Lemma 1: Let σ be a schedule with $\sigma[q] = k$, $\sigma[q + c] = i$, where $q \in N_n$ and $c \in N_{n-q}$. Let $\underline{\sigma}$ be an imaginary schedule that $\underline{\sigma}[l] = \sigma[l]$ $\forall l \in \{N_n \setminus \{q, q + c\}\}$ and two imaginary jobs J'_k and J'_i are in positions q and $q + c$, respectively. The processing and setup times of J'_k and J'_i on $M_1, M_2, \ldots, M_{m+1}$ are $p_{i1}, s_{i1}, p_{k2}, s_{k2}, \ldots, p_{k(m+1)}, s_{k(m+1)}$ and $p_{k1}, s_{k1}, p_{i2}, s_{i2}, \ldots, p_{i(m+1)}, s_{i(m+1)}$, respectively. The deadlines of J'_k and J'_i are d_k and d_i, respectively. If $p_{i1} + s_{i1} \leq p_{k1} + s_{k1}$, then $TT(\underline{\sigma}) \leq TT(\sigma)$.

Proof: Since $\forall l \in N_{q-1}$, $\sigma[l] = \underline{\sigma}[l]$, $f_j(\underline{\sigma}_{[l]}) = f_j(\sigma_{[l]})$ $\forall j \in N_{m+1}$ and $\forall l \in N_{q-1}$. $\forall l \in N_{q-1}$, $d_{\sigma[l]} = d_{\underline{\sigma}[l]}$, thus $TT(\underline{\sigma}, l) = TT(\sigma, l)$.

Since $p_{i1} + s_{i1} \leq p_{k1} + s_{k1}$, $f_1(\underline{\sigma}_{[q]}) \leq f_1(\sigma_{[q]})$. The processing and setup times of J'_k on $M_2, M_3, \ldots, M_{m+1}$ are the same as those of J_k, $f_d(\underline{\sigma}_{[q]}) = f_d(\sigma_{[q]})$ $\forall d = 2, \ldots, m$. Thus, $f_{m+1}(\underline{\sigma}_{[q]}) = \max\{\max_{j\in N_m}\{f_j(\underline{\sigma}_{[q]})\}, f_{m+1}(\underline{\sigma}_{[q-1]}) + s_{k(m+1)}\} + p_{k(m+1)} \leq \max\{\max_{j\in N_m}\{f_j(\sigma_{[q]})\}, f_{m+1}(\sigma_{[q-1]}) + s_{k(m+1)}\} + p_{k(m+1)} = f_{m+1}(\sigma_{[q]})$. Similarly, $f_{m+1}(\underline{\sigma}_{[l]}) \leq f_{m+1}(\sigma_{[l]})$ $\forall l = q + 1, \ldots, q + c - 1$. $\forall l = q, \ldots, q + c - 1$, $d_{\sigma[l]} = d_{\underline{\sigma}[l]}$, thus $TT(\underline{\sigma}, l) \leq TT(\sigma, l)$ $\forall l = q, \ldots, q + c - 1$.

Since the processing and setup times of J'_i on $M_1, M_2, \ldots, M_{m+1}$ are $p_{k1}, s_{k1}, p_{i2}, s_{i2}, \ldots, p_{i(m+1)}, s_{i(m+1)}$, $f_d(\underline{\sigma}_{[q+c]}) = f_d(\sigma_{[q+c]})$ $\forall d \in N_m$. Thus, $f_{m+1}(\underline{\sigma}_{[q+c]}) = \max\{\max_{j\in N_m}\{f_j(\underline{\sigma}_{[q+c]})\}, f_{m+1}(\underline{\sigma}_{[q+c-1]}) + s_{i(m+1)}\} + p_{i(m+1)} \leq \max\{\max_{j\in N_m}\{f_j(\sigma_{[q+c]})\}, f_{m+1}(\sigma_{[q+c-1]}) + s_{i(m+1)}\} + p_{i(m+1)} = f_{m+1}(\sigma_{[\tau+c]})$. Similarly, $f_{m+1}(\underline{\sigma}_{[l]}) \leq f_{m+1}(\sigma_{[l]})$ $\forall l = q + c + 1, \ldots, |\sigma|$. $\forall l = q + c, \ldots, |\sigma|$, $d_{\sigma[l]} = d_{\underline{\sigma}[l]}$, thus $TT(\underline{\sigma}, l) \leq TT(\sigma, l)$.

