Counting Irreducible Representations of General Linear Groups and Unitary Groups

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August 11, 2025

Outline

- Introduce the counting method developed by Dan Barbasch, Jia-Jun Ma, Binyong Sun, and Chen-Bo Zhu in their paper: Special unipotent representations of real classical groups: Counting and reduction. J. Eur. Math. Soc. (2025).
- Show it's application in counting the irreducible representations of general linear groups and unitary groups.

Background

- G: connected reductive algebraic group defined over $\mathbb{R}$;
- G is a real Lie group together with a Lie group homorphism $\iota : G \to \mathrm{G}(\mathbb{R})$ with open image and finite kernel;
- $\mathfrak{g}$, $\mathfrak{g}_0$ are Lie algebras of $\mathrm{G}(\mathbb{C})$, G;
- ${}^a\mathfrak{h}$: the abstract Caratan subalgebra of $\mathfrak{g}$, with root lattice $Q_\mathfrak{g}$, weight group $Q^\mathfrak{g}$, and analalytic weight lattice Q_ι ($Q_\mathfrak{g} \subseteq Q_\iota \subseteq Q^\mathfrak{g} \subseteq {}^a\mathfrak{h}^*$);
- W: the abstract Weyl group of $\mathfrak{g}$ act on ${}^a\mathfrak{h}$;
- $\mathrm{Rep}(G)$: the category of Casselman-Wallach representations of G;
- $\mathrm{Irr}(G)$: set of isomorphism classes of irreducible objects in $\mathrm{Rep}(G)$.

- There is a partition of $\mathrm{Irr}(G)$ with respect to infinitesimal characters:

$$\mathrm{Irr}(G) = \bigsqcup_{\lambda \in W \backslash {}^a\mathfrak{h}^*} \mathrm{Irr}_\lambda(G),$$

according to work of Harish-Chandra, each set $\mathrm{Irr}_\lambda(G)$ is finite;

- According to complex associated variety (or called annihilator variety, which is always the closure of a unique nilpotent orbits by results of [Borho, Brylinski], [Joseph]), there is a further partition of $\mathrm{Irr}_\lambda(G)$:

$$\mathrm{Irr}_\lambda(G) = \bigsqcup_{\mathcal{O} \in \mathrm{G}(\mathbb{C}) \backslash \mathrm{Nil}(\mathfrak{g})} \mathrm{Irr}_\lambda(G; \mathcal{O}).$$

Goal: Describe the size of each set $\mathrm{Irr}_\lambda(G; \mathcal{O})$ in terms of combinatorial data.

Counting Formula

Theorem (Barbasch, Ma, Sun, Zhu)

$$\sharp(\mathrm{Irr}_\nu(G;\mathcal{O})) \leq \sum_{\sigma\in\mathrm{Irr}(W(\Lambda);\mathcal{O})} [1_{W_\nu} : \sigma]\cdot[\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))],$$

where 1_{W_ν} denotes the trivial representation of the stabilizer W_ν of ν in W. The equality holds if the Coxeter group $W(\Lambda)$ has no simple factor of type F_4, E_6, E_7, or E_8, and G is linear or isomorphic to a real metaplectic group.

In their paper, they use this formula to count the number of special unipotent representations in order to construct them via theta correspondence.

Double Cells, Special Representations

- Springer correspondence: $\sigma \in \mathrm{Irr}(W) \rightsquigarrow (\mathcal{O}, \mathcal{L})$;
- For the Weyl group W, Lusztig define a class of special representations: special nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}) \rightsquigarrow \sigma_{\mathcal{O}}$ (via Springer correspondence).
- Lusztig also defined an equivalence relation on $\mathrm{Irr}(W)$, each equivalence is called a double cell, $\mathrm{Irr}^{sp}(W) \leftrightarrow \{\text{double cells}\}$.
- There is a j-induction (also called the truncated induction) operation: $j_{W(\Lambda)}^{W} : \{\text{special representations } W(\Lambda)\} \to \{\text{Springer representations of } W\}$.

Definition

For a nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g})$.

