$$\sigma_{PI} = \frac{4\pi^2 e^2}{c}\,\omega\,|\hat{z}_{fi}|^2$$

$$|i\rangle = R_i(r)\,Y_i(\vartheta\varphi) \qquad i = n\,l\,m$$

$$|f\rangle = R_f(r)\,Y_f(\vartheta\varphi) \qquad f = E'\,l'\,m'$$

$$\hat{z} = r\cos\vartheta = \sqrt{\frac{4\pi}{3}}\,r\,Y_{10}$$

$$\hat{z}_{fi} = \underbrace{\langle R_f|r|R_i\rangle}_{r_{fi}} \cancel{\sqrt{\frac{4\pi}{3}}}\; \underbrace{\overset{m'=m}{\langle Y_{l'm'}|Y_{10}|Y_{lm}\rangle}}_{\sqrt{\frac{2l+1}{2l'+1}}\cancel{\sqrt{\frac{3}{4\pi}}}\,C^{l'0}_{l010}\,C^{l'm}_{lm10}} \Rightarrow$$

$$z_{fi} = r_{fi}\sqrt{\frac{2l+1}{2l'+1}}\,C^{l'0}_{l010}\,C^{l'm}_{lm10}$$

$$R_i = \frac{\chi_i}{r}\;;\; R_f = \frac{\chi_f}{r} \Rightarrow -\frac{\hbar^2}{2m}\chi'' + \left(-\frac{1}{r} + \frac{\hbar^2 l(l+1)}{2mr^2}\right)\chi = E\chi$$

χ_i - bound state

χ_f - continuum Coulomb state with normalization

$$\chi_f \xrightarrow[r\to\infty]{} \sqrt{\frac{2}{\pi k}}\,\sin\left(kr + \frac{1}{k}\ln r + \sigma_c\right)$$

Summation over m' and averaging over m $|z_{fi}|^2$ gives us

$$\frac{1}{2l+1}\sum_{m\,m'}|z_{fi}|^2 = |r_{fi}|^2\,\frac{1}{3}\,\left|C^{l'0}_{l010}\right|^2 \Rightarrow \sigma_{PI} = \frac{4\pi^2 e^2\omega}{3c}\left|C^{l'0}_{l010}\right|^2 |r_{fi}|^2$$