Since $TT(\underline{\sigma}, l) \leq TT(\sigma, l)$ holds $\forall l \in N_{|\sigma|}$, $TT(\underline{\sigma}) = \sum_{j\in N_{|\underline{\sigma}|}} TT(\underline{\sigma}, j) \leq TT(\sigma) = \sum_{j\in N_{|\sigma|}} TT(\sigma, j)$.

Lemma 2: Let σ be a schedule with $\sigma[q] = k$, $\sigma[q + c] = i$, where $q \in N_n$ and $c \in N_{n-q}$. Let $\underline{\sigma}$ be an imaginary schedule that $\underline{\sigma}[l] = \sigma[l]$ $\forall l \in \{N_{|\sigma|} \setminus \{q, q + c\}\}$ and two imaginary jobs J'_k and J'_i are in positions q and $q + c$, respectively. The deadlines of J'_k and J'_i are d_i and d_k, respectively. The processing and setup times of J'_k and J'_i on $M_1, M_2, \ldots, M_{m+1}$ are the same with those of J_k and J_i, respectively. If $d_k \geq d_i$, then $TT(\underline{\sigma}) \leq TT(\sigma)$.

Proof: Since the processing and setup times of any job in σ are the same with those of the job in the same position in $\underline{\sigma}$, $\forall l \in N_{|\sigma|}$, $f_{m+1}(\underline{\sigma}_{[l]}) = f_{m+1}(\sigma_{[l]})$. $\forall l \in \{N_{|\sigma|} \setminus \{q, q + c\}\}$, $d_{\sigma[l]} = d_{\underline{\sigma}[l]}$, thus $TT(\underline{\sigma}, l) = TT(\sigma, l)$. There are six cases for the distribution of d_i, d_k, $f_{m+1}(\sigma_{[q]})$, and $f_{m+1}(\sigma_{[q+c]})$.

Case (1): $d_i \leq d_k \leq f_{m+1}(\sigma_{[q]})$. $TT(\sigma, q) + TT(\sigma, q + c) = f_{m+1}(\sigma_{[q]}) - d_k + f_{m+1}(\sigma_{[q+c]}) - d_i$. $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c) = f_{m+1}(\underline{\sigma}_{[q]}) - d_i + f_{m+1}(\underline{\sigma}_{[q+c]}) - d_k$. Thus, $TT(\sigma, q) + TT(\sigma, q + c) = TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c)$.

Case (2): $d_k \geq d_i \geq f_{m+1}(\sigma_{[q+c]})$. $TT(\sigma, q) + TT(\sigma, q + c) = 0$. $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c) = 0$. Thus, $TT(\sigma, q) + TT(\sigma, q + c) = TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c)$.

Case (3): $f_{m+1}(\sigma_{[q]}) \leq d_i \leq d_k \leq f_{m+1}(\sigma_{[q+c]})$. $TT(\sigma, q) + TT(\sigma, q + c) = 0 + f_{m+1}(\sigma_{[q+c]}) - d_i = f_{m+1}(\sigma_{[q+c]}) - d_i$. $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c) = 0 + f_{m+1}(\underline{\sigma}_{[q+c]}) - d_k = f_{m+1}(\underline{\sigma}_{[q+c]}) - d_k$. Since $d_i \leq d_k$ and $f_{m+1}(\sigma_{[q+c]}) = f_{m+1}(\underline{\sigma}_{[q+c]})$, $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c) \leq TT(\sigma, q) + TT(\sigma,$

$q + c$).

Case (4): $d_i \leq f_{m+1}(\sigma_{[q]}) \leq d_k \leq f_{m+1}(\sigma_{[q+c]})$. $TT(\sigma, q) + TT(\sigma, q + c) = 0 + f_{m+1}(\sigma_{[q+c]}) - d_i = f_{m+1}(\sigma_{[q+c]}) - d_i$. $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c) = f_{m+1}(\underline{\sigma}_{[q]}) - d_i + f_{m+1}(\underline{\sigma}_{[q+c]}) - d_k$. Since $f_{m+1}(\underline{\sigma}_{[q]}) = f_{m+1}(\sigma_{[q]}) \leq d_k$, $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c) \leq TT(\sigma, q) + TT(\sigma, q + c)$.

