# Counting Irreducible Representations of General Linear Groups and Unitary Groups


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# Outline

- Introduce the counting method developed by Dan Barbasch, Jia-Jun Ma, Binyong Sun, and Chen-Bo Zhu in their paper: Special unipotent representations of real classical groups: Counting and reduction.
- Show its application in counting the irreducible representations of general linear groups over $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, or a real unitary groups.

# Language for today

- $\mathrm{G}$: connected reductive algebraic group defined over $\mathbb{R}$;
- $G$ is a real Lie group together with a Lie group homorphism $\iota : G \to \mathrm{G}(\mathbb{R})$ with open image and finite kernel;
- $\mathfrak{g}$, $\mathfrak{g}_0$ are Lie algebras of $\mathrm{G}(\mathbb{C})$, $G$;
- ${}^a\mathfrak{h} = \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$: the abstract Cartan subalgebra of $\mathfrak{g}$, with root lattice $Q_{\mathfrak{g}}$, weight group $Q^{\mathfrak{g}}$, and analytic weight lattice $Q_\iota$ ($Q_{\mathfrak{g}} \subseteq Q_\iota \subseteq Q^{\mathfrak{g}} \subseteq {}^a\mathfrak{h}^*$);
- $W$: the abstract Weyl group of $\mathfrak{g}$ acts on ${}^a\mathfrak{h}^*$;
- Let $\nu \in {}^a\mathfrak{h}^*$, denote by $\Lambda = \nu + Q_\iota$ the translate of analytic lattice $Q_\iota$ by $\nu$;
- $\Delta$, $\Delta(\Lambda)$ be the root system and integral root system respectively, the integral Weyl group is $W(\Lambda)$.

- $\mathrm{Rep}(G)$ is the category of Casselman-Wallach representations of $G$, whose Grothendieck group (with $\mathbb{C}$-coefficient) is denoted by $\mathcal{K}(G)$;
- $\mathrm{Irr}(G)$ is the set of isomorphism classes of irreducible objects in $\mathrm{Rep}(G)$;
- $\mathrm{Rep}_\nu(G)$ is the category of Casselman-Wallach representations of $G$ with generalized infinitesimal character $\nu$;
- $\mathrm{Irr}_\nu(G)$ is the set of isomorphism classes of irreducible objects in $\mathrm{Rep}_\nu(G)$;
- For any irreducible representation $\mathrm{V} \in \mathrm{Irr}(G)$, recall that $I = \mathrm{Ann}(\mathrm{V}) \subseteq \mathrm{U}(\mathfrak{g})$ is a primitive ideal, its associated variety $\mathrm{AV}(I) \subseteq \mathfrak{g}^*$ is the closure of a unique nilpotent orbit (Borho-Brylinski and Joseph).
- The complex associated variety (annihilator variety) of $\mathrm{V}$ is $\mathrm{AV}_{\mathbb{C}}(\mathrm{V}) = \mathrm{AV}(I) \subseteq \mathfrak{g}^*$.

# Goals

- The set $\mathrm{Irr}(G)$ admits a partition according to infinitesimal characters:
$$\mathrm{Irr}(G) = \bigsqcup_{\lambda \in W \backslash {}^a\mathfrak{h}^*} \mathrm{Irr}_\lambda(G).$$
By early work of Harish-Chandra, each subset $\mathrm{Irr}_\lambda(G)$ is finite.
- Furthermore, according to complex associated variety, the set $\mathrm{Irr}_\nu(G)$ admits a finer partition:
$$\mathrm{Irr}_\nu(G) = \bigsqcup_{\mathcal{O} \in \mathrm{G}(\mathbb{C}) \backslash \mathrm{Nil}(\mathfrak{g})} \mathrm{Irr}_\nu(G; \mathcal{O}).$$
Here $\mathrm{Irr}_\nu(G; \mathcal{O})$ is the subset of $\mathrm{Irr}_\nu(G)$ consists of irreducible representations with complex associated variety $\overline{\mathcal{O}}$.

