Counting Irreducible Representations of General Linear Groups and Unitary Groups

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Outline

- Introduce the counting method developed by Dan Barbasch, Jia-Jun Ma, Binyong Sun, and Chen-Bo Zhu in their paper: Special unipotent representations of real classical groups: Counting and reduction.

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- Show its application in counting the irreducible representations of general linear groups over $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, or a real unitary groups.

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- Let $\nu \in {}^a\mathfrak{h}^*$, denote by $\Lambda = \nu + Q_\iota$ the translate of analytic lattice Q_ι by ν;
- Δ, $\Delta(\Lambda)$ be the root system and integral root system respectively, the integral Weyl group is $W(\Lambda)$.

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- $\mathrm{Irr}_\nu(G)$ is the set of isomorphism classes of irreducible objects in $\mathrm{Rep}_\nu(G)$;
- For any irreducible representation $\mathrm{V} \in \mathrm{Irr}(G)$, recall that $I = \mathrm{Ann}(\mathrm{V}) \subseteq \mathrm{U}(\mathfrak{g})$ is a primitive ideal, its associated variety $\mathrm{AV}(I) \subseteq \mathfrak{g}^*$ is the closure of a unique nilpotent orbit (Borho-Brylinski and Joseph).

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- The complex associated variety (annihilator variety) of V is $\mathrm{AV}_{\mathbb{C}}(\mathrm{V}) = \mathrm{AV}(I) \subseteq \mathfrak{g}^*$.

Goals

- The set $\mathrm{Irr}(G)$ admits a partition according to infinitesimal characters:

$$\mathrm{Irr}(G) = \bigsqcup_{\lambda \in W \backslash {}^a\mathfrak{h}^*} \mathrm{Irr}_\lambda(G).$$

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- Furthermore, according to complex associated variety, the set $\mathrm{Irr}_\nu(G)$ admits a finer partition:

$$\mathrm{Irr}_\nu(G) = \bigsqcup_{\mathcal{O} \in \mathrm{G}(\mathbb{C}) \backslash \mathrm{Nil}(\mathfrak{g})} \mathrm{Irr}_\nu(G; \mathcal{O}).$$

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Goal: Describe the size of each set $\mathrm{Irr}_\lambda(G; \mathcal{O})$ in terms of some combinatorial data.

Counting Formula

Theorem (Barbasch, Ma, Sun, Zhu)

$$\sharp(\mathrm{Irr}_\nu(G;\mathcal{O})) \leq \sum_{\sigma\in\mathrm{Irr}(W(\Lambda);\mathcal{O})} [1_{W_\nu} : \sigma]\cdot[\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))],$$

where 1_{W_ν} denotes the trivial representation of the stabilizer W_ν of ν in W. The equality holds if the Coxeter group $W(\Lambda)$ has no simple factor of type F_4, E_6, E_7, or E_8, and G is linear or isomorphic to a real metaplectic group.

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In their paper, they use this formula to count the number of special unipotent representations to construct them via theta correspondence.

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- Lusztig also defined an equivalence relation on $\mathrm{Irr}(W)$, each equivalence class is called a double cell, $\mathrm{Irr}^{sp}(W) \leftrightarrow \{\text{double cells}\}$.
- There is a j-induction (also called the truncated induction) operation, $j_{W(\Lambda)}^{W} : \{\text{special representations } W(\Lambda)\} \to \{\text{Springer representations of } W\}$.

Definition

For a nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g})$.

- $\mathrm{Irr}^{\mathrm{sp}}(W(\Lambda); \mathcal{O}) = \{\sigma \in \mathrm{Irr}(W(\Lambda)) \,|\, \sigma \text{ is special}, j_{W(\Lambda)}^{W}(\sigma) = \sigma_{\mathcal{O}}\}$;

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$\mathrm{Irr}(W(\Lambda); \mathcal{O})$ is a union of several double cells.

