2 Modules

2.4 Homological dimensions and semisimple rings

1.

$$\begin{tikzcd}
P_{n+1} \arrow[r] & P_n \arrow[r, "d_n"] \arrow[d, "\varphi_n"'] & \cdots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\varphi_1"] & P_0 \arrow[r, "d_0"] \arrow[d, dashed, "\varphi_0"'] \arrow[dr, dotted, "\varphi d_0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
P'_{n+1} \arrow[r] & P'_n \arrow[r, "d'_n"] & \cdots \arrow[r, "d'_3"] & P'_2 \arrow[r, "d'_2"] & P'_1 \arrow[r, "d'_1"] & P'_0 \arrow[r, "d'_0"] & M' \arrow[r] & 0
\end{tikzcd}$$

Since P_0 is projective and d'_0 is surjective, there is φ_0 such that $d'_0\varphi_0 = \varphi d_0$. $d'_0\varphi_0 d_1 = d_1 d_0 \varphi = 0$, then $Im\varphi_0 d_1 \subseteq Kerd'_0 = Imd'_1$, thus $\varphi_0 d_1 : P_1 \to Imd'_1$. While $P'_1 \xrightarrow{d'_1} Imd'_1 \longrightarrow 0$ is surjective, there is $\varphi_1 : P_1 \to P'_1$ such that $d'_1\varphi_1 = d_1\varphi_0$. If there is $\varphi_n : P_n \to P'_n$ such that $d'_n\varphi_n = \varphi_{n-1}d_n$, then $d'_n\varphi_n d_{n+1} = \varphi_{n-1}d_n d_{n+1} = 0$, thus there is $Im\varphi_n \circ d_{n+1} \subseteq Kerd'_n = Imd'_{n+1}$. Since $d'_{n+1} : P'_{n+1} \to Imd'_{n+1}$ is surjective and P_{n+1} is projective, there is $\varphi_{n+1} : P_{n+1} \to P'_{n+1}$ such that $d'_{n+1}\varphi_{n+1} = \varphi_n d_{n+1}$.

2.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "d_0"] \arrow[d, "\varphi"'] \arrow[dr, dotted, "d'_0\varphi"] & E_0 \arrow[r, "d_1"] \arrow[d, dashed, "\varphi_0"] & E_1 \arrow[r, "d_2"] \arrow[d, "\varphi_1"] & E_2 \arrow[r, "d_3"] & \cdots \arrow[r, "d_n"] & E_n \arrow[r] \arrow[d, "\varphi_n"'] & P_{n+1} \\
0 \arrow[r] & M' \arrow[r, "d'_0"] & E'_0 \arrow[r, "d'_1"] & E'_1 \arrow[r, "d'_2"] & E'_2 \arrow[r, "d'_3"] & \cdots \arrow[r, "d'_n"] & E'_n \arrow[r] & P'_{n+1}
\end{tikzcd}$$

Since E'_0 is injective and d_0 is injective, there is φ_0 such that $\varphi_0 d_0 = d'_0\varphi$. $d'_1\varphi_0 d_0 = d'_1 d'_0 \varphi = 0$, then $\varphi_0 Kerd_1 = Im\varphi_0 d_0 \subseteq Kerd'_1$, thus $d'_1\varphi_0 : Kerd_1 \to E'_1$. While $i : Kerd_1 \to E_1$ is a embedding, there is $\varphi_1 : E_1 \to E'_1$ such that $d'_1\varphi_1 = d_1\varphi_0$. If there is $\varphi_n : P_n \to P'_n$ such that $\varphi_n d_n = d'_n\varphi_{n-1}$, then $d'_{n+1}\varphi_n d_n = d'_{n+1}d'_n\varphi_{n-1} = 0$, thus there is $\varphi_n Kerd_{n+1} = Im\varphi_n \circ d_n \subseteq Kerd'_{n+1}$. Since $i : Kerd_{n+1} \to E_{n+1}$ is a embedding and E'_{n+1} is injective, there is $\varphi_{n+1} : E_{n+1} \to E'_{n+1}$ such that $\varphi_{n+1}d_{n+1} = d'_{n+1}\varphi_n$.