**Goal:** Describe the size of each set $\mathrm{Irr}_\lambda(G; \mathcal{O})$ in terms of some combinatorial data.

# Counting Formula

**Theorem (Barbasch, Ma, Sun, Zhu)**

$$\sharp(\mathrm{Irr}_\nu(G;\mathcal{O})) \leq \sum_{\sigma\in\mathrm{Irr}(W(\Lambda);\mathcal{O})} [1_{W_\nu}:\sigma]\cdot[\sigma:\mathrm{Coh}_\Lambda(\mathcal{K}(G))],$$

where $1_{W_\nu}$ denotes the trivial representation of the stabilizer $W_\nu$ of $\nu$ in $W$. The equality holds if the Coxeter group $W(\Lambda)$ has no simple factor of type $F_4$, $E_6$, $E_7$, or $E_8$, and $G$ is linear or isomorphic to a real metaplectic group.

In their paper, they use this formula to count the number of special unipotent representations to construct them via theta correspondence.

# Double cells, Special representations

- Springer correspondence: $\sigma \in \mathrm{Irr}(W) \rightsquigarrow (\mathcal{O}, \mathcal{L})$; If $\mathcal{L} = 1$ we call the corresponding representation a Springer representation;
- For the Weyl group $W$, Lusztig define a class of special representations, which are Springer representations corresponding to a special nilpotent orbit.
- Lusztig also defined an equivalence relation on $\mathrm{Irr}(W)$, each equivalence class is called a double cell, $\mathrm{Irr}^{sp}(W) \leftrightarrow \{\text{double cells}\}$.
- There is a $j$-induction (also called the truncated induction) operation, $j_{W(\Lambda)}^{W} : \{\text{special representations } W(\Lambda)\} \to \{\text{Springer representations of } W\}$.

## Definition

For a nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g})$.

- $\mathrm{Irr}^{\mathrm{sp}}(W(\Lambda);\mathcal{O}) = \{\sigma \in \mathrm{Irr}(W(\Lambda)) \,|\, \sigma \text{ is special}, j_{W(\Lambda)}^{W}(\sigma) = \sigma_{\mathcal{O}}\}$;
- $\mathrm{Irr}(W(\Lambda);\mathcal{O}) = \{\sigma \in \mathrm{Irr}(W(\Lambda)) \,|\, \text{exist } \sigma_0 \in \mathrm{Irr}^{\mathrm{sp}}(W(\Lambda);\mathcal{O}), \sigma \approx \sigma_0, \}$

$\mathrm{Irr}(W(\Lambda);\mathcal{O})$ is a union of several double cells.

# Coherent continuation representations

$\mathcal{R}_{\mathrm{hol}}(\mathrm{G}(\mathbb{C}))$: Grothendieck ring of finite-dimensional holomorphic representations of $\mathrm{G}(\mathbb{C})$.
$\mathcal{K}(G)$ has a $\mathcal{R}_{\mathrm{hol}}(\mathrm{G}(\mathbb{C}))$ module structure via tensor product.

## Coherent family

Let $\Lambda = \nu + Q_\iota \subseteq {}^a\mathfrak{h}^*$, a $\Lambda$-coherent family is a map

$$\Phi : \Lambda \to \mathcal{K}(G),$$

such that:

- For any $\mu \in \Lambda$, $\Phi(\mu) \in \mathcal{K}_\mu(G)$;
- For any $F \in \mathcal{R}_{\mathrm{hol}}(\mathrm{G}(\mathbb{C}))$ and $\mu \in \Lambda$, $F \cdot (\Phi(\mu)) = \sum_{\lambda \in \Delta(F)} \Phi(\mu + \lambda)$ (where $\Delta(F)$ is the set of weights of $F$ counted multiplicity).

## Coherent continuation representation

Let $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ denote the complex vector space of all coherent families on $\Lambda$. It is a representation of $W(\Lambda)$ under the action

$$(w \cdot \Psi)(\nu) = \Psi(w^{-1}\nu),$$

for any $w \in W(\Lambda)$, $\Psi \in \mathrm{Coh}_\Lambda(\mathcal{K}(G))$, $\nu \in \Lambda$.

For any $\mu \in \Lambda$ we have the evaluation map

$$\begin{array}{rlcl} \mathrm{ev}: & \mathrm{Coh}_\Lambda(\mathcal{K}(G)) & \longrightarrow & \mathcal{K}_\mu(G) \\ & \Psi & \longmapsto & \Psi(\mu) \end{array}$$

## Theorem (Schmid, Zuckerman)

$\mathrm{ev}$ is surjective for each $\mu \in \Lambda$, and bijective when $\mu$ is regular.

