

# Counting Irreducible Representations of General Linear Groups and Unitary Groups

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- $\text{Rep}(G)$  is the category of Casselman-Wallach representations of  $G$ , with the Grothendieck group  $\mathcal{K}(G)$ ;
- $\text{Irr}(G)$  is the set of isomorphism classes of irreducible objects in  $\text{Rep}(G)$ ,  $\text{Irr}_{\lambda}(G)$  is the subset with infinitesimal character  $\lambda \in {}^a\mathfrak{h}^*$ .

- The set  $\text{Irr}(G)$  admits a partition according to infinitesimal characters:

$$\text{Irr}(G) = \bigsqcup_{\lambda \in \mathcal{W} \setminus {}^a\mathfrak{h}^*} \text{Irr}_\lambda(G).$$

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- Further, using the notion of the complex associated variety (also called the annihilator variety, which is always the **closure of a unique nilpotent orbit** by results of Borho-Brylinski and Joseph), the set  $\text{Irr}_\lambda(G)$  admits a finer partition:

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**Goal:** Describe the size of each set  $\text{Irr}_\lambda(G; \mathcal{O})$  in terms of some combinatorial data.

Theorem (Barbasch, Ma, Sun, Zhu)

$$\#(\mathrm{Irr}_\nu(G; \mathcal{O})) \leq \sum_{\sigma \in \mathrm{Irr}(W(\Lambda); \mathcal{O})} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))],$$

where  $1_{W_\nu}$  denotes the trivial representation of the stabilizer  $W_\nu$  of  $\nu$  in  $W$ . The **equality holds** if the Coxeter group  $W(\Lambda)$  has no simple factor of type  $F_4$ ,  $E_6$ ,  $E_7$ , or  $E_8$ , and  $G$  is linear or isomorphic to a real metaplectic group.

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In their paper, they use this formula to count the number of special unipotent representations to construct them via theta correspondence.

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- There is a  **$j$ -induction** (also called the truncated induction) operation,  $j_{W(\Lambda)}^W : \{\text{special representations } W(\Lambda)\} \rightarrow \{\text{Springer representations of } W\}$ .

## Definition

For a nilpotent orbit  $\mathcal{O} \in \text{Nil}(\mathfrak{g})$ .

- $\text{Irr}^{\text{sp}}(W(\Lambda); \mathcal{O}) = \{\sigma \in \text{Irr}(W(\Lambda)) \mid \sigma \text{ is special, } j_{W(\Lambda)}^W(\sigma) = \sigma_{\mathcal{O}}\};$

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$\text{Irr}(W(\Lambda); \mathcal{O})$  is a union of several double cells.

# Coherent continuation representations

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## Coherent family

Let  $\Lambda = \nu + Q_{\ell} \subseteq {}^a\mathfrak{h}^*$ , a  **$\Lambda$ -coherent family** is a map

$$\Phi : \Lambda \rightarrow \mathcal{K}(G),$$

such that:

- for any  $\nu \in \Lambda$ ,  $\Phi(\nu) \in \mathcal{K}_{\nu}(G)$ ;
- for any  $F \in \mathcal{R}_{\text{hol}}(G(\mathbb{C}))$  and  $\nu \in \Lambda$ ,  
 $F \cdot (\Phi(\nu)) = \sum_{\mu \in \Delta(F)} \Phi(\nu + \mu)$  (where  $\Delta(F)$  is the set of weights of  $F$  counted multiplicity).

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### Coherent continuation representation

Let  $\text{Coh}_\Lambda(\mathcal{K}(G))$  denote the complex vector space of all coherent families on  $\Lambda$ . It is a representation of  $W(\Lambda)$  under the action

$$(w \cdot \Psi)(\nu) = \Psi(w^{-1}\nu),$$

for any  $w \in W(\Lambda)$ ,  $\Psi \in \text{Coh}_\Lambda(\mathcal{K}(G))$ ,  $\nu \in \Lambda$ .

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For any  $\nu \in \Lambda$  we have the evaluation map

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### Theorem (Schmid, Zuckerman)

$\text{ev}$  is surjective for each  $\nu \in \Lambda$ , and bijective when  $\nu$  is regular.

