Roulette

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0.1 Summary

Let's assume we want to bet for the round i^{th} in a game and our target profit is $\$p$ for each round until we win. So far we lost in the previous rounds. We call The amount of bet in i^{th} round, b_i:

$$b_i = 2 \times b_{i-1}$$

Or we can say:

$$b_i = (2^{i-1}) \times p$$

In other words, if our previous bet was x, we should bet $2 \times x$ this time!

If we want to afford to lose $n-1$ times in a row and win in the n^{th} round, our budget should be:

$$budget = (2^n - 1) \times p$$

0.2 Expected value

The American roulette has 37 pockets and the Canadian has 36. We use n for the number of pockets in the wheel. Let's assume we bet on c numbers. For example for betting on number 1, $c = 1$ and for a red number $c = 18$. The payout is $36 - c$ to c. That means for every $\$c$ bet, we get $\$36 - \$c$. We also receive the original $\$c$. So our total balance for \$1 if we would win is:

$$\begin{aligned} &\frac{36 - c + c}{c} \\ &= \frac{36}{c} \end{aligned}$$

If we lose, we lose \$1. So the expected value is:

$$\begin{aligned} E &= \frac{36-c}{c} \times \frac{c}{n} - \frac{c}{c} \times \frac{n-c}{n} \\ &= \frac{36-c}{n} - \frac{n-c}{n} \\ &= \frac{36-n}{n} \\ &= \frac{36}{n} - 1 \end{aligned}$$

As you can see the expected value is not related to our choice! If we have 36 pockets, it's zero and we can call it a fair game. In Canada it's $\frac{36}{37} - 1 = -0.027$ and in the US it's $\frac{36}{38} - 1 = -0.053$. In other words, in Canada on average we should lose 3 cents per dollar and in US, 5 cents per dollar.

As an example, let's assume we have 3 chips with the value of $\$\frac{1}{3}$. We bet one of them on black, the other on an even number and the other on a number greater than 18. Note that all of our bets are in the same round. we assume $p = \frac{c}{n}$. The expected value is:

$$
\begin{aligned}
E = &\underbrace{\frac{3}{3} \times p^3}_{\text{3 wins}} \\
&+ \underbrace{(\frac{2}{3} - \frac{1}{3}) \times \binom{3}{2} \times p^2 \times (1-p)}_{\text{2 wins and 1 loss}} \\
&+ \underbrace{(\frac{1}{3} - \frac{2}{3}) \times \binom{3}{2} \times p \times (1-p)^2}_{\text{1 win and 2 losses}} \\
&+ \underbrace{(-\frac{3}{3}) \times (1-p)^3}_{\text{3 losses}} \\
&= p^3 + \frac{1}{3} \times 3 \times p^2 \times (1-p) - \frac{1}{3} \times 3 \times p \times (1-p)^2 - (1-p)^3 \\
&= p^3 + p^2 \times (1-p) - p \times (1-p)^2 - (1-p)^3 \\
&= 2 \times p - 1
\end{aligned}
$$

Note that we should multiply the probability by $\binom{3}{2}$ when we have 2 wins and 1 loss. Because the status of the chips is one of the three sequences (W means win and L means loss):

$$
\begin{gathered}
WWL \\
WLW \\
WWL
\end{gathered}
$$

The probablity of each of these sequenes is $p^2 \times (1-p)$. We use the same logic for 2 losses and 1 win.

Now let's assume we have 2 chips with the value of $\$\frac{1}{2}$. We bet one of them on black and the other on an even number in the same round. The expected value is:

$$
\begin{aligned}
E = &(\frac{1}{2} + \frac{1}{2}) \times p^2 \\
&+ (\frac{1}{2} - \frac{1}{2}) \times \binom{2}{1} \times p * (1-p) \\
&+ (-\frac{1}{2} - \frac{1}{2}) \times (1-p)^2 \\
&= p^2 - (1-p)^2 \\
&= 2 \times p - 1
\end{aligned}
$$

Now let's assume we have a chip of value \$1 and we bet on a black number.

