : \mathbb{Z}_6 \to \mathbb{Z}_6/I$ is a canonical homomorphism and $\alpha : \mathbb{Z}_6/I \to \mathbb{Z}_3$ is an isomorphism where $\alpha(\overline{0}) = \alpha(\overline{3}) = \overline{0}$, $\alpha(\overline{1}) = \alpha(\overline{4}) = \overline{1}$ and $\alpha(\overline{2}) = \alpha(\overline{5}) = \overline{2}$. Let $\varphi = \alpha \cdot \pi : \mathbb{Z}_6 \to \mathbb{Z}_3$, then

   (1) $\varphi(\overline{2}) = \overline{2}$ and $\varphi(\overline{4}) = \overline{1}$ are invertible.

   (2) $\overline{1} = \varphi(\overline{4})\varphi(\overline{2})^{-1}, \overline{2} = \varphi(\overline{2})\varphi(\overline{4})^{-1}, \overline{0} = \varphi(\overline{0})\varphi(\overline{2})^{-1}$.

   (3) $\varphi(\overline{0}) = \varphi(\overline{3}) = 0$ if only if $0 \cdot 2 = 3 \cdot 2 = 0$.

   Hence $R[S^{-1}] \cong \mathbb{Z}_3$.

3. Since $R_P = \{\frac{a}{s} | a \in R, s \notin P\}$, $PR_P = \{\frac{ab}{s} | a \in P, b \in R, s \notin P\} = \{\frac{a}{s} | a \in P, s \notin P\}$. For any $\frac{a}{s}, \frac{b}{t} \in PR_P$ where $a, b \in P$, $\frac{a}{s} - \frac{b}{t} = \frac{at-bs}{st} \in PR_P$. For any $\frac{c}{t'} \in PR_P$, $\frac{a}{s} \cdot \frac{c}{t'} = \frac{ac}{st'} = \frac{c}{t'} \cdot \frac{a}{s} \in PR_P$. Hence PR_P is an ideal of R_P. If $R_P \rhd I \supsetneqq PR_P$, then there is $\frac{a}{s} \in I$ where $a \notin P$, then $\frac{a}{s} \cdot \frac{s}{a} = 1 \in I$, thus $R_P = I$, therefore PR_P is maximal. Let M be another maximal ideal of R_P, if $M \nsubseteq PR_P$, then there is $\frac{a}{s} \in M \backslash PR_P$, therefore $1 = \frac{a}{s} \cdot \frac{s}{a} \in M$, thus $M = R_P$. Hence $M \subseteq PR_P$ for any $M \neq R_P$. Thus PR_P is an unique maximal ideal of R_P.

4. For any $r_1, r_2 \in S_*$, there is $s_1, s_2 \in S$ such that $(r_1, s_1), (r_2, s_2) \in T$, then $(r_1, s_1)(r_2, s_2) = (r_1 r_2, s_1 s_2) \in T$, while $s_1 s_2 \in S$, thus $r_1 r_2 \in S_*$. Let $\varphi_S : R \to R[S^{-1}]$ and $: \varphi_T R[S^{-1}] \to R[S^{-1}][T^{-1}]$ are quotient ring homomorphisms, then $\varphi = \varphi_T \circ \varphi_S : R \to R[S^{-1}][T^{-1}]$ is a ring homomorphism.

   (a) For any $a \in S_*$, there is $s \in S$ such that $(a, s) \in T$. Since $(s, s^2)(s^2, s) = (s, s) \in R[S^{-1}]$, (s^2, s) is invertible in $R[S^{-1}]$, then $\varphi_T((s^2, s))$ is invertible. As $(a, s) \in T$, $\varphi_T((a, s))$ is invertible. Hence $\varphi(a) = \varphi_T((as, s)) = \varphi_T((a, s)(s^2, s)) = \varphi_T((a, s))\varphi_T((s^2, s))$ is invertible in $R[S^{-1}][T^{-1}]$.

   (b) For any $((a, s), (r, t)) \in R[S^{-1}][T^{-1}]$, then $((a, s), (r, t)) = \varphi_T((a_1, s_1))\varphi_T((r_1, t_1))^{-1}$ $\varphi(s) = ((rs^2, st), (r, t)) = \varphi_T((s^2, s))$ where $s \in S$, then $\varphi(s)^{-1} =$

$\varphi_T((s,s^2)) = ((sr, s^2t),(r,t))$. Since $(a_1,s_1),(r_1,t_1) \in R[S^{-1}]$, then $(a_1,s_1) = \varphi_S(a_2)\varphi_S(s_2)^{-1}$ and $(r_1,t_1) = \varphi_S(r_2)\varphi_S(t_2)^{-1}$.