## Theorem(Vogan's green book)

Suppose  $\nu \in {}^a\mathfrak{h}^*$  dominant,  $M \in \mathcal{K}_\nu(G)$  is an irreducible representation. Then there exist a unique coherent family  $\overline{\Psi}$  characterised by the following properties:

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We can also define basal subrepresentations, which are subrepresentations of  $\text{Coh}_\Lambda(\mathcal{K}(G))$  spanned by a subset of  $\mathcal{B}$ .

- For any subset  $\mathcal{S}$  of  $\text{Coh}_\Lambda(\mathcal{K}(G))$ , denote by  $\langle \mathcal{S} \rangle$  the minimal basal subrepresentation containing  $\mathcal{S}$ .

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### Cell representations

Let  $\mathcal{C}$  be a cell in  $\mathcal{B}$  and put  $\overline{\mathcal{C}} = \langle \mathcal{C} \rangle \cap \mathcal{B}$ . Define the cell representation attached to  $\mathcal{C}$  by

$$\text{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C}) := \langle \overline{\mathcal{C}} \rangle / \langle \overline{\mathcal{C}} \setminus \mathcal{C} \rangle$$

- There is also a coherent continuation theory ( $\text{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ ) for **highest weight representations** of complex reductive Lie algebra  $\mathfrak{g}$ , similarly there is a notion of **two-side cells** (Kazhdan-Lusztig calls), which is 1 – 1 correspondence with special representations in  $\text{Irr}(W(\Lambda))$ .
- Every Harish-Chandra cell corresponds to a Kazhdan-Lusztig cell (via comparing their annihilator ideals); furthermore, we have a map  $\{\text{Harish} - \text{Chandra cells}\} \rightarrow \text{Irr}^{sp}(W(\Lambda)), \mathcal{C} \mapsto \sigma_{\mathcal{C}}$ .
- In fact every representation in a Harish-Chandra cell  $\mathcal{C}$  has the **same complex associated variety**  $\overline{\mathcal{O}_{\mathcal{C}}}$ , here  $\mathcal{O}_{\mathcal{C}}$  is the nilpotent orbit corresponds to  $j_{W(\Lambda)}^W \sigma_{\mathcal{C}}$ .

### Conjecture

The set  $\{\sigma \in \text{Irr}(W(\Lambda)) \mid \sigma \text{ occurs in } \text{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C})\}$  is contained in the double cell containing the special representation  $\sigma_{\mathcal{C}}$ .

# BMSZ's proof of the counting formula

BMSZ proved that this conjecture holds under some technical assumptions ( $W(\Lambda)$  has no simple factor of type  $F_4$ ,  $E_6$ ,  $E_7$ , or  $E_8$ , and  $G$  is linear or isomorphic to a real metaplectic group).

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$$\begin{aligned}\sharp(\mathrm{Irr}_\nu(G)) &= \dim \mathcal{K}_\nu(G) \stackrel{\text{Vogan}}{=} \dim \mathrm{Coh}_\Lambda(\mathcal{K}(G))_{W_\nu} \\ &= [1_{W_\nu} : \mathrm{Coh}_\Lambda(\mathcal{K}(G))] \\ &= \sum_{\sigma \in \mathrm{Irr}(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))],\end{aligned}$$



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If  $S$  is a Zariski closed  $G(\mathbb{C})$ -stable subset of  $\mathrm{Nil}(\mathfrak{g})$ , then

$$\begin{aligned}\#(\mathrm{Irr}_{\nu,S}(G)) &= \sum_{\sigma \in \mathrm{Irr}(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_{\Lambda,S}(\mathcal{K}(G))] \\ &\stackrel{\text{Conjecture}}{=} \sum_{\sigma \in \mathrm{Irr}_S(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))].\end{aligned}$$