- $\mathrm{Irr}^{\mathrm{sp}}(W(\Lambda); \mathcal{O}) = \{\sigma \in \mathrm{Irr}(W(\Lambda)) \,|\, \sigma \text{ is special}, j_{W(\Lambda)}^{W}(\sigma) = \sigma_{\mathcal{O}}\}$;
- $\mathrm{Irr}(W(\Lambda); \mathcal{O}) = \{\sigma \in \mathrm{Irr}(W(\Lambda)) \,|\, \text{exist } \sigma_0 \in \mathrm{Irr}^{\mathrm{sp}}(W(\Lambda); \mathcal{O}), \sigma \approx \sigma_0, \}$

$\mathrm{Irr}(W(\Lambda); \mathcal{O})$ is a union of several double cells.

Coherent Continuation Representation

$\mathcal{R}_{\mathrm{hol}}(\mathrm{G}(\mathbb{C}))$: Grothendieck ring of finite-dimensional holomorphic representations of $\mathrm{G}(\mathbb{C})$.
$\mathcal{K}(G)$: Grothendieck group of $\mathrm{Rep}(G)$ which has a $\mathcal{R}_{\mathrm{hol}}(\mathrm{G}(\mathbb{C}))$ module structure via tensor product.

Coherent family

Let $\Lambda = \nu + Q_\iota \subseteq {}^a\mathfrak{h}^*$, a Λ-coherent family is a map

$$\Phi : \Lambda \to \mathcal{K}(G),$$

such that:

- for any $\nu \in \Lambda$, $\Phi(\nu) \in \mathcal{K}_\nu(G)$;
- for any $F \in \mathcal{R}_{\mathrm{hol}}(\mathrm{G}(\mathbb{C}))$ and $\nu \in \Lambda$,
  $F \cdot (\Phi(\nu)) = \sum_{\mu \in \Delta(F)} \Phi(\nu + \mu)$ (where $\Delta(F)$ is the set of weights of F counted multiplicity).

Let $W(\Lambda) \subseteq W$ denote the integral Weyl group with respect to Λ.

Coherent continuation representation

Let $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ denote the complex vector space of all coherent families on Λ. It is a representation of $W(\Lambda)$ under the action

$$(w \cdot \Psi)(\nu) = \Psi(w^{-1}\nu),$$

for any $w \in W(\Lambda)$, $\Psi \in \mathrm{Coh}_\Lambda(\mathcal{K}(G))$, $\nu \in \Lambda$.

For any $\nu \in \Lambda$ we have the evaluation map

$$\begin{array}{rccc} \mathrm{ev}: & \mathrm{Coh}_\Lambda(\mathcal{K}(G)) & \longrightarrow & \mathcal{K}_\nu \\ & \Psi & \longmapsto & \Psi(\nu) \end{array}$$

Theorem (Schmid, Zuckerman)

ev is surjective for each $\nu \in \Lambda$, and bijective when ν is regular.

Harish-Chandra Cells

Theorem(Vogan's green book)

Suppose $\nu \in {}^a\mathfrak{h}^*$ dominant, $M \in \mathcal{K}_\nu(G)$ is an irreducible representation representation. Then there exist a unique coherent family $\overline{\Psi}$ characterised by the following properties:

- $\overline{\Psi}(\nu) = M$;
- If μ is dominant, then $\overline{\Psi}(\mu)$ is irreducible or zero.

There is a basis $\mathcal{B} = \{\overline{\Psi_i}\}$ of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ such that for any regular dominant μ, $\overline{\Psi_i}(\mu)$ is an irreducible representation with infinitesimal character ν.

We call $(\mathrm{Coh}_\Lambda(\mathcal{K}(G)), \mathcal{B})$ a basal representation.

We can also define basal subrepresentations.

- For any subset $\mathcal{S}$ of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$, denote by $\langle \mathcal{S} \rangle$ the minimal basal subrepresentation containing $\mathcal{S}$.
- Define an equivalence relation on $\mathcal{B}$ by: $\overline{\Psi_i} \approx \overline{\Psi_j}$ if and only if $\left\langle \overline{\Psi_i} \right\rangle = \left\langle \overline{\Psi_j} \right\rangle$.
- The equivalence classes of $\mathcal{B}$ with respect to this relation are called Harish-Chandra cells.