Case (5): $f_{m+1}(\sigma_{[q]}) \leq d_i \leq f_{m+1}(\sigma_{[q+c]}) \leq d_k$. $TT(\sigma, q) + TT(\sigma, q + c) = 0 + f_{m+1}(\sigma_{[q+c]}) - d_i = f_{m+1}(\sigma_{[q+c]}) - d_i$. $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c) = 0$. Thus, $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c) \leq TT(\sigma, q) + TT(\sigma, q + c)$.

Case (6): $d_i \leq f_{m+1}(\sigma_{[q]}) \leq f_{m+1}(\sigma_{[q+c]}) \leq d_k$. $TT(\sigma, q) + TT(\sigma, q + c) = 0 + f_{m+1}(\sigma_{[q+c]}) - d_i = f_{m+1}(\sigma_{[q+c]}) - d_i$. $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c) = f_{m+1}(\underline{\sigma}_{[q]}) - d_i + f_{m+1}(\underline{\sigma}_{[q+c]}) - d_k$. Since $f_{m+1}(\underline{\sigma}_{[q]}) = f_{m+1}(\sigma_{[q]}) \leq d_k$, $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c) \leq TT(\sigma, q) + TT(\sigma, q + c)$.

Hence, $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + c) \leq TT(\sigma, q) + TT(\sigma, q + c)$ holds for all these six cases. Thus, $TT(\underline{\sigma}) = \sum_{j\in N_{|\underline{\sigma}|}} TT(\underline{\sigma}, j) \leq TT(\sigma) = \sum_{j\in N_{|\sigma|}} TT(\sigma, j)$.

Lemma 3: Let σ be a schedule, $q \in N_{|\sigma|}$, and $p_{min} = \min\{p_{\sigma[i](m+1)} \mid i \in \{q+1, ..., |\sigma|\}\}$. Let $\underline{\sigma}$ be an imaginary schedule that $\underline{\sigma}[l] = \sigma[l]$, $\forall l \in N_q$, and $\forall k \in \{q+1, ..., |\sigma|\}$, $J'_{\sigma[k]}$ is an imaginary job. The deadline of $J'_{\sigma[k]}$ is $d_{\sigma[k]}$. The processing and setup times of J'_k on $M_1, M_2, \ldots, M_{m+1}$ are $p_{\sigma[k]1}$, $s_{\sigma[k]1}$, $p_{\sigma[k]2}$, $s_{\sigma[k]2}$, …, p_{min}, $s_{\sigma[k](m+1)}$, respectively. Then, $TT(\underline{\sigma}) \leq TT(\sigma)$.

Proof: Since $\forall l \in N_q$, $\sigma[l] = \underline{\sigma}[l]$, $f_j(\underline{\sigma}_{[l]}) = f_j(\sigma_{[l]})$ $\forall j \in N_{m+1}$ and $\forall l \in N_q$. $\forall l \in N_q$, $d_{\sigma[l]} = d_{\underline{\sigma}[l]}$, thus $TT(\underline{\sigma}, l) = TT(\sigma, l)$.

Since the processing and setup times of $J'_{\sigma[q+1]}$ on $M_1, M_2, \ldots, M_m$ are the same with those of $J_{\sigma[q+1]}$, $f_j(\underline{\sigma}_{[q+1]}) = f_j(\sigma_{[q+1]})$ $\forall j \in N_m$. Since $p_{min} \leq p_{\sigma[q+1](m+1)}$, $f_{m+1}(\underline{\sigma}_{[q+1]}) = \max\{\max_{j\in N_m}\{f_j(\underline{\sigma}_{[q+1]})\}, f_{m+1}(\underline{\sigma}_{[q]}) + s_{\sigma[q+1](m+1)}\} + p_{min} \leq f_{m+1}(\sigma_{[q+1]}) = \max\{\max_{j\in N_m}\{f_j(\sigma_{[q+1]})\}, f_{m+1}(\sigma_{[q]}) + s_{\sigma[q+1](m+1)}\} + p_{\sigma[q+1](m+1)}$. Since $d_{\sigma[q+1]} = d_{\underline{\sigma}[q+1]}$, $TT(\underline{\sigma}, q+1) \leq TT(\sigma, q+1)$. Similarly, $TT(\underline{\sigma}, l) \leq TT(\sigma, l)$ $\forall l = q + 2, \ldots, |\sigma|$.

