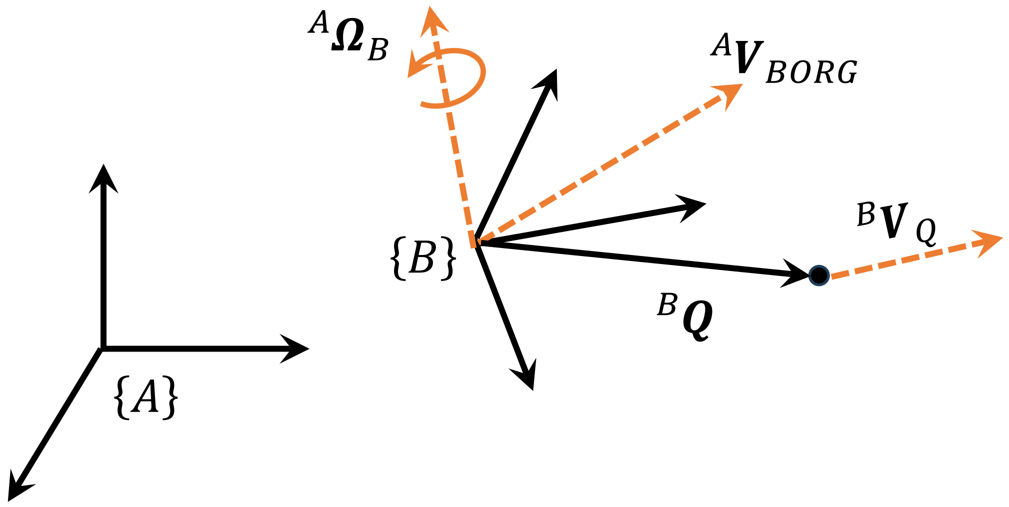
机器人建模与控制

第7章 机器人动力学

7.1 速度和加速度的传递

7.1.1 线速度和角速度的传递

- 点$\boldsymbol{Q}$以线速度${}^{B}\boldsymbol{V}_{Q}$相对于坐标系$\{B\}$运动
- $\{B\}$的原点以线速度${}^{A}\boldsymbol{V}_{BORG}$相对于坐标系$\{A\}$运动
- $\{B\}$以角速度${}^{A}\boldsymbol{\Omega}_{B}$绕坐标系$\{A\}$运动



- 线速度的传递关系为：

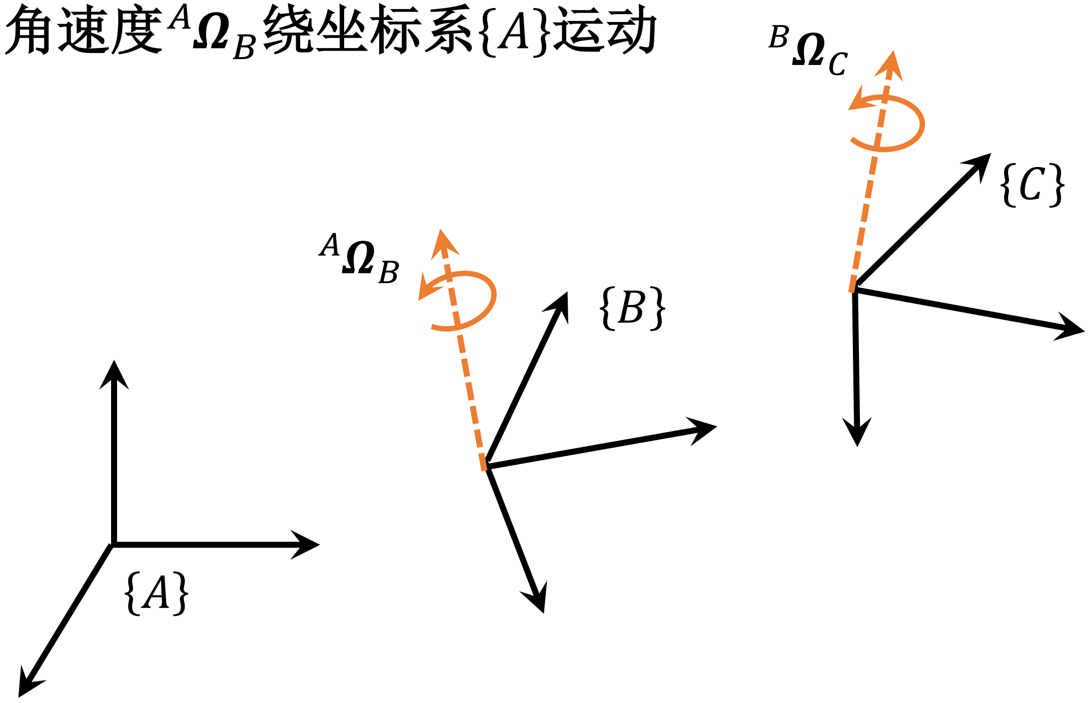
$$ {}^{A}\boldsymbol{V}_{Q} = {}^{A}\boldsymbol{V}_{BORG} + {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{V}_{Q} + {}^{A}\boldsymbol{\Omega}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q} $$

注意：需要用到角速度${}^{A}\boldsymbol{\Omega}_{B}$

7.1 速度和加速度的传递

7.1.1 线速度和角速度的传递

- 坐标系$\{C\}$以角速度${}^{B}\boldsymbol{\Omega}_{C}$绕坐标系$\{B\}$运动
- $\{B\}$以角速度${}^{A}\boldsymbol{\Omega}_{B}$绕坐标系$\{A\}$运动



- 坐标系$\{C\}$绕坐标系$\{A\}$运动的角速度为：

$$ {}^{A}\boldsymbol{\Omega}_{C} = {}^{A}\boldsymbol{\Omega}_{B} + {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{\Omega}_{C} $$

7.1.2 线加速度的传递

- 线加速度的传递可通过对线速度传递关系式的求导获得

- 特殊情况：坐标系$\{A\}$的原点和坐标系$\{B\}$的原点重合，有：

$$^{A}\boldsymbol{V}_Q = {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{V}_Q + {}^{A}\boldsymbol{\Omega}_B \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q}$$

求导

$$^{A}\dot{\boldsymbol{V}}_Q = \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{V}_Q\right) + {}^{A}\dot{\boldsymbol{\Omega}}_B \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q} + {}^{A}\boldsymbol{\Omega}_B \times \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q}\right)$$

- 注意到：$^{A}\boldsymbol{V}_Q = \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q}\right) = {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{V}_Q + {}^{A}\boldsymbol{\Omega}_B \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q}$

- 同理有：$\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{V}_Q\right) = {}^{A}_{B}\boldsymbol{R}\,{}^{B}\dot{\boldsymbol{V}}_Q + {}^{A}\boldsymbol{\Omega}_B \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{V}_Q$

$$\begin{aligned}
^{A}\dot{\boldsymbol{V}}_Q &= {}^{A}_{B}\boldsymbol{R}\,{}^{B}\dot{\boldsymbol{V}}_Q + {}^{A}\boldsymbol{\Omega}_B \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{V}_Q + {}^{A}\dot{\boldsymbol{\Omega}}_B \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q} + {}^{A}\boldsymbol{\Omega}_B \times \left({}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{V}_Q + {}^{A}\boldsymbol{\Omega}_B \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q}\right) \\
&= {}^{A}_{B}\boldsymbol{R}\,{}^{B}\dot{\boldsymbol{V}}_Q + 2\,{}^{A}\boldsymbol{\Omega}_B \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{V}_Q + {}^{A}\dot{\boldsymbol{\Omega}}_B \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q} + {}^{A}\boldsymbol{\Omega}_B \times \left({}^{A}\boldsymbol{\Omega}_B \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q}\right)
\end{aligned}$$

7.1 速度和加速度的传递

7.1.2 线加速度的传递

- 一般情况：如果坐标系$\{A\}$的原点和坐标系$\{B\}$的原点不重合
- 需加上$\{B\}$原点的线加速度

$$^{A}\dot{\boldsymbol{V}}_{Q} = {}^{A}\dot{\boldsymbol{V}}_{BORG} + {}^{A}_{B}\boldsymbol{R}\,{}^{B}\dot{\boldsymbol{V}}_{Q} + 2\,{}^{A}\boldsymbol{\Omega}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{V}_{Q} + {}^{A}\dot{\boldsymbol{\Omega}}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q} + {}^{A}\boldsymbol{\Omega}_{B} \times \left({}^{A}\boldsymbol{\Omega}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q}\right)$$

- 如果矢量$^{B}\boldsymbol{Q}$保持不动
- 即： $^{B}\boldsymbol{V}_{Q} = \boldsymbol{0}, {}^{B}\dot{\boldsymbol{V}}_{Q} = \boldsymbol{0}$

$$^{A}\dot{\boldsymbol{V}}_{Q} = {}^{A}\dot{\boldsymbol{V}}_{BORG} + {}^{A}\dot{\boldsymbol{\Omega}}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q} + {}^{A}\boldsymbol{\Omega}_{B} \times \left({}^{A}\boldsymbol{\Omega}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q}\right)$$

7.1.3 角加速度的传递

- 类似的，角加速度的传递关系可以通过对角速度传递关系式求导得到

$$ {}^{A}\boldsymbol{\Omega}_{C} = {}^{A}\boldsymbol{\Omega}_{B} + {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{\Omega}_{C} $$

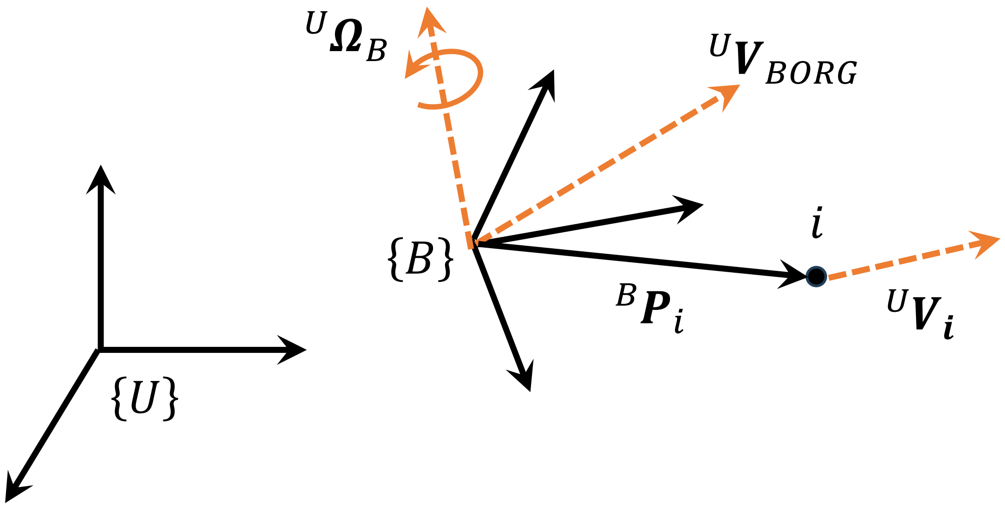
求导

$$ {}^{A}\dot{\boldsymbol{\Omega}}_{C} = {}^{A}\dot{\boldsymbol{\Omega}}_{B} + \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{\Omega}_{C}\right) $$

$$ \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{\Omega}_{C}\right) = {}^{A}_{B}\boldsymbol{R}\,{}^{B}\dot{\boldsymbol{\Omega}}_{C} + {}^{A}\boldsymbol{\Omega}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{\Omega}_{C} $$

$$ {}^{A}\dot{\boldsymbol{\Omega}}_{C} = {}^{A}\dot{\boldsymbol{\Omega}}_{B} + {}^{A}_{B}\boldsymbol{R}\,{}^{B}\dot{\boldsymbol{\Omega}}_{C} + {}^{A}\boldsymbol{\Omega}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{\Omega}_{C} $$

7.2 刚体的惯性张量与欧拉方程



- 考虑多个质点连接形成刚体
- 设质点i的质量为m_i，则刚体的总质量$m = \sum_i m_i$
- 考虑该刚体的联体坐标系$\{B\}$。在惯性坐标系$\{U\}$中，有：

$$^{U}\boldsymbol{V}_i = {}^{U}\boldsymbol{V}_{BORG} + {}^{U}\boldsymbol{\Omega}_B \times {}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i$$

- $^{U}\boldsymbol{V}_i$表示质点i在$\{U\}$中的速度

- 对速度$^{U}\boldsymbol{V}_i$求导获得其加速度：

$$^{U}\dot{\boldsymbol{V}}_i = {}^{U}\dot{\boldsymbol{V}}_{BORG} + {}^{U}\dot{\boldsymbol{\Omega}}_B \times {}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i + {}^{U}\boldsymbol{\Omega}_B \times \left({}^{U}\boldsymbol{\Omega}_B \times {}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i\right)$$

- 注意：由于是刚体，$^{B}\dot{\boldsymbol{P}}_i = \boldsymbol{0}$

- 作用在质点i上的力：

$$^{U}(\boldsymbol{f}_i) = m_i\,{}^{U}\dot{\boldsymbol{V}}_i = m_i\left({}^{U}\dot{\boldsymbol{V}}_{BORG} + {}^{U}\dot{\boldsymbol{\Omega}}_B \times {}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i + {}^{U}\boldsymbol{\Omega}_B \times \left({}^{U}\boldsymbol{\Omega}_B \times {}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i\right)\right)$$

7.2 刚体的惯性张量与欧拉方程

- 作用在质点$\boldsymbol{i}$上的力矩：

$$ {}^{U}({}^{B}\boldsymbol{N}_i) = {}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i \times {}^{U}(\boldsymbol{f}_i) = m_i\,{}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i \times \left({}^{U}\dot{\boldsymbol{V}}_{BORG} + {}^{U}\dot{\boldsymbol{\Omega}}_B \times {}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i + {}^{U}\boldsymbol{\Omega}_B \times \left({}^{U}\boldsymbol{\Omega}_B \times {}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i\right)\right) $$

- 作用在整个刚体上的总力矩：

$$ \begin{aligned} {}^{U}({}^{B}\boldsymbol{N}) &= \sum_i {}^{U}({}^{B}\boldsymbol{N}_i) = \sum_i m_i\,{}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i \times \left({}^{U}\dot{\boldsymbol{V}}_{BORG} + {}^{U}\dot{\boldsymbol{\Omega}}_B \times {}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i + {}^{U}\boldsymbol{\Omega}_B \times \left({}^{U}\boldsymbol{\Omega}_B \times {}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i\right)\right) \\ &= \sum_i m_i\,{}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i \times {}^{U}\dot{\boldsymbol{V}}_{BORG} + \sum_i m_i\,{}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i \times \left({}^{U}\dot{\boldsymbol{\Omega}}_B \times {}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i\right) \\ &\quad + \sum_i m_i\,{}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i \times \left({}^{U}\boldsymbol{\Omega}_B \times \left({}^{U}\boldsymbol{\Omega}_B \times {}^{U}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{P}_i\right)\right) \end{aligned} $$