# Harish-Chandra cells

## Theorem

Suppose $\nu \in {}^a\mathfrak{h}^*$ dominant, $M \in \mathcal{K}_\nu(G)$ is an irreducible representation. Then there exist a unique coherent family $\overline{\Psi}$ characterised by the following properties:

- $\overline{\Psi}(\nu) = M$;
- If $\mu$ is dominant, then $\overline{\Psi}(\mu)$ is irreducible or zero.

There is a basis $\mathcal{B} = \{\overline{\Psi}_i\}$ of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ such that for any regular dominant $\mu$, $\overline{\Psi}_i(\mu)$ is an irreducible representation with infinitesimal character $\mu$ (there is also a basis $\Psi_i$ of standard modules).
We view $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ as a basal representation with basal elements $\mathcal{B}$.
We can also define basal subrepresentations, which are subrepresentations of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ spanned by a subset of $\mathcal{B}$.

- For any subset $\mathcal{S}$ of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$, denote by $\langle \mathcal{S} \rangle$ the minimal basal subrepresentation containing $\mathcal{S}$.
- Define an equivalence relation on $\mathcal{B}$ by: $\overline{\Psi}_i \approx \overline{\Psi}_j$ if and only if $\langle \overline{\Psi}_i \rangle = \langle \overline{\Psi}_j \rangle$.
- The equivalence classes of $\mathcal{B}$ under this relation are called Harish-Chandra cells.

**Cell representations**

Let $\mathcal{C}$ be a cell in $\mathcal{B}$ and put $\overline{\mathcal{C}} = \langle \mathcal{C} \rangle \cap \mathcal{B}$. Define the cell representation attached to $\mathcal{C}$ by

$$\mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C}) := \langle \overline{\mathcal{C}} \rangle / \langle \overline{\mathcal{C}} \backslash \mathcal{C} \rangle$$

It is easy to see that the image of $\mathcal{C}$ under this quotient form a basis of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C})$.

# Kazhdan-Lusztig cells

- Consider the category $\mathrm{Rep}(\mathfrak{g}, \mathfrak{b})$ of finite generated $\mathrm{U}(\mathfrak{g})$-module which is locally finite over $\mathrm{U}(\mathfrak{b})$, denote its Grothendieck group by $\mathcal{K}(\mathfrak{g}, \mathfrak{b})$;
- The Grothendieck ring $\mathcal{R}(\mathfrak{g}, Q_\iota)$ ($\simeq \mathcal{R}_{\mathrm{hol}}(\mathrm{G}(\mathbb{C}))$) acts on $\mathcal{K}(\mathfrak{g}, \mathfrak{b})$ via tensor product, so we can also define a coherent family $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$;
- It has two basis $\{\Psi_w \mid w \in W\}$, $\{\overline{\Psi}_w \mid w \in W\}$ where $\Psi_w(\mu) = \mathrm{M}(w\mu)$, $\overline{\Psi}_w(\mu) = \mathrm{L}(w\mu)$ for regular dominant $\mu \in {}^a\mathfrak{h}^*$;
- Define the $W \times W(\Lambda)$ action explicitly by

$$(w_1, w_2) \cdot \Psi_w = \Psi_{w_1 w w_2^{-1}} \text{ for all } w_1 \in W, w_2 \in W(\Lambda);$$

- We view $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ as a basal representation with basal elements $\{\overline{\Psi}_w\}$.

- The cells of $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$ under $W \times W(\Lambda)$ action is called a Kazhdan-Lusztig cells (two-side cells);
- Each cell representation $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))(\mathcal{C})$ contains a unique special representation of $W(\Lambda)$ called $\sigma_{\mathcal{C}}$, this induce a bijection between Kazhdan-Lusztig cells and special representations of $W(\Lambda)$;
- As a representation of $W \times W(\Lambda)$,

$$\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))(\mathcal{C}) \simeq \sum_{\sigma \text{ in the double cell contain } \sigma_{\mathcal{C}}} (\mathrm{Ind}_{W(\Lambda)}^{W}\sigma) \otimes \sigma.$$