Since $\forall l \in N_q$, $TT(\underline{\sigma}, l) = TT(\sigma, l)$ and $\forall l \in \{q+1, ..., |\sigma|\}$, $TT(\underline{\sigma}, l) \leq TT(\sigma, l)$ Thus, $TT(\underline{\sigma}) \leq TT(\sigma)$.

Lemma 4: Let σ be a schedule, $q \in N_{|\sigma|}$, and $s_{min} = \min\{s_{\sigma[i](m+1)} \mid i \in \{q+1, ..., |\sigma|\}\}$. Let $\underline{\sigma}$ be an imaginary schedule that $\underline{\sigma}[l] = \sigma[l]$, $\forall l \in N_q$, and $\forall k \in \{q+1, ..., |\sigma|\}$, $J'_{\sigma[k]}$ is an imaginary job. The deadline of $J'_{\sigma[k]}$ is $d_{\sigma[k]}$. The processing and setup times of J'_k on $M_1, M_2, \ldots, M_{m+1}$ are $p_{\sigma[k]1}$, $s_{\sigma[k]1}$, $p_{\sigma[k]2}$, $s_{\sigma[k]2}$, …, $p_{\sigma[k](m+1)}$, s_{min}, respectively. Then, $TT(\underline{\sigma}) \leq TT(\sigma)$.

Proof: Since $\forall l \in N_q$, $\sigma[l] = \underline{\sigma}[l]$, $f_j(\underline{\sigma}_{[l]}) = f_j(\sigma_{[l]})$ $\forall j \in N_{m+1}$ and $\forall l \in N_q$. $\forall l \in N_q$, $d_{\sigma[l]} = d_{\underline{\sigma}[l]}$, thus $TT(\underline{\sigma}, l) = TT(\sigma, l)$.

Since the processing and setup times of $J'_{\sigma[q+1]}$ on $M_1, M_2, \ldots, M_m$ are the same with those of $J_{\sigma[q+1]}$, $f_j(\underline{\sigma}_{[q+1]}) = f_j(\sigma_{[q+1]})$ $\forall j \in N_m$. Since $s_{min} \leq s_{\sigma[q+1](m+1)}$, $f_{m+1}(\underline{\sigma}_{[q+1]}) = \max\{\max_{j\in N_m}\{f_j(\underline{\sigma}_{[q+1]})\}, f_{m+1}(\underline{\sigma}_{[q]}) + s_{min}\} + p_{\sigma[q+1](m+1)} \leq f_{m+1}(\sigma_{[q+1]}) = \max\{\max_{j\in N_m}\{f_j(\sigma_{[q+1]})\}, f_{m+1}(\sigma_{[q]}) + s_{\sigma[q+1](m+1)}\} + p_{\sigma[q+1](m+1)}$. Since $d_{\sigma[q+1]} = d_{\underline{\sigma}[q+1]}$, $TT(\underline{\sigma}, q+1) \leq TT(\sigma, q+1)$. Similarly, $TT(\underline{\sigma}, l) \leq TT(\sigma, l)$ $\forall l = q + 2, \ldots, |\sigma|$.

Since $\forall l \in N_q$, $TT(\underline{\sigma}, l) = TT(\sigma, l)$ and $\forall l \in \{q+1, ..., |\sigma|\}$, $TT(\underline{\sigma}, l) \leq TT(\sigma, l)$

Thus, $TT(\underline{\sigma}) \leq TT(\sigma)$.

Lemma 5: Let σ be a schedule. Then, the finish time of the next job in σ_u to be arranged after σ is no earlier than $f(\sigma, 1) = mst(\sigma, m+1) + \min_{j\in\sigma'}\{p_{j(m+1)}\}$.

Proof: The start time of the next job in σ_u to be arranged after σ is no less than $st(\sigma, j, m+1)$ and the processing time of the next job in σ_u is no less than $\min_{j\in\sigma'}\{p_{j(m+1)}\}$, thus the finish time of the next job in σ_u to be arranged is no less than $f(\sigma, 1)$.

