Counting Irreducible Representations of General Linear Groups and Unitary Groups

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Outline

- Introduce the counting method developed by Dan Barbasch, Jia-Jun Ma, Binyong Sun, and Chen-Bo Zhu.
- Show it's application in counting the irreducible representations of general linear groups and unitary groups.

Background

- G: connected reductive algebraic group defined over $\mathbb{R}$;
- G is a real Lie group together with a Lie group homorphism $\iota : G \to \mathrm{G}(\mathbb{R})$ with open image and finite kernel;
- $\mathfrak{g}$, $\mathfrak{g}_0$ are Lie algebras of $\mathrm{G}(\mathbb{C})$, G;
- ${}^a\mathfrak{h}$: the abstract Caratan subalgebra of $\mathfrak{g}$, with root lattice $Q_{\mathfrak{g}}$, weight group $Q^{\mathfrak{g}}$, and analalytic weight lattice Q_ι $(Q_{\mathfrak{g}} \subseteq Q_\iota \subseteq Q^{\mathfrak{g}} \subseteq {}^a\mathfrak{h}^*)$;
- W: the abstract Weyl group of $\mathfrak{g}$ act on ${}^a\mathfrak{h}$;
- $\mathrm{Rep}(G)$: the category of Casselman-Wallach representations of G;
- $\mathrm{Irr}(G)$: set of isomorphism classes of irreducible objects in $\mathrm{Rep}(G)$.

- There is a partition of $\mathrm{Irr}(G)$ with respect to infinitesimal characters:

$$\mathrm{Irr}(G) = \bigsqcup_{\lambda \in W \backslash {}^a\mathfrak{h}^*} \mathrm{Irr}_\lambda(G),$$

according to work of Harish-Chandra, each set $\mathrm{Irr}_\lambda(G)$ is finite;
- According to complex associated variety (annihilator variety), there is a further partition of $\mathrm{Irr}_\lambda(G)$:

$$\mathrm{Irr}_\lambda(G) = \bigsqcup_{\mathcal{O} \in \mathrm{G}(\mathbb{C}) \backslash \mathrm{Nil}(\mathfrak{g})} \mathrm{Irr}_\lambda(G; \mathcal{O}).$$

Goal: Describe the size of each set $\mathrm{Irr}_\lambda(G; \mathcal{O})$ in terms of combinatorial data.

Counting Formula

Theorem (Barbasch, Ma, Sun, Zhu)

$$\sharp(\mathrm{Irr}_\nu(G;\mathcal{O})) \leq \sum_{\sigma\in\mathrm{Irr}(W(\Lambda);\mathcal{O})} [1_{W_\nu} : \sigma]\cdot[\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))],$$

where 1_{W_ν} denotes the trivial representation of the stabilizer W_ν of ν in W. The equality holds if the Coxeter group $W(\Lambda)$ has no simple factor of type F_4, E_6, E_7, or E_8, and G is linear or isomorphic to a real metaplectic group.

In their paper, they use this formula to count the number of special unipotent representations.

Double Cells, Special Representations

Coherent Continuation Representation

$\mathcal{R}_{\mathrm{hol}}(\mathrm{G}(\mathbb{C}))$: Grothendieck ring of finite-dimensional holomorphic representations of $\mathrm{G}(\mathbb{C})$.
$\mathcal{K}(G)$: Grothendieck group of $\mathrm{Rep}(G)$ which has a $\mathcal{R}_{\mathrm{hol}}(\mathrm{G}(\mathbb{C}))$ module structure via tensor product.

Coherent family

Let $\Lambda = \nu +_{Q_\iota} \subseteq {}^a\mathfrak{h}^*$, a Λ-coherent family is a map

$$\Phi : \Lambda \to \mathcal{K}(G),$$

such that:

- for any $\nu \in \Lambda$, $\Phi(\nu) \in \mathcal{K}_\nu(G)$,
- for any $F \in \mathcal{R}_{\mathrm{hol}}(\mathrm{G}(\mathbb{C}))$ and $\nu \in \Lambda$, $F \cdot (\Phi(\nu)) = \sum_{\mu \in \Delta(F)} \Phi(\nu + \mu)$ (where $\Delta(F)$ is the set of weights of F counted multiplicity).