$$\begin{aligned}
((a,s),(r,t)) &= \varphi_T((a_1,s_1))\varphi_T((r_1,t_1))^{-1} \\
&= \varphi_T(\varphi_S(a_2)\varphi_S(s_2)^{-1})\varphi_T(\varphi_S(r_2)\varphi_S(t_2)^{-1}) \\
&= \varphi(a_2)\varphi(s_2)^{-1}\varphi(r_2)\varphi(t_2)^{-1} \\
&= \varphi(a_2)\varphi_T((s_2,s_2^2)(r_2s_2,s_2))\varphi(t_2)^{-1} \\
&= \varphi(a_2)\varphi_T((r_2s_2,s_2)(s_2,s_2^2))\varphi(t_2)^{-1} \\
&= \varphi(a_2)\varphi(r_2)\varphi(s_2)^{-1}\varphi(t_2)^{-1} \\
&= \varphi(a_2r_2)\varphi(t_2s_2)^{-1}.
\end{aligned}$$

(c) For any $a \in R$, if $\varphi(a) = 0 = \varphi_T \circ \varphi_S(a)$ then $\varphi_S(a)(r,t) = 0$ where $(r,t) \in T$, thus $\varphi_S(a)(rt,t)(t,t^2) = 0 = \varphi_S(a)\varphi_S(r)\varphi_S(t)^{-1}$, therefore $\varphi_S(ar) = 0$. Hence there is $s \in S$ such that $ars = 0$. Since $(r,t) \in T$, $(rs,ts) = (r,t) \in T$, then $rs \in S_*$.

Hence $(R[S^{-1}][T_{-1}],\varphi)$ is a quotient ring with respect to S_*, thus $R[S^{-1}][T_{-1}] \cong R[S_*^{-1}]$.

5. Assume that $(R_i[S^{-1}] = D_i,\psi_i)$ are quotient rings of R_i with respect to S_i, then $\psi_2\varphi : R_1 \to D_2$ is a ring homomorphism satisfies

   (a) For any $0 \neq a \in R_1$, $0 \neq \varphi(a) \in R_2$, then $\psi_2(\varphi(a)) = (\psi_2\varphi)(a)$ is invertible in D_2.

   (b) For any $x \in D_2$

$$\begin{aligned}
x &= \psi_2(a)\psi_2(b)^{-1} \\
&= (\psi_2(\varphi(\varphi^{-1}(a))))(\psi_2(\varphi(\varphi^{-1}(b))))^{-1} \\
&= (\psi_2\varphi)(\varphi^{-1}(a))(\psi_2\varphi)(\varphi^{-1}(b))^{-1}.
\end{aligned}$$

   (c) If $\psi_2\varphi(a) = 0$, then $\varphi(a)\cdot b = 0$ where $0 \neq b \in S_2$, thus $\varphi(a)\varphi(\varphi^{-1}(b)) = \varphi(a\varphi^{-1}(b)) = 0$, therefore $a\varphi^{-1}(b) = 0$.

Hence $(D_2,\psi_2\varphi)$ is a quotient ring of R_1 with respect to S_1. Thus $\sigma : D_1 \cong D_2$ where $\sigma(\psi_1(a)\psi_1(b)^{-1}) = \psi_2\varphi(a)(\psi_2\varphi(b))^{-1}$, that is φ can extend to σ.

6. Assume that $A \nsubseteq \bigcup_{\substack{i=1 \\ i\neq i_0}}^{n} I_i$ for any $1 \le i_0 \le n$, then there is $a_{i_0} \in A \setminus \bigcup_{\substack{i=1 \\ i\neq i_0}}^{n} I_i$, thus $a \in I_{i_0}$. If $i = 2$, $A = (A\cap I_1)\cup(A\cap I_2)$. If $A\cap I_1 \subseteq A\cap I_2$, then $A = A\cap I_2$, therefore $A \subseteq I_2$; similarly, if $A\cap I_2 \subseteq A\cap I_1$, then $A \subseteq I_1$. If $A\cap I_1 \nsubseteq A\cap I_2$ and $A\cap I_2 \nsubseteq A\cap I_1$, then there are $a \in (A\cap I_1)\setminus(A\cap I_2)$ and $(A\cap I_2)\setminus(A\cap I_1)$, thus $a+b \in A \subseteq I_1\cup I_2$. If $a+b \in I_1$, then $b \in I_1$, it is a contradiction; if $a+b \in I_2$, then $a \in I_2$, it is a contradiction. If $n > 2$, we can assume that $I_k, k > 2$ are prime ideals, then $a_3 + a_1a_2a_4\cdots a_n \in A =$

$\bigcup_{i=1}^{n} I_i$, thus $a_3 + a_1a_2a_4\cdots a_n \in I_j$. When $j \neq 3$, then $a_1a_2a_4\cdots a_n \in I_j$, therefore $a_3 \in I_j$, it is a contradiction; when $j = 3$, then $a_3 \in I_3$, therefore $a_1a_2a_4\cdots a_n \in I_3$, since I_3 is prime, then there is $a_{i_0} \in I_3$ where $i_0 \neq 3$, it is a contradiction. Hence $A \subseteq I_i$ for some i.