注意叉乘不满足乘法结合律，不能将后面的括号去掉

7.2 刚体的惯性张量与欧拉方程

- 如果将联体坐标系$\{B\}$的原点选在刚体质心上，则有：

$$\sum_i m_i{}^B\boldsymbol{P}_i = \boldsymbol{0}$$

- 为强调联体坐标系原点在刚体质心上这一情况，下面用$\{C\}$替代$\{B\}$

- $\sum_i m_i{}^B\boldsymbol{P}_i = \boldsymbol{0}$，则有$\sum_i m_i{}^U_B\boldsymbol{R}{}^B\boldsymbol{P}_i \times {}^U\dot{\boldsymbol{V}}_{BORG} = \boldsymbol{0}$
- 总力矩可简化为：

$${}^U({}^C\boldsymbol{N}) = \sum_i m_i{}^U_C\boldsymbol{R}{}^C\boldsymbol{P}_i \times \left({}^U\dot{\boldsymbol{\Omega}}_C \times {}^U_C\boldsymbol{R}{}^C\boldsymbol{P}_i\right) + \sum_i m_i{}^U_C\boldsymbol{R}{}^C\boldsymbol{P}_i \times \left({}^U\boldsymbol{\Omega}_C \times \left({}^U\boldsymbol{\Omega}_C \times {}^U_C\boldsymbol{R}{}^C\boldsymbol{P}_i\right)\right)$$

- 计算${}^U({}^C\boldsymbol{N})$在坐标系$\{C\}$中的表示：

$${}^C({}^C\boldsymbol{N}) = {}^C\boldsymbol{N} = {}^C_U\boldsymbol{R}{}^U({}^C\boldsymbol{N})$$

$$= {}^C_U\boldsymbol{R}\left(\sum_i m_i{}^U_C\boldsymbol{R}{}^C\boldsymbol{P}_i \times \left({}^U\dot{\boldsymbol{\Omega}}_C \times {}^U_C\boldsymbol{R}{}^C\boldsymbol{P}_i\right) + \sum_i m_i{}^U_C\boldsymbol{R}{}^C\boldsymbol{P}_i \times \left({}^U\boldsymbol{\Omega}_C \times \left({}^U\boldsymbol{\Omega}_C \times {}^U_C\boldsymbol{R}{}^C\boldsymbol{P}_i\right)\right)\right)$$

7.2 刚体的惯性张量与欧拉方程

- ${}^{C}\boldsymbol{N}$的第一项

$$ {}_{U}^{C}\boldsymbol{R}\left(\sum_{i} m_i\, {}_{C}^{U}\boldsymbol{R}\,{}^{C}\boldsymbol{P}_i \times \left({}^{U}\dot{\boldsymbol{\Omega}}_C \times {}_{C}^{U}\boldsymbol{R}\,{}^{C}\boldsymbol{P}_i\right)\right) = \sum_{i} m_i\, {}_{U}^{C}\boldsymbol{R}\,{}_{C}^{U}\boldsymbol{R}\,{}^{C}\boldsymbol{P}_i \times \left({}_{U}^{C}\boldsymbol{R}\left({}^{U}\dot{\boldsymbol{\Omega}}_C \times {}_{C}^{U}\boldsymbol{R}\,{}^{C}\boldsymbol{P}_i\right)\right) $$

$$ = \sum_{i} m_i\, {}^{C}\boldsymbol{P}_i \times \left({}_{U}^{C}\boldsymbol{R}\left({}^{U}\dot{\boldsymbol{\Omega}}_C \times {}_{C}^{U}\boldsymbol{R}\,{}^{C}\boldsymbol{P}_i\right)\right) = \sum_{i} m_i\, {}^{C}\boldsymbol{P}_i \times \left({}_{U}^{C}\boldsymbol{R}\,{}^{U}\dot{\Omega}_C \times {}_{U}^{C}\boldsymbol{R}\,{}_{C}^{U}\boldsymbol{R}\,{}^{C}\boldsymbol{P}_i\right) $$

$$ = \sum_{i} m_i\, {}^{C}\boldsymbol{P}_i \times \left({}^{C}\left({}^{U}\dot{\boldsymbol{\Omega}}_C\right) \times {}^{C}\boldsymbol{P}_i\right) = \sum_{i} -m_i\, {}^{C}\boldsymbol{P}_i \times \left({}^{C}\boldsymbol{P}_i \times {}^{C}\left({}^{U}\dot{\boldsymbol{\Omega}}_C\right)\right) $$

$$ = \sum_{i} -m_i\, {}^{C}\boldsymbol{P}_i^{\wedge}\, {}^{C}\boldsymbol{P}_i^{\wedge}\, {}^{C}\left({}^{U}\dot{\boldsymbol{\Omega}}_C\right) = \sum_{i} -m_i \left({}^{C}\boldsymbol{P}_i^{\wedge}\right)^2\, {}^{C}\dot{\boldsymbol{\omega}}_C $$

- 遵循之前符号规则，这里用$\dot{\boldsymbol{\omega}}_C = {}^{U}\dot{\boldsymbol{\Omega}}_C$，表示该（联体）质心坐标系在惯性坐标系$\{U\}$中的角加速度

7.2 刚体的惯性张量与欧拉方程

- ${}^{C}\boldsymbol{N}$的第二项

$$ {}^{C}_{U}\boldsymbol{R}\left(\sum_i m_i {}^{U}_{C}\boldsymbol{R}{}^{C}\boldsymbol{P}_i \times \left({}^{U}\boldsymbol{\Omega}_C \times \left({}^{U}\boldsymbol{\Omega}_C \times {}^{U}_{C}\boldsymbol{R}{}^{C}\boldsymbol{P}_i\right)\right)\right) $$

$$ = \sum_i m_i {}^{C}_{U}\boldsymbol{R}{}^{U}_{C}\boldsymbol{R}{}^{C}\boldsymbol{P}_i \times \left({}^{C}_{U}\boldsymbol{R}\left({}^{U}\boldsymbol{\Omega}_C \times \left({}^{U}\boldsymbol{\Omega}_C \times {}^{U}_{C}\boldsymbol{R}{}^{C}\boldsymbol{P}_i\right)\right)\right) $$

$$ = \sum_i m_i {}^{C}\boldsymbol{P}_i \times \left({}^{C}_{U}\boldsymbol{R}{}^{U}\boldsymbol{\Omega}_C \times {}^{C}_{U}\boldsymbol{R}\left({}^{U}\boldsymbol{\Omega}_C \times {}^{U}_{C}\boldsymbol{R}{}^{C}\boldsymbol{P}_i\right)\right) $$

$$ = \sum_i m_i {}^{C}\boldsymbol{P}_i \times \left({}^{C}({}^{U}\boldsymbol{\Omega}_C) \times \left({}^{C}_{U}\boldsymbol{R}{}^{U}\boldsymbol{\Omega}_C \times {}^{C}_{U}\boldsymbol{R}{}^{U}_{C}\boldsymbol{R}{}^{C}\boldsymbol{P}_i\right)\right) $$

$$ = \sum_i m_i {}^{C}\boldsymbol{P}_i \times \left({}^{C}({}^{U}\boldsymbol{\Omega}_C) \times \left({}^{C}({}^{U}\boldsymbol{\Omega}_C) \times {}^{C}\boldsymbol{P}_i\right)\right) $$

$$ = \sum_i -m_i {}^{C}({}^{U}\boldsymbol{\Omega}_C) \times \left({}^{C}\boldsymbol{P}_i \times \left({}^{C}\boldsymbol{P}_i \times {}^{C}({}^{U}\boldsymbol{\Omega}_C)\right)\right) $$

$$ = \sum_i -m_i {}^{C}\boldsymbol{\omega}_C^{\wedge}({}^{C}\boldsymbol{P}_i^{\wedge})^2\, {}^{C}\boldsymbol{\omega}_C $$

- 遵循之前符号规则，这里用$\boldsymbol{\omega}_C = {}^{U}\boldsymbol{\Omega}_C$，表示该（联体）质心坐标系在惯性坐标系$\{U\}$中的角速度

7.2 刚体的惯性张量与欧拉方程

$$^{C}\boldsymbol{N} = \sum_{i} -m_i ({}^{C}\boldsymbol{P}_i^{\wedge})^2 \, {}^{C}\dot{\boldsymbol{\omega}}_C + \sum_{i} -m_i \, {}^{C}\boldsymbol{\omega}_C^{\wedge} ({}^{C}\boldsymbol{P}_i^{\wedge})^2 \, {}^{C}\boldsymbol{\omega}_C$$

记 $^{C}\boldsymbol{I} = \sum_i -m_i ({}^{C}\boldsymbol{P}_i^{\wedge})^2$

$$^{C}\boldsymbol{N} = {}^{C}\boldsymbol{I} \, {}^{C}\dot{\boldsymbol{\omega}}_C + {}^{C}\boldsymbol{\omega}_C \times {}^{C}\boldsymbol{I} \, {}^{C}\boldsymbol{\omega}_C$$

- 该式为旋转刚体的欧拉方程
- 欧拉方程描述了作用在刚体上的力矩 $^{C}\boldsymbol{N}$ 与刚体旋转角速度 $^{C}\boldsymbol{\omega}_C$ 和角加速度 $^{C}\dot{\boldsymbol{\omega}}_C$ 之间的关系
- $^{C}\boldsymbol{I}$ 称为刚体的惯性张量（**inertia tensor**），或旋转惯性矩阵（**rotational inertia matrix**）

7.2 刚体的惯性张量与欧拉方程

- 可以看出，${}^{C}\boldsymbol{I}$是一个**3 × 3矩阵**
- 如将${}^{C}\boldsymbol{P}_i$完整记为$[x_i, y_i, z_i]^{\mathrm{T}}$，${}^{C}\boldsymbol{I}$矩阵各元素为：

$$
{}^{C}\boldsymbol{I} = \begin{bmatrix} \sum m_i\left(y_i^2+z_i^2\right) & -\sum m_i\, x_i y_i & -\sum m_i\, x_i z_i \\ -\sum m_i\, x_i y_i & \sum m_i\left(x_i^2+z_i^2\right) & -\sum m_i\, y_i z_i \\ -\sum m_i\, x_i z_i & -\sum m_i\, y_i z_i & \sum m_i\left(x_i^2+y_i^2\right) \end{bmatrix}
$$

- 记为:

$$
{}^{C}\boldsymbol{I} =: \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}
$$

7.2 刚体的惯性张量与欧拉方程

- 考虑质量连续分布的刚体
- 用密度函数$\rho(x,y,z)$和微分单元体dV的乘积替代点质量，用积分运算替代求和运算，可得：

$$
{}^{C}\boldsymbol{I} =: \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}
$$

$$
I_{xx} = \int_{\mathcal{B}} (y^2 + z^2)\rho(x,y,z)\, dV
$$

$$
I_{yy} = \int_{\mathcal{B}} (x^2 + z^2)\rho(x,y,z)\, dV
$$

$$
I_{zz} = \int_{\mathcal{B}} (x^2 + y^2)\rho(x,y,z)\, dV
$$

$$
I_{xy} = \int_{\mathcal{B}} xy\rho(x,y,z)\, dV
$$

$$
I_{xz} = \int_{\mathcal{B}} xz\rho(x,y,z)\, dV
$$

$$
I_{yz} = \int_{\mathcal{B}} yz\rho(x,y,z)\, dV
$$

- 惯性张量是一个对称矩阵
- 惯性张量中的对角元素I_{xx}、I_{yy}和I_{zz}称为**惯性矩（mass moments of inertia）**
- 非对角元素I_{xy}、I_{xz}和I_{yz}称为**惯性积（mass product of inertia）**

- 例7-1:考虑如图中的质量为m，长度为l，宽度为w，高度为h的长方体连杆。连杆的质量是均匀分布的。建立如图所示的（原点）位于长方体连杆质心的联体坐标系$\{C\}$。计算该连杆在$\{C\}$下的惯性张量。

解： 该连杆的密度$\rho=\dfrac{m}{hlw}$。

（1）惯性矩：

$$I_{xx}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{l}{2}}^{\frac{l}{2}}\left(\int_{-\frac{w}{2}}^{\frac{w}{2}}(y^2+z^2)\rho dx\right)dydz$$

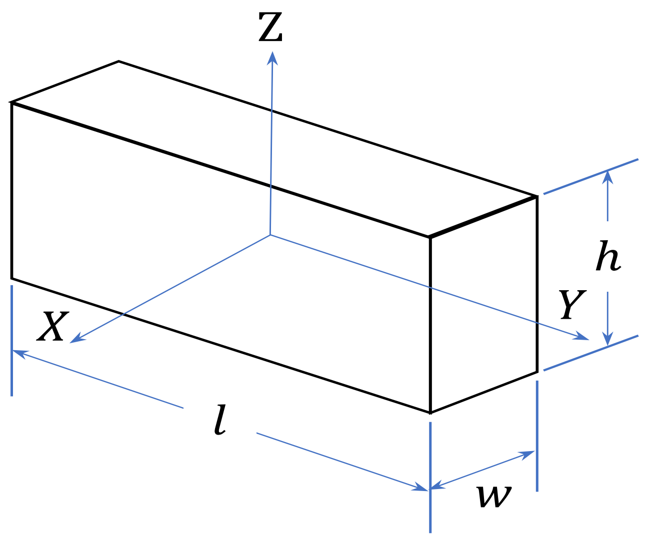
$$=\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{l}{2}}^{\frac{l}{2}}w(y^2+z^2)\rho\, dydz=w\rho\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{l}{2}}^{\frac{l}{2}}d\left(\frac{y^3}{3}+z^2y\right)dz$$

$$=w\rho\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{l^3}{12}+z^2l\right)dz=w\rho\int_{-\frac{h}{2}}^{\frac{h}{2}}d\left(\frac{l^3z}{12}+\frac{z^3l}{3}\right)=w\rho\left(\frac{l^3h}{12}+\frac{h^3l}{12}\right)$$

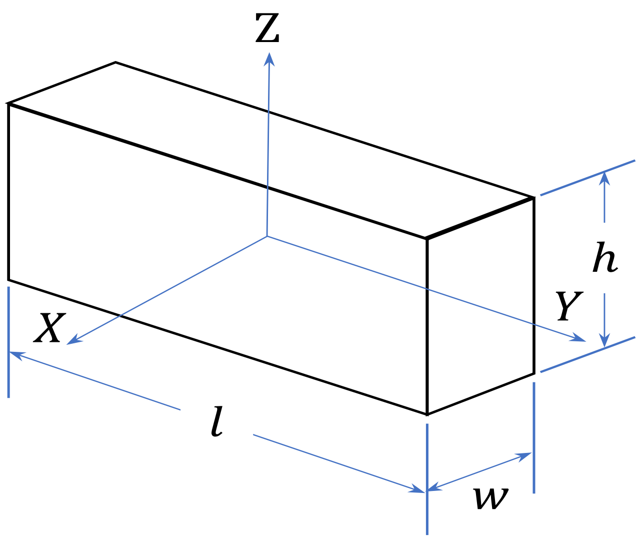
$$=\frac{m}{12}(l^2+h^2)$$

类似的，可计算得：

$$I_{yy}=\frac{m}{12}(h^2+w^2),I_{zz}=\frac{m}{12}(w^2+l^2)$$



7.2 刚体的惯性张量与欧拉方程



（2）惯性积：

$$I_{xy}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{l}{2}}^{\frac{l}{2}}\left(\int_{-\frac{w}{2}}^{\frac{w}{2}}xy\rho dx\right)dydz$$

$$=\rho\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{l}{2}}^{\frac{l}{2}}\left(\int_{-\frac{w}{2}}^{\frac{w}{2}}d\frac{x^2y}{2}\right)dydz=0$$

类似的，可计算得：

$$I_{yz}=0, I_{yz}=0$$

（3）该连杆在$\{C\}$下的惯性张量：

$$^{C}\boldsymbol{I}=\begin{bmatrix}\frac{m}{12}(l^2+h^2) & 0 & 0\\ 0 & \frac{m}{12}(h^2+w^2) & 0\\ 0 & 0 & \frac{m}{12}(w^2+l^2)\end{bmatrix}$$