Coherent continuation representations

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Coherent family

Let $\Lambda = \nu + Q_\iota \subseteq {}^a\mathfrak{h}^*$, a Λ-coherent family is a map

$$\Phi : \Lambda \to \mathcal{K}(G),$$

such that:

- For any $\mu \in \Lambda$, $\Phi(\mu) \in \mathcal{K}_\mu(G)$;
- For any $F \in \mathcal{R}_{\mathrm{hol}}(\mathrm{G}(\mathbb{C}))$ and $\mu \in \Lambda$, $F \cdot (\Phi(\mu)) = \sum_{\lambda \in \Delta(F)} \Phi(\mu + \lambda)$ (where $\Delta(F)$ is the set of weights of F counted multiplicity).

Coherent continuation representation

Let $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ denote the complex vector space of all coherent families on Λ. It is a representation of $W(\Lambda)$ under the action

$$(w \cdot \Psi)(\nu) = \Psi(w^{-1}\nu),$$

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For any $\mu \in \Lambda$ we have the evaluation map

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Theorem (Schmid, Zuckerman)

ev is surjective for each $\mu \in \Lambda$, and bijective when μ is regular.

Harish-Chandra cells

Theorem

Suppose $\nu \in {}^a\mathfrak{h}^*$ dominant, $M \in \mathcal{K}_\nu(G)$ is an irreducible representation. Then there exist a unique coherent family $\overline{\Psi}$ characterised by the following properties:

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There is a basis $\mathcal{B} = \{\overline{\Psi}_i\}$ of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ such that for any regular dominant μ, $\overline{\Psi}_i(\mu)$ is an irreducible representation with infinitesimal character μ (there is also a basis Ψ_i of standard modules).

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We can also define basal subrepresentations, which are subrepresentations of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ spanned by a subset of $\mathcal{B}$.

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Cell representations

Let $\mathcal{C}$ be a cell in $\mathcal{B}$ and put $\overline{\mathcal{C}} = \langle \mathcal{C} \rangle \cap \mathcal{B}$. Define the cell representation attached to $\mathcal{C}$ by

$$\mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C}) := \left\langle \overline{\mathcal{C}} \right\rangle / \left\langle \overline{\mathcal{C}} \backslash \mathcal{C} \right\rangle$$

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It is easy to see that the image of $\mathcal{C}$ under this quotient form a basis of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C})$.

Kazhdan-Lusztig cells

- Consider the category $\mathrm{Rep}(\mathfrak{g}, \mathfrak{b})$ of finite generated $\mathrm{U}(\mathfrak{g})$-module which is locally finite over $\mathrm{U}(\mathfrak{b})$, denote its Grothendieck group by $\mathcal{K}(\mathfrak{g}, \mathfrak{b})$;

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- The Grothendieck ring $\mathcal{R}(\mathfrak{g}, Q_\iota)$ ($\simeq \mathcal{R}_{\mathrm{hol}}(\mathrm{G}(\mathbb{C}))$) acts on $\mathcal{K}(\mathfrak{g}, \mathfrak{b})$ via tensor product, so we can also define a coherent family $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$;

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- Define the $W \times W(\Lambda)$ action explicitly by

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- As a representation of $W \times W(\Lambda)$,

$$\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))(\mathcal{C}) \simeq \sum_{\sigma \text{ in the double cell contain } \sigma_{\mathcal{C}}} (\mathrm{Ind}_{W(\Lambda)}^{W} \sigma) \otimes \sigma.$$

Comparison of HC cells and KL cells

- For every Harish-Chandra cell we can attach a Kazhdan-Lusztig cell to it (via comparing their annihilator ideals); furthermore, we have a map $\{\text{Harish-Chandra cells}\} \to \mathrm{Irr}^{sp}(W(\Lambda))$, $\mathcal{C} \mapsto \sigma_{\mathcal{C}}$.

