Roulette

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0.1 Summary

Let's assume we want to bet for the round i^{th} in a game and our target profit is \$$p$ for each round until we win. So far we lost in the previous rounds. We call The amount of bet in i^{th} round, b_i:

$$b_i = 2 \times b_{i-1}$$

Or we can say:

$$b_i = (2^{i-1}) \times p$$

In other words, if our previous bet was x, we should bet $2 \times x$ this time!

If we want to afford to lose $n-1$ times in a row and win in the n^{th} round, our budget should be:

$$budget = (2^n - 1) \times p$$

0.2 Expected value

The American roulette has 37 pockets and the Canadian has 36. We use n for the number of pockets in the wheel. Let's assume we bet on c numbers. For example for betting on number 1, $c = 1$ and for a red number $c = 18$. The payout is $36 - c$ to c. That means for every \$$c$ bet, we get \$36 − \$$c$. We also receive the original \$$c$. So our total balance for \$1 if we would win is:

$$\begin{aligned} &\frac{36 - c + c}{c} \\ &= \frac{36}{c} \end{aligned}$$

If we lose, we lose \$1. So the expected value is:

$$\begin{aligned} E &= \frac{36-c}{c} \times \frac{c}{n} - \frac{c}{c} \times \frac{n-c}{n} \\ &= \frac{36-c}{n} - \frac{n-c}{n} \\ &= \frac{36-n}{n} \\ &= \frac{36}{n} - 1 \end{aligned}$$

As you can see the expected value is not related to our choice! If we have 36 pockets, it's zero and we can call it a fair game. In Canada it's $\frac{36}{37} - 1 = -0.027$ and in the US it's $\frac{36}{38} - 1 = -0.053$. In other words, in Canada on average we should lose 3 cents per dollar and in US, 5 cents per dollar.

As an example, let's assume we have 3 chips with the value of \$$\frac{1}{3}$. We bet one of them on black, the other on an even number and the other on a number greater than 18. Note that all of our bets are in the same round. we assume $p = \frac{c}{n}$. The expected value is:

$$
\begin{aligned}
E = & \underbrace{\frac{3}{3} \times p^3}_{\text{3 wins}} \\
& + \underbrace{(\frac{2}{3} - \frac{1}{3}) \times \binom{3}{2} \times p^2 \times (1-p)}_{\text{2 wins and 1 loss}} \\
& + \underbrace{(\frac{1}{3} - \frac{2}{3}) \times \binom{3}{2} \times p \times (1-p)^2}_{\text{1 win and 2 losses}} \\
& + \underbrace{(-\frac{3}{3}) \times (1-p)^3}_{\text{3 losses}} \\
& = p^3 + \frac{1}{3} \times 3 \times p^2 \times (1-p) - \frac{1}{3} \times 3 \times p \times (1-p)^2 - (1-p)^3 \\
& = p^3 + p^2 \times (1-p) - p \times (1-p)^2 - (1-p)^3 \\
& = 2 \times p - 1
\end{aligned}
$$

Note that we should multiply the probability by $\binom{3}{2}$ when we have 2 wins and 1 loss. Because the status of the chips is one of the three sequences (W means win and L means loss):

$$
\begin{gathered}
WWL \\
WLW \\
WWL
\end{gathered}
$$

The probablity of each of these sequenes is $p^2 \times (1-p)$. We use the same logic for 2 losses and 1 win.

Now let's assume we have 2 chips with the value of $\$\frac{1}{2}$. We bet one of them on black and the other on an even number in the same round. The expected value is:

$$
\begin{aligned}
E = & (\frac{1}{2} + \frac{1}{2}) \times p^2 \\
& + (\frac{1}{2} - \frac{1}{2}) \times \binom{2}{1} \times p * (1-p) \\
& + (-\frac{1}{2} - \frac{1}{2}) \times (1-p)^2 \\
& = p^2 - (1-p)^2 \\
& = 2 \times p - 1
\end{aligned}
$$

Now let's assume we have a chip of value \$1 and we bet on a black number.

The expected value is:

$$E = 1 \times p - 1 \times (1 - p)$$
$$= 2 \times p - 1$$

So it doesn't matter what we choose, we are going to lose $2 \times p - 1 = -0.027$ per dollar in Canada! It's almost 3 cents per dollar.