# Classical groups considered

Label $\star$	Classical Lie Group $G$	Complex Lie Group $G(\mathbb{C})$
$A^{\mathbb{R}}$	$GL_n(\mathbb{R})$	$GL_n(\mathbb{C})$
$A^{\mathbb{H}}$	$GL_{\frac{n}{2}}(\mathbb{H})$ ( $n$ is even)	$GL_n(\mathbb{C})$
$A^{\mathbb{C}}$	$GL_n(\mathbb{C})$	$GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$
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Identification:

$${}^a\mathfrak{h}^* = \begin{cases} \mathbb{C}^n, & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A\}; \\ \mathbb{C}^n \times \mathbb{C}^n, & \text{if } \star = A^{\mathbb{C}}. \end{cases}$$

$$Q_{\iota} = \begin{cases} \mathbb{Z}^n \subseteq \mathbb{C}^n = {}^a\mathfrak{h}^*, & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A\}; \\ \mathbb{Z}^n \times \mathbb{Z}^n \subseteq \mathbb{C}^n \times \mathbb{C}^n = {}^a\mathfrak{h}^*, & \text{if } \star = A^{\mathbb{C}}, \end{cases}$$

## Painted Young Diagram (type $A^{\mathbb{R}}$ )

A painting on a Young diagram  $\iota$  of type  $A^{\mathbb{R}}$  is a map (we place a symbol in each box)

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A painted Young diagram is a pair  $(\iota, \mathcal{P})$  consisting of a Young diagram  $\iota$  and a painting  $\mathcal{P}$  on  $\iota$ . Denote by  $P_{A^{\mathbb{R}}}(\iota)$  the set of paintings on  $\iota$  of type  $A^{\mathbb{R}}$ .



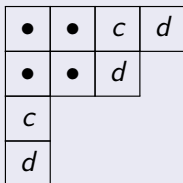
## Example

The following represents a painted Young diagram.

•	•	$c$	$d$
•	•	$d$	
$c$			
$d$			

## Example

The following represents a painted Young diagram.



Each of the following does not represent a painted Young diagram.



## Assigned Young Diagram

For a Young diagram  $\iota$ , and a partition  $[d_1, \dots, d_k]$  of  $|\iota|$ . An assignment of type  $[d_1, d_2, \dots, d_N]$  on  $\iota$  is a map

$$\mathcal{Q} : \text{Box}(\iota) \rightarrow \{1, 2, \dots, N\}$$

With the following properties

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With the following properties

- for each  $i \in \{1, 2, \dots, N\}$ , the preimage  $P^{-1}(i)$  has exactly  $d_i$  elements;
- for each  $1 \leq n \leq N$ , if we remove the boxes assigned with  $\{n+1, \dots, N\}$ , the reminder still constitutes a Young diagram;
- each positive integer occurs at most once in each column.

An assigned Young diagram of type  $[d_1, d_2, \dots, d_N]$  is a pair  $(\iota, Q)$  consisting of a Young diagram  $\iota$  and an assignment  $Q$  of type  $[d_1, d_2, \dots, d_N]$  on  $\iota$ . Denote by  $A_{[d_1, d_2, \dots, d_N]}(\iota)$  the set of all assignments on  $\iota$  of type  $[d_1, d_2, \dots, d_N]$ .

# Counting result for $GL_n(\mathbb{R})$ : The integral case

$G = GL_n(\mathbb{R})$  ( $n \in \mathbb{N}$ ).  $\mathcal{O}$  is a nilpotent orbits in  $\mathfrak{g}$ , denote the corresponding Young diagram by  $\iota(\mathcal{O})$ .

If  $\nu \in {}^a\mathfrak{h}^* = \mathbb{C}^n$  is integral, its coordinates can be permuted such that

$$\nu = (\underbrace{\lambda_1, \dots, \lambda_1}_{d_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{d_2}, \dots, \underbrace{\lambda_k, \dots, \lambda_k}_{d_k}) \in \mathbb{C}^n,$$

where  $[d_1, d_2, \dots, d_k]$  is a partition of  $n$ , and the  $\lambda_i \in \mathbb{C}$  satisfy the condition  $\lambda_i - \lambda_j \in \mathbb{Z} \setminus \{0\}$  for any  $i \neq j$ .