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- It is a well-known fact that every representation in a Harish-Chandra cell $\mathcal{C}$ has the same complex associated variety $\overline{\mathcal{O}_{\mathcal{C}}}$, where $\mathcal{O}_{\mathcal{C}}$ is the nilpotent orbit corresponding to $j_{W(\Lambda)}^{W}\sigma_{\mathcal{C}}$.

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Conjecture

The set $\{\sigma \in \mathrm{Irr}(W(\Lambda)) \mid \sigma \text{ occurs in } \mathrm{Coh}_{\Lambda}(\mathcal{K}(G))(\mathcal{C})\}$ is contained in the double cell containing the special representation $\sigma_{\mathcal{C}}$.

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BMSZ proved this conjecture holds under some technical assumptions ($W(\Lambda)$ has no simple factor of type F_4, E_6, E_7, or E_8, and G is linear or isomorphic to a real metaplectic group).

BMSZ's proof of the counting formula

If S is a Zariski closed $\mathrm{G}(\mathbb{C})$-stable subset of $\mathrm{Nil}(\mathfrak{g})$, then

$$\begin{aligned}
\sharp(\mathrm{Irr}_{\nu,S}(G)) &= \dim \mathcal{K}_{\nu,S}(G) = \dim \mathrm{Coh}_{\Lambda}(\mathcal{K}_S(G))_{W_\nu} \\
&= [1_{W_\nu} : \mathrm{Coh}_{\Lambda}(\mathcal{K}_S(G))] \\
&= \sum_{\sigma \in \mathrm{Irr}(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_{\Lambda}(\mathcal{K}_{\mathrm{S}}(G))] \\
&\overset{\textit{Conjecture}}{=\!=\!=} \sum_{\sigma \in \mathrm{Irr}_S(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_{\Lambda}(\mathcal{K}(G))].
\end{aligned}$$

Classical groups considered

Label $\star$	Classical Lie Group G	Complex Lie Group $\mathrm{G}(\mathbb{C})$
$A^{\mathbb{R}}$	$\mathrm{GL}_n(\mathbb{R})$	$\mathrm{GL}_n(\mathbb{C})$
$A^{\mathbb{H}}$	$\mathrm{GL}_{\frac{n}{2}}(\mathbb{H})$ (n is even)	$\mathrm{GL}_n(\mathbb{C})$
$A^{\mathbb{C}}$	$\mathrm{GL}_n(\mathbb{C})$	$\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$
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Identification:

$$
{}^a\mathfrak{h}^* = \begin{cases} \mathbb{C}^n, & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A\}; \\ \mathbb{C}^n \times \mathbb{C}^n, & \text{if } \star = A^{\mathbb{C}}. \end{cases}
$$

$$
Q_\iota = \begin{cases} \mathbb{Z}^n \subseteq \mathbb{C}^n = {}^a\mathfrak{h}^*, & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A\}; \\ \mathbb{Z}^n \times \mathbb{Z}^n \subseteq \mathbb{C}^n \times \mathbb{C}^n = {}^a\mathfrak{h}^*, & \text{if } \star = A^{\mathbb{C}}, \end{cases}
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Painted Young Diagram (type $A^{\mathbb{R}}$)

A painting on a Young diagram ι of type $A^{\mathbb{R}}$ is a map (we place a symbol in each box)

$$\mathcal{P} : \mathrm{Box}(\iota) \to \{\bullet, c, d\}$$

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Combinatorial notions

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A painting on a Young diagram $\imath$ of type $A^{\mathbb{R}}$ is a map (we place a symbol in each box)

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With the following properties

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Painted Young Diagram (type $A^{\mathbb{R}}$)

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A painted Young diagram is a pair $(\iota, \mathcal{P})$ consisting of a Young diagram ι and a painting $\mathcal{P}$ on ι. Denote by $\mathrm{P}_{A^{\mathbb{R}}}(\iota)$ the set of paintings on ι of type $A^{\mathbb{R}}$.

Example

The following represents a painted Young diagram.