Cell representations

Let $\mathcal{C}$ be a cell in $\mathcal{B}$ and put $\overline{\mathcal{C}} = \langle \mathcal{C} \rangle \cap \mathcal{B}$. Define the cell representation attached to $\mathcal{C}$ by

$$\mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C}) := \left\langle \overline{\mathcal{C}} \right\rangle / \left\langle \overline{\mathcal{C}} \backslash \mathcal{C} \right\rangle$$

- There is also a coherent continuation theory $(\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b})))$ for highest weight representations of complex Lie algebra $\mathfrak{g}$, similarly there is a notion of (two-side) Kazhdan-Lusztig cells, which is $1-1$ correspondence with special representations in $\mathrm{Irr}(W(\Lambda))$.
- Every Harish-Chandra cell corresponds to a Kazhdan-Lusztig cell (via compairing their annihilator ideals), so we have a map $\{Harish - Chandra\ cells\} \to \mathrm{Irr}^{sp}(W(\Lambda))$, $\mathcal{C} \mapsto \sigma_{\mathcal{C}}$.
- In fact every representation in a Harish-Chandra cell $\mathcal{C}$ has the complex associated variety $\overline{\mathcal{O}_{\mathcal{C}}}$, here $\mathcal{O}_{\mathcal{C}}$ is the nilpotent orbit corresponds to $j_{W(\Lambda)}^{W}\sigma_{\mathcal{C}}$.

Hypothesis

The set $\{\sigma \in \mathrm{Irr}(W(\Lambda)) \mid \sigma \text{ occurs in } \mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C})\}$ is contained in the double cell containing the special representation $\sigma_{\mathcal{C}}$.

BMSZ's Proof

BMSZ proved that this hypothesis holds under some technical assumptions ($W(\Lambda)$ has no simple factor of type F_4, E_6, E_7, or E_8, and G is linear or isomorphic to a real metaplectic group.).

$$\begin{aligned}\sharp(\mathrm{Irr}_\nu(G)) = \dim \mathcal{K}_\nu(G) &\xlongequal{\textit{Vogan}} \dim \mathrm{Coh}_\Lambda(\mathcal{K}(G))_{W_\nu} \\ &= [1_{W_\nu} : \mathrm{Coh}_\Lambda(\mathcal{K}(G))] \\ &= \sum_{\sigma \in \mathrm{Irr}(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))],\end{aligned}$$

If S is a Zariski closed $\mathrm{G}(\mathbb{C})$-stable subset of $\mathrm{Nil}(\mathfrak{g})$, then

$$\begin{aligned}\sharp(\mathrm{Irr}_{\nu,S}(G)) &= \sum_{\sigma \in \mathrm{Irr}(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_{\Lambda,S}(\mathcal{K}(G))] \\ &\xlongequal{\textit{Hypothesis}} \sum_{\sigma \in \mathrm{Irr}_S(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))].\end{aligned}$$

Classical Groups Considered

Label $\star$	Classical Lie Group G	Complex Lie Group $\mathrm{G}(\mathbb{C})$
$A^{\mathbb{R}}$	$\mathrm{GL}_n(\mathbb{R})$	$\mathrm{GL}_n(\mathbb{C})$
$A^{\mathbb{H}}$	$\mathrm{GL}_{\frac{n}{2}}(\mathbb{H})$ $(n \text{ is even})$	$\mathrm{GL}_n(\mathbb{C})$
$A^{\mathbb{C}}$	$\mathrm{GL}_n(\mathbb{C})$	$\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$
A	$\mathrm{U}(p, q)$	$\mathrm{GL}_n(\mathbb{C})$ $(n = p + q)$

Identification:

$$
{}^a\mathfrak{h}^* = \begin{cases} = \mathbb{C}^n, & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A\}; \\ = \mathbb{C}^n \times \mathbb{C}^n, & \text{if } \star = A^{\mathbb{C}}. \end{cases}
$$

Painted Young Diagram (type $A^{\mathbb{R}}$)

A painting on a Young diagram $\imath$ of type $A^{\mathbb{R}}$ is a map (we place a symbol in each box)

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, c, d\}$$

with the following properties

- if we remove the boxes painted with $\{d\}, \{c, d\}$, the remainder still constitutes a Young diagram;
- every column of $\imath$ has at most one box painted with c, and has at most one box painted with d;
- every row of $\imath$ has an even number of boxes painted with $\bullet$.