0.3 Chance of winning

In each round the change of winning in Canada is $\frac{18}{37} \simeq 49\%$ and the chance of loss is $\frac{19}{37} \simeq 51\%$. In the US the chance of winning is $\frac{18}{38} \simeq 47\%$ and the chance of loss is $\frac{20}{38} \simeq 53\%$.

Please refer to this MIT lecture For a more detailed explanation.

0.3.1 Playing only one round

Let's assume we only play 1 round. No matter what is the result, we leave casino at the end of the first round. In Canada $p = \frac{18}{37}$ and in the US $p = \frac{18}{38}$. In other words we only play for 1 to 1 payout like betting on colors.

One chip

The probability of win is p and the loss is $1 - p$.

Country	Win	Loss
Canada	49%	51%
US	47%	53%

Two chips

Let's say we have two chips with the same value. We bet the first chip on black and the second on an even number. If one of the chips wins and the other loses, it's a tie. So we don't gain or lose money. So we have:

$$\text{Win} = p^2$$
$$\text{Loss} = (1 - p)^2$$
$$\text{Tie} = \binom{2}{1} \times p \times (1 - p) = 2 \times p \times (1 - p)$$

So we have:

Country	Win	Loss	Tie
Canada	24%	26%	50%
US	22%	28%	50%

Note that in Canada the chance of not losing money is 74% and in the US is 72%.

Three chips

Let's say we have 3 chips with the same value. We bet the first chip on black and the second on an even number and the third on the first 18 numbers.

At least 2 chips should be in win status to get money; otherwise we lose money. Unlike previous case, there is no tie here. So we have:

$$\text{Win} = p^3 + \binom{3}{2} \times p^2 \times (1-p)$$
$$\text{Loss} = (1-p)^3 + \binom{3}{2} \times p \times (1-p)^2$$

So we have:

Country	Win	Loss
Canada	48%	52%
US	46%	54%

As you can see this is worse than 1 chip!

Four chips

Let's assume we have chip L with value $\$d$ and three chips H_1, H_2 and H_3 with value $\$2d$. We bet L on column 1 with probability $q = \frac{12}{n}$. The payout is 2 to 1. We bet the other chips as before with payout 1 to 1. We can win (T), lose (L) or tie (T). The chance column is based on Canadian casinos:

H_1	H_2	H_3	L	Wins	Balance	Result	Chance
W	W	W	W	4	8d	W	4%
W	W	W	L	3	5d	W	8%
L	W	W	W	3	4d	W	12%
L	W	W	L	2	d	W	25%
L	L	W	W	2	0	T	12%
L	L	W	L	1	-3d	L	26%
L	L	L	W	1	-4d	L	4%
L	L	L	L	0	-7d	L	9%

For example for winning all the chips our balance will be:

$$\begin{aligned}\text{Balance} &= \underbrace{3 \times -2d - d}_{\text{bet}} + \underbrace{3 \times 4d}_{H_1 \text{ to } H_3} + \underbrace{d + 2d}_{\text{L}} \\ &= 8d\end{aligned}$$

Th probabilities are:

$$\begin{aligned}
\text{Win} &= \underbrace{p^3 \times q}_{\text{4 wins}} \\
&+ \underbrace{p^3 \times (1-q)}_{\text{3 wins}} \\
&+ \underbrace{\binom{3}{2} \times p^2 \times (1-p) \times q}_{\text{3 wins}} \\
&+ \underbrace{\binom{3}{2} \times p^2 \times (1-p) \times (1-q)}_{\text{2 wins}}
\end{aligned}$$

For tie:

$$\text{Tie} = \underbrace{\binom{3}{1} \times p * (1-p)^2 \times q}_{\text{2 wins}}$$

For loss:

$$\begin{aligned}
\text{Loss} &= \underbrace{\binom{3}{1} \times p \times (1-p)^2 \times (1-q)}_{\text{1 win}} \\
&+ \underbrace{(1-p)^3 \times q}_{\text{1 win}} \\
&+ \underbrace{(1-p)^3 \times (1-q)}_{\text{0 win}}
\end{aligned}$$

So we have:

Country	Win	Loss	Tie
Canada	48%	40%	12%
US	46%	42%	12%

In Canada the chance of not losing money is 60% and in the US is 58%. We can also calculate the expected value. We assume $d = \frac{1}{7}$ so the value of all 4 chips is 1 dollar. Again we are assuming it's a Canadian casino:

$$\begin{aligned}
E(\text{win}) &= (3 \times \frac{2}{7} + \frac{2}{7}) \times p^3 \times q \\
&+ (3 \times \frac{2}{7} - \frac{1}{7}) \times p^3 \times (1-q) \\
&+ (2 \times \frac{2}{7} - \frac{2}{7} + \frac{2}{7}) \times 3 \times p^2 \times (1-p) \times q \\
&+ (2 \times \frac{2}{7} - \frac{2}{7} - \frac{1}{7}) \times 3 \times p^2 \times (1-p) \times (1-q) \\
&= 0.201
\end{aligned}$$

and for the loss:

$$
\begin{aligned}
E(\text{loss}) &= (\frac{2}{7} - 2 \times \frac{2}{7} - \frac{1}{7}) \times 3 \times p \times (1-p)^2 \times (1-q) \\
&+ (-3 \times \frac{2}{7} + \frac{2}{7}) \times (1-p)^3 \times q \\
&+ (-3 \times \frac{2}{7} - \frac{1}{7}) \times (1-p)^3 \times (1-q) \\
&= -0.228
\end{aligned}
$$

So in total we have:

$$E = E(\text{win}) + E(\text{loss}) = -0.027$$

This is exactly $E = \frac{36}{n} - 1$. No matter what choices we make, we are going to lose money on average.

0.4 General case

Let's assume we are in round i^{th} of a game. We can calculate the balance for round i that we call it b_i. We want to make p unrealized profit/loss for this game, assuming we eventually win in this round

0.4.1 Mathematical induciton

We define b_i as the amount of bet for round i in such a way that if we win round i, our total profit for this game would be p. We use mathematical induciton and we assume we know how to solve b_{i-1}.

When $b_{i-1} < 0$

Let's assume we will win at the end of round i and our profit will be p. In round $i-1$ we lost b_{i-1} so we just need to spend b_{i-1} to cancel it. We use mathematical hypotheis and spend another b_{i-1} to cancel losses for rounds 1 to $i-2$ and gain p at the end of the round (assuming we will win):

$$
\begin{aligned}
b_i &= \sum_{j=0}^{i-1} b_j - p \\
&= \underbrace{\sum_{j=0}^{i-2} b_j - p}_{b_{i-1}} + b_{i-1} \\
&= 2 \times b_{i-1}
\end{aligned}
$$

When $b_{i-1} \geq 0$

Note that if $b_{i-1} \geq p$, we've already achieved our goal and no need to continue the game for the profit p. We may increase the target profit to a more ambition one later.

Let's assume $b_{i-1} > 0 \wedge b_{i-1} < p$ and we will win at the end of round i and our profit will be p:

$$\begin{aligned} b_{i-1} + \underbrace{x}_{bet} \underbrace{-2 \times x}_{rewards} &= p \\ \implies x &= b_{i-1} - p \end{aligned}$$

So we just need to bet on $b_{i-1} - p$ and if we win this round our profit will be p.

0.4.2 General Formula

$$b_i = \begin{cases} 2 \times b_{i-1} & b_{i-1} < 0 \\ b_{i-1} - p & b_{i-1} \geq 0 \wedge b_{i-1} < p \\ b_{i-1} & b_{i-1} \geq 0 \wedge b_{i-1} \geq p \\ b_0 - p & i = 1 \end{cases}$$

Note that we define $b_1 = b_0 - p$. Usually $b_0 = 0$, but if in the round k we decide to change p to p'. Then we can initiate a new game with $b_0 = b_k$ and profit p'.

0.5 Losing all previous rounds

Let's consider the worst case scenario and we are going to lose in n rounds in a row:

$$\begin{aligned} b_0 &= 0 \\ b_1 &= b_0 - p = -p \\ b_2 &= 2 \times b_1 = -2p \\ b_3 &= 2 \times b_2 = -4p \\ b_4 &= 2 \times b_3 = -8p \\ &\vdots \\ b_n &= 2 \times b_{n-1} = -2^{n-1} \times p \end{aligned}$$

0.6 Budget for n rounds

Now let's calculate what should be our budget if we can afford to lose in $n-1$ rounds and win at round n. We assume r_i is our loss in round i^{th} for $1 \leq i \leq n{-}1$

$$\begin{aligned} budget &= -\sum_{i=0}^{n} b_i \\ &= \sum_{i=0}^{n} 2^{i-1} \times p \\ &= (2^n - 1) \times p \end{aligned}$$

Note that we use the following formula to get the budget:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$