Corollary 2: Let σ be a schedule, σ_p be a permutation of σ_u whose jobs are sorted by the processing times on M_{m+1} in ascending order, and $\sigma_p[k, m+1]$ denote the processing time of the kth job in σ_s on M_{m+1}. Then, the finish time of the second job after σ is no earlier than $f(\sigma, 2) = f(\sigma, 1) + \sigma_p[2, m+1]$; and the finish time of the kth job after σ is no earlier than $f(\sigma, k) = f(\sigma, k\text{-}1) + \sigma_p[k, m+1]$.

Proof: The start time of the next job in σ_u to be arranged after σ is no less than $mst(\sigma, m+1)$ and the sum of processing time of the next two jobs in σ_u is no less than $\sigma_p[1, m+1] + \sigma_p[2, m+1]$, thus the finish time of the second job after σ is no earlier than $mst(\sigma, m+1) + \sigma_p[1, m+1] + \sigma_p[2, m+1] = f(\sigma, 1) + \sigma_p[2, m+1]$. Similarly, the finish time of the kth job after σ is no earlier than $mst(\sigma, m+1) + \sigma_p[1, m+1] + ... + \sigma_p[k\text{-}1, m+1] + \sigma_p[k, m+1] = f(\sigma, k\text{-}1) + \sigma_p[k, m+1]$.

Lemma 6: Given a schedule σ, let $d(\sigma^*, k)$ denote the deadline of the kth job in σ^* arranged after σ. If $\exists k, j \in N_{|\sigma_u|}$ such that $k < j$ and $d(\sigma^*, k) > d(\sigma^*, j)$, then

$$\max\{f(\sigma, k) - d(\sigma^*, k), 0\} + \max\{f(\sigma, j) - d(\sigma^*, j), 0\} \geq$$
$$\max\{f(\sigma, k) - d(\sigma^*, j), 0\} + \max\{f(\sigma, j) - d(\sigma^*, k), 0\}.$$

Proof: There are six cases for the distribution of $d(\sigma^*, k)$, $d(\sigma^*, j)$, $f(\sigma, k)$, and $f(\sigma, j)$, i.e., Case (1): $d(\sigma^*, j) \leq d(\sigma^*, k) \leq f(\sigma, k)$; Case (2): $d(\sigma^*, k) \geq d(\sigma^*, j) \geq f(\sigma, j)$; Case (3): $f(\sigma, k) \leq d(\sigma^*, j) \leq d(\sigma^*, k) \leq f(\sigma, j)$; Case (4): $d(\sigma^*, j) \leq f(\sigma, k) \leq d(\sigma^*, k) \leq f(\sigma, j)$; Case (5): $f(\sigma, k) \leq d(\sigma^*, j) \leq f(\sigma, j) \leq d(\sigma^*, k)$; and Case (6): $d(\sigma^*, j) \leq f(\sigma, k) \leq f(\sigma, j) \leq d(\sigma^*, k)$. The proof of them is similar to that of the six cases in Lemma 2, and hence omitted.

Theorem 2: Let σ be a schedule, $TT(\sigma\sigma^*) \geq LB_2(\sigma_1)$.

Proof: Let $f(\sigma, \sigma^*, k)$ ($d(\sigma^*, k)$) denote the finish time (deadline) of the kth job in σ^* arranged after σ. $TT(\sigma\sigma^*) = TT(\sigma) + \sum_{k\in N_{|\sigma_u|}}\max\{f(\sigma, \sigma^*, k) - d(\sigma^*, k), 0\}$. Since $f(\sigma, \sigma^*, k) \geq f(\sigma, k)$ holds $\forall j \in N_{|\sigma_u|}$, $\sum_{k\in N_{|\sigma_u|}}\max\{f(\sigma, \sigma^*, k) - d(\sigma^*, k), 0\} \geq \sum_{k\in N_{|\sigma_u|}}\max\{f(\sigma, k) - d(\sigma^*, k), 0\}$. By Corollary 3, we know that $\sum_{k\in N_{|\sigma_u|}}\max\{f(\sigma, k) - d(\sigma^*, k), 0\} \geq \sum_{k\in N_{|\sigma_u|}}\max\{f(\sigma, k) - \sigma_d[k], 0\}$. Hence, $TT(\sigma\sigma^*) \geq LB_2(\sigma_1)$.