A painted Young diagram is a pair $(\imath, \mathcal{P})$ consisting of a Young diagram $\imath$ and a painting $\mathcal{P}$ on $\imath$. Denote by $\mathrm{P}_{A^{\mathbb{R}}}(\imath)$ the set of paintings on $\imath$ of type $A^{\mathbb{R}}$.

Example

The following represents a painted Young diagram.

$$\begin{array}{|c|c|c|c|}\hline \bullet & \bullet & c & d \\ \hline \bullet & \bullet & d \\ \cline{1-3} c \\ \cline{1-1} d \\ \cline{1-1}\end{array}$$

Each of the followings does not represent a painted Young diagram.

$$\begin{array}{|c|c|}\hline d & c \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline c & d \\ \hline c \\ \cline{1-1}\end{array}$$

Assigned Young Diagram

For a Young diagram ι, and a partition $[d_1, \cdots, d_k]$ of $|\iota|$. An assignment of type $[d_1, d_2, \cdots, d_N]$ on ι is a map

$$\mathcal{Q} : \mathrm{Box}(\iota) \to \{1, 2, \cdots, N\}$$

with the following properties

- for each $i \in \{1, 2, \cdots, N\}$, the preimage $\mathcal{P}^{-1}(i)$ has exactly d_i elements;
- for each $1 \leq n \leq N$, if we remove the boxes assigned with $\{N + 1, \cdots, |\iota|\}$, the reminder still constitutes a Young diagram;
- each positive integer occurs at most once in each column.

An assigned Young diagram of type $[d_1, d_2, \cdots, d_N]$ is a pair $(\iota, \mathcal{Q})$ consisting of a Young diagram ι and an assignment $\mathcal{Q}$ of type $[d_1, d_2, \cdots, d_N]$ on ι. Denote by $\mathrm{A}_{[d_1, d_2, \cdots, d_N]}(\iota)$ the set of all assignments on ι of type $[d_1, d_2, \cdots, d_N]$.

Counting Result for $\mathrm{GL}_n(\mathbb{R})$, The Integral Case

$G = \mathrm{GL}_n(\mathbb{R})$ ($n \in \mathbb{N}$). $\mathcal{O}$ is a nilpotent orbits in $\mathfrak{g}$, denote the corresponding Young diagram by $\iota(\mathcal{O})$.
If $\nu \in {}^a\mathfrak{h}^* = \mathbb{C}^n$ is integral, its coordinates can be permuted such that

$$\nu = (\underbrace{\lambda_1, \cdots, \lambda_1}_{d_1}, \underbrace{\lambda_2, \cdots, \lambda_2}_{d_2}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{d_k}) \in \mathbb{C}^n,$$

where $[d_1, d_2, \cdots, d_k]$ is a partition of n, and the $\lambda_i \in \mathbb{C}$ satisfy the condition $\lambda_i - \lambda_j \in \mathbb{Z} \setminus \{0\}$ for any $i \neq j$.

Theorem

$$\sharp(\mathrm{Irr}_\nu(\mathrm{GL}_n(\mathbb{R}); \mathcal{O})) = \sharp\left(\mathrm{P}_{A^{\mathbb{R}}}(\iota(\mathcal{O}))\right) \cdot \sharp\left(\mathrm{A}_{[d_1, \cdots, d_k]}(\iota(\mathcal{O}))\right).$$

Sketch of the Calculation

Coherent continuation representations can be explicitly computed via the Hecke algebra module structure described by Lusztig and Vogan.

$$\mathrm{Coh}_\Lambda(\mathcal{K}(\mathrm{GL}_n(\mathbb{R}))) = \bigoplus_{2r+i\le n} \mathrm{Ind}_{\mathrm{W}_r\times \mathrm{S}_i \times \mathrm{S}_{n-2r-i}}^{\mathrm{S}_n} \epsilon \otimes 1 \otimes 1.$$

The computation makes essential use of Pieri's rule and Frobenius reciprocity .