# Comparison of HC cells and KL cells

- For every Harish-Chandra cell we can attach a Kazhdan-Lusztig cell to it (via comparing their annihilator ideals); furthermore, we have a map $\{\text{Harish-Chandra cells}\} \to \mathrm{Irr}^{sp}(W(\Lambda))$, $\mathcal{C} \mapsto \sigma_{\mathcal{C}}$.
- It is a well-known fact that every representation in a Harish-Chandra cell $\mathcal{C}$ has the same complex associated variety $\overline{\mathcal{O}_{\mathcal{C}}}$, where $\mathcal{O}_{\mathcal{C}}$ is the nilpotent orbit corresponding to $j_{W(\Lambda)}^{W}\sigma_{\mathcal{C}}$.

**Conjecture**

The set $\{\sigma \in \mathrm{Irr}(W(\Lambda)) \mid \sigma \text{ occurs in } \mathrm{Coh}_{\Lambda}(\mathcal{K}(G))(\mathcal{C})\}$

is contained in the double cell containing the special representation $\sigma_{\mathcal{C}}$.

BMSZ proved this conjecture holds under some technical assumptions ($W(\Lambda)$ has no simple factor of type $F_4$, $E_6$, $E_7$, or $E_8$, and $G$ is linear or isomorphic to a real metaplectic group).

# BMSZ's proof of the counting formula

If $S$ is a Zariski closed $\mathrm{G}(\mathbb{C})$-stable subset of $\mathrm{Nil}(\mathfrak{g})$, then

$$
\begin{aligned}
\sharp(\mathrm{Irr}_{\nu,S}(G)) &= \dim \mathcal{K}_{\nu,S}(G) = \dim \mathrm{Coh}_{\Lambda}(\mathcal{K}_S(G))_{W_\nu} \\
&= [1_{W_\nu} : \mathrm{Coh}_{\Lambda}(\mathcal{K}_S(G))] \\
&= \sum_{\sigma \in \mathrm{Irr}(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_{\Lambda}(\mathcal{K}_{\mathrm{S}}(G))] \\
&\overset{\text{Conjecture}}{=\!=\!=} \sum_{\sigma \in \mathrm{Irr}_S(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_{\Lambda}(\mathcal{K}(G))].
\end{aligned}
$$

# Classical groups considered

| Label $\star$ | Classical Lie Group $G$ | Complex Lie Group $\mathrm{G}(\mathbb{C})$ |
|---|---|---|
| $A^{\mathbb{R}}$ | $\mathrm{GL}_n(\mathbb{R})$ | $\mathrm{GL}_n(\mathbb{C})$ |
| $A^{\mathbb{H}}$ | $\mathrm{GL}_{\frac{n}{2}}(\mathbb{H})$ ($n$ is even) | $\mathrm{GL}_n(\mathbb{C})$ |
| $A^{\mathbb{C}}$ | $\mathrm{GL}_n(\mathbb{C})$ | $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ |
| $A$ | $\mathrm{U}(p, q)$ | $\mathrm{GL}_n(\mathbb{C})$ ($n = p + q$) |

Identification:

$$
{}^a\mathfrak{h}^* = \begin{cases} \mathbb{C}^n, & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A\}; \\ \mathbb{C}^n \times \mathbb{C}^n, & \text{if } \star = A^{\mathbb{C}}. \end{cases}
$$

$$
Q_\iota = \begin{cases} \mathbb{Z}^n \subseteq \mathbb{C}^n = {}^a\mathfrak{h}^*, & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A\}; \\ \mathbb{Z}^n \times \mathbb{Z}^n \subseteq \mathbb{C}^n \times \mathbb{C}^n = {}^a\mathfrak{h}^*, & \text{if } \star = A^{\mathbb{C}}, \end{cases}
$$

# Combinatorial notions

## Painted Young Diagram (type $A^{\mathbb{R}}$)

A painting on a Young diagram $\imath$ of type $A^{\mathbb{R}}$ is a map (we place a symbol in each box)

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, c, d\}$$

With the following properties

- if we remove the boxes painted with $\{d\}, \{c, d\}$, the remainder still constitutes a Young diagram;
- every column of $\imath$ has at most one box painted with $c$, and has at most one box painted with $d$;
- every row of $\imath$ has an even number of boxes painted with $\bullet$.