3. According to Theorem 2.2.5, $T^m \cong T^n$ as R-module if and only if $m = n$.

4. If R is a division, according to Theorem 2.2.4, every left R-module is free. Conversely, if every left R-module is free, then R is semisimple, thus $R = M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r)$ where D_i is a division. If $r > 1$, since $(a_1, \cdots, a_{r-1})(0, \cdots, 0, a_r) = 0$ where $a_i \in M_{n_i}(D_i)$, $M_{n_r}(D_r)$ is not a free R-module, thus $r = 1$, i.e. $R = M_{n_1}(D_1)$. If $n_1 > 1$, similarly to Example 2.3.1, we can proof that $M = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_{n_1} \end{pmatrix} \middle| a_i \in D_i \right\}$ is not a free R-module, thus $n_1 = 1$, i.e. $R = D_1$ is a division.

5. (1) $Ann(M) \neq \emptyset$ for $0 \in Ann(M)$.

   (2) For any $a, b \in Ann(M)$, then $aM = bM = 0$, thus $(a-b)m = 0$, hence $a - b \in Ann(M)$.

   (3) For any $a \in Ann(M)$ and any $r \in R$, $a(rm) \subset aM = 0$ and $(ra)m = r(am) = r\dot{0} = 0$, thus $ar \in Ann(M)$ and $ra \in Ann(M)$.

   Hence $Ann(M)$ is an ideal of R.

6. Suppose ${}_RS$ is simple and $Ann(S) = 0$, then $D_1 = End_R(S)$ is a division and ${}_RS_{D_1^{OP}}$ is a bimodule where D_1^{OP} is a division. Suppose $D = D_1^{OP}$. For any $r \in R$, let $l_r : S \to S$, $l_r(v) = rv$, it is obvious that l_r is a D-linear map. Let $\varphi : R \to End_D(S)$, $\varphi(r) = l_r$. For any $v \in S$, $l_{r_1+r_2}(v) = r_1v + r_2v = (l_{r_1} + l_{r_2})(v)$, then $\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2)$. For any $r_1, r_2 \in R$ and any $v \in S$, $l_{r_1r_2}(v) = r_1r_2v = (l_{r_1} \circ l_{r_2})(v)$, then $l_{r_1r_2} = l_{r_1} \circ l_{r_2}$, i.e. $\varphi(r_1r_2) = \varphi(r_1)\varphi(r_2)$. If $r \in Ker\varphi$, then $l_r(v) = rv = 0$ for any $v \in S$, thus $r \in Ann(S) = 0$. Hence φ is a monomorphism of rings.

7. Since R contains identity, there is a maximal ideal M, then R/M is a simple R-module.

   (1) Let $I = \{a \in R | a(R/M) = 0\} = \{a \in R | aR \subseteq M\} = Ann(R/M)$ is an ideal of R. Let $R/I \times R/M \to R/M$, $(a + I, b + M) \mapsto ab + M$. If $(a_1 + I, b_1 + M) = (a_2 + I, b_2 + M)$, then $a_1 - a_2 \in I, b_1 - b_2 \in M$, thus $a_1b_1 + M = a_1(b_1 - b_2) + a_1b_2 + M = (a_1 - a_2)b_2 + a_2b_2 + M = a_2b_2 + M$, hence the above action is well-defined. It is obvious that R/M also is a R/I simple module and faithful module. Therefore R/I is a left semiprimitive ring.

   (2) $R = M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r)$, let $T_i = \left\{ \left(\begin{array}{c} a_1 \\ \vdots \\ a_{n_1} \end{array} \right) \middle| a_i \in D_i \right\}$, then $S = T_1 \oplus \cdots \oplus T_r$ is a faithful semisimple left R-module.

   (3) $\mathbb{Z}$ is a left semiprimitive ring, $S = \sum\limits_{p \text{ } is \text{ } primitive} \oplus \mathbb{Z}_p$, for any $a \in Ann(S)$, then $p \mid a$ for any prime number, thus $a = 0$.

8. Let S_i be faithful semisimple R/I_i-module, then $S_1 \oplus S_2$ is a semisimple $R/(I_1 \cap I_2)$-module and $Ann(S_1 \oplus S_2) = Ann(S_1) \cap Ann(S_2) = 0$, thus $R/(I_1 \cap I_2)$ is left semiprimitive.