$\bullet$	$\bullet$	c	d
$\bullet$	$\bullet$	d	
c			
d			

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$$\begin{array}{|c|c|c|c|}
\hline
\bullet & \bullet & c & d \\
\hline
\bullet & \bullet & d & \multicolumn{1}{c}{} \\
\cline{1-3}
c & \multicolumn{3}{c}{} \\
\cline{1-1}
d & \multicolumn{3}{c}{} \\
\cline{1-1}
\end{array}$$

Each of the following does not represent a painted Young diagram.

$$\begin{array}{|c|c|}
\hline
d & c \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|}
\hline
c & d \\
\hline
c & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}$$

Assigned Young Diagram

For a Young diagram ι, and a partition $[d_1, \cdots, d_k]$ of $|\iota|$. An assignment of type $[d_1, d_2, \cdots, d_N]$ on ι is a map

$$\mathcal{Q} : \mathrm{Box}(\iota) \to \{1, 2, \cdots, N\}$$

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- each positive integer occurs at most once in each column.

An assigned Young diagram of type $[d_1, d_2, \cdots, d_N]$ is a pair $(\iota, \mathcal{Q})$ consisting of a Young diagram ι and an assignment $\mathcal{Q}$ of type $[d_1, d_2, \cdots, d_N]$ on ι. Denote by $\mathrm{A}_{[d_1, d_2, \cdots, d_N]}(\iota)$ the set of all assignments on ι of type $[d_1, d_2, \cdots, d_N]$.

Counting result for $\mathrm{GL}_n(\mathbb{R})$: The integral case

$G = \mathrm{GL}_n(\mathbb{R})$ ($n \in \mathbb{N}$). $\mathcal{O}$ is a nilpotent orbits in $\mathfrak{g}$, denote the corresponding Young diagram by $\iota(\mathcal{O})$.
If $\nu \in {}^a\mathfrak{h}^* = \mathbb{C}^n$ is integral, its coordinates can be permuted such that

$$\nu = (\underbrace{\lambda_1, \cdots, \lambda_1}_{d_1}, \underbrace{\lambda_2, \cdots, \lambda_2}_{d_2}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{d_k}) \in \mathbb{C}^n,$$

where $[d_1, d_2, \cdots, d_k]$ is a partition of n, and the $\lambda_i \in \mathbb{C}$ satisfy the condition $\lambda_i - \lambda_j \in \mathbb{Z} \setminus \{0\}$ for any $i \neq j$.

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Theorem

$$\sharp(\mathrm{Irr}_\nu(\mathrm{GL}_n(\mathbb{R}); \mathcal{O})) = \sharp\left(\mathrm{P}_{A^{\mathbb{R}}}(\iota(\mathcal{O}))\right) \cdot \sharp\left(\mathrm{A}_{[d_1, \cdots, d_k]}(\iota(\mathcal{O}))\right).$$

Sketch of the calculation

Coherent continuation representations can be explicitly computed via the Hecke algebra module structure described by Lusztig and Vogan.

$$\mathrm{Coh}_{\Lambda}(\mathcal{K}(\mathrm{GL}_n(\mathbb{R}))) = \bigoplus_{2r+i\leq n} \mathrm{Ind}_{\mathrm{W}_r\times \mathrm{S}_i \times \mathrm{S}_{n-2r-i}}^{\mathrm{S}_n} \epsilon \otimes 1 \otimes 1.$$

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The computation makes essential use of *Pieri's rule* and Frobenius reciprocity.

$$\begin{aligned}\sharp(\mathrm{Irr}_{\nu}(\mathrm{GL}_n(\mathbb{R});\mathcal{O})) &= [1_{W_\nu} : \sigma_{\mathcal{O}}] \cdot [\sigma_{\mathcal{O}} : \mathrm{Coh}_{\Lambda}(\mathcal{K}(\mathrm{GL}_n(\mathbb{R})))] \\ &= [\sigma_{\mathcal{O}} : \mathrm{Ind}_{W_\nu}^{W} 1_{W_\nu}] \cdot [\sigma_{\mathcal{O}} : \mathrm{Coh}_{\Lambda}(\mathcal{K}(\mathrm{GL}_n(\mathbb{R})))] \\ &= \sharp\left(\mathrm{A}_{[d_1,\cdots,d_k]}(\iota(\mathcal{O}))\right) \cdot \sharp\left(\mathrm{P}_{A^{\mathbb{R}}}(\iota(\mathcal{O}))\right).\end{aligned}$$

