Compound Interest

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0.1 Compound interst with only principal

Let's assume we want to invest with p amount for principal. The anual interest rate is r. The anual interest rate is distributed n times per year. We want to see how much our invevstment grows over t years:

$$p' = p \times (1 + \frac{r}{n})^{n \times t}$$

Let's explain the formula with an example. If $n = 12$ and $t = 1$, then:

$$\begin{aligned}
p_0 &= p \\
p_1 &= p_0 \times (1 + \frac{r}{n}) = p \times (1 + \frac{r}{n}) \\
p_2 &= p_1 \times (1 + \frac{r}{n}) = p \times (1 + \frac{r}{n})^2 \\
&\vdots \\
p_{12} &= p_{11} \times (1 + \frac{r}{n}) = p \times (1 + \frac{r}{n})^{12}
\end{aligned}$$

0.2 Compound interest with principal and recurring contribution

We introduced a new variable. Let's assume m is the number of times that we contribute c in a year.

0.2.1 When $n = 1 \wedge m = 1$

Let's assume $n = 1$ and we have a recurring contribution of c once every year ($m = 1$). Let's assume e_i is c as well as its growth if we start investing it in year i for $t - i$ years:

$$e_i = c \times (1 + r)^{t-i}$$

We assumed the contribution started at the end of the year. If it's at the beginning of the year, we call it b_i:

$$b_i = c \times (1 + r)^{t-i+1}$$

The total growth is:

$$t_e = p \times (1 + r)^t + \sum_{i=1}^{t} e_i$$

$$t_b = p \times (1 + r)^t + \sum_{i=1}^{t} b_i$$

Let's calculate the second part of t_e:

$$\begin{aligned}
\sum_{i=1}^{t} e_i &= \sum_{i=1}^{t} c \times (1+r)^{t-i} \\
&= \sum_{i=0}^{t-1} c \times (1+r)^{i} \\
&= c \times \sum_{i=0}^{t-1} (1+r)^{i} \\
&= c \times (1 + (1+r)^1 + (1+r)^2 + \cdots + (1+r)^{t-1}) \\
&= c \times \frac{(1+r)^t - 1}{(1+r) - 1} \\
&= c \times \frac{(1+r)^t - 1}{r}
\end{aligned}$$

Note that we use the fact that $\sum_{i=0}^{t-1} (1+r)^i$ is geometric series. So we have:

$$t_e = p \times (1+r)^t + c \times \frac{(1+r)^t - 1}{r}$$

With a similar approach we can calcualte t_b:

$$\begin{aligned}
\sum_{i=1}^{t} b_i &= \sum_{i=1}^{t} c \times (1+r)^{t-i+1} \\
&= \sum_{i=1}^{t} c \times (1+r)^{i} \\
&= c \times ((1+r)^1 + (1+r)^2 + \cdots + (1+r)^{t}) \\
&= c \times (1+r) \times (1 + (1+r)^1 + (1+r)^2 + \cdots + (1+r)^{t-1}) \\
&= c \times (1+r) \times \frac{(1+r)^t - 1}{r}
\end{aligned}$$

So we have:

$$t_b = p \times (1+r)^t + c \times (1+r) \times \frac{(1+r)^t - 1}{r}$$

0.2.2 General formula for recurring contribution

Here is the formula:

$$
\begin{aligned}
t_e &= p \times (1 + \frac{r}{n})^{n \times t} + \sum_{i=1}^{m \times t} c \times (1 + \frac{r}{n})^{n \times t - i \times \frac{n}{m}} \\
t_b &= p \times (1 + \frac{r}{n})^{n \times t} + \sum_{i=0}^{m \times t - 1} c \times (1 + \frac{r}{n})^{n \times t - i \times \frac{n}{m}}
\end{aligned}
$$

Let's explain it with some examples.