A painted Young diagram is a pair $(\imath, \mathcal{P})$ consisting of a Young diagram $\imath$ and a painting $\mathcal{P}$ on $\imath$. Denote by $\mathrm{P}_{A^{\mathbb{R}}}(\imath)$ the set of paintings on $\imath$ of type $A^{\mathbb{R}}$.

## Example

The following represents a painted Young diagram.

$$\begin{array}{|c|c|c|c|}
\hline
\bullet & \bullet & c & d \\
\hline
\bullet & \bullet & d \\
\cline{1-3}
c \\
\cline{1-1}
d \\
\cline{1-1}
\end{array}$$

Each of the following does not represent a painted Young diagram.

$$\begin{array}{|c|c|}
\hline
d & c \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|}
\hline
c & d \\
\hline
c \\
\cline{1-1}
\end{array}$$

## Assigned Young Diagram

For a Young diagram $\iota$, and a partition $[d_1, \cdots, d_k]$ of $|\iota|$. An assignment of type $[d_1, d_2, \cdots, d_N]$ on $\iota$ is a map

$$\mathcal{Q} : \mathrm{Box}(\iota) \to \{1, 2, \cdots, N\}$$

With the following properties

- for each $i \in \{1, 2, \cdots, N\}$, the preimage $\mathcal{P}^{-1}(i)$ has exactly $d_i$ elements;
- for each $1 \leq n \leq N$, if we remove the boxes assigned with $\{n+1, \cdots, N\}$, the reminder still constitutes a Young diagram;
- each positive integer occurs at most once in each column.

An assigned Young diagram of type $[d_1, d_2, \cdots, d_N]$ is a pair $(\iota, \mathcal{Q})$ consisting of a Young diagram $\iota$ and an assignment $\mathcal{Q}$ of type $[d_1, d_2, \cdots, d_N]$ on $\iota$. Denote by $\mathrm{A}_{[d_1, d_2, \cdots, d_N]}(\iota)$ the set of all assignments on $\iota$ of type $[d_1, d_2, \cdots, d_N]$.

# Counting result for $\mathrm{GL}_n(\mathbb{R})$: The integral case

$G = \mathrm{GL}_n(\mathbb{R})$ ($n \in \mathbb{N}$). $\mathcal{O}$ is a nilpotent orbits in $\mathfrak{g}$, denote the corresponding Young diagram by $\iota(\mathcal{O})$.
If $\nu \in {}^a\mathfrak{h}^* = \mathbb{C}^n$ is integral, its coordinates can be permuted such that

$$\nu = (\underbrace{\lambda_1, \cdots, \lambda_1}_{d_1}, \underbrace{\lambda_2, \cdots, \lambda_2}_{d_2}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{d_k}) \in \mathbb{C}^n,$$

where $[d_1, d_2, \cdots, d_k]$ is a partition of $n$, and the $\lambda_i \in \mathbb{C}$ satisfy the condition $\lambda_i - \lambda_j \in \mathbb{Z} \setminus \{0\}$ for any $i \neq j$.

**Theorem**

$$\sharp(\mathrm{Irr}_\nu(\mathrm{GL}_n(\mathbb{R}); \mathcal{O})) = \sharp\left(\mathrm{P}_{A^{\mathbb{R}}}(\iota(\mathcal{O}))\right) \cdot \sharp\left(\mathrm{A}_{[d_1, \cdots, d_k]}(\iota(\mathcal{O}))\right).$$

# Sketch of the calculation

Coherent continuation representations can be explicitly computed via the Hecke algebra module structure described by Lusztig and Vogan.

$$\mathrm{Coh}_{\Lambda}(\mathcal{K}(\mathrm{GL}_n(\mathbb{R}))) = \bigoplus_{2r+i\leq n} \mathrm{Ind}_{\mathrm{W}_r\times \mathrm{S}_i \times \mathrm{S}_{n-2r-i}}^{\mathrm{S}_n} \epsilon \otimes 1 \otimes 1.$$

The computation makes essential use of *Pieri's rule* and Frobenius reciprocity.