Example: Minimal representations

Let $\nu \in {}^a\mathfrak{h}^*$ be a regular integral infinitesimal character, and let $\mathcal{O}_{\min}$ denote the minimal nilpotent orbit.

There are $n-1$ assignments of type $[\underbrace{1, 1, \ldots, 1}_{n}]$ on it, given by

$$\begin{array}{|c|c|} \hline 1 & i{+}1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} i & \multicolumn{1}{c}{} \\ \cline{1-1} i{+}2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} n & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \text{for } 1 \le i \le n-1.$$

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Since there are 4 distinct paintings on $\iota_{\mathcal{O}}$, we obtain exactly 4(n-1) minimal representations with the fixed infinitesimal character ν.

Non-integral case

For an arbitrary $\nu \in {}^a\mathfrak{h}^*$, its coordinates can be permuted such that

$$\nu = (\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_r) \in \mathbb{C}^n,$$

$$\boldsymbol{\lambda}_i = (\underbrace{\lambda_{i,1}, \cdots, \lambda_{i,1}}_{d_{i,1}}, \cdots, \underbrace{\lambda_{i,k_i}, \cdots, \lambda_{i,k_i}}_{d_{i,k_i}}) \in \mathbb{C}^{e_i} \quad (e_i \geq 1),$$

where each $\boldsymbol{\lambda}_i$ is integral but $(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j) \in \mathbb{C}^{e_i+e_j}$ is not integral for any $i \neq j$, $[d_{i,1}, \cdots d_{i,k_i}]$ is a partition of e_i, and the condition $\lambda_{i,p} - \lambda_{i,q} \in \mathbb{Z} \setminus \{0\}$ holds for any $p \neq q$.

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Theorem

$$\sharp(\mathrm{Irr}_\nu(G; \mathcal{O})) = \sum_{\substack{(\iota_1, \cdots, \iota_r) \in \mathrm{YD}_{e_1} \times \cdots \times \mathrm{YD}_{e_r} \\ \iota_1 \overset{r}{\sqcup} \iota_2 \cdots \overset{r}{\sqcup} \iota_r = \iota(\mathcal{O})}} \prod_{i=1}^{r} \sharp(\mathrm{Irr}_{\boldsymbol{\lambda}_i}(\mathrm{GL}_{e_i}(\mathbb{R}); \mathcal{O}_{\iota_i}))$$

Key observation

There are natural isomorphism of $W(\Lambda)$-representations:

$$\mathrm{Coh}_{\Lambda_1}(\mathcal{K}(\mathrm{GL}_{e_1}(\mathbb{R}))) \otimes \cdots \otimes \mathrm{Coh}_{\Lambda_r}(\mathcal{K}(\mathrm{GL}_{e_r}(\mathbb{R}))) \to \mathrm{Coh}_{\Lambda}(\mathcal{K}(\mathrm{GL}_{n}(\mathbb{R}
$$
$$\Psi_1 \otimes \cdots \otimes \Psi_r \mapsto \Psi$$

where $\Psi(\mu) = \mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{R})} \Psi_1(\mu) \otimes \cdots \otimes \Psi_r(\mu)$ is also a standard module for regular dominant $\mu \in {}^a\mathfrak{h}^*$.
Actually, this isomorphism also takes irreducible objects to irreducible objects.