Lemma 7: Let σ be a schedule. Then, the finish time of the next job in σ_u to be arranged after σ is no earlier than $F(\sigma, 1) = I(\sigma, m+1) + \min_{j\in\sigma'}\{p_{j(m+1)} + s_{j(m+1)}\}$.

Proof: The idle time of M_{m+1} from $f_{m+1}(\sigma)$ to the start time of the next job in σ_u to be arranged after σ on M_{m+1} is no less than $I(\sigma, m+1)$ and the sum of processing time and setup time of the next job in σ_u is no less than $\min_{j\in\sigma'}\{p_{j(m+1)} + s_{j(m+1)}\}$, thus the finish time of the next job in σ_u to be arranged is no less than $F(\sigma, 1)$.

Corollary 4: Let σ be a schedule, σ_s be a permutation of σ_u whose jobs are sorted by the sums of processing time and setup time on M_{m+1} in ascending order, and $\sigma_s[k, m+1]$ denote the sum of processing time and setup time of the kth job in σ_s on M_{m+1}. Then, the finish time of the second job after σ is no earlier than $F(\sigma, 2) = F(\sigma, 1) + \sigma_s[2, m+1]$; and the finish time of the kth job after σ is no earlier than $F(\sigma, k) = F(\sigma, k\text{-}1) + \sigma_s[k, m+1]$.

Proof: The idle time of M_{m+1} from $f_{m+1}(\sigma)$ to the start time of the next job in σ_u to be arranged after σ on M_{m+1} is no less than $I(\sigma, m+1)$ and the sum of processing times and setup times of the next two jobs in σ_u is no less than $\sigma_s[1, m+1] + \sigma_s[2, m+1]$, thus the finish time of the second job after σ is no earlier than $I(\sigma, m+1) + \sigma_s[1, m+1] + \sigma_s[2, m+1] = F(\sigma, 1) + \sigma_s[2, m+1]$. Similarly, the finish time of the kth job after σ is no earlier than $I(\sigma, m+1) + \sigma_s[1, m+1] + ... +\sigma_s[k\text{-}1, m+1] +\sigma_s[k, m+1] = F(\sigma, k\text{-}1) + \sigma_s[k, m+1]$.

Theorem 3: Let σ be a schedule, $TT(\sigma\sigma^*) \geq LB_3(\sigma_1)$.

Proof: Let $f(\sigma, \sigma^*, k)$ ($d(\sigma^*, k)$) denote the finish time (deadline) of the kth job in σ^* arranged after σ. $TT(\sigma\sigma^*) = TT(\sigma) + \sum_{k\in N_{|\sigma_u|}}\max\{f(\sigma, \sigma^*, k) - d(\sigma^*, k), 0\}$. Since $f(\sigma, \sigma^*, k) \geq F(\sigma, k)$ holds $\forall j \in N_{|\sigma_u|}$, $\sum_{k\in N_{|\sigma_u|}}\max\{f(\sigma, \sigma^*, k) - d(\sigma^*, k), 0\} \geq \sum_{k\in N_{|\sigma_u|}}\max\{F(\sigma, k) - d(\sigma^*, k), 0\}$. By Corollary 5, we know that $\sum_{k\in N_{|\sigma_u|}}\max\{F(\sigma, k) - d(\sigma^*, k), 0\} \geq \sum_{k\in N_{|\sigma_u|}}\max\{F(\sigma, k) - \sigma_d[k], 0\}$. Hence, $TT(\sigma\sigma^*) \geq LB_3(\sigma_1)$.

Dominance rule 1: If $f(\underline{\sigma}_{[q+1]}) \leq f(\sigma_{[q+1]})$, $f(\underline{\sigma}_{[q]}) \leq f(\sigma_{[q]})$, and $f(\underline{\sigma}_{[q]}) \geq d_j$, then $TT(\underline{\sigma}) \leq TT(\sigma)$.

Proof: By Lemma 9, to prove $TT(\underline{\sigma}) \leq TT(\sigma)$, we only need to prove $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + 1) \leq TT(\sigma, q) + TT(\sigma, q + 1)$. Consider the following two cases:

Case 1: $f(\sigma_{[q]}) < d_k$. In this case, we have $d_j \leq f(\underline{\sigma}_{[q]}) \leq f(\sigma_{[q]}) < d_k$.

$TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + 1) = \max(f(\underline{\sigma}_{[q]}) - d_j, 0) + \max(f(\underline{\sigma}_{[q+1]}) - d_k, 0)$.