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## Theorem

$$\sharp(\text{Irr}_\nu(GL_n(\mathbb{R}); \mathcal{O})) = \sharp(P_{A^{\mathbb{R}}}(\iota(\mathcal{O}))) \cdot \sharp(A_{[d_1, \dots, d_k]}(\iota(\mathcal{O}))).$$

# Sketch of the calculation

Coherent continuation representations can be explicitly computed via the Hecke algebra module structure described by Lusztig and Vogan.

$$\mathrm{Coh}_\Lambda(\mathcal{K}(\mathrm{GL}_n(\mathbb{R}))) = \bigoplus_{2r+i \leq n} \mathrm{Ind}_{W_r \times S_i \times S_{n-2r-i}}^{S_n} \epsilon \otimes 1 \otimes 1.$$

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The computation makes essential use of **Pieri's rule and Frobenius reciprocity**.

$$\begin{aligned} \sharp(\mathrm{Irr}_\nu(\mathrm{GL}_n(\mathbb{R}); \mathcal{O})) &= [1_{W_\nu} : \sigma_{\mathcal{O}}] \cdot [\sigma_{\mathcal{O}} : \mathrm{Coh}_\Lambda(\mathcal{K}(\mathrm{GL}_n(\mathbb{R})))] \\ &= [\sigma_{\mathcal{O}} : \mathrm{Ind}_{W_\nu}^W 1_{W_\nu}] \cdot [\sigma_{\mathcal{O}} : \mathrm{Coh}_\Lambda(\mathcal{K}(\mathrm{GL}_n(\mathbb{R})))] \\ &= \sharp(A_{[d_1, \dots, d_k]}(\iota(\mathcal{O}))) \cdot \sharp(P_{A^\mathbb{R}}(\iota(\mathcal{O}))). \end{aligned}$$

## Example: Minimal representations

Let  $\nu \in {}^a\mathfrak{h}^*$  be a regular integral infinitesimal character, and let  $\mathcal{O}_{\min}$  denote the **minimal nilpotent orbit**.

There are  $n - 1$  assignments of type  $\underbrace{[1, 1, \dots, 1]}_n$  on it, given by

1	$i+1$
2	
$\vdots$	
$i$	
$i+2$	
$\vdots$	
$n$	

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 $\quad i+1 \quad \text{for } 1 \leq i \leq n - 1.$

Since there are 4 distinct paintings on  $\iota_{\mathcal{O}}$ , we obtain exactly  **$4(n-1)$  minimal representations** with the fixed infinitesimal character  $\nu$ .

# Non-integral case

For an arbitrary  $\nu \in {}^a\mathfrak{h}^*$ , its coordinates can be permuted such that

$$\nu = (\lambda_1, \dots, \lambda_r) \in \mathbb{C}^n,$$

$$\lambda_i = (\underbrace{\lambda_{i,1}, \dots, \lambda_{i,1}}_{d_{i,1}}, \dots, \underbrace{\lambda_{i,k_i}, \dots, \lambda_{i,k_i}}_{d_{i,k_i}}) \in \mathbb{C}^{e_i} \quad (e_i \geq 1),$$

where each  $\lambda_i$  is integral but  $(\lambda_i, \lambda_j) \in \mathbb{C}^{e_i+e_j}$  is not integral for any  $i \neq j$ ,  $[d_{i,1}, \dots, d_{i,k_i}]$  is a partition of  $e_i$ , and the condition  $\lambda_{i,p} - \lambda_{i,q} \in \mathbb{Z} \setminus \{0\}$  holds for any  $p \neq q$ .

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## Theorem

$$\#(\text{Irr}_\nu(G; \mathcal{O})) = \sum_{\substack{(\iota_1, \dots, \iota_r) \in \text{YD}_{e_1} \times \dots \times \text{YD}_{e_r} \\ \iota_1 \sqcup^r \iota_2 \dots \sqcup^r \iota_r = \iota(\mathcal{O})}} \prod_{i=1}^r \#(\text{Irr}_{\lambda_i}(\text{GL}_{e_i}(\mathbb{R}); \mathcal{O}_{\iota_i}))$$

There are natural isomorphism of  $W(\Lambda)$ -representations:

$$\mathrm{Coh}_{\Lambda_1}(\mathcal{K}(\mathrm{GL}_{e_1}(\mathbb{R}))) \otimes \cdots \otimes \mathrm{Coh}_{\Lambda_r}(\mathcal{K}(\mathrm{GL}_{e_r}(\mathbb{R}))) \rightarrow \mathrm{Coh}_{\Lambda}(\mathcal{K}(\mathrm{GL}_n(\mathbb{R})))$$
$$\overline{\Psi}_1 \otimes \cdots \otimes \overline{\Psi}_r \mapsto \overline{\Psi}$$

where  $\overline{\Psi}(\mu) = \mathrm{Ind}_{\mathcal{P}}^{\mathrm{GL}_n(\mathbb{R})} \overline{\Psi}_1(\mu) \otimes \cdots \otimes \overline{\Psi}_r(\mu)$  is still irreducible for regular dominant  $\mu \in {}^a\mathfrak{h}^*$ .