7. Let $S = \{a \in R | a \text{ is not a zero} - \text{divisor}\}$, $\Omega = \{P \lhd | P \cap S = \emptyset\}$, then $\Omega \neq \emptyset$ for $0 \in \Omega$. If $P_1 \subseteq \cdots \subseteq P_n \subseteq \cdots$ in Ω, then $P = \bigcup P_i \in \Omega$. According to Zorn's Lemma, there is a maximal element $P \in \Omega$, then P is a prime ideal. In fact, assume that $ab \in P$ where $a, b \in R$, if $a \notin P$ and $b \notin P$, then $(P + Ra) \cap S \neq 0$ and $(P + Rb) \cap S \neq 0$, therefore there are $s_1 = p_1 + r_1a \in S$ and $s_2 = p_2 + r_2b \in S$ where $p_i \in P$. Thus $s_1s_2 = p_1p_2 + p_1r_2b + p_2r_1a + r_1r_2ab \in P \cap S$, it is a contradiction. Therefore $a \in P$ or $b \in P$, hence P is a prime ideal. While $P \subseteq S \cup B = R$ and $S \cap B = \emptyset$, then $P \subseteq B$.

8. Similarly method of Exercise 2.6.7.

9. Let P is a maximal ideal and A, B be two left ideals such that $AB \subseteq P$, then $(AR)(BR) \subseteq ABR \subseteq P$, therefore $AR + P \supseteq P$ and $BR + P \supseteq P$ are ideals of R. If $A \nsubseteq P$ and $B \nsubseteq P$, then $R = AR + P = BR + P$, therefore $R = R^2 = (AR + P)(BR + P) \subseteq P$, it is a contradiction. Hence P is prime.

10. $\Rightarrow$: If $p|ab$, then $ab = rp \in Rp$, therefore $a \in Rp$ or $b \in Rp$, that is $a|p$ or $b|p$. Since Rp is a prime ideal, $Rp \neq R$, then p is not a unit. If p is not a zero-divisor, then p is a prime element.

    $\Leftarrow$: If p is a prime element, then $Rp \neq R$. For any $a, b \in R$, if $ab \in Rp$, then $p|ab$, therefore $p|a$ or $p|b$, that is $a \in Rp$ or $b \in Rp$. Hence Rp is a prime ideal.

11. For any non zero-divisor $r \in R$, then
$$\begin{tikzcd} Rr \arrow[r, "\lambda"] \arrow[d, "g"'] & R \arrow[dl, "f"] \\ R & \end{tikzcd}$$
such that $f \circ \lambda = f$ where $g(ar) = a$(Since $a_1r = a_2r$, $(a_1 - a_2)r = 0$, then $a_1 = a_2$), thus $a = g(r) = f(r) = rf(1)$, therefore there is $a \in R$ such that $1 = ra$. Similarly there is $b \in R$ such that $br = 1$, then $a = 1 \cdot a = b(ra) = b$. Hence R is a ring of quotient.

12. (1)
$$\begin{aligned} Hom_{R[S^{-1}]}(M[S^{-1}], N) &= Hom_{R[S^{-1}]}(M \otimes_R R[S^{-1}], N) \\ &\cong Hom_R(M, Hom_{R[S^{-1}]}(R[S^{-1}], N)) \\ &\cong Hom_R(M, N). \end{aligned}$$

(2) Assume that M is an injective right $R[S^{-1}]$-module, $f : L \to N$ is a right R-module monomorphism, since ${}_RR[s^{-1}]$ is flat, $f \otimes id : L \otimes_R R[S^{-1}] \to N \otimes_R R[S^{-1}]$ is a monomorphism. As M is an injective right $R[S^{-1}]$-module,

$$\begin{tikzcd}
Hom_{R[S^{-1}]}(N[S^{-1}], M) \arrow[r] \arrow[d, "\cong"'] & Hom_{R[S^{-1}]}(L[S^{-1}], M) \arrow[r] \arrow[d, "\cong"] & 0 , \\
Hom_R(N, M) \arrow[r] & Hom_R(L, M) \arrow[r] & 0
\end{tikzcd}$$

thus M_R is an injective module. Conversely, If $R[S^{-1}]$-module M is injective as a right R-module, for any $0 \longrightarrow L \longrightarrow M$ right $R[S^{-1}]$-module monomorphism, then

$$\begin{tikzcd}
& 0 & 0 \\
0 \arrow[r] & L \otimes_{R[S^{-1}]} R[S^{-1}] \arrow[r, "f_1"] \arrow[u] & N \otimes_{R[S^{-1}]} R[S^{-1}] \arrow[u] \\
& L \otimes_R R[S^{-1}] \arrow[r, "f_2"] \arrow[u, "\varphi_1"] & N \otimes_R R[S^{-1}] \arrow[u, "\varphi_2"']
\end{tikzcd} .$$