$$\begin{aligned}\sharp(\mathrm{Irr}_\nu(\mathrm{GL}_n(\mathbb{R});\mathcal{O})) &= [1_{W_\nu} : \sigma_{\mathcal{O}}]\cdot[\sigma_{\mathcal{O}} : \mathrm{Coh}_{\Lambda}(\mathcal{K}(\mathrm{GL}_n(\mathbb{R})))] \\ &= [\sigma_{\mathcal{O}} : \mathrm{Ind}_{W_\nu}^{W} 1_{W_\nu}]\cdot[\sigma_{\mathcal{O}} : \mathrm{Coh}_{\Lambda}(\mathcal{K}(\mathrm{GL}_n(\mathbb{R})))] \\ &= \sharp\left(\mathrm{A}_{[d_1,\cdots,d_k]}(\iota(\mathcal{O}))\right)\cdot \sharp\left(\mathrm{P}_{A^{\mathbb{R}}}(\iota(\mathcal{O}))\right).\end{aligned}$$

# Example: Minimal representations

Let $\nu \in {}^a\mathfrak{h}^*$ be a regular integral infinitesimal character, and let $\mathcal{O}_{\min}$ denote the minimal nilpotent orbit.

There are $n-1$ assignments of type $[\underbrace{1, 1, \ldots, 1}_{n}]$ on it, given by

$$\begin{array}{|c|c|} \hline 1 & i{+}1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} i & \multicolumn{1}{c}{} \\ \cline{1-1} i{+}2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} n & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \text{for } 1 \leq i \leq n-1.$$

Since there are 4 distinct paintings on $\iota_{\mathcal{O}}$, we obtain exactly 4(n-1) minimal representations with the fixed infinitesimal character $\nu$.

# Non-integral case

For an arbitrary $\nu \in {}^a\mathfrak{h}^*$, its coordinates can be permuted such that

$$\nu = (\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_r) \in \mathbb{C}^n,$$

$$\boldsymbol{\lambda}_i = (\underbrace{\lambda_{i,1}, \cdots, \lambda_{i,1}}_{d_{i,1}}, \cdots, \underbrace{\lambda_{i,k_i}, \cdots, \lambda_{i,k_i}}_{d_{i,k_i}}) \in \mathbb{C}^{e_i} \quad (e_i \geq 1),$$

where each $\boldsymbol{\lambda}_i$ is integral but $(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j) \in \mathbb{C}^{e_i+e_j}$ is not integral for any $i \neq j$, $[d_{i,1}, \cdots d_{i,k_i}]$ is a partition of $e_i$, and the condition $\lambda_{i,p} - \lambda_{i,q} \in \mathbb{Z} \setminus \{0\}$ holds for any $p \neq q$.

## Theorem

$$\sharp(\mathrm{Irr}_\nu(G; \mathcal{O})) = \sum_{\substack{(\iota_1, \cdots, \iota_r) \in \mathrm{YD}_{e_1} \times \cdots \times \mathrm{YD}_{e_r} \\ \iota_1 \overset{r}{\sqcup} \iota_2 \cdots \overset{r}{\sqcup} \iota_r = \iota(\mathcal{O})}} \prod_{i=1}^{r} \sharp(\mathrm{Irr}_{\boldsymbol{\lambda}_i}(\mathrm{GL}_{e_i}(\mathbb{R}); \mathcal{O}_{\iota_i}))$$

# Key observation

There are natural isomorphism of $W(\Lambda)$-representations:

$$\mathrm{Coh}_{\Lambda_1}(\mathcal{K}(\mathrm{GL}_{e_1}(\mathbb{R}))) \otimes \cdots \otimes \mathrm{Coh}_{\Lambda_r}(\mathcal{K}(\mathrm{GL}_{e_r}(\mathbb{R}))) \to \mathrm{Coh}_{\Lambda}(\mathcal{K}(\mathrm{GL}_n(\mathbb{R})))$$
$$\Psi_1 \otimes \cdots \otimes \Psi_r \mapsto \Psi$$

where $\Psi(\mu) = \mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{R})} \Psi_1(\mu) \otimes \cdots \otimes \Psi_r(\mu)$ is also a standard module for regular dominant $\mu \in {}^a\mathfrak{h}^*$.
Actually, this isomorphism also takes irreducible objects to irreducible objects.

