题3.2

求矩阵函数$\boldsymbol{AX}^{-1}\boldsymbol{B}$和$\boldsymbol{AXB}$的Jacobian矩阵。

对于$\boldsymbol{AXB}$，方法一：
采用Jacobian矩阵的定义（教材P145）

$$D_{\boldsymbol{X}}\boldsymbol{F}(\boldsymbol{X})=\frac{\partial \mathrm{vec}(\boldsymbol{F}(\boldsymbol{X}))}{\partial \mathrm{vec}(\mathbf{X})^{\top}}=\begin{bmatrix}\frac{\partial f_{11}}{\partial x_{11}} & \frac{\partial f_{11}}{\partial x_{21}} & \cdots & \frac{\partial f_{11}}{\partial x_{mn}}\\ \frac{\partial f_{21}}{\partial x_{11}} & \frac{\partial f_{21}}{\partial x_{21}} & \cdots & \frac{\partial f_{21}}{\partial x_{mn}}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_{pq}}{\partial x_{11}} & \frac{\partial f_{pq}}{\partial x_{21}} & \cdots & \frac{\partial f_{pq}}{\partial x_{mn}}\end{bmatrix}$$

假定$\boldsymbol{A}\in\mathbb{R}^{p\times m}$,$\boldsymbol{X}\in\mathbb{R}^{m\times n}$,$\boldsymbol{B}\in\mathbb{R}^{n\times q}$

$$\frac{\partial f_{kl}}{\partial x_{ij}}=\frac{\partial(\boldsymbol{AXB})_{kl}}{\partial x_{ij}}=\frac{\partial(\sum_{u=1}^{m}\sum_{v=1}^{n}a_{ku}x_{uv}b_{vl})}{\partial x_{ij}}=a_{ki}b_{jl}$$

所以

$$D_{\boldsymbol{X}}(\boldsymbol{AXB})=\boldsymbol{B}^{T}\otimes\boldsymbol{A}$$

对于$\boldsymbol{AXB}$，方法二：
教材P162结论：

$$d\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{A}(d\boldsymbol{X})\boldsymbol{B} + \boldsymbol{C}(d\boldsymbol{X}^T)\boldsymbol{D}$$

$$\Longleftrightarrow D_{\boldsymbol{X}}\boldsymbol{F}(\boldsymbol{X}) = (\boldsymbol{B}^T \otimes \boldsymbol{A}) + (\boldsymbol{D}^T \otimes \boldsymbol{C})\boldsymbol{K}_{mn}$$

$$d(\boldsymbol{AXB}) = \boldsymbol{A}(d\boldsymbol{X})\boldsymbol{B}$$

对于$\boldsymbol{AX}^{-1}\boldsymbol{B}$，见教材P153

$$d(\boldsymbol{X}^{-1}) = -\boldsymbol{X}^{-1}(d\boldsymbol{X})\boldsymbol{X}^{-1}$$

那么

$$d(\boldsymbol{AX}^{-1}\boldsymbol{B}) = \boldsymbol{A}(d\boldsymbol{X}^{-1})\boldsymbol{B} = -\boldsymbol{AX}^{-1}(d\boldsymbol{X})\boldsymbol{X}^{-1}\boldsymbol{B}$$

所以

$$\boldsymbol{D_x}(\boldsymbol{AX}^{-1}\boldsymbol{B}) = -(\boldsymbol{X}^{-1}\boldsymbol{B})^T \otimes (\boldsymbol{AX}^{-1})$$