Combinatorial data

Painted Young diagram (type A)

A painting on a Young diagram ι of type A is a map (we place a symbol in each box)

$$\mathcal{P} : \mathrm{Box}(\iota) \to \{\bullet, s, r\}$$

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A painted Young diagram is a pair $(\iota, \mathcal{P})$ consisting of a Young diagram ι and a painting $\mathcal{P}$ on ι. Denote by $\mathrm{P}_A(\iota)$ the set of paintings on ι of type A.

Let ι be a Young diagram and $\mathcal{P}$ be a painting on ι of type A. Define the signature of $\mathcal{P}$ to be the pair of non-negative integers

$$(p_{\mathcal{P}}, q_{\mathcal{P}}) := \left(\frac{\sharp(\mathcal{P}^{-1}(\bullet))}{2} + \sharp(\mathcal{P}^{-1}(s)), \frac{\sharp(\mathcal{P}^{-1}(\bullet))}{2} + \sharp(\mathcal{P}^{-1}(r)) \right),$$

for every $p, q \in \mathbb{N}$ such that $p + q = |\iota|$, we define

$$\mathrm{P}_A^{p,q}(\iota) := \{\mathcal{P} \in \mathrm{P}_A(\iota) \,|\, (p_{\mathcal{P}}, q_{\mathcal{P}}) = (p, q)\}.$$

Counting result for $\mathrm{U}(p, q)$: The integral case

$G = \mathrm{U}(p, q)$ $(p, q \in \mathbb{N})$. $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$, denote the corresponding Young diagram by $\iota(\mathcal{O})$.
If $\nu \in {}^a\mathfrak{h}^* = \mathbb{C}^n$ $(n = p + q)$ is integral, that is, the differences of its coordinates are integral, then its coordinates can be permuted such that

$$\nu = (\underbrace{\lambda_1, \cdots, \lambda_1}_{d_1}, \underbrace{\lambda_2, \cdots, \lambda_2}_{d_2}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{d_k}) \in \mathbb{C}^n,$$

where $[d_1, d_2, \cdots, d_k]$ is a partition of n, and the $\lambda_i \in \mathbb{C}$ satisfy the condition $\lambda_i - \lambda_j \in \mathbb{Z} \backslash \{0\}$ for any $i \neq j$.

Theorem

If $\lambda_1 \in \frac{n-1}{2} + \mathbb{Z}$, then

$$\sharp(\mathrm{Irr}_\nu(\mathrm{U}(p,q);\mathcal{O})) = \sharp\left(\mathrm{P}_A^{p,q}(\iota(\mathcal{O}))\right) \cdot \sharp\left(\mathrm{A}_{[d_1,\cdots,d_k]}(\iota(\mathcal{O}))\right).$$

If $\lambda_1 \in \frac{n}{2} + \mathbb{Z}$, then

$$\sharp(\mathrm{Irr}_\nu(\mathrm{U}(p,q);\mathcal{O})) = \sharp\left(\mathrm{A}_{[d_1,\cdots,d_k]}(\iota(\mathcal{O}))\right) \cdot \delta_{p,q}.$$

Otherwise, $\sharp(\mathrm{Irr}_\nu(\mathrm{U}(p,q);\mathcal{O})) = 0$.

Example: Generic representations

Let $G = \mathrm{U}(n,n)$, and let $\mathcal{O}_{\mathrm{prin}}$ denote the principal nilpotent orbit.

If the coordinates of ν are all half-integers, there is only one assignment on $\iota(\mathcal{O}_{\mathrm{prin}})$, and exactly two paintings:

$$\boxed{\bullet}\boxed{\bullet}\boxed{\cdots}\boxed{\bullet}\boxed{\bullet}\boxed{\bullet} \quad \text{and} \quad \boxed{\bullet}\boxed{\bullet}\boxed{\cdots}\boxed{\bullet}\boxed{s}\boxed{r}$$

Hence, there are exactly 2 generic representations with this infinitesimal character.

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If the coordinates of ν are all integers, then there is only one assignment. Therefore, there is only 1 generic representation with this infinitesimal character.

Thank you