Consider
$$\begin{tikzcd}
Hom_{R[S^{-1}]}(N[S^{-1}], M) \arrow[r, "f_2^*"] & Hom_{R[S^{-1}]}(L[S^{-1}], M) \arrow[r] & 0 , \\
Hom_{R[S^{-1}]}(N, M) \arrow[r, "f_1^*"'] \arrow[u, "\varphi_2^*"] & Hom_{R[S^{-1}]}(L, M) \arrow[u, "\varphi_1^*"'] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

then $\varphi_1^*(g) = g\varphi_1 \in Hom_{R[S^{-1}]}(L[S^{-1}], M)$, $\forall g \in Hom_{R[S^{-1}]}(L, M)$. Since f_2^* is epimorphic, there is $h \in Hom_{R[S^{-1}]}(N[S^{-1}], M)$ such that $g\varphi_1 = hf_2 = f_2^*(h)$. Then we have $h\varphi_2^{-1} \in Hom_{R[S^{-1}]}(N, M)$ such that $\varphi_1^* f_1^*(h\varphi_2^{-1}) = f_2^* \varphi_2^*(h\varphi_2^{-1}) = hf_2 = \varphi_1^*(g)$, since φ_1^* is monomorphic, thus $f_1^*(h\varphi_2^{-1}) = g$, that f_1^* is epimorphic. Hence M is injective.

13. If $x \in Ker(\mu_M)$, then $(1, x) = 0 = (s_1, 0)$ where $s_1 \in S$ and $0 \in M$, therefore there is $c \in S$ and $d \in R$ such that $cx = 0$, thus $Ker(\mu_M) \subseteq \{x \in M | sx = 0 \; for \; some \; s \in S\}$. Contrary, if $sx = 0$ for $s \in S$ and $x \in M$, then $(1, x) = (s, 0) = 0$. Hence $Ker(\mu_M) = \{x \in M | sx = 0 \; for \; some \; s \in S\}$.

14. $\Leftarrow$: For any submodule N of M, then $0 \to N_P \to M_P$ for $0 \to N \to M$. Since $M_P = 0$, $N_P = 0$. If $M \neq 0$, then there is $0 \neq m \in M$ such that $N = Rm \leqslant M$, therefore $(Rm)_P = 0$ for any maximal ideal P.

While $I = \{r \in R | rm = 0\} \lhd R$, then there is P_1 such that $I \subseteq P_1$ and $(Rm)_{P_1} = 0$, therefore $m \in Ker(\mu)$ where $\mu : Rm \to [(R \setminus P)^{-1}]Rm$, $\mu(x) = (1, x)$. According to Exercise 2.6.13, there is $s \in R \setminus P$ such that $sm = 0$, then $s \in I \subseteq P$, it is a contradiction. Hence $M = 0$.

$\Rightarrow$: It is obvious.

15. Consider $0 \longrightarrow M_1 \cap M_2 \longrightarrow M_1 \oplus M_2 \longrightarrow M_1 + M_2 \longrightarrow 0$, then

$$\begin{tikzcd}
0 \arrow[r] & R[S^{-1}] \otimes_R (M_1 \cap M_2) \arrow[r] \arrow[d] & R[S^{-1}] \otimes_R (M_1 \oplus M_2) \arrow[r] \arrow[d, "\cong"] & R[S^{-1}] \otimes_R (M_1 + M_2) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & {[S^{-1}]M_1 \cap [S^{-1}]M_2} \arrow[r] & {[S^{-1}]M_1 \oplus [S^{-1}]M_2} \arrow[r] & {[S^{-1}]M_1 + [S^{-1}]M_2} \arrow[r] & 0
\end{tikzcd}$$

Since $(s, x) + (t, y) = (cs, cx + dy)$ where $cs = dt$, $[S^{-1}]M_1 + [S^{-1}]M_2 \subseteq [S^{-1}](M_1 + M_2)$. As $(s, x+y) = (s, x) + (s, y)$ where $x \in M_1, y \in M_2, s \in S$, $[S^{-1}]M_1 + [S^{-1}]M_2 = [S^{-1}](M_1 + M_2)$. According to the diagram above, $[S^{-1}]M_1 \cap [S^{-1}]M_2 \cong R[S^{-1}] \otimes_R (M_1 \cap M_2)$.