Let $W(\Lambda) \subseteq W$ denote the integral Weyl group with respect to Λ.

Coherent continuation representation

Let $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ denote the complex vector space of all coherent families on Λ. It is a representation of $W(\Lambda)$ under the action

$$(w \cdot \Psi)(\nu) = \Psi(w^{-1}\nu),$$

for any $w \in W(\Lambda)$, $\Psi \in \mathrm{Coh}_\Lambda(\mathcal{K}(G))$, $\nu \in \Lambda$.

For any $\nu \in \Lambda$ we have the evaluation map

$$\begin{array}{rccc} \mathrm{ev}: & \mathrm{Coh}_\Lambda(\mathcal{K}(G)) & \longrightarrow & \mathcal{K}_\nu \\ & \Psi & \longmapsto & \Psi(\nu) \end{array}$$

Theorem (Schmid, Zuckerman)

ev is surjective for each $\nu \in \Lambda$, and bijective when ν is regular.

Harish-Chandra Cells

Theorem(Vogan's green book)

Suppose $\nu \in {}^a\mathfrak{h}^*$ dominant, $M \in \mathcal{K}_\nu(G)$ is an irreducible representation representation. Then there exist a unique coherent family $\overline{\Psi}$ characterised by the following properties:

- $\overline{\Psi}(\nu) = M$;
- If μ is dominant, then $\overline{\Psi}(\mu)$ is irreducible or zero.

There is a basis $\mathcal{B} = \{\overline{\Psi_i}\}$ of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ such that for any regular dominant μ, $\overline{\Psi_i}(\mu)$ is an irreducible representation with infinitesimal character ν.
We call $(\mathrm{Coh}_\Lambda(\mathcal{K}(G)), \mathcal{B})$ a basal representation .
We can also define basal subrepresentations.

For any subset $\mathcal{S}$ of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$, denote by $\langle \mathcal{S} \rangle$ the minimal basal subrepresentation containing $\mathcal{S}$.

Define an equivalence relation on $\mathcal{B}$ by: $\overline{\Psi_i} \approx \overline{\Psi_j}$ if and only if $\left\langle \overline{\Psi_i} \right\rangle = \left\langle \overline{\Psi_j} \right\rangle$.

The equivalence classes of $\mathcal{B}$ with respect to this relation are called Harish-Chandra cells.

Cell representations

Let $\mathcal{C}$ be a cell in $\mathcal{B}$ and put $\overline{\mathcal{C}} = \langle \mathcal{C} \rangle \cap \mathcal{B}$. Define the cell representation attached to $\mathcal{C}$ by

$$\mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C}) := \langle \overline{\mathcal{C}} \rangle \,/\, \langle \overline{\mathcal{C}} \backslash \mathcal{C} \rangle$$

Hypothesis

The set $\{\sigma \in \mathrm{Irr}(W(\Lambda)) \mid \sigma \text{ occurs in } \mathrm{Coh}_{\Lambda}(\mathcal{K}(G))(\mathcal{C})\}$ is contained in the double cell containing the special representation $\sigma_{\mathcal{C}}$.

BMSZ's Proof

$$\begin{aligned}\sharp(\mathrm{Irr}_\nu(G)) &= \dim \mathcal{K}_\nu(G) \xlongequal{\textit{Vogan}} \dim \mathrm{Coh}_\Lambda(\mathcal{K}(G))_{W_\nu} \\ &= [1_{W_\nu} : \mathrm{Coh}_\Lambda(\mathcal{K}(G))] \\ &= \sum_{\sigma \in \mathrm{Irr}(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))],\end{aligned}$$

If S is a Zariski closed $\mathrm{G}(\mathbb{C})$-stable subset of $\mathrm{Nil}(\mathfrak{g})$, then

$$\begin{aligned}\sharp(\mathrm{Irr}_{\nu,S}(G)) &= \sum_{\sigma \in \mathrm{Irr}(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_{\Lambda,S}(\mathcal{K}(G))] \\ &\xlongequal{\textit{Hypothesis}} \sum_{\sigma \in \mathrm{Irr}_S(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))].\end{aligned}$$

Combinatorial Notations