   Let $\Omega = \{I \text{ } is \text{ } an \text{ } ideal \text{ } of \text{ } R \mid R/I \text{ } is \text{ } left \text{ } semiprimitive\}$, define $I_1 \leq I_2$ if $I_1 \supseteq I_2$. According to Exercise 2.4.7(1), $\Omega \neq \emptyset$. If there is a ascending chain $I_1 \leq I_2 \leq \cdots \leq \cdots$ in Ω, let S_i is a faithful semisimple R/I_i-module, then $S = \sum_{i=1}^{\infty} \oplus S_i$ is a simisimple module. $Ann(S) = \cap Ann(S_i) = \cap_{i=1}^{\infty} I_i \geq I_j (\forall j)$, then $Ann(S) \in \Omega$. By Zorn's Lemma, there is a maximal element J such that R/J is a left semiprimitive ring. If there is another I such that R/I is a left semiprimitive ring, then $R/(J \cap I)$ is a left semiprimitive ring. Thus $J \cap I \geq J$, this means that $J \cap I = J$, i.e. $J \subseteq I$. Hence J is a minimal ideal such that R/J is a left semiprimitive ring.

9. Suppose J is the Jacobson radical of R, then there is a faithful semisimple left R/J-module $T = \sum\limits_{i\in\Lambda} \oplus S_i$. It is obvious that T is a semisimple R-module. $J = Ann(T) = Ann(\sum\limits_{i\in\Lambda} \oplus S_i) = \bigcap\limits_{i\in\Lambda} Ann(S_i)$ and S_i is a faithful simple $R/Ann(S_i)$-module, then $Ann(S_i)$ is a left primitive ideal, thus J is the intersection of some left primitive ideal. For any left primitive ideal P, since $R/(J \cap P)$ also is a left semiprimitive ring, $J \cap P = J$, then $J \subseteq P$, thus $J \subseteq \cap\{P \mid P\ is\ a\ left\ primitive\ ideal\} \subseteq \bigcap\limits_{i\in\Lambda} Ann(S_i) = J$.

10. For any $z \in J$, if $R(1-z) \neq R$, then there is a maximal left ideal M such that $R(1-z) \subseteq M \subseteq R$, then $1 - z \in M$, while $z \in J \subseteq M$, then $1 \in M$, thus $M = R$, it is contradiction. Hence $R(1-z) = R$, i.e. $z \in J$ is left quasi-regular.

11. For any $z \in J$, if $R(1-az) \neq R$ for some $a \in R$, then there is a maximal left ideal M such that $R(1-az) \subseteq M \subseteq R$. Considering simple module R/M, $az(R/M) = 0$, then $az \in M$, while $1 - az \in M$, thus $1 \in M$, hence $M = R$, it is contradiction. Therefore $R(1-az) = R$, i.e. az is left quasi-regular. Conversely, if az is left quasi-regular for every $a \in R$ and $z \notin J$, then there is a simple module R/M such that $z(R/M) = zR + M/M$. In particular, $z \in M$, Since M is a maximal module, $Rz + M = R$, then there is $a \in R, m \in M$ such that $az + m = 1$. Since $Rm = R(1-az) = R$, $M = R$, it is contradiction. Hence $z \in J$.

12. Suppose $M = Rm_1 + \cdots + Rm_r$ is a finitely generated left R-module, if $M \neq 0$, there is a maximal submodule N such that M/N is simple, then $0 = J(M/N) = JM + N/N = M/N$,it is contradiction. Hence $M = 0$.

13. Suppose P is a primitive ideal of commutative ring R, then R/P is a primitive ring, therefore there is a faithful simple R/P-module M. Let $Q \lhd R$ is a maximal ideal, then $(Q/P)M$ is a proper submodule of M. Since M is simple, $(Q/P)M = 0$, then $Q/P \subseteq Ann(M) = 0$, hence $Q = P$. Thus P is a maximal ideal. Conversely, If P is a maximal ideal of R, then R/P is a faithful simple R/P-module, hence P is a primitive ideal.

14. For any $g \in G$, $gc_ig^{-1} = \sum\limits_{x\in C_i} gxg^{-1} = \sum\limits_{x\in C_i} x = c_i$, then $gc_i = c_ig$, thus $c_1, \cdots, c_r \in C(F[G])$. For any $a = k_1g_1 + \cdots + k_ng_n \in C(F[G])$ where $n = |G|$, then $a = a_1 + \cdots + a_r$ where $a_i = \sum\limits_{x_i\in C_i} k_ix_i$. For any $g \in G$, $gag^{-1} = a$, then $ga_ig^{-1} = \sum\limits_{x_i\in C_i} k_igx_ig^{-1} = a_i$. Since $C_i = \{gxg^{-1} | g \in G\}$ for any $x \in C_i$, $a_i = \sum\limits_{x_i\in C_i} k_ix_i = u_ic_i$, hence $a \in Span\{c_1, \cdots, c_r\}$.