$$\begin{aligned}
\sharp(\mathrm{Irr}_\nu(\mathrm{GL}_n(\mathbb{R}); \mathcal{O})) &= [1_{W_\nu} : \sigma_{\mathcal{O}}] \cdot [\sigma_{\mathcal{O}} : \mathrm{Coh}_\Lambda(\mathcal{K}(\mathrm{GL}_n(\mathbb{R})))] \\
&= [\sigma_{\mathcal{O}} : \mathrm{Ind}_{W_\nu}^{W} 1_{W_\nu}] \cdot [\sigma_{\mathcal{O}} : \mathrm{Coh}_\Lambda(\mathcal{K}(\mathrm{GL}_n(\mathbb{R})))] \\
&= \sharp\left(\mathrm{A}_{[d_1,\cdots,d_k]}(\iota(\mathcal{O}))\right) \cdot \sharp\left(\mathrm{P}_{A^{\mathbb{R}}}(\iota(\mathcal{O}))\right).
\end{aligned}$$

Example: Minimal Representations

Let $\nu \in {}^a\mathfrak{h}^*$ be a regular integral infinitesimal character, and let $\mathcal{O}_{\min}$ denote the minimal nilpotent orbit.

There are $n-1$ assignments of type $[\underbrace{1,1,\ldots,1}_{n}]$ on it, given by

$$\begin{array}{|c|c|} \hline 1 & i{+}1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} i & \multicolumn{1}{c}{} \\ \cline{1-1} i{+}2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} n & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \text{for } 1 \leq i \leq n-1.$$

Since there are 4 distinct paintings on each of the $n-1$ assignments, we obtain exactly 4(n-1) minimal representations with the fixed infinitesimal character ν.

Non-integral Case

Theorem

For an arbitrary $\nu \in {}^a\mathfrak{h}^*$, its coordinates can be permuted such that

$$\nu = (\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_r) \in \mathbb{C}^n,$$

$$\boldsymbol{\lambda}_i = (\underbrace{\lambda_{i,1}, \cdots, \lambda_{i,1}}_{d_{i,1}}, \cdots, \underbrace{\lambda_{i,k_i}, \cdots, \lambda_{i,k_i}}_{d_{i,k_i}}) \in \mathbb{C}^{e_i} \quad (e_i \geq 1),$$

where each $\boldsymbol{\lambda}_i$ is integral but $(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j) \in \mathbb{C}^{e_i+e_j}$ is not integral for any $i \neq j$, $[d_{i,1}, \cdots d_{i,k_i}]$ is a partition of e_i, and the condition $\lambda_{i,p} - \lambda_{i,q} \in \mathbb{Z} \setminus \{0\}$ holds for any $p \neq q$. Then

$$\sharp(\mathrm{Irr}_\nu(G; \mathcal{O})) = \sum_{\substack{(\iota_1, \cdots, \iota_r) \in \mathrm{YD}_{e_1} \times \cdots \times \mathrm{YD}_{e_r} \\ \iota_1 \overset{r}{\sqcup} \iota_2 \cdots \overset{r}{\sqcup} \iota_r = \iota(\mathcal{O})}} \prod_{i=1}^{r} \sharp(\mathrm{Irr}_{\boldsymbol{\lambda}_i}(\mathrm{GL}_{e_i}(\mathbb{R}); \mathcal{O}_{\iota_i})).$$

Key Observation

There are natural isomorphism of $W(\Lambda)$-representations:

$$\mathrm{Coh}_{\Lambda_1}(\mathcal{K}(\mathrm{GL}_{e_1}(\mathbb{R}))) \otimes \cdots \otimes \mathrm{Coh}_{\Lambda_r}(\mathcal{K}(\mathrm{GL}_{e_r}(\mathbb{R}))) \to \mathrm{Coh}_{\Lambda}(\mathcal{K}(\mathrm{GL}_n(\mathbb{R}$$
$$\overline{\Psi_1} \otimes \cdots \otimes \overline{\Psi_r} \mapsto \overline{\Psi}$$

where $\overline{\Psi}(\mu) = \mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{R})} \overline{\Psi_1}(\mu) \otimes \cdots \otimes \overline{\Psi_r}(\mu)$ is still irreducible for regular dominant $\mu \in {}^a\mathfrak{h}^*$.