$n = 12 \wedge m = 4 \wedge t = 1$

Here is an overall overview of interests and contributions:

$$
\overbrace{r_1, r_2, r_3}^{c_1}, \overbrace{r_4, r_5, r_6}^{c_2}, \overbrace{r_7, r_8, r_9}^{c_3}, \overbrace{r_{10}, r_{11}, r_{12}}^{c_4}
$$

As we defined it before e_i is the i^{th} contribution of c and its growth. We contribute it at the end of the itnerval. On the other hand, b_i is the i^{th} contribution of c and its growth in which we started the contribution at the beginning of the inetrval. We can verify e_1 and b_1. The former starts from r_4 and the latter from r_1:

$$
\begin{aligned}
e_1 &= c \times (1 + \frac{r}{12})^{12 \times 1 - 1 \times \frac{12}{4}} \\
&= c \times (1 + \frac{r}{12})^{9} \\
b_1 &= c \times (1 + \frac{r}{12})^{12 \times 1 - 0 \times \frac{12}{4}} \\
&= c \times (1 + \frac{r}{12})^{12}
\end{aligned}
$$

$n = 4 \wedge m = 12 \wedge t = 1$

Here is an overall overview of interests and contributions:

$$
\overbrace{c_1, c_2, c_3}^{r_1}, \overbrace{c_4, c_5, c_6}^{r_2}, \overbrace{c_7, c_8, c_9}^{r_3}, \overbrace{c_{10}, c_{11}, c_{12}}^{r_4}
$$

The interests of e_1 starts at c_2 and the interests of b_1 starts at c_1. This is a bit different than the other case because we contribute more than ocne per interest interval. Let's first consider the following fact:

$$
\begin{aligned}
p &= (1 + \frac{r}{n})^{\frac{1}{3}} \\
&\implies c \times p^3 = c \times (1 + \frac{1}{n})
\end{aligned}
$$

This fact helps us to find out e_1 and b_1:

$$
\begin{aligned}
e_1 &= c \times (1+\frac{r}{n})^{1-\frac{1}{3}} \times (1+\frac{r}{n})^{4-1} \\
&= c \times (1+\frac{r}{n})^{4-1\times\frac{1}{3}} \\
b_1 &= c \times (1+\frac{r}{n})^{4-0\times\frac{1}{3}}
\end{aligned}
$$

Now lets use geometric series sum formula to simplify t_e and t_b:

$$
\begin{aligned}
&\sum_{i=1}^{m\times t} c \times (1+\frac{r}{n})^{n\times t - i \times \frac{n}{m}} \\
&= \sum_{i=0}^{mt-1} c \times (1+\frac{r}{n})^{\frac{n}{m}\times i} \\
&= c \times \sum_{i=0}^{mt-1} (1+\frac{r}{n})^{\frac{n}{m}\times i} \\
&= c \times \frac{((1+\frac{r}{n})^{\frac{n}{m}})^{m\times t} - 1}{(1+\frac{r}{n})^{\frac{n}{m}} - 1} \\
&= c \times \frac{(1+\frac{r}{n})^{n\times t} - 1}{(1+\frac{r}{n})^{\frac{n}{m}} - 1}
\end{aligned}
$$

So we have:

$$
t_e = p \times (1+\frac{r}{n})^{n\times t} + c \times \frac{(1+\frac{r}{n})^{n\times t} - 1}{(1+\frac{r}{n})^{\frac{n}{m}} - 1}
$$

Now let's focus on t_b:

$$
\begin{aligned}
&\sum_{i=0}^{m\times t-1} c \times (1+\frac{r}{n})^{n\times t - i \times \frac{n}{m}} \\
&= \sum_{i=1}^{m\times t} c \times (1+\frac{r}{n})^{\frac{n}{m}\times i} \\
&= c \times \sum_{i=1}^{m\times t} (1+\frac{r}{n})^{\frac{n}{m}\times i} \\
&= c \times (1+\frac{r}{n})^{\frac{n}{m}} \times \sum_{i=0}^{mt-1} (1+\frac{r}{n})^{\frac{n}{m}\times i} \\
&= c \times (1+\frac{r}{n})^{\frac{n}{m}} \times \frac{(1+\frac{r}{n})^{n\times t} - 1}{(1+\frac{r}{n})^{\frac{n}{m}} - 1}
\end{aligned}
$$

Now we have t_b:

$$t_b = p \times (1 + \frac{r}{n})^{n \times t} + c \times (1 + \frac{r}{n})^{\frac{n}{m}} \times \frac{(1 + \frac{r}{n})^{n \times t} - 1}{(1 + \frac{r}{n})^{\frac{n}{m}} - 1}$$