15. (1) For any left ideal I of $M_n(D)$, then I is a left vector space over D, $D \times I \to I$, $(a, A) \mapsto aE\dot{A}$, $M_n(D)$ is a left vector space of dimension n^2 over D. If $I_1 \geq I_2$ where I_1, I_2 are left module, then $dim(I_1) \geq dim(I_2)$. Thus $M_n(D)$ is a left Artinian ring.

    (2) If $R = M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r)$, I is a left ideal of R, then $I = I_1 \oplus \cdots \oplus I_r$ where $I_i \lhd M_{n_i}(D_i)$. According to (1), $M_{n_i}(D_i)$ is a left Artinian ring, thus R is a left Artinian ring. Let $S_i = \left\{ \left(\begin{array}{c} a_1 \\ \vdots \\ a_{n_i} \end{array} \right) \middle| a_i \in D_i \right\}$, then $S_1 + \cdots + S_r$ is a faithful semisimple module, Thus R is semiprimitive.

    Conversely, let $\Omega = \{I \lhd R | I\ is\ the\ intersection\ of\ finite\ maximal\ left\ ideal\}$, then $\Omega \neq \emptyset$. Define $I_1 \leq I_2$ if $I_1 \supseteq I_2$. For any ascending chain $I_1 \leq I_2 \leq \cdots \leq I_n \leq \cdots$, then $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$, since R is left Artinian, there is n such that $I_n = I_{n+1} = \cdots$, then $I_n \geq I_i$ for any i. By Zorn's Lemma, there is a minimal element $I_0 \in \Omega$, for any maximal left ideal M, $I_0 \cap M = I_0$, thus I_0 is intersection of all maximal left ideal, i.e. Jacobson radical. Suppose $I_0 = M_1 \cap \cdots \cap M_n = 0$ where M_i be maximal left ideal. Let $\varphi : R \to R/M_1 \oplus \cdots \oplus R/M_n$, $\varphi(r) = (r + M_1, \cdots, r + M_n)$, then φ is a monomorphism of R-module. While $R/M_1 \oplus \cdots \oplus R/M_n$ is a simple module, $R \cong Im\varphi$ is a simple module. Hence R is a semisimple ring.

16. $J^n = J^{n+1} = \cdots$ since R is Artinian. Suppose $J^n \neq 0$ and $I = J^n$, then $I^2 = I$, $\{Ia \neq 0 | a \neq 0\}$ has a minimal element Ia. Since $I^2 a = Ia \neq 0$, there is $b \in I$ such that $Iba \neq 0$ and $Iba \subseteq Ia$. By the minimality of Ia, we have $Iba = Ia$, then $ba \in Ia = Iba$, thus there is $c \in I$ such that $cba = ba$, hence $(1 - c)ba = 0$. Since $c \in J$, $R(1 - c) = R$, then$ba = 0$, it is contradiction. Therefore $I = 0$.

17. Considering $\mathbb{C}[G] = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$, since $dim_{\mathbb{C}}(\mathbb{C}[G]) = r$ where r is the number of conjugate classes, $n_1^2 + n_2^2 + \cdots + n_r^2 = |G|$. Simple module of $\mathbb{C}[G]$ are: $T_i = \left\{ \left(\begin{array}{c} a_1 \\ \vdots \\ a_{n_i} \end{array} \right) \middle| a_i \in D_i \right\}$ $(i = 1, \cdots, r)$. If $G = S_3$, the conjugate classes are: $\{(1)\}, \{(12), (13), (23)\}, \{(123), (132)\}$, then there are three irreducible representation, and $n_1^2 + n_2^2 + n_3^2 = 6$, thus $n_1 = n_2 = 1, n_3 = 2$. The submodule of dimension 1 of $\mathbb{C}[G]$ is $V = C$ and $\mathbb{C}[G] \to End_{\mathbb{C}}\mathbb{C} = \mathbb{C}^*$ is a homomorphism, then there are two homomorphisms of group from S_3 to $\mathbb{C}^*$: $\sigma \mapsto 1$ and $\sigma \mapsto sgn\sigma$. Hence we get the table:

	$\overline{(1)}$	$\overline{(12)}$	$\overline{(123)}$
r_1	1	1	1
r_2	1	-1	1
r_3	2	0	-1