Painted Young Diagram (type A)

A painting on a Young diagram $\imath$ of type A is a map (we place a symbol in each box)

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, s, r\}$$

with the following properties

- if we remove the boxes painted with $\{s\}, \{s, r\}$, the remainder still constitutes a Young diagram;
- every row of $\imath$ has at most one box painted with c, and has at most one box painted with d;
- every row of $\imath$ has an even number of boxes painted with $\bullet$.

A painted Young diagram is a pair $(\imath, \mathcal{P})$ consisting of a Young diagram $\imath$ and a painting $\mathcal{P}$ on $\imath$. Denote by $\mathrm{P}_A(\imath)$ the set of paintings on $\imath$ of type A.

Let ι be a Young diagram and $\mathcal{P}$ is a painting on ι of type A. Define the signature of $\mathcal{P}$ to be the pair of non-negative integers

$$(p_{\mathcal{P}}, q_{\mathcal{P}}) := \left(\frac{\sharp(\mathcal{P}^{-1}(\bullet))}{2} + \sharp(\mathcal{P}^{-1}(s)), \frac{\sharp(\mathcal{P}^{-1}(\bullet))}{2} + \sharp(\mathcal{P}^{-1}(r)) \right),$$

for every $p, q \in \mathbb{N}$ such that $p + q = |\iota|$, we define

$$\mathrm{P}_A^{p,q}(\iota) := \{\mathcal{P} \in \mathrm{P}_A(\iota) \,|\, (p_{\mathcal{P}}, q_{\mathcal{P}}) = (p, q)\}.$$

Counting Result for $\mathrm{U}(p,q)$, The Integral Case

$G = \mathrm{U}(p,q)$ $(p,q \in \mathbb{N})$. $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$, denote the corresponding Young diagram by $\iota(\mathcal{O})$.
If $\nu \in {}^a\mathfrak{h}^* = \mathbb{C}^n$ $(n = p+q)$ is integral, that is, the differences of its coordinates are integral, then its coordinates can be permuted such that

$$\nu = (\underbrace{\lambda_1, \cdots, \lambda_1}_{d_1}, \underbrace{\lambda_2, \cdots, \lambda_2}_{d_2}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{d_k}) \in \mathbb{C}^n,$$

where $[d_1, d_2, \cdots, d_k]$ is a partition of n, and the $\lambda_i \in \mathbb{C}$ satisfy the condition $\lambda_i - \lambda_j \in \mathbb{Z}\backslash\{0\}$ for any $i \neq j$.

Theorem

If $\lambda_1 \in \frac{n-1}{2} + \mathbb{Z}$, then

$$\sharp(\mathrm{Irr}_\nu(\mathrm{U}(p,q);\mathcal{O})) = \sharp\left(\mathrm{P}_A^{p,q}(\iota(\mathcal{O}))\right) \cdot \sharp\left(\mathrm{A}_{[d_1,\cdots,d_k]}(\iota(\mathcal{O}))\right).$$

If $\lambda_1 \in \frac{n}{2} + \mathbb{Z}$, then

$$\sharp(\mathrm{Irr}_\nu(\mathrm{U}(p,q);\mathcal{O})) = \sharp\left(\mathrm{A}_{[d_1,\cdots,d_k]}(\iota(\mathcal{O}))\right) \delta_{p,q}.$$

Otherwise, $\sharp(\mathrm{Irr}_\nu(\mathrm{U}(p,q);\mathcal{O})) = 0$.

Example: Generic Representations

Let $G = \mathrm{U}(n, n)$, and let $\mathcal{O}_{\mathrm{prin}}$ denote the principal nilpotent orbit.

If the coordinates of ν are all half-integers, there is only one assignment on $\iota(\mathcal{O}_{\mathrm{prin}})$, and exactly two paintings:

$$\begin{array}{|c|c|c|c|c|c|}\hline \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|c|}\hline \bullet & \bullet & \cdots & \bullet & s & r \\ \hline \end{array}$$

Hence, there are exactly 2 generic representations with this infinitesimal character.

If the coordinates of ν are all integers, then there is only one assignment and only one painting. Therefore, there is exactly 1 generic representation with this infinitesimal character.

Thank you