- 注意：惯性张量也可以定义在非质心坐标系中
- 例7-2:考虑如图中的质量为m，长度为l，宽度为w，高度为h的长方体连杆。连杆的质量是均匀分布的。建立如图所示的（原点）位于长方体连杆一顶点的联体坐标系$\{B\}$。计算该连杆在$\{B\}$下的惯性张量。

解：该连杆的密度$\rho=\frac{m}{hlw}$。

（1）惯性矩：

$$I_{xx}=\int_0^h\int_0^l\left(\int_0^w(y^2+z^2)\rho dx\right)dydz$$

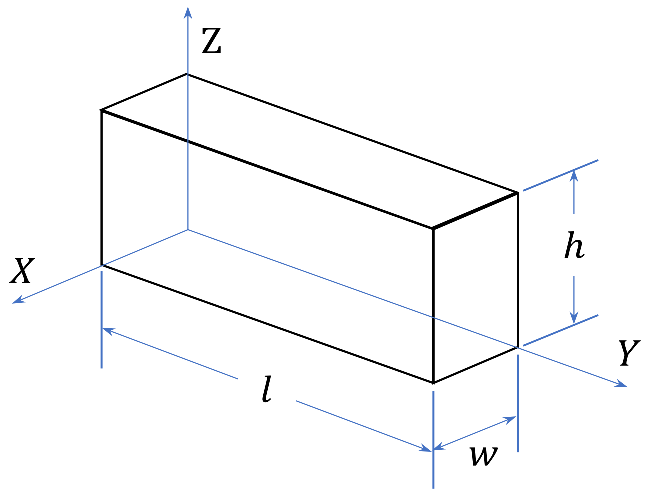
$$=\int_0^h\int_0^l w(y^2+z^2)\rho\,dydz=w\rho\int_0^h\int_0^l d\left(\frac{y^3}{3}+z^2y\right)dz$$

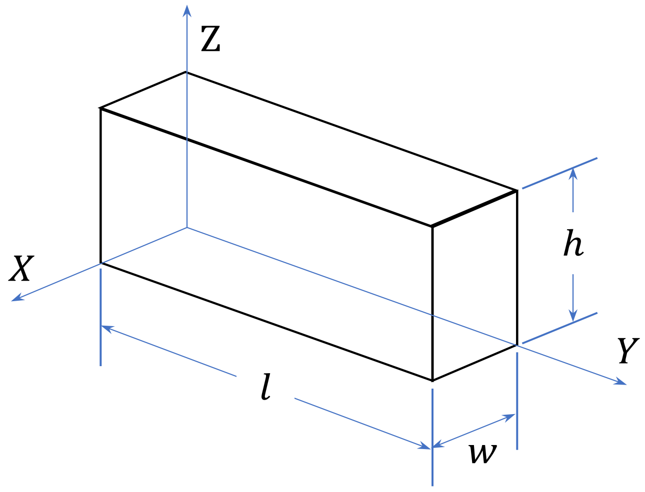
$$=w\rho\int_0^h\left(\frac{l^3}{3}+z^2l\right)dz=w\rho\int_0^h d\left(\frac{l^3z}{3}+\frac{z^3l}{3}\right)$$

$$=w\rho\left(\frac{l^3h}{3}+\frac{h^3l}{3}\right)=\frac{m}{3}(l^2+h^2)$$

类似的，可计算得：

$$I_{yy}=\frac{m}{3}(h^2+w^2),I_{zz}=\frac{m}{3}(w^2+l^2)$$





（2）惯性积：

$$I_{xy}=\int_0^h\int_0^l\left(\int_0^w xy\rho dx\right)dydz$$
$$=\rho\int_0^h\int_0^l\left(\int_0^w d\frac{x^2y}{2}\right)dydz$$
$$=\rho\int_0^h\int_0^l\frac{w^2y}{2}dydz=\rho\int_0^h\int_0^l d\frac{w^2y^2}{4}dz=\rho\int_0^h\frac{w^2l^2}{4}dz$$
$$=\rho\frac{w^2l^2h}{4}=\frac{m}{4}wl$$

类似的，可计算得：

$$I_{yz}=\frac{m}{4}lh,I_{xz}=\frac{m}{4}hw$$

（3）该连杆在$\{B\}$下的惯性张量：

$$^{B}\boldsymbol{I}=\begin{bmatrix}\frac{m}{3}(l^2+h^2) & -\frac{m}{4}wl & -\frac{m}{4}hw\\ -\frac{m}{4}wl & \frac{m}{3}(h^2+w^2) & -\frac{m}{4}lh\\ -\frac{m}{4}hw & -\frac{m}{4}lh & \frac{m}{3}(w^2+l^2)\end{bmatrix}$$

7.3 牛顿-欧拉迭代动力学方程

- 利用牛顿-欧拉法求解动力学方程分两个阶段：

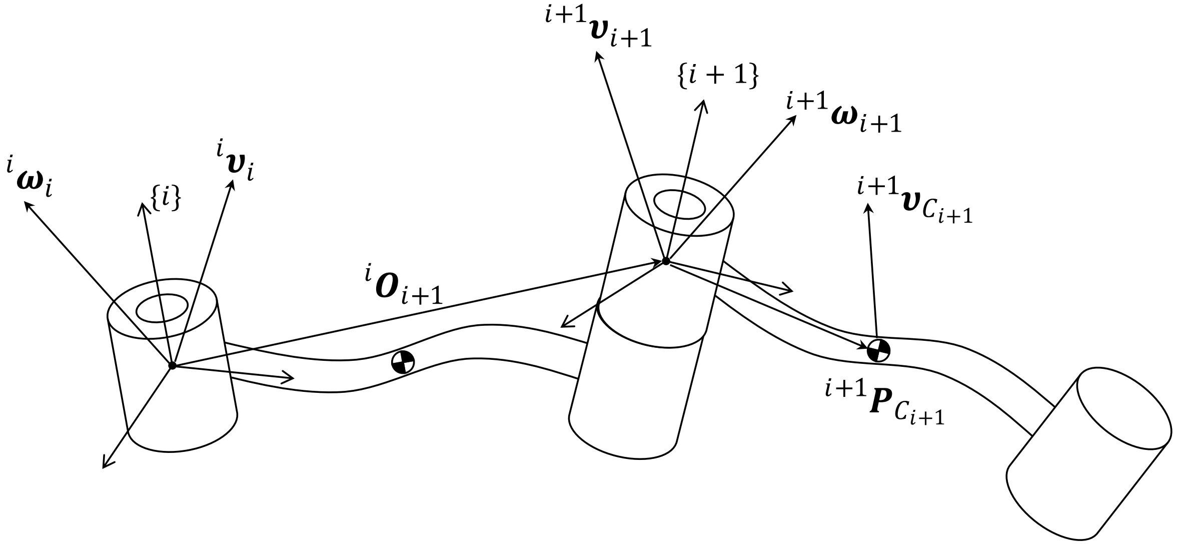
  - **向外迭代**：从（虚拟）的连杆0开始，依次计算连杆1到N联体坐标系角速度以及加速度（线加速度和角加速度），同时利用连杆i联体坐标系的角速度和加速度计算连杆i质心的加速度，并利用牛顿方程和欧拉方程求取作用在连杆上的力和力矩

  - **向内迭代**：从连杆N开始，根据力平衡方程和力矩平衡方程，依次计算出连杆$N-1$到连杆1上的力，同时计算出产生这些力和力矩所需的（转动型关节）关节力矩或（平动型关节）关节力

- 注意：下面牛顿-欧拉迭代动力学方程的讨论中**忽略了摩擦**的影响，即假设各关节均无摩擦

7.3.1 向外迭代：速度和加速度的计算

（1）连杆的速度传递:



- 考虑一般的连杆坐标系$\{i\}$及其相邻连杆坐标系$\{i+1\}$，已知：

$$^{i}\boldsymbol{\omega}_{i+1} = {}^{i}\boldsymbol{\omega}_{i} + {}^{i}_{i+1}\boldsymbol{R}\dot{\theta}_{i+1}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1}$$

$$^{i}\boldsymbol{v}_{i+1} = {}^{i}\boldsymbol{v}_{i} + {}^{i}\boldsymbol{\omega}_{i} \times {}^{i}\boldsymbol{O}_{i+1}$$

- 左乘$^{i+1}_{i}\boldsymbol{R}$：

$$^{i+1}\boldsymbol{\omega}_{i+1} = {}^{i+1}_{i}\boldsymbol{R}\,{}^{i}\boldsymbol{\omega}_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1}$$

$$^{i+1}\boldsymbol{v}_{i+1} = {}^{i+1}_{i}\boldsymbol{R}\left({}^{i}\boldsymbol{v}_{i} + {}^{i}\boldsymbol{\omega}_{i} \times {}^{i}\boldsymbol{O}_{i+1}\right)$$

7.3 牛顿-欧拉迭代动力学方程

7.3.1 向外迭代：速度和加速度的计算

（2）连杆的角加速度传递:

- 角加速度传递公式：

$$ {}^{A}\dot{\boldsymbol{\Omega}}_{C} = {}^{A}\dot{\boldsymbol{\Omega}}_{B} + {}^{A}_{B}\boldsymbol{R}\,{}^{B}\dot{\boldsymbol{\Omega}}_{C} + {}^{A}\boldsymbol{\Omega}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{\Omega}_{C} $$

- 令$\{\boldsymbol{A}\} = \{\boldsymbol{0}\}$，$\{\boldsymbol{B}\} = \{\boldsymbol{i}\}$，$\{\boldsymbol{C}\} = \{\boldsymbol{i}+\boldsymbol{1}\}$：

$$ \dot{\boldsymbol{\omega}}_{i+1} = \dot{\boldsymbol{\omega}}_{i} + {}^{0}_{i}\boldsymbol{R}\,{}^{i}\dot{\boldsymbol{\Omega}}_{i+1} + \boldsymbol{\omega}_{i} \times {}^{0}_{i}\boldsymbol{R}\,{}^{i}\boldsymbol{\Omega}_{i+1} $$

- ${}^{i}\boldsymbol{\Omega}_{i+1} = \dot{\theta}_{i+1}\,{}^{i}_{i+1}\boldsymbol{R}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1}$

- 注意： ${}^{i}\boldsymbol{\Omega}_{i+1}$和 ${}^{i}\boldsymbol{\omega}_{i+1}$具有不同的物理意义

7.3 牛顿-欧拉迭代动力学方程

$$\dot{\boldsymbol{\omega}}_{i+1} = \dot{\boldsymbol{\omega}}_i + {}^0_i\boldsymbol{R}\,{}^i\dot{\boldsymbol{\Omega}}_{i+1} + \boldsymbol{\omega}_i \times {}^0_i\boldsymbol{R}\,{}^i\boldsymbol{\Omega}_{i+1}$$

7.3.1 向外迭代：速度和加速度的计算

$${}^i\boldsymbol{\Omega}_{i+1} = \dot{\theta}_{i+1}\,{}^i_{i+1}\boldsymbol{R}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1}$$

- 求导得：

$${}^i\dot{\boldsymbol{\Omega}}_{i+1} = \ddot{\theta}_{i+1}\,{}^i_{i+1}\boldsymbol{R}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1}$$

- 代入$\dot{\boldsymbol{\omega}}_{i+1}$式：

$$\begin{aligned}\dot{\boldsymbol{\omega}}_{i+1} &= \dot{\boldsymbol{\omega}}_i + {}^0_i\boldsymbol{R}\ddot{\theta}_{i+1}\,{}^i_{i+1}\boldsymbol{R}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1} + \boldsymbol{\omega}_i \times {}^0_i\boldsymbol{R}\dot{\theta}_{i+1}\,{}^i_{i+1}\boldsymbol{R}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1} \\ &= \dot{\boldsymbol{\omega}}_i + \ddot{\theta}_{i+1}\,{}^0_{i+1}\boldsymbol{R}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1} + \boldsymbol{\omega}_i \times \dot{\theta}_{i+1}\,{}^0_{i+1}\boldsymbol{R}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1}\end{aligned}$$

- 两边乘上${}^i_0\boldsymbol{R}$：

$${}^i\dot{\boldsymbol{\omega}}_{i+1} = {}^i\dot{\boldsymbol{\omega}}_i + \ddot{\theta}_{i+1}\,{}^i_{i+1}\boldsymbol{R}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1} + {}^i\boldsymbol{\omega}_i \times \dot{\theta}_{i+1}\,{}^i_{i+1}\boldsymbol{R}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1}$$

- 为得到$\{i\}$到$\{i+1\}$的递归式
- 两边再乘上${}^{i+1}_i\boldsymbol{R}$：

$${}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} = {}^{i+1}_i\boldsymbol{R}\,{}^i\dot{\boldsymbol{\omega}}_i + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1} + {}^{i+1}_i\boldsymbol{R}\,{}^i\boldsymbol{\omega}_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{\boldsymbol{Z}}_{i+1}$$

- 对于平动型关节，因为$\dot{\theta}_{i+1} = 0$，$\ddot{\theta}_{i+1} = 0$，上式可简化为：

$${}^{i+1}\dot{\boldsymbol{\omega}}_{i+1} = {}^{i+1}_i\boldsymbol{R}\,{}^i\dot{\boldsymbol{\omega}}_i$$

7.3 牛顿-欧拉迭代动力学方程

7.3.1 向外迭代：速度和加速度的计算

（3）连杆的线加速度传递：

- 线加速度传递公式

$$^{A}\dot{V}_{Q} = {}^{A}\dot{V}_{BORG} + {}^{A}_{B}R\,{}^{B}\dot{V}_{Q} + 2\,{}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_{B} \times \left({}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q\right)$$

- 令$\{A\} = \{0\}$，$\{B\} = \{i\}$，Q为$\{i+1\}$原点：

$$\dot{v}_{i+1} = \dot{v}_{i} + {}^{0}_{i}R\,{}^{i}\dot{V}_{i+1} + 2\omega_{i} \times {}^{0}_{i}R\,{}^{i}V_{i+1} + \dot{\omega}_{i} \times {}^{0}_{i}R\,{}^{i}O_{i+1} + \omega_{i} \times \left(\omega_{i} \times {}^{0}_{i}R\,{}^{i}O_{i+1}\right)$$

- 两边同乘上${}^{i}_{0}R$：

$$^{i}\dot{v}_{i+1} = {}^{i}\dot{v}_{i} + {}^{i}\dot{V}_{i+1} + 2\,{}^{i}\omega_{i} \times {}^{i}V_{i+1} + {}^{i}\dot{\omega}_{i} \times {}^{i}O_{i+1} + {}^{i}\omega_{i} \times \left({}^{i}\omega_{i} \times {}^{i}O_{i+1}\right)$$

- 两边再乘上${}^{i+1}_{i}R$，可得到$\{i\}$到$\{i+1\}$的递归式：

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left[{}^{i}\dot{v}_{i} + {}^{i}\dot{\omega}_{i} \times {}^{i}O_{i+1} + {}^{i}\omega_{i} \times \left({}^{i}\omega_{i} \times {}^{i}O_{i+1}\right)\right] + {}^{i+1}_{i}R\left[{}^{i}\dot{V}_{i+1} + 2\,{}^{i}\omega_{i} \times {}^{i}V_{i+1}\right]$$

7.3 牛顿-欧拉迭代动力学方程

7.3.1 向外迭代：速度和加速度的计算

- 对于平动型关节：

$$ {}^{i+1}_{i}\boldsymbol{R}\,{}^{i}\boldsymbol{V}_{i+1} = {}^{i+1}\boldsymbol{V}_{i+1} = \dot{d}_{i+1}\,{}^{i+1}\widehat{\boldsymbol{Z}}_{i+1},\quad {}^{i+1}_{i}\boldsymbol{R}\,{}^{i}\dot{\boldsymbol{V}}_{i+1} = {}^{i+1}\dot{\boldsymbol{V}}_{i+1} = \ddot{d}_{i+1}\,{}^{i+1}\widehat{\boldsymbol{Z}}_{i+1},\quad {}^{i+1}\boldsymbol{\omega}_{i+1} = {}^{i+1}_{i}\boldsymbol{R}\,{}^{i}\boldsymbol{\omega}_{i} $$