## Painted Young diagram (type $A$ )

A painting on a Young diagram  $\iota$  of type  $A$  is a map (we place a symbol in each box)

$$\mathcal{P} : \text{Box}(\iota) \rightarrow \{\bullet, s, r\}$$

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Let  $\iota$  be a Young diagram and  $\mathcal{P}$  be a painting on  $\iota$  of type  $A$ .  
 Define the signature of  $\mathcal{P}$  to be the pair of non-negative integers

$$(p_{\mathcal{P}}, q_{\mathcal{P}}) := \left( \frac{\sharp(\mathcal{P}^{-1}(\bullet))}{2} + \sharp(\mathcal{P}^{-1}(s)), \frac{\sharp(\mathcal{P}^{-1}(\bullet))}{2} + \sharp(\mathcal{P}^{-1}(r)) \right),$$

for every  $p, q \in \mathbb{N}$  such that  $p + q = |\iota|$ , we define

$$P_A^{p,q}(\iota) := \{\mathcal{P} \in P_A(\iota) \mid (p_{\mathcal{P}}, q_{\mathcal{P}}) = (p, q)\}.$$

# Counting result for $U(p, q)$ : The integral case

$G = U(p, q)$  ( $p, q \in \mathbb{N}$ ).  $\mathcal{O}$  is a nilpotent orbit in  $\mathfrak{g}$ , denote the corresponding Young diagram by  $\iota(\mathcal{O})$ .

If  $\nu \in {}^a\mathfrak{h}^* = \mathbb{C}^n$  ( $n = p + q$ ) is integral, that is, the differences of its coordinates are integral, then its coordinates can be permuted such that

$$\nu = (\underbrace{\lambda_1, \dots, \lambda_1}_{d_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{d_2}, \dots, \underbrace{\lambda_k, \dots, \lambda_k}_{d_k}) \in \mathbb{C}^n,$$

where  $[d_1, d_2, \dots, d_k]$  is a partition of  $n$ , and the  $\lambda_i \in \mathbb{C}$  satisfy the condition  $\lambda_i - \lambda_j \in \mathbb{Z} \setminus \{0\}$  for any  $i \neq j$ .

## Theorem

If  $\lambda_1 \in \frac{n-1}{2} + \mathbb{Z}$ , then

$$\sharp(\text{Irr}_\nu(\mathcal{U}(p, q); \mathcal{O})) = \sharp(P_A^{p,q}(\iota(\mathcal{O}))) \cdot \sharp(A_{[d_1, \dots, d_k]}(\iota(\mathcal{O}))) .$$

If  $\lambda_1 \in \frac{n}{2} + \mathbb{Z}$ , then

$$\sharp(\text{Irr}_\nu(\mathcal{U}(p, q); \mathcal{O})) = \sharp(A_{[d_1, \dots, d_k]}(\iota(\mathcal{O}))) \cdot \delta_{p,q}.$$

Otherwise,  $\sharp(\text{Irr}_\nu(\mathcal{U}(p, q); \mathcal{O})) = 0$ .



## Example: Generic representations

Let  $G = \mathrm{U}(n, n)$ , and let  $\mathcal{O}_{\mathrm{prin}}$  denote the principal nilpotent orbit.

If the coordinates of  $\nu$  are all **half-integers**, there is only one assignment on  $\iota(\mathcal{O}_{\mathrm{prin}})$ , and exactly two paintings:

$$\begin{array}{|c|c|c|c|c|c|} \hline \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|c|} \hline \bullet & \bullet & \cdots & \bullet & s & r \\ \hline \end{array}$$

Hence, there are exactly **2 generic representations** with this infinitesimal character.

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If the coordinates of  $\nu$  are all **integers**, then there is only one assignment. Therefore, there is only **1 generic representation** with this infinitesimal character.

Thank  
you