$TT(\sigma, q) + TT(\sigma, q + 1) = \max(f(\sigma_{[q]}) - d_k, 0) + \max(f(\sigma_{[q+1]}) - d_j, 0)$.

Since $d_j < d_k$ and $f(\sigma_{[q]}) < f(\sigma_{[q+1]})$, it can be easily proved that $TT(\sigma, q) + TT(\sigma, q + 1) > \max(f(\sigma_{[q]}) - d_j, 0) + \max(f(\sigma_{[q+1]}) - d_k, 0)$. The proof process is similar to that of Lemma 2 and hence omitted. Since $f(\underline{\sigma}_{[q]}) \leq f(\sigma_{[q]})$ and $f(\underline{\sigma}_{[q+1]}) \leq f(\sigma_{[q+1]})$, $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + 1) \leq \max(f(\sigma_{[q]}) - d_j, 0) + \max(f(\sigma_{[q+1]}) - d_k, 0)$. Hence, $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + 1) \leq TT(\sigma, q) + TT(\sigma, q + 1)$.

Case 2: $f(\sigma_{[q]}) \geq d_k$. Since $f(\underline{\sigma}_{[q+1]}) \geq f(\sigma_{[q]})$, $f(\underline{\sigma}_{[q+1]}) \geq d_k$. Since $f(\sigma_{[q+1]}) \geq f(\underline{\sigma}_{[q]})$ and $f(\underline{\sigma}_{[q]}) \geq d_j$, $f(\sigma_{[q+1]}) \geq d_j$. Hence,

$$\begin{aligned}TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + 1) &= \max(f(\underline{\sigma}_{[q]}) - d_j, 0) + \max(f(\underline{\sigma}_{[q+1]}) - d_k, 0)\\ &= f(\underline{\sigma}_{[q]}) - d_j + f(\underline{\sigma}_{[q+1]}) - d_k.\\ TT(\sigma, q) + TT(\sigma, q + 1) &= \max(f(\sigma_{[q]}) - d_k, 0) + \max(f(\sigma_{[q+1]}) - d_j, 0)\\ &= f(\sigma_{[q]}) - d_k + f(\sigma_{[q+1]}) - d_j.\end{aligned}$$

Since $f(\underline{\sigma}_{[q+1]}) \leq f(\sigma_{[q+1]})$ and $f(\underline{\sigma}_{[q]}) \leq f(\sigma_{[q]})$, $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + 1) \leq TT(\sigma, q) + TT(\sigma, q + 1)$.

Dominance rule 2: If (a) $f(\underline{\sigma}_{[q+1]}) \leq d_k$ and (b) $f(\underline{\sigma}_{[q+1]}) \leq \min_{x\in\{\sigma\backslash\sigma_{[q+1]}\}}\{\max_{1\leq k\leq m}\{\sum_{y\in\sigma_{[q+1]}}(p_{yk} + s_{yk}) + p_{xk} + s_{xk}\} - s_{x(m+1)}\}$, then $TT(\underline{\sigma}) \leq TT(\sigma)$.

Proof: Since $f(\sigma_{[q+1]}) \geq f(\underline{\sigma}_{[q]})$, $TT(\sigma, q + 1) = \max(f(\sigma_{[q+1]}) - d_j, 0) \geq \max(f(\underline{\sigma}_{[q]}) - d_j, 0) = TT(\underline{\sigma}, q)$. Similarly, we have $TT(\underline{\sigma}, q + 1) \geq TT(\sigma, q)$. Since ($a$), $TT(\underline{\sigma}, q + 1) = 0$. Hence, $TT(\sigma, q) = 0$. Hence, $TT(\sigma, q) + TT(\sigma, q + 1) = TT(\sigma, q + 1) \geq TT(\underline{\sigma}, q) = TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + 1)$.

Since (b), $f(\sigma_{[q+2]}) \geq f(\underline{\sigma}_{[q+2]})$. Similarly, we have $f(\sigma_{[l]}) \geq f(\underline{\sigma}_{[l]})$ $\forall l\in\{q+3, q+4, |\sigma|\}$. Hence, $TT(\sigma, l) \geq TT(\underline{\sigma}, l)$ $\forall l\in\{q+2, q+3, |\sigma|\}$.