# Combinatorial data

## Painted Young diagram (type $A$)

A painting on a Young diagram $\iota$ of type $A$ is a map (we place a symbol in each box)

$$\mathcal{P} : \mathrm{Box}(\iota) \to \{\bullet, s, r\}$$

With the following properties

- if we remove the boxes painted with $\{s\}, \{s, r\}$, the remainder still constitutes a Young diagram;
- every row of $\iota$ has at most one box painted with $c$, and has at most one box painted with $d$;
- every row of $\iota$ has an even number of boxes painted with $\bullet$.

A painted Young diagram is a pair $(\iota, \mathcal{P})$ consisting of a Young diagram $\iota$ and a painting $\mathcal{P}$ on $\iota$. Denote by $\mathrm{P}_A(\iota)$ the set of paintings on $\iota$ of type $A$.

Let $\iota$ be a Young diagram and $\mathcal{P}$ be a painting on $\iota$ of type $A$. Define the signature of $\mathcal{P}$ to be the pair of non-negative integers

$$(p_{\mathcal{P}}, q_{\mathcal{P}}) := \left( \frac{\sharp(\mathcal{P}^{-1}(\bullet))}{2} + \sharp(\mathcal{P}^{-1}(s)), \frac{\sharp(\mathcal{P}^{-1}(\bullet))}{2} + \sharp(\mathcal{P}^{-1}(r)) \right),$$

for every $p, q \in \mathbb{N}$ such that $p + q = |\iota|$, we define

$$\mathrm{P}_A^{p,q}(\iota) := \{\mathcal{P} \in \mathrm{P}_A(\iota) \,|\, (p_{\mathcal{P}}, q_{\mathcal{P}}) = (p, q)\}.$$

# Counting result for $\mathrm{U}(p, q)$: The integral case

$G = \mathrm{U}(p, q)$ $(p, q \in \mathbb{N})$. $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$, denote the corresponding Young diagram by $\iota(\mathcal{O})$.
If $\nu \in {}^a\mathfrak{h}^* = \mathbb{C}^n$ $(n = p + q)$ is integral, that is, the differences of its coordinates are integral, then its coordinates can be permuted such that

$$\nu = (\underbrace{\lambda_1, \cdots, \lambda_1}_{d_1}, \underbrace{\lambda_2, \cdots, \lambda_2}_{d_2}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{d_k}) \in \mathbb{C}^n,$$

where $[d_1, d_2, \cdots, d_k]$ is a partition of $n$, and the $\lambda_i \in \mathbb{C}$ satisfy the condition $\lambda_i - \lambda_j \in \mathbb{Z} \backslash \{0\}$ for any $i \neq j$.

## Theorem

If $\lambda_1 \in \frac{n-1}{2} + \mathbb{Z}$, then

$$\sharp(\mathrm{Irr}_\nu(\mathrm{U}(p,q);\mathcal{O})) = \sharp\left(\mathrm{P}_A^{p,q}(\iota(\mathcal{O}))\right) \cdot \sharp\left(\mathrm{A}_{[d_1,\cdots,d_k]}(\iota(\mathcal{O}))\right).$$

If $\lambda_1 \in \frac{n}{2} + \mathbb{Z}$, then

$$\sharp(\mathrm{Irr}_\nu(\mathrm{U}(p,q);\mathcal{O})) = \sharp\left(\mathrm{A}_{[d_1,\cdots,d_k]}(\iota(\mathcal{O}))\right) \cdot \delta_{p,q}.$$

Otherwise, $\sharp(\mathrm{Irr}_\nu(\mathrm{U}(p,q);\mathcal{O})) = 0$.

# Example: Generic representations

Let $G = \mathrm{U}(n, n)$, and let $\mathcal{O}_{\mathrm{prin}}$ denote the principal nilpotent orbit.

If the coordinates of $\nu$ are all half-integers, there is only one assignment on $\iota(\mathcal{O}_{\mathrm{prin}})$, and exactly two paintings:

$$\boxed{\bullet}\boxed{\bullet}\boxed{\cdots}\boxed{\bullet}\boxed{\bullet}\boxed{\bullet} \quad \text{and} \quad \boxed{\bullet}\boxed{\bullet}\boxed{\cdots}\boxed{\bullet}\boxed{s}\boxed{r}$$

Hence, there are exactly 2 generic representations with this infinitesimal character.

If the coordinates of $\nu$ are all integers, then there is only one assignment. Therefore, there is only 1 generic representation with this infinitesimal character.

Thank you