MULTIVARIABLE CALCULUS HT20 SHEET 6
Divergence theorem. Examples. Consequences.

1. Let C be a closed, positively oriented curve in $\mathbb{R}^2$ bounding a region D. Show that

$$\text{area of } D = \frac{1}{2}\int_C x\,\mathrm{d}y - y\,\mathrm{d}x.$$

Hence find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

2. Let $D \subseteq \mathbb{R}^2$ be a closed, boundary region with smooth boundary ∂D, and f be a smooth function defined in D. By applying Green's theorem in the plane with suitable functions P and Q, show that

$$\iint_D \nabla^2 f\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial D} \frac{\partial f}{\partial n}\,\mathrm{d}s.$$

3. Let R be the region $1 < a < r < b$, where r is the distance from the origin in $\mathbb{R}^2$. Find a solution of the boundary-value problem

$$\nabla^2 f + 1 = 0 \quad \text{in } R, \qquad \frac{\partial f}{\partial n} + f = 0 \quad \text{on } \partial R,$$

which is a function of r only. Show that this is the only solution, even within the class of not necessarily radial functions.

4. The temperature $T(r,\theta)$ in an annulus $a \leqslant r \leqslant b$ satisfies $\nabla^2 T = 1$ inside the annulus. On the inner boundary $\partial T/\partial n = k$, where $k > 0$ and the outer boundary is insulated.

(i) Use Exercise 2 to show the uniqueness, up to a constant, of any solution to this boundary value problem.

(ii) Find all circularly symmetric solutions $T(r)$ to

$$\nabla^2 T = \frac{\mathrm{d}^2 T}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}T}{\mathrm{d}r} = 1$$

in the annulus.

(iii) For what value of k is there a circularly symmetric solution to this boundary value problem? Interpret this value physically.

5. Let R be the region $x^2/a^2 + y^2/b^2 + z^2/c^2 \leqslant 1$ with boundary ∂R and $a, b, c > 0$. Suppose $u(x,y,z)$ satisfies $\nabla^2 u = -1$ in R and $u = 0$ on ∂R.

(i) Show that the solution u is unique.

(ii) Show that the solution u is a quadratic function of x, y, z and evaluate

$$\iint_{\partial R} \nabla u \cdot \mathrm{d}\mathbf{S}.$$

6. (Optional) *Differentiation under the integral sign* relates to the theorem that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I f(x,t)\,\mathrm{d}x = \int_I \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x,$$

which holds, under quite general hypotheses, for a function $f(x,t)$ and an interval $I \subseteq \mathbb{R}$.

(i) By differentiating with respect to a, reproduce a solution to Sheet 1, Exercise 1.

(ii) Let $a \in \mathbb{R}$. Determine and solve a differential equation involving

$$I(a) = \int_{-\infty}^{\infty} e^{-x^2}\cos 2ax\,\mathrm{d}x$$

and hence show that $I(a) = \sqrt{\pi}e^{-a^2}$.

(iii) A compressible fluid of density $\rho(x,t)$ moves with velocity $u(x,t)$ in and out of an interval $I = [\alpha, \beta]$. Explain why

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\alpha}^{\beta}\rho(x,t)\,\mathrm{d}t = \rho(\alpha,t)u(\alpha,t) - \rho(\beta,t)u(\beta,t),$$

interpreting each term physically. Hence derive the continuity equation (Sheet 5, Exercise 6).