- ${}^{i+1}\dot{\boldsymbol{v}}_{i+1}$式可表示为：

$$ {}^{i+1}\dot{\boldsymbol{v}}_{i+1} = {}^{i+1}_{i}\boldsymbol{R}\left[{}^{i}\dot{\boldsymbol{v}}_{i} + {}^{i}\dot{\boldsymbol{\omega}}_{i} \times {}^{i}\boldsymbol{O}_{i+1} + {}^{i}\boldsymbol{\omega}_{i} \times \left({}^{i}\boldsymbol{\omega}_{i} \times {}^{i}\boldsymbol{O}_{i+1}\right)\right] + \ddot{d}_{i+1}\,{}^{i+1}\widehat{\boldsymbol{Z}}_{i+1} + 2\,{}^{i+1}\boldsymbol{\omega}_{i+1} \times \dot{d}_{i+1}\,{}^{i+1}\widehat{\boldsymbol{Z}}_{i+1} $$

- 对于转动型关节：

$$ {}^{i}\boldsymbol{V}_{i+1} = \boldsymbol{0},\quad {}^{i}\dot{\boldsymbol{V}}_{i+1} = \boldsymbol{0} $$

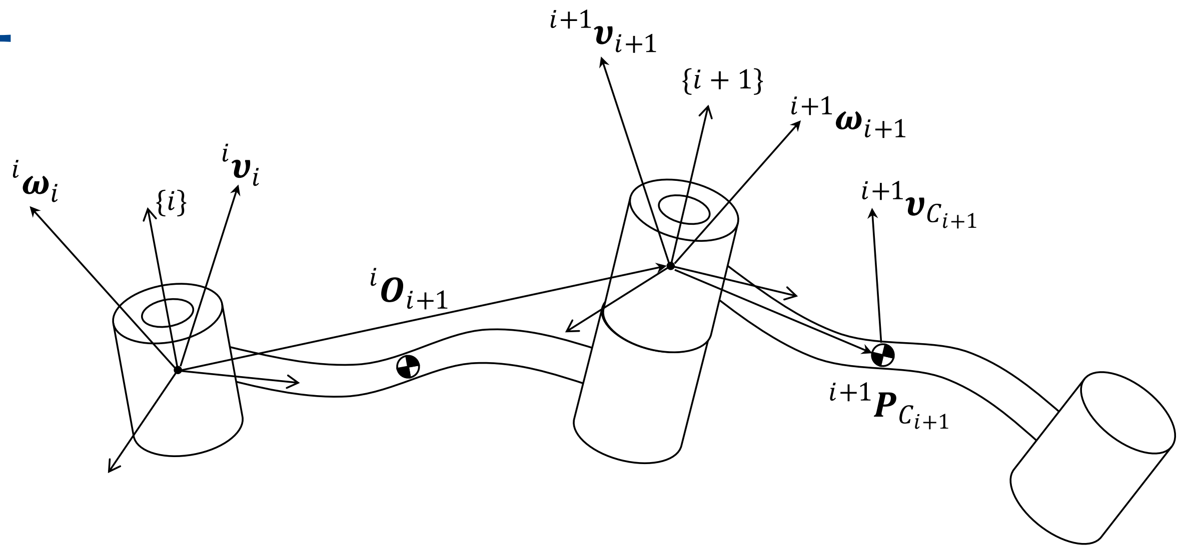
- ${}^{i+1}\dot{\boldsymbol{v}}_{i+1}$式可简化为：

$$ {}^{i+1}\dot{\boldsymbol{v}}_{i+1} = {}^{i+1}_{i}\boldsymbol{R}\left[{}^{i}\dot{\boldsymbol{v}}_{i} + {}^{i}\dot{\boldsymbol{\omega}}_{i} \times {}^{i}\boldsymbol{O}_{i+1} + {}^{i}\boldsymbol{\omega}_{i} \times \left({}^{i}\boldsymbol{\omega}_{i} \times {}^{i}\boldsymbol{O}_{i+1}\right)\right] $$

7.3 牛顿-欧拉迭代动力学方程

7.3.1 向外迭代：速度和加速度的计算

（4）连杆质心的线加速度传递：



- 为了计算作用在连杆质心上的力，还需要计算连杆质心的加速度
- 线加速度传递公式：

$$^{A}\dot{\boldsymbol{V}}_{Q} = {}^{A}\dot{\boldsymbol{V}}_{BORG} + {}^{A}_{B}\boldsymbol{R}\,{}^{B}\dot{\boldsymbol{V}}_{Q} + 2\,{}^{A}\boldsymbol{\Omega}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{V}_{Q} + {}^{A}\dot{\boldsymbol{\Omega}}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q} + {}^{A}\boldsymbol{\Omega}_{B} \times \left({}^{A}\boldsymbol{\Omega}_{B} \times {}^{A}_{B}\boldsymbol{R}\,{}^{B}\boldsymbol{Q}\right)$$

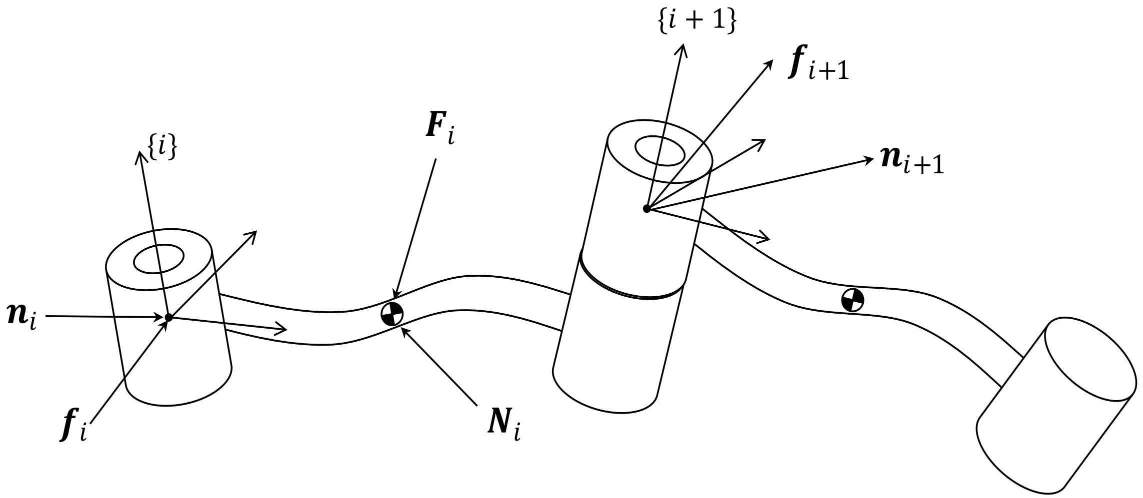
- 选取$\{A\} = \{0\}$，$\{B\} = \{i\}$，$\boldsymbol{Q}$为连杆i质心，表示为$\boldsymbol{C}_i$
- 因为$^{B}\boldsymbol{V}_{Q} = {}^{i}\boldsymbol{V}_{C_i} = \boldsymbol{0}$，$^{B}\dot{\boldsymbol{V}}_{Q} = {}^{i}\dot{\boldsymbol{V}}_{C_i} = \boldsymbol{0}$，有：

$$\dot{\boldsymbol{v}}_{C_i} = \dot{\boldsymbol{v}}_i + \dot{\boldsymbol{\omega}}_i \times {}^{0}_{i}\boldsymbol{R}\,{}^{i}\boldsymbol{P}_{C_i} + \boldsymbol{\omega}_i \times \left(\boldsymbol{\omega}_i \times {}^{0}_{i}\boldsymbol{R}\,{}^{i}\boldsymbol{P}_{C_i}\right)$$

- 两边同乘上${}^{i}_{0}\boldsymbol{R}$：

$$^{i}\dot{\boldsymbol{v}}_{C_i} = {}^{i}\dot{\boldsymbol{v}}_i + {}^{i}\dot{\boldsymbol{\omega}}_i \times {}^{i}\boldsymbol{P}_{C_i} + {}^{i}\boldsymbol{\omega}_i \times \left({}^{i}\boldsymbol{\omega}_i \times {}^{i}\boldsymbol{P}_{C_i}\right)$$

7.3 牛顿-欧拉迭代动力学方程



- 由前面的计算，我们可以从连杆0开始，向外迭代计算连杆1至连杆n的质心坐标系的线加速度、角速度和角加速度

- 接着利用牛顿-欧拉公式，可计算作用在连杆i质心上的惯性力和力矩：

$$\boldsymbol{F}_i = m_i \dot{\boldsymbol{v}}_{C_i}$$

$$^{C_i}\boldsymbol{N}_i = {}^{C_i}\boldsymbol{I}_i {}^{C_i}\dot{\boldsymbol{\omega}}_i + {}^{C_i}\boldsymbol{\omega}_i \times {}^{C_i}\boldsymbol{I}_i {}^{C_i}\boldsymbol{\omega}_i$$

- 坐标系$\{C_i\}$的原点位于连杆质心，可以选取坐标系$\{C_i\}$的各坐标轴方向与原连杆坐标系$\{i\}$方向相同，则上式中${}^{C_i}\dot{\boldsymbol{\omega}}_i = {}^{i}\dot{\boldsymbol{\omega}}_i$，${}^{C_i}\boldsymbol{\omega}_i = {}^{i}\boldsymbol{\omega}_i$
- 在连杆坐标系$\{i\}$中表示$\boldsymbol{F}_i$和${}^{C_i}\boldsymbol{N}_i$，则有

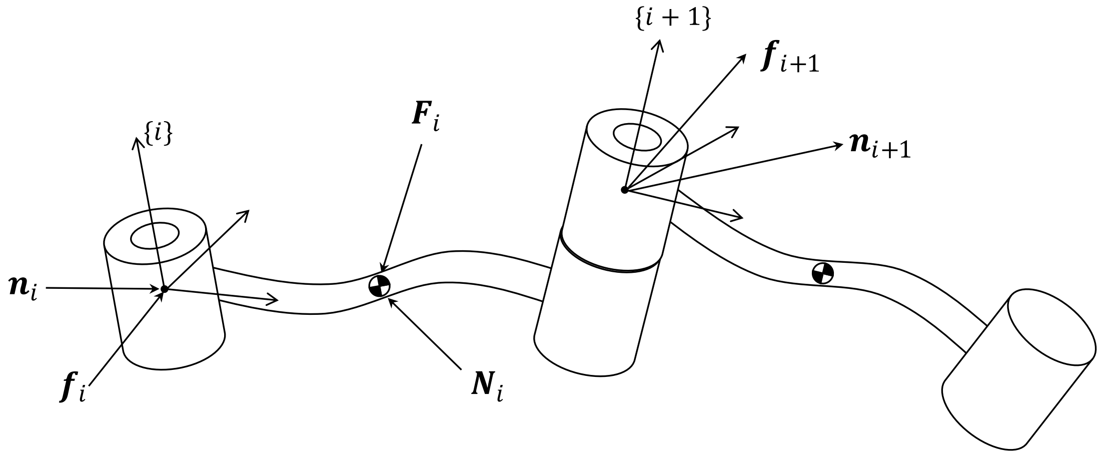
$$^{i}\boldsymbol{F}_i = m_i {}^{i}\dot{\boldsymbol{v}}_{C_i}$$

$$^{i}\boldsymbol{N}_i = {}^{C_i}\boldsymbol{I}_i {}^{i}\dot{\boldsymbol{\omega}}_i + {}^{i}\boldsymbol{\omega}_i \times {}^{C_i}\boldsymbol{I}_i {}^{i}\boldsymbol{\omega}_i$$

- 这里${}^{i}\boldsymbol{N}_i = {}^{i}({}^{C_i}\boldsymbol{N}_i)$

7.3 牛顿-欧拉迭代动力学方程

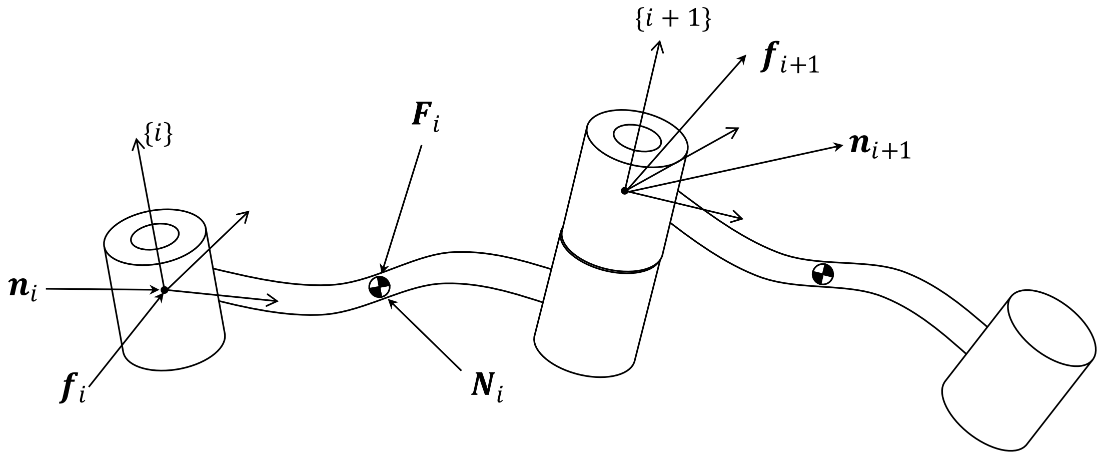
7.3.2 向内迭代：力和力矩的计算



- 右图为典型连杆在无重力状态下的受力情况
- 考虑连杆i，先考虑其力平衡方程
  - 作用于$\{i\}$原点的力向量$\boldsymbol{f}_i$表示连杆$i-1$施加在连杆i上的力，力矩向量$\boldsymbol{n}_i$表示连杆$i-1$施加在连杆i上的力矩
  - 除了惯性力${}^{i}\boldsymbol{F}_i$，连杆i还受到连杆$i-1$施加在连杆i上的${}^{i}\boldsymbol{f}_i$，这里左上标表示这两个力表示在坐标系$\{i\}$下
  - 由于连杆i对连杆$i+1$有一作用力$\boldsymbol{f}_{i+1}$，所以连杆$i+1$对连杆i有一反作用力$-\boldsymbol{f}_{i+1}$，同样的，用${}^{i+1}\boldsymbol{f}_{i+1}$在坐标系$\{i+1\}$下表示$\boldsymbol{f}_{i+1}$
- 将所有作用于连杆i上的力向量相加，得到力平衡方程：

$$ {}^{i}\boldsymbol{F}_i = {}^{i}\boldsymbol{f}_i - {}_{i+1}^{\ \ i}\boldsymbol{R}\,{}^{i+1}\boldsymbol{f}_{i+1} $$