Dominance rule 3: If (a) $f(\underline{\sigma}_{[q]}) \geq d_j$, ($b$) $f(\sigma_{[q]}) \geq d_k$, (c) $f(\sigma_{[q]}) \geq f(\underline{\sigma}_{[q]})$, ($d$) $p_{k(m+1)} \leq p_{j(m+1)}$, and ($e$) $s_{k(m+1)} \leq s_{j(m+1)}$, then $TT(\underline{\sigma}) \leq TT(\sigma)$.

Proof: By (a) and (b), $f(\sigma_{[q+1]}) \geq f(\underline{\sigma}_{[q]}) \geq d_j$, and $f(\underline{\sigma}_{[q+1]}) \geq f(\sigma_{[q]}) \geq d_k$. Thus, $TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + 1) = f(\underline{\sigma}_{[q]}) - d_j + f(\underline{\sigma}_{[q+1]}) - d_k$, and $TT(\sigma, q) + TT(\sigma, q + 1) = f(\sigma_{[q]}) - d_k + f(\sigma_{[q+1]}) - d_j$.

$$f(\sigma_{[q+1]}) = \max\{\max_{1\leq k\leq m}\{\textstyle\sum_{y\in\sigma_{[q+1]}}(p_{yk} + s_{yk})\}, f(\sigma_{[q]}) + s_{j(m+1)}\} + p_{j(m+1)}.$$

$$f(\underline{\sigma}_{[q+1]}) = \max\{\max_{1\leq k\leq m}\{\textstyle\sum_{y\in\sigma_{[q+1]}}(p_{yk} + s_{yk})\}, f(\underline{\sigma}_{[q]}) + s_{k(m+1)}\} + p_{k(m+1)}.$$

By (c), (d), and (e), $f(\sigma_{[q+1]}) \geq f(\underline{\sigma}_{[q+1]})$.

Hence, $TT(\sigma, q) + TT(\sigma, q + 1) \geq TT(\underline{\sigma}, q) + TT(\underline{\sigma}, q + 1)$.

Since $f(\underline{\sigma}_{[q+1]}) \leq f(\sigma_{[q+1]})$, by Lemma 9, $TT(\underline{\sigma}, l) \leq TT(\sigma, l)$ $\forall l \in \{q+2, q+3, ..., N_{|\sigma|}\}$.

Dominance rule 4: If (a) $f(\underline{\sigma}_{[q]}) \geq d_j$, ($b$) $f(\sigma_{[q]}) \geq f(\underline{\sigma}_{[q]})$, ($c$) $p_{k(m+1)} \leq p_{j(m+1)}$, and ($d$) $s_{k(m+1)} \leq s_{j(m+1)}$, then $TT(\underline{\sigma}) \leq TT(\sigma)$.

Proof: Consider the following two cases:

Case (1): $f(\sigma_{[q]}) \geq d_k$. By dominance rule 3, $TT(\underline{\sigma}) \leq TT(\sigma)$.

Case (2): $f(\sigma_{[q]}) < d_k$. By (a) and (b), $d_j \leq f(\underline{\sigma}_{[q]}) \leq f(\sigma_{[q]}) \leq d_k$.

$$f(\sigma_{[q+1]}) = \max\{\max_{1\leq k\leq m}\{\textstyle\sum_{y\in\sigma_{[q+1]}}(p_{yk} + s_{yk})\}, f(\sigma_{[q]}) + s_{j(m+1)}\} + p_{j(m+1)}.$$

$$f(\underline{\sigma}_{[q+1]}) = \max\{\max_{1\leq k\leq m}\{\textstyle\sum_{y\in\sigma_{[q+1]}}(p_{yk} + s_{yk})\}, f(\underline{\sigma}_{[q]}) + s_{k(m+1)}\} + p_{k(m+1)}.$$

By (b), (c) and (d), $f(\sigma_{[q+1]}) \geq f(\underline{\sigma}_{[q+1]})$.

Since $f(\sigma_{[q+1]}) \geq f(\underline{\sigma}_{[q+1]})$, $f(\sigma_{[q]}) \geq f(\underline{\sigma}_{[q]})$, and $f(\underline{\sigma}_{[q]}) \geq d_j$, by dominance rule 1, $TT(\underline{\sigma}) \leq TT(\sigma)$.