The expected value is:

$$\begin{aligned} E &= 1 \times p - 1 \times (1-p) \\ &= 2 \times p - 1 \end{aligned}$$

So it doesn't matter what we choose, we are going to lose $2 \times p - 1 = -0.027$ per dollar in Canada! It's almost 3 cents per dollar.

0.3 Chance of winning

In each round the change of winning in Canada is $\frac{18}{37} \simeq 49\%$ and the chance of loss is $\frac{19}{37} \simeq 51\%$. In the US the chance of winning is $\frac{18}{38} \simeq 47\%$ and the chance of loss is $\frac{20}{38} \simeq 53\%$.

Please refer to this MIT lecture For a more detailed explanation.

0.4 General case

Let's assume we are in round i^{th} of a game. We can calculate the balance for round i that we call it b_i. We want to make p unrealized profit/loss for this game, assuming we eventually win in this round

0.4.1 Mathematical induciton

We define b_i as the amount of bet for round i in such a way that if we win round i, our total profit for this game would be p. We use mathematical induciton and we assume we know how to solve b_{i-1}.

When $b_{i-1} < 0$

Let's assume we will win at the end of round i and our profit will be p. In round $i-1$ we lost b_{i-1} so we just need to spend b_{i-1} to cancel it. We use mathematical hypotheis and spend another b_{i-1} to cancel losses for rounds 1 to $i-2$ and gain p at the end of the round (assuming we will win):

$$\begin{aligned} b_i &= \sum_{j=0}^{i-1} b_j - p \\ &= \underbrace{\sum_{j=0}^{i-2} b_j - p}_{b_{i-1}} + b_{i-1} \\ &= 2 \times b_{i-1} \end{aligned}$$

When $b_{i-1} \geq 0$

Note that if $b_{i-1} \geq p$, we've already achieved our goal and no need to continue the game for the profit p. We may increase the target profit to a more ambition one later.

Let's assume $b_{i-1} > 0 \land b_{i-1} < p$ and we will win at the end of round i and our profit will be p:

$$
\begin{aligned}
b_{i-1} + \underbrace{x}_{bet} \underbrace{-2 \times x}_{rewards} &= p \\
\implies x &= b_{i-1} - p
\end{aligned}
$$

So we just need to bet on $b_{i-1} - p$ and if we win this round our profit will be p.

0.4.2 General Formula

$$
b_i = \begin{cases} 2 \times b_{i-1} & b_{i-1} < 0 \\ b_{i-1} - p & b_{i-1} \geq 0 \land b_{i-1} < p \\ b_{i-1} & b_{i-1} \geq 0 \land b_{i-1} \geq p \\ b_0 - p & i = 1 \end{cases}
$$

Note that we define $b_1 = b_0 - p$. Usually $b_0 = 0$, but if in the round k we decide to change p to p'. Then we can initiate a new game with $b_0 = b_k$ and profit p'.

0.5 Losing all previous rounds

Let's consider the worst case scenario and we are going to lose in n rounds in a row:

$$
\begin{aligned}
b_0 &= 0 \\
b_1 &= b_0 - p = -p \\
b_2 &= 2 \times b_1 = -2p \\
b_3 &= 2 \times b_2 = -4p \\
b_4 &= 2 \times b_3 = -8p \\
&\vdots \\
b_n &= 2 \times b_{n-1} = -2^{n-1} \times p
\end{aligned}
$$

0.6 Budget for n rounds

Now let's calculate what should be our budget if we can afford to lose in $n-1$ rounds and win at round n. We assume r_i is our loss in round i^{th} for $1 \leq i \leq n-1$

$$\begin{aligned} budget &= -\sum_{i=0}^{n} b_i \\ &= \sum_{i=0}^{n} 2^{i-1} \times p \\ &= (2^n - 1) \times p \end{aligned}$$

Note that we use the following formula to get the budget:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$