7.3.2 向内迭代：力和力矩的计算



- 下面考虑作用在连杆$\boldsymbol{i}$质心处的力矩平衡方程
  - 在坐标系$\{i\}$中表示，除了力矩${}^{i}\boldsymbol{N}_i$，连杆$\boldsymbol{i}$还受到连杆$i-1$施加在连杆i上的力矩${}^{i}\boldsymbol{n}_i$
  - 由于连杆i对连杆$i+1$有一作用力矩${}^{i}\boldsymbol{n}_{i+1}$，所以连杆$i+1$对连杆$i$有一反作用力矩$-{}^{i+1}\boldsymbol{n}_{i+1}$，这里${}^{i}\boldsymbol{n}_{i+1}={}^{i}_{i+1}\boldsymbol{R}\,{}^{i+1}\boldsymbol{n}_{i+1}$
  - 存在${}^{i}\boldsymbol{f}_i$和$-{}^{i}\boldsymbol{f}_{i+1}$在连杆$i$质心处产生的力矩
- 将所有力矩向量相加，得到力矩平衡方程：

$$ {}^{i}\boldsymbol{N}_i={}^{i}\boldsymbol{n}_i+\left(-{}^{i}\boldsymbol{P}_{C_i}\right)\times{}^{i}\boldsymbol{f}_i+\left({}^{i}\boldsymbol{O}_{i+1}-{}^{i}\boldsymbol{P}_{C_i}\right)\times\left(-{}^{i}\boldsymbol{f}_{i+1}\right)-{}^{i}\boldsymbol{n}_{i+1} $$

7.3 牛顿-欧拉迭代动力学方程

7.3.2 向内迭代：力和力矩的计算

- 将力平衡方程与力矩平衡方程整理为连杆$i+1$到连杆i的的迭代形式：
  - 力平衡方程

$$ {}^{i}\boldsymbol{f}_{i} = {}_{i+1}^{\ \ i}\boldsymbol{R}\,{}^{i+1}\boldsymbol{f}_{i+1} + {}^{i}\boldsymbol{F}_{i} $$

  - 力矩平衡方程

$$ {}^{i}\boldsymbol{n}_{i} = {}^{i}\boldsymbol{N}_{i} + {}_{i+1}^{\ \ i}\boldsymbol{R}\,{}^{i+1}\boldsymbol{n}_{i+1} + {}^{i}\boldsymbol{P}_{C_i} \times {}^{i}\boldsymbol{f}_{i} + \left({}^{i}\boldsymbol{O}_{i+1} - {}^{i}\boldsymbol{P}_{C_i}\right) \times {}^{i}\boldsymbol{f}_{i+1} $$

- 由于${}^{i}\boldsymbol{f}_{i} = {}^{i}\boldsymbol{f}_{i+1} + {}^{i}\boldsymbol{F}_{i}$，有：

$$ \begin{aligned} {}^{i}\boldsymbol{n}_{i} &= {}^{i}\boldsymbol{N}_{i} + {}_{i+1}^{\ \ i}\boldsymbol{R}\,{}^{i+1}\boldsymbol{n}_{i+1} + {}^{i}\boldsymbol{P}_{C_i} \times \left({}^{i}\boldsymbol{f}_{i+1} + {}^{i}\boldsymbol{F}_{i}\right) + \left({}^{i}\boldsymbol{O}_{i+1} - {}^{i}\boldsymbol{P}_{C_i}\right) \times {}^{i}\boldsymbol{f}_{i+1} \\ &= {}^{i}\boldsymbol{N}_{i} + {}_{i+1}^{\ \ i}\boldsymbol{R}\,{}^{i+1}\boldsymbol{n}_{i+1} + {}^{i}\boldsymbol{P}_{C_i} \times {}^{i}\boldsymbol{F}_{i} + {}^{i}\boldsymbol{O}_{i+1} \times {}^{i}\boldsymbol{f}_{i+1} \\ &= {}^{i}\boldsymbol{N}_{i} + {}_{i+1}^{\ \ i}\boldsymbol{R}\,{}^{i+1}\boldsymbol{n}_{i+1} + {}^{i}\boldsymbol{P}_{C_i} \times {}^{i}\boldsymbol{F}_{i} + {}^{i}\boldsymbol{O}_{i+1} \times {}_{i+1}^{\ \ i}\boldsymbol{R}\,{}^{i+1}\boldsymbol{f}_{i+1} \end{aligned} $$

- 用上述方程对连杆依次求解，从连杆N开始向内递推直至机器人基座

7.3.2 向内迭代：力和力矩的计算

- 对于转动型关节$\boldsymbol{i}$，为产生力矩$\boldsymbol{n}_i$，所需的关节力矩：

$$\tau_i = {}^{i}\boldsymbol{n}_i^{T}\,{}^{i}\widehat{\boldsymbol{Z}}_i$$

- 对于平动型关节$\boldsymbol{i}$，为产生力$\boldsymbol{f}_i$，所需的关节力：

$$\tau_i = {}^{i}\boldsymbol{f}_i^{T}\,{}^{i}\widehat{\boldsymbol{Z}}_i$$

- 注意：上面的讨论并没有谈及重力
- 这是因为我们可以考虑惯性系中连杆坐标系$\{0\}$以加速度$\boldsymbol{G}$运动，即 ${}^{0}\dot{\boldsymbol{v}}_0 = \boldsymbol{G}$，这里$\boldsymbol{G}$与重力矢量大小相等，方向相反，其产生的效果就与重力作用的效果是一样的

7.3 牛顿-欧拉迭代动力学方程

7.3.2 向内迭代：力和力矩的计算

- 例7-3：计算如图所示平面机器人的动力学方程，假设每个连杆的质量都集中在连杆末端。

解：

连杆质心的位置矢量

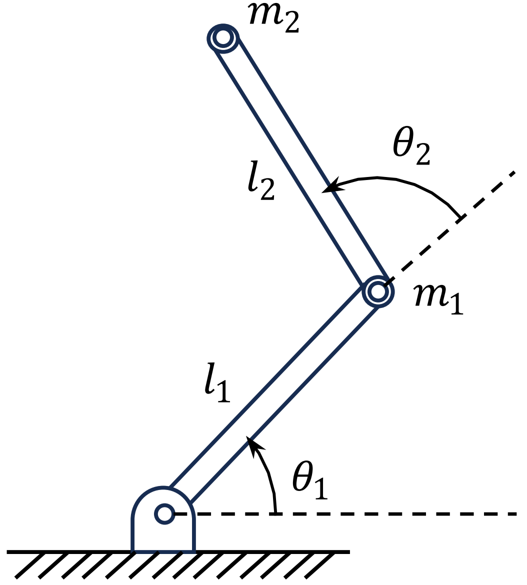
$$^{1}\boldsymbol{P}_{C_1} = l_1\hat{\boldsymbol{X}}_1 = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}$$

$$^{2}\boldsymbol{P}_{C_2} = l_2\hat{\boldsymbol{X}}_2 = \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix}$$

连杆质心的惯性张量

$$^{C_1}\boldsymbol{I}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$^{C_2}\boldsymbol{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



7.3 牛顿-欧拉迭代动力学方程

7.3.2 向内迭代：力和力矩的计算

无力作用于末端执行器上，即：

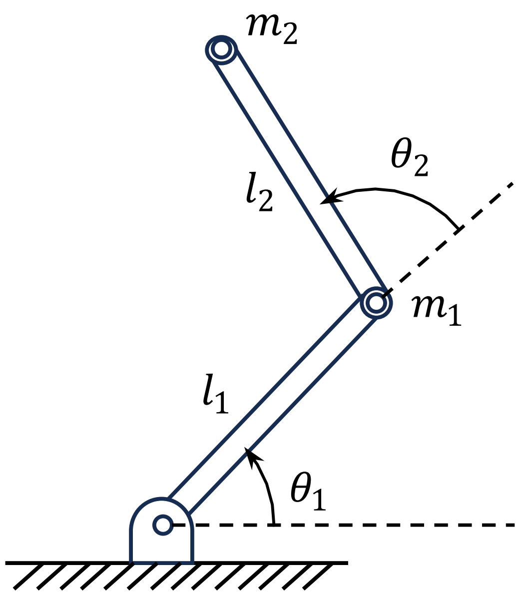
$$ {}^{3}\boldsymbol{f}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, {}^{3}\boldsymbol{n}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

机器人基座保持不动

$$ {}^{0}\boldsymbol{\omega}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, {}^{0}\dot{\boldsymbol{\omega}}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

考虑重力

$$ {}^{0}\dot{\boldsymbol{v}}_0 = g\hat{\boldsymbol{Y}}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} $$



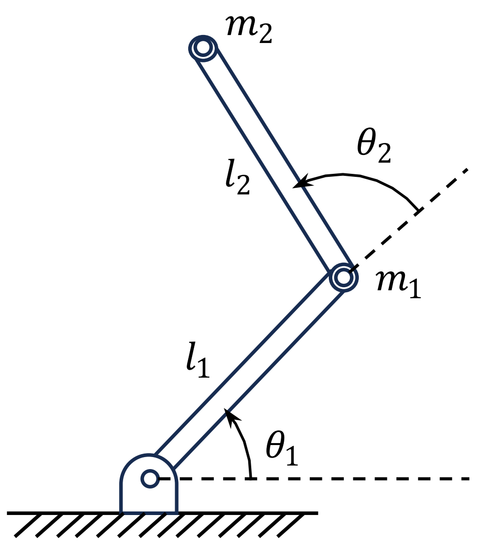
连杆间的相对转动

$$ {}_{i+1}^{\;\;i}\boldsymbol{R} = \begin{bmatrix} c_{i+1} & -s_{i+1} & 0 \\ s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, {}_{\;i}^{i+1}\boldsymbol{R} = \begin{bmatrix} c_{i+1} & s_{i+1} & 0 \\ -s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

7.3 牛顿-欧拉迭代动力学方程

7.3.2 向内迭代：力和力矩的计算

➢ 向外迭代：连杆0到连杆1（$i=0$）：



$$ {}^{1}\boldsymbol{\omega}_1 = {}^{1}_{0}\boldsymbol{R}\,{}^{0}\boldsymbol{\omega}_0 + \dot{\theta}_1\,{}^{1}\hat{\boldsymbol{Z}}_1 = {}^{1}_{0}\boldsymbol{R}\begin{bmatrix}0\\0\\0\end{bmatrix} + \dot{\theta}_1\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix} $$

$$ {}^{1}\dot{\boldsymbol{\omega}}_1 = {}^{1}_{0}\boldsymbol{R}\,{}^{0}\dot{\boldsymbol{\omega}}_0 + {}^{1}_{0}\boldsymbol{R}\,{}^{0}\boldsymbol{\omega}_0 \times \dot{\theta}_1\,{}^{1}\hat{\boldsymbol{Z}}_1 + \ddot{\theta}_1\,{}^{1}\hat{\boldsymbol{Z}}_1 $$

$$ = {}^{1}_{0}\boldsymbol{R}\begin{bmatrix}0\\0\\0\end{bmatrix} + {}^{1}_{0}\boldsymbol{R}\begin{bmatrix}0\\0\\0\end{bmatrix} \times \dot{\theta}_1\,{}^{1}\hat{\boldsymbol{Z}}_1 + \ddot{\theta}_1\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\\ddot{\theta}_1\end{bmatrix} $$

$$ {}^{1}\dot{\boldsymbol{v}}_1 = {}^{1}_{0}\boldsymbol{R}\left({}^{0}\dot{\boldsymbol{\omega}}_0 \times {}^{0}\boldsymbol{O}_1 + {}^{0}\boldsymbol{\omega}_0 \times \left({}^{0}\boldsymbol{\omega}_0 \times {}^{0}\boldsymbol{O}_1\right) + {}^{0}\dot{\boldsymbol{v}}_0\right) = \begin{bmatrix}gs_1\\gc_1\\0\end{bmatrix} $$

$$ {}^{1}\dot{\boldsymbol{v}}_{C_1} = {}^{1}\dot{\boldsymbol{\omega}}_1 \times {}^{1}\boldsymbol{P}_{C_1} + {}^{1}\boldsymbol{\omega}_1 \times \left({}^{1}\boldsymbol{\omega}_1 \times {}^{1}\boldsymbol{P}_{C_1}\right) + {}^{1}\dot{\boldsymbol{v}}_1 = \begin{bmatrix}-l_1\dot{\theta}_1^2 + gs_1\\ l_1\ddot{\theta}_1 + gc_1\\ 0\end{bmatrix} $$

$$ {}^{1}\boldsymbol{F}_1 = m_1\,{}^{1}\dot{\boldsymbol{v}}_{C_1} = \begin{bmatrix}-m_1 l_1\dot{\theta}_1^2 + m_1 g s_1\\ m_1 l_1\ddot{\theta}_1 + m_1 g c_1\\ 0\end{bmatrix} $$

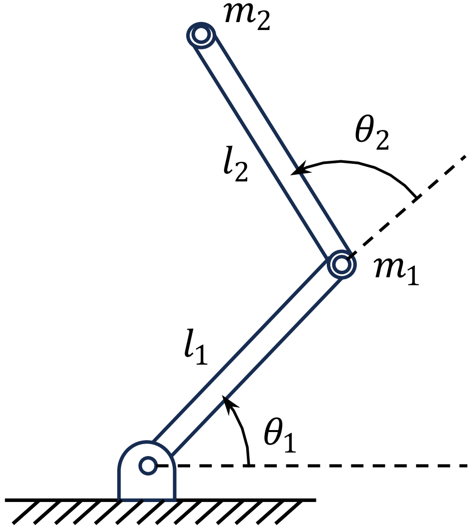
$$ {}^{1}\boldsymbol{N}_1 = {}^{C_1}\boldsymbol{I}_1\,{}^{1}\dot{\boldsymbol{\omega}}_1 + {}^{1}\boldsymbol{\omega}_1 \times {}^{C_1}\boldsymbol{I}_1\,{}^{1}\boldsymbol{\omega}_1 = \begin{bmatrix}0\\0\\0\end{bmatrix} $$

式中，$ {}^{C_1}\boldsymbol{I}_1 = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} $

7.3 牛顿-欧拉迭代动力学方程

7.3.2 向内迭代：力和力矩的计算

➢ 向外迭代：连杆1到连杆2（$i=1$）：



$${}^2\boldsymbol{\omega}_2 = {}^2_1\boldsymbol{R}\,{}^1\boldsymbol{\omega}_1 + \dot\theta_2\,{}^2\widehat{\boldsymbol{Z}}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1+\dot\theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_{12} \end{bmatrix}$$

为简化表达式，这里用$\dot\theta_{12}=\dot\theta_1+\dot\theta_2$

$${}^2\dot{\boldsymbol{\omega}}_2 = {}^2_1\boldsymbol{R}\,{}^1\dot{\boldsymbol{\omega}}_1 + {}^2_1\boldsymbol{R}\,{}^1\boldsymbol{\omega}_1 \times \dot\theta_2\,{}^2\widehat{\boldsymbol{Z}}_2 + \ddot\theta_2\,{}^2\widehat{\boldsymbol{Z}}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot\theta_{12} \end{bmatrix}$$

$${}^2\dot{\boldsymbol{v}}_2 = {}^2_1\boldsymbol{R}\left({}^1\dot{\boldsymbol{\omega}}_1 \times {}^1\boldsymbol{O}_2 + {}^1\boldsymbol{\omega}_1 \times ({}^1\boldsymbol{\omega}_1 \times {}^1\boldsymbol{O}_2) + {}^1\dot{\boldsymbol{v}}_1\right) = \begin{bmatrix} l_1\ddot\theta_1 s_2 - l_1\dot\theta_1^2 c_2 + g s_{12} \\ l_1\ddot\theta_1 c_2 + l_1\dot\theta_1^2 s_2 + g c_{12} \\ 0 \end{bmatrix}$$

$${}^2\dot{\boldsymbol{v}}_{C_2} = {}^2\dot{\boldsymbol{\omega}}_2 \times {}^2\boldsymbol{P}_{C_2} + {}^2\boldsymbol{\omega}_2 \times \left({}^2\boldsymbol{\omega}_2 \times {}^2\boldsymbol{P}_{C_2}\right) + {}^2\dot{\boldsymbol{v}}_2 = \begin{bmatrix} -l_2\dot\theta_{12}^2 \\ l_2\ddot\theta_{12} \\ 0 \end{bmatrix} + \begin{bmatrix} l_1\ddot\theta_1 s_2 - l_1\dot\theta_1^2 c_2 + g s_{12} \\ l_1\ddot\theta_1 c_2 + l_1\dot\theta_1^2 s_2 + g c_{12} \\ 0 \end{bmatrix}$$

$${}^2\boldsymbol{F}_2 = m_2\,{}^2\dot{\boldsymbol{v}}_{C_2}$$
$$= \begin{bmatrix} m_2 l_1\ddot\theta_1 s_2 - m_2 l_1\dot\theta_1^2 c_2 - m_2 l_2\dot\theta_{12}^2 + m_2 g s_{12} \\ m_2 l_1\ddot\theta_1 c_2 + m_2 l_1\dot\theta_1^2 s_2 + m_2 l_2\ddot\theta_{12} + m_2 g c_{12} \\ 0 \end{bmatrix}$$

$${}^2\boldsymbol{N}_2 = {}^{C_2}\boldsymbol{I}_2\,{}^2\dot{\boldsymbol{\omega}}_2 + {}^2\boldsymbol{\omega}_2 \times {}^{C_2}\boldsymbol{I}_2\,{}^2\boldsymbol{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

式中，$^{C_2}\boldsymbol{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7.3 牛顿-欧拉迭代动力学方程

7.3.2 向内迭代：力和力矩的计算

- 向内迭代：连杆3到连杆2（$i=2$）：

$$
{}^{2}\boldsymbol{f}_2 = {}^{2}_{3}\boldsymbol{R}\,{}^{3}\boldsymbol{f}_3 + {}^{2}\boldsymbol{F}_2 = \begin{bmatrix} m_2 l_1 \ddot{\theta}_1 s_2 - m_2 l_1 \dot{\theta}_1^2 c_2 - m_2 l_2 \dot{\theta}_{12}^2 + m_2 g s_{12} \\ m_2 l_1 \ddot{\theta}_1 c_2 + m_2 l_1 \dot{\theta}_1^2 s_2 + m_2 l_2 \ddot{\theta}_{12} + m_2 g c_{12} \\ 0 \end{bmatrix}
$$

$$
{}^{2}\boldsymbol{n}_2 = {}^{2}\boldsymbol{N}_2 + {}^{2}_{3}\boldsymbol{R}\,{}^{3}\boldsymbol{n}_3 + {}^{2}\boldsymbol{P}_{C_2} \times {}^{2}\boldsymbol{F}_2 + {}^{2}\boldsymbol{O}_3 \times {}^{2}_{3}\boldsymbol{R}\,{}^{3}\boldsymbol{f}_3
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 \ddot{\theta}_1 c_2 + m_2 l_1 l_2 \dot{\theta}_1^2 s_2 + m_2 l_2^2 \ddot{\theta}_{12} + m_2 l_2 g c_{12} \end{bmatrix}
$$

$$
\tau_2 = {}^{2}\boldsymbol{n}_2^{\mathrm{T}}\,{}^{2}\hat{\boldsymbol{Z}}_2
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 \ddot{\theta}_1 c_2 + m_2 l_1 l_2 \dot{\theta}_1^2 s_2 + m_2 l_2^2 \ddot{\theta}_{12} + m_2 l_2 g c_{12} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
= m_2 l_1 l_2 \ddot{\theta}_1 c_2 + m_2 l_1 l_2 \dot{\theta}_1^2 s_2 + m_2 l_2^2 \ddot{\theta}_{12} + m_2 l_2 g c_{12}
$$

7.3.2 向内迭代：力和力矩的计算

➢ 向内迭代：连杆2到连杆1（$i=1$）：

$$
{}^{1}\boldsymbol{f}_1 = {}^{1}_{2}\boldsymbol{R}\,{}^{2}\boldsymbol{f}_2 + {}^{1}\boldsymbol{F}_1 = \begin{bmatrix} -m_{12}l_1\dot{\theta}_1^2 - m_2l_2c_2\dot{\theta}_{12}^2 - m_2l_2s_2\ddot{\theta}_{12} + m_{12}gs_1 \\ m_{12}l_1\ddot{\theta}_1 - m_2l_2s_2\dot{\theta}_{12}^2 + m_2l_2c_2\ddot{\theta}_{12} + m_{12}gc_1 \\ 0 \end{bmatrix}
$$

为简化表示$m_{12} = m_1 + m_2$

$$
{}^{1}\boldsymbol{n}_1 = {}^{1}\boldsymbol{N}_1 + {}^{1}_{2}\boldsymbol{R}\,{}^{2}\boldsymbol{n}_2 + {}^{1}\boldsymbol{P}_{C_1} \times {}^{1}\boldsymbol{F}_1 + {}^{1}\boldsymbol{O}_2 \times {}^{1}_{2}\boldsymbol{R}\,{}^{2}\boldsymbol{f}_2
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ m_2l_1l_2(\ddot{\theta}_1 + \ddot{\theta}_{12})c_2 + m_2l_1l_2(\dot{\theta}_1^2 - \dot{\theta}_{12}^2)s_2 + m_2l_2^2\ddot{\theta}_{12} + m_2l_2gc_{12} + m_{12}l_1^2\ddot{\theta}_1 + m_{12}l_1gc_1 \end{bmatrix}
$$

$$
\tau_1 = {}^{1}\boldsymbol{n}_1^{\mathrm{T}}\,{}^{1}\hat{\boldsymbol{Z}}_1
$$

$$
= m_2l_1l_2(\ddot{\theta}_1 + \ddot{\theta}_{12})c_2 + m_2l_1l_2(\dot{\theta}_1^2 - \dot{\theta}_{12}^2)s_2 + m_2l_2^2\ddot{\theta}_{12} + m_2l_2gc_{12} + m_{12}l_1^2\ddot{\theta}_1 + m_{12}l_1gc_1
$$

7.4 机器人动力学方程的拉格朗日方法

- 牛顿-欧拉方法是基于动力学方程以及作用在连杆之间约束力和力矩的分析之上的
- 拉格朗日力学是基于能量项对系统变量及时间微分的动力学方法
- 对于一个机器人来说，这两种方法得到的运动方程是相同的

7.4 机器人动力学方程的拉格朗日方法

- 一个机械结构系统的动能和势能的差值称为拉格朗日函数，表示为：

$$\mathcal{L}\left(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}}\right)=k\left(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}}\right)-u(\boldsymbol{\Phi})$$

- 关节向量$\boldsymbol{\Phi}$为广义坐标
- $k\left(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}}\right)$：系统动能
- $u(\boldsymbol{\Phi})$：系统势能

- 机器人的动力学方程可表示为：

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{\Phi}}}-\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Phi}}=\boldsymbol{\xi}$$

- $\boldsymbol{\xi}$是非保守力/力矩向量
- 它包括关节力/力矩向量$\boldsymbol{\tau}=[\tau_1\ldots\tau_N]^T$、摩擦力/力矩向量$\boldsymbol{B}\dot{\boldsymbol{\Phi}}$、末端执行器与环境接触而引起的关节负荷力/力矩向量$\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{\Phi})\boldsymbol{F}$

7.4 机器人动力学方程的拉格朗日方法

- 假设机器人末端执行器与环境不接触，末端执行器与环境的接触力/力矩向量$\boldsymbol{F}=\boldsymbol{0}$
- 机器人的动力学方程可进一步表示为：

$$\mathcal{L}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})=k(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})-u(\boldsymbol{\Phi}) \quad + \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{\Phi}}}-\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Phi}}=\xi \quad + \quad \xi=\boldsymbol{\tau}-\boldsymbol{B}\dot{\boldsymbol{\Phi}}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial k}{\partial\dot{\boldsymbol{\Phi}}}-\frac{\partial k}{\partial\boldsymbol{\Phi}}+\frac{\partial u}{\partial\boldsymbol{\Phi}}=\boldsymbol{\tau}-\boldsymbol{B}\dot{\boldsymbol{\Phi}}}$$

式中，$\boldsymbol{B}=\mathrm{diag}(b_1,\ldots,b_N)$，$b_i$为折算到关节$i$的粘滞摩擦参数

7.4.1 动能的计算

- 考虑N连杆机器人，其中k_i表示连杆i的动能
- 连杆$\boldsymbol{i}$的动能：

$$k_i = \frac{1}{2} m_i \boldsymbol{v}_{C_i}^{\mathrm{T}} \boldsymbol{v}_{C_i} + \frac{1}{2} {}^{i}\boldsymbol{\omega}_i^{\mathrm{T}} \, {}^{C_i}\boldsymbol{I}_i \, {}^{i}\boldsymbol{\omega}_i$$

$$= \frac{1}{2} m_i \boldsymbol{v}_{C_i}^{\mathrm{T}} \boldsymbol{v}_{C_i} + \frac{1}{2} \boldsymbol{\omega}_i^{T} \, {}_{i}^{0}\boldsymbol{R} \, {}^{C_i}\boldsymbol{I}_i \, {}_{i}^{0}\boldsymbol{R}^{\mathrm{T}} \boldsymbol{\omega}_i$$

- ${}^{i}\boldsymbol{\omega}_i = {}_{0}^{i}\boldsymbol{R} \, \boldsymbol{\omega}_i = {}_{i}^{0}\boldsymbol{R}^{\mathrm{T}} \boldsymbol{\omega}_i$
- 整个机器人的动能是各个连杆动能之和：

$$k = \sum_{i=1}^{n} k_i$$

7.4 机器人动力学方程的拉格朗日方法

$$k_i = \frac{1}{2} m_i \boldsymbol{v}_{C_i}^{\mathrm{T}} \boldsymbol{v}_{C_i} + \frac{1}{2} \boldsymbol{\omega}_i^{T}\, {}_i^0\boldsymbol{R}^{C_i}\boldsymbol{I}_i\, {}_i^0\boldsymbol{R}^{\mathrm{T}} \boldsymbol{\omega}_i$$

- 运用前面微分运动学部分引入的雅可比矩阵，可由关节变量计算$\boldsymbol{v}_{C_i}$以及$\boldsymbol{\omega}_i$：

$$\boldsymbol{v}_{C_i} = \boldsymbol{J}_P^{(i)} \dot{\boldsymbol{\Phi}}, \qquad \boldsymbol{\omega}_i = \boldsymbol{J}_O^{(i)} \dot{\boldsymbol{\Phi}}$$

- 将各连杆的动能相加，并注意$\boldsymbol{J}_P^{(i)}$、$\boldsymbol{J}_O^{(i)}$和${}_i^0\boldsymbol{R}$都依赖于$\boldsymbol{\Phi}$，就得到机器人的总动能：

$$k(\boldsymbol{\Phi}, \dot{\boldsymbol{\Phi}}) = \sum_{i=1}^{N} k_i(\boldsymbol{\Phi}, \dot{\boldsymbol{\Phi}}) = \frac{1}{2} \dot{\boldsymbol{\Phi}}^{\mathrm{T}} \boldsymbol{M}(\boldsymbol{\Phi}) \dot{\boldsymbol{\Phi}}$$

- 对称矩阵$\boldsymbol{M}(\boldsymbol{\Phi})$称为惯性矩阵：

$$\boldsymbol{M}(\boldsymbol{\Phi}) = \sum_{i=1}^{N} \left(m_i \left(\boldsymbol{J}_P^{(i)}\right)^{\mathrm{T}} \boldsymbol{J}_P^{(i)} + \left(\boldsymbol{J}_O^{(i)}\right)^{\mathrm{T}} {}_i^0\boldsymbol{R}^{C_i}\boldsymbol{I}_i\, {}_i^0\boldsymbol{R}^{\mathrm{T}} \boldsymbol{J}_O^{(i)} \right)$$

- 因为机器人的总动能非负，且仅在$\dot{\boldsymbol{\Phi}} = \boldsymbol{0}$时总动能为零，所以惯性矩阵是一个正定矩阵

7.4.2 势能的计算

- ${}^0\boldsymbol{g}$表示世界坐标系中的重力加速度向量
- 例如，如果以y轴为竖直向上方向，则${}^0\boldsymbol{g}=[0,-g,0]^{\mathrm{T}}$
- $\boldsymbol{P}_{C_i}$是连杆质心的位置矢量
- 连杆i的势能：

$$u_i=-m_i{}^0\boldsymbol{g}^{\mathrm{T}}\,{}^0\boldsymbol{P}_{C_i}$$

- 操作臂的总势能：

$$u=\sum_{i=1}^{n}u_i$$

- 将各连杆势能相加，并注意 ${}^0\boldsymbol{P}_{C_i}$依赖于$\boldsymbol{\Phi}$，就得到机器人的总势能：

$$u(\boldsymbol{\Phi})=\sum_{i=1}^{N}u_i(\boldsymbol{\Phi})=-\sum_{i=1}^{N}m_i{}^0\boldsymbol{g}^{T}\,{}^0\boldsymbol{P}_{C_i}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial k}{\partial \dot{\boldsymbol{\Phi}}}-\frac{\partial k}{\partial \boldsymbol{\Phi}}+\frac{\partial u}{\partial \boldsymbol{\Phi}}=\boldsymbol{\tau}-\boldsymbol{B}\dot{\boldsymbol{\Phi}}$$

7.4.3 完整的拉格朗日动力学方程

- 由前述推得的机器人的总动能方程和总势能方程可得到完整的机器人动力学方程

$$k=\frac{1}{2}\dot{\boldsymbol{\Phi}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{\Phi})\dot{\boldsymbol{\Phi}}=\frac{1}{2}\begin{bmatrix}\dot{\phi}_1\\ \dot{\phi}_2\\ \vdots\\ \dot{\phi}_i\\ \vdots\\ \dot{\phi}_N\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}m_{11} & m_{12} & \cdots & m_{1j} & \cdots & m_{1N}\\ m_{21} & m_{22} & \cdots & m_{2j} & \cdots & m_{2N}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ m_{i1} & m_{i2} & \cdots & m_{ij} & \cdots & m_{iN}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ m_{N1} & m_{N2} & \cdots & m_{Nj} & \cdots & m_{NN}\end{bmatrix}\begin{bmatrix}\dot{\phi}_1\\ \dot{\phi}_2\\ \vdots\\ \dot{\phi}_j\\ \vdots\\ \dot{\phi}_N\end{bmatrix}$$

- $m_{ij}=m_{ij}(\boldsymbol{\Phi})$是矩阵$\boldsymbol{M}(\boldsymbol{\Phi})$第$i$行第$j$列元素
- 利用展开后的总动能表达式，可计算得到：

$$①\quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial k}{\partial \dot{\phi}_i}=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\begin{bmatrix}0\\ 0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}m_{11} & m_{12} & \cdots & m_{1j} & \cdots & m_{1N}\\ m_{21} & m_{22} & \cdots & m_{2j} & \cdots & m_{2N}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ m_{i1} & m_{i2} & \cdots & m_{ij} & \cdots & m_{iN}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ m_{N1} & m_{N2} & \cdots & m_{Nj} & \cdots & m_{NN}\end{bmatrix}\begin{bmatrix}\dot{\phi}_1\\ \dot{\phi}_2\\ \vdots\\ \dot{\phi}_j\\ \vdots\\ \dot{\phi}_N\end{bmatrix}+\frac{1}{2}\begin{bmatrix}\dot{\phi}_1\\ \dot{\phi}_2\\ \vdots\\ \dot{\phi}_j\\ \vdots\\ \dot{\phi}_N\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}m_{11} & m_{12} & \cdots & m_{1i} & \cdots & m_{1N}\\ m_{21} & m_{22} & \cdots & m_{2i} & \cdots & m_{2N}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ m_{j1} & m_{j2} & \cdots & m_{ji} & \cdots & m_{jN}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ m_{N1} & m_{N2} & \cdots & m_{Ni} & \cdots & m_{NN}\end{bmatrix}\begin{bmatrix}0\\ 0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}\right\}$$

7.4 机器人动力学方程的拉格朗日方法

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial k}{\partial \dot{\boldsymbol{\Phi}}}-\frac{\partial k}{\partial \boldsymbol{\Phi}}+\frac{\partial u}{\partial \boldsymbol{\Phi}}=\boldsymbol{\tau}-\boldsymbol{B}\dot{\boldsymbol{\Phi}}$$

- $\boldsymbol{M}(\boldsymbol{\Phi})$是对称矩阵，有：

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial k}{\partial \dot{\phi}_i}=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\begin{bmatrix}0\\0\\\vdots\\1\\\vdots\\0\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}m_{11} & m_{12} & \cdots & m_{1j} & \cdots & m_{1N}\\ m_{21} & m_{22} & \cdots & m_{2j} & \cdots & m_{2N}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ m_{i1} & m_{i2} & \cdots & m_{ij} & \cdots & m_{iN}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ m_{N1} & m_{N2} & \cdots & m_{Nj} & \cdots & m_{NN}\end{bmatrix}\begin{bmatrix}\dot{\phi}_1\\\dot{\phi}_2\\\vdots\\\dot{\phi}_j\\\vdots\\\dot{\phi}_N\end{bmatrix}\right\}=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\begin{bmatrix}m_{i1}\\m_{i2}\\\vdots\\m_{ij}\\\vdots\\m_{iN}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\dot{\phi}_1\\\dot{\phi}_2\\\vdots\\\dot{\phi}_j\\\vdots\\\dot{\phi}_N\end{bmatrix}\right\}$$

$$=\begin{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}m_{i1}\\\frac{\mathrm{d}}{\mathrm{d}t}m_{i2}\\\vdots\\\frac{\mathrm{d}}{\mathrm{d}t}m_{ij}\\\vdots\\\frac{\mathrm{d}}{\mathrm{d}t}m_{iN}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\dot{\phi}_1\\\dot{\phi}_2\\\vdots\\\dot{\phi}_j\\\vdots\\\dot{\phi}_N\end{bmatrix}+\begin{bmatrix}m_{i1}\\m_{i2}\\\vdots\\m_{ij}\\\vdots\\m_{iN}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\ddot{\phi}_1\\\ddot{\phi}_2\\\vdots\\\ddot{\phi}_j\\\vdots\\\ddot{\phi}_N\end{bmatrix}=\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial m_{ij}}{\partial \phi_k}\dot{\phi}_k\dot{\phi}_j+\sum_{j=1}^{N}m_{ij}\ddot{\phi}_j$$

$$\frac{\mathrm{d}}{\mathrm{d}t}m_{ij}=\sum_{k=1}^{N}\frac{\partial m_{ij}}{\partial \phi_k}\dot{\phi}_k$$

7.4 机器人动力学方程的拉格朗日方法

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial k}{\partial \dot{\boldsymbol{\Phi}}}-\frac{\partial k}{\partial \boldsymbol{\Phi}}+\frac{\partial u}{\partial \boldsymbol{\Phi}}=\boldsymbol{\tau}-\boldsymbol{B}\dot{\boldsymbol{\Phi}}$$

② $$\frac{\partial k}{\partial \phi_i}=\frac{1}{2}\begin{bmatrix}\dot{\phi}_1\\ \dot{\phi}_2\\ \vdots\\ \dot{\phi}_j\\ \vdots\\ \dot{\phi}_N\end{bmatrix}^T\begin{bmatrix}\frac{\partial m_{11}}{\partial \phi_i} & \frac{\partial m_{12}}{\partial \phi_i} & \cdots & \frac{\partial m_{1k}}{\partial \phi_i} & \cdots & \frac{\partial m_{1N}}{\partial \phi_i}\\ \frac{\partial m_{21}}{\partial \phi_i} & \frac{\partial m_{22}}{\partial \phi_i} & \cdots & \frac{\partial m_{2k}}{\partial \phi_i} & \cdots & \frac{\partial m_{2N}}{\partial \phi_i}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ \frac{\partial m_{j1}}{\partial \phi_i} & \frac{\partial m_{j2}}{\partial \phi_i} & \cdots & \frac{\partial m_{jk}}{\partial \phi_i} & \cdots & \frac{\partial m_{jN}}{\partial \phi_i}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ \frac{\partial m_{N1}}{\partial \phi_i} & \frac{\partial m_{N2}}{\partial \phi_i} & \cdots & \frac{\partial m_{Nk}}{\partial \phi_i} & \cdots & \frac{\partial m_{NN}}{\partial \phi_i}\end{bmatrix}\begin{bmatrix}\dot{\phi}_1\\ \dot{\phi}_2\\ \vdots\\ \dot{\phi}_k\\ \vdots\\ \dot{\phi}_N\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\dot{\phi}_1\\ \dot{\phi}_2\\ \vdots\\ \dot{\phi}_j\\ \vdots\\ \dot{\phi}_N\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\sum_{k=1}^{N}\frac{\partial m_{1k}}{\partial \phi_i}\dot{\phi}_k\\ \sum_{k=1}^{N}\frac{\partial m_{2k}}{\partial \phi_i}\dot{\phi}_k\\ \vdots\\ \sum_{k=1}^{N}\frac{\partial m_{jk}}{\partial \phi_i}\dot{\phi}_k\\ \vdots\\ \sum_{k=1}^{N}\frac{\partial m_{Nk}}{\partial \phi_i}\dot{\phi}_k\end{bmatrix}$$

$$=\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial m_{jk}}{\partial \phi_i}\dot{\phi}_k\,\dot{\phi}_j$$

③ $$\frac{\partial u}{\partial \phi_i}=-\sum_{j=1}^{N}m_j\,{}^0\boldsymbol{g}^T\frac{\partial\,{}^0\boldsymbol{P}_{C_j}}{\partial \phi_i}=g_i(\boldsymbol{\Phi})$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial k}{\partial \dot{\boldsymbol{\Phi}}}-\frac{\partial k}{\partial \boldsymbol{\Phi}}+\frac{\partial u}{\partial \boldsymbol{\Phi}}=\boldsymbol{\tau}-\boldsymbol{B}\dot{\boldsymbol{\Phi}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial k}{\partial \dot{\phi}_i}=\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial m_{ij}}{\partial \phi_k}\dot{\phi}_k\dot{\phi}_j+\sum_{j=1}^{N}m_{ij}\ddot{\phi}_j \quad - \quad \frac{\partial k}{\partial \phi_i}=\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial m_{jk}}{\partial \phi_i}\dot{\phi}_k\dot{\phi}_j \quad + \quad \frac{\partial u}{\partial \phi_i}=-\sum_{j=1}^{N}m_j{}^0\boldsymbol{g}^T\frac{\partial {}^0\boldsymbol{P}_{C_j}}{\partial \phi_i}=g_i(\boldsymbol{\Phi})$$

$$\sum_{j=1}^{N}m_{ij}\ddot{\phi}_j+\sum_{j=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial m_{ij}}{\partial \phi_k}-\frac{1}{2}\frac{\partial m_{jk}}{\partial \phi_i}\right)\dot{\phi}_k\dot{\phi}_j+g_i(\boldsymbol{\Phi})=\tau_i-b_i\dot{\phi}_i, i=1,2,\cdots,N$$

- 可证： $\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial m_{ij}}{\partial \phi_k}\dot{\phi}_k\dot{\phi}_j=\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial m_{ij}}{\partial \phi_k}+\frac{\partial m_{ik}}{\partial \phi_j}\right)\dot{\phi}_k\dot{\phi}_j$

- 则有：$\sum_{j=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial m_{ij}}{\partial \phi_k}-\frac{1}{2}\frac{\partial m_{jk}}{\partial \phi_i}\right)\dot{\phi}_k\dot{\phi}_j=\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{1}{2}\left(\frac{\partial m_{ij}}{\partial \phi_k}+\frac{\partial m_{ik}}{\partial \phi_j}-\frac{\partial m_{jk}}{\partial \phi_i}\right)\dot{\phi}_k\dot{\phi}_j=\sum_{j=1}^{N}\sum_{k=1}^{N}c_{kji}\dot{\phi}_k\dot{\phi}_j$

$$c_{kji}=\frac{1}{2}\left(\frac{\partial m_{ij}}{\partial \phi_k}+\frac{\partial m_{ik}}{\partial \phi_j}-\frac{\partial m_{jk}}{\partial \phi_i}\right)$$ 称为（第一类）Christoffel符号

7.4 机器人动力学方程的拉格朗日方法

$$c_{kji}=\frac{1}{2}\left(\frac{\partial m_{ij}}{\partial \phi_k}+\frac{\partial m_{ik}}{\partial \phi_j}-\frac{\partial m_{jk}}{\partial \phi_i}\right) \Rightarrow c_{jki}=\frac{1}{2}\left(\frac{\partial m_{ik}}{\partial \phi_j}+\frac{\partial m_{ij}}{\partial \phi_k}-\frac{\partial m_{kj}}{\partial \phi_i}\right)$$

- 因为$\boldsymbol{M}(\boldsymbol{\Phi})$是对称矩阵，有：

$$m_{jk}=m_{kj}$$

- 可以发现：

$$c_{jki}=c_{kji}$$

- 利用（第一类）Christoffel符号，拉格朗日动力学方程可写成更简洁的形式：

$$\sum_{j=1}^{N} m_{ij}\ddot{\phi}_j+\sum_{j=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial m_{ij}}{\partial \phi_k}-\frac{1}{2}\frac{\partial m_{jk}}{\partial \phi_i}\right)\dot{\phi}_k\dot{\phi}_j+g_i(\boldsymbol{\Phi})=\tau_i-b_i\dot{\phi}_i, i=1,2,\cdots,N$$

$$\Downarrow$$

$$\sum\nolimits_{j=1}^{N} m_{ij}\ddot{\phi}_j+\sum\nolimits_{j=1}^{N}\sum\nolimits_{k=1}^{N} c_{kji}\dot{\phi}_k\dot{\phi}_j+g_i(\boldsymbol{\Phi})=\tau_i-b_i\dot{\phi}_i, i=1,2,\cdots,N$$

7.4 机器人动力学方程的拉格朗日方法

$$\sum_{j=1}^{N} m_{ij}\ddot{\phi}_j + \sum_{j=1}^{N}\sum_{k=1}^{N} c_{kji}\dot{\phi}_k\dot{\phi}_j + g_i(\Phi) = \tau_i - b_i\dot{\phi}_i, i = 1,2,\cdots,N$$

- 将$i = 1,2,\cdots,N$所有等式写成如下的矩阵形式：

$$\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1j} & \cdots & m_{1N} \\ m_{21} & m_{22} & \cdots & m_{2j} & \cdots & m_{2N} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ m_{i1} & m_{i2} & \cdots & m_{ij} & \cdots & m_{iN} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ m_{N1} & m_{N2} & \cdots & m_{Nj} & \cdots & m_{NN} \end{bmatrix} \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \vdots \\ \ddot{\phi}_j \\ \vdots \\ \ddot{\phi}_N \end{bmatrix} + \begin{bmatrix} b_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & b_N \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \vdots \\ \dot{\phi}_i \\ \vdots \\ \dot{\phi}_N \end{bmatrix} + \begin{bmatrix} \sum_k c_{k11}\dot{\phi}_k & \sum_k c_{k21}\dot{\phi}_k & \cdots & \sum_k c_{kj1}\dot{\phi}_k & \cdots & \sum_k c_{kN1}\dot{\phi}_k \\ \sum_k c_{k12}\dot{\phi}_k & \sum_k c_{k22}\dot{\phi}_k & \cdots & \sum_k c_{kj2}\dot{\phi}_k & \cdots & \sum_k c_{kN2}\dot{\phi}_k \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \sum_k c_{k1i}\dot{\phi}_k & \sum_k c_{k2i}\dot{\phi}_k & \cdots & \sum_k c_{kji}\dot{\phi}_k & \cdots & \sum_k c_{kNi}\dot{\phi}_k \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \sum_k c_{k1N}\dot{\phi}_k & \sum_k c_{k2N}\dot{\phi}_k & \cdots & \sum_k c_{kjN}\dot{\phi}_k & \cdots & \sum_k c_{kNN}\dot{\phi}_k \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \vdots \\ \dot{\phi}_j \\ \vdots \\ \dot{\phi}_N \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_i \\ \vdots \\ g_N \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_i \\ \vdots \\ \tau_N \end{bmatrix}$$

$$\boldsymbol{M}(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \boldsymbol{C}(\boldsymbol{\Phi}, \dot{\boldsymbol{\Phi}})\dot{\boldsymbol{\Phi}} + \boldsymbol{B}\dot{\boldsymbol{\Phi}} + \boldsymbol{G}(\boldsymbol{\Phi}) = \boldsymbol{\tau}$$

- 矩阵$\boldsymbol{C}$的第(i,j)项元素被定义为：

$$c_{ij} = \sum_{k=1}^{N} c_{kji}\dot{\phi}_k$$

7.4.4 动力学方程的性质

- 将$\boldsymbol{M}(\boldsymbol{\Phi})$的第$(i,j)$项元素对时间求导，有：

$$\dot{m}_{ij}=\sum_{k=1}^{N}\frac{\partial m_{ij}}{\partial \phi_k}\dot{\phi}_k$$

- 则矩阵$\dot{\boldsymbol{M}}(\boldsymbol{\Phi})-2\boldsymbol{C}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})$的第$(i,j)$项元素是:

$$\dot{m}_{ij}-2c_{ij}=\sum_{k=1}^{N}\left(\frac{\partial m_{ij}}{\partial \phi_k}-2c_{kji}\right)\dot{\phi}_k=\sum_{k=1}^{N}\left(\frac{\partial m_{ij}}{\partial \phi_k}-\left(\frac{\partial m_{ij}}{\partial \phi_k}+\frac{\partial m_{ik}}{\partial \phi_j}-\frac{\partial m_{jk}}{\partial \phi_i}\right)\right)\dot{\phi}_k$$

$$=\sum_{k=1}^{N}\left(\frac{\partial m_{jk}}{\partial \phi_i}-\frac{\partial m_{ik}}{\partial \phi_j}\right)\dot{\phi}_k$$

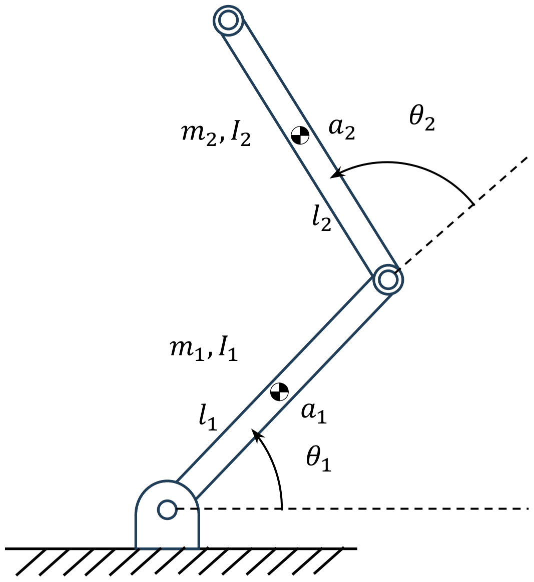
- 类似的，矩阵$\dot{\boldsymbol{M}}(\boldsymbol{\Phi})-2\boldsymbol{C}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})$的第$(j,i)$项元素是:

$$\dot{m}_{ji}-2c_{ji}=\sum_{k=1}^{N}\left(\frac{\partial m_{ik}}{\partial \phi_j}-\frac{\partial m_{jk}}{\partial \phi_i}\right)\dot{\phi}_k$$

- 可以看出$\dot{m}_{ij}-2c_{ij}=-(\dot{m}_{ji}-2c_{ji})$，矩阵$\dot{\boldsymbol{M}}(\boldsymbol{\Phi})-\boldsymbol{2C}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})$是反对称的
- <u>我们会在后面机器人控制部分用到这一性质</u>

7.4 机器人动力学方程的拉格朗日方法

- **例:计算如图所示的两连杆平面机器人的动力学方程（忽略摩擦）**
- **其中:**
  - a_1和a_2分别表示连杆1和连杆2的长度
  - l_1和l_2分别表示连杆1和连杆2质心到各自关节轴的距离
  - m_1和m_2分别表示连杆1和连杆2的质量
  - I_1和I_2分别表示连杆1和连杆2对穿过各自质心并指向纸外的轴线的转动惯量
  - τ_1和τ_2分别表示作用在关节1和关节2上的关节力矩



拉格朗日方法

$$\boldsymbol{M}(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \boldsymbol{C}\left(\boldsymbol{\Phi}, \dot{\boldsymbol{\Phi}}\right)\dot{\boldsymbol{\Phi}} + \boldsymbol{B}\dot{\boldsymbol{\Phi}} + \boldsymbol{G}(\boldsymbol{\Phi}) = \boldsymbol{\tau}$$

7.4 机器人动力学方程的拉格朗日方法

$$\boldsymbol{M}(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}}+\boldsymbol{C}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})\dot{\boldsymbol{\Phi}}+\boldsymbol{B}\dot{\boldsymbol{\Phi}}+\boldsymbol{G}(\boldsymbol{\Phi})=\boldsymbol{\tau}$$

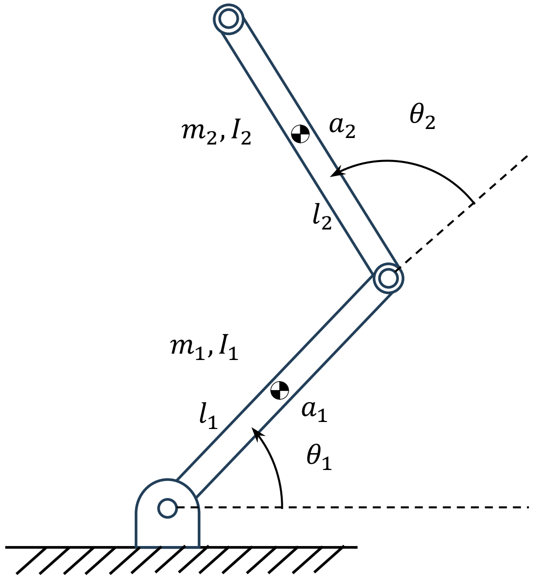
解：① 惯性矩阵$\boldsymbol{M}(\boldsymbol{\Phi})$的计算：

$$\boldsymbol{M}(\boldsymbol{\Phi})=\sum_{i=1}^{2}\left(m_i\left(\boldsymbol{J}_P^{(i)}\right)^{\mathrm{T}}\boldsymbol{J}_P^{(i)}+\left(\boldsymbol{J}_O^{(i)}\right)^{\mathrm{T}}{}_i^0\boldsymbol{R}\,{}^{C_i}\boldsymbol{I}_i\,{}_i^0\boldsymbol{R}^{\mathrm{T}}\boldsymbol{J}_O^{(i)}\right)$$

连杆1和2雅可比矩阵、旋转矩阵和惯性张量：

$$\boldsymbol{J}_P^{(1)}=\begin{bmatrix}-l_1s_1 & 0\\ l_1c_1 & 0\\ 0 & 0\end{bmatrix},\boldsymbol{J}_O^{(1)}=\begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 0\end{bmatrix},{}_1^0\boldsymbol{R}=\begin{bmatrix}c_1 & -s_1 & 0\\ s_1 & c_1 & 0\\ 0 & 0 & 1\end{bmatrix},{}^{C_1}\boldsymbol{I}_1=\begin{bmatrix}* & 0 & 0\\ 0 & * & 0\\ 0 & 0 & I_1\end{bmatrix}$$

$$\boldsymbol{J}_P^{(2)}=\begin{bmatrix}-a_1s_1-l_2s_{12} & -l_2s_{12}\\ a_1c_1+l_2c_{12} & l_2c_{12}\\ 0 & 0\end{bmatrix},\boldsymbol{J}_O^{(2)}=\begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 1\end{bmatrix},{}_2^0\boldsymbol{R}=\begin{bmatrix}c_{12} & -s_{12} & 0\\ s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{bmatrix},{}^{C_2}\boldsymbol{I}_2=\begin{bmatrix}* & 0 & 0\\ 0 & * & 0\\ 0 & 0 & I_2\end{bmatrix}$$



目标式中：

$$m_1\left(\boldsymbol{J}_P^{(1)}\right)^{\mathrm{T}}\boldsymbol{J}_P^{(1)}=m_1\begin{bmatrix}l_1^2 & 0\\ 0 & 0\end{bmatrix}$$

$$\left(\boldsymbol{J}_O^{(1)}\right)^{\mathrm{T}}{}_1^0\boldsymbol{R}\,{}^{C_1}\boldsymbol{I}_1\,{}_1^0\boldsymbol{R}^{\mathrm{T}}\boldsymbol{J}_O^{(1)}=\begin{bmatrix}I_1 & 0\\ 0 & 0\end{bmatrix}$$

$$m_2\left(\boldsymbol{J}_P^{(2)}\right)^{\mathrm{T}}\boldsymbol{J}_P^{(2)}=m_2\begin{bmatrix}a_1^2+l_2^2+2a_1l_2c_2 & l_2^2+a_1l_2c_2\\ l_2^2+a_1l_2c_2 & l_2^2\end{bmatrix}$$

$$\left(\boldsymbol{J}_O^{(2)}\right)^{\mathrm{T}}{}_2^0\boldsymbol{R}\,{}^{C_2}\boldsymbol{I}_2\,{}_2^0\boldsymbol{R}^{\mathrm{T}}\boldsymbol{J}_O^{(2)}=\begin{bmatrix}I_2 & I_2\\ I_2 & I_2\end{bmatrix}$$

$$\boldsymbol{M}(\boldsymbol{\Phi})=\begin{bmatrix}m_{11}(\theta_2) & m_{12}(\theta_2)\\ m_{21}(\theta_2) & m_{22}\end{bmatrix}$$

$$\begin{cases}m_{11}=I_1+m_1l_1^2+I_2+m_2(a_1^2+l_2^2+2a_1l_2c_2)\\ m_{12}=m_{21}=I_2+m_2(l_2^2+a_1l_2c_2)\\ m_{22}=I_2+m_2l_2^2\end{cases}$$

$$\boldsymbol{M}(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}}+\boldsymbol{C}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})\dot{\boldsymbol{\Phi}}+\boldsymbol{B}\dot{\boldsymbol{\Phi}}+\boldsymbol{G}(\boldsymbol{\Phi})=\boldsymbol{\tau}$$

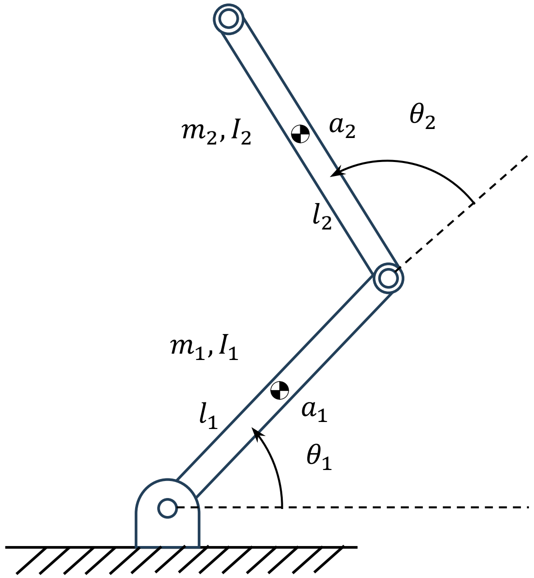
② 矩阵$\boldsymbol{C}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})$的计算：

计算 Christoffel 符号：$c_{kji}=\frac{1}{2}\left(\frac{\partial m_{ij}}{\partial \phi_k}+\frac{\partial m_{ik}}{\partial \phi_j}-\frac{\partial m_{jk}}{\partial \phi_i}\right)$

$$c_{111}=\frac{1}{2}\frac{\partial m_{11}}{\partial \theta_1}=0 \qquad c_{121}=c_{211}=\frac{1}{2}\frac{\partial m_{11}}{\partial \theta_2}=-m_2a_1l_2s_2=h \qquad c_{221}=\frac{\partial m_{12}}{\partial \theta_2}-\frac{1}{2}\frac{\partial m_{22}}{\partial \theta_1}=h$$

$$c_{112}=\frac{\partial m_{21}}{\partial \theta_1}-\frac{1}{2}\frac{\partial m_{11}}{\partial \theta_2}=-h \qquad c_{122}=c_{212}=\frac{1}{2}\frac{\partial m_{22}}{\partial \theta_1}=0 \qquad c_{222}=\frac{1}{2}\frac{\partial m_{22}}{\partial \theta_2}=0$$

$$\boldsymbol{C}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})=\begin{bmatrix} h\dot{\theta}_2 & h(\dot{\theta}_1+\dot{\theta}_2) \\ -h\dot{\theta}_1 & 0 \end{bmatrix}$$



③ 重力矢量$\boldsymbol{G}(\boldsymbol{\Phi})$的计算：

$$g_i(\boldsymbol{\Phi})=\frac{\partial u}{\partial \theta_i}=-\sum_{j=1}^{2} m_j\,{}^0\boldsymbol{g}^T\frac{\partial\, {}^0\boldsymbol{P}_{C_j}}{\partial \theta_i}$$

式中，${}^0\boldsymbol{g}=[0 \quad -g \quad 0]^{\mathrm{T}}$

$$\frac{\partial\, {}^0\boldsymbol{P}_{C_1}}{\partial \theta_1}=\begin{bmatrix}-l_1s_1\\ l_1c_1\\ 0\end{bmatrix},\frac{\partial\, {}^0\boldsymbol{P}_{C_2}}{\partial \theta_1}=\begin{bmatrix}-a_1s_1-l_2s_{12}\\ a_1c_1+l_2c_{12}\\ 0\end{bmatrix},\frac{\partial\, {}^0\boldsymbol{P}_{C_1}}{\partial \theta_2}=\begin{bmatrix}0\\0\\0\end{bmatrix},\frac{\partial\, {}^0\boldsymbol{P}_{C_2}}{\partial \theta_2}=\begin{bmatrix}-l_2s_{12}\\ l_2c_{12}\\ 0\end{bmatrix}$$

$$\boldsymbol{G}(\boldsymbol{\Phi})=\begin{bmatrix}g_1(\boldsymbol{\Phi})\\ g_2(\boldsymbol{\Phi})\end{bmatrix}=\begin{bmatrix}m_1gl_1c_1+m_2g(a_1c_1+l_2c_{12})\\ m_2gl_2c_{12}\end{bmatrix}$$

7.4 机器人动力学方程的拉格朗日方法

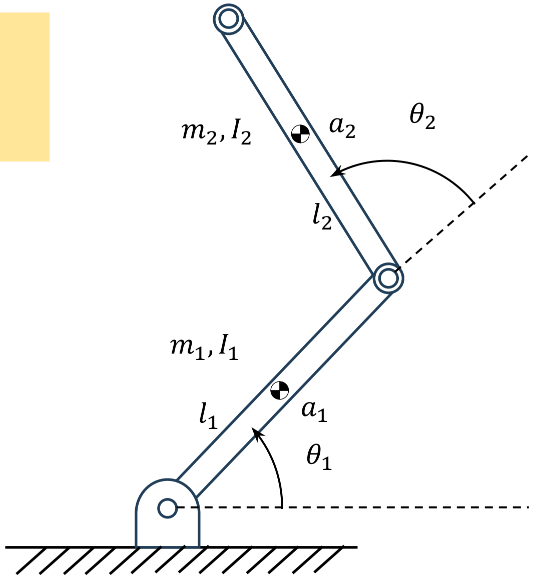
$$\boldsymbol{M}(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \boldsymbol{C}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})\dot{\boldsymbol{\Phi}} + \boldsymbol{B}\dot{\boldsymbol{\Phi}} + \boldsymbol{G}(\boldsymbol{\Phi}) = \boldsymbol{\tau}$$

$$\boldsymbol{M}(\boldsymbol{\Phi}) = \begin{bmatrix} m_{11}(\theta_2) & m_{12}(\theta_2) \\ m_{21}(\theta_2) & m_{22} \end{bmatrix}$$

$$\begin{cases} m_{11} = I_1 + m_1 l_1^2 + I_2 + m_2(a_1^2 + l_2^2 + 2a_1 l_2 c_2) \\ m_{12} = m_{21} = I_2 + m_2(l_2^2 + a_1 l_2 c_2) \\ m_{22} = I_2 + m_2 l_2^2 \end{cases}$$

$$\boldsymbol{C}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}}) = \begin{bmatrix} h\dot{\theta}_2 & h(\dot{\theta}_1 + \dot{\theta}_2) \\ -h\dot{\theta}_1 & 0 \end{bmatrix}$$

$$\boldsymbol{G}(\boldsymbol{\Phi}) = \begin{bmatrix} g_1(\boldsymbol{\Phi}) \\ g_2(\boldsymbol{\Phi}) \end{bmatrix} = \begin{bmatrix} m_1 g l_1 c_1 + m_2 g(a_1 c_1 + l_2 c_{12}) \\ m_2 g l_2 c_{12} \end{bmatrix}$$



将$\boldsymbol{M}(\boldsymbol{\Phi})$，$\boldsymbol{C}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})$和$\boldsymbol{G}(\boldsymbol{\Phi})$代入

$$\boldsymbol{M}(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \boldsymbol{C}(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})\dot{\boldsymbol{\Phi}} + \boldsymbol{B}\dot{\boldsymbol{\Phi}} + \boldsymbol{G}(\boldsymbol{\Phi}) = \boldsymbol{\tau}$$

$$\left(I_1 + m_1 l_1^2 + I_2 + m_2(a_1^2 + l_2^2 + 2a_1 l_2 c_2)\right)\ddot{\theta}_1 + \left(I_2 + m_2(l_2^2 + a_1 l_2 c_2)\right)\ddot{\theta}_2 - 2m_2 a_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 a_1 l_2 s_2 \dot{\theta}_2^2 + \left(m_1 g l_1 c_1 + m_2 g(a_1 c_1 + l_2 c_{12})\right) = \tau_1$$

$$\left(I_2 + m_2(l_2^2 + a_1 l_2 c_2)\right)\ddot{\theta}_1 + (I_2 + m_2 l_2^2)\ddot{\theta}_2 + m_2 a_1 l_2 s_2 \dot{\theta}_1^2 + m_2 g l_2 c_{12} = \tau_2$$

Thanks !