Compound Interest

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0.1 Compound interst with only principal

Let's assume we want to invest with p amount for principal. The anual interest rate is r. The anual interest rate is distributed n times per year. We want to see how much our invevstment grows over t years:

$$p' = p \times (1 + \frac{r}{n})^{n \times t}$$

Let's explain the formula with an example. If $n = 12$ and $t = 1$, then:

$$\begin{aligned}
p_0 &= p \\
p_1 &= p_0 \times (1 + \frac{r}{n}) = p \times (1 + \frac{r}{n}) \\
p_2 &= p_1 \times (1 + \frac{r}{n}) = p \times (1 + \frac{r}{n})^2 \\
&\vdots \\
p_{12} &= p_{11} \times (1 + \frac{r}{n}) = p \times (1 + \frac{r}{n})^{12}
\end{aligned}$$

0.2 Compound interest with principal and recurring contribution

We introduced a new variable. Let's assume m is the number of times that we contribute c in a year.

0.2.1 When $n = 1 \wedge m = 1$

Let's assume $n = 1$ and we have a recurring contribution of c once every year ($m = 1$). Let's assume e_i is c as well as its growth if we start investing it in year i for $t - i$ years:

$$e_i = c \times (1 + r)^{t-i}$$

We assumed the contribution started at the end of the year. If it's at the beginning of the year, we call it b_i:

$$b_i = c \times (1 + r)^{t-i+1}$$

The total growth is:

$$\begin{aligned}
t_e &= p \times (1 + r)^t + \sum_{i=1}^{t} e_i \\
t_b &= p \times (1 + r)^t + \sum_{i=1}^{t} b_i
\end{aligned}$$

Let's calculate the second part of t_e:

$$\begin{aligned}
\sum_{i=1}^{t} e_i &= \sum_{i=1}^{t} c \times (1+r)^{t-i} \\
&= \sum_{i=0}^{t-1} c \times (1+r)^{i} \\
&= c \times \sum_{i=0}^{t-1} (1+r)^{i} \\
&= c \times (1 + (1+r)^1 + (1+r)^2 + \cdots + (1+r)^{t-1}) \\
&= c \times \frac{(1+r)^t - 1}{(1+r) - 1} \\
&= c \times \frac{(1+r)^t - 1}{r}
\end{aligned}$$

Note that we use the fact that $\sum_{i=0}^{t-1} (1+r)^i$ is geometric series. So we have:

$$t_e = p \times (1+r)^t + c \times \frac{(1+r)^t - 1}{r}$$

With a similar approach we can calcualte t_b:

$$\begin{aligned}
\sum_{i=1}^{t} b_i &= \sum_{i=1}^{t} c \times (1+r)^{t-i+1} \\
&= \sum_{i=1}^{t} c \times (1+r)^{i} \\
&= c \times ((1+r)^1 + (1+r)^2 + \cdots + (1+r)^{t}) \\
&= c \times (1+r) \times (1 + (1+r)^1 + (1+r)^2 + \cdots + (1+r)^{t-1}) \\
&= c \times (1+r) \times \frac{(1+r)^t - 1}{r}
\end{aligned}$$

So we have:

$$t_b = p \times (1+r)^t + c \times (1+r) \times \frac{(1+r)^t - 1}{r}$$

0.2.2 General formula for recurring contribution

It's a bit difficult to calculate it very accurately. However, The following formula is reasonably accurate:

$$t_e = p \times (1 + \frac{r}{n})^{n \times t} + \sum_{i=1}^{m \times t} c \times (1 + \frac{r}{n})^{n \times t - i \times \frac{n}{m}}$$
$$t_b = p \times (1 + \frac{r}{n})^{n \times t} + \sum_{i=0}^{m \times t - 1} c \times (1 + \frac{r}{n})^{n \times t - i \